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# Micro–macro asymptotic approach applied to heterogeneous elastic micropolar media. Analysis of some examples

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In this work, the derivation of the effective properties for heterogeneous micropolar media with periodic structure using the two-scale asymptotic homogenization method (AHM) is reported. Analytical expressions for the local problems and the effective coefficients are explicitly described. As a particular case, periodic laminated composites are also analyzed, focusing on centro-symmetric Cosserat composites with isotropic and cubic constituents. Also, closed-form formulae of the effective properties are obtained for both constituent symmetries, and numerical values are reported and discussed. The resulting composite belongs to the orthotropic symmetry under rotations of  $90^\circ$  about the unitary vector  $e_3$ , i.e., it has eighteen effective independent properties: nine stiffness and nine torques. As a limit case, a comparison between classical and Cosserat effective elastic properties is shown for a laminated composite with isotropic constituents. Finally, the engineering moduli of centro-symmetric laminated Cosserat materials with isotropic and cubic constituents are reported, and the numerical values are analyzed.

## 1. Introduction

Nowadays, the development of macro–micro-mechanical models capable of describing the structure-properties relationship of heterogeneous complex materials plays an important role. The *a priori* estimation of the global material response is of great help for different engineering applications. It illustrates the ways the micro-structure, the coupling effects, the constituent parameters, and the volume percentage of their phases, among others, can be manipulated to obtain appropriated properties.

In the framework of the generalized continuum, micropolar or Cosserat media with coupled stresses, a series of works have addressed the estimation of effective properties of heterogeneous Cosserat materials using linear or nonlinear micro-continuum models (Lazar and Kirchner, 2005; Trovalusci et al., 2015; Abreu et al., 2018; Rueger et al., 2019; Rizzi et al., 2021; Nika, 2021). These works have emerged from the theory developed by the Cosserat brothers at the beginning of the last century (Cosserat and Cosserat, 1909), who extended Voigt’s work

on generalized non-symmetric elasticity theory Voigt (1887). Cosserat continua have been seen in granular and fibrous composite materials (Lakes, 2001; Bleyer, 2018), cellular and bone structures (Park and Lakes, 1986; Lakes et al., 1990; Lakes, 1995; Rosenber et al., 2002; Tekoglu and Onck, 2005; Liu and Su, 2009; Beltran-Fernández et al., 2010), foams (Diebels and Steeb, 2002; Rueger and Lakes, 2016, 2019; Skrzat and Eremeyev, 2020), masonry (Masiani and Trovalusci, 1996; Stefanou et al., 2008; Trovalusci and Pau, 2014; Leonetti et al., 2019), metamaterials (Forest et al., 2001), among others. State of the art, reviews and the basis foundations related to micropolar and generalized coupled stress theories are found in Toupin (1962), Eringen (1966, 1999), Nowacki (1974, 1986), Maugin and Metrikine (2010), Markert (2011), Altenbach and Eremeyev (2013), Eremeyev et al. (2013) and Maugin (2013).

In the scientific literature, there exist some papers in which the homogenization theory is applied to generalized continua or Cosserat media. For example: An homogenization scheme based on polynomial

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expansion is proposed by Forest and Sab (1998) in order to compute the effective properties of 2D Cauchy medium by minimizing the elastic strain energy with respect to displacement fields. The construction of an effective generalized continuum model replacing the heterogeneous Cauchy medium by a homogeneous Cosserat continuum is reported by Forest (1998) and Forest and Sab (1998). Later, in Forest et al. (2001), a heterogeneous linear elastic Cosserat media with periodic microstructure is analyzed using a multiscale homogenization method. In particular, two schemes are implemented depending on the hierarchy of three characteristic lengths: the size  $l$  of the unit cell, the Cosserat intrinsic length  $l_c$  of the constituents, and the characteristic length  $L$  of the composite. Extensions of Forest et al. (2001) work are proposed in subsequent studies to construct an effective generalized continuum Forest (2002), Forest and Trinh (2011) and a discrete Cosserat media Sab and Pradel (2009). In this last contribution, the homogenization of beam lattice is addressed analogous to the homogenization of discrete particles media. Cosserat composite materials are also studied by homogenization schemes in Trovalusci and Masiani (2003) and Liu and Hu (2004). Other approaches based on the various homogenization procedures are also implemented to find the micropolar moduli, see for instance (Ehlers et al., 2003; Larsson and Diebels, 2007; Larsson and Zhang, 2007; Branke et al., 2009). On the other hand, Bigoni and Drugan (2007) derived closed-form formulas for heterogeneous Cosserat-elastic materials via homogenization, where a dilute suspension of inclusions (spherical in 3D and circular cylindrical in 2D) embedded in an isotropic matrix is assumed. Further, in Altenbach et al. (2010), 1D Cosserat beam models and 2D Mindlin plate models are considered by different homogenization techniques. Li et al. (2011) employed the generalized Hill's lemma for micro-macro homogenization modeling of heterogeneous gradient-enhanced Cosserat continuum. In the works of Dos Reis and Ganghoffer (2011, 2012) and Goda et al. (2012, 2013, 2014), micropolar anisotropic constitutive models are constructed for repetitive lattices and trabecular structures by the asymptotic homogenization techniques. Gorbachev and Emel'yanov (2014) and Gorbachev and Emel'yanov (2021) used an integral formulation for the displacements and microrotations as a constructive method for the homogenization process in a heterogeneous Cosserat body. Recently, homogenization methods have been applied to disordered Cosserat-type materials without assuming any spatial periodicity of the microstructures (Trovalusci et al., 2015) and a broad class of architected materials and chiral Cosserat composites subject to such micropolar effects (Reda et al., 2021; Alavi et al., 2021).

In the present work, the formal description of the two-scale asymptotic homogenization method (AHM) implemented for periodic heterogeneous elastic media by Bakhvalov and Panasenko (1989), Pobedrya (1984), Sanchez-Palencia (1985) and Castillero et al. (1998) is extended to linear elastic Cosserat media. The AHM procedure is developed to obtain the statements of the local problems and the homogenized problem, as well as the effective properties for the 3D linear elastic Cosserat media. In addition, the analytical expressions of the effective coefficients are reported for infinite multi-laminated Cosserat composites with laminate distribution perpendicular to the  $x_3$  axis and for centro-symmetric laminated Cosserat composites. The local problem solutions are characterized by the volume fraction and the properties of the constituents. Numerical results are shown and discussed for two examples of centro-symmetric bi-laminated composites with different symmetries of constituents: isotropic and cubic. The relationships between the micropolar and the classical effective moduli are established when isotropic constituents are assumed. Finally, the formulas of the effective engineering moduli of laminated Cosserat materials with isotropic and cubic constituent materials are reported.

The main contributions of this work are focused on the development of a fully AHM scheme to find the effective properties of periodic 3D elastic Cosserat media that are not necessarily restricted to the centro-symmetric criteria through double scale asymptotic expansions (micro and macro scales) for the displacements and the microrotations in terms

of the small parameter  $\epsilon$ , see Eq. (6). In particular, centro-symmetric multi-laminated Cosserat composites with isotropic and cubic constituents are studied. Gorbachev and Emel'yanov (2014) developed a similar procedure, but they applied an integral formulation for the solution of static and elastic boundary-value problems on heterogeneous bodies for displacements and microrotations. In the present work, numerical results, engineering constants, comparison between micropolar and classic elastic media, derivation of effective coefficients for laminated with isotropic and cubic constituents are presented, which are not provided in Gorbachev and Emel'yanov (2014). Furthermore, the effective properties reported here differ from those reported by Forest et al. (2001) since they assume centro-symmetric materials considering that the coupling moduli ( $B_{ijpq}$ ) are zeros. Both approaches are similar if we consider centro-symmetric constituents where the constitutive relation reported in Eq. (3) is relaxed or constrained, and the effective properties match with those reported in Forest et al. (2001). In this sense, the present work can be considered as a generalization or an extension of the two papers mentioned above.

## 2. Heterogeneous problem formulation and basic equations for micropolar media. Statement of the problem

A three-dimensional micropolar continuum (Cosserat continuum) is considered as a periodic domain  $\Omega$  with an infinitely smooth boundary surface  $\partial\Omega$  in the Cartesian coordinate system  $\{x_1, x_2, x_3\}$ . For a linear heterogeneous micropolar continuum  $\Omega$ , the governing equations are defined by a system of partial differential equations through the linear and angular equilibrium equations,

$$\sigma_{ji,j} + f_i = 0, \quad \mu_{ji,j} + \epsilon_{ijk} \sigma_{jk} + g_i = 0, \quad \text{in } \Omega, \quad (1)$$

together with the boundary conditions on  $\partial\Omega$

$$u_i |_{\partial\Omega_1} = 0, \quad \sigma_{ji} n_j |_{\partial\Omega_2} = F_i, \quad \omega_i |_{\partial\Omega_3} = 0, \quad \mu_{ji} n_j |_{\partial\Omega_4} = G_i, \quad (2)$$

where  $i, j, k = 1, 2, 3$  and the subsets  $\partial\Omega_1, \partial\Omega_2, \partial\Omega_3$ , and  $\partial\Omega_4$  of the  $\partial\Omega$  boundary partition are disjoint, such as,  $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3 \cup \partial\Omega_4$ . Here,  $\sigma_{ji}$ ,  $\mu_{ji}$  and  $\epsilon_{ijk}$  are the components of the stress tensor, the couple stress tensor, and the Levi-Civita tensor, respectively.  $f_i$  and  $g_i$  define the components of the body force and moment, respectively. In addition,  $u = u_i$  is the displacement field vector and  $\omega = \omega_i$  is the microrotation field vector, independent of the displacement field. Also,  $n_j$  is the unit outer normal vector to  $\partial\Omega$  and the functions  $F_i$  and  $G_i$  are the components of the surface forces and torques, respectively. The comma notation represents the partial derivate relative to the  $x_j$  component.

Furthermore, the linear constitutive equations are defined by

$$\sigma_{ji} = C_{ijmn} e_{nm} + B_{ijmn} \psi_{nm}, \quad \mu_{ji} = B_{ijmn} e_{nm} + D_{ijmn} \psi_{nm}, \quad (3)$$

where  $C_{ijmn}$ , and  $D_{ijmn}$  ( $m, n = 1, 2, 3$ ) are the elastic and torque moduli, respectively, with 45 components each and  $B_{ijmn}$  is the coupling moduli with 81 components. The second order tensors  $e_{nm}$  and  $\psi_{nm}$  represent the asymmetric strain and the couple strain, respectively. Also, in Eq. (3), the material properties  $C_{ijmn}$ ,  $D_{ijmn}$  and  $B_{ijmn}$  are supposed to be infinitely differentiable, rapidly oscillating and satisfy the symmetry conditions

$$C_{ijmn} = C_{mnij}, \quad D_{ijmn} = D_{mnij}. \quad (4)$$

In addition, the micropolar deformations are fully described by the asymmetric strain ( $e_{nm}$ ) and the couple strain ( $\psi_{nm}$ ) tensors, namely

$$e_{nm} = u_{m,n} + \epsilon_{mns} \omega_s, \quad \psi_{nm} = \omega_{m,n}. \quad (5)$$

Eq. (1) together with the boundary conditions given in Eq. (2) and Eqs. (3)–(5), define the classical boundary value problem associated with the linear theory of micropolar elasticity, whose coefficients are rapidly oscillating. Fundamentals of micropolar elasticity theory can be seen in Toupin (1962), Nowacki (1974), Altenbach and Eremeyev (2013) and Eremeyev et al. (2013).

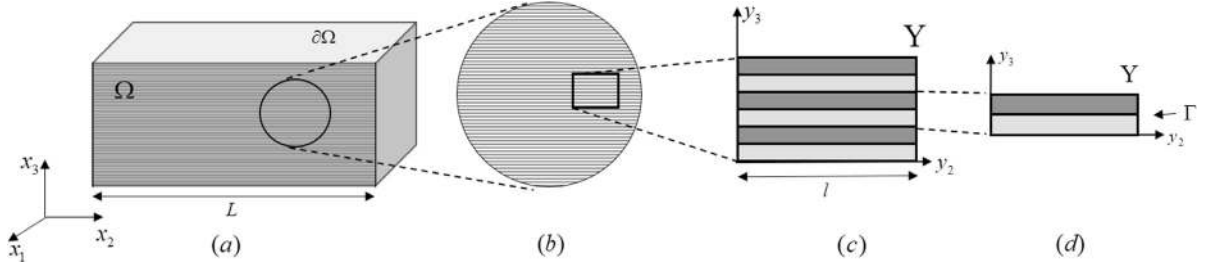


Fig. 1. (a) Heterogeneous laminated Cosserat composite; (b) blow-up of periodic structure; (c) the cross-section of the periodic structure  $Y$  at the plane  $O_{y_2y_3}$  of a laminated composite.

### 3. Asymptotic homogenization method: Local problems, homogeneous problem and effective coefficients

The homogenized local problems over the periodic unit cell  $Y$ , the homogeneous problem and the effective properties of a Cosserat media are derived from Eqs. (1)–(5) by means of the well-known AHM (Sanchez-Palencia, 1980, 1985; Pobedrya, 1984; Bakhvalov and Panasenko, 1989) through two-scale asymptotic expansion for  $u_m$  and  $\omega_m$ , as follows

$$\begin{aligned} u_m &= \varepsilon^0 u_m^{(0)}(\mathbf{x}, \mathbf{y}) + \varepsilon^1 u_m^{(1)}(\mathbf{x}, \mathbf{y}) + \varepsilon^2 u_m^{(2)}(\mathbf{x}, \mathbf{y}) + \dots, \\ \omega_m &= \varepsilon^0 \omega_m^{(0)}(\mathbf{x}, \mathbf{y}) + \varepsilon^1 \omega_m^{(1)}(\mathbf{x}, \mathbf{y}) + \varepsilon^2 \omega_m^{(2)}(\mathbf{x}, \mathbf{y}) + \dots, \end{aligned} \quad (6)$$

where the terms  $u_m^{(i)}(\mathbf{x}, \mathbf{y})$  and  $\omega_m^{(i)}(\mathbf{x}, \mathbf{y})$  ( $i = 0, 1, 2, \dots$ ) are infinitely differentiable functions and  $Y$ -periodic functions with respect to  $\mathbf{y}$ . The superscript ( $i$ ) denotes the  $i$ th term in the expansions. In addition, the two scales,  $\mathbf{x} = \{x_1, x_2, x_3\}$  (macro or slow variable) and  $\mathbf{y} = \{y_1, y_2, y_3\}$  (micro or fast variable) characterize the macroscopic or global behavior of the composite and the heterogeneities at microscopic or local level, respectively. Both scales are related by  $\mathbf{y} = \mathbf{x}/\varepsilon$ , where  $\varepsilon = l/L \ll 1$  is a small geometric parameter (see, Fig. 1) which defines the ratio between the characteristic dimension of the representative volume element ( $l$ ) and the representative length of composite ( $L$ ) used to measure the composite's properties of interest. Also, as a natural process of homogenization, the material moduli  $C_{ijmn}(\mathbf{y})$ ,  $B_{ijmn}(\mathbf{y})$  and  $D_{ijmn}(\mathbf{y})$  are functions on the local variable, which means that there is an intrinsic dependency of the material moduli on the scale parameter  $\varepsilon$ , and therefore on  $l$ , but for simplicity, in the present work we write  $C_{ijmn}$ ,  $B_{ijmn}$  and  $D_{ijmn}$ . Notice that, in Forest et al. (2001), the homogenization theory is applied to centro-symmetric micropolar composites using two schemes, denoted by HS1 and HS2. The scheme HS1 considers a Cosserat length  $l_c$  and it is used a homogenization scheme as  $l_c \sim l \ll L$ , whereas the scheme HS2 is applied when  $l_c \sim L$ . That is, HS2 corresponds to the situation we are dealing with.

Because of the scales separation, we have that,

$$\frac{\partial f^\varepsilon(\mathbf{x}, \mathbf{y})}{\partial x_j} = f(\mathbf{x}, \mathbf{y})_{,j} + \varepsilon^{-1} f(\mathbf{x}, \mathbf{y})_{|j}, \quad (7)$$

where  $f(\mathbf{x}, \mathbf{y})_{,j} = \partial f(\mathbf{x}, \mathbf{y})/\partial x_j$  and  $f(\mathbf{x}, \mathbf{y})_{|j} = \partial f(\mathbf{x}, \mathbf{y})/\partial y_j$ .

Then, applying Eq. (7) into  $u_m$  and  $\omega_m$  (see, Eq. (6)) we have

$$\begin{aligned} u_{m,n}(\mathbf{x}) &= \varepsilon^{-1} u_{m|n}^{(0)}(\mathbf{x}, \mathbf{y}) + \sum_{i=0}^{+\infty} \varepsilon^i \left[ u_{m,n}^{(i)}(\mathbf{x}, \mathbf{y}) + u_{m|n}^{(i+1)}(\mathbf{x}, \mathbf{y}) \right], \\ \omega_{m,n}(\mathbf{x}) &= \varepsilon^{-1} \omega_{m|n}^{(0)}(\mathbf{x}, \mathbf{y}) + \sum_{i=0}^{+\infty} \varepsilon^i \left[ \omega_{m,n}^{(i)}(\mathbf{x}, \mathbf{y}) + \omega_{m|n}^{(i+1)}(\mathbf{x}, \mathbf{y}) \right], \end{aligned} \quad (8)$$

and in  $C_{ijmn}(\mathbf{y})$ ,  $B_{ijmn}(\mathbf{y})$  and  $D_{ijmn}(\mathbf{y})$  we get

$$C_{ijmn,j} = \varepsilon^{-1} C_{ijmn|j}, \quad B_{ijmn,j} = \varepsilon^{-1} B_{ijmn|j}, \quad D_{ijmn,j} = \varepsilon^{-1} D_{ijmn|j}. \quad (9)$$

because the material properties are assumed to be  $\varepsilon Y$ -periodic in  $\Omega$ .

From now on, the dependency related to  $\mathbf{x}$  and  $\mathbf{y}$  is omitted in order to simplify the expressions, unless otherwise stated.

Let us start by replacing Eqs. (3)–(6) into Eqs. (1) and (2), and then, we apply the differentiation rule (Eq. (7)) neglecting the second order

or higher terms, as a result, after grouping by the powers of  $\varepsilon$ , the explicit form of the system given in Eqs. (1) and (2) can be rewritten as

$$\begin{aligned} \varepsilon^{-2} & \left[ C_{ijmn|j} u_{m|n}^{(0)} + C_{ijmn} u_{m|n}^{(0)} + B_{ijmn|j} \omega_{m|n}^{(0)} + B_{ijmn} \omega_{m|n}^{(0)} \right] + \\ & \varepsilon^{-1} \left[ C_{ijmn|j} \left( u_{m,n}^{(0)} + u_{m|n}^{(1)} + \varepsilon_{mns} \omega_s^{(0)} \right) \right. \\ & \left. + C_{ijmn} \left( 2u_{m|n,j}^{(0)} + u_{m|n,j}^{(1)} + \varepsilon_{mns} \omega_{s|j}^{(0)} \right) + \right. \\ & \left. B_{ijmn|j} \left( \omega_{m,n}^{(0)} + \omega_{m|n}^{(1)} \right) + B_{ijmn} \left( 2\omega_{m|n,j}^{(0)} + \omega_{m|n,j}^{(1)} \right) \right] + \\ \varepsilon^0 & \left[ C_{ijmn} \left[ u_{m,n}^{(0)} + 2u_{m|n,j}^{(1)} + u_{m|n,j}^{(2)} + \varepsilon_{mns} \left( \omega_{s,j}^{(0)} + \omega_{s|j}^{(1)} \right) \right] \right. \\ & \left. + B_{ijmn|j} \left( \omega_{m,n}^{(1)} + \omega_{m|n}^{(2)} \right) + \right. \\ & \left. C_{ijmn|j} \left( u_{m,n}^{(1)} + u_{m|n}^{(2)} + \varepsilon_{mns} \omega_s^{(1)} \right) \right. \\ & \left. + B_{ijmn} \left( \omega_{m,n}^{(0)} + 2\omega_{m|n,j}^{(1)} + \omega_{m|n,j}^{(2)} \right) + f_i \right] = 0, \end{aligned} \quad (10)$$

and

$$\begin{aligned} \varepsilon^{-2} & \left[ B_{ijmn|j} u_{m|n}^{(0)} + B_{ijmn} u_{m|n}^{(0)} + D_{ijmn|j} \omega_{m|n}^{(0)} + D_{ijmn} \omega_{m|n}^{(0)} \right] + \\ & \varepsilon^{-1} \left[ B_{ijmn|j} \left( u_{m,n}^{(0)} + u_{m|n}^{(1)} + \varepsilon_{mns} \omega_s^{(0)} \right) \right. \\ & \left. + B_{ijmn} \left( 2u_{m|n,j}^{(0)} + u_{m|n,j}^{(1)} + \varepsilon_{mns} \omega_{s|j}^{(0)} \right) + \right. \\ & \left. D_{ijmn|j} \left( \omega_{m,n}^{(0)} + \omega_{m|n}^{(1)} \right) + D_{ijmn} \left( 2\omega_{m|n,j}^{(0)} + \omega_{m|n,j}^{(1)} \right) \right. \\ & \left. + \varepsilon_{ijk} \left( C_{kjmn} u_{m|n}^{(0)} + B_{kjmn} \omega_{m|n}^{(0)} \right) \right] + \\ \varepsilon^0 & \left[ B_{ijmn} \left[ u_{m,n}^{(0)} + 2u_{m|n,j}^{(1)} + u_{m|n,j}^{(2)} + \varepsilon_{mns} \left( \omega_{s,j}^{(0)} + \omega_{s|j}^{(1)} \right) \right] \right. \\ & \left. + D_{ijmn|j} \left( \omega_{m,n}^{(1)} + \omega_{m|n}^{(2)} \right) + \right. \\ & \left. B_{ijmn|j} \left( u_{m,n}^{(1)} + u_{m|n}^{(2)} + \varepsilon_{mns} \omega_s^{(1)} \right) + D_{ijmn} \left( \omega_{m,n}^{(0)} + 2\omega_{m|n,j}^{(1)} + \omega_{m|n,j}^{(2)} \right) \right. \\ & \left. + \varepsilon_{ijk} C_{kjmn} \left( u_{m,n}^{(0)} + u_{m|n}^{(1)} + \varepsilon_{mns} \omega_s^{(0)} \right) + \varepsilon_{ijk} B_{kjmn} \left( \omega_{m,n}^{(0)} + \omega_{m|n}^{(1)} \right) + g_i \right] = 0. \end{aligned} \quad (11)$$

where  $f_{m,nj} = \frac{\partial^2 f_m}{\partial x_j \partial x_n}$ ,  $f_{m|n,j} = \frac{\partial^2 f_m}{\partial x_j \partial y_n}$ ,  $f_{m,n|j} = \frac{\partial^2 f_m}{\partial y_j \partial x_n}$ , and  $f_{m|n|j} = \frac{\partial^2 f_m}{\partial y_j \partial y_n}$ .

From Eqs. (10) and (11), a sequence of problems defined by a system of partial differential equations arises, according to the powers of  $\varepsilon$  small parameter. Each contribution is assumed equal to zero for all powers of  $\varepsilon$ . Subsequently, the resulting problems are solved under suitable conditions in order to guarantee the  $Y$ -periodic solution. Only, the powers  $-2$ ,  $-1$  and  $0$  of  $\varepsilon$  are enough for finding the local problems, effective coefficients and the homogenized problem.

The terms corresponding to  $\varepsilon^{-2}$  can be written as a system of partial differential equations, as follows

$$\begin{aligned} \left( C_{ijmn} u_{m|n}^{(0)} + B_{ijmn} \omega_{m|n}^{(0)} \right)_{|j} &= 0, \\ \left( B_{ijmn} u_{m|n}^{(0)} + D_{ijmn} \omega_{m|n}^{(0)} \right)_{|j} &= 0. \end{aligned} \quad (12)$$

where the unknowns  $u_{m|n}^{(0)}$  and  $\omega_{m|n}^{(0)}$  are defined as a function of  $\mathbf{x}$  and  $\mathbf{y}$ . Then, it can be proved that the terms  $u_m^{(0)}$  and  $\omega_m^{(0)}$  are independent

functions of the local variable  $\mathbf{y}$  from Eq. (12), i.e.,

$$u_m^{(0)}(\mathbf{x}, \mathbf{y}) \equiv u_m(\mathbf{x}), \quad \omega_m^{(0)}(\mathbf{x}, \mathbf{y}) \equiv \omega_m(\mathbf{x}). \quad (13)$$

Similarly, the terms corresponding to  $\varepsilon^{-1}$  can be written by a system of partial differential equations that result from Eqs. (10) and (11) and considering that the derivatives of  $u_m^{(0)}(\mathbf{x})$  and  $\omega_m^{(0)}(\mathbf{x})$  are null with respect to the fast variable  $\mathbf{y}$  (see, Eq. (13)), as follows

$$\begin{aligned} & \left( C_{ijmn} u_{m|n}^{(1)} \right)_{|j} + C_{ijmn|j} \left( u_{m,n}^{(0)} + \varepsilon_{mns} \omega_s^{(0)} \right) \\ & + \left( B_{ijmn} \omega_{m|n}^{(1)} \right)_{|j} + B_{ijmn|j} \omega_{m,n}^{(0)} = 0, \\ & \left( B_{ijmn} u_{m|n}^{(1)} \right)_{|j} + B_{ijmn|j} \left( u_{m,n}^{(0)} + \varepsilon_{mns} \omega_s^{(0)} \right) \\ & + \left( D_{ijmn} \omega_{m|n}^{(1)} \right)_{|j} + D_{ijmn|j} \omega_{m,n}^{(0)} = 0. \end{aligned} \quad (14)$$

Analogous to the system in Eq. (12) related to  $\varepsilon^{-2}$ , a solution of Eq. (14) is found in the class of Y-periodic functions with respect to  $\mathbf{y}$ . Therefore, it is expected to have solutions in a similar way.

Firstly, the strains and couple strains expressions associate with the Cosserat theory of elasticity (see, Eq. (5)) are rewritten in two-scale series expansion form. So, by substituting Eq. (6) into Eq. (5), we have

$$\begin{aligned} e_{nm} &= \varepsilon^{-1} u_{m|n}^{(0)} + \varepsilon^0 \left( u_{m,n}^{(0)} + \varepsilon_{mns} \omega_s^{(0)} + u_{m|n}^{(1)} \right) \\ &+ \varepsilon \left( u_{m,n}^{(1)} + \varepsilon_{mns} \omega_s^{(1)} + u_{m|n}^{(2)} \right) + \dots, \end{aligned} \quad (15)$$

$$\psi_{nm} = \omega_{m,n} = \varepsilon^{-1} \omega_{m|n}^{(0)} + \varepsilon^0 \left( \omega_{m,n}^{(0)} + \omega_{m|n}^{(1)} \right) + \varepsilon \left( \omega_{m,n}^{(1)} + \omega_{m|n}^{(2)} \right) + \dots,$$

where  $u_{m|n}^{(0)}$  and  $\omega_{m|n}^{(0)}$  are null (see, Eq. (13)) and the  $k$ th terms ( $k \geq 0$ ) are given by

$$e_{nm}^{(k)} = u_{m|n}^{(k+1)} + u_{m,n}^{(k)} + \varepsilon_{mns} \omega_s^{(k)}, \quad \psi_{nm}^{(k)} = \omega_{m|n}^{(k+1)} + \omega_{m,n}^{(k)}. \quad (16)$$

The first terms  $e_{nm}^{(0)}$  and  $\psi_{nm}^{(0)}$  ( $k = 0$ ) are given by the expressions

$$e_{nm}^{(0)} = u_{m|n}^{(1)} + u_{m,n}^{(0)} + \varepsilon_{mns} \omega_s^{(0)} \quad \text{and} \quad \psi_{nm}^{(0)} = \omega_{m|n}^{(1)} + \omega_{m,n}^{(0)}. \quad (17)$$

Due to the linearity of the system (Eq. (14)), a solution for  $u_m^{(1)}(\mathbf{x}, \mathbf{y})$  and  $\omega_m^{(1)}(\mathbf{x}, \mathbf{y})$  is admitted in the class of Y-periodic functions with respect to  $\mathbf{y}$  through the variable separation method (see, for instance, Otero et al. (1997)), as follows:

$$\begin{aligned} u_m^{(1)}(\mathbf{x}, \mathbf{y}) &= {}_{pq} \hat{N}_m(\mathbf{y}) \left( u_{p,q}^{(0)} + \varepsilon_{pqk} \omega_k^{(0)} \right) + {}_{pq} \hat{U}_m(\mathbf{y}) \omega_{p,q}^{(0)} + \tilde{u}_m^{(1)}, \\ \omega_m^{(1)}(\mathbf{x}, \mathbf{y}) &= {}_{pq} \hat{V}_m(\mathbf{y}) \left( u_{p,q}^{(0)} + \varepsilon_{pqk} \omega_k^{(0)} \right) + {}_{pq} \hat{M}_m(\mathbf{y}) \omega_{p,q}^{(0)} + \tilde{\omega}_m^{(1)}, \end{aligned} \quad (18)$$

where  ${}_{pq} \hat{N}_m(\mathbf{y})$ ,  ${}_{pq} \hat{U}_m(\mathbf{y})$ ,  ${}_{pq} \hat{V}_m(\mathbf{y})$ , and  ${}_{pq} \hat{M}_m(\mathbf{y})$  ( $p, q = 1, 2, 3$ ) are Y-periodic functions on  $\mathbf{y}$ , which are defined as local functions. The terms  $\tilde{u}_m^{(1)}$  and  $\tilde{\omega}_m^{(1)}$  are constant vectors.

In addition, as the strains are related to microrotations whereas the couple strains are not related to the displacements, the functions  ${}_{pq} \hat{N}_{m|n}$  and  ${}_{pq} \hat{U}_{m|n}$  can be redefined as

$${}_{pq} \hat{N}_{m|n} = {}_{pq} N_{m|n} + \varepsilon_{mnk} {}_{pq} V_k, \quad {}_{pq} \hat{U}_{m|n} = {}_{pq} U_{m|n} + \varepsilon_{mnk} {}_{pq} M_k, \quad (19)$$

where the  ${}_{pq} N_m$  and  ${}_{pq} U_m$   ${}_{pq}$ -displacements and the  ${}_{pq} V_k$  and  ${}_{pq} M_k$   ${}_{pq}$ -microrotations are Y-periodic functions too.

Now, replacing Eq. (19) into Eq. (18), and then, the resulting expressions into Eq. (14) and collecting with respect to  $u_{p,q}^{(0)}$  and  $\varepsilon_{pqk} \omega_k^{(0)}$  and  $\omega_{p,q}^{(0)}$ , we have

$$\begin{aligned} & \left[ C_{ijpq} + C_{ijmn} ({}_{pq} N_{m|n} + \varepsilon_{mnk} {}_{pq} V_k) + B_{ijmn} {}_{pq} V_{m|n} \right]_{|j} \left( u_{p,q}^{(0)} + \varepsilon_{pqk} \omega_k^{(0)} \right) + \\ & \left[ B_{ijpq} + C_{ijmn} ({}_{pq} U_{m|n} + \varepsilon_{mnk} {}_{pq} M_k) + B_{ijmn} {}_{pq} M_{m|n} \right]_{|j} \omega_{p,q}^{(0)} = 0, \\ & \left[ B_{ijpq} + B_{ijmn} ({}_{pq} N_{m|n} + \varepsilon_{mnk} {}_{pq} V_k) + D_{ijmn} {}_{pq} V_{m|n} \right]_{|j} \left( u_{p,q}^{(0)} + \varepsilon_{pqk} \omega_k^{(0)} \right) + \\ & \left[ D_{ijpq} + B_{ijmn} ({}_{pq} U_{m|n} + \varepsilon_{mnk} {}_{pq} M_k) + D_{ijmn} {}_{pq} M_{m|n} \right]_{|j} \omega_{p,q}^{(0)} = 0. \end{aligned} \quad (20)$$

Next, as  $u_{p,q}^{(0)} + \varepsilon_{pqk} \omega_k^{(0)}$  and  $\omega_{p,q}^{(0)}$  are not null in Eq. (20), then, the strains satisfy

$$\begin{aligned} & \left( C_{ijpq} + C_{ijmn} ({}_{pq} N_{m|n} + \varepsilon_{mnk} {}_{pq} V_k) + B_{ijmn} {}_{pq} V_{m|n} \right)_{|j} = 0, \\ & \left( B_{ijpq} + B_{ijmn} ({}_{pq} U_{m|n} + \varepsilon_{mnk} {}_{pq} M_k) + D_{ijmn} {}_{pq} V_{m|n} \right)_{|j} = 0, \end{aligned} \quad (21)$$

and the couple strains

$$\begin{aligned} & \left( B_{ijpq} + C_{ijmn} ({}_{pq} U_{m|n} + \varepsilon_{mnk} {}_{pq} M_k) + B_{ijmn} {}_{pq} M_{m|n} \right)_{|j} = 0, \\ & \left( D_{ijpq} + B_{ijmn} ({}_{pq} U_{m|n} + \varepsilon_{mnk} {}_{pq} M_k) + D_{ijmn} {}_{pq} M_{m|n} \right)_{|j} = 0. \end{aligned} \quad (22)$$

Equations (21) and (22) are the problems on the periodic cell Y related to the micropolar theory of elasticity (so-called local problems), which we denote as  ${}_{pq} \mathcal{L}^1$  and  ${}_{pq} \mathcal{L}^2$ , respectively. The solutions of the local problems are important for the computation of the effective properties.

From Eqs. (10) and (11), the terms corresponding to  $\varepsilon^0$  can be rewritten as a system of partial differential equations, as follows

$$\begin{aligned} & \left( C_{ijmn} u_{m|n}^{(2)} + C_{ijmn} u_{m,n}^{(1)} + C_{ijmn} \varepsilon_{mns} \omega_s^{(1)} \right)_{|j} \\ & + \left( B_{ijmn} \omega_{m|n}^{(2)} + B_{ijmn} \omega_{m,n}^{(1)} \right)_{|j} + \\ & C_{ijmn} u_{m|n,j}^{(1)} + C_{ijmn} \left( u_{m,n,j}^{(0)} + \varepsilon_{mns} \omega_{s,j}^{(0)} \right) \\ & + B_{ijmn} \omega_{m|n,j}^{(1)} + B_{ijmn} \omega_{m,n,j}^{(0)} + f_i = 0, \\ & \left( B_{ijmn} u_{m|n}^{(2)} + B_{ijmn} u_{m,n}^{(1)} + B_{ijmn} \varepsilon_{mns} \omega_s^{(1)} \right)_{|j} \\ & + \left( D_{ijmn} \omega_{m|n}^{(2)} + D_{ijmn} \omega_{m,n}^{(1)} \right)_{|j} + \\ & B_{ijmn} u_{m|n,j}^{(1)} + B_{ijmn} \left( u_{m,n,j}^{(0)} + \varepsilon_{mns} \omega_{s,j}^{(0)} \right) + D_{ijmn} \omega_{m|n,j}^{(1)} + D_{ijmn} \omega_{m,n,j}^{(0)} + \\ & \varepsilon_{ijk} \left[ C_{kjmn} u_{m|n}^{(1)} + C_{kjmn} \left( u_{m,n}^{(0)} + \varepsilon_{mns} \omega_s^{(0)} \right) \right. \\ & \left. + B_{kjmn} \omega_{m|n}^{(1)} + B_{kjmn} \omega_{m,n}^{(0)} \right] + g_i = 0, \end{aligned} \quad (23)$$

then, applying the average operator  $\langle \bullet \rangle_Y = \int_Y (\bullet) d\mathbf{y}$  into Eq. (23) and considering the y-periodicity of the involved functions, it yields

$$\begin{aligned} & \left\langle C_{ijmn} u_{m|n,j}^{(1)} + C_{ijmn} \left( u_{m,n,j}^{(0)} + \varepsilon_{mns} \omega_{s,j}^{(0)} \right) \right. \\ & \left. + B_{ijmn} \omega_{m|n,j}^{(1)} + B_{ijmn} \omega_{m,n,j}^{(0)} \right\rangle_Y + f_i = 0, \\ & \left\langle B_{ijmn} u_{m|n,j}^{(1)} + B_{ijmn} \left( u_{m,n,j}^{(0)} + \varepsilon_{mns} \omega_{s,j}^{(0)} \right) \right. \\ & \left. + D_{ijmn} \omega_{m|n,j}^{(1)} + D_{ijmn} \omega_{m,n,j}^{(0)} \right\rangle_Y + \\ & \varepsilon_{ijk} \left\langle C_{kjmn} u_{m|n}^{(1)} + C_{kjmn} \left( u_{m,n}^{(0)} + \varepsilon_{mns} \omega_s^{(0)} \right) \right. \\ & \left. + B_{kjmn} \omega_{m|n}^{(1)} + B_{kjmn} \omega_{m,n}^{(0)} \right\rangle_Y + g_i = 0. \end{aligned} \quad (24)$$

Finally, replacing  $u_m^{(1)}$  and  $\omega_m^{(1)}$  (see, Eq. (18)) into Eq. (24), and grouping terms conveniently, we obtain the homogenized system, as follows

$$\begin{aligned} & C_{ijpq}^* \left( u_{p,q}^{(0)} + \varepsilon_{pqk} \omega_k^{(0)} \right)_j + B_{ijpq}^* \omega_{p,q}^{(0)} + f_i = 0, \\ & B_{ijpq}^* \left( u_{p,q}^{(0)} + \varepsilon_{pqk} \omega_k^{(0)} \right)_j + D_{ijpq}^* \omega_{p,q}^{(0)} \\ & + \varepsilon_{ijl} \left[ C_{ljpq}^* \left( u_{p,q}^{(0)} + \varepsilon_{pqk} \omega_k^{(0)} \right) + B_{ljpq}^* \omega_{p,q}^{(0)} \right] + g_i = 0, \end{aligned} \quad (25)$$

where  $u_p^{(0)}$  and  $\omega_p^{(0)}$  are the system solution, and the coefficients  $C_{ijpq}^*$ ,  $B_{ijpq}^*$  and  $D_{ijpq}^*$  represent the effective properties of a periodic Cosserat medium, which are defined as follows

$$C_{ijpq}^* = \langle C_{ijpq} + C_{ijmn} ({}_{pq} N_{m|n} + \varepsilon_{mnk} {}_{pq} V_k) + B_{ijmn} {}_{pq} V_{m|n} \rangle_Y, \quad (26)$$

$$B_{ijpq}^* = \langle B_{ijpq} + C_{ijmn} ({}_{pq} U_{m|n} + \varepsilon_{mnk} {}_{pq} M_k) + B_{ijmn} {}_{pq} M_{m|n} \rangle_Y, \quad (27)$$

$$B_{ijpq}^* = \langle B_{ijpq} + B_{ijmn} ({}_{pq} N_{m|n} + \varepsilon_{mnk} {}_{pq} V_k) + D_{ijmn} {}_{pq} V_{m|n} \rangle_Y, \quad (28)$$

$$D_{ijpq}^* = \langle D_{ijpq} + B_{ijmn} ({}_{pq} U_{m|n} + \varepsilon_{mnk} {}_{pq} M_k) + D_{ijmn} {}_{pq} M_{m|n} \rangle_Y. \quad (29)$$

The effective properties (Eqs. (26)–(29)) coincide with those reported by Gorbachev and Emel'yanov (2014), see, Eqs. (3.27)–(3.29), page 77, and also with Forest et al. (2001), see Eq. (46), page 4594, for

centro-symmetric micropolar materials. The latter is obtained when the constitutive relations (Eq. (3)) are constrained to the case  $B_{ijpq} = 0$ , then Eqs. (26)–(29) become

$$C_{ijpq}^* = \langle C_{ijpq} + C_{ijmn} N_{m|n} \rangle_Y, \quad D_{ijpq}^* = \langle D_{ijpq} + D_{ijmn} M_{m|n} \rangle_Y.$$

Herein, it is important to note that the homogenized system (Eq. (25)) subject to the boundary conditions

$$\begin{aligned} u_p^{(0)}|_{\partial\bar{\Omega}_1} = 0, \quad \sigma_{ji}^{(0)} n_j|_{\partial\bar{\Omega}_2} = F_i^{(0)}, \quad \omega_p^{(0)}|_{\partial\bar{\Omega}_3} = 0, \\ \mu_{ji}^{(0)} n_j|_{\partial\bar{\Omega}_4} = G_i^{(0)}, \end{aligned} \quad (30)$$

where  $i, j, k, p = \overline{1, 2, 3}$ , represents the homogenized problem formulation, defined on  $\bar{\Omega}$ , which is equivalent to the boundary value problem (Eqs. (1)–(2)) of a periodic Cosserat media. Here,  $F_i^{(0)}$  and  $G_i^{(0)}$  are infinitely differential functions and  $\partial\bar{\Omega} = \partial\bar{\Omega}_1 \cup \partial\bar{\Omega}_2 \cup \partial\bar{\Omega}_3 \cup \partial\bar{\Omega}_4$ .

In addition, the effective properties formulation (Eqs. (26)–(29)) depend on the local  $pq$ -displacements ( ${}_{pq}N_m$  and  ${}_{pq}U_m$ ) and the local  $pq$ -microrotations ( ${}_{pq}V_m$  and  ${}_{pq}M_m$ ) relative to the  ${}_{pq}\mathcal{L}^1$  and  ${}_{pq}\mathcal{L}^2$  local problems, therefore, they need to be determined.

This way, the  ${}_{pq}\mathcal{L}^1$  and  ${}_{pq}\mathcal{L}^2$  local problems are given by the systems Eqs. (21) and (22) subject to the corresponding homogenized perfect contact conditions and null average conditions for the local functions, respectively, as defined below:

The  ${}_{pq}\mathcal{L}^1$  local problems allow to find the  $y$ -periodic local functions  ${}_{pq}N_m$  and  ${}_{pq}V_m$  in the periodic cell  $Y$  through the solution of the problem:

$$\begin{aligned} (C_{ijpq} + C_{ijmn} ({}_{pq}N_{m|n} + \epsilon_{mnk} {}_{pq}V_k) + B_{ijmn} {}_{pq}V_{m|n})|_j = 0, \quad \text{in } Y, \\ (B_{ijpq} + B_{ijmn} ({}_{pq}N_{m|n} + \epsilon_{mnk} {}_{pq}V_k) + D_{ijmn} {}_{pq}V_{m|n})|_j = 0, \quad \text{in } Y, \\ [[{}_{pq}N_m]] = 0, \quad [[{}_{pq}V_m]] = 0, \quad \text{over } \Gamma, \\ [[{}_{pq}\sigma_{3i}^1 n_j]] = - [[C_{ijpq}]] n_j, \quad [[{}_{pq}\mu_{ji}^1 n_j]] = - [[B_{ijpq}]] n_j, \quad \text{over } \Gamma, \end{aligned} \quad (31)$$

where

$$\begin{aligned} {}_{pq}\sigma_{ji}^1 = C_{ijmn} ({}_{pq}N_{m|n} + \epsilon_{mnk} {}_{pq}V_k) + B_{ijmn} {}_{pq}V_{m|n}, \\ {}_{pq}\mu_{ji}^1 = B_{ijmn} ({}_{pq}N_{m|n} + \epsilon_{mnk} {}_{pq}V_k) + D_{ijmn} {}_{pq}V_{m|n}. \end{aligned}$$

Analogously, in the  ${}_{pq}\mathcal{L}^2$  local problems, the  $y$ -periodic local functions  ${}_{pq}U_m$  and  ${}_{pq}M_m$  are sought in the periodic cell  $Y$ , which result from the problem:

$$\begin{aligned} (B_{ijpq} + C_{ijmn} ({}_{pq}U_{m|n} + \epsilon_{mnk} {}_{pq}M_k) + B_{ijmn} {}_{pq}M_{m|n})|_j = 0, \quad \text{in } Y, \\ (D_{ijpq} + B_{ijmn} ({}_{pq}U_{m|n} + \epsilon_{mnk} {}_{pq}M_k) + D_{ijmn} {}_{pq}M_{m|n})|_j = 0, \quad \text{in } Y, \\ [[{}_{pq}U_m]] = 0, \quad [[{}_{pq}M_m]] = 0, \quad \text{over } \Gamma, \\ [[{}_{pq}\sigma_{ji}^2 n_j]] = - [[B_{ijpq}]] n_j, \quad [[{}_{pq}\mu_{ji}^2 n_j]] = - [[D_{ijpq}]] n_j, \quad \text{over } \Gamma, \end{aligned} \quad (32)$$

where

$$\begin{aligned} {}_{pq}\sigma_{ji}^2 = C_{ijmn} ({}_{pq}U_{m|n} + \epsilon_{mnk} {}_{pq}M_k) + B_{ijmn} {}_{pq}M_{m|n}, \\ {}_{pq}\mu_{ji}^2 = B_{ijmn} ({}_{pq}U_{m|n} + \epsilon_{mnk} {}_{pq}M_k) + D_{ijmn} {}_{pq}M_{m|n}. \end{aligned}$$

Additionally, the following conditions over  $Y$  are required to guarantee the existence and uniqueness of the local problem solutions,

$$\langle {}_{pq}N_m \rangle_Y = 0, \quad \langle {}_{pq}V_m \rangle_Y = 0, \quad \langle {}_{pq}U_m \rangle_Y = 0, \quad \langle {}_{pq}M_m \rangle_Y = 0. \quad (33)$$

where  $\langle \bullet \rangle_Y = (1/|Y|) \int_Y \bullet dy$  is the average operator of  $(\bullet)$  over the periodic cell  $Y$ . Also, the double brackets symbol  $[[f]]$  denotes the jump of the function  $f$  across the interface surface  $\Gamma$  and  $n_j$  is the unit outer normal vector to  $\Gamma$ .

#### 4. Effective coefficients of multi-laminated Cosserat media

In this section, the local problems (Eqs. (31)–(33)) and the effective coefficients (Eqs. (26)–(29)) are reformulated for a heterogeneous finite periodic laminated Cosserat composite  $\Omega$  with boundary  $\partial\Omega$ . Thus, a laminate composite characterized by a parallelepiped generated by repetitions of the periodic cell  $Y$  is considered, in which the layered

direction is along the  $x_3$ -axis, see Fig. 1. Herein, the Cosserat material properties  $C_{ijkl}$ ,  $B_{ijkl}$  and  $D_{ijkl}$  only depend on the coordinate  $x_3$  and they satisfy the symmetry conditions of Eqs. (4).

The periodic cell  $Y = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : 0 \leq y_i \leq l_i\}$  with  $i = 1, 2, 3$  is assumed to be a bi-laminated composite, where  $l_i$  is the cell length in the  $y_i$  direction and  $L$  is the plate thickness. The interface region  $\Gamma$  between the layers is considered perfect, i.e., the layers are in welded contact so that the displacement, stress, microrotation and couple stress are continuous across the interface.

In this framework, the unknown local functions  ${}_{pq}N_m$ ,  ${}_{pq}U_m$ ,  ${}_{pq}V_m$ , and  ${}_{pq}M_m$  only depend on  $y_3$  as well. Therefore, the local problems (Eqs. (31)–(33)) in  $Y$  turn into a system of ordinary integro-differential equation under perfect contact conditions, as can be seen below:

The  ${}_{pq}\mathcal{L}^1$  local problems

$$\begin{aligned} (C_{i3pq} + C_{i3m3} {}_{pq}N'_m + C_{i3mn} \epsilon_{mnk} {}_{pq}V_k + B_{i3m3} {}_{pq}V'_m)' = 0, \quad \text{in } Y, \\ (B_{i3pq} + B_{i3m3} {}_{pq}N'_m + B_{i3mn} \epsilon_{mnk} {}_{pq}V_k + D_{i3m3} {}_{pq}V'_m)' = 0, \quad \text{in } Y, \\ [[{}_{pq}N_m]] = 0, \quad [[{}_{pq}V_m]] = 0 \quad \text{over } \Gamma, \\ [[{}_{pq}\sigma_{3i}^1 n_j]] = - [[C_{i3pq}]] n_3, \quad [[{}_{pq}\mu_{3i}^1 n_j]] = - [[B_{i3pq}]] n_3 \quad \text{over } \Gamma, \end{aligned} \quad (34)$$

where  ${}_{pq}\sigma_{3i}^1 = C_{i3m3} {}_{pq}N'_m + C_{i3mn} \epsilon_{mnk} {}_{pq}V_k + B_{i3m3} {}_{pq}V'_m$  and  ${}_{pq}\mu_{3i}^1 = B_{i3m3} {}_{pq}N'_m + B_{i3mn} \epsilon_{mnk} {}_{pq}V_k + D_{i3m3} {}_{pq}V'_m$ .

Analogously, the  ${}_{pq}\mathcal{L}^2$  local problems

$$\begin{aligned} (B_{i3pq} + C_{i3m3} {}_{pq}U'_m + C_{i3mn} \epsilon_{mnk} {}_{pq}M_k + B_{i3m3} {}_{pq}M'_m)' = 0, \quad \text{in } Y, \\ (D_{i3pq} + B_{i3m3} {}_{pq}U'_m + B_{i3mn} \epsilon_{mnk} {}_{pq}M_k + D_{i3m3} {}_{pq}M'_m)' = 0, \quad \text{in } Y, \\ [[{}_{pq}U_m]] = 0, \quad [[{}_{pq}M_m]] = 0 \quad \text{over } \Gamma, \\ [[{}_{pq}\sigma_{3i}^2 n_j]] = - [[B_{i3pq}]] n_3, \quad [[{}_{pq}\mu_{3i}^2 n_j]] = - [[D_{i3pq}]] n_3 \quad \text{over } \Gamma, \end{aligned} \quad (35)$$

where  ${}_{pq}\sigma_{3i}^2 = C_{i3mn} {}_{pq}U'_m + C_{i3mn} \epsilon_{mnk} {}_{pq}M_k + B_{i3m3} {}_{pq}M'_m$  and  ${}_{pq}\mu_{3i}^2 = B_{i3m3} {}_{pq}U'_m + B_{i3mn} \epsilon_{mnk} {}_{pq}M_k + D_{i3m3} {}_{pq}M'_m$ .

Additionally, null average conditions over  $Y$  are required to guarantee the unique solution, i.e.,  $\langle {}_{pq}N_m \rangle_Y = 0$ ,  $\langle {}_{pq}V_m \rangle_Y = 0$ ,  $\langle {}_{pq}U_m \rangle_Y = 0$  and  $\langle {}_{pq}M_m \rangle_Y = 0$ . In Eqs. (34)–(35), the prime indicate the ordinary derivative of the function with respect to  $y_3$ .

Consequently, from Eqs. (26)–(29), the corresponding effective properties for a periodic laminate Cosserat composite are

$$C_{ijpq}^* = \langle C_{ijpq} + C_{ijm3} ({}_{pq}N'_m + \epsilon_{m3k} {}_{pq}V_k) + B_{ijm3} {}_{pq}V'_m \rangle_Y, \quad (36)$$

$$B_{ijpq}^* = \langle B_{ijpq} + C_{ijm3} ({}_{pq}U'_m + \epsilon_{m3k} {}_{pq}M_k) + B_{ijm3} {}_{pq}M'_m \rangle_Y, \quad (37)$$

$$B_{ijpq}^* = \langle B_{ijpq} + B_{ijm3} ({}_{pq}N'_m + \epsilon_{m3k} {}_{pq}V_k) + D_{ijm3} {}_{pq}V'_m \rangle_Y, \quad (38)$$

$$D_{ijpq}^* = \langle D_{ijpq} + B_{ijm3} ({}_{pq}U'_m + \epsilon_{m3k} {}_{pq}M_k) + D_{ijm3} {}_{pq}M'_m \rangle_Y. \quad (39)$$

As can be seen, the analytical formulas of the effective coefficients (Eqs. (36)–(39)) depend on the local functions  ${}_{pq}N'_m$ ,  ${}_{pq}V'_m$ ,  ${}_{pq}U'_m$ ,  ${}_{pq}M'_m$ , the constituent material properties and the volumes through the average operator on the periodic unit cell  $Y$ . Therefore, once the solutions of the  ${}_{pq}\mathcal{L}^1$  and  ${}_{pq}\mathcal{L}^2$  local problems have been determined, the effective coefficients of a Cosserat laminated media can be computed by replacing the local functions and their derivatives in them. Details of the local problem solutions are shown in Appendix.

#### 4.1. Effective properties of centro-symmetric laminated Cosserat composites

Now, the analytical formulas of centro-symmetric laminated Cosserat composite are determined. A periodic Cosserat material is defined as centro-symmetric if  $-I$  ( $I$  is the second order identity tensor) is a symmetry transformation of its constitutive law. In this case,  $B_{ijrs}$  are not considered due to the symmetry condition  $B_{ijrs} = B_{rsij}$  not being fulfilled. This condition implies that the stresses and the couple stresses no longer depend on the microcurvatures and strains, respectively, because  $B_{ijrs}$  is related to the coupling between stresses and microcurvatures and between couple stresses and strains.

**Table 1**

Local problems and associated Cosserat effective properties.

$_{11}\mathcal{L}^1$	$_{22}\mathcal{L}^1$	$_{33}\mathcal{L}^1$	$_{12}\mathcal{L}^1$	$_{13}\mathcal{L}^1$	$_{23}\mathcal{L}^1$	$_{21}\mathcal{L}^1$	$_{31}\mathcal{L}^1$	$_{32}\mathcal{L}^1$
$C_{1111}^*$	$C_{1122}^*$	$C_{1133}^*$	0	0	0	0	0	0
$C_{2211}^*$	$C_{2222}^*$	$C_{2233}^*$	0	0	0	0	0	0
$C_{3311}^*$	$C_{3322}^*$	$C_{3333}^*$	0	0	0	0	0	0
0	0	0	$C_{1212}^*$	0	0	$C_{1221}^*$	0	0
0	0	0	0	$C_{1313}^*$	0	0	$C_{1331}^*$	0
0	0	0	0	0	$C_{2323}^*$	0	0	$C_{2332}^*$
0	0	0	$C_{2112}^*$	0	0	$C_{2121}^*$	0	0
0	0	0	0	$C_{3113}^*$	0	0	$C_{3131}^*$	0
0	0	0	0	0	$C_{3223}^*$	0	0	$C_{3232}^*$
$_{11}\mathcal{L}^2$	$_{22}\mathcal{L}^2$	$_{33}\mathcal{L}^2$	$_{12}\mathcal{L}^2$	$_{13}\mathcal{L}^2$	$_{23}\mathcal{L}^2$	$_{21}\mathcal{L}^2$	$_{31}\mathcal{L}^2$	$_{32}\mathcal{L}^2$
$D_{1111}^*$	$D_{1122}^*$	$D_{1133}^*$	0	0	0	0	0	0
$D_{2211}^*$	$D_{2222}^*$	$D_{2233}^*$	0	0	0	0	0	0
$D_{3311}^*$	$D_{3322}^*$	$D_{3333}^*$	0	0	0	0	0	0
0	0	0	$D_{1212}^*$	0	0	$D_{1221}^*$	0	0
0	0	0	0	$D_{1313}^*$	0	0	$D_{1331}^*$	0
0	0	0	0	0	$D_{2323}^*$	0	0	$D_{2332}^*$
0	0	0	$D_{2112}^*$	0	0	$D_{2121}^*$	0	0
0	0	0	0	$D_{3113}^*$	0	0	$D_{3131}^*$	0
0	0	0	0	0	$D_{3223}^*$	0	0	$D_{3232}^*$

Then, let us write the condition explicitly  $B_{ijpq} = 0$ , therefore applying this condition into Eq. (A.12) we have

$${}_{pq}V'_m = 0. \quad (40)$$

In order to find the local function  ${}_{pq}V_m$ , it is necessary to integrate Eq. (40) assuming that  $\langle {}_{pq}V_m \rangle = 0$ , such as

$${}_{pq}V_m = \int_0^L {}_{pq}V'_m dy - \left\langle \int_0^L {}_{pq}V'_m dy \right\rangle \equiv 0. \quad (41)$$

Taking into account  $B_{ijpq} = 0$  and Eqs. (40) and (41), the expressions Eqs. (A.11) and (A.16) become

$${}_{pq}N'_m = C_{m3l3}^{-1} \langle C_{l3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} C_{d3pq} \rangle - C_{m3l3}^{-1} C_{l3pq}, \quad (42)$$

$${}_{pq}M'_m = D_{m3l3}^{-1} \langle D_{l3k3}^{-1} \rangle \langle D_{k3d3}^{-1} D_{d3pq} \rangle - D_{m3l3}^{-1} D_{l3pq}. \quad (43)$$

Finally, replacing Eqs. (40)–(43) into the statements Eqs. (36)–(39), the effective properties of centro-symmetric micropolar laminated composites can be found as follows,

$$C_{ijpq}^* = \left\langle C_{ijpq} + C_{ijm3} C_{m3l3}^{-1} \left( \langle C_{l3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} C_{d3pq} \rangle - C_{l3pq} \right) \right\rangle_Y, \quad (44)$$

$$D_{ijpq}^* = \left\langle D_{ijpq} + D_{ijm3} D_{m3l3}^{-1} \left( \langle D_{l3k3}^{-1} \rangle \langle D_{k3d3}^{-1} D_{d3pq} \rangle - D_{l3pq} \right) \right\rangle_Y, \quad (45)$$

$$B_{ijpq}^* = 0. \quad (46)$$

It is important to remark that the expression of  $C_{ijpq}^{\text{eff}}$  (Eq. (44)) coincide with the formula (Eq. 1.12, page 145) of Pobodrya (1984), considering a classical Cauchy elastic problem.

Details of Cosserat centro-symmetric materials can be found in Eringen (1999) and Zheng and Spencer (1993). The correspondence between the non-null Cosserat effective coefficients and the local problems is given in Table 1.

## 5. Numerical results

In this section, the effective properties (Eqs. (44)–(46)) are reduced for a centro-symmetric laminated Cosserat composite with isotropic or cubic constituents. In addition, the numerical values of the effective properties for a centro-symmetric bi-laminated Cosserat composite are computed for both constituent symmetries, and the results are analyzed and discussed. As a limit case, a comparison is reported between classical and Cosserat effective elastic properties for a laminated composite with isotropic constituents. Finally, the analytical formulas of the effective engineering moduli are reported for centro-symmetric bi-laminated Cosserat composites with isotropic and cubic constituent materials.

Since the constituent materials are considered centro-symmetric, then, the orthotropic symmetry and all orthotropic sub-classes, i.e., transversely isotropic, cubic, and isotropic, are also considered centro-symmetric, see Zheng and Spencer (1993).

### 5.1. Centro-symmetric laminated Cosserat composites with isotropic constituents

Assuming that each layer possesses isotropic symmetry, the constituent properties are rewritten as a function of six independent constants, see for instance Nowacki (1986) and Hassanpour and Hepler (2015), in the form

$$C_{ijpq} = C_{1122} \delta_{ij} \delta_{pq} + C_{1212} \delta_{ip} \delta_{jq} + C_{1221} \delta_{iq} \delta_{jp}, \quad (47)$$

$$D_{ijpq} = D_{1122} \delta_{ij} \delta_{pq} + D_{1212} \delta_{ip} \delta_{jq} + D_{1221} \delta_{iq} \delta_{jp}, \quad (48)$$

where  $B_{ijpq}$  is vanished and  $\delta_{ij}$  is the Kronecker delta tensor and  $i, j, p, q = 1, 2, 3$ .

In addition, the following restrictions derived from the positive-definite quadratic form of the internal energy are assumed (see, Eringen (1999))

$$\begin{aligned} C_{1212} + C_{1221} > 0, \quad C_{1212} - C_{1221} > 0, \quad 3C_{1122} + C_{1212} + C_{1221} > 0, \\ D_{1212} + D_{1221} > 0, \quad D_{1212} - D_{1221} > 0, \quad 3D_{1122} + D_{1212} + D_{1221} > 0, \end{aligned} \quad (49)$$

and the stiffness  $C_{ijpq}$  and torque  $D_{ijpq}$  moduli are defined in matrix form as

$$C_{ijpq} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 & 0 & 0 & 0 \\ & C_{1111} & C_{1122} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & C_{1111} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & C_{1212} & 0 & 0 & C_{1221} & 0 & 0 \\ & & & & C_{1212} & 0 & 0 & C_{1221} & 0 \\ & & & & & \text{SYM} & C_{1212} & 0 & 0 \\ & & & & & & & C_{1212} & 0 \\ & & & & & & & & C_{1212} \end{pmatrix} \quad (50)$$

and

$$D_{ijpq} = \begin{pmatrix} D_{1111} & D_{1122} & D_{1122} & 0 & 0 & 0 & 0 & 0 & 0 \\ & D_{1111} & D_{1122} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & D_{1111} & 0 & 0 & 0 & 0 & 0 \\ & & & & D_{1212} & 0 & 0 & D_{1221} & 0 \\ & & & & & D_{1212} & 0 & 0 & D_{1221} \\ & & & & & & D_{1212} & 0 & 0 \\ & & & & & & & \text{SYM} & D_{1212} \\ & & & & & & & & & D_{1212} \\ & & & & & & & & & & D_{1212} \end{pmatrix}, \quad (51)$$

where  $C_{1111} = C_{1122} + C_{1212} + C_{1221}$  and  $D_{1111} = D_{1122} + D_{1212} + D_{1221}$ , see Eringen (1999) and Eremeyev and Pietraszkiewicz (2012). The  $C_{ijpq}$  and  $D_{ijpq}$  matrices are characterized by three independent components each: ( $C_{1122}, C_{1212}, C_{1221}$ ) for  $C_{ijpq}$  and ( $D_{1122}, D_{1212}, D_{1221}$ ) for  $D_{ijpq}$ . An analysis for the linear isotropic equations associated to the theory of micropolar elasticity and his representations associated to the Hassanpour and Hepler (2015).

Then, replacing Eqs. (47) and (48) into Eqs. (44) and (46), the analytical expressions of the non-null effective properties  $C_{ijpq}^*$  and  $D_{ijpq}^*$

for centro-symmetric laminated Cosserat composites with isotropic constituents are found as follows:

$$\begin{aligned}
C_{1111}^* &= C_{2222}^* = \langle C_{1122} + C_{1212} + C_{1221} \rangle \\
&\quad - \langle C_{1122}^2 (C_{1122} + C_{1212} + C_{1221})^{-1} \rangle + \\
\langle C_{1122} (C_{1122} + C_{1212} + C_{1221})^{-1} \rangle \langle (C_{1122} + C_{1212} + C_{1221})^{-1} \rangle^{-1}, \\
C_{3333}^* &= \langle (C_{1122} + C_{1212} + C_{1221})^{-1} \rangle^{-1}, \\
C_{1122}^* &= \langle C_{1122} \rangle - \langle C_{1122}^2 (C_{1122} + C_{1212} + C_{1221})^{-1} \rangle + \\
\langle C_{1122} (C_{1122} + C_{1212} + C_{1221})^{-1} \rangle \langle (C_{1122} + C_{1212} + C_{1221})^{-1} \rangle^{-1}, \quad (52) \\
C_{1212}^* &= C_{2121}^* = \langle C_{1212} \rangle, \quad C_{1313}^* = C_{2323}^* = \langle C_{1212}^{-1} \rangle^{-1}, \\
C_{1133}^* &= C_{2233}^* = \langle C_{1122} (C_{1122} + C_{1212} + C_{1221})^{-1} \rangle \\
&\quad \langle (C_{1122} + C_{1212} + C_{1221})^{-1} \rangle^{-1}, \\
C_{3131}^* &= \langle C_{1212} \rangle + \langle C_{1221} C_{1212}^{-1} \rangle^2 \langle C_{1212}^{-1} \rangle^{-1} - \langle C_{1221}^2 C_{1212}^{-1} \rangle, \\
C_{1221}^* &= \langle C_{1221} \rangle, \quad C_{1331}^* = C_{2332}^* = \langle C_{1221} C_{1212}^{-1} \rangle \langle C_{1212}^{-1} \rangle^{-1},
\end{aligned}$$

and

$$\begin{aligned}
D_{1111}^* &= D_{2222}^* = \langle D_{1122} + D_{1212} + D_{1221} \rangle \\
&\quad - \langle D_{1122}^2 (D_{1122} + D_{1212} + D_{1221})^{-1} \rangle + \\
\langle D_{1122} (D_{1122} + D_{1212} + D_{1221})^{-1} \rangle \langle (D_{1122} + D_{1212} + D_{1221})^{-1} \rangle^{-1}, \\
D_{3333}^* &= \langle (D_{1122} + D_{1212} + D_{1221})^{-1} \rangle^{-1}, \\
D_{1122}^* &= \langle D_{1122} \rangle - \langle D_{1122}^2 (D_{1122} + D_{1212} + D_{1221})^{-1} \rangle + \\
\langle D_{1122} (D_{1122} + D_{1212} + D_{1221})^{-1} \rangle \langle (D_{1122} + D_{1212} + D_{1221})^{-1} \rangle^{-1}, \quad (53) \\
D_{1212}^* &= D_{2121}^* = \langle D_{1212} \rangle, \quad D_{1313}^* = D_{2323}^* = \langle D_{1212}^{-1} \rangle^{-1}, \\
D_{1133}^* &= D_{2233}^* = \langle D_{1122} (D_{1122} + D_{1212} + D_{1221})^{-1} \rangle \\
&\quad \langle (D_{1122} + D_{1212} + D_{1221})^{-1} \rangle^{-1}, \\
D_{3131}^* &= \langle D_{1212} \rangle + \langle D_{1221} D_{1212}^{-1} \rangle^2 \langle D_{1212}^{-1} \rangle^{-1} - \langle D_{1221}^2 D_{1212}^{-1} \rangle, \\
D_{1221}^* &= \langle D_{1221} \rangle, \quad D_{1331}^* = D_{2332}^* = \langle D_{1221} D_{1212}^{-1} \rangle \langle D_{1212}^{-1} \rangle^{-1}.
\end{aligned}$$

In Eqs. (52) and (53), the symbol  $\langle f \rangle$  is the Voigt's average (average operator) of the property  $f$ . Also, in case of a bi-laminated composite,  $\langle f \rangle = f^{(1)}V_1 + f^{(2)}V_2$  where  $V_1$  and  $V_2$  represents the volume fractions per unit length occupied by the layer 1 and 2, respectively; such as,  $V_1 + V_2 = 1$ , see Fig. 1(d).

As it can be observed in Eqs. (52) and (53), the resulting homogenized material has orthotropic symmetry restricted with the invariance of stiffness and torques under rotations of  $90^\circ$  about the unitary vector  $e_3$ , i.e., it is described by eighteen independent effective properties, see Eremeyev and Pietraszkiewicz (2012). Also, it can be noticed that the effective coefficients given in Eq. (52) match with those reported by Pobedrya (1984) (Eq. 1.19, page 147) when  $C_{1212} = C_{1221}$ . This limit case allows reproducing the effective coefficients of laminate composites with isotropic constituents in the framework of classical Cauchy elasticity, as in Pobedrya (1984). In addition, it is worth mentioning that the expressions of these coefficients (Eq. (52)) do not match with those reported by Emel'yanov (2016) because different constitutive relations for  $C_{ijpq}$  and  $D_{ijpq}$  are used. We follow the same reported by Nowacki (1974) and Lazar and Kirchner (2005).

From now on, the non-null effective properties for a bi-laminated Cosserat composite with isotropic material constituents are shown as a function of the volume fraction. The values of the material properties for the numerical computations are established by the data given in Table 2 and the relations  $C_{1122} \equiv \lambda$ ,  $(C_{1212} + C_{1221})/2 \equiv \mu$ ,  $(C_{1212} - C_{1221})/2 \equiv \alpha$ ,  $D_{1122} \equiv \beta$ ,  $(D_{1212} + D_{1221})/2 \equiv \gamma$ , and  $(D_{1212} - D_{1221})/2 \equiv \epsilon$ , which represent the Lamé coefficient, the Lamé shear modulus, the micropolar couple modulus and the remaining ones define the Cosserat or micropolar elastic constants. Following these relations an equivalent representation of Eqs. (47) and (48) can be obtained (see, Lazar and Kirchner (2005)). Constituent materials with a full set of micropolar elastic constants measured are reported by Hassanpour and Heppler (2015). It is important to note that, despite several studies carried

**Table 2**

Constitutive material properties.

Material properties	$\lambda$ (MPa)	$\mu$ (MPa)	$\alpha$ (MPa)	$\beta$ (N)	$\gamma$ (N)	$\epsilon$ (N)
Syntactic foam (hollow glass spheres in epoxy resin)	2097	1033	114.8	-2.91	4.364	-0.133
Dense polyurethane (high dense polyurethane foam)	762.7	104	4.333	-26.65	39.98	4.504

out in micropolar media, further research is needed focused on the determination of the material constants.

Fig. 2 plots the effective elastic (Figs. 2(a) and (b)) and torque (Figs. 2(c) and (d)) properties for a homogenized bi-laminated Cosserat composite (layer 1/layer 2) = (Syntactic foam (SyF)/Dense polyurethane foam (PUF)) as a function of  $V_1$  volume fraction. From Figs. 2(a) and (b), it can be noticed that all effective elastic properties have a monotone increasing behavior, e.g.,  $C_{1111}^*$  and  $C_{1122}^*$  in a linear manner in almost whole volume fraction,  $C_{1212}^*$  and  $C_{1221}^*$  as linear functions and the other ones in a convex form. Also, the latter ones have a higher growth rate as  $V_1$  approaches 1. On the other hand, from Fig. 2(c) and (d), it is worthwhile to mention that there is a different effective behavior. When  $V_1$  increases, the effective torque properties decrease except for  $D_{1122}^*$  and  $D_{1133}^*$ . These last two properties have the peculiarity of being negative. In particular,  $D_{1133}^*$  grows in a concave form for all  $V_1$  values and  $D_{1122}^*$  has a minimum for  $V_1 = 0.092$ . This negative behavior is due to the  $D_{1122}$  Cosserat twist coefficient influence, which is negative for both constituents.  $D_{1122}$  is negative for micropolar isotropic solids (see, Hassanpour and Heppler (2015)).

## 5.2. Comparison between Cauchy and Cosserat effective properties

As mentioned before, micropolar media can be reduced to classical or Cauchy media. In the classical theory of linear elasticity, the effective moduli  $D_{ijpq}^*$  do not exist, and the effective stiffness moduli  $C_{ijpq}^*$  are defined by five independent constants, which corresponds to a homogenized material with transversely isotropic symmetry. Therefore, the effective properties of laminated Cosserat composites with isotropic constituents can be reduced to the classical effective stiffness properties making  $C_{1212}^* = C_{1221}^*$ ,  $D_{1122}^* = 0$  and  $D_{1212}^* = D_{1221}^* = 0$ .

Then, from Eqs. (52) and (53), the stiffness moduli are given by:

$$\begin{aligned}
C_{1111}^C &= C_{2222}^C = \langle C_{1122} + 2C_{1212} \rangle - \langle C_{1122}^2 (C_{1122} + 2C_{1212})^{-1} \rangle + \\
&\quad \langle C_{1122} (C_{1122} + 2C_{1212})^{-1} \rangle \langle (C_{1122} + 2C_{1212})^{-1} \rangle^{-1}, \\
C_{3333}^C &= \langle (C_{1122} + 2C_{1212})^{-1} \rangle^{-1}, \\
C_{1122}^C &= \langle C_{1122} \rangle - \langle C_{1122}^2 (C_{1122} + 2C_{1212})^{-1} \rangle + \\
&\quad \langle C_{1122} (C_{1122} + 2C_{1212})^{-1} \rangle \langle (C_{1122} + 2C_{1212})^{-1} \rangle^{-1}, \\
C_{1133}^C &= C_{2233}^C = \langle C_{1122} (C_{1122} + 2C_{1212})^{-1} \rangle \langle (C_{1122} + 2C_{1212})^{-1} \rangle^{-1}, \\
C_{1212}^C &= \langle C_{1212} \rangle, \quad C_{1313}^C = C_{2323}^C = \langle C_{1212}^{-1} \rangle^{-1}, \quad (54)
\end{aligned}$$

where  $C_{1212}^C = (C_{1111}^C - C_{1122}^C)/2$ . Here, the notation  $C_{ijpq}^C$  represents the classical or Cauchy effective property (effective stiffness moduli) and they satisfy the classical symmetry conditions  $C_{ijpq}^C = C_{pqij}^C = C_{ijqp}^C = C_{jipq}^C$ . The resulting composite has hexagonal symmetry. Notice that, a good agreement is obtained with the expressions reported by Pobedrya (1984), page 147.

On the other hand, if we assume that all the material parameters given in Table 2 are null except  $C_{1122} = \lambda$  and  $C_{1212} = \mu$ , it can be obtained that the effective Cosserat and Cauchy properties satisfy that  $C_{1111}^* = C_{1111}^C$ ,  $C_{2222}^* = C_{2222}^C$ ,  $C_{3333}^* = C_{3333}^C$ ,  $C_{1122}^* = C_{1122}^C$  and  $C_{1133}^* = C_{1133}^C$ , whereas  $C_{1212}^* = C_{2121}^* \neq C_{1212}^C$  and  $C_{1313}^* \neq C_{1313}^C$ , from a direct comparison between Eqs. (52) and (54). Hence, the differences between the Cosserat and Cauchy effective elastic moduli need to be



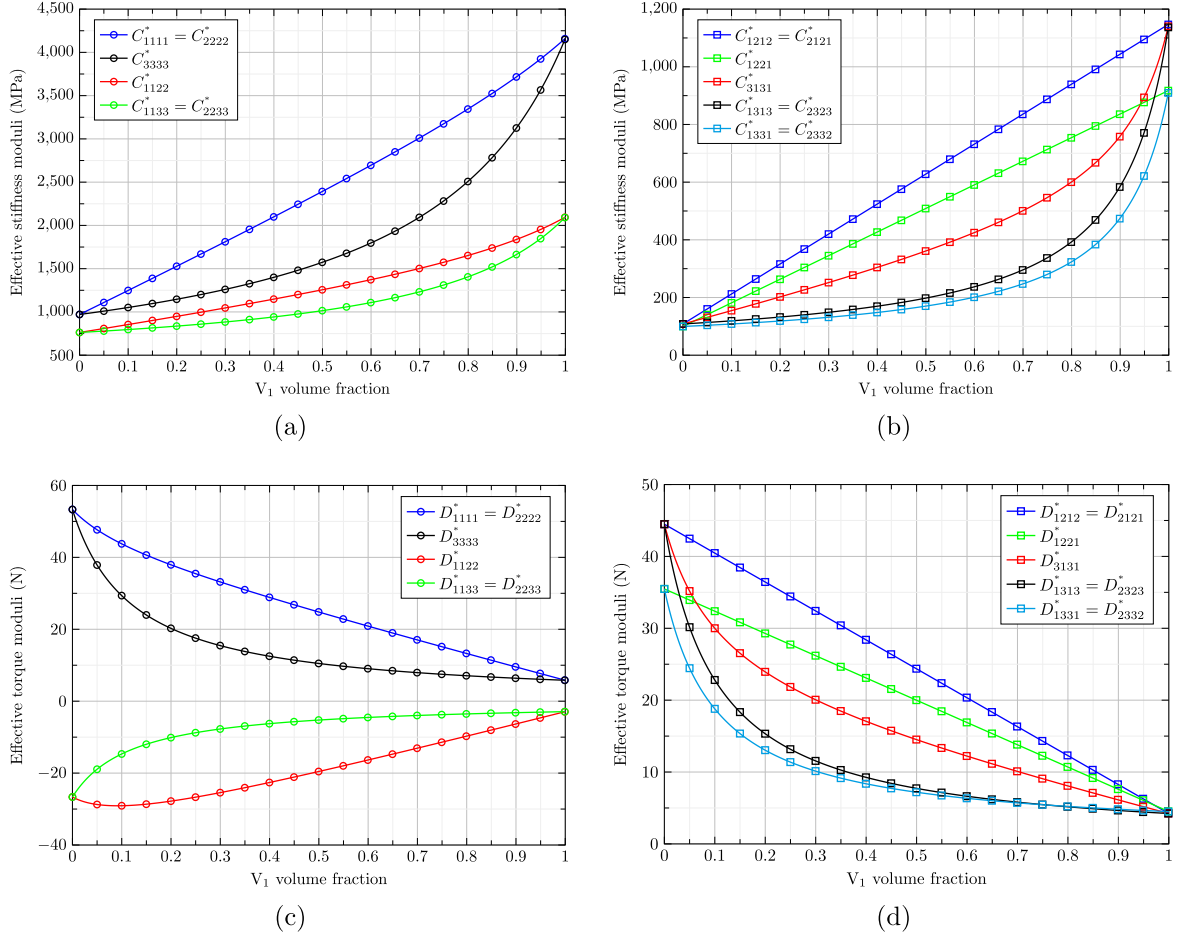


Fig. 2. Effective elastic and torque properties for a homogenized bi-laminated Cosserat composite (SyF/PUF) with isotropic constituents as function of  $V_1$  volume fraction.

Table 3

Classical ( $C_{ijpq}^C$ ) and Cosserat ( $C_{ijpq}^*$ ) elastic effective properties for a bi-laminated composite with isotropic constituents.

Effective properties (GPa)							
$V_1$	$C_{1212}^* = C_{2121}^*$	$C_{1212}^C$	$C_{1221}^*$	$C_{3131}^*$	$C_{1313}^*$	$C_{1313}^C$	$C_{1331}^*$
0	0.10833	0.10400	0.09967	0.10833	0.10833	0.10400	0.09967
0.1	0.21228	0.19690	0.18152	0.15451	0.11912	0.11428	0.10816
0.2	0.31623	0.28980	0.26337	0.20217	0.13230	0.12681	0.11854
0.3	0.42017	0.38270	0.34523	0.25187	0.14875	0.14243	0.13149
0.4	0.52412	0.47560	0.42708	0.30445	0.16987	0.16243	0.14812
0.5	0.62807	0.56850	0.50893	0.36138	0.19798	0.18897	0.17026
0.6	0.73201	0.66140	0.59079	0.42522	0.23725	0.22589	0.20118
0.7	0.83596	0.75430	0.67264	0.50110	0.29594	0.28072	0.24740
0.8	0.93991	0.84720	0.75449	0.60091	0.39321	0.37071	0.32399
0.9	1.04385	0.94010	0.83635	0.75980	0.58576	0.54562	0.47562
1.0	1.14780	1.03300	0.91820	1.14780	1.14780	1.03300	0.91820

shown for only unequal properties; see Table 3. Fig. 2(a) and (b) can be used to analyze the Cauchy effective property due to equality relations.

Table 3 illustrates the values of the Cosserat and Cauchy elastic properties for a bi-laminated composite with isotropic constituents as a function of  $V_1$  volume fraction. The Cauchy elastic moduli is calculated considering the material properties given in Table 2. Notice that the effective properties  $C_{1212}^*$ ,  $C_{1221}^*$  and  $C_{1212}^C$  are positive and increase linearly as  $V_1$  increases. Also  $C_{1221}^* < C_{1212}^C < C_{1212}^*$  is satisfied in the whole volume fraction interval. A similar behavior occurs for  $C_{1313}^*$ ,

Table 4

Deviation between the effective Cauchy ( $C_{ijpq}^C$ ) and Cosserat ( $C_{ijpq}^*$ ) elastic properties and relative errors.

Deviations (GPa) and relative errors:					
Error 1 = $ \tilde{C}_{1313}^*/C_{1313}^C  \times 100\%$ , Error 2 = $ \tilde{C}_{3131}^*/C_{3131}^*  \times 100\%$					
$V_1$	$\tilde{C}_{1212}^*$	$\tilde{C}_{1313}^*$	Error 1	$\tilde{C}_{1313}^*$	Error 2
0.1	0	-0.001075	0.69594	0.000636	0.55643
0.2	0	-0.006131	3.03255	0.001393	1.09861
0.3	0	-0.010699	4.24773	0.002309	1.62122
0.4	0	-0.014563	4.78325	0.003437	2.11592
0.5	0	-0.017361	4.80407	0.004855	2.56887
0.6	0	-0.018429	4.33400	0.006675	2.95521
0.7	0	-0.016411	3.27496	0.009055	3.22551
0.8	0	-0.008045	1.33873	0.012105	3.26535
0.9	0	0.016937	2.22917	0.014929	2.73616

$C_{1331}^*$  and  $C_{1313}^C$ , but they increase in a convex form and we have that  $C_{1331}^* < C_{1313}^C < C_{1313}^* < C_{3131}^*$ . The differences between Cauchy and Cosserat elastic properties are more noticeable for larger volume fractions.

From Table 3, it is also worthy to mention that  $C_{1212}^C$  are related to the average of  $C_{1212}^*$  and  $C_{1221}^*$ . Similar behavior can be remarkable for  $C_{1313}^C$  and  $C_{3131}^*$  through the average between  $C_{1313}^*$  and  $C_{1331}^*$ , and between  $C_{1212}^C$  and  $C_{1313}^C$ , respectively. From numerical experiments we can confirm that the deviation is linked to the mean values through the following relations

$$\tilde{C}_{1212}^* = C_{1212}^C - \frac{C_{1212}^* + C_{1221}^*}{2}, \quad (55)$$

$$\bar{C}_{1313}^* = C_{1313}^C - \frac{C_{1313}^* + C_{1331}^*}{2}, \quad (56)$$

$$\bar{C}_{3131}^* = C_{3131}^* - \frac{C_{1212}^C + C_{1313}^C}{2}, \quad (57)$$

and the results are illustrated in Table 4. An analysis of the relative error is also shown.

From Table 4, it is concluded that  $\bar{C}_{1212}^*$ ,  $\bar{C}_{1313}^*$  and  $\bar{C}_{3131}^*$  are close to zero and they represent an error less than 5% of the  $C_{1221}^C$ ,  $C_{1313}^C$  and  $C_{3131}^*$ , respectively. Thus, in this case, a relation can be established between the classical and the Cosserat elastic moduli for the isotropic case as follows,

$$\begin{aligned} C_{1212}^C &= (C_{1212}^* + C_{1221}^*)/2, & C_{1313}^C &\approx (C_{1313}^* + C_{1331}^*)/2, \\ C_{3131}^* &\approx (C_{1212}^C + C_{1313}^C)/2. \end{aligned} \quad (58)$$

### 5.3. Centro-symmetric laminated Cosserat composites with cubic constituents

Now, we consider that each layer possesses cubic symmetry, then, the constituent properties can be rewritten as a function of eight independent constants, in the form

$$\begin{aligned} C_{ijpq} &= C_{1122}\delta_{ij}\delta_{pq} + C_{1212}\delta_{ip}\delta_{jq} + C_{1221}\delta_{iq}\delta_{jp} \\ &+ (C_{1111} - C_{1122} - C_{1212} - C_{1221})\delta_{ijpq}, \end{aligned} \quad (59)$$

$$\begin{aligned} D_{ijpq} &= D_{1122}\delta_{ij}\delta_{pq} + D_{1212}\delta_{ip}\delta_{jq} + D_{1221}\delta_{iq}\delta_{jp} \\ &+ (D_{1111} - D_{1122} - D_{1212} - D_{1221})\delta_{ijpq}, \end{aligned} \quad (60)$$

where  $B_{ijpq}$  is null and  $\delta_{ij}$  is the Kronecker delta tensor and the four order tensor  $\delta_{ijrs}$  is defined as follows:  $\delta_{ijrs} = 1$  if  $i = j = r = s$ , else  $\delta_{ijrs} = 0$ . As in Section 5.1, an equivalent representation of Eqs. (59) and (60) can be obtained through the relations  $C_{1122} = \lambda$ ,  $(C_{1212} + C_{1221})/2 = \mu$ ,  $(C_{1111} - C_{1122} - C_{1212} - C_{1221}) = \eta$ ,  $(C_{1212} - C_{1221})/2 = \alpha$ ,  $D_{1122} = \beta$ ,  $(D_{1212} + D_{1221})/2 = \gamma$ ,  $(D_{1212} - D_{1221})/2 = \epsilon$ , and  $(D_{1111} - D_{1122} - D_{1212} - D_{1221}) = \nu$ , where  $\lambda$  and  $\mu$  are the Lamé parameters,  $\eta$  is the classical cubic constant or anisotropy constant,  $\alpha$  is the micropolar couple modulus, and  $\beta, \gamma, \epsilon$ , and  $\nu$  represent the additional micropolar elastic constants introduced in micropolar theory.

Similarly to the isotropic case, restrictions derived from the positive-definite quadratic form of the internal energy are assumed, see Eringen (1999), i.e.,

$$\begin{aligned} C_{1212} + C_{1221} &> 0, & C_{1212} - C_{1221} &> 0, \\ C_{1111} - C_{1122} &> 0, & 2C_{1122} + C_{1111} &> 0, \\ D_{1212} + D_{1221} &> 0, & D_{1212} - D_{1221} &> 0, \\ D_{1111} - D_{1122} &> 0, & 2D_{1122} + D_{1111} &> 0, \end{aligned} \quad (61)$$

and the stiffness  $C_{ijpq}$  and torque  $D_{ijpq}$  moduli have the same matrix form of Eqs. (50) and (51) but  $C_{1111} \neq C_{1122} + C_{1212} + C_{1221}$  and  $D_{1111} \neq D_{1122} + D_{1212} + D_{1221}$ .

In this case, the material symmetry group related to constituents cubic symmetry has rotations of  $90^\circ$  about all orthogonal axes with unitary vectors  $\mathbf{e}_k$  ( $k = 1, 2, 3$ ). Some details of the mathematical foundations, material symmetry regulations, and stability of micropolar media are given in Eringen (1999) and Eremeyev and Pietraszkiewicz (2012). The explicit structure of  $C_{ijpq}$  and  $D_{ijpq}$  tensors are reported by Zheng and Spencer (1993) for fourteen symmetry groups.

Then, replacing Eqs. (59) and (60) into Eqs. (44)–(46), the non-null effective properties  $C_{ijpq}^*$  and  $D_{ijpq}^*$  of centro-symmetric laminated

Cosserat materials with cubic constituents can be determined, as follows:

$$\begin{aligned} C_{1111}^* &= C_{2222}^* = \langle C_{1111} \rangle + \langle C_{1122} C_{1111}^{-1} \rangle \langle C_{1111}^{-1} \rangle - \langle C_{1122}^2 C_{1111}^{-1} \rangle, \\ C_{3333}^* &= \langle C_{1111}^{-1} \rangle^{-1}, \\ C_{1122}^* &= \langle C_{1122} \rangle + \langle C_{1122} C_{1111}^{-1} \rangle \langle C_{1111}^{-1} \rangle - \langle C_{1122}^2 C_{1111}^{-1} \rangle, \\ C_{1133}^* &= C_{2233}^* = \langle C_{1122} C_{1111}^{-1} \rangle \langle C_{1111}^{-1} \rangle^{-1}, \\ C_{1212}^* &= C_{2121}^* = \langle C_{1212} \rangle, & C_{1313}^* &= C_{2323}^* = \langle C_{1212}^{-1} \rangle^{-1}, \\ C_{3131}^* &= \langle C_{1212} \rangle + \langle C_{1221} C_{1212}^{-1} \rangle \langle C_{1212}^{-1} \rangle - \langle C_{1221}^2 C_{1212}^{-1} \rangle, \\ C_{1221}^* &= \langle C_{1221} \rangle, & C_{1331}^* &= C_{2332}^* = \langle C_{1221} C_{1212}^{-1} \rangle \langle C_{1212}^{-1} \rangle^{-1}, \end{aligned} \quad (62)$$

and

$$\begin{aligned} D_{1111}^* &= D_{2222}^* = \langle D_{1111} \rangle + \langle D_{1122} D_{1111}^{-1} \rangle \langle D_{1111}^{-1} \rangle - \langle D_{1122}^2 D_{1111}^{-1} \rangle, \\ D_{3333}^* &= \langle D_{1111}^{-1} \rangle^{-1}, \\ D_{1122}^* &= \langle D_{1122} \rangle + \langle D_{1122} D_{1111}^{-1} \rangle \langle D_{1111}^{-1} \rangle - \langle D_{1122}^2 D_{1111}^{-1} \rangle, \\ D_{1133}^* &= D_{2233}^* = \langle D_{1122} D_{1111}^{-1} \rangle \langle D_{1111}^{-1} \rangle^{-1}, \\ D_{1212}^* &= D_{2121}^* = \langle D_{1212} \rangle, & D_{1313}^* &= D_{2323}^* = \langle D_{1212}^{-1} \rangle^{-1}, \\ D_{3131}^* &= \langle D_{1212} \rangle + \langle D_{1221} D_{1212}^{-1} \rangle \langle D_{1212}^{-1} \rangle - \langle D_{1221}^2 D_{1212}^{-1} \rangle, \\ D_{1221}^* &= \langle D_{1221} \rangle, & D_{1331}^* &= D_{2332}^* = \langle D_{1221} D_{1212}^{-1} \rangle \langle D_{1212}^{-1} \rangle^{-1}. \end{aligned} \quad (63)$$

From Eq. (62), the analytical formulas for a cubic elastic Cauchy composite are obtained when  $\alpha = 0$  (i.e.,  $C_{1212} = C_{1221}$ ), then, the effective properties reduce to six properties only, which matches with those reported in Eq. (35) by Castillero et al. (1998). Also, it is important to mention that the resulting composite is described by eighteen independent effective properties (nine are stiffness and nine are torque); therefore, it belongs to an orthotropic symmetry group with invariance under rotations of  $90^\circ$  about the unitary vector  $\mathbf{e}_3$ :  $\mathbf{O} = \{\mathbf{e}_3 \otimes \mathbf{e}_3 \mp \mathbf{e}_3 \times \mathbf{I}\}$ , see for instance Eremeyev and Pietraszkiewicz (2012).

From now on, the dimensionless effective stiffness  $C_{ijpq}^*/C_{1111}^{(1)}$  (Fig. 3(a) and (b)) and torque  $D_{ijpq}^*/D_{1111}^{(1)}$  (Fig. 3(c) and (d)) moduli are computed for a homogenized bi-laminated composite as a function of the  $V_1$  volume fraction. The computations are carried out using Eqs. (62) and (63) for fictitious constituents materials; such that, for the material constituent 1, we have

$$\begin{aligned} \frac{C_{1122}^{(1)}}{C_{1111}^{(1)}} &= 0.65, & \frac{C_{1212}^{(1)}}{C_{1111}^{(1)}} &= 0.35, & \frac{C_{1221}^{(1)}}{C_{1111}^{(1)}} &= 0.30, & \frac{D_{1122}^{(1)}}{D_{1111}^{(1)}} &= -0.4, \\ \frac{D_{1212}^{(1)}}{D_{1111}^{(1)}} &= 0.70, & \frac{D_{1221}^{(1)}}{D_{1111}^{(1)}} &= 0.60, \end{aligned}$$

and for the material constituent 2

$$\begin{aligned} \frac{C_{1111}^{(2)}}{C_{1111}^{(1)}} &= 0.45, & \frac{C_{1122}^{(2)}}{C_{1111}^{(1)}} &= 0.20, & \frac{C_{1212}^{(2)}}{C_{1111}^{(1)}} &= 0.026, & \frac{C_{1212}^{(2)}}{C_{1111}^{(1)}} &= 0.024, \\ \frac{D_{1111}^{(2)}}{D_{1111}^{(1)}} &= 10.0, & \frac{D_{1122}^{(2)}}{D_{1111}^{(1)}} &= -3.7, & \frac{D_{1212}^{(2)}}{D_{1111}^{(1)}} &= 7.5, & \frac{D_{1212}^{(2)}}{D_{1111}^{(1)}} &= 6.3. \end{aligned}$$

The fictitious constituents materials are assumed due to the lack of experimental data for the complete characterization of Cosserat cubic-symmetric materials. Therefore, the values of the defined ratios are not taken randomly. They are considered under the following conditions:

- (i) The restrictions in Eq. (61) are satisfied.
- (ii) The material properties of a first layer has greater elastic properties, i.e.,  $|C_{1122}^{(1)}| > |C_{1122}^{(2)}|$ ,  $|C_{1212}^{(1)} + C_{1221}^{(1)}| > |C_{1212}^{(2)} + C_{1221}^{(2)}|$ ,  $|C_{1212}^{(1)} - C_{1221}^{(1)}| > |C_{1212}^{(2)} - C_{1221}^{(2)}|$ , and  $|C_{1111}^{(1)} - C_{1122}^{(1)} - C_{1212}^{(1)} - C_{1221}^{(1)}| > |C_{1111}^{(2)} - C_{1122}^{(2)} - C_{1212}^{(2)} - C_{1221}^{(2)}|$ .
- (iii)  $0 < C_{ijpq}^{(\alpha)}/C_{1111}^{(1)} < 1$  ( $\alpha = 1, 2$ ), then as expected, it is satisfied that  $0 < C_{ijpq}^*/C_{1111}^{(1)} < 1$ .

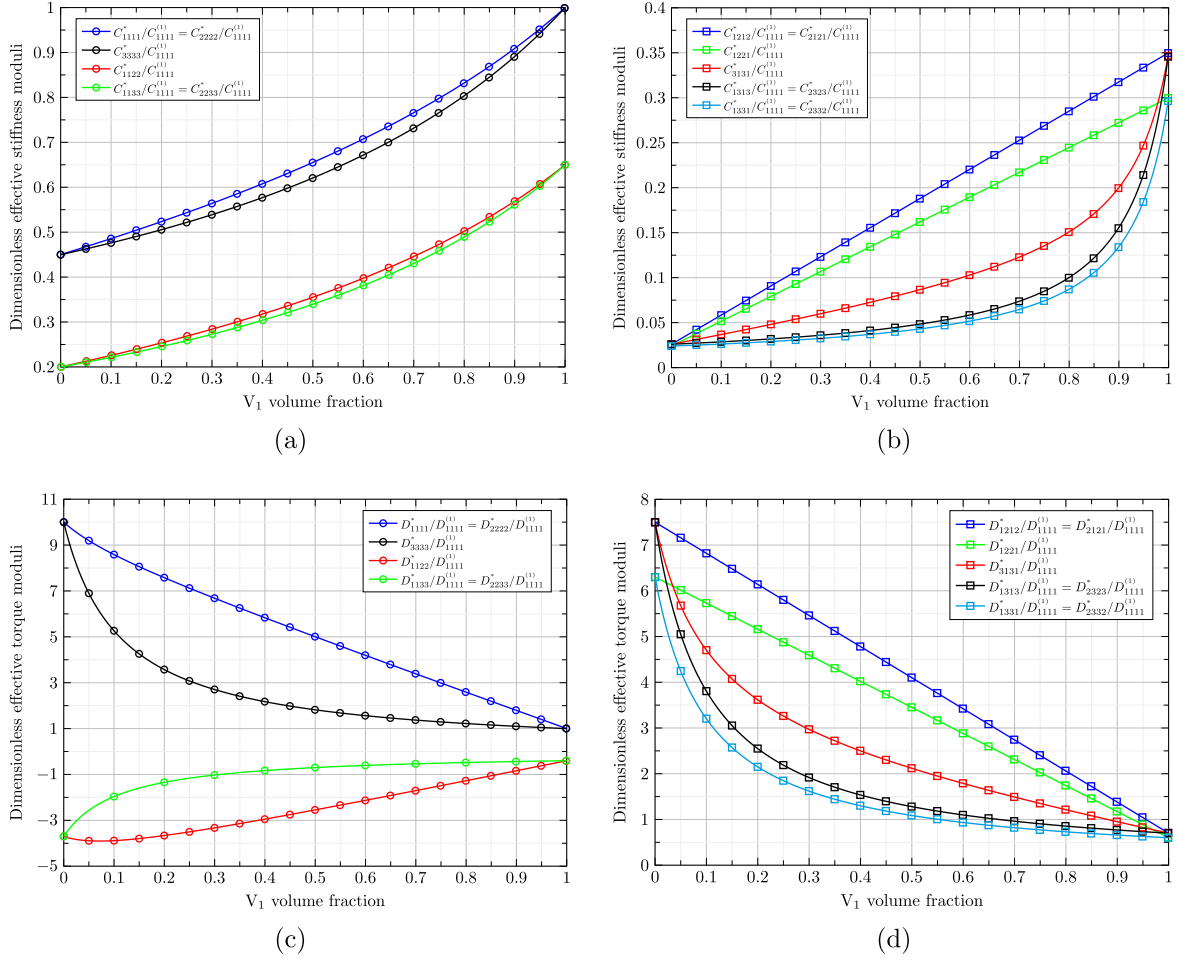


Fig. 3. Dimensionless effective elastic and torque properties for a homogenized bi-laminate Cosserat composite with cubic constituents as a function of  $V_1$  volume fraction.

- (iv) The torque properties of the first layer are lower than the second one, i.e.,  $|D_{1122}^{(1)}| < |D_{1122}^{(2)}|$ ,  $|D_{1212}^{(1)} + D_{1221}^{(1)}| < |D_{1212}^{(2)} + D_{1221}^{(2)}|$ ,  $|D_{1212}^{(1)} - D_{1221}^{(1)}| < |D_{1212}^{(2)} - D_{1221}^{(2)}|$ , and  $|D_{1111}^{(1)} - D_{1122}^{(1)} - D_{1212}^{(1)} - D_{1221}^{(1)}| < |D_{1111}^{(2)} - D_{1122}^{(2)} - D_{1212}^{(2)} - D_{1221}^{(2)}|$ .
- (v) The negative values are due to  $D_{1122}^{(2)}$  is defined negative.

Figs. 3(a) and (b) display that effective elastic properties behave as monotonically increasing convex functions in the whole interval for  $V_1$ , except  $C_{1212}^*$  and  $C_{1221}^*$  that linearly behave. In Figs. 3(c) and (d), a different behavior is presented for the effective torque properties. Here, the effective torque properties are monotone decreasing and concave functions in the whole interval except  $D_{1122}^*$  and  $D_{1133}^*$ . The property  $D_{1122}^*$  has a minimum value when  $V_1 \approx 0.092$  and then increases as the  $V_1$  volume fraction increases. The property  $D_{1133}^*$  is a monotone increasing and concave function. Both  $D_{1122}^*$  and  $D_{1133}^*$  have negative values for every value of  $V_1$ . The negative behavior of  $D_{1122}^*$  and  $D_{1133}^*$  is due to the  $D_{1122}$  Cosserat twist coefficient that is negative for both constituents.

#### 5.4. Engineering moduli of laminated Cosserat materials with isotropic and cubic constituents

Let us start by recalling the classical linear elasticity theory for solid materials, in which the engineering moduli are found, i.e., Young's modulus, shear modulus, and Poisson's ratio. The relationship between the engineering constants and the elastic constants of the stiffness

matrix is also a topic of interest, as it is reported by Hayes and Shuvalov (1998), Devorak (2013), and others.

In the theory of micropolar elasticity, local rotations and displacements are assumed at each point, whereas only displacements are considered in the classical linear elasticity theory. Therefore, other engineering constants are added, such as the torsional Young's modulus, the micropolar twist (Poisson) ratio, among others. These engineering constants can be defined as a function of the effective stiffness and torque moduli.

In this section, the engineering moduli are determined for a homogenized laminated Cosserat composite with isotropic and cubic constituents.

From Eq. (3), it can be concluded that the strain-stress relationships for a centro-symmetric Cosserat material are given by the uncoupled equations

$$e_{nm} = S_{ijmn} \sigma_{ji}, \quad \psi_{nm} = T_{ijmn} \mu_{ji}, \quad (64)$$

where  $S_{ijmn} = C_{ijmn}^{-1}$  and  $T_{ijmn} = D_{ijmn}^{-1}$  ( $i, j, m, n = 1, 2, 3$ ).

Then, as mentioned before in Sections 5.1 and 5.3, the homogenized laminated composite has orthotropic symmetry with invariance under rotations of  $90^\circ$  when isotropic (Section 5.1) and cubic (Section 5.3) constituents are assumed. Therefore, the components of the effective

compliance matrix satisfy

$$\begin{aligned}
S_{1111}^* &= S_{2222}^* = \frac{C_{1111}^* C_{3333}^* - (C_{1122}^*)^2}{(C_{1111}^* - C_{1122}^*)(C_1^* - 2(C_{1122}^*)^2)}, \\
S_{3333}^* &= \frac{C_{1111}^* + C_{1122}^*}{C_1^* - 2(C_{1133}^*)^2}, \\
S_{1122}^* &= \frac{(C_{1133}^*)^2 - C_{1122}^* C_{3333}^*}{(C_{1111}^* - C_{1122}^*)(C_1^* - 2(C_{1133}^*)^2)}, \\
S_{1133}^* &= S_{2233}^* = -\frac{C_{1133}^*}{C_1^* - 2(C_{1133}^*)^2}, \\
S_{1212}^* &= S_{2121}^* = \frac{C_{1212}^*}{C_2^*}, \quad S_{1221}^* = -\frac{C_{1221}^*}{C_2^*}, \quad S_{1313}^* = S_{2323}^* = \frac{C_{3232}^*}{C_3^*}, \\
S_{1331}^* &= S_{2332}^* = -\frac{C_{1331}^*}{C_3^*}, \quad S_{3131}^* = S_{3232}^* = \frac{C_{1313}^*}{C_3^*},
\end{aligned} \tag{65}$$

where  $C_1^* = C_{1111}^* C_{3333}^* + C_{1122}^* C_{3333}^*$ ,  $C_2^* = (C_{1212}^*)^2 - (C_{1221}^*)^2$  and  $C_3^* = C_{1313}^* C_{3232}^* - (C_{1331}^*)^2$ .

Consequently, the components of  $S_{ijmn}^*$  are written in terms of the effective engineering moduli, as follows:

$$\begin{aligned}
S_{iiii}^* &= \frac{e_{ii}^*}{\sigma_{ii}^*} = \frac{1}{sE_i^*}, \quad S_{ijij}^* = \frac{e_{ij}^*}{\sigma_{ij}^*} = -\frac{s\nu_{ij}^*}{sE_j^*} \quad (i \neq j), \\
S_{ijij}^* &= \frac{e_{ij}^*}{\sigma_{ij}^*} = \frac{1}{sG_{ij}^*} \quad (i \neq j), \quad S_{ijji}^* = \frac{e_{ij}^*}{\sigma_{ji}^*} = -\frac{s\zeta_{ijij}^*}{sG_{ji}^*} = -\frac{s\zeta_{ijji}^*}{sG_{ij}^*} \quad (i \neq j),
\end{aligned} \tag{66}$$

where  $i, j = 1, 2, 3$  and no summation by repeated Latin indices is assumed. In addition,  $sE_i^* \equiv sE_{ii}^* = \sigma_{ii}^*/e_{ii}^*$  is the classical Young's modulus along the  $x_i$ -direction according to the Voigt's notation,  $sG_{ij}^* = \sigma_{ij}^*/e_{ij}^*$  is the classical shear modulus on the  $Ox_i x_j$  plane,  $s\nu_{ij}^* = -e_{jj}^*/e_{ii}^*$  is the Poisson's ratio (the ratio between orthogonal strains directed in the principal direction), and  $s\zeta_{ijpq}^* = -e_{pq}^*/e_{ij}^*$  is the shear-strain ratio (the ratio between strains directed in the shear direction). The subscript  $S$  means that the engineering constants results from the compliance matrix.

Then, the independent engineering moduli written as functions of the components of the effective stiffness matrix are given as follows:  
Effective Young's moduli:

$$\begin{aligned}
sE_1^* &= sE_2^* = \frac{(C_{1111}^* - C_{1122}^*)(C_{1111}^* C_{3333}^* + C_{1122}^* C_{3333}^* - 2(C_{1122}^*)^2)}{C_{1111}^* C_{3333}^* - (C_{1122}^*)^2}, \\
sE_3^* &= \frac{C_{1111}^* C_{3333}^* + C_{1122}^* C_{3333}^* - 2(C_{1133}^*)^2}{C_{1111}^* + C_{1122}^*}.
\end{aligned} \tag{67}$$

Effective shear moduli

$$\begin{aligned}
sG_{12}^* &= sG_{21}^* = C_{1212}^* - \frac{(C_{1221}^*)^2}{C_{1212}^*}, \quad sG_{13}^* = sG_{23}^* = C_{1313}^* - \frac{(C_{1331}^*)^2}{C_{3232}^*}, \\
sG_{32}^* &= sG_{31}^* = C_{3232}^* - \frac{(C_{1331}^*)^2}{C_{1313}^*}.
\end{aligned} \tag{68}$$

Effective Poisson's ratios:

$$s\nu_{21}^* = \frac{(C_{1133}^*)^2 - C_{1122}^* C_{3333}^*}{(C_{1133}^*)^2 - C_{1111}^* C_{3333}^*}, \quad s\nu_{31}^* = s\nu_{32}^* = \frac{C_{1133}^*}{C_{1111}^* + C_{1122}^*}. \tag{69}$$

Effective shear-strain ratios:

$$s\zeta_{2112}^* = \frac{C_{1221}^*}{C_{1212}^*}, \quad s\zeta_{3113}^* = s\zeta_{3223}^* = \frac{C_{1331}^*}{C_{1313}^*}. \tag{70}$$

The Poisson's ratios ( $s\nu_{12}^*$ ,  $s\nu_{13}^*$ ,  $s\nu_{23}^*$ ) and shear-strain ratios ( $s\zeta_{1221}^*$ ,  $s\zeta_{1331}^*$ ,  $s\zeta_{2332}^*$ ) are not null but neither independent since they can be

expressed as functions of the previously given, i.e.,

$$\begin{aligned}
s\nu_{12}^* &= s\nu_{21}^*, \quad s\nu_{13}^* = s\nu_{23}^* = \frac{sE_1^*}{sE_3^*} s\nu_{31}^*, \quad s\zeta_{1221}^* = s\zeta_{2112}^*, \\
s\zeta_{1331}^* &= s\zeta_{2332}^* = \frac{sG_{13}^*}{sG_{32}^*} s\zeta_{3113}^*.
\end{aligned}$$

In a similar manner, it is possible to find the effective components  $T_{ijpq}^*$ , inverse of the torque matrix, as

$$\begin{aligned}
T_{1111}^* &= T_{2222}^* = \frac{D_{1111}^* D_{3333}^* - (D_{1122}^*)^2}{(D_{1111}^* - D_{1122}^*)(D_1^* - 2(D_{1122}^*)^2)}, \\
T_{3333}^* &= \frac{D_{1111}^* + D_{1122}^*}{D_1^* - 2(D_{1133}^*)^2}, \\
T_{1122}^* &= \frac{(D_{1133}^*)^2 - D_{1122}^* D_{3333}^*}{(D_{1111}^* - D_{1122}^*)(D_1^* - 2(D_{1133}^*)^2)}, \\
T_{1133}^* &= T_{2233}^* = -\frac{D_{1133}^*}{D_1^* - 2(D_{1133}^*)^2}, \\
T_{1212}^* &= T_{2121}^* = \frac{D_{1212}^*}{D_2^*}, \quad T_{1221}^* = -\frac{D_{1221}^*}{D_2^*}, \quad T_{1313}^* = T_{2323}^* = \frac{D_{3232}^*}{D_3^*}, \\
T_{1331}^* &= T_{2332}^* = -\frac{D_{1331}^*}{D_3^*}, \quad T_{3131}^* = T_{3232}^* = \frac{D_{1313}^*}{D_3^*},
\end{aligned} \tag{71}$$

where  $D_1^* = D_{1111}^* D_{3333}^* + D_{1122}^* D_{3333}^*$ ,  $D_2^* = (D_{1212}^*)^2 - (D_{1221}^*)^2$  and  $D_3^* = D_{1313}^* D_{3232}^* - (D_{1331}^*)^2$ , and thus, the relationships with the corresponding engineering constants are defined by

$$\begin{aligned}
T_{iiii}^* &= \frac{e_{ii}^*}{\sigma_{ii}^*} = \frac{1}{T E_i^*}, \quad T_{ijij}^* = \frac{e_{ij}^*}{\sigma_{ij}^*} = -\frac{T\nu_{ij}^*}{T E_j^*} \quad (i \neq j), \\
T_{ijij}^* &= \frac{e_{ij}^*}{\sigma_{ij}^*} = \frac{1}{T G_{ij}^*} \quad (i \neq j), \quad T_{ijji}^* = \frac{e_{ij}^*}{\sigma_{ji}^*} = -\frac{T\zeta_{ijij}^*}{T G_{ji}^*} = -\frac{T\zeta_{ijji}^*}{T G_{ij}^*} \quad (i \neq j),
\end{aligned} \tag{72}$$

where  $i, j = 1, 2, 3$  and no summation by repeated Latin indices is assumed. Also  $T E_i^* \equiv T E_{ii}^* = \sigma_{ii}^*/e_{ii}^*$  is the micropolar tortile or torsional Young's modulus along the  $x_i$ -direction according to the Voigt's notation,  $T G_{ij}^* = \sigma_{ij}^*/e_{ij}^*$  is the torsional shear modulus on the  $Ox_i x_j$  plane,  $T\nu_{ij}^* = -e_{jj}^*/e_{ii}^*$  is the twist Poisson's ratio and  $T\zeta_{ijpq}^* = -e_{pq}^*/e_{ij}^*$  represents the twist shear-strain ratio. The subscript  $T$  means that the engineering constants result from torque compliance matrix. A brief description of the duality between the terminologies used in classical and micropolar theories of elasticity can be found in Hassanpour and Hepler (2015).

Then, the independent engineering moduli as a function of the components of the effective torque matrix are given as follows:

Effective torsional Young's moduli:

$$\begin{aligned}
T E_1^* &= T E_2^* = \frac{(D_{1111}^* - D_{1122}^*)(D_{1111}^* D_{3333}^* + D_{1122}^* D_{3333}^* - 2(D_{1122}^*)^2)}{D_{1111}^* D_{3333}^* - (D_{1122}^*)^2}, \\
T E_3^* &= \frac{D_{1111}^* D_{3333}^* + D_{1122}^* D_{3333}^* - 2(D_{1133}^*)^2}{D_{1111}^* + D_{1122}^*}.
\end{aligned} \tag{73}$$

Effective torsional shear moduli:

$$\begin{aligned}
T G_{12}^* &= T G_{21}^* = D_{1212}^* - \frac{(D_{1221}^*)^2}{D_{1212}^*}, \quad T G_{13}^* = T G_{23}^* = D_{1313}^* - \frac{(D_{1331}^*)^2}{D_{3232}^*}, \\
T G_{32}^* &= T G_{31}^* = D_{3232}^* - \frac{(D_{1331}^*)^2}{D_{1313}^*}.
\end{aligned} \tag{74}$$

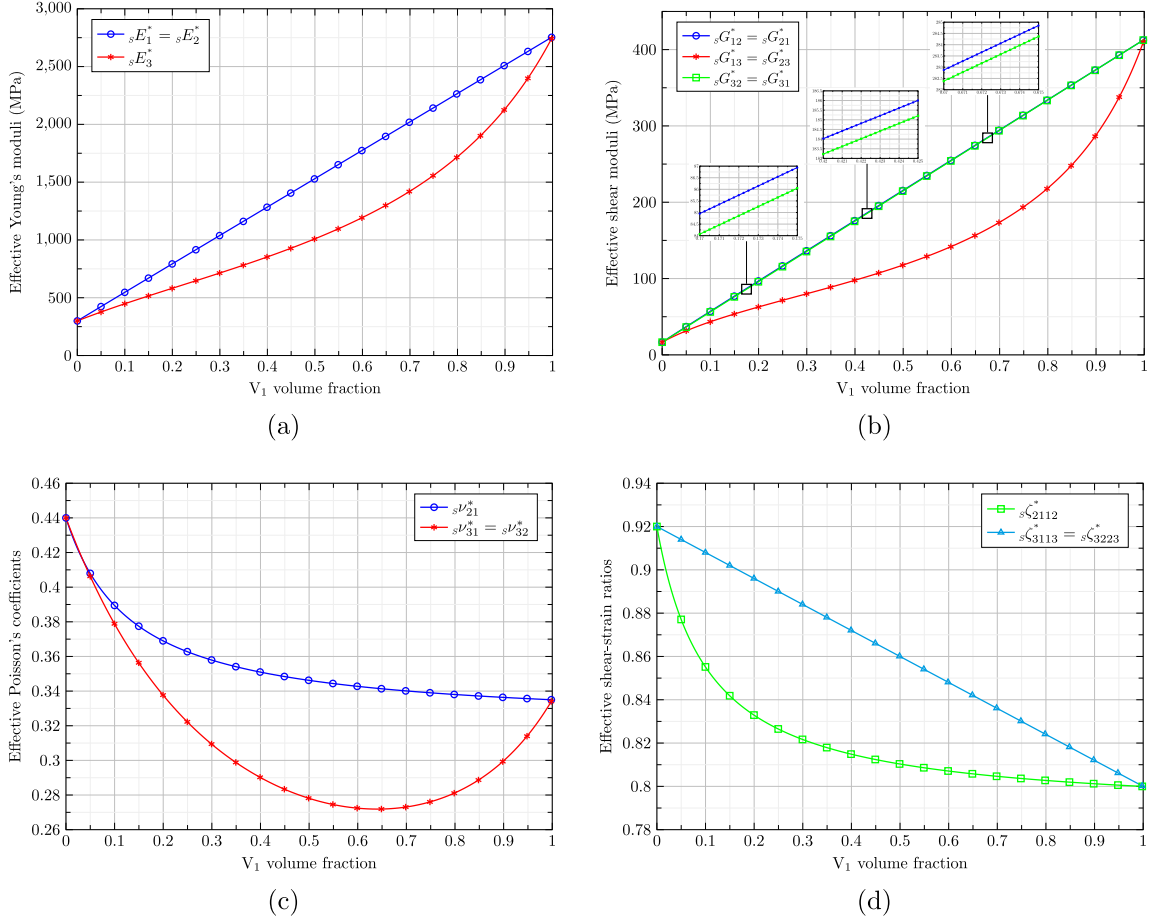


Fig. 4. Effective engineering moduli relative to the stiffness for a homogenized bi-laminated Cosserat composites with isotropic constituents as a function of  $V_1$  volume fraction: (a) Young's moduli, (b) Shear moduli, (c) Poisson's coefficients, and (d) Shear-strain ratios.

Effective twist Poisson's ratios:

$$T\nu_{21}^* = \frac{(D_{1133}^*)^2 - D_{1122}^* D_{3333}^*}{(D_{1133}^*)^2 - D_{1111}^* D_{3333}^*}, \quad T\nu_{31}^* = T\nu_{32}^* = \frac{D_{1133}^*}{D_{1111}^* + D_{1122}^*}. \quad (75)$$

Effective twist shear-strain ratios:

$$T\zeta_{2113}^* = \frac{D_{1221}^*}{D_{1212}^*}, \quad T\zeta_{3113}^* = T\zeta_{3223}^* = \frac{D_{1331}^*}{D_{1313}^*}. \quad (76)$$

where

$$T\nu_{12}^* = T\nu_{21}^*, \quad T\nu_{13}^* = T\nu_{23}^* = \frac{T E_1^*}{T E_3^*} T\nu_{31}^*, \quad T\zeta_{1221}^* = T\zeta_{2112}^*,$$

$$T\zeta_{1331}^* = T\zeta_{2332}^* = \frac{T G_{13}^*}{T G_{32}^*} T\zeta_{3113}^*.$$

Figs. 4–7 display the effective engineering moduli of centro-symmetric bi-laminated micropolar composites with isotropic (Figs. 4 and 5) and cubic (Figs. 6 and 7) constituents as a function of the  $V_1$  volume fraction. In both cases, as previously mentioned, the effective composite belongs to an orthotropic symmetry group with invariance under rotations of  $90^\circ$ , see Eremeyev and Pietraszkiewicz (2012). Therefore, the effective composite is defined by eighteen independent engineering constants, which can be derived using Eqs. (67)–(70) and (73)–(76). In addition, the values of  $C_{ijpq}^*$  and  $D_{ijpq}^*$  are obtained from Eqs. (52)–(53) for isotropic constituents, and from Eqs. (62)–(63) for cubic constituents. Their results are illustrated in Figs. 2 and 3 of Sections 5.1 and 5.3, respectively.

As it is observed in Fig. 4, all the effective engineering constants are continuous smooth functions in the whole analysis interval, and

they exhibit interesting behaviors. The effective Young's and shear moduli (Fig. 4(a) and (b)) have an increasing behavior as  $V_1$  increases. Moreover,  $sE_3^*$  and  $sG_{13}^*$  behave in a convex way. As expected, this is because the elastic material properties of the first layer (SyF) are bigger than those of the second one (PUF). In addition, the effective shear modulus  $sG_{12}^*$  and  $sG_{32}^*$  are close to each other with a deviation less than 1%, so that  $sG_{32}^* \leq sG_{12}^*$ . The effective Poisson's ratio  $s\nu_{21}^*$  is a decreasing function in the whole composite domain, whereas  $s\nu_{31}^*$  has a minimum value when for  $V_1 = 0.64$  (see Fig. 4(c)). Also, comparing them, the highest values are obtained for  $s\nu_{21}^*$ . The shear-strain ratios are decreasing functions in the whole interval (see Fig. 4(d)).

On the other hand, the effective torsional Young's moduli (Fig. 5(a)) and the micropolar shear moduli (Fig. 5(b)) are linearly decreasing as  $V_1$  increases, except  $TG_{13}^*$  which describes an inflexion point at  $V_1 = 0.680545$ . Notice that the effective torsional Young's moduli are negative when  $V_1 \geq 0.84$  approximately, and their direction changes. Also, the effective torsional Young's moduli  $TE_1^* = TE_2^*$  and  $TE_3^*$  have very approximate values. Therefore, to observe differences between them, a  $10^{-5}$  precision of its values is needed. More details can be seen in the figures' zoomed portions. In addition, the effective micropolar twist Poisson's ratios  $T\nu_{21}^*$  and  $T\nu_{31}^* = T\nu_{32}^*$  (Fig. 5(c)) are negative and so close to  $-1$ , and  $T\nu_{31}^* = T\nu_{32}^*$  reaches a maximum value when  $V_1 = 0.4$ . On the other hand, the effective twist shear-strain ratios (Fig. 5(d)) are positive and increasing functions, so that  $T\zeta_{3113}^* \geq T\zeta_{2112}^*$  for each  $V_1$  value.

In Figs. 6 and 7, a similar behavior is observed in almost all the effective engineering constants to those illustrated in Figs. 4 and 5. Here, it is assumed that the effective composite has constituents

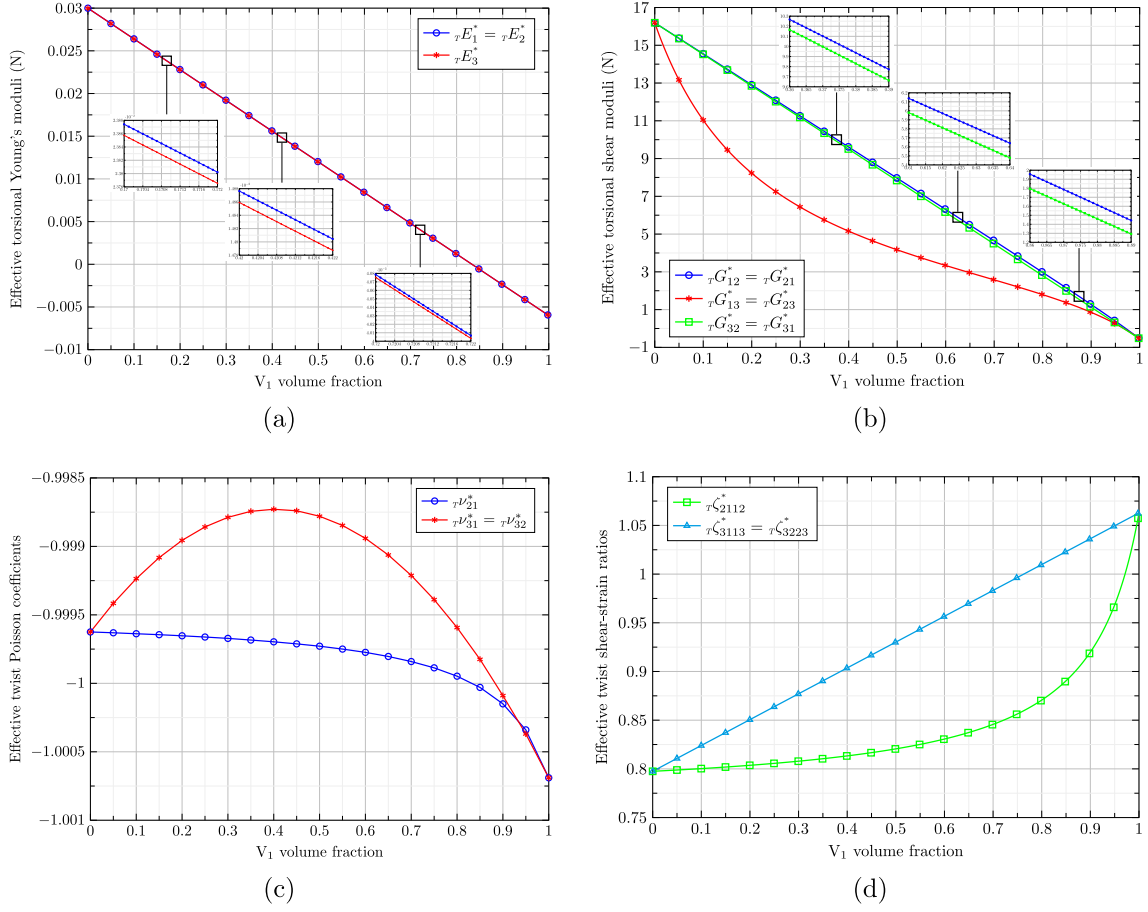


Fig. 5. Effective engineering moduli relative to the torques for a homogenized bi-laminated Cosserat composites with isotropic constituents as a function of  $V_1$  volume fraction: (a) Torsional Young's moduli, (b) Torsional shear moduli, (c) Twist Poisson's coefficients, and (d) Twist shear-strain ratios.

with cubic symmetry. In this case, the most interesting behaviors are observed in: (i) the effective Poisson's coefficients (Fig. 6(c)) that are increasing, unlike the case of isotropic constituents where they are decreasing as  $V_1$  increases, and (ii) the effective twist Poisson's ratios (Fig. 7(c)) are far from  $-1$  for cubic constituents and they are near to  $-1$  when isotropic constituents are assumed. They cannot be considered constant. In addition,  $r\nu_{31}^*$  reaches a maximum value equal to  $-0.284226$  when  $V_1 = 0.477$ .

### 5.5. Limit cases

In this section, we determine the expressions corresponding to the engineering modules for isotropic and cubic solids, based on the results derived of the homogenization process. It is important to mention that we do not obtain the expressions for micropolar laminated composites with isotropic and cubic constituents, but for simple monolithic isotropic and cubic materials.

From Eqs. (67)–(70) and (73)–(76), it is possible to find the engineering moduli for isotropic and cubic materials. With this purpose, the stiffness and torque matrices must be defined with equal number of independent properties: three for the isotropic case and four for the cubic case.

In order to find the analytical expressions of the engineering moduli of isotropic materials as functions of stiffness and torques, it is needed to know that the stiffness  $C_{ijpq}$  and torques  $D_{ijpq}$  matrices satisfy the form of Eqs. (50) and (51), respectively, where  $C_{1111} = C_{1122} + C_{1212} + C_{1221}$  and  $D_{1111} = D_{1122} + D_{1212} + D_{1221}$ , see Eringen (1999) and Eremeyev and Pietraszkiewicz (2012). Then, taking into account the symmetry conditions for isotropic Cosserat materials (Eqs. (50) and

(51)) and applying them in the Eqs. (67)–(70) and (73)–(76), the corresponding engineering moduli are obtained, such that: Young's modulus:

$${}_sE = {}_sE_1 = {}_sE_2 = {}_sE_3 = \frac{(C_{1111})^2 + C_{1111}C_{1122} - 2(C_{1122})^2}{C_{1111} + C_{1122}}. \quad (77)$$

Shear modulus:

$${}_sG = {}_sG_{12} = {}_sG_{13} = {}_sG_{32} = \frac{(C_{1212})^2 - (C_{1221})^2}{C_{1212}} = \frac{{}_sE(1 - {}_s\xi)}{1 + {}_s\nu}. \quad (78)$$

Poisson's ratio:

$${}_s\nu = {}_s\nu_{21} = {}_s\nu_{31} = {}_s\nu_{32} = \frac{C_{1122}}{C_{1111} + C_{1122}}. \quad (79)$$

Shear-strain ratio:

$${}_s\xi = {}_s\xi_{2112} = {}_s\xi_{3113} = {}_s\xi_{3223} = \frac{C_{1221}}{C_{1212}}. \quad (80)$$

Torsional Young's modulus:

$${}_TE = {}_TE_1 = {}_TE_2 = {}_TE_3 = \frac{(D_{1111})^2 + D_{1111}D_{1122} - 2(D_{1122})^2}{D_{1111} + D_{1122}}. \quad (81)$$

Torsional shear moduli:

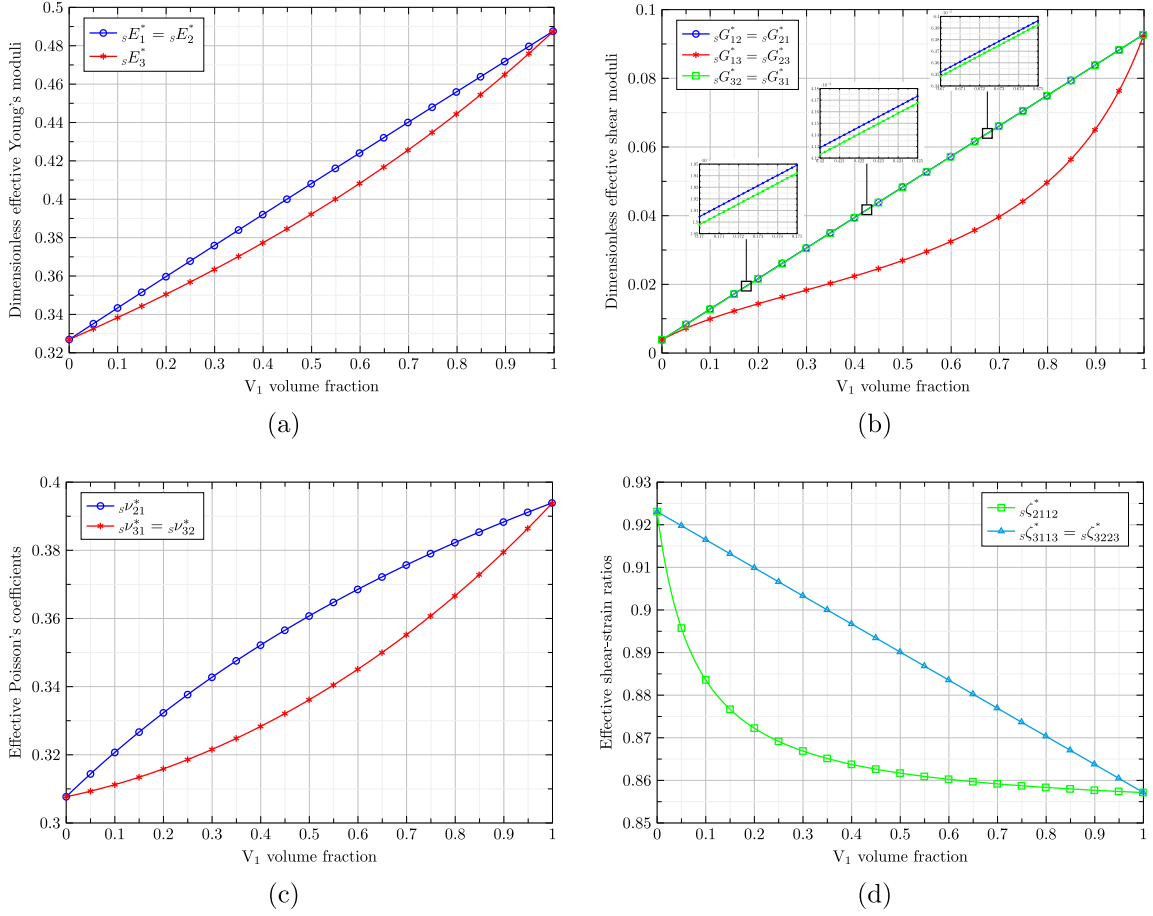
$${}_TG = {}_TG_{12} = {}_TG_{13} = {}_TG_{32} = \frac{(D_{1212})^2 - (D_{1221})^2}{D_{1212}} = \frac{{}_TE(1 - {}_T\xi)}{1 + {}_T\nu}. \quad (82)$$

Twist Poisson's ratio:

$${}_T\nu = {}_T\nu_{21} = {}_T\nu_{31} = {}_T\nu_{32} = \frac{D_{1122}}{D_{1111} + D_{1122}}. \quad (83)$$

Twist shear-strain ratio:

$${}_T\xi = {}_T\xi_{2112} = {}_T\xi_{3113} = {}_T\xi_{3223} = \frac{D_{1221}}{D_{1212}}. \quad (84)$$



**Fig. 6.** Effective engineering moduli relative to the stiffness for a homogenized bi-laminated Cosserat composites with cubic constituents as a function of  $V_1$  volume fraction: (a) Young's moduli, (b) Shear moduli, (c) Poisson's coefficients, and (d) Shear-strain ratios.

As mentioned before, micropolar media can be reduced to Cauchy media. Therefore, the formulas of the engineering moduli for isotropic materials are easy to determine assuming that  $C_{1212} = (C_{1111} - C_{1122})/2$ ,  $C_{1111} - C_{1122} - C_{1212} = 0$  and all effective torque moduli are null. Then, the Young's, Poisson's ratio and shear modulus are given by,

$$E = \frac{(C_{1111})^2 + C_{1111}C_{1122} - 2(C_{1122})^2}{C_{1111} + C_{1122}}, \quad \nu = \frac{C_{1122}}{C_{1111} + C_{1122}}, \quad (85)$$

$$G = \frac{C_{1111} - C_{1122}}{2} = \frac{E}{2(1 + \nu)}.$$

The expressions in Eq. (85) coincide with those reported by Jones (1999) in Eqs. (2.38), page 67, and by Royer and Dieulesaint (2000) in Fig. 3.8, page 140.

On the other hand, in the present work, the analytical expressions of the engineering moduli for a cubic centro-symmetric laminated composite are found replacing the symmetric conditions for cubic Cosserat materials into Eqs. (67)–(70) and (73)–(76), hence, we obtain the same expression as in Eqs. (77) and (84). Here, the difference is that the effective stiffness and torque matrices satisfy  $C_{1111} \neq C_{1122} + C_{1212} + C_{1221}$  and  $D_{1111} \neq D_{1122} + D_{1212} + D_{1221}$ , as in Eqs. (50) and (51), therefore, there are eight independent engineering moduli for cubic materials in contrast with the six independent moduli obtained for isotropic ones.

## 6. Conclusions

In this work, a brief introduction to the micropolar theory of elasticity and the basic equations for a micropolar (Cosserat) media are given. After that, based on the asymptotic homogenization method, the

general analytical formulas of the homogenized local problems over the periodic unit cell  $Y$ , the homogeneous problem, and the effective properties of a Cosserat media are derived. In addition, the local problems and the effective properties of multi-laminated Cosserat media are determined. In particular, the effective coefficients for centro-symmetric laminated Cosserat composites with isotropic or cubic constituents as a function of the material properties and the constituents volume fractions are provided. The homogenized Cosserat material belongs to an orthotropic symmetry group restricted with invariance under rotations by  $90^\circ$ , which is defined by nine effective stiffness and nine effective torque properties. Also, the effective engineering moduli related to the stiffness and torques are provided for centro-symmetric bi-laminated Cosserat composites with isotropic or cubic constituents (two effective Young's moduli, three effective shear moduli, two effective Poisson's ratios, two effective shear-strain ratios, two effective torsional Young's moduli, three effective torsional shear moduli, two effective twist Poisson's ratios, and two effective twist shear-strain ratios). Finally, numerical results are presented and discussed.

## CRedit authorship contribution statement

**R. Rodríguez-Ramos:** Conceptualization, Formal analysis, Investigation, Methodology, Software, Validation, Writing – original draft, Writing – review & editing. **V. Yanes:** Conceptualization, Formal analysis, Investigation, Methodology, Software, Validation, Writing – original draft, Writing – review & editing. **Y. Espinosa-Almeyda:** Conceptualization, Formal analysis, Investigation, Methodology, Software, Validation, Writing – original draft, Writing – review & editing. **J.A.**

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## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix. Solution of the local problems

In this appendix, the fundamental steps of the mathematical derivation for the  ${}_{pq}\mathcal{L}^1$  local problem solution is developed. An analogous procedure can be applied to find the  ${}_{pq}\mathcal{L}^2$  local problem solution, so the latter will be omitted, only the solutions are given.

The solution of the  ${}_{pq}\mathcal{L}^1$  local problem (Eq. (34)) consists in finding the periodic local functions  ${}_{pq}N_m$  and  ${}_{pq}V_m$ .

First, we integrate the first expression in Eq. (34) respect to  $y_3$ , which leads to

$$C_{i3pq} + C_{i3m3} {}_{pq}N'_m + C_{i3mn} \epsilon_{mnk} {}_{pq}V_k + B_{i3m3} {}_{pq}V'_m = {}_{pq}A_i, \quad (\text{A.1})$$

where  ${}_{pq}A_i$  is the integration constant that needs to be found.

Next, solving for  ${}_{pq}N'_m$  from Eq. (A.1) and applying the average operator, so that  $\langle {}_{pq}N'_m \rangle = 0$ , we get that  ${}_{pq}A_i$

$${}_{pq}A_i = \langle C_{i3m3}^{-1} \rangle^{-1} (\langle C_{i3m3}^{-1} C_{i3pq} \rangle + \langle C_{i3m3}^{-1} C_{i3mn} \epsilon_{mnk} {}_{pq}V_k \rangle + \langle C_{i3m3}^{-1} C_{i3mp3} {}_{pq}V'_k \rangle). \quad (\text{A.2})$$

Then, from Eqs. (A.1) and (A.2), the local function  ${}_{pq}N'_m$  can be written as a function of  ${}_{pq}V_m$  and  ${}_{pq}V'_m$ , as follows

$${}_{pq}N'_m = \left[ C_{m3i3}^{-1} \langle C_{i3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} C_{d3pq} \rangle - C_{m3i3}^{-1} C_{i3pq} \right] + \left[ C_{m3i3}^{-1} \langle C_{i3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} C_{d3ab} \epsilon_{abc} {}_{pq}V_c \rangle - C_{m3i3}^{-1} C_{i3ab} \epsilon_{abc} {}_{pq}V_c \right] + \left[ C_{m3i3}^{-1} \langle C_{i3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} B_{d3c3} {}_{pq}V'_c \rangle - C_{m3i3}^{-1} B_{i3c3} {}_{pq}V'_c \right]. \quad (\text{A.3})$$

In Eq. (A.3), the average value of each term is null. Therefore, re-writing Eq. (A.3), we have

$${}_{pq}N'_m = r_{mpq} + \hat{r}_{mpq}^{(1)} + \hat{r}_{mpq}^{(2)}, \quad (\text{A.4})$$

where

$$\begin{aligned} r_{mpq} &= C_{m3i3}^{-1} \langle C_{i3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} C_{d3pq} \rangle - C_{m3i3}^{-1} C_{i3pq}, \\ \hat{r}_{mpq}^{(1)} &= C_{m3i3}^{-1} \langle C_{i3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} C_{d3ab} \epsilon_{abc} {}_{pq}V_c \rangle - C_{m3i3}^{-1} C_{i3ab} \epsilon_{abc} {}_{pq}V_c, \\ \hat{r}_{mpq}^{(2)} &= C_{m3i3}^{-1} \langle C_{i3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} C_{d3c3} {}_{pq}V'_c \rangle - C_{m3i3}^{-1} B_{i3c3} {}_{pq}V'_c. \end{aligned} \quad (\text{A.5})$$

Similarly, following the above procedure in the second expression of Eq. (34), but for finding  ${}_{pq}V'_m$ , we have the relation of the local function  ${}_{pq}V'_m$  as a function of  ${}_{pq}N'_m$ , as follows

$${}_{pq}V'_m = D_{m3i3}^{-1} \langle D_{i3k3}^{-1} \rangle^{-1} \langle D_{k3d3}^{-1} (B_{d3pq} + B_{d3ab} \epsilon_{abc} {}_{pq}V_c + B_{d3c3} {}_{pq}N'_c) \rangle - D_{m3i3}^{-1} B_{i3pq} - D_{m3i3}^{-1} B_{i3ab} \epsilon_{abc} {}_{pq}V_c - D_{m3i3}^{-1} B_{i3c3} {}_{pq}N'_c, \quad (\text{A.6})$$

then, replacing Eq. (A.4) into Eq. (A.6) and grouping conveniently, we get that

$$\begin{aligned} {}_{pq}V'_m &= \left[ D_{m3i3}^{-1} \langle D_{i3k3}^{-1} \rangle^{-1} \langle D_{k3d3}^{-1} (B_{d3pq} + B_{d3c3} r_{cpq}) \rangle - D_{m3i3}^{-1} (B_{i3pq} + B_{i3c3} r_{cpq}) \right] + \\ &\left[ D_{m3i3}^{-1} \langle D_{i3k3}^{-1} \rangle^{-1} \langle D_{k3d3}^{-1} (B_{d3ab} \epsilon_{abc} {}_{pq}V_k + B_{d3c3} \hat{r}_{cpq}^{(1)}) \rangle - (B_{i3ab} \epsilon_{abc} {}_{pq}V_k + B_{i3c3} \hat{r}_{cpq}^{(1)}) \right] + \\ &\left[ D_{m3i3}^{-1} \langle D_{i3k3}^{-1} \rangle^{-1} \langle D_{k3d3}^{-1} B_{d3c3} \hat{r}_{cpq}^{(2)} \rangle - D_{m3i3}^{-1} B_{i3c3} \hat{r}_{cpq}^{(2)} \right]. \end{aligned} \quad (\text{A.7})$$

In particular, the average value of each term in Eq. (A.7) is null too, then, we can rewrite Eq. (A.7) as in Eq. (A.4), so that,

$${}_{pq}V'_m = e_{mpq} + \hat{e}_{mpq}^{(1)} + \hat{e}_{mpq}^{(2)}, \quad (\text{A.8})$$

where

$$\begin{aligned} e_{mpq} &= D_{m3i3}^{-1} \langle D_{i3k3}^{-1} \rangle^{-1} \langle D_{k3d3}^{-1} (B_{d3pq} + B_{d3c3} r_{cpq}) \rangle - D_{m3i3}^{-1} (B_{i3pq} + B_{i3c3} r_{cpq}), \\ \hat{e}_{mpq}^{(1)} &= D_{m3i3}^{-1} \left[ \langle D_{i3k3}^{-1} \rangle^{-1} \langle D_{k3d3}^{-1} (B_{d3ab} \epsilon_{abc} {}_{pq}V_k + B_{d3c3} \hat{r}_{cpq}^{(1)}) \rangle - B_{i3ab} \epsilon_{abc} {}_{pq}V_k - B_{i3c3} \hat{r}_{cpq}^{(1)} \right], \\ \hat{e}_{mpq}^{(2)} &= D_{m3i3}^{-1} \langle D_{i3k3}^{-1} \rangle^{-1} \langle D_{k3d3}^{-1} B_{d3c3} \hat{r}_{cpq}^{(2)} \rangle - D_{m3i3}^{-1} B_{i3c3} \hat{r}_{cpq}^{(2)}. \end{aligned} \quad (\text{A.9})$$

The structure of Eqs. (A.3) and (A.7) is assumed to be

$$\begin{aligned} \mathbf{N}'(y) &= \mathbf{a}^{(1)} + [\mathbf{a}^{(2)} \langle \mathbf{a}^{(3)} \mathbf{V} \rangle - \mathbf{a}^{(3)} \mathbf{V}] + [\mathbf{a}^{(2)} \langle \mathbf{a}^{(3)} \mathbf{V}' \rangle - \mathbf{a}^{(3)} \mathbf{V}'], \\ \mathbf{V}'(y) &= \mathbf{b}^{(1)} + [\mathbf{b}^{(2)} \langle \mathbf{b}^{(3)} \mathbf{V} \rangle - \mathbf{b}^{(3)} \mathbf{V}] + [\mathbf{b}^{(2)} \langle \mathbf{b}^{(3)} \mathbf{V}' \rangle - \mathbf{b}^{(3)} \mathbf{V}'], \end{aligned} \quad (\text{A.10})$$

where  $\mathbf{a}^{(n)} \equiv \mathbf{a}^{(n)}(y)$  and  $\mathbf{b}^{(n)} \equiv \mathbf{b}^{(n)}(y)$  ( $n = 1, 2, 3$ ) depend on  $y_3$ , and the symbols  $\mathbf{N}$  and  $\mathbf{V}$  represent the local functions  ${}_{pq}N_m$  and  ${}_{pq}V_m$ , respectively. Here the indexes are omitted for the sake of simplicity.

As can be seen in Eqs. (A.3) and (A.7), and therefore Eq. (A.10), the corresponding terms to  $\mathbf{a}^{(2)}(y)$  and  $\mathbf{b}^{(2)}(y)$  are near to 1, therefore, the second and third terms of Eq. (A.10) can be assumed as deviations from the mean value of  $\mathbf{V}$  and  $\mathbf{V}'$ , respectively. Thus, the main contribution in Eq. (A.10) is given by the first terms. Under this consideration, in order to find the local functions  ${}_{pq}N'_m$  and  ${}_{pq}V'_m$ , we only consider, the first terms in Eqs. (A.3) and (A.7), i.e.,

$${}_{pq}N'_m(y_3) \approx r_{mpq} = C_{m3i3}^{-1} \langle C_{i3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} C_{d3pq} \rangle - C_{m3i3}^{-1} C_{i3pq}, \quad (\text{A.11})$$

$${}_{pq}V'_m(y_3) \approx e_{mpq} = D_{m3i3}^{-1} \langle D_{i3k3}^{-1} \rangle^{-1} \langle D_{k3d3}^{-1} (B_{d3pq} + B_{d3c3} r_{cpq}) \rangle - D_{m3i3}^{-1} (B_{i3pq} + B_{i3c3} r_{cpq}), \quad (\text{A.12})$$

then, integrating  ${}_{pq}N'_m(y_3)$  and  ${}_{pq}V'_m(y_3)$  we have the approximate solutions, as follows

$$\begin{aligned} {}_{pq}N_m(y_3) &= \int_0^{y_3} {}_{pq}N'_m(z) dz - \left\langle \int_0^{y_3} {}_{pq}N'_m(z) dz \right\rangle \approx \\ &\int_0^{y_3} \left( C_{m3i3}^{-1} \langle C_{i3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} C_{d3pq} \rangle - C_{m3i3}^{-1} C_{i3pq} \right) dz, \end{aligned} \quad (\text{A.13})$$



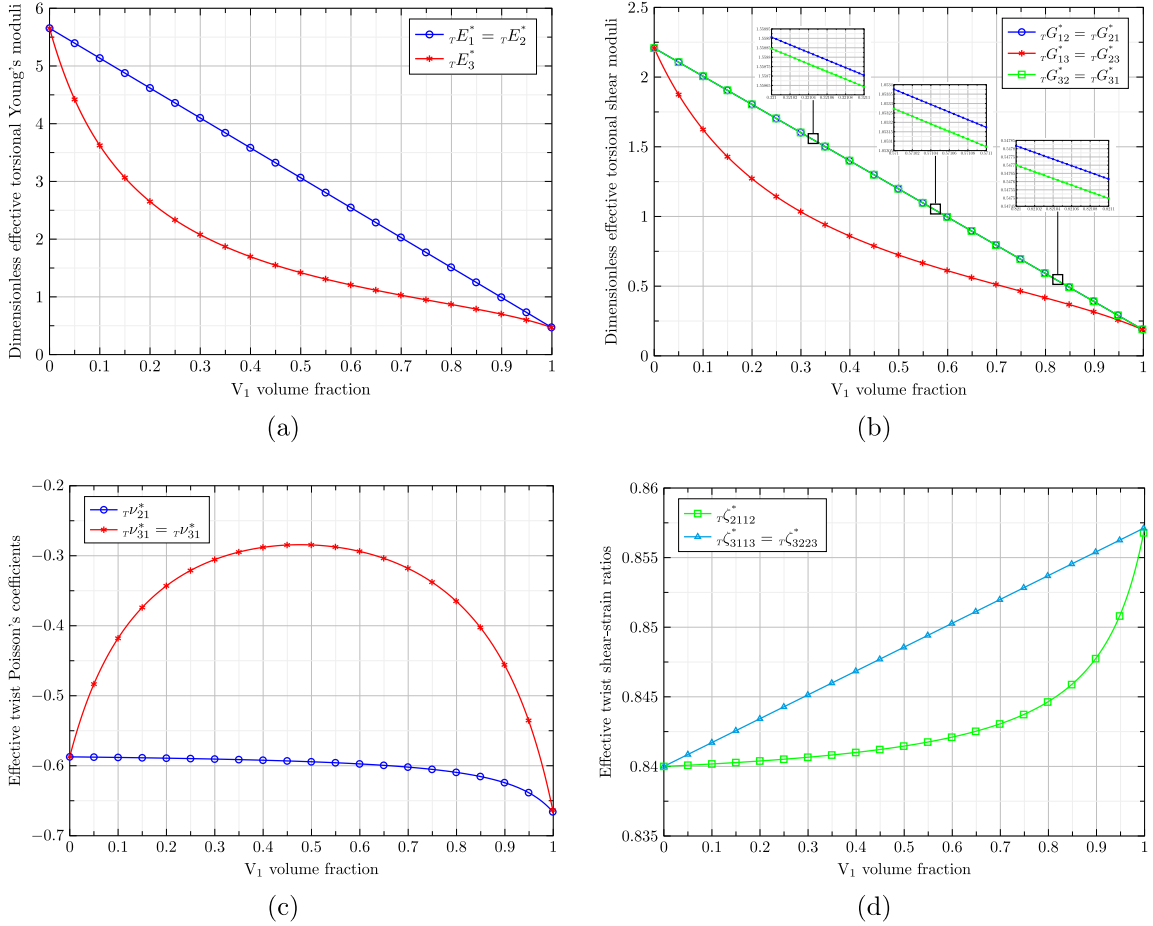


Fig. 7. Effective engineering moduli relative to the torques for a homogenized bi-laminated Cosserat composite with cubic constituents as a function of  $V_1$  volume fraction: (a) Torsional Young's moduli, (b) Torsional shear moduli, (c) Twist Poisson's coefficients, and (d) Twist shear-strain ratios.

$$\begin{aligned}
 {}_{pq}V_m(y_3) &= \int_0^{y_3} {}_{pq}V'_m(z) dz - \left\langle \int_0^{y_3} {}_{pq}V'_m(z) dz \right\rangle \approx \\
 &\int_0^{y_3} \left( D_{m3l3}^{-1} \left[ \langle D_{l3k3}^{-1} \rangle^{-1} \langle D_{k3d3}^{-1} (B_{d3pq} + B_{d3c3} r_{cpq}) \rangle \right. \right. \\
 &\left. \left. - (B_{l3pq} + B_{l3c3} r_{cpq}) \right] \right) dz. \quad (A.14)
 \end{aligned}$$

where  $r_{cpq} = C_{c3l3}^{-1} \langle C_{l3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} C_{d3pq} \rangle - C_{c3l3}^{-1} C_{l3pq}$ .

On the other hand, the approximately solutions of the  ${}_{pq}\mathcal{L}^2$  local problem (Eq. (35)), i.e.,  ${}_{pq}U_m$  and  ${}_{pq}M_m$ , are figured out here as follows

$${}_{pq}U'_m(y_3) = C_{m3l3}^{-1} \langle C_{l3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} B_{d3pq} \rangle - C_{m3l3}^{-1} B_{l3pq}, \quad (A.15)$$

$$\begin{aligned}
 {}_{pq}M'_m(y_3) &= D_{m3l3}^{-1} \langle D_{l3k3}^{-1} \rangle \langle D_{k3d3}^{-1} (D_{d3pq} + B_{d3c3} s_{cpq}) \rangle \\
 &- D_{m3l3}^{-1} (D_{l3pq} + B_{l3c3} s_{cpq}), \quad (A.16)
 \end{aligned}$$

then, integrating  ${}_{pq}U'_m(y_3)$  and  ${}_{pq}M'_m(y_3)$ , we have

$$\begin{aligned}
 {}_{pq}U_m(y_3) &= \int_0^{y_3} {}_{pq}U'_m(z) dz - \left\langle \int_0^{y_3} {}_{pq}U'_m(z) dz \right\rangle \approx \\
 &\int_0^{y_3} \left( C_{m3l3}^{-1} \langle C_{l3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} B_{d3pq} \rangle - C_{m3l3}^{-1} B_{l3pq} \right) dz, \quad (A.17)
 \end{aligned}$$

$$\begin{aligned}
 {}_{pq}M_m(y_3) &= \int_0^{y_3} {}_{pq}M'_m(z) dz - \left\langle \int_0^{y_3} {}_{pq}M'_m(z) dz \right\rangle \approx \\
 &\int_0^{y_3} \left( D_{m3l3}^{-1} \left[ \langle D_{l3k3}^{-1} \rangle \langle D_{k3d3}^{-1} (D_{d3pq} + B_{d3c3} s_{cpq}) \rangle \right. \right. \\
 &\left. \left. - (D_{l3pq} + B_{l3c3} s_{cpq}) \right] \right) dz, \quad (A.18)
 \end{aligned}$$

where  $s_{cpq} = C_{c3l3}^{-1} \langle C_{l3k3}^{-1} \rangle^{-1} \langle C_{k3d3}^{-1} B_{d3pq} \rangle - C_{c3l3}^{-1} B_{l3pq}$ .

## References

- Abreu, R., Thomas, C., Durand, S., 2018. Effect of observed micropolar motions on wave propagation in deep Earth minerals. *Phys. Earth Planet. Inter.* 276, 215–225.
- Alavi, S., Nasimsobhan, M., Ganghoffer, J., Sinoimeri, A., Sadighi, M., 2021. Chiral Cosserat model for architected materials constructed by homogenization. *Meccanica* 56, 2547–2574.
- Altenbach, J., Altenbach, H., Eremeyev, V.A., 2010. On generalized Cosserat type theories of plates and shells. A short review and bibliography. *Arch. Appl. Mech.* 80 (1), 73–92.
- Altenbach, H., Eremeyev, V.A., 2013. *Generalized Continua - From the Theory to Engineering Applications*, CISM International Centre for Mechanical Sciences, first ed. Springer, Vienna.
- Bakhvalov, N., Panasenko, G., 1989. *Homogenisation: Averaging Processes in Periodic Media*, first ed. In: *Mathematics and its Applications*, Kluwer Academic (Soviet Series).
- Beltran-Fernández, J., Hernández-Gómez, L., Urriolaigoitia-Calderón, G., González-Rebatú, A., Urriolaigoitia-Sosa, G., 2010. Biomechanics and numerical evaluation of cervical porcine models considering compressive loads using 2-D classic computer tomography CT, 3-D scanner and 3-D computed tomography. *Appl. Mech. Mater.* 24–25, 287–295.
- Bigoni, D., Drugan, W., 2007. Analytical derivation of Cosserat moduli via homogenization of heterogeneous elastic materials. *J. Appl. Mech.* 74 (4), 741–753.
- Bleyer, J., 2018. Multiphase continuum models for fiber-reinforced materials. *J. Mech. Phys. Solids* 121, 198–233.
- Branke, D., Brummund, J., Haasemann, G., Ulbricht, V., 2009. Obtaining Cosserat material parameters by homogenization of a Cauchy continuum. *Proc. Appl. Math.* 9, 425–426.
- Castillero, J.B., Otero, J.A., Ramos, R.R., Bourgeat, A., 1998. Asymptotic homogenization of laminated piezocomposite materials. *Int. J. Solids Struct.* 35 (5-6), 527–541.
- Cosserat, E., Cosserat, F., 1909. *Théorie des Corps Déformables*. A. Hermann et Fils, Paris, [in French].
- Devorak, G.J., 2013. *Micromechanics of Composite Materials*, first ed. Springer, Dordrecht, The Netherlands.

- Diebels, S., Steeb, H., 2002. The size effect in foams and its theoretical and numerical investigation. *Proc. R. Soc. Lond. Ser. A* 458, 2869–2883.
- Dos Reis, F., Ganghoffer, J., 2011. Construction of micropolar continua from the homogenization of repetitive planar lattices. In: Altenbach, H., Maugin, G., Erofeev, V. (Eds.), *Mechanics of Generalized Continua*. In: *Advanced Structured Materials*, Springer, Berlin, Heidelberg, pp. 193–217.
- Dos Reis, F., Ganghoffer, J., 2012. Construction of micropolar continua from the asymptotic homogenization of beam lattices. *Comput. Struct.* 112, 345–363.
- Ehlers, W., Ramm, E., Diebels, S., d'Addetta, G., 2003. From particle ensembles to Cosserat continua: homogenization of contact forces towards stresses and couple stresses. *Int. J. Solids Struct.* 40, 6681–6702.
- Emel'yanov, A.N., 2016. Efficient Performance in the Theory of Elasticity with Moment (Ph.D. thesis). Faculty of Mechanics and Mathematics. Moscow State University "M.V. Lomonosov", [in Russian].
- Eremeyev, V., Lebedev, L., Altenbach, H., 2013. *Foundations of Micropolar Mechanics*, first ed. Springer, Vienna.
- Eremeyev, V., Pietraszkiewicz, W., 2012. Material symmetry group of the non-linear polar-elastic continuum. *Int. J. Solids Struct.* 49 (14), 1993–2005.
- Eringen, A.C., 1966. Linear theory of micropolar elasticity. *J. Math. Mech.* JSTOR 15 (6), 909–923.
- Eringen, A.C., 1999. *Microcontinuum Field Theories I: Foundations and Solids*, first ed. Springer, New York.
- Forest, S., 1998. Mechanics of generalized continua : Construction by homogenization. *J. Phys.* IV 8, 39–484.
- Forest, S., 2002. Homogenization methods and mechanics of generalized continua - part 2. *Theor. Appl. Mech.* 28–29, 113–144.
- Forest, S., Padel, F., Sab, K., 2001. Asymptotic analysis of heterogeneous Cosserat media. *Int. J. Solids Struct.* 38, 4585–4608.
- Forest, S., Sab, K., 1998. Cosserat overall modeling of heterogeneous media. *Mech. Res. Commun.* 25 (4), 449–454.
- Forest, S., Trinh, D., 2011. Generalized continua and non-homogeneous boundary conditions in homogenisation methods. *J. Appl. Math. Mech.* ZAMM 91 (2), 90–109.
- Goda, I., Assadi, M., Belouettar, S., Ganghoffer, J.F., 2012. A micropolar anisotropic constitutive model of cancellous bone from discrete homogenization. *J. Mech. Behav. Biomed. Mater.* 16, 87–108.
- Goda, I., Assadi, M., Ganghoffer, J., 2013. Cosserat anisotropic models of trabecular bone from the homogenization of the trabecular structure: 2D and 3D framework. *Adv. Struct. Mater.* 111–141.
- Goda, I., Assadi, M., Ganghoffer, J.F., 2014. A 3D elastic micropolar model of vertebral trabecular bone from lattice homogenization of the bone microstructure. *Biomech. Model. Mechanobiol.* 13 (1), 53–83.
- Gorbachev, V., Emel'yanov, A., 2014. Homogenization of the equations of the Cosserat theory of elasticity of inhomogeneous bodies. *Mech. Solids* 49 (1), 73–82.
- Gorbachev, V., Emel'yanov, A., 2021. Homogenization of problems of Cosserat theory of elasticity of composites. Additional materials. In: *Intern. Scientific Symposium in Problems of Mechanics of Deformable Solids Dedicated to A.A. Il'yushin on the Occasion of his 100th Birthday* 49. pp. 81–88, [in Russian] (1).
- Hassanpour, S., Heppler, G.R., 2015. Micropolar elasticity theory: a survey of linear isotropic equations, representative notations, and experimental investigations. *Math. Mech. Solids* 22, 224–242.
- Hayes, M., Shuvalov, A., 1998. On the extreme values of Young's modulus, the shear modulus, and Poisson's ratio for cubic materials. *J. Appl. Mech.* 65, 786–787.
- Jones, R.M., 1999. *Mechanics of Composite Materials*, second ed. Taylor & Francis Inc.
- Lakes, R., 1995. On the torsional properties of single osteons. *J. Biomech.* 28, 1409–1410.
- Lakes, R., 2001. Elastic and viscoelastic behaviour of chiral materials. *Int. J. Mech. Sci.* 43 (7), 1579–1589.
- Lakes, R., Nakamura, S., Behiri, J., Bonfield, W., 1990. Fracture mechanics of bone with short cracks. *J. Biomech.* 23 (10), 967–975.
- Larsson, R., Diebels, S., 2007. A second-order homogenization procedure for multi-scale analysis based on micropolar kinematics. *Int. J. Numer. Methods Eng.* 69 (12), 2485–2512.
- Larsson, R., Zhang, Y., 2007. Homogenization of microsystem interconnects based on micropolar theory and discontinuous kinematics. *J. Mech. Phys. Solids* 55 (4), 819–841.
- Lazar, M., Kirchner, H.O.K., 2005. Cosserat (micropolar) elasticity in stroh form. *Int. J. Solids Struct.* 42 (20), 5377–5398.
- Leonetti, L., Fantuzzi, N., Trovalusci, P., Tornabene, F., 2019. Scale effects in orthotropic composite assemblies as micropolar continua: A comparison between weak- and strong-form finite element solutions. *Materials* 12 (5), 758.
- Li, X., Zhang, J., Zhang, X., 2011. Micro-macro homogenization of gradient-enhanced Cosserat media. *Eur. J. Mech. A Solids* 30, 362–372.
- Liu, X., Hu, G., 2004. Inclusion problem of microstretch continuum. *Int. J. Eng. Sci.* 42, 849–860.
- Liu, S., Su, W., 2009. Effective couple-stress continuum model of cellular solids and size effects analysis. *Int. J. Solids Struct.* 46, 2787–2799.
- Markert, B., 2011. *Advances in Extended and Multifield Theories for Continua*, first ed. In: *Lecture Notes in Applied and Computational Mechanics*, Springer, Berlin, Heidelberg.
- Masiani, R., Trovalusci, P., 1996. Cosserat and Cauchy materials as continuum models of brick masonry. *Meccanica* 31, 421–432.
- Maugin, G., 2013. *Continuum Mechanics Through the Twentieth Century. A Concise Historical Perspective*, first ed. Springer, Dordrecht.
- Maugin, G.A., Metrikine, A.V., 2010. *Mechanics of Generalized Continua: One Hundred Years After the Cosserats*, first ed. Springer, New York.
- Nika, G., 2021. Derivation of effective models from heterogeneous Cosserat media via periodic unfolding. *Ric. Mat.* <http://dx.doi.org/10.1007/s11587-021-00610-3>.**
- Nowacki, W., 1974. *The Linear Theory of Micropolar Elasticity, Micropolar Elasticity*, first ed.
- Nowacki, W., 1986. *Theory of Asymmetric Elasticity*, first ed. Pergamon-Press.
- Otero, J., Bravo-Castillero, J., Rodríguez-Ramos, 1997. Homogenization of heterogeneous piezoelectric medium. *Mech. Res. Commun.* 24 (1), 75–84.
- Park, H., Lakes, R., 1986. Cosserat micromechanics of human bone: Strain redistribution by a hydration sensitive constituent. *J. Biomech.* 19 (5), 385–397.
- Pobedrya, B., 1984. *Mechanics of Composite Materials [in Russian]*, first ed. Izd-vo MGU, Moscow.
- Reda, H., Alavi, S., Nasimsobhan, M., Ganghoffer, J., 2021. Homogenization towards chiral Cosserat continua and applications to enhanced Timoshenko beam theories. *Mech. Mater.* 155, 103728.
- Rizzi, G., Hütter, G., Madeo, A., Neff, P., 2021. Analytical solutions of the simple shear problem for micromorphic models and other generalized continua. *Arch. Appl. Mech.* 91, 2237–2254.
- Rosenber, J., Cimrman, R., Novacek, V., 2002. Modelling based on the orthotropic micropolar continuum. *Eng. Mech. Svatka*.
- Royer, D., Dieulesaint, E., 2000. *Elastic Waves in Solids I*, first ed. Springer-Verlag, Berlin Germany.
- Rueger, Z., Ha, C., Lakes, R., 2019. Cosserat elastic lattices. *Meccanica* 54, 1983–1999.
- Rueger, Z., Lakes, R., 2016. Experimental Cosserat elasticity in open-cell polymer foam. *Phil. Mag.* 96, 93–111.
- Rueger, Z., Lakes, R., 2019. Experimental study of elastic constants of a dense foam with weak Cosserat coupling. *J. Elasticity* 137, 101–115.
- Sab, K., Pradel, F., 2009. Homogenisation of periodic Cosserat media. *Int. J. Comput. Appl. Technol.* 34 (1), 60–71.
- Sanchez-Palencia, E., 1980. *Non-Homogeneous Media and Vibration Theory*, first ed. Springer-Verlag, Moscow.
- Sanchez-Palencia, E., 1985. *Homogenization Techniques for Composite Media*, first ed. Springer-Verlag, Moscow.
- Skrzat, A., Eremeyev, V., 2020. On the effective properties of foams in the framework of the couple stress theory. *Contin. Mech. Thermodyn.* 32, 1779–1801.
- Stefanou, I., Sulem, J., Vardoulakis, I., 2008. Three-dimensional Cosserat homogenization of masonry structures: elasticity. *Acta Geotech.* 3, 71–83.
- Tekoglu, C., Onck, P.R., 2005. Size effects in the mechanical behavior of cellular materials. *J. Mater. Sci.* 40, 5911–5917.
- Toupin, R., 1962. Elastic materials with couple-stresses. *Arch. Ration. Mech. Anal.* 11 (1), 385–414.
- Trovalusci, P., Masiani, R., 2003. Non-linear micropolar and classical continua for anisotropic discontinuous materials. *Int. J. Solids Struct.* 40 (5), 1281–1297.
- Trovalusci, P., Ostojca-Starzewski, M., Laura De Bellis, M., Murrall, A., 2015. Scale-dependent homogenization of random composites as micropolar continua. *Eur. J. Mech. A Solids* 49, 396–407.
- Trovalusci, P., Pau, A., 2014. Derivation of microstructured continua from lattice systems via principle of virtual works: the case of masonry-like materials as micropolar, second gradient and classical continua. *Acta Mech.* 225, 157–177.
- Voigt, W., 1887. Theoretische studien über die elastizitätsverhältnisse der krystalle. *Abh. Math. Cl. K. Ges. Wiss. Göttingen* 34, 3–51.
- Zheng, Q.S., Spencer, A.J.M., 1993. On the canonical representation for kronecker powers of orthogonal tensors with application to material symmetry problems. *Int. J. Eng. Sci.* 31 (4), 617–635.