On the evolutionary velocity-discrete Boltzmann equation

Leif Arkeryd, Anne Nouri

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Mathematical Sciences, 41296 Göteborg, Sweden,
arkeryd@chalmers.se
Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France,
anne.nouri@univ-amu.fr

Abstract. We consider the Cauchy problem for the velocity-discrete Boltzmann equations in any dimension. The velocity-discrete Boltzmann equations can be considered as approximations of the original velocity-continuous Boltzmann equation. The particles move with velocities in a given finite set. We prove global in time existence of mild solutions to the Cauchy problem for initial data with finite mass, entropy and entropy production. The proof is based on the introduction of sets of particular characteristics and a new strong compactness property of the integrated collision frequency. The sets of particular characteristics are constructed in order that the density and its associated collision frequency are bounded from above. Their complements are of arbitrarily small measure. The strong compactness property of the integrated collision frequency is based on the Kolmogorov-Riesz theorem. This replaces the compactness of velocity averages in the continuous velocity case, not available when the velocities are discrete.

1 Introduction.

The Boltzmann equation is a fundamental mathematical model in the kinetic theory of gases. Replacing its continuum of velocities with a discrete set of velocites is a simplification preserving the essential features of free flow and quadratic collision term. Besides this fundamental aspect they can approximate the Boltzmann equation with any given accuracy [9], and are thereby useful for approximations and numerics. In the quantum realm the

discrete Boltzmann equation can also be more directly connected to microscopic quasi/particle models.
A velocity-discrete model of a kinetic gas in $\mathbb{R}^d$ is a system of partial differential equations having the form,

$$\big(\partial_t + v_i \cdot \nabla_z\big) f_i = Q_i(f,f), \quad 1 \le i \le p,$$

where $f = (f_i)_{1\le i\le p}$ and $f_i = f_i(t,z)$ are the phase space densities at time t, position z, at given velocities $v_i \in \mathbb{R}^d$, $1 \le i \le p$. The collision operator $Q = (Q_i)_{1\le i\le p}$ with gain part Q^+, loss part Q^-, and collision frequency ν, is given by

$$Q_i(f,f) = Q_i^+(f,f) - Q_i^-(f,f), \quad Q_i^+(f,f) = \sum_{j,m,q=1}^{p} \Gamma_{ij}^{mq} f_m f_q,$$

$$\nu_i(f) = \sum_{j,m,q=1}^{p} \Gamma_{ij}^{mq} f_j, \qquad Q_i^-(f,f) = f_i\, \nu_i(f), \quad i = 1, ..., p.$$

The collision coefficients satisfy

$$\Gamma_{ij}^{mq} = \Gamma_{ji}^{mq} = \Gamma_{mq}^{ij} \ge 0, \quad \Gamma_{ij}^{ij} = 0. \tag{1.1}$$

If a collision coefficient Γ_{ij}^{mq} is non-zero, then the conservation laws for momentum and energy,

$$v_i + v_j = v_m + v_q, \quad |v_i|^2 + |v_j|^2 = |v_m|^2 + |v_q|^2, \tag{1.2}$$

are satisfied. The velocity-discrete model is called normal [12] if any solution of the equations

$$\Psi(v_i) + \Psi(v_j) = \Psi(v_m) + \Psi(v_q),$$

where the indices $(i,j;m,q)$ take all possible values satisfying $\Gamma_{ij}^{mq} > 0$, is given by

$$\Psi(v) = a + b \cdot v + c|v|^2, \quad \text{for some constants } (a,b,c) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}.$$

The paper studies the Cauchy problem in the torus $\mathbb{T}^d$ of $\mathbb{R}^d$, i.e.

$$(\partial_t + v_i \cdot \nabla_z) f_i(t,z) = Q_i(f,f)(t,z), \quad t > 0, \quad z \in \mathbb{T}^d, \tag{1.3}$$

$$f_i(0,z) = f_{0i}(z), \quad z \in \mathbb{T}^d, \tag{1.4}$$

for a given initial datum f_0. Solutions are considered in L^1 in one of the following equivalent forms;
the exponential multiplier form,

$$f_i(t,z) = f_{0i}(z - tv_i)e^{-\int_{-t}^{0} \nu_i(f)(t+s,z+sv_i)ds} + \int_{-t}^{0} Q_i^+(f,f)(t+s,z+sv_i)e^{-\int_s^0 \nu_i(f)(t+r,z+rv_i)dr}ds, \quad \text{a.a. } t>0, \quad z\in\mathbb{T}^d, \quad 1\le i\le p, \tag{1.5}$$

the mild form,

$$f_i(t,z) = f_{0i}(z - tv_i) + \int_{-t}^{0} Q_i(f,f)(t+s,z+sv_i)ds, \quad \text{a.a. } t>0, \quad z\in\mathbb{T}^d, \quad 1\le i\le p, \tag{1.6}$$

the renormalized form [14],

$$(\partial_t + v_i\cdot\nabla_z)\ln(1+f_i) = \frac{Q_i(f,f)}{1+f_i}, \quad f_i(0,\cdot) = f_{0i}, \quad 1\le i\le p, \tag{1.7}$$

in the sense of distributions.

For any $T>0$, denote by $L^1_+([0,T]\times\mathbb{T}^d)$ the set of space-periodic non-negative integrable functions. The main result of the present paper is

Theorem 1.1
Consider a non-negative initial value f_0, with mass, entropy and entropy production bounded,

$$\int_{\mathbb{T}^d} f_{0i}(1+\ln f_{0i})(z)dz \le c_0, \quad 1\le i\le p, \tag{1.8}$$

$$\int_{\mathbb{T}^d} \sum_{i,j,m,q=1}^{p} \Gamma_{ij}^{mq}\big(f_{0i}f_{0j} - f_{0m}f_{0q}\big)\ln\frac{f_{0i}f_{0j}}{f_{0m}f_{0q}}(z)dz \le c_0\,, \tag{1.9}$$

for some $c_0>0$. For any $T>0$ there exists a mild solution in $\big(L^1_+([0,T]\times\mathbb{T}^d)\big)^p$ to the Cauchy problem (1.3)-(1.4).

A general frame for velocity-discrete models of the Boltzmann equation has been developped in [15] and [11]. Contrary to the velocity-continuous Boltzmann equation, there was up to now no global in time existence results

for the Cauchy problem of general velocity-discrete models. Existence and uniqueness of global solutions to the Cauchy problem with small initial data is proven in [10]. Convergence of two particular velocity-discrete schemes to the Boltzmann equation is proven in [19]. Most mathematical results for velocity-discrete models of the Boltzmann equation have been obtained for specific models like the Carleman or Broadwell models, or in one space dimension. An overview is given in [16]. Half-space problems [4] and weak shock waves [5] for discrete velocity models have also been studied. A discussion of normal discrete velocity models can be found in [7]. In two dimensions, special classes of solutions to the Broadwell model are given in [6], [8] and [17]. In [13] the existence of continuous solutions to the two-dimensional stationary Broadwell model in a rectangle for continuous boundary data is proven. The paper [1] solves that problem in an L^1-setting. Existence of stationary solutions to the general velocity-discrete Boltzmann equation in the plane is proven in [2]-[3].

The proof of Theorem 1.1 starts with a priori-estimates for (1.3)-(1.4) and the control of mass, energy and entropy. The recourse to renormalized solutions is classical [14]. After having constructed approximate solutions to the problem, a compactness argument in the strong L^1 topology is required in order to pass to the limit in the collision terms. Averaging lemmas as for the continuous velocity Boltzmann equation can not be used when there is only a finite number of velocities. Instead, we prove a weaker result, still relevant for our purpose, i.e. the strong L^1-compactness of the collision frequency integrated along appropriate characteristics. Compactness is proven using the Kolmogorov-Riesz theorem [18], [20]. Compared to the proofs in the stationary frame in the plane [2]-[3] where the same compactness is used, with the two directions of two interacting velocities, a supplementary argument has to be introduced in order to deal with the $d-1$ supplementary directions that are involved. This is performed using the geometric properties of the problem. The present approximation scheme has strongly converging integrated collision frequency and weak convergence of the gain term. Together they imply that the weak limit of the gain term equals the gain term of the weak limit.
Approximations for bounded collision operators are constructed in Section 2. Some of their properties are studied in Section 3. 'Good' i-characteristics are stressed here where the i^{th} component of the phase density and its associated collision frequency are bounded from above. The complement is of arbitrarily small measure. Section 4 is devoted to the proof of the key lemma stating the L^1 compactness of the collision frequency integrated along

characteristics. The passage to the limit in the equations satisfied by the approximations is performed in Section 5.

2 Approximations.

All along the paper, c_0 will denote constants only depending on f_0. Let

$$Q^k = Q^{k+} - Q^{k-}, \quad Q_i^{k+}(F,F) = \sum_{j,m,q=1}^{p} \Gamma_{ij}^{mq} \frac{F_m}{1+\frac{F_m}{k}} \frac{F_q}{1+\frac{F_q}{k}},$$

$$Q_i^{k-}(F,F) = F_i\, \nu_i(F), \qquad \nu_i(F) = \sum_{j,m,q=1}^{p} \Gamma_{ij}^{mq} \frac{F_j}{\big(1+\frac{F_i}{k}\big)\big(1+\frac{F_j}{k}\big)}.$$

Lemma 2.1
For every $k \in \mathbb{N}^*$, there is a unique solution $F^k \in \big(L^1_+([0,T]\times\mathbb{T}^d)\big)^p$ to

$$\partial_t F_i^k + v_i \cdot \nabla_z F_i^k = Q_i^k(F^k, F^k), \tag{2.1}$$

$$F_i^k(0,\cdot) = f_{0i}^k, \quad 1 \le i \le p. \tag{2.2}$$

Proof of Lemma 2.1
Denote by $a \wedge b$ the minimum of two real numbers a and b. Denote by

$$f_{0i}^k = f_{0i} \wedge k, \quad 1 \le i \le p.$$

Let $\mathcal{S}$ be the map defined on $(L^1_+([0,T]\times\mathbb{T}^d)^p$ by $\mathcal{S}(f) = F$, where $F = (F_i)_{1\le i\le p}$ is the solution of

$$\big(\partial_t + v_i\cdot\nabla_z\big)F_i = \sum_{j,m,q=1}^{p} \Gamma_{ij}^{mq}\Big(\frac{F_m}{1+\frac{F_m}{k}}\frac{f_q}{1+\frac{f_q}{k}} - \frac{F_i}{1+\frac{F_i}{k}}\frac{f_j}{1+\frac{f_j}{k}}\Big), \tag{2.3}$$

$$F_i(0,\cdot) = f_{0i}^k. \tag{2.4}$$

$F = \mathcal{S}(f)$ can be obtained as the limit in $\big(L^1_+([0,T]\times\mathbb{T}^d)\big)^p$ of the sequence $(F^n)_{n\in\mathbb{N}}$ defined by $F^0 = 0$ and

$$\big(\partial_t + v_i\cdot\nabla_z\big)F_i^{n+1} = \sum_{j,m,q=1}^{p} \Gamma_{ij}^{mq}\Big(\frac{F_m^n}{1+\frac{F_m^n}{k}}\frac{f_q}{1+\frac{f_q}{k}} - \frac{F_i^{n+1}}{1+\frac{F_i^n}{k}}\frac{f_j}{1+\frac{f_j}{k}}\Big), \tag{2.5}$$

$$F_i^{n+1}(0,\cdot) = f_{0i}^k, \qquad n \in \mathbb{N}. \tag{2.6}$$

F^{n+1} can be written in the following exponential form,

$$F_i^{n+1}(t,z) = f_{0i}^k(z-tv_i)e^{-\sum_{j,m,q}\Gamma_{ij}^{mq}\int_{-t}^{0}\frac{f_j}{(1+\frac{F_i^n}{k})(1+\frac{f_j}{k})}(t+s,z+sv_i)ds} + \sum_{j,m,q=1}^{p}\Gamma_{ij}^{mq}\int_{-t}^{0}\frac{F_m^n}{1+\frac{F_m^n}{k}}\frac{f_q}{1+\frac{f_q}{k}}(t+s,z+sv_i) \\ e^{-\sum_{j,m,q}\Gamma_{ij}^{mq}\int_{s}^{0}\frac{f_j}{(1+\frac{F_i^n}{k})(1+\frac{f_j}{k})}(t+r,z+rv_i)dr}\,ds, \quad 1\le i\le p. \tag{2.7}$$

The sequence $(F^n)_{n\in\mathbb{N}}$ is monotone. Indeed, by the exponential form of F_i^1,

$$F_i^0 \le F_i^1, \quad 1\le i\le p.$$

If $F_i^n \le F_i^{n+1}$, $1\le i\le p$, then it follows from the exponential form that $F_i^{n+1}\le F_i^{n+2}$. Moreover,

$$\sum_{i=1}^{p}\big(\partial_t + v_i\cdot\nabla_z\big)F_i^{n+1} = \sum_{j,m,q=1}^{p}\Gamma_{ij}^{mq}\frac{(F_m^n - F_m^{n+1})}{1+\frac{F_m^n}{k}}\frac{f_q}{1+\frac{f_q}{k}} \le 0,$$

so that

$$\sum_{i=1}^{p}\int_{\mathbb{T}^d}F_i^{n+1}(t,z)dz \le \sum_{i=1}\int_{\mathbb{T}^d}f_{0i}^k(z)dz, \quad t\in[0,T]. \tag{2.8}$$

By the monotone convergence theorem, $(F^n)_{n\in\mathbb{N}}$ converges in $\big(L^1([0,T]\times\mathbb{T}^d)\big)^p$ to a solution F of (2.3)-(2.4). The solution of (2.3)-(2.4) is unique in the set of non-negative functions. Indeed, let $G=(G_i)_{1\le i\le p}$ be a non-negative solution of (2.3)-(2.4). By induction it holds that

$$\forall n\in\mathbb{N}, \quad F_i^n\le G_i, \quad 1\le i\le p, \tag{2.9}$$

where (2.9) holds for $n=0$, since $G_i\ge 0$, $1\le i\le p$. Assume (2.9) holds for n. Using the exponential form of F_i^{n+1} implies $F_i^{n+1}\le G_i$. Consequently,

$$F_i\le G_i, \quad 1\le i\le p. \tag{2.10}$$

Moreover, subtracting the partial differential equations satisfied by G_i from the partial differential equations satisfied by F_i, $1\le i\le p$, and integrating

the resulting equation on $[0,T]\times\mathbb{T}^d$, it results

$$\sum_{i=1}^{p}\int_{\mathbb{T}^d}(G_i-F_i)(t,z)dz=0,\quad t\in[0,T]. \tag{2.11}$$

It follows from (2.10)-(2.11) that $G=F$.
The map $\mathcal{S}$ is a contraction in $(L^1([0,T_1]\times\mathbb{T}^d))^p$ for a small enough time T_1 only depending on k. Hence by a Banach fixed point theorem, there is a unique fixed point for $\mathcal{S}$ in $\big(L^1([0,T_1]\times\mathbb{T}^d)\big)^p$. The procedure can be repeated on $[T_1,2T_1]$, ..., which gives a solution F to (2.1)-(2.2) in $\big(L^1_+(0,T]\times\mathbb{T}^d)\big)^p$. ■

3 Properties of the approximations

Denote by $\mathcal{F}_i^k=\frac{F_i^k}{1+\frac{F_i^k}{k}}$.

Lemma 3.1
Mass and energy are conserved by F^k.
A variant $\sum_{i=1}^{p}\int_{\mathbb{T}^d}F_i^k\ln\mathcal{F}_i^k(\cdot,z)dz$ of entropy, and entropy dissipation

$$\int_{[0,T]\times\mathbb{T}^d}\sum_{i,j,m,q=1}^{p}\Gamma_{ij}^{mq}\big(\mathcal{F}_i^k\mathcal{F}_j^k-\mathcal{F}_m^k\mathcal{F}_q^k\big)\ln\frac{\mathcal{F}_i^k\mathcal{F}_j^k}{\mathcal{F}_m^k\mathcal{F}_q^k}(t,z)dtdz\,, \tag{3.1}$$

are uniformly bounded. $(F^k)_{k\in\mathbb{N}^*}$ is weakly compact in L^1.

Proof of Lemma 3.1.
The conservations of mass and energy are straightforward. Denote (3.1) by D^k. Multiply (2.1) with $\ln\frac{F_i^k}{1+\frac{F_i^k}{k}}$, add the equations in i, integrate the resulting equation on $[0,t]\times\mathbb{T}^d$ and use (1.8). Hence

$$\sum_{i=1}^{p}\int_{\mathbb{T}^d}\Big(F_i^k\ln F_i^k-k(1+\frac{F_i^k}{k})\ln(1+\frac{F_i^k}{k})\Big)(t,z)dz$$
$$+\int_{[0,t]\times\mathbb{T}^d}\sum_{ijmq}\Gamma_{ij}^{mq}\big(\mathcal{F}_i^k\mathcal{F}_j^k-\mathcal{F}_m^k\mathcal{F}_q^k\big)\ln\frac{\mathcal{F}_i^k\mathcal{F}_j^k}{\mathcal{F}_m^k\mathcal{F}_q^k}\le c_0,\quad t\in[0,T].$$

This implies the uniform boundedness of $(D_k)_{k\in\mathbb{N}^*}$. Moreover,

$$k\int_{\mathbb{T}^d}\ln(1+\frac{F_i^k}{k})(t,z)dz\le\int_{\mathbb{T}^d}F_i^k(t,z)dz\le c_0.$$

Hence

$$\sum_{i=1}^p\int_{\mathbb{T}^d}F_i^k\ln\frac{F_i^k}{1+\frac{F_i^k}{k}}(t,z)dz\le c_0,\quad t\in[0,T]. \tag{3.2}$$

And so,

$$\begin{aligned}&\sum_{i=1}^p\int_{z\in\mathbb{T}^d;F_i^k(t,z)\le k}F_i^k\ln F_i^k(t,z)dz\\&+\ln\frac{k}{2}\int_{z\in\mathbb{T}^d;F_i^k(t,z)\ge k}F_i^k(t,z)(t,z)\le c_0,\quad t\in[0,T].\end{aligned}$$

Together with the uniform boundedness of its mass, it implies the weak L^1 compactness of $(F^k)_{k\in\mathbb{N}^*}$. ∎

For any $i\in\{1,\cdots,p\}$, denote by I_j the subset of $\{1,\cdots,p\}$ such that $j\in I_j$ if and only if there is $\Gamma_{ij}^{mq}>0$ for some $(m,q)\in\{1,\cdots,p\}^2$.

Lemma 3.2
Let $T>0$ be given. For every $\epsilon>0$, there is a subset $\Omega_i^{k,\epsilon}$ of measure smaller than $c_0\epsilon$ of i-characteristics of $[0,T]\times\mathbb{T}^d$,

$$\{(t+s,z+sv_i);\quad(t,z)\in[0,T],\times\mathbb{T}^d,\ s\in[-t,T-t]\},$$

such that

$$F_i^k(t+s,z+sv_i)\le\frac{1}{\epsilon}\exp(\frac{1}{\epsilon}), \tag{3.3}$$

$$\text{and}\quad\int_{-t}^{T-t}F_j^k(t+s,z+sv_i)ds\le\frac{1}{\epsilon},\quad(t+s,z+sv_i)\notin\Omega_{i\epsilon}^k,\quad j\in J_i. \tag{3.4}$$

Proof of Lemma 3.2.
It follows from

$$\frac{d}{ds}F_i^k(t+s,z+sv_i)e^{\int_0^s\nu_i^k(F^k)(t+r,z+rv_i)dr}\ge0,\quad s\in[-t,0],\quad z\in\mathbb{T}^d,$$

that

$$F_i^k(t+s,z+sv_i) \le F_i^k(T,z+(T-t)v_i)e^{\sum_{j,m,q}\Gamma_{ij}^{mq}\int_{-t}^{T-t}F_j^k(t+r,z+rv_i)dr}, \qquad t\in[0,T],\quad z\in\mathbb{T}^d,\quad s\in[-t,T-t].$$

The bound $\int_{\mathbb{T}^2} F_i^k(T,Z)dZ \le c_0$ implies that the measure of the set A of $Z\in\mathbb{T}^d$ such that $F_i^k(T,Z) > \frac{1}{\epsilon}$ is smaller than $c_0\epsilon$. Consequently

$$\mid \{(t,z)\in[0,T]\times\mathbb{T}^d;\quad z+(T-t)v_i\in A\} \mid \le c_0\epsilon.$$

Analogously, $\int_{[0,T]\times\mathbb{T}^d} F_j^k(r,z)dzdr \le c_0$, $j\in J_i$, implies

$$\mid \{(t,z)\in[0,T]\times\mathbb{T}^d; \int_{-t}^{T-t} F_j^k(t+r,z+rv_i)dr > \frac{1}{\epsilon}\} \mid \le c_0\epsilon.$$

The lemma follows. ∎

Denote by

$$\chi_i^{k,\epsilon} \quad \text{the characteristic function of the complement of } \Omega_i^{k,\epsilon}. \tag{3.5}$$

4 Compactness of the collision frequency integrated along characteristics.

Lemma 4.1
For every $i\in\{1,\cdots,p\}$, the integrated collision frequency sequence associated to F_i^k,

$$\sum_{j\in J_i}\Big(\int_{-t}^{0}(\sum_{m,q}\Gamma_{ij}^{mq})F_j^k(t+s,z+sv_i)ds\Big)_{k\in\mathbb{N}^*}$$

is compact in L^1.

Proof of Lemma 4.1.
Let $i\in\{1,\cdots,p\}$. Equation (1.3) can be rewritten as

$$\big(\partial_t+(v_n-v_i)\cdot\nabla_z\big)f_n(t,z+tv_i) = Q_n(f,f)(t,z+tv_0),\quad 1\le n\le p,$$

Consider $(v_n-v_i)_{1\le n\le p}$ as the set of interacting velocities, and $\big(f_n(t,z+tv_i)\big)_{1\le n\le p}$ as the new unknown. It satisfies (1.3). Moreover, $v_i=0$

and $\partial_t f_i = Q_i(f,f)$.
Let $j \in J_i$. Let us prove the L^1 compactness of $\big(\int_{-t}^0 F_j^k(t+s,z)ds\big)_{k\in\mathbb{N}^*}$. The uniform equi-integrability of $(\int_{-t}^0 F_j^k(t+s,z)ds)_{k\in\mathbb{N}^*}$ in the $(1,0)$ direction is straightforward. Its uniform equi-integrability in the $(1,v_j)$ direction is proven using the solution formula for F_j^k. Indeed, by the weak L^1 compactness of $(F^k)_{k\in\mathbb{N}^*}$, it is sufficient to prove the uniform equi-integrability of $(\int_{-t}^0 \chi_j^{k,\epsilon} F_j^k(t+s,z)ds)_{j\in\mathbb{N}}$ for ϵ small enough. Moreover,

$$\begin{aligned}
&\int_{-t}^0 \big((\chi_j^{k,\epsilon}F_j^k)(t+s+h,z+hv_j) - (\chi_j^{k,\epsilon}F_j^k)(t+s,z)\big)ds \\
&= \int_{-t}^0\int_0^h \big(\chi_j^{k,\epsilon}Q_j^k(F^k,F^k)\big)(t+s+r,z+rv_j)dr\,ds, \qquad (4.1)
\end{aligned}$$

so that, by the uniform boundedness of (3.1), and (3.3),

$$\begin{aligned}
&\int |\int_{-t}^0 \big((\chi_j^{k,\epsilon}F_j^k)(t+s+h,z+hv_j) - (\chi_j^{k,\epsilon}F_j^k)(t+s,z)\big)ds|dt\,dz \\
&\le \frac{c}{\ln\Lambda} + \frac{c\Lambda e^{\frac{1}{\epsilon}}}{\epsilon}\int\int_{-t}^0\int_0^h F_i^k(t+s+r,z+rv_j)dr\,ds\,dt\,dz \\
&\le \frac{c}{\ln\Lambda} + \frac{ch\Lambda e^{\frac{1}{\epsilon}}}{\epsilon}. \qquad (4.2)
\end{aligned}$$

Choosing Λ large enough, then h small enough closes the argument.
Complete v_j by $u_1,\cdots,u_{d-1}$ so that $(v_j,u_1,\cdots,u_{d-1})$ be a basis of $\mathbb{R}^d$. The proof of Lemma 4.1 will be closed by proving the uniform equi-integrability of $\big(\int_{-t}^0 F_j^k(t+s,z)ds\big)_{k\in\mathbb{N}^*}$ in the $(0,u_n)$ directions, $1\le n\le d-1$. Let $u\in\{u_1,\cdots,u_{d-1}\}$. It is sufficient to prove the uniform equi-integrability of $\big(\int_{-t}^0 \big(\chi_j^{k,\epsilon}F_j^k\big)(t+s,z)ds\big)_{k\in\mathbb{N}^*}$ in the $(0,u)$ direction. Using the solution formula for F_j^k,

$$\begin{aligned}
&\int_{-t}^0 \big(\chi_j^{k,\epsilon}F_j^k(t+s,z+hu) - \chi_j^{k,\epsilon}F_j^k(t+s,z)\big)ds \\
&= \int_{-t}^0 \big(\chi_j^{k,\epsilon}f_{0j}^k(z+hu-(t+s)v_j) - \chi_j^{k,\epsilon}f_{0j}^k(z-(t+s)v_j)\big)ds \\
&+ \int_{\mathcal{T}_h(t,z)} \chi_j^{k,\epsilon}Q_j^k(F^k,F^k) - \int_{\mathcal{T}(t,z)} \chi_j^{k,\epsilon}Q_j^k(F^k,F^k), \qquad (4.3)
\end{aligned}$$

where $\mathcal{T}(t,z)$ (resp. $\mathcal{T}_h(t,z)$) are the triangles defined by

$$\begin{aligned}
&\mathcal{T}(t,z) = \{(t+s+r,z+rv_j);\quad s\in[-t,0],\quad r\in[-t-s,0]\}, \\
&(\text{resp. } \mathcal{T}_h(t,z) = \{(t+s+r,z+rv_j+hu);\ s\in[-t,0],\ r\in[-t-s,0]\})\,.
\end{aligned}$$

When integrated with respect to $(t,z) \in [0,T] \times \Omega$, the absolute value of the second line of (4.3) tends to zero with h.

Let $A(t,z)$, $C(0,z)$, $D(0,z+tv_j)$ be the vertices of $\mathcal{T}(t,z)$, and $B(t,z+hu)$, $E(0,z+hu)$, $F(0,z+tv_j+hu)$ be the vertices of $\mathcal{T}_h(t,z)$.

Denote by D_h the trapezoidal volume between $\mathcal{T}(t,z)$ and $\mathcal{T}_h(t,z)$ connected by the segments $[A,B]$, $[C,E]$ and $[D,F]$.

Consider the $(1,u)$ characteristics starting at $\mathcal{T}(t,z)$. $\mathcal{T}(t,z)$ splits into

$$\mathcal{T}(t,z) = A_h \cup \big(\mathcal{T}(t,z) \setminus A_h\big), \tag{4.4}$$

where the forward in time $(1,u)$ characteristics starting at A_h cross $ABFD$ before reaching $\mathcal{T}_h(t,z)$. Denote by $\bar{A}_h$ the intersection of these forward $(1,u)$ characteristics with $ABFD$. $\mathcal{T}_h(t,z)$ splits into

$$\mathcal{T}_h(t,z) = A'_h \cup \big(\mathcal{T}_h(t,z) \setminus A'_h\big), \tag{4.5}$$

where the backward in time $(1,u)$ characteristics starting at A'_h cross $CDFE$ before reaching $\mathcal{T}(t,z)$. Denote by $A_{h,0}$ the intersection of these backward $(1,u)$ characteristics with $CDFE$. It holds that

$$\begin{aligned}
&\int_{\mathcal{T}_h(t,z)} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) - \int_{\mathcal{T}(t,z)} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) \\
&= \int_{\mathcal{T}_h(t,z)\setminus A'_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{A'_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{\bar{A}_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) \\
&- \Big(\int_{\mathcal{T}(t,z)\setminus A_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{A_{h,0}} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{A_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) \Big) \\
&- \int_{\bar{A}_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{A_{h,0}} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) \\
&= \int_{D_h} \partial_s \big(\chi_j^{k,\epsilon} Q_j^k(F^k,F^k)\big)(t+s,z+su) - \int_{\bar{A}_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{A_{h,0}} \chi_j^{k,\epsilon} Q_j^k(f_0^k,f_0^k) \\
&= \int_{D_h} \Big(\partial_t \big(\chi_j^{k,\epsilon} Q_j^k(F^k,F^k)\big) + u \cdot \nabla_z \big(\chi_j^{k,\epsilon} Q_j^k(F^k,F^k)\big) \Big) \\
&- \int_{\bar{A}_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{A_{h,0}} \chi_j^{k,\epsilon} Q_j^k(f_0^k,f_0^k).
\end{aligned} \tag{4.6}$$

Here the formal derivative ∂_s which is removed in (4.10) can be made rigorous by inserting a mollifier.
Consider the $(1,-u)$ characteristics starting at $\mathcal{T}(t,z)$. $\mathcal{T}(t,z)$ splits into

$$\mathcal{T}(t,z) = B_{h,0} \cup \big(\mathcal{T}(t,z) \setminus B_{h,0}\big), \tag{4.7}$$

where the forward in time $(1,-u)$ characteristics starting at B_h cross $CDFE$ before reaching $\mathcal{T}_h(t,z)$. Denote by $\bar{B}_{h,0}$ the intersection of these $(1,-u)$ characteristics with $CDFE$. $\mathcal{T}_h(t,z)$ splits into

$$\mathcal{T}_h(t,z) = B'_h \cup \big(\mathcal{T}(t,z) \setminus B'_h\big), \tag{4.8}$$

where the backward in time $(1,-u)$ characteristics starting at B'_h cross $ABFD$ before reaching $\mathcal{T}(t,z)$. Denote by $\bar{B}_h$ the intersection of these backward $(1,-u)$ characteristics with $ABFD$. It holds that

$$\begin{aligned}
&\int_{\mathcal{T}_h(t,z)} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) - \int_{\mathcal{T}(t,z)} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) \\
&= \int_{\mathcal{T}_h(t,z)\setminus B'_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{B'_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{\bar{B}_{h,0}} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) \\
&- \Big(\int_{\mathcal{T}(t,z)\setminus B_{h,0}} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{\bar{B}_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{B_{h,0}} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) \Big) \\
&- \int_{\bar{B}_{h,0}} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{\bar{B}_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) \\
&= \int_{D_h} \partial_s \big(\chi_j^{k,\epsilon} Q_j^k(F^k,F^k))(t+s,z-su)\big) - \int_{\bar{B}_{h,0}} \chi_j^{k,\epsilon} Q_j^k(f_0^k,f_0^k) + \int_{\bar{B}_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) \\
&= \int_{D_h} \Big(\partial_t \big(\chi_j^{k,\epsilon} Q_j^k(F^k,F^k)\big) - u \cdot \nabla_z \big(\chi_j^{k,\epsilon} Q_j^k(F^k,F^k)\big) \Big) \\
&- \int_{\bar{B}_{h,0}} \chi_j^{k,\epsilon} Q_j^k(f_0^k,f_0^k) + \int_{\bar{B}_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k).
\end{aligned} \tag{4.9}$$

Adding (4.6) and (4.9) implies that

$$\begin{aligned}
&\int_{\mathcal{T}_h(t,z)} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) - \int_{\mathcal{T}(t,z)} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k)\\
&= \int_{D_h} \partial_t\big(\chi_j^{k,\epsilon} Q_j^k(F^k,F^k)\big)\\
&+ \frac{1}{2}\Big(- \int_{\bar{A}_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{\bar{B}_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k)\\
&\qquad + \int_{A_{h,0}} \chi_j^{k,\epsilon} Q_j^k(f_0^k,f_0^k) - \int_{\bar{B}_{h,0}} \chi_j^{k,\epsilon} Q_j^k(f_0^k,f_0^k)\Big)\\
&= \int_{ABFD} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) - \int_{CDFE} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k)\\
&+ \frac{1}{2}\Big(- \int_{\bar{A}_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k) + \int_{\bar{B}_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k)\\
&\qquad + \int_{A_{h,0}} \chi_j^{k,\epsilon} Q_j^k(f_0^k,f_0^k) - \int_{\bar{B}_{h,0}} \chi_j^{k,\epsilon} Q_j^k(f_0^k,f_0^k)\Big). \qquad (4.10)
\end{aligned}$$

Moreover,

$$\lim_{h\to 0} \int |\int_{ABFD} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k)|dtdz = 0 \qquad (4.11)$$

by the weak L^1 compactness of $(\chi_j^{k,\epsilon} Q_j^{\pm k}(F^k,F^k))_{k\in\mathbb{N}}$. Analogously,

$$\lim_{h\to 0} \int |\int_{\bar{A}_h\cup\bar{B}_h} \chi_j^{k,\epsilon} Q_j^k(F^k,F^k)|dtdz = 0,$$

and

$$\lim_{h\to 0} \int |\int_{A_{h,0}\cup\bar{B}_{h,0}} \chi_j^{k,\epsilon} Q_j^k(f_0^k,f_0^k)|dtdz = 0.$$

This proves the uniform equi-integrability of $\big(\int_{-t}^0 F_j^k(t+s,z)ds\big)_{k\in\mathbb{N}^*}$ in the $(0,u)$ direction. Hence there is translational equi-continuity of $(\int_{-t}^0 F_j^k(t+s,z)ds)_{k\in\mathbb{N}^*}$, $j\in J_i$. This closes the proof of Lemma 4.1. ∎

5 Passage to the limit

Denote by f the weak L^1 limit when $k \to +\infty$ of $(F^k)_{k\in\mathbb{N}^*}$ up to a subsequence. Denote by F^ϵ (resp. f_0^ϵ) the weak L^1 limit of $(\chi^{k,\epsilon}F^k)_{k\in\mathbb{N}^*}$ (resp. $(\chi^{k,\epsilon}f_0^k)_{k\in\mathbb{N}^*}$) up to a subsequence.

Lemma 5.1
For any $\beta \in]0,1[$, there is a subset $\omega_\beta = (\omega_i)_{1\le i\le p}$ of $\big([0,T]\times\mathbb{T}^d\big)^p$ consisting of characteristics $\{(s, z+sv_i); s\in[0,T]\}$, with $|\omega_i| \le \beta$, such that

$$f_i(t+s, z+sv_i) \le \frac{c}{\beta}e^{\frac{1}{\beta}}, \ (t,z)\notin\omega_\beta, \ s\in[-t,T-t], \ 1\le i\le p. \quad (5.1)$$

Proof of Lemma 5.1
Let $1\le i\le p$ and $\beta>0$. Passing to the limit when $k\to+\infty$ in

$$\begin{aligned}&\chi_i^{k,\epsilon}F_i^k(t+s,z+sv_i)\\ &\le \chi_i^{k,\epsilon}F_i^k(T,z+(T-t)v_i)e^{\chi_i^{k,\epsilon}(t,z)\int_t^T\nu_i^k(F^k)(r,z+(r-t)v_i)dr},\\ &\qquad (t,z)\in[0,T]\times\mathbb{T}^d, \quad s\in[-t,T-t], \end{aligned} \quad (5.2)$$

and taking into account that by Lemma 4.1 $\big(\int_t^T\nu_i^k(F^k)(r,z+(r-t)v_i)ds\big)_{k\in\mathbb{N}^*}$ is a strongly convergent sequence in L^1, it holds that

$$F_i^\epsilon(t+s,z+sv_i) \le F_i^\epsilon(T,z+(T-t)v_i)e^{\int_t^T\nu_i(f)(r,z+(r-t)v_i)ds}. \quad (5.3)$$

It follows from $\int_{[0,T]\times\mathbb{T}^d}\nu_i(f)(t,z)dzdt\le c_0$ that

$$|\ \{(t,z)\in[0,T]\times\mathbb{T}^d; \int_t^T\nu_i(f)(r,z+(r-t)v_i)dr > \frac{c}{\beta}\}\ | \le \frac{\beta}{2}.$$

Consequently there is some subset ω_{i1} of $[0,T]\times\mathbb{T}^d$ of measure smaller than $\frac{\beta}{2}$ such that

$$F_i^\epsilon(t,z) \le e^{\frac{c}{\beta}}F_i^\epsilon(T,z+(T-t)v_i), \quad (t,z)\notin\omega_{i1}. \quad (5.4)$$

Passing to the limit in (5.4) when $\epsilon\to 0$ by monotonicity arguments, implies that

$$f_i(t,z) \le e^{\frac{c}{\beta}}f_i(T,z+(T-t)v_i), \quad (t,z)\notin\omega_{i1}. \quad (5.5)$$

Hence there is $\omega_{i2} \subset [0,T] \times \mathbb{T}^d$ of measure smaller than $\frac{\beta}{2}$ such that

$$f_i(t,z) \leq \frac{c}{\beta} e^{\frac{c}{\beta}}, \quad (t,z) \notin \omega_{i1} \cup \omega_{i2}. \tag{5.6}$$

Define ω_i as $\omega_i = \omega_{i1} \cup \omega_{i2}$ and $\omega = (\omega_i)_{1 \leq i \leq p}$. ■

Denote by χ_β the characteristic function of the complement of ω_β.
Take a test function $\psi \in C^1([0,T] \times \mathbb{T}^d)$.
Proving that f is a mild solution to (1.3)-(1.4) is equivalent to proving that for $\beta > 0$,

$$\begin{aligned}
&\int \chi_{\beta i} \psi f_i(t,z) dt dz - \int \chi_{\beta i} \psi(0, z - t v_i) f_{0i}(z - t v_i) dt dz \\
&- \int \Big(\int_{-t}^{0} f_i \, (\partial_t + v_i \cdot \nabla_z)(\chi_{\beta i} \psi)(t+s, z + s v_i) ds \Big) dt dz \\
&= \int \Big(\int_{-t}^{0} \chi_{\beta i} \psi \, Q_i(f,f)(t+s, z + s v_i) ds \Big) dt dz, \quad 1 \leq i \leq p.
\end{aligned} \tag{5.7}$$

Lemma 5.2
The weak L^1 limit f of a subsequence of $(F^k)_{k \in \mathbb{N}^*}$ is a mild solution to the Cauchy problem (1.3)-(1.4).

Proof of Lemma 5.2.
F_i^k satisfies the equation

$$\begin{aligned}
&\int \chi_{\beta i} \psi \chi_i^{k,\epsilon} F_i^k(t,z) dt dz - \int \chi_{\beta i} \psi \chi_i^{k,\epsilon}(0, z - t v_i) f_{0i}^k(z - t v_i) dt dz \\
&- \int \Big(\int_{-t}^{0} \chi_i^{k,\epsilon} F_i^k \, (\partial_t + v_i \cdot \nabla_z)(\chi_{\beta i} \psi)(t+s, z + s v_i) ds \Big) dt dz \\
&= \int \Big(\int_{-t}^{0} \chi_{\beta i} \psi \chi_i^{k,\epsilon} Q_i^k(F^k, F^k)(t+s, z + s v_i) ds \Big) dt dz.
\end{aligned} \tag{5.8}$$

Passing to the limit when $k \to +\infty$ in the two first lines of (5.8) is straightforward. Their limit is

$$\begin{aligned}
&\int \chi_{\beta i} \psi F_i^\epsilon(t,z) dt dz - \int \chi_{\beta i} \psi(0, z - t v_i) f_{0i}^\epsilon(z - t v_i) dt dz \\
&- \int \Big(\int_{-t}^{0} F_i^\epsilon \, (\partial_t + v_i \cdot \nabla_z)(\chi_{\beta i} \psi)(t+s, z + s v_i) ds \Big) dt dz.
\end{aligned} \tag{5.9}$$

The passage to the limit when $k \to +\infty$ in any term of the loss term of (5.8) follows from an integration by parts

$$\begin{aligned}
&\int_{-t}^{0} \chi_{\beta i}\psi\chi_i^{k,\epsilon}\frac{F_i^k}{1+\frac{F_i^k}{k}}\frac{F_j^k}{1+\frac{F_j^k}{k}}(t+s,z+sv_i)ds \\
&= \chi_{\beta i}\psi\chi_i^{k,\epsilon}(0,z-tv_i)\frac{f_{0i}^k}{1+\frac{f_{0i}^k}{k}}(z-tv_i)\int_{-t}^{0}\frac{F_j^k}{1+\frac{F_j^k}{k}}(t+s,z+sv_i)ds \\
&+ \int_{-t}^{0}\frac{d}{ds}\big(\chi_{\beta i}\psi\big)\chi_i^{k,\epsilon}\frac{F_i^k}{1+\frac{F_i^k}{k}}(t+s,z+sv_i)\int_{s}^{0}\frac{F_j^k}{1+\frac{F_j^k}{k}}(t+r,z+rv_i)dr \\
&+ \int_{-t}^{0}\chi_{\beta i}\psi\frac{d}{ds}\big(\chi_i^{k,\epsilon}\frac{F_i^k}{1+\frac{F_i^k}{k}}\big)(t+s,z+sv_i) \\
&\qquad\qquad \big(\chi_i^{k,\epsilon}(t,z)\int_{s}^{0}\frac{F_j^k}{1+\frac{F_j^k}{k}}(t+r,z+srv_i)dr\big)\,ds. \qquad (5.10)
\end{aligned}$$

Passing to the limit when $k \to +\infty$ in the second and third lines of (5.10) is straightforward. Their limit is

$$\begin{aligned}
&\chi_{\beta i}\psi(0,z-tv_i)f_{0i}^{\epsilon}(z-tv_i)\int_{-t}^{0}f_j(t+s,z+sv_i)ds \\
&+\int_{-t}^{0}\frac{d}{ds}\big(\chi_{\beta i}\psi\big)F_i^{\epsilon}(t+s,z+sv_i)\int_{s}^{0}f_j(t+r,z+rv_i)dr. \qquad (5.11)
\end{aligned}$$

By (3.1)-(3.3), $\big(\frac{d}{ds}\big(\chi_i^{k,\epsilon}\frac{F_i^k}{1+\frac{F_i^k}{k}}\big)(t+s,z+sv_i)\big)_{k\in\mathbb{N}^*}$ converges weakly in L^1 to $\frac{d}{ds}F_i^{\epsilon}(t+s,z+sv_i)$. Moreover, $\big(\chi_i^{k,\epsilon}(t,z)\int_s^0\frac{F_j^k}{1+\frac{F_j^k}{k}}(t+r,z+rv_i)dr\big)_{k\in\mathbb{N}^*}$ is uniformly bounded in L^∞ and converges strongly in L^1 to some $G_{i,j}^{\epsilon}$, such that $(G_{i,j}^{\epsilon})_{\epsilon\in]0,1[}$ increasingly converges in L^1 to $\int_s^0 f_j(t+r,z+rv_i)dr$ when

$\epsilon \to 0$. And so,

$$\lim_{k\to+\infty} \int \Big(\int_{-t}^{0} \chi_{\beta i}\psi\chi_i^{k,\epsilon} \frac{F_i^k}{1+\frac{F_i^k}{k}} \frac{F_j^k}{1+\frac{F_j^k}{k}}(t+s,z+sv_i)ds\Big)dtdz$$
$$= \int \Big(\chi_{\beta i}\psi(0,z-tv_i) f_{0i}^{\epsilon}(z-tv_i) \int_{-t}^{0} f_j(t+s,z+sv_i)ds\Big)dtdz$$
$$+ \int \Big(\int_{-t}^{0} \frac{d}{ds}\big(\chi_{\beta i}\psi\big) F_i^{\epsilon}(t+s,z+sv_i) \int_{s}^{0} f_j(t+r,z+rv_i)dr\Big)dtdz$$
$$+ \int_{-t}^{0} \chi_{\beta i}\psi\Big(\frac{d}{ds}F_i^{\epsilon}(t+s,z+sv_i)\Big) G_{i,j}^{\epsilon}(s,t,z)\,ds. \tag{5.12}$$

By monotonicity arguments in L^1, the limit when $\epsilon \to 0$ in (5.12) can be performed, leading to the limit

$$\int \Big(\chi_{\beta i}\psi(0,z-tv_i) f_{0i}(z-tv_i) \int_{-t}^{0} f_j(t+s,z+sv_i)ds\Big)dtdz$$
$$+ \int \Big(\int_{-t}^{0} \frac{d}{ds}\big(\chi_{\beta i}\psi\big) f_i(t+s,z+sv_i) \int_{s}^{0} f_j(t+r,z+rv_i)dr\Big)dtdz$$
$$+ \int_{-t}^{0} \chi_{\beta i}\psi\Big(\frac{d}{ds}f_i(t+s,z+sv_i)\Big) \int_{s}^{0} f_j(t+r,z+rv_i)dr\,ds$$

$$= \int \Big(\int_{-t}^{0} \chi_{\beta i}\psi f_i f_j(t+s,z+sv_i)ds\Big)dtdz. \tag{5.13}$$

Similarly as for the loss term, the passage to the limit when $k \to +\infty$ of terms composing the gain term of $\chi_i^{k,\epsilon} Q_i^k(F^k,F^k)$ such as

$$\chi_i^{k,\epsilon}(t,z) \int_{-t}^{0} \frac{F_m^k}{1+\frac{F_m^k}{k}} \frac{F_q^k}{1+\frac{F_q^k}{k}}(t+s,z+sv_i)ds,$$

can be handled as follows.

$$\begin{aligned}
&\chi_i^{k,\epsilon}(t,z)\int_{-t}^{0}\frac{F_m^k}{1+\frac{F_m^k}{k}}\frac{F_q^k}{1+\frac{F_q^k}{k}}(t+s,z+sv_i)ds\\
&=\chi_i^{k,\epsilon}(t,z)\int_{-t}^{0}(1-\chi_m^{k,\epsilon})\frac{F_m^k}{1+\frac{F_m^k}{k}}\frac{F_q^k}{1+\frac{F_q^k}{k}}(t+s,z+sv_i)ds\\
&-(1-\chi_i^{k,\epsilon})(t,z)\int_{-t}^{0}\chi_m^{k,\epsilon}\frac{F_m^k}{1+\frac{F_m^k}{k}}\frac{F_q^k}{1+\frac{F_q^k}{k}}(t+s,z+sv_i)ds\\
&+\int_{-t}^{0}\chi_m^{k,\epsilon}\frac{F_m^k}{1+\frac{F_m^k}{k}}\frac{F_q^k}{1+\frac{F_q^k}{k}}(t+s,z+sv_i)ds. \qquad (5.14)
\end{aligned}$$

When integrated over $[0,T]\times\mathbb{T}^d$, the second (resp. third) line of (5.14) tends to zero when $\epsilon\to 0$ uniformly with respect to $k\in\mathbb{N}^*$ since

$$\begin{aligned}
&(t,z)\to\chi_i^{k,\epsilon}(t,z)\int_{-t}^{0}\frac{F_m^k}{1+\frac{F_m^k}{k}}\frac{F_q^k}{1+\frac{F_q^k}{k}}(t+s,z+sv_i)ds,\\
&(\text{resp. } (t,z)\to\int_{-t}^{0}\chi_m^{k,\epsilon}\frac{F_m^k}{1+\frac{F_m^k}{k}}\frac{F_q^k}{1+\frac{F_q^k}{k}}(t+s,z+sv_i)ds)
\end{aligned}$$

is weakly compact in L^1 and the measure of $\Omega_m^{k,\epsilon}$ (resp. $\Omega_i^{k,\epsilon}$) is of order ϵ. When integrated over $[0,T]\times\mathbb{T}^d$, the last line of (5.14) can be rewritten as

$$\begin{aligned}
&\int\Big(\int_{-t}^{0}\chi_{\beta i}\psi\chi_m^{k,\epsilon}\frac{F_m^k}{1+\frac{F_m^k}{k}}\frac{F_q^k}{1+\frac{F_q^k}{k}}(t+s,z+sv_i)ds\Big)dt\,dz\\
&=\int_0^T\int_{-t}^{0}\int_{\mathbb{T}^2}\chi_{\beta i}\psi\chi_m^{k,\epsilon}\frac{F_m^k}{1+\frac{F_m^k}{k}}\frac{F_q^k}{1+\frac{F_q^k}{k}}(t+s,Z)dZ\,ds\,dt\\
&=\int\Big(\int_{-t}^{0}\chi_{\beta i}\psi\chi_m^{k,\epsilon}\frac{F_m^k}{1+\frac{F_m^k}{k}}\frac{F_q^k}{1+\frac{F_q^k}{k}}(t+s,z+sv_m)ds\Big)dt\,dz. \qquad (5.15)
\end{aligned}$$

The passage to the limit when $k\to+\infty$ in (5.15) can be performed like in the loss term with a supplementary intermediate regularization of $\chi_{\beta i}$. Its next limit when $\epsilon\to 0$ is

$$\int\Big(\int_{-t}^{0}\chi_{\beta i}\psi f_m f_q(t+s,z+sv_m)ds\Big)dt\,dtdz. \qquad (5.16)$$

And so, f satisfies (5.7). ∎

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