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Ewa Bednarczuk, Giovanni Bruccola, Gabriele Scrivanti, The Hung Tran. Calculus rules for proximal ϵ -subdifferentials and inexact proximity operators for weakly convex functions. 2022. hal-03909061

HAL Id: hal-03909061

<https://hal.science/hal-03909061>

Preprint submitted on 21 Dec 2022

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Calculus rules for proximal ε -subdifferentials and inexact proximity operators for weakly convex functions

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Abstract—We investigate inexact proximity operators for weakly convex functions. To this aim we derive sum rules for proximal ε -subdifferentials, by incorporating the moduli of weak convexity of the functions into the respective formulas. This allows us to investigate inexact proximity operators for weakly convex functions in terms of proximal ε -subdifferentials.

KEYWORDS: weakly convex functions, criticality, proximal operator, inexactness, inexact proximal operator, sum rule for proximal ε -subdifferentials

I. INTRODUCTION

Proximal operators are a fundamental tool in constructing algorithms for solving large-scale convex optimisation problems [10]. Responding to the need for solving optimization problems with convex objectives which do not fall into the class of typical convex objective functions appearing in data analysis (see *e.g.* the webpage [9]), a number of inexact (approximate) proximal operators have been introduced, see *e.g.* [28], [31], [20].

It is our aim to investigate inexact proximal operators for a class of functions which is larger than the one of convex functions: in the present work, we focus on weakly convex functions, which have been appearing in current models in data science problems in a rapidly growing number.

a) Weak convexity: Weak convexity can be considered as a special case of the general notions of γ -paraconvexity and $\alpha(\cdot)$ -paraconvexity that were studied by, among others, Jourani and Rolewicz [16], [23], [26]. For a general characterisation in Hilbert spaces, weakly convex functions can be expressed as the difference between a convex function and a quadratic function. This class includes all the convex functions and all the smooth (but not necessarily convex) functions with a Lipschitz continuous gradient, together with many other interesting non-convex functions. Examples of weakly convex functions appearing in data analysis can be found in [11] and in [6]. The growing interest in the use of this class of function in many fields of applications suggested the necessity of a careful analysis of their properties in

terms of subdifferentials and proximal operators, which is the core of the present work.

b) Proximal Subdifferentials: It has already been observed that the concept of subdifferential which is particularly suitable when defining criticality for weakly convex functions is that of *proximal subdifferential* (see *e.g.* [13], [12]). There exists a vast literature devoted to proximal subdifferentials, see *e.g.* in the finite dimensional case, the monograph by Rockafellar and Wets [21], in Hilbert spaces the work by Bernards and Thibault [5]. In these monographs and papers, the proximal subdifferential at a given x_0 is defined locally, in the sense that there exists a neighbourhood V of x_0 and a constant $C \geq 0$, such that for every $x \in V$,

$$\langle x^*, x - x_0 \rangle \leq f(x) - f(x_0) + C\|x - x_0\|^2. \quad (1)$$

In our developments we make use of a property that holds in the class of paraconvex functions, called *globalisation property*. Precisely, in the class of paraconvex functions, if (1) holds, then it holds globally over the whole space (see Def. 1 and Prop. 1). More on the *globalization property* can be found in [24].

We focus our analysis on the more general notion of proximal ε -subdifferentials, which represents a useful tool allowing to take into account inexactness and perturbations in the resolution of optimisation problems.

c) Contribution: Our contribution addresses the following issues.

- 1) We provide sufficient and necessary conditions for the sum rule of the global proximal ε -subdifferentials for the sum of two ρ -weakly convex functions (see Theo. 2 and Theo. 3) (Section III).
- 2) By using the above theorems, in Prop. 4 and Prop. 5, we investigate the relationship between the ε -proximal operator of a ρ -weakly convex function f and the ε -proximal subdifferential of f (Section IV).
- 3) The notion of inexact (approximate) proximal point that we infer can be related to *Type-1* and *Type-2*

approximations proposed in [28], [20] in the convex settings (Section IV).

In the present work, we incorporate and make a consistent use of the modulus of proximal subdifferentiability and of the modulus of weak convexity ρ into the calculus rules for proximal ε -subdifferentials.

II. PRELIMINARIES

Before focusing on the class of weakly convex functions, we introduce a more general notion of γ -paraconvexity and the corresponding notion of (γ, C) -subdifferential as presented in [16]. The class of γ -paraconvex functions has been studied by [23]. For $\gamma = 2$, we obtain weakly convex functions.

Definition 1 (γ -Paraconvexity): Let \mathcal{X} be a normed vector space. A function $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ is said to be γ -paraconvex if there exists a positive constant C such that for $\lambda \in [0, 1]$, $(\forall(x, y) \in \mathcal{X}^2)$, the following inequality holds:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + C\lambda(1-\lambda)\|x-y\|^\gamma.$$

When $\alpha : [0, +\infty) \rightarrow [0, +\infty)$ is a non-decreasing function with $\lim_{t \downarrow 0} \frac{\alpha(t)}{t} = 0$, a function $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ is called $\alpha(\cdot)$ -paraconvex if there exists a constant $C > 0$ such that for $\lambda \in [0, 1]$

$$\begin{aligned} (\forall(x, y) \in \mathcal{X}^2) \quad & f(\lambda x + (1-\lambda)y) \\ & \leq \lambda f(x) + (1-\lambda)f(y) + C \min\{\lambda, 1-\lambda\} \alpha(\|x-y\|) \end{aligned}$$

This class has been introduced by Rolewicz under the name of $\alpha(\cdot)$ -strongly paraconvex functions and investigated in a series of papers by Jourani [15], [16] and Rolewicz [23], [26]. When $\alpha(\|x-y\|) = \|x-y\|^\gamma$ the notion of $\alpha(\cdot)$ -paraconvexity coincides with the one of γ -paraconvexity, (see [25, Lemma 5]). In Hilbert spaces, when

$$\limsup_{t \downarrow 0} \frac{\alpha(t)}{t^2} < +\infty,$$

then a $\alpha(\cdot)$ -paraconvex function is a difference of a convex and a quadratic function and is called *weakly convex* (see [26]).

Definition 2 ((γ, C) -Subdifferential [16, Def. 3.1]): Let \mathcal{X} be a normed vector space. By X^* we denote the dual space of all continuous linear functionals defined on X . Let $\gamma > 0$ and $C > 0$. Let $x_0 \in \mathcal{X}$ and $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ be finite at x_0 . A point $x^* \in \mathcal{X}^*$ is said to be a (γ, C) -subgradient of f at x_0 if there exists a neighbourhood V of x_0 such that

$$(\forall x \in V) \quad \langle x^*, x - x_0 \rangle \leq f(x) - f(x_0) + C\|x - x_0\|^\gamma.$$

The set of all (γ, C) -subgradients of f at x_0 is denoted by $\partial_{(\gamma, C)}^{Loc} f(x_0)$ and it is referred to as

(γ, C) -Subdifferential. Whenever $\partial_{(\gamma, C)}^{Loc} f(x_0) \neq \emptyset$, we say that f is *proximally C -subdifferentiable* at x_0 .

Proposition 1 ([16, Prop. 3.1]): Let \mathcal{X} be a normed space. Let $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ be γ -paraconvex with $\gamma > 1$. Then there exists $C > 0$ such that

$$\partial_{(\gamma, C)}^{Loc} f(x_0) = \partial_{(\gamma, C)} f(x_0)$$

where

$$\begin{aligned} \partial_{(\gamma, C)} f(x_0) & := \{x^* \in \mathcal{X}^* \mid \langle x^*, x - x_0 \rangle \leq \\ & f(x) - f(x_0) + C\|x - x_0\|^\gamma \forall x \in \mathcal{X}\}. \end{aligned}$$

The constant C can be chosen equal to the one appearing in the definition of paraconvexity (Def. 1).

Occasionally, we will refer to the constant C appearing in the definition of proximal subdifferentiability as the *modulus of proximal subdifferentiability*.

In the sequel we will use the global $(2, C)$ -subdifferential of f at x which will be referred to as *proximal subdifferential*: (see e.g. [21], [5]).

Definition 3 (Global Proximal Subdifferential): Let \mathcal{X} be a normed vector space. Let $f : \mathcal{X} \rightarrow (-\infty, +\infty]$, $\rho \geq 0$ and let $x \in \text{dom } f$. Then the *proximal subdifferential* of f at x_0 with constant $C > 0$ is defined as the set

$$\begin{aligned} \partial_{(2, C)} f(x_0) & := \{x^* \in \mathcal{X}^* \mid f(x) \geq f(x_0) \\ & + \langle x^*, x - x_0 \rangle - C\|x - x_0\|^2, \forall x \in \mathcal{X}\}. \end{aligned} \quad (2)$$

In view of (2), $\partial_{(2, 0)}$ denotes the subdifferential in the sense of convex analysis. For simplicity, in this case we will use the notation $\partial_0 \equiv \partial_{(2, 0)}$.

When investigating inexact proximal points, the following concept of proximal ε -subdifferentials is used.

Definition 4 (Global proximal ε -subdifferentials): Let \mathcal{X} be a normed vector space and $\varepsilon \geq 0$. The global proximal ε -subdifferentials of a function $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ at $x_0 \in \mathcal{X}$ for $C \geq 0$ is defined as follows:

$$\begin{aligned} \partial_{(2, C)}^\varepsilon f(x_0) & = \{v \in \mathcal{X}^* \mid \text{s.t. } f(x) - f(x_0) \geq \\ & \langle v, x - x_0 \rangle - C\|x - x_0\|^2 - \varepsilon \forall x \in \mathcal{X}\}. \end{aligned}$$

Clearly, for every $\varepsilon' \geq \varepsilon$ and $C' \geq C$ we have the following inclusion

$$\partial_{(2, C)}^\varepsilon f(x_0) \subseteq \partial_{(2, C')}^{\varepsilon'} f(x_0). \quad (3)$$

In Hilbert spaces, a weakly convex function f in the sense of Def. 1 for $\gamma = 2$ and $C = \rho/2$ can be characterised by the fact that $f(\cdot) + \rho/2\|\cdot\|^2$ is a convex function. A proof can be obtained by directly adapting the finite-dimensional proof given in [7, Prop. 1.1.3]. Such a function will then be referred to as a ρ -weakly convex and ρ is known as *modulus of weak convexity*. A variant of Prop. 1 corresponding to $\gamma = 2$ and C not necessarily coinciding with the weak convexity

parameter can be found in [29].

For any set-valued mapping $M : \mathcal{X} \rightrightarrows \mathcal{X}$, we will use the notation $\text{dom } M$ to indicate the set

$$\text{dom } M := \{x \in \mathcal{X} \mid M(x) \neq \emptyset\},$$

while for a function $f : \mathcal{X} \rightarrow (-\infty, +\infty]$, the notation $\text{dom } f$ will indicate the set

$$\text{dom } f := \{x \in \mathcal{X} \mid f(x) < +\infty\}.$$

Proposition 2: Let \mathcal{X} be a Hilbert space. Let $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ be a lower semicontinuous and ρ -weakly convex function with $\rho \geq 0$. Then for every $\varepsilon \geq 0$

$$\text{dom } \partial_{(2, \rho/2)}^\varepsilon f = \text{dom } \partial_0^\varepsilon (f + \frac{\rho}{2} \|\cdot\|^2)$$

and

$$\text{dom } \partial_{(2, \rho/2)}^\varepsilon f \subset \text{dom } f.$$

Moreover, for every $\varepsilon > 0$

$$\text{dom } \partial_{(2, \rho/2)}^\varepsilon f = \text{dom } f.$$

Proof: We start by showing that for any $x_0 \in \text{dom } \partial_0^\varepsilon (f + \frac{\rho}{2} \|\cdot\|^2)$ and $\varepsilon \geq 0$, we have that

$$\partial_0^\varepsilon (f + \frac{\rho}{2} \|\cdot\|^2)(x_0) - \rho x_0 = \partial_{(2, \rho/2)}^\varepsilon f(x_0). \quad (4)$$

Indeed, for any $v \in \partial_0^\varepsilon (f + \frac{\rho}{2} \|\cdot\|^2)(x_0)$ and $x_0 \in \text{dom } \partial_0^\varepsilon (f + \frac{\rho}{2} \|\cdot\|^2)$, we have

$$\begin{aligned} f(x) + \frac{\rho}{2} \|x\|^2 - f(x_0) - \frac{\rho}{2} \|x_0\|^2 &\geq \langle v, x - x_0 \rangle - \varepsilon \\ \iff f(x) - f(x_0) &\geq \\ \langle v - \rho x_0, x - x_0 \rangle - \frac{\rho}{2} \|x - x_0\|^2 - \varepsilon & \end{aligned}$$

which is equivalent to the fact that $v - \rho x_0 \in \partial_{(2, \rho/2)}^\varepsilon f(x_0)$ and proves (4). Hence, $\text{dom } \partial_0^\varepsilon (f + \frac{\rho}{2} \|\cdot\|^2) = \text{dom } \partial_{(2, \rho/2)}^\varepsilon f$. Since ∂_0^ε corresponds to the ε -subdifferential for convex functions and function $(f + \frac{\rho}{2})(\cdot)$ is convex, we have that for all $\varepsilon > 0$

$$\text{dom } \partial_0^\varepsilon (f + \frac{\rho}{2} \|\cdot\|^2) = \text{dom } (f + \frac{\rho}{2} \|\cdot\|^2)$$

(see [2, Cor. 2.81]). Since $\text{dom } f = \text{dom } (f + \frac{\rho}{2} \|\cdot\|^2)$, the assertion follows. ■

In Prop. 2, we include the assumption of lower semicontinuity on f because it is required by [2, Cor. 2.81].

Definition 5 (ε -solution): Let \mathcal{X} be a normed space. Let $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ be a proper function that is bounded from below. Then, for any $\varepsilon \geq 0$, the element x_ε is said to be an ε -solution to the minimisation problem

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad f(x)$$

if the following condition is satisfied:

$$(\forall x \in \mathcal{X}) \quad f(x_\varepsilon) \leq f(x) + \varepsilon.$$

Definition 6 (ε -C-critical point): Let \mathcal{X} be a normed space and $\varepsilon \geq 0$. Let $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ be a proper function. A point $x \in \mathcal{X}$ is said to be a ε -C-critical point of f if $0 \in \partial_{(2, C)}^\varepsilon f(x)$. The set of ε -C-critical points is identified as

$$\varepsilon\text{-crit}_C f := \{x \in \mathcal{X} \mid 0 \in \partial_{(2, C)}^\varepsilon f(x)\}.$$

When f is ρ -weakly convex, it is of particular interest to consider ε - $\rho/2$ -critical points and then we write " ε -critical points" and use the notation $\varepsilon\text{-crit}$.

Remark 1 (Fermat's Rule): We highlight that ε -C-criticality is a necessary condition for a point to be a ε -solution. Notice that, by Prop. 2, $\text{dom } f = \text{dom } \partial_{(2, C)}^\varepsilon h$. If $x_\varepsilon \in \text{dom } f$ is an ε -solution of f , then

$$\begin{aligned} (\forall x \in \mathcal{X}) \quad f(x) &\geq f(x_\varepsilon) - \varepsilon \\ &\geq f(x_\varepsilon) - C \|x - x_\varepsilon\|^2 - \varepsilon \end{aligned}$$

for every $C \geq 0$. This implies $0 \in \partial_{(2, C)}^\varepsilon f(x_\varepsilon)$.

III. CALCULUS RULES

In the literature, there exist numerous results providing calculus rules for the Fréchet, the limiting and the proximal subdifferentials, see *e.g.* [16], [17], [19], [18], [30] and many others. The main result of the present section is stated in Theo. 2, where we provide the conditions for a sum rule for the global proximal ε -subdifferentials (in the sense of Def. 4) of the sum of two weakly convex functions. The proposed result allows to extend the sum rule in [16, Theo. 5.1] – proved for exact proximal subdifferentials in normed spaces – to proximal ε -subdifferentials in Hilbert spaces: the interesting aspect of such rule is that it allows to keep track of the modulus of weak convexity.

The following notion of ρ -conjugate function will be used in the proof of Theo. 2.

Definition 7: Let \mathcal{X} be a Hilbert space. Let $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ be a proper function. For every $\rho \geq 0$ the function $(h)_\rho^* : \mathcal{X} \rightarrow [-\infty, +\infty]$ defined as

$$(f)_\rho^*(u) := \sup_{y \in \mathcal{X}} \left\{ -\frac{\rho}{2} \|y\|^2 + \langle u, y \rangle - f(y) \right\}$$

is called ρ -conjugate of f at $u \in \mathcal{X}$ (when $\rho = 0$ we obtain the definition of the conjugate as defined in convex analysis and in this case we omit the subscript).

We recall the following result, which is an important fact in view of the proof of Theo. 2.

Theorem 1 ([22, Theo. 3]): Let \mathcal{X} be a Hilbert space. Let $f_0, f_1 : \mathcal{X} \rightarrow (-\infty, +\infty]$ be proper convex

functions. Assume that $\text{dom } f_0 \cap \text{dom } f_1$ contains a point at which either f_0 or f_1 is continuous. Then, for all $s, x \in \mathcal{X}$ we have

$$(a) \quad (f_0 + f_1)^*(s) = \min_{\substack{p_0, p_1 \in \mathcal{X} \\ s = p_0 + p_1}} \{f_0^*(p_0) + f_1^*(p_1)\}$$

$$(b) \quad \partial_0(f_0 + f_1)(x) = \partial_0 f_0(x) + \partial_0 f_1(x)$$

Remark 2: By [3, Theo. 15.3 (Attouch–Brézis Theorem)], [1], (a) of Theo. 1 can be proved under the assumption that f_0 and f_1 are convex proper lsc functions such that the conical hull of $\text{dom } f_0 - \text{dom } f_1$ is a closed linear subspace, i.e.,

$$0 \in \text{sri}(\text{dom } f_0 - \text{dom } f_1),$$

where sri denotes the strong relative interior, see [3, Def. 6.9]. The regularity assumption in [1] is more general than the one in [22] (see [1, Remark 1.3]). However, in some cases it is easier to verify the regularity condition requested in [22].

The following proposition provides an important auxiliary fact used in the proof of Theo. 2.

Proposition 3: Let \mathcal{X} be a Hilbert space. For $i = 0, 1$, let function $f_i : \mathcal{X} \rightarrow (-\infty, +\infty]$ be proper lower semicontinuous and ρ_i -weakly convex on \mathcal{X} with $\rho_i \geq 0$. Assume that $\text{dom } f_0 \cap \text{dom } f_1$ contains a point at which either f_0 or f_1 is continuous. Then the following holds: for any $s \in \text{dom}(f_0 + f_1)^*_\rho$, there exist $p_0, p_1 \in \mathcal{X}$ such that $s = p_0 + p_1$ and

$$(f_0 + f_1)^*_{\rho_0 + \rho_1}(s) = (f_0)^*_{\rho_0}(p_0) + (f_1)^*_{\rho_1}(p_1).$$

Proof: We have that $\bar{f}_0(\cdot) = f_0(\cdot) + \rho_0/2 \|\cdot\|^2$ and $\bar{f}_1(\cdot) = f_1(\cdot) + \rho_1/2 \|\cdot\|^2$ are convex. By Theo. 1, there exist p_0, p_1 such that $s = p_0 + p_1$ and

$$(\bar{f}_0 + \bar{f}_1)^*(s) = (\bar{f}_0)^*(p_0) + (\bar{f}_1)^*(p_1)$$

Notice that for $i = 0, 1$

$$\begin{aligned} \bar{f}_i^*(\cdot) &:= \sup_{y \in \mathcal{X}} \{\langle \cdot, y \rangle - \bar{f}_i(y)\} \\ &= \sup_{y \in \mathcal{X}} \left\{ \langle \cdot, y \rangle - f_i(y) - \frac{\rho_i}{2} \|y\|^2 \right\} \\ &= (f_i)^*_{\rho_i}(\cdot) \end{aligned}$$

so

$$(\bar{f}_0 + \bar{f}_1)^*(s) = (f_0)^*_{\rho_0}(p_0) + (f_1)^*_{\rho_1}(p_1)$$

and in conclusion

$$(f_0 + f_1)^*_{\rho_0 + \rho_1}(s) = (f_0)^*_{\rho_0}(p_0) + (f_1)^*_{\rho_1}(p_1). \quad \blacksquare$$

Now we are ready to prove the following sum rule for proximal ε -subdifferentials $\partial_{(2, \rho/2)}^\varepsilon$. This result generalises [14, Theo. 3.1.1] and [32, Theo. 2.8.7], which are formulated for convex functions and convex

subdifferentials. An important aspect of our result – which will be used below in the analysis of proximal operators – is that it allows to keep track of the modulus of proximal subdifferentiability (as related to the modulus of weak convexity of the functions involved).

Theorem 2 (Sum Rule for ε -subdifferential): Let \mathcal{X} be a Hilbert space. For $i = 0, 1$, let function $f_i : \mathcal{X} \rightarrow (-\infty, +\infty]$ be proper lower semicontinuous and ρ_i -weakly convex on \mathcal{X} with $\rho_i \geq 0$. Then, for all $x \in \text{dom } f_0 \cap \text{dom } f_1$ and for all $\varepsilon_0, \varepsilon_1 \geq 0$ we have

$$\partial_{(2, \rho_0/2)}^{\varepsilon_0} f_0(x) + \partial_{(2, \rho_1/2)}^{\varepsilon_1} f_1(x) \subseteq \partial_{(2, \rho/2)}^\varepsilon (f_0 + f_1)(x) \quad (5)$$

for all $\varepsilon \geq \varepsilon_0 + \varepsilon_1$ and for all $\rho \geq \rho_0 + \rho_1$. The equality

$$\begin{aligned} \partial_{(2, (\rho_0 + \rho_1)/2)}^\varepsilon (f_0 + f_1)(x) = \\ \bigcup_{\varepsilon_0, \varepsilon_1 \mid \varepsilon_0 + \varepsilon_1 \leq \varepsilon} \partial_{(2, \rho_0/2)}^{\varepsilon_0} f_0(x) + \partial_{(2, \rho_1/2)}^{\varepsilon_1} f_1(x) \end{aligned} \quad (6)$$

holds when $\text{dom } f_0 \cap \text{dom } f_1$ contains a point at which either $f_0 + \rho_0/2 \|\cdot\|^2$ or $f_1(\cdot) + \rho_1/2 \|\cdot\|^2$ is continuous.

Proof: For $x \in \mathcal{X}$, if $w \in \partial_{(2, \rho_0/2)}^{\varepsilon_0} f_0(x)$, $v \in \partial_{(2, \rho_1/2)}^{\varepsilon_1} f_1(x)$, then it is clear that $w + v \in \partial_{(2, (\rho_0 + \rho_1)/2)}^\varepsilon (f_0 + f_1)(x)$. Hence the inclusion (5) is satisfied.

To prove the equality in (6), let us consider $x \in \text{dom } f_0 \cap \text{dom } f_1$ and $u \in \partial_{(2, \rho/2)}^\varepsilon (f_0 + f_1)(x)$, where $\rho = \rho_0 + \rho_1$. By [4, Theo. 2.4.ii, Eq. (5)] we have

$$\begin{aligned} (f_0 + f_1)(x) + (f_0 + f_1)^*_\rho(u + \rho x) \\ \leq -\frac{\rho}{2} \|x\|^2 + \langle u + \rho x, x \rangle + \varepsilon \end{aligned} \quad (7)$$

The inequality in (7) implies that $u + \rho x \in \text{dom}(f_0 + f_1)^*_\rho$. By applying Prop. 3, there exist two elements $p_0, p_1 \in \mathcal{X}$ such that $u + \rho x = p_0 + p_1$ and

$$(f_0 + f_1)^*_\rho(u + \rho x) = f_0^*(p_0) + (f_1)^*(p_1)$$

so that (7) can be rewritten as

$$\begin{aligned} (f_0 + f_1)(x) + (f_0)^*_{\rho_0}(p_0) + (f_1)^*_{\rho_1}(p_1) \\ \leq -\frac{\rho}{2} \|x\|^2 + \langle p_0 + p_1, x \rangle + \varepsilon \end{aligned}$$

for all $x \in \text{dom } f_0 \cap \text{dom } f_1$. We now define the following values

$$\varepsilon_0 := (f_0)(x) + (f_0)^*_{\rho_0}(p_0) - \langle p_0, x \rangle + \frac{\rho_0}{2} \|x\|^2 \geq 0 \quad (8)$$

$$\varepsilon_1 := (f_1)(x) + (f_1)^*_{\rho_1}(p_1) - \langle p_1, x \rangle + \frac{\rho_1}{2} \|x\|^2 \geq 0 \quad (9)$$

which are positive in view of the definition of ρ -conjugate. Notice that (8) and (9) can be rewritten as

$$\begin{aligned}\varepsilon_0 &= f_0(x) + (f_0)_{\rho_0}^* ((p_0 - \rho_0 x) + \rho_0 x) \\ &\quad - \langle (p_0 - \rho_0 x) + \rho_0 x, x \rangle + \frac{\rho_0}{2} \|x\|^2 \\ \varepsilon_1 &= f_1(x) + (f_1)_{\rho_1}^* ((p_1 - \rho_1 x) + \rho_1 x) \\ &\quad - \langle (p_1 - \rho_1 x) + \rho_1 x, x \rangle + \frac{\rho_1}{2} \|x\|^2\end{aligned}$$

from which, by applying [4, Theo. 2.4.ii, Eq. (5)] in a similar fashion as in (7), we obtain

$$\begin{aligned}(p_0 - \rho_0 x) &\in \partial_{(2, \rho_0/2)}^{\varepsilon_0} f_0(x) \\ (p_1 - \rho_1 x) &\in \partial_{(2, \rho_1/2)}^{\varepsilon_1} f_1(x)\end{aligned}$$

which completes the proof. \blacksquare

Remark 3: When $\varepsilon = 0$, the sum rule presented in the theorem above can be shown to hold for γ -paraconvex functions, $\gamma > 1$, defined over complete metric spaces (see [16, Theo. 5.1, Cor. 5.1]). The sum rule in [16, Theo. 5.1, Cor. 5.1] also keeps track of the modulus of subdifferentiability. Our effort in Theo. 2 is to extend this result to the more general notion of ε -subdifferentials and, in order to do so, we exploited the structure of Hilbert spaces. It is worth noticing that, as in the case of the sum rule for $\partial_{(\gamma, C)}^\varepsilon$, also in the case of the sum rule for $\partial_{(\gamma, C)}^\varepsilon$ we are able to control the modulus of proximal subdifferentiability.

Remark 4: Let $f_i(x)$, $i = 0, 1$ be a ρ_i -weakly convex function. If there exist p_0, p_1 such that $p_0 + p_1 = u \in \partial_{(2, (\rho_0 + \rho_1)/2)}^\varepsilon (f_0 + f_1)(x)$ and

$$p_0 \in \partial_{(2, \rho_0/2)}^{\varepsilon_0} f_0(x), \quad p_1 \in \partial_{(2, \rho_1/2)}^{\varepsilon_1} f_1(x)$$

then by (3), for all $\varepsilon'_i > \varepsilon_i$, $i = 0, 1$, we have

$$p_0 \in \partial_{(2, \rho_0/2)}^{\varepsilon'_0} f_0(x), \quad p_1 \in \partial_{(2, \rho_1/2)}^{\varepsilon'_1} f_1(x).$$

In particular, we can set $\varepsilon'_1 + \varepsilon'_2 = \varepsilon$ in (6).

In the following theorem we show that, in presence of differentiable functions, the notion of proximal ε -subdifferentials allows to infer an inclusion which involves the gradient of the differentiable function.

Theorem 3: Let \mathcal{X} be a Hilbert space. Let $f_0 : \mathcal{X} \rightarrow (-\infty, +\infty]$ be proper, convex and differentiable with a L_0 -Lipschitz continuous gradient on the whole space \mathcal{X} . Let $f_1 : \mathcal{X} \rightarrow (-\infty, +\infty]$ be proper and ρ -weakly convex on \mathcal{X} with $\rho \geq 0$. We then have the following inclusion

$$\partial_{(2, \rho/2)}^\varepsilon (f_0 + f_1)(x) \subset \nabla(f_0)(x) + \partial_{(2, \rho/2 + L_0/2)}^\varepsilon f_1(x)$$

for all $x \in \text{dom } f_1$ such that $\partial_{(2, \rho/2)}^\varepsilon f_1(x) \neq \emptyset$.

Proof: In view of [3, Lemma 2.64], both f_0 and $-f_0$ are proximally L_0 -subdifferentiable on \mathcal{X} . Precisely, for f_0 we have that for every $x, y \in \mathcal{X}$

$$\begin{aligned}(-f_0)(y) - (-f_0)(x) &\geq \\ \langle \nabla(-f_0)(x) \mid y - x \rangle - \frac{L_0}{2} \|y - x\|^2.\end{aligned}\tag{10}$$

Let us choose $x \in \text{dom } f_1$ and take $v \in \partial_{(2, \rho/2)}^\varepsilon (f_0 + f_1)(x)$. For every $y \in \mathcal{X}$, by adding

$$(f_1 + f_0)(y) - (f_1 + f_0)(x) \geq \langle v \mid y - x \rangle - \frac{\rho}{2} \|y - x\|^2 - \varepsilon$$

and (10) we get

$$\partial_{(2, \rho/2)}^\varepsilon (f_1 + f_0)(x) + \nabla(-f_0)(x) \subset \partial_{(2, \rho/2 + L_0/2)}^\varepsilon f_1(x)$$

i.e.

$$\partial_{(2, \rho/2)}^\varepsilon (f_1 + f_0)(x) \subset \partial_{(2, \rho/2 + L_0/2)}^\varepsilon f_1(x) + \nabla(f_0)(x).$$

Remark 5: We highlight the following facts. \blacksquare

- For $\varepsilon > 0$, if f_1 is also lower semicontinuous, Theo. 3 holds for all $x \in \text{dom } f_1$ by Prop. 2.
- Theo. 3 still holds when f_1 is convex, i.e. $\rho = 0$, but not for convex subdifferential. In fact the inclusion becomes

$$\partial_0^\varepsilon (f_0 + f_1)(x) \subset \nabla(f_0)(x) + \partial_{(2, L_0/2)}^\varepsilon f_1(x).$$

IV. INEXACT PROXIMAL MAPS

In general settings, the computation of the proximal map needs to be addressed as an independent optimisation problem. Some practical examples involves non-convex ℓ_p -seminorms (i.e. $p \in (0, 1)$) or the convex ℓ_p -norms (i.e. $p \geq 1$), unless p takes some specific values [8]. Another example is given by the combination of a sparsity-promoting functions with a non-orthogonal linear operator, as in the case of the popular discrete Total Variation functional [27] (and its non-convex modifications), which has been extensively used in the context of image and signal processing. In these cases, at each point, the proximal map is defined up to a certain degree of accuracy and in the framework of proximal algorithms, it is important to carry out a convergence analysis that takes this fact into account. In order to do so, we consider the concept of ε -solution for an optimisation problem (see Def. 5) and the related notion of ε -proximal point.

Definition 8 (ε -proximal point): Let \mathcal{X} be a normed vector space. Let function $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ be proper and bounded from below. Then for all $y \in \mathcal{X}$ and for all $\varepsilon \geq 0$, any ε -solution to the proximal minimization problem

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad f(x) + \frac{1}{2} \|x - y\|^2, \tag{11}$$

is said to be a ε -proximal point for f at y . The set of all ε -proximal points of f at y is denoted as

$$\varepsilon\text{-prox}_h(y) := \{x \in \mathcal{X} \mid x \text{ is a } \varepsilon\text{-solution of (11)}\}$$

In the following result, we provide a relationship between the ε -proximal operator and the ε -proximal subdifferentials of weakly convex function, using the sum rule from Theo. 3. Specifically, in Prop. 4, we keep track of the constant C of the $(2, C)$ -subdifferential of f .

Proposition 4: Let \mathcal{X} be a Hilbert space. Let $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ be a proper, lower semicontinuous ρ -weakly convex function that is bounded from below on \mathcal{X} . Let $\varepsilon \geq 0$, $\alpha > 0$. Then for every $y \in \mathcal{X}$, $x_\varepsilon \in \varepsilon\text{-prox}_{\alpha f}(y)$ implies

$$\frac{y - x_\varepsilon}{\alpha} \in \partial_{(2, \rho/2 + 1/(2\alpha))}^\varepsilon f(x_\varepsilon). \quad (12)$$

If $1/\alpha > \rho$, we obtain the equivalence

$$(y \in \mathcal{X}) \quad x_\varepsilon \in \varepsilon\text{-prox}_{\alpha f}(y) \\ \iff \frac{y - x_\varepsilon}{\alpha} \in \partial_{(2, \rho/2 + 1/(2\alpha))}^\varepsilon f(x_\varepsilon).$$

Proof: By the definition of ε -proximal point and Remark 1 we have

$$x_\varepsilon \in \varepsilon\text{-prox}_{\alpha f}(y) \\ \implies 0 \in \partial_{(2, \rho/2)}^\varepsilon \left(\frac{1}{2\alpha} \|\cdot - y\|^2 + f(\cdot) \right) (x_\varepsilon).$$

The assumptions in Theo. 3 are satisfied since $f_0(\cdot) = \frac{1}{2\alpha} \|\cdot - y\|^2$ is differentiable on the whole space and its gradient has a Lipschitz constant $L_0 = 1/\alpha$, hence we also have the inclusion

$$0 \in \nabla \left(\frac{1}{2\alpha} \|\cdot - y\|^2 \right) (x_\varepsilon) + \partial_{(2, \rho/2 + 1/(2\alpha))}^\varepsilon f(x_\varepsilon) \\ = \frac{x_\varepsilon - y}{\alpha} + \partial_{(2, \rho/2 + 1/(2\alpha))}^\varepsilon f(x_\varepsilon)$$

which is equivalent to

$$\frac{y - x_\varepsilon}{\alpha} \in \partial_{(2, \rho/2 + 1/(2\alpha))}^\varepsilon f(x_\varepsilon). \quad \blacksquare$$

Remark 6: We highlight that (12) is related to the notion of Type-2 approximation of the proximal point that is proposed in [20], [28] in the convex settings. In other words, by using the ε -proximal subdifferential instead of the (convex) ε -subdifferential, we can obtain a Type 2 approximation of the proximal point directly from Def. 8. This is due to Theo. 3, at the expense of increasing the modulus of proximal subdifferentiability by $1/(2\alpha)$.

Remark 7 (ε -Subdifferential of a quadratic function): Let \mathcal{X} be a Hilbert space. As a consequence of [14, Formula 1.2.5], for a function of the form

$h_0(x) = \frac{1}{2\alpha} \|x - y\|^2$ for some $y \in \mathcal{X}$ and $\alpha > 0$, we have

$$\partial_0^\varepsilon f_0(x) = \left\{ \frac{x - y}{\alpha} + \frac{e}{\alpha} \mid \frac{1}{2\alpha} \|e\|^2 \leq \varepsilon \right\}.$$

In view of Remark 7 and Theo. 2, we can provide another interpretation for the ε -proximal points of a ρ -weakly convex function in terms of proximal ε -subdifferentials, where this time we are able to keep track of the modulus of weak convexity of the function.

Proposition 5: Let \mathcal{X} be a Hilbert space. Let $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ be a proper, lower semicontinuous ρ -weakly convex function that is bounded from below on \mathcal{X} and let $\varepsilon \geq 0$, $\alpha > 0$. If $x_\varepsilon \in \varepsilon\text{-prox}_{\alpha f}(y)$, then there exist $\varepsilon_0, \varepsilon_1 \geq 0$ with $\varepsilon_0 + \varepsilon_1 \leq \varepsilon$ and there exists $e \in \mathcal{X}$ with $\frac{\|e\|^2}{2\alpha} \leq \varepsilon_0$ such that

$$\frac{y - x_\varepsilon - e}{\alpha} \in \partial_{(2, \rho/2)}^{\varepsilon_1} f(x_\varepsilon). \quad (13)$$

If $1/\alpha > \rho$, we obtain the equivalence

Proof: By definition of ε -proximal point we have

$$x_\varepsilon \in \varepsilon\text{-prox}_{\alpha f}(y) \\ \implies 0 \in \partial_{(2, \rho/2)}^\varepsilon \left(\frac{1}{2\alpha} \|\cdot - y\|^2 + f(\cdot) \right) (x_\varepsilon).$$

We can now apply Theo. 2, according to which there exist $\varepsilon_0, \varepsilon_1 \geq 0$ with $\varepsilon_0 + \varepsilon_1 \leq \varepsilon$ such that

$$0 \in \partial_0^{\varepsilon_0} \left(\frac{1}{2\alpha} \|\cdot - y\|^2 \right) (x_\varepsilon) + \partial_{(2, \rho/2)}^{\varepsilon_1} f(x_\varepsilon). \quad (14)$$

By applying Remark 7, we infer that there exists $e \in \mathcal{X}$ with $\frac{\|e\|^2}{2\alpha} \leq \varepsilon_0$ such that

$$\frac{x_\varepsilon - y}{\alpha} + \frac{e}{\alpha} \in \partial_0^{\varepsilon_0} \left(\frac{1}{2\alpha} \|\cdot - y\|^2 \right) (x_\varepsilon)$$

which implies

$$\frac{y - x_\varepsilon - e}{\alpha} \in \partial_{(2, \rho/2)}^{\varepsilon_1} f(x_\varepsilon). \quad \blacksquare$$

Remark 8: The interpretation provided by proposition 5 is related to the notion of Type-1 approximation of the proximal point that is proposed in [20], [28] for convex functions.

Remark 9: Notice that (13) from Prop. 5 implies the inclusion (12) from Prop. 4, which is based on Theo. 3. By definition of proximal ε -subdifferentials, (13) is equivalent to

$$(x \in \mathcal{X}) \quad f(x) - f(x_\varepsilon) \geq \\ \left\langle \frac{y - x_\varepsilon}{\alpha}, x - x_\varepsilon \right\rangle - \frac{\rho}{2} \|x - x_\varepsilon\|^2 - \left\langle \frac{e}{\alpha}, x - x_\varepsilon \right\rangle - \varepsilon_1.$$

We consider the following estimation

$$\left\langle \frac{e}{\alpha}, x - x_\varepsilon \right\rangle \leq \varepsilon_0 + \frac{1}{2\alpha} \|x - x_\varepsilon\|^2$$

which stems from Cauchy-Schwarz and Young's inequality. It follows that

$$f(x) - f(x_\varepsilon) \geq \left\langle \frac{y - x_\varepsilon}{\alpha}, x - x_\varepsilon \right\rangle - \left(\frac{\rho}{2} + \frac{1}{2\alpha} \right) \|x - x_\varepsilon\|^2 - \varepsilon$$

that is equivalent to (12)

$$\frac{y - x_\varepsilon}{\alpha} \in \partial_{(2, \rho/2 + 1/(2\alpha))}^\varepsilon f(x_\varepsilon).$$

The inclusion in (14) further leads to the following corollary which is a generalisation of [20, Lemma 2] from convex to proximal ε -subdifferentials.

Corollary 1: Let \mathcal{X} be a Hilbert space. Let $f : \mathcal{X} \rightarrow (-\infty, +\infty]$ be a proper, lower semicontinuous ρ -weakly convex function that is bounded from below on \mathcal{X} and let $\varepsilon \geq 0$. If $x_\varepsilon \in \varepsilon\text{-prox}_{\alpha f}(x)$, then there exist $e \in \mathcal{X}$ with $\frac{\|e\|^2}{2\alpha} \leq \varepsilon$ such that

$$\frac{y - x_\varepsilon - e}{\alpha} \in \partial_{(2, \rho/2)}^\varepsilon f(x_\varepsilon)$$

Proof: The proof is equivalent to the one from Prop. 5, with the only difference that we exploit the fact that the inclusion in (14) always implies the following inclusion

$$0 \in \partial_0^\varepsilon \left(\frac{1}{2\alpha} \|\cdot - y\|^2 \right) (x_\varepsilon) + \partial_{(2, \rho/2)}^\varepsilon f(x_\varepsilon)(x_\varepsilon).$$

by (3) and the fact that ε_0 and ε_1 from Theo. 2 are always smaller than ε . ■

V. CONCLUSIONS

We discussed inexact proximal operators (in the sense of Def. 8) for weakly convex functions defined on Hilbert spaces and their relationships to proximal ε -subdifferentials. We highlighted the main differences and similarities with the fully convex settings. An important feature of the obtained results is that in Prop. 4 and Prop. 5 we are able to control the moduli of proximal ε -subdifferentiability of f (in relation to the moduli of weak convexity). Such result could contribute to the convergence analysis of proximal algorithms for solving optimization problems containing weakly convex functions.

Acknowledgments This work has been supported by the ITN-ETN project TraDE-OPT funded by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 861137. This work represents only the author's view and the European Commission is not responsible for any use that may be made of the information it contains.

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