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# On the protected nodes in exponential recursive trees

Mehri Javanian\*      Rafik Aguech†

August 14, 2023

## Abstract

The exponential recursive trees model several kinds of networks. At each step of growing of these trees, each node independently attracts a new node with probability  $p$ , or fails to do with probability  $1 - p$ . Here, we investigate the number of protected nodes, total path length of protected nodes, and a mean study of the protected node profile of such trees.

## 1 Introduction

A social networking site is an Internet-based platform which people use to build social relationships with friends, family, colleagues, customers, or clients. Social networking can have a social purpose, a business purpose, or both, through sites like Facebook, Twitter, LinkedIn, Instagram, TikTok, Snapchat, Pinterest, Reddit, Tumblr, Telegram, WhatsApp and YouTube. These sites allow people and corporations to connect with one another so they can develop relationships very quickly and so they can share information, ideas, and messages. Social networking has become a significant base for marketers seeking to engage customers. In order to model certain aspects of fast-growing networks, the references [4] and [8] introduce exponential binary trees and exponential recursive trees, respectively.

A rooted tree grown on  $n + 1$  nodes labeled distinctly with the numbers  $1, 2, \dots, n, n + 1$ ; is a *recursive tree* of age  $n$ ,  $T_n$ , that is built by attaching, at the  $n$ th step, the new node  $n + 1$  to one node of a recursive tree of age  $n - 1$ ,  $T_{n-1}$ , according to some distribution on the set  $\{1, \dots, n\}$  and independently of the structure of the tree  $T_{n-1}$ . As an example, Figure 1 shows a sequence of recursive trees growing from a single node labeled with 1,  $T_0$ , into a tree of age 8,  $T_8$ , in 8 steps. See a survey of results of recursive trees in [11].

The recursive trees are slow-growing where one node is added at each step. So, the recursive tree models cannot be suitable for fast-growing phenomena (e.g., the Corona virus spreads very quickly from person to person). In [8], a fast-growing analogue of recursive trees

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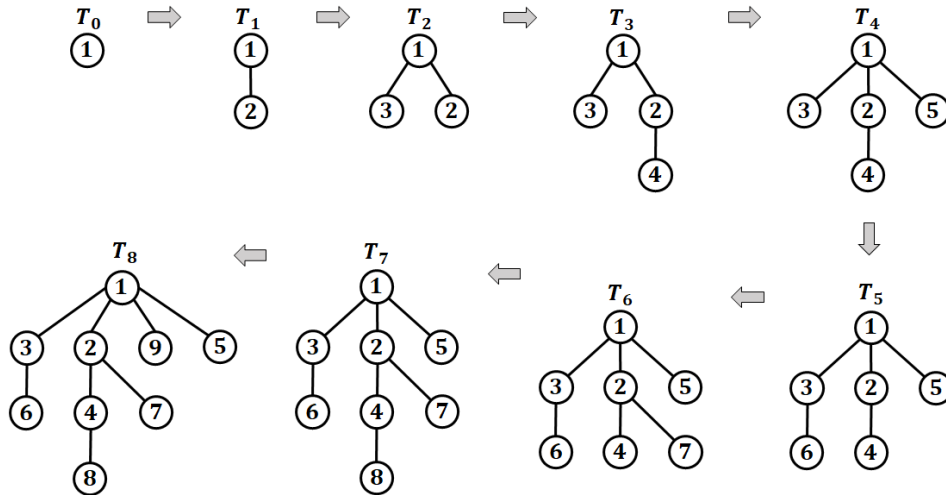


Figure 1: The evolution of a recursive tree in 8 steps.

has been defined as follows: Initially,  $T_0$  is a tree of a single root node. For  $n \geq 1$ , at the  $n$ th step, a tree of age  $n$ ,  $T_n$ , is constructed when every node of the tree of age  $n - 1$ ,  $T_{n-1}$ , independently attracts a child with probability  $p \in (0, 1)$ , or not to attract with probability  $q := 1 - p$ . After the  $n$ th step, the obtained tree is called *exponential recursive tree (ERT) of index  $p$* . The trees in Figure 2 (a) illustrate a sequence of ERT of index  $p$  growing from  $T_0$  into  $T_7$ , a tree of probability  $p^{19}q^{14}$  in the 7th step.

By a *protected node* in a rooted tree, we mean a node that is not a leaf and not all of its children are leaves, for instance see Figure 2 (b). For many types of random trees, protected nodes have been investigated, see e.g. [3, 5, 6, 7]. In this paper, we study the number of protected nodes in exponential recursive trees (ERT). Here, we present the asymptotic expectation, variance and characterizing of the limiting distribution of the number of protected nodes. Via contraction method, we also show the convergence in distribution for the sum of depths of all protected nodes, i.e., the total path length of all protected nodes in ERT of index  $p$ . Finally, we derive the expectation of protected node profile, i.e., the number of protected nodes at the same level.

## 2 Setting and Preliminary Lemmas

Let  $T_n$  be an exponential recursive tree of age  $n$  and index  $p$ . Define

- $X_n :=$  the number of protected nodes in  $T_n$ ,
- $\mathcal{R}_n :=$  the event that the root of  $T_n$  is protected,
- $|T_n| :=$  the number of nodes in  $T_n$  (the size of  $T_n$ ).

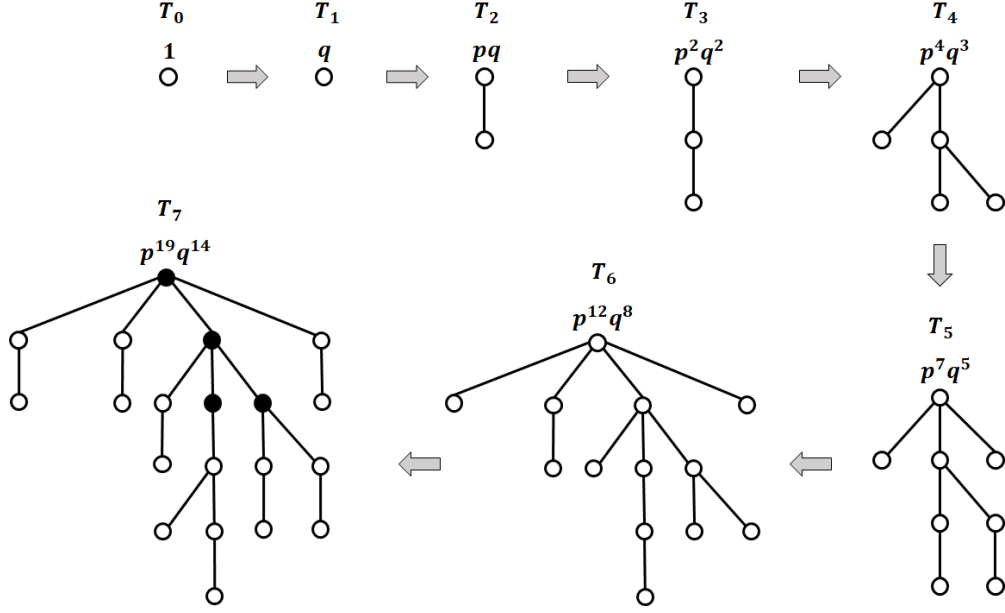


Figure 2: **(a)** The evolution of an exponential recursive tree (ERT) of index  $p$  in 7 steps. The probability of an ERT appears above it. **(b)** In  $T_7$ , protected nodes are shown in black.

At the first step, if the root of  $T_0$  fails to attract a child, then  $T_1$  will be a root and after  $n-1$  steps the root of  $T_1$  will produce an exponential recursive tree  $T'_{n-1}$ . Alternatively, if the root of  $T_0$  attracts a child, then, in  $n-1$  steps, the child and the root will independently develop two exponential recursive trees  $T''_{n-1}$  and  $T'''_{n-1}$ , respectively. Let us indicate the number of protected nodes in  $T'_{n-1}$ ,  $T''_{n-1}$  and  $T'''_{n-1}$  by  $X'_{n-1}$ ,  $X''_{n-1}$  and  $X'''_{n-1}$ , respectively. Therefore,  $X'_{n-1}$ ,  $X''_{n-1}$  and  $X'''_{n-1}$  are independent copies of  $X_{n-1}$ . We denote the events that the trees  $T'_{n-1}$ ,  $T''_{n-1}$  and  $T'''_{n-1}$  have protected roots by  $\mathcal{R}'_{n-1}$ ,  $\mathcal{R}''_{n-1}$  and  $\mathcal{R}'''_{n-1}$ , respectively.

If  $\mathbb{1}_A$  denotes the indicator function of an event  $A$ , then we define

$$\mathbb{I} := \mathbb{1}_{\{|T_1|=2\}}, \quad \mathbb{J}_n := \mathbb{1}_{\{|T'_n|=1\} \cap \{|T''_n| \geq 2\}}, \quad \mathbb{G}_n := \mathbb{1}_{\{|T''_n|=1\} \cap \mathcal{R}'''_n}. \quad (1)$$

The proof of our results requires to prove the following Lemmas.

**Lemma 1.** *By the above setting, for  $n \geq 2$ , the probability of  $\mathcal{R}_n$  is*

$$\mathbb{P}(\mathcal{R}_n) = \sum_{k=1}^{n-1} pq^k(1-q^k) \prod_{i=k+1}^{n-1} (1-pq^i). \quad (2)$$

*Proof.* We can observe that

$$\begin{aligned} \mathcal{R}_n &= \left[ \{|T''_{n-1}| \geq 2\} \cap \mathcal{R}'''_{n-1} \cap \{|T_1| = 2\} \right] \cup \left[ \mathcal{R}'_{n-1} \cap \{|T_1| = 1\} \right] \\ &\cup \left[ \{|T''_{n-1}| \geq 2\} \cap \{|T'''_{n-1}| = 1\} \cap \{|T_1| = 2\} \right] \end{aligned}$$

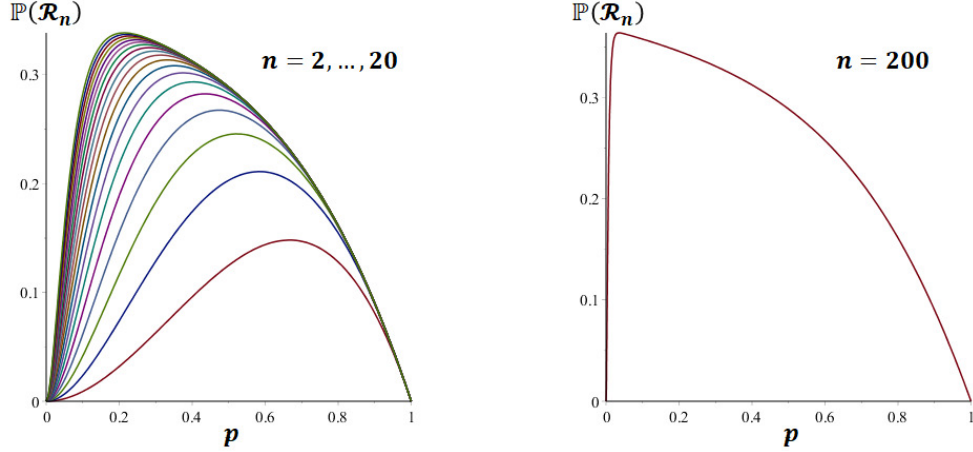


Figure 3: Graphs of  $\mathbb{P}(\mathcal{R}_n)$  versus  $p$ , for  $n = 2, \dots, 20$  and  $n = 200$ . For each  $p$ ,  $\mathbb{P}(\mathcal{R}_n)$  is increasing in  $n$ . E.g., the bottom and top curves are the graphs of  $\mathbb{P}(\mathcal{R}_2)$  and  $\mathbb{P}(\mathcal{R}_{20})$ .

is credible for  $n \geq 2$ . Since  $\mathbb{P}(|T_n''| \geq 2) = 1 - \mathbb{P}(|T_n''| = 1) = 1 - q^n$  and the trees  $T_{n-1}''$  and  $T_{n-1}'''$  are developed independently, then

$$\begin{aligned} \mathbb{P}(\mathcal{R}_n) &= p(1 - q^{n-1})\mathbb{P}(\mathcal{R}_{n-1}''') + q\mathbb{P}(\mathcal{R}_{n-1}'') + pq^{n-1}(1 - q^{n-1}) \\ &= (1 - pq^{n-1})\mathbb{P}(\mathcal{R}_{n-1}) + pq^{n-1}(1 - q^{n-1}). \end{aligned}$$

Iterating the recurrence we obtain the claim. See Figure 3 as a plot for  $\mathbb{P}(\mathcal{R}_n)$ .  $\square$

**Lemma 2.** *By the above setting, we have the distributional equation*

$$X_n \stackrel{d}{=} X_{n-1}'(1 - \mathbb{I}) + (X_{n-1}'' + X_{n-1}''' + \mathbb{J}_{n-1} - \mathbb{G}_{n-1})\mathbb{I}, \quad (3)$$

where the symbol  $\stackrel{d}{=}$  denotes the equality in distribution.

*Proof.* In order to construct an exponential recursive tree  $\mathcal{T}_n$ , a tree distributed like  $T_n$ , we can attach the root of the tree  $T_{n-1}'''$  to the root of the tree  $T_{n-1}''$  by adding an edge. Consequently, if  $|T_{n-1}''| \geq 2$  and  $T_{n-1}'''$  is a leaf, then the number of protected nodes in  $\mathcal{T}_n$  is increased by 1; as the attaching causes the root of  $\mathcal{T}_n$  to be protected. From the other point of view, if  $|T_{n-1}''| = 1$  and the root of  $T_{n-1}'''$  is protected, then this protection is lost after the attaching. Hence we obtain the assertion by (1).  $\square$

**Lemma 3.** For  $m \geq 1$ , we have the following recurrences for  $\mathbb{E}[X_n^m]$  and  $\mathbb{E}[X_n^m \mathbb{1}_{\mathcal{R}_n}]$ :

$$\begin{aligned}\mathbb{E}[X_n^m] &= (p+1)\mathbb{E}[X_{n-1}^m] + p \sum_{k=1}^{m-1} \binom{m}{k} \mathbb{E}[X_{n-1}^k] \mathbb{E}[X_{n-1}^{m-k}] + pq^{n-1} \sum_{k=1}^{m-1} \binom{m}{k} \mathbb{E}[X_{n-1}^k] \\ &\quad + pq^{n-1} \sum_{k=1}^{m-1} \binom{m}{k} (-1)^{m-k} \mathbb{E}[X_{n-1}^k \mathbb{1}_{\mathcal{R}_{n-1}}] + pq^{n-1} (1 - q^{n-1} + (-1)^m \mathbb{P}(\mathcal{R}_{n-1})), \\ \mathbb{E}[X_n^m \mathbb{1}_{\mathcal{R}_n}] &= (1 - pq^{n-1}) \mathbb{E}[X_{n-1}^m \mathbb{1}_{\mathcal{R}_{n-1}}] + p \sum_{k=1}^m \binom{m}{k} \mathbb{E}[X_{n-1}^k] \mathbb{E}[X_{n-1}^{m-k} \mathbb{1}_{\mathcal{R}_{n-1}}] \\ &\quad + pq^{n-1} \sum_{k=1}^m \binom{m}{k} \mathbb{E}[X_{n-1}^k] + pq^{n-1} (1 - q^{n-1}).\end{aligned}$$

*Proof.* Raise both sides of (3) to the  $m$ th power. So we get

$$X_n^m \stackrel{d}{=} (X'_{n-1})^m (1 - \mathbb{I}) + \sum_{\substack{i,j,k,l \geq 0 \\ i+j+k+l=m}} \binom{m}{i,j,k,l} (X''_{n-1})^i (X'''_{n-1})^j \mathbb{J}_{n-1}^k (-\mathbb{G}_{n-1})^l \mathbb{I}.$$

Now, separate the second term with  $(k=0, l \geq 1)$ ,  $(k \geq 1, l=0)$  and  $(k=0, l=0)$ . Then

$$\begin{aligned}X_n^m &\stackrel{d}{=} (X'_{n-1})^m (1 - \mathbb{I}) + \sum_{\substack{i,j \geq 0, l \geq 1 \\ i+j+l=m}} \binom{m}{i,j,l} (X''_{n-1})^i (X'''_{n-1})^j (-\mathbb{G}_{n-1})^l \mathbb{I} \\ &\quad + \sum_{\substack{i,j \geq 0, k \geq 1 \\ i+j+k=m}} \binom{m}{i,j,k} (X''_{n-1})^i (X'''_{n-1})^j (\mathbb{J}_{n-1})^k \mathbb{I} + \sum_{\substack{i,j \geq 0 \\ i+j=m}} \binom{m}{i,j} (X''_{n-1})^i (X'''_{n-1})^j \mathbb{I}. \quad (4)\end{aligned}$$

Note that, for  $i, j, k, l \geq 1$ ,

$$\begin{aligned}(\mathbb{G}_{n-1})^l &= \mathbb{G}_{n-1} = \mathbb{1}_{\{|T''_{n-1}|=1\}} \mathbb{1}_{\mathcal{R}'''_{n-1}}, & (X''_{n-1})^i \mathbb{1}_{\{|T''_{n-1}|=1\}} &= 0 \\ (\mathbb{J}_{n-1})^k &= \mathbb{J}_{n-1} = \mathbb{1}_{\{|T''_{n-1}| \geq 2\}} \mathbb{1}_{\{|T'''_{n-1}|=1\}}, & (X'''_{n-1})^j \mathbb{1}_{\{|T'''_{n-1}|=1\}} &= 0.\end{aligned}$$

Therefore, the equation (4) can be simplified as

$$\begin{aligned}X_n^m &\stackrel{d}{=} (X'_{n-1})^m (1 - \mathbb{I}) + \sum_{j=0}^{m-1} \binom{m}{j} (X'''_{n-1})^j (-1)^{m-j} \mathbb{1}_{\{|T''_{n-1}|=1\}} \mathbb{1}_{\mathcal{R}'''_{n-1}} \mathbb{I} \\ &\quad + \sum_{i=0}^{m-1} \binom{m}{i} (X''_{n-1})^i \mathbb{1}_{\{|T''_{n-1}| \geq 2\}} \mathbb{1}_{\{|T'''_{n-1}|=1\}} \mathbb{I} + \sum_{i=0}^m \binom{m}{i} (X''_{n-1})^i (X'''_{n-1})^{m-i} \mathbb{I}. \quad (5)\end{aligned}$$

Take the expectation of (5) and observe the independence in  $T'_{n-1}$ ,  $T''_{n-1}$  and  $T'''_{n-1}$ , to write

$$\begin{aligned}\mathbb{E}[X_n^m] &= q\mathbb{E}[(X'_{n-1})^m] + p\mathbb{E}[(X'''_{n-1})^m] + p\mathbb{E}[(X''_{n-1})^m] + p\sum_{i=1}^{m-1}\binom{m}{i}\mathbb{E}[(X''_{n-1})^i]\mathbb{E}[(X'''_{n-1})^{m-i}] \\ &\quad + p\sum_{i=1}^{m-1}\binom{m}{i}(-1)^{m-i}\mathbb{E}[(X'''_{n-1})^i\mathbb{1}_{\mathcal{R}'''_{n-1}}]\mathbb{E}[\mathbb{1}_{\{|T''_{n-1}|=1\}}] + (-1)^m p\mathbb{E}[\mathbb{1}_{\mathcal{R}'''_{n-1}}]\mathbb{E}[\mathbb{1}_{\{|T''_{n-1}|=1\}}] \\ &\quad + p\sum_{i=1}^{m-1}\binom{m}{i}\mathbb{E}[(X''_{n-1})^i\mathbb{1}_{\{|T''_{n-1}|\geq 2\}}]\mathbb{E}[\mathbb{1}_{\{|T'''_{n-1}|=1\}}] + p\mathbb{E}[\mathbb{1}_{\{|T''_{n-1}|\geq 2\}}]\mathbb{E}[\mathbb{1}_{\{|T'''_{n-1}|=1\}}].\end{aligned}$$

This yields the first recurrence of the Lemma, by the identical distribution in the subtrees and

$$\mathbb{E}[(X''_{n-1})^i\mathbb{1}_{\{|T''_{n-1}|\geq 2\}}] = \mathbb{E}[(X''_{n-1})^i(1 - \mathbb{1}_{\{|T''_{n-1}|=1\}})] = \mathbb{E}[(X''_{n-1})^i], \quad i \geq 1.$$

By the definitions of the subtrees  $T'_{n-1}$ ,  $T''_{n-1}$  and  $T'''_{n-1}$ , observe that

$$\begin{aligned}X_n^m \mathbb{1}_{\mathcal{R}_n} &\stackrel{d}{=} (X'_{n-1})^m \mathbb{1}_{\mathcal{R}'_{n-1}}(1 - \mathbb{1}) + (X''_{n-1} + 1)^m \mathbb{1}_{\{|T''_{n-1}|=1\}} \mathbb{1}_{\{|T''_{n-1}|\geq 2\}} \mathbb{1} \\ &\quad + (X''_{n-1} + X'''_{n-1})^m \mathbb{1}_{\mathcal{R}'''_{n-1}} \mathbb{1}_{\{|T''_{n-1}|\geq 2\}} \mathbb{1}.\end{aligned}$$

Taking the expectation of this equation, it implies the second recurrence of the Lemma.  $\square$

### 3 The Expectation and Variance

In this section, we obtain the expectation and variance of  $X_n$ , using Lemma 1 and Lemma 3.

**Theorem 4.** *Let  $X_n$  be the number of protected nodes in an exponential recursive tree of age  $n$  and index  $p$ . For  $q := 1 - p$ ,*

$$\begin{aligned}\mathbb{E}[X_n] &= \frac{p}{q} \left( \sum_{j=2}^n (p+1)^{-j} q^j \left( 1 - q^{j-1} - \sum_{k=1}^{j-2} p q^k (1 - q^k) \prod_{i=k+1}^{j-2} (1 - p q^i) \right) \right) (p+1)^n \\ &=: \mu_{n,p} (p+1)^n, \quad (\mu_{n,p} < 1 \text{ for } n \geq 1 \text{ and } p \in (0, 1)).\end{aligned}\tag{6}$$

See Figure 4 for some graphs of  $\mu_{n,p}$ . For  $n \geq 1$ ,  $\mu_{n,p} < \mu_{n+1,p}$ . So  $\mu_p := \lim_{n \rightarrow \infty} \mu_{n,p}$  exists.

*Proof.* In Lemma 3, set  $m = 1$  to obtain the recurrence

$$\mathbb{E}[X_n] = (p+1)\mathbb{E}[X_{n-1}] + p q^{n-1}(1 - q^{n-1}) - p q^{n-1} \mathbb{P}(\mathcal{R}_{n-1}),$$

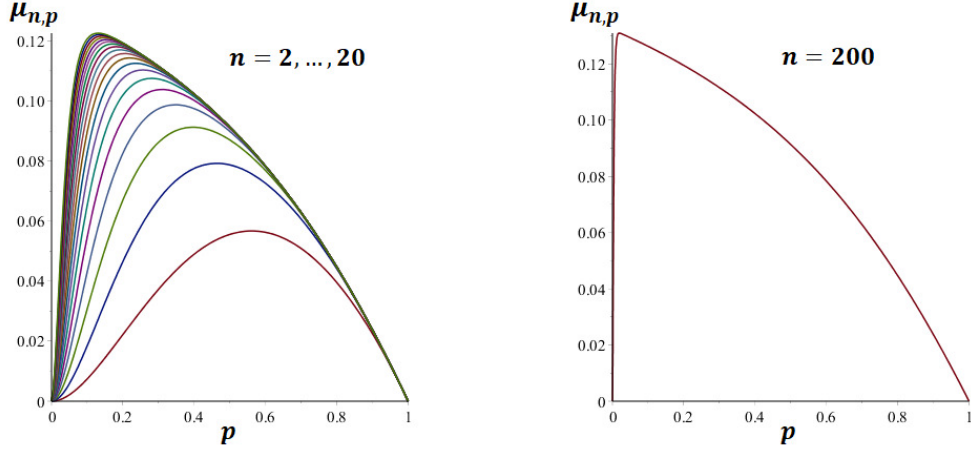


Figure 4: Graphs of  $\mu_{n,p}$  versus  $p \in (0, 1)$ , for  $n = 2, \dots, 20$  and  $n = 200$ .

with initial condition  $\mathbb{E}[X_1] = 0$ . Iterating this formula, we find

$$\begin{aligned} \mathbb{E}[X_n] &= \sum_{j=0}^{n-2} (p+1)^j p q^{n-j-1} (1 - q^{n-j-1} - \mathbb{P}(\mathcal{R}_{n-j-1})) \\ &= \frac{p}{q} \left( \sum_{j=2}^n (p+1)^{-j} q^j (1 - q^{j-1} - \mathbb{P}(\mathcal{R}_{j-1})) \right) (p+1)^n \end{aligned} \quad (7)$$

Substituting  $\mathbb{P}(\mathcal{R}_{j-1})$  from (2) into (7), we get the exact solution in the Theorem.  $\square$

In the second moment recurrence, we need  $\mathbb{E}[X_n \mathbb{1}_{\mathcal{R}_n}]$  that is given in the following lemma.

**Lemma 5.** *Let  $Y_n := X_n \mathbb{1}_{\mathcal{R}_n}$ , then*

$$\begin{aligned} \mathbb{E}[Y_n] &= \left( p \sum_{k=1}^{n-1} \mu_{n-k,p} (p+1)^{-k} (q^{n-k} + \mathbb{P}(\mathcal{R}_{n-k})) \prod_{i=n-k+1}^{n-1} (1 - pq^i) \right) (p+1)^n + \mathbb{P}(\mathcal{R}_n) \\ &=: \alpha_{n,p} (p+1)^n + \mathbb{P}(\mathcal{R}_n) = \alpha_p (p+1)^n + \mathcal{O}(1), \end{aligned} \quad (8)$$

where  $\alpha_p := \lim_{n \rightarrow \infty} \alpha_{n,p}$  (see Figure 5) and the functions  $\mathbb{P}(\mathcal{R}_n)$  and  $\mu_{n,p}$  are obtained in Lemma 1 and Theorem 4, respectively.

*Proof.* In Lemma 3, set  $m = 1$  to find the recurrence

$$\mathbb{E}[Y_n] = (1 - pq^{n-1})\mathbb{E}[Y_{n-1}] + p(q^{n-1} + \mathbb{P}(\mathcal{R}_{n-1}))\mathbb{E}[X_{n-1}] + pq^{n-1}(1 - q^{n-1})$$



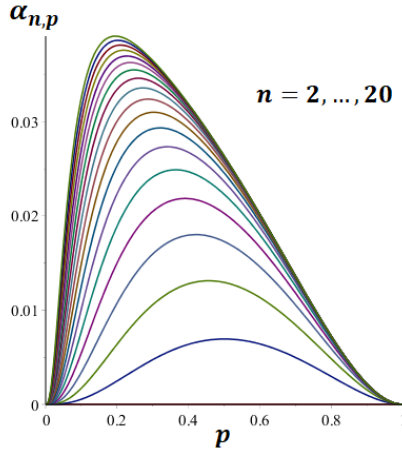


Figure 5: Graphs of  $\alpha_{n,p}$  versus  $p \in (0, 1)$ , for  $n = 2, \dots, 20$ .

with initial condition  $\mathbb{E}[Y_1] = 0$ . Iterating this formula, we obtain

$$\begin{aligned} \mathbb{E}[Y_n] &= \sum_{k=1}^{n-1} \left( p(q^k + \mathbb{P}(\mathcal{R}_k))\mathbb{E}[X_k] + pq^k(1 - q^k) \right) \prod_{i=k+1}^{n-1} (1 - pq^i) \\ &= p \sum_{k=1}^{n-1} \mu_{k,p}(p+1)^k (q^k + \mathbb{P}(\mathcal{R}_k)) \prod_{i=k+1}^{n-1} (1 - pq^i) + \mathbb{P}(\mathcal{R}_n). \end{aligned}$$

This implies the claim. □

**Theorem 6.** *By the above definitions for  $\mathbb{P}(\mathcal{R}_n)$ ,  $\mu_{n,p}$  and  $\alpha_{n,p}$ , we have*

$$\begin{aligned} \mathbb{E}[X_n^2] &= \sum_{j=1}^{n-1} \left( 2p\mu_{j,p}^2(p+1)^{-n+j-1} + 2pq^j(\mu_{j,p} - \alpha_{j,p})(p+1)^{-n-1} \right. \\ &\quad \left. + pq^j(1 - q^j - \mathbb{P}(\mathcal{R}_j))(p+1)^{-n-j-1} \right) (p+1)^{2n} =: \beta_{n,p}(p+1)^{2n}. \end{aligned}$$

See Figure 6 for some graphs of  $\beta_{n,p}$  with  $\beta_p := \lim_{n \rightarrow \infty} \beta_{n,p}$ .

*Proof.* In Lemma 3, set  $m = 2$  to find the recurrence

$$\begin{aligned} \mathbb{E}[X_n^2] &= (p+1)\mathbb{E}[X_{n-1}^2] + 2p\mathbb{E}^2[X_{n-1}] + 2pq^{n-1}\mathbb{E}[X_{n-1}] \\ &\quad - 2pq^{n-1}\mathbb{E}[X_{n-1}\mathbb{1}_{\mathcal{R}_{n-1}}] + pq^{n-1}(1 - q^{n-1} + \mathbb{P}(\mathcal{R}_{n-1})), \end{aligned}$$

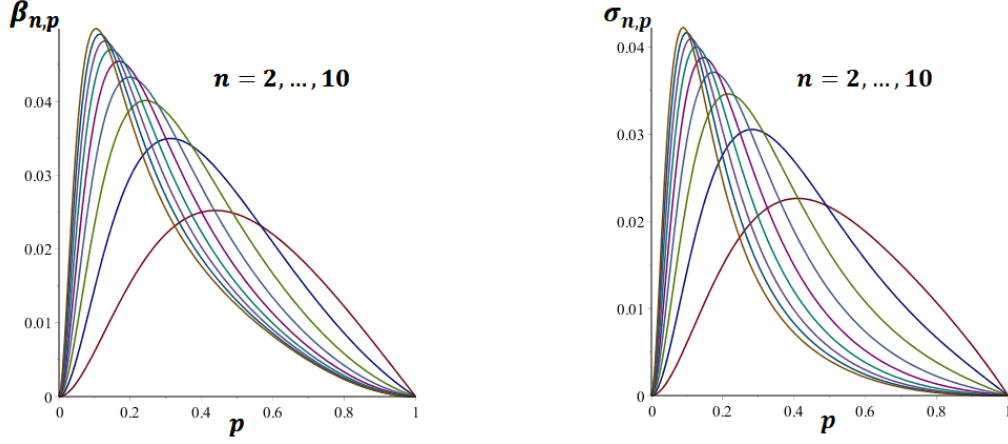


Figure 6: Graphs of  $\beta_{n,p}$  and  $\sigma_{n,p}$  versus  $p \in (0, 1)$ , for  $n = 2, \dots, 10$ .

with boundary condition  $\mathbb{E}[X_1^2] = 0$ . This standard linear recurrence has the solution

$$\begin{aligned} \mathbb{E}[X_n^2] = \sum_{j=0}^{n-2} (p+1)^j & \left( 2p\mathbb{E}^2[X_{n-j-1}] + 2pq^{n-j-1}\mathbb{E}[X_{n-j-1}] \right. \\ & \left. - 2pq^{n-j-1}\mathbb{E}[Y_{n-j-1}] + pq^{n-j-1}(1 - q^{n-j-1} + \mathbb{P}(\mathcal{R}_{n-j-1})) \right). \end{aligned} \quad (9)$$

After substituting (2), (6), (8) in (9), we have

$$\begin{aligned} \mathbb{E}[X_n^2] = \sum_{j=0}^{n-2} & \left( 2p\mu_{n-j-1,p}^2(p+1)^{-j-2} + 2pq^{n-j-1}(\mu_{n-j-1,p} - \alpha_{n-j-1,p})(p+1)^{-n-1} \right. \\ & \left. + pq^{n-j-1}(p+1)^{-2n+j}(1 - q^{n-j-1} - \mathbb{P}(\mathcal{R}_{n-j-1})) \right) (p+1)^{2n}. \end{aligned}$$

This yields the claimed result by changing  $n - j - 1 \rightarrow j$  in the range of the above sum.  $\square$

**Corollary 7.** *By the above definitions for  $\mu_{n,p}$  and  $\beta_{n,p}$ , we have*

$$\text{Var}[X_n] = (\beta_{n,p} - \mu_{n,p}^2)(p+1)^{2n} =: \sigma_{n,p}(p+1)^{2n}.$$

See Figure 6 for some graphs of  $\sigma_{n,p}$  with  $\lim_{n \rightarrow \infty} \sigma_{n,p} =: \sigma_p^2 = \beta_p - \mu_p^2$ .

*Proof.* Using  $\text{Var}[X_n] = \mathbb{E}[X_n^2] - \mathbb{E}^2[X_n]$  and Theorem 6, the proof is straightforward.  $\square$

## 4 Convergence in Distribution

Here, we characterize the limiting distribution of  $\frac{X_n}{(p+1)^n}$ , i.e., a scaled version of  $X_n$ , the number of protected nodes in an ERT of age  $n$  and index  $p$ , by its moments.

**Theorem 8.** Let  $X_n$  be the number of protected nodes in an exponential recursive tree of age  $n$  and index  $p$ . We have the convergence in distribution

$$\frac{X_n}{(p+1)^n} \xrightarrow{\mathcal{D}} X_*,$$

where the limiting random variable  $X_*$  has moments  $b_m := \mathbb{E}[X_*^m]$  defined inductively by

$$b_m = \frac{p}{(p+1)^m - (p+1)} \sum_{i=1}^{m-1} \binom{m}{i} b_i b_{m-i}, \quad m \geq 2,$$

with  $b_1 = \mathbb{E}[X_*] = \mu_p = \lim_{n \rightarrow \infty} \mu_{n,p}$  at the basis of the induction.

*Proof.* Scale the two recurrences in Lemma 3 by  $(p+1)^{nm}$  to get

$$\begin{aligned} \mathbb{E}\left[\left(\frac{X_n}{(p+1)^n}\right)^m\right] &= \frac{1}{(p+1)^{m-1}} \mathbb{E}\left[\left(\frac{X_{n-1}}{(p+1)^{n-1}}\right)^m\right] \\ &+ \frac{p}{(p+1)^m} \sum_{k=1}^{m-1} \binom{m}{k} \mathbb{E}\left[\left(\frac{X_{n-1}}{(p+1)^{n-1}}\right)^k\right] \mathbb{E}\left[\left(\frac{X_{n-1}}{(p+1)^{n-1}}\right)^{m-k}\right] \\ &+ pq^{n-1} \sum_{k=1}^{m-1} \binom{m}{k} \frac{1}{(p+1)^{(m-k)n+k}} \mathbb{E}\left[\left(\frac{X_{n-1}}{(p+1)^{n-1}}\right)^k\right] \\ &+ pq^{n-1} \sum_{k=1}^{m-1} \binom{m}{k} \frac{(-1)^{m-k}}{(p+1)^{(m-k)n+k}} \mathbb{E}\left[\left(\frac{X_{n-1}}{(p+1)^{n-1}}\right)^k \mathbb{1}_{\mathcal{R}_{n-1}}\right] \\ &+ \frac{1}{(p+1)^{nm}} \cdot pq^{n-1} (1 - q^{n-1} + (-1)^m \mathbb{P}(\mathcal{R}_{n-1})), \end{aligned} \quad (10)$$

$$\begin{aligned} \mathbb{E}\left[\left(\frac{X_n}{(p+1)^n}\right)^m \mathbb{1}_{\mathcal{R}_n}\right] &= \frac{1 - pq^{n-1}}{(p+1)^m} \mathbb{E}\left[\left(\frac{X_{n-1}}{(p+1)^{n-1}}\right)^m \mathbb{1}_{\mathcal{R}_{n-1}}\right] + \frac{1}{(p+1)^{nm}} \cdot pq^{n-1} (1 - q^{n-1}) \\ &+ \frac{p}{(p+1)^m} \sum_{k=1}^m \binom{m}{k} \mathbb{E}\left[\left(\frac{X_{n-1}}{(p+1)^{n-1}}\right)^k\right] \mathbb{E}\left[\left(\frac{X_{n-1}}{(p+1)^{n-1}}\right)^{m-k} \mathbb{1}_{\mathcal{R}_{n-1}}\right] \\ &+ pq^{n-1} \sum_{k=1}^m \binom{m}{k} \frac{1}{(p+1)^{(m-k)n+k}} \mathbb{E}\left[\left(\frac{X_{n-1}}{(p+1)^{n-1}}\right)^k\right]. \end{aligned} \quad (11)$$

According to the two recurrences (10) and (11) with finite limits

$$\lim_{n \rightarrow \infty} \mathbb{E}\left[\frac{X_n}{(p+1)^n}\right] = \mu_p, \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{E}\left[\frac{X_n}{(p+1)^n} \mathbb{1}_{\mathcal{R}_n}\right] = \alpha_p,$$

we see that  $b_m := \lim_{n \rightarrow \infty} \mathbb{E}\left[\left(\frac{X_n}{(p+1)^n}\right)^m\right]$  exists, by induction on  $m$ , and we get

$$(p+1)^m b_m = (p+1) b_m + p \sum_{i=1}^{m-1} \binom{m}{i} b_i b_{m-i}, \quad m \geq 2.$$

Using this recurrence and  $b_1 = \mu_p < 1$ , we can conclude that  $\frac{b_m}{m!} < 1$  by induction on  $m$  (this induction is shown in page 7 of [8] and in page 12 of [1], as well). Subsequently, for  $|z| < 1$ , the series  $\sum_{m=0}^{\infty} \frac{b_m}{m!} z^m$  converges. Therefore, by Theorem 30.1 in [2],  $\frac{X_n}{(p+1)^n}$  converges in distribution to a unique limit  $X_*$ .  $\square$

## 5 The Total path length

Let denote by  $I_n$  be the total path length of protected nodes in an exponential recursive tree after  $n$  steps: is the sum of depths of all protected nodes. At step one if the root recruits a new child then the total path length at step  $n$  is the total path length of the root at step  $n-1$  (for the rest of steps) plus the total path length of the subtree with root the first child of the root of the tree  $T_n$  and we have to add to all protected nodes of this subtree 1 (the distance between the first child and the root), but if the root does not recruits a child at step 1, then the total path length at step  $n$  will be the same as the total path length at a step  $n-1$ . Let  $I_n$  be the total path length of protected nodes in an exponential recursive tree at time  $n$ , the following lemma holds

**Lemma 9.**

$$I_n \stackrel{d}{=} \mathbb{I} \left( I_{n-1}^{(1)} + I_{n-1}^{(2)} + X_{n-1} \right) + (1 - \mathbb{I}) \tilde{I}_{n-1}^{(1)} \quad (12)$$

where  $I_n^{(1)}$ ,  $(I_n^{(2)}, X_n)$ ,  $\tilde{I}_n^{(1)}$  and  $\mathbb{I}$  are independent,  $I_n^{(i)} \stackrel{d}{=} I_n$  for  $i = 1, 2$ ,  $\mathbb{I}$  is a Bernoulli random variable with success probability  $p$ , and  $X_n$  is the number of protected nodes of an exponential recursive tree at age  $n$ .

**Remark 1.** The internal path length can be obtained by another method, if we denote for all  $k$  by

$$\mathcal{N}_k = \{ \text{at step } k \text{ the root recruits a new child} \},$$

then,  $I_n$  satisfies, almost surely,

$$I_n = \sum_{k=1}^n \mathbb{I}_{\mathcal{N}_k} I_{n-k} + X_n - \mathbb{I}_{\mathcal{R}_n}, \quad (13)$$

where  $I_{n-k}$  and  $\mathcal{N}_k$  are independent.

### 5.1 Mean of $I_n$

Define, for all integer  $n$ ,  $x_n := \mathbb{E}[X_n]$  and  $i_n := \mathbb{E}[I_n]$ . From the Lemma 9, we deduce that

$$i_n = p(2i_{n-1} + x_{n-1}) + q i_{n-1} = (1+p)i_{n-1} + p x_{n-1}. \quad (14)$$

**Proposition 10.** *The mean  $i_n$  of  $I_n$  is given by*

$$\mathbb{E}[I_n] = p (1 + p)^n \sum_{k=1}^n \mu_{k,p},$$

and

$$\lim_n \mathbb{E} \left[ \frac{I_n}{n (1 + p)^n} \right] = p \mu_p,$$

where  $\mu_{k,p}$  and  $\mu_p$  are given respectively in Theorem 4.

*Proof.* By iteration, we conclude that the solution of the recurrence (14) is

$$i_n = p \sum_{k=1}^{n-1} (1 + p)^{n-k} x_k.$$

Since, for all  $k$ ,  $x_k = \mu_{k,p}(1 + p)^k$  and  $\lim_n \mu_{n,p} := \mu_p$ , the claim of the lemma follows.  $\square$

## 5.2 Second moment of $I_n$

To obtain the variance of  $I_n$  we need to compute, at first,  $b_n := \mathbb{E}[I_n X_n]$ .

**Lemma 11.** *Using the notations (1), we have*

$$b_n = (1 + p) b_{n-1} + 2 p x_{n-1} i_{n-1} + p q^{n-1} (i_{n-1} - \mathbb{E} [I_{n-1} \mathbb{I}_{\mathcal{R}_{n-1}}]).$$

*Proof.* The sequence  $X_n$  satisfies

$$X_n = \mathbb{I} \left( X_{n-1}^{(1)} + X_{n-1}^{(2)} + \mathbb{J}_{n-1} - \mathbb{G}_{n-1} \right) + (1 - \mathbb{I}) \tilde{X}_{n-1}^{(1)}, \quad (15)$$

where

$$\mathbb{J}_n = \mathbb{I}_{\{|T_n^{(1)}|=1\}} \mathbb{I}_{\{|T_n^{(2)}|\geq 2\}}, \quad \mathbb{G}_n = \mathbb{I}_{\{|T_n^{(2)}|=1\}} \mathbb{I}_{\mathcal{R}_n^{(1)}}.$$

Multiplying (12) by (15) and recall that  $\mathbb{I} := \mathbb{1}_{\{|T_1|=2\}}$ , we deduce,

$$\begin{aligned} X_n I_n &= \mathbb{I} \left( X_{n-1}^{(1)} I_{n-1}^{(1)} + X_{n-1}^{(2)} I_{n-1}^{(1)} + \mathbb{J}_{n-1} I_{n-1}^{(1)} - \mathbb{G}_{n-1} I_{n-1}^{(1)} \right) \\ &\quad + \mathbb{I} \left( X_{n-1}^{(1)} I_{n-1}^{(2)} + X_{n-1}^{(2)} I_{n-1}^{(2)} + \mathbb{J}_{n-1} I_{n-1}^{(2)} - \mathbb{G}_{n-1} I_{n-1}^{(2)} \right) \\ &\quad + \mathbb{I} \left( X_{n-1}^{(1)} X_{n-1}^{(2)} + X_{n-1}^{(2)} X_{n-1}^{(2)} + \mathbb{J}_{n-1} X_{n-1}^{(2)} - \mathbb{G}_{n-1} X_{n-1}^{(2)} \right) \\ &\quad + (1 - \mathbb{I}) \tilde{X}_{n-1}^{(1)} \tilde{I}_{n-1}^{(1)}. \end{aligned}$$

On one hand, observe that, almost surely,

$$\begin{aligned}\mathbb{J}_{n-1} X_{n-1}^{(2)} &= I_{n-1}^{(1)} \mathbb{J}_{n-1} = \mathbb{G}_{n-1} I_{n-1}^{(2)} = (I_{n-1}^{(2)} \mathbb{I}_{\{|T_{n-1}^{(2)}|=1\}}) \mathbb{I}_{\mathcal{R}_{n-1}^{(1)}} = 0, \\ \mathbb{G}_{n-1} X_{n-1}^{(2)} &= \mathbb{G}_{n-1} I_{n-1}^{(2)} = (I_{n-1}^{(2)} \mathbb{I}_{\{|T_{n-1}^{(2)}|=1\}}) \mathbb{I}_{\mathcal{R}_{n-1}^{(1)}} = 0.\end{aligned}$$

We deduce, almost surely, that

$$\mathbb{J}_{n-1} I_{n-1}^{(1)} = \mathbb{G}_{n-1} I_{n-1}^{(2)} = \mathbb{J}_{n-1} X_{n-1}^{(2)} = \mathbb{G}_{n-1} X_{n-1}^{(2)} = 0.$$

On the other hand, from

$$\begin{aligned}I_{n-1}^{(2)} \mathbb{J}_{n-1} &= (I_{n-1}^{(2)} \mathbb{I}_{\{|T_{n-1}^{(2)}|\geq 2\}}) \mathbb{I}_{\{|T_{n-1}^{(1)}|=1\}} = I_{n-1}^{(2)} \mathbb{I}_{\{|T_{n-1}^{(1)}|=1\}}, \\ I_{n-1}^{(1)} \mathbb{G}_{n-1} &= (I_{n-1}^{(1)} \mathbb{I}_{\mathcal{R}_{n-1}^{(1)}}) \mathbb{I}_{\{|T_{n-1}^{(2)}|=1\}},\end{aligned}$$

we obtain

$$\begin{aligned}\mathbb{E}[I_{n-1}^{(2)} \mathbb{J}_{n-1}] &= \mathbb{E}[I_{n-1}^{(2)}] \mathbb{E}[\mathbb{I}_{\{|T_{n-1}^{(1)}|=1\}}] = i_{n-1} q^{n-1}, \\ \mathbb{E}[I_{n-1}^{(1)} \mathbb{G}_{n-1}] &= \mathbb{E}[I_{n-1}^{(1)} \mathbb{I}_{\mathcal{R}_{n-1}^{(1)}}] \mathbb{E}[\mathbb{I}_{\{|T_{n-1}^{(2)}|=1\}}] = q^{n-1} \mathbb{E}[I_{n-1}^{(1)} \mathbb{I}_{\mathcal{R}_{n-1}^{(1)}}].\end{aligned}$$

Then

$$\begin{aligned}b_n &= p \left( b_{n-1} + x_{n-1} i_{n-1} + \mathbb{E}[I_{n-1}^{(1)} \mathbb{J}_{n-1}] - \mathbb{E}[I_{n-1}^{(1)} \mathbb{G}_{n-1}] \right) \\ &\quad + p \left( x_{n-1} i_{n-1} + b_{n-1} + \mathbb{E}[I_{n-1}^{(2)} \mathbb{J}_{n-1}] - \mathbb{E}[I_{n-1}^{(2)} \mathbb{G}_{n-1}] \right) \\ &\quad + x_{n-1}^2 + \mathbb{E}[(X_{n-1}^{(2)})^2] - \mathbb{E}[\mathbb{G}_{n-1} X_{n-1}^{(2)}] + q b_{n-1} \\ &= (1+p) b_{n-1} + 2p x_{n-1} i_{n-1} + p \mathbb{E}[I_{n-1}^{(1)} \mathbb{J}_{n-1}] \\ &\quad + p \left( \mathbb{E}[I_{n-1}^{(2)} \mathbb{J}_{n-1}] - \mathbb{E}[\mathbb{G}_{n-1} X_{n-1}^{(2)}] - \mathbb{E}[I_{n-1}^{(2)} \mathbb{G}_{n-1}] - \mathbb{E}[I_{n-1}^{(1)} \mathbb{G}_{n-1}] \right).\end{aligned}$$

So the assertion of the lemma follows.  $\square$

To obtain a closed form of  $b_n$ , we need to compute  $\mathbb{E}[I_n \mathbb{I}_{\mathcal{R}_n}]$ .

**Lemma 12.**

$$\begin{aligned}\mathbb{E}[I_n \mathbb{I}_{\mathcal{R}_n}] &= q \mathbb{E}[I_{n-1} \mathbb{I}_{\mathcal{R}_{n-1}}] + p q^{n-1} (i_{n-1} + x_{n-1}) + p \mathbb{E}[I_{n-1}] \mathbb{P}(\mathcal{R}_{n-1}) \\ &\quad + p (1 - q^{n-1}) \mathbb{E}[I_{n-1} \mathbb{I}_{\mathcal{R}_{n-1}}] + p x_{n-1} \mathbb{P}(\mathcal{R}_{n-1}). \\ &= (1 - p q^{n-1}) \mathbb{E}[I_{n-1} \mathbb{I}_{\mathcal{R}_{n-1}}] + (p \mathbb{P}(\mathcal{R}_{n-1}) + p q^{n-1}) (i_{n-1} + x_{n-1}) \\ &= \sum_{k=0}^{n-1} \prod_{j=k+1}^n (1 - p q^{j-1}) (p \mathbb{P}(\mathcal{R}_k) + p q^k) (i_k + x_k).\end{aligned}$$

*Proof.* Since

$$\begin{aligned} \mathcal{R}_n &= \left[ \{|T_1| = 1\} \cap \tilde{\mathcal{R}}_{n-1}^{(1)} \right] \cup \left[ \{|T_1| = 2\} \cap \mathcal{R}_{n-1}^{(1)} \cap \{|T_{n-1}^{(2)}| \geq 2\} \right] \\ &\cup \left[ \{|T_1| = 2\} \cap \left\{ |T_{n-1}^{(1)}| = 1 \right\} \cap \left\{ |T_{n-1}^{(2)}| \geq 2 \right\} \right]. \end{aligned}$$

we have

$$\mathbb{I}_{\mathcal{R}_n} = (1 - \mathbb{I}) \mathbb{I}_{\tilde{\mathcal{R}}_{n-1}^{(1)}} + \mathbb{I}_{\{|T_{n-1}^{(1)}|=1\}} \mathbb{I}_{\{|T_{n-1}^{(2)}| \geq 2\}} \mathbb{I} + \mathbb{I}_{\mathcal{R}_{n-1}^{(1)}} \mathbb{I}_{\{|T_{n-1}^{(2)}| \geq 2\}} \mathbb{I},$$

Multiplying by  $I_n$ , we obtain

$$\begin{aligned} I_n \mathbb{I}_{\mathcal{R}_n} &= (1 - \mathbb{I}) \tilde{I}_{n-1}^{(1)} \mathbb{I}_{\tilde{\mathcal{R}}_{n-1}^{(1)}} + \mathbb{I}_{\{|T_{n-1}^{(1)}|=1\}} \mathbb{I}_{\{|T_{n-1}^{(2)}| \geq 2\}} (I_{n-1}^{(2)} + X_{n-1}^{(2)}) \mathbb{I} \\ &+ (I_{n-1}^{(2)} + I_{n-1}^{(1)} + X_{n-1}^{(2)}) \mathbb{I}_{\mathcal{R}_{n-1}^{(1)}} \mathbb{I}_{\{|T_{n-1}^{(2)}| \geq 2\}} \mathbb{I}. \end{aligned}$$

The result of the lemma is obtained by taking the expectation of this equation.  $\square$

Using Lemmas 11 and 12, we deduce

**Proposition 13.** *The second moment of  $I_n$  satisfies the following recursion*

$$\mathbb{E}[I_n^2] = (p + 1) \mathbb{E}[I_{n-1}^2] + 2p (i_{n-1}^2 + i_{n-1} x_{n-1} + b_{n-1}) + p \mathbb{E}[X_{n-1}^2],$$

with as general solution

$$\mathbb{E}[I_n^2] = p \sum_{k=1}^{n-1} (p + 1)^{n-k} (2(i_k^2 + i_k x_k + b_k) + \mathbb{E}[X_k^2]).$$

### 5.3 Convergence in Distribution of $I_n$

The aim of this section is to use the contraction method in [10] or the multivariate contraction method in [9] to state the limiting distribution of  $\hat{I}_n := I_n/n(1+p)^n$ .

To apply the contraction method, we set up some notation as follows: For a random variable  $X$ , we write  $X \sim \lambda$  if the distribution of  $X$  is  $\lambda$ , i.e. the law  $\mathcal{L}(X)$  of  $X$  is  $\lambda$ . The symbol  $\|X\|_2 := (\mathbb{E}[|X|^2])^{1/2}$  denotes the usual  $L_2$ -norm of  $X$ . The Wasserstein-metric  $\ell_2$  is defined on the space of probability distributions with existing second moments by

$$\ell_2(X, Y) := \ell_2(\mathcal{L}(X), \mathcal{L}(Y)) := \ell_2(\lambda, \nu) := \inf\{\|X - Y\|_2 : X \sim \lambda, Y \sim \nu\}.$$

By  $\mathcal{M}_2$  the space of all probability distributions  $\lambda$  with mean  $p\mu_p$  (as in Proposition 10) and finite second moment is denoted. The metric space  $(\mathcal{M}_2, \ell_2)$  is complete and convergence in  $\ell_2$  is equivalent to convergence in distribution plus convergence of the second moments.

By equation (12) we have

$$\begin{aligned} \frac{I_n}{n(1+p)^n} &\stackrel{d}{=} \frac{\mathbb{I}}{1+p} \cdot \frac{n-1}{n} \cdot \left( \frac{I_{n-1}^{(1)}}{(n-1)(1+p)^{n-1}} + \frac{I_{n-1}^{(2)}}{(n-1)(1+p)^{n-1}} \right) \\ &\quad + \frac{1-\mathbb{I}}{1+p} \cdot \frac{n-1}{n} \cdot \frac{\tilde{I}_{n-1}^{(1)}}{(n-1)(1+p)^{n-1}} + \mathbb{I} \cdot \frac{X_{n-1}}{n(1+p)^n}. \end{aligned} \quad (16)$$

**Theorem 14.** *Let  $I_n$  and  $X_n$  be the total path length and the number of protected nodes of an exponential recursive tree at age  $n$ , respectively. The normalized total path length  $\hat{I}_n := I_n/n(1+p)^n$  satisfies the distributional recursion*

$$\hat{I}_n \stackrel{d}{=} \frac{\mathbb{I}}{1+p} \cdot \frac{n-1}{n} \cdot (\hat{I}_{n-1}^{(1)} + \hat{I}_{n-1}^{(2)}) + \frac{1-\mathbb{I}}{1+p} \cdot \frac{n-1}{n} \cdot \hat{I}_{n-1}^{(1)} + \mathbb{I} \cdot \frac{X_{n-1}}{n(1+p)^n}, \quad (17)$$

where  $\hat{I}_n^{(1)}$ ,  $(\hat{I}_n^{(2)}, X_n)$ ,  $\hat{I}_n^{(1)}$  and  $\mathbb{I}$  are independent,  $\hat{I}_n^{(i)} \stackrel{d}{=} \hat{I}_n$  for  $i = 1, 2$ , and  $\mathbb{I}$  is a Bernoulli random variable with success probability  $p$ , then  $\hat{I}_n \xrightarrow{\mathcal{D}} \hat{I}$ , as  $n \rightarrow \infty$ , where the random variable  $\hat{I}$  is the unique distributional fixed-point of

$$\hat{I} \stackrel{d}{=} \frac{\mathbb{I}}{1+p} (\hat{I}^{(1)} + \hat{I}^{(2)}) + \frac{1-\mathbb{I}}{1+p} \hat{I}^{(1)}, \quad (18)$$

with  $\hat{I}^{(i)}$ ,  $i = 1, 2$ , are independent copies of  $\hat{I}$  and independent of  $\mathbb{I}$ .

*Proof.* The equation (17) is an immediate consequence of the equation (16), where we define  $\hat{I}_n^{(3)} := \tilde{I}_n^{(1)}/n(1+p)^n$ . Using Theorem 8 and Slutsky's theorem, we have  $\frac{X_{n-1}}{n(1+p)^{n-1}} \xrightarrow{\mathcal{D}} 0$ , and then convergence in probability  $\frac{X_{n-1}}{n(1+p)^{n-1}} \xrightarrow{\mathcal{P}} 0$ . That is,

$$\hat{I}_n \stackrel{d}{=} \frac{\mathbb{I}}{1+p} \cdot \frac{n-1}{n} \cdot (\hat{I}_{n-1}^{(1)} + \hat{I}_{n-1}^{(2)}) + \frac{1-\mathbb{I}}{1+p} \cdot \frac{n-1}{n} \cdot \hat{I}_{n-1}^{(1)} + o_p(1), \quad (19)$$

where  $o_p(1)$  denotes a quantity tending to zero in probability. For purposes of convergence we can, and hence will, ignore the  $o_p(1)$  term.

Consider the well-defined transformation  $T : \mathcal{M}_2 \rightarrow \mathcal{M}_2$ ,

$$T(\lambda) := \mathcal{L} \left( \frac{\mathbb{I}}{1+p} (\hat{I}_\lambda^{(1)} + \hat{I}_\lambda^{(2)}) + \frac{1-\mathbb{I}}{1+p} \hat{I}_\lambda^{(1)} \right), \quad (20)$$

where  $\hat{I}_\lambda^{(i)} := \hat{I}^{(i)}$ ,  $i = 1, 2$ , and  $\mathbb{I}$  are independent and  $\hat{I}^{(i)}$  have  $\lambda$  as distribution.

At a first step, we have to prove that the transformation  $T$  has a unique fixed point with respect to the  $\ell_2$ -metric. Let  $\lambda, \nu \in \mathcal{M}_2$  be given. By (20), we have

$$\begin{aligned} T(\lambda) &= \mathcal{L} \left( \frac{\mathbb{I}}{1+p} (\hat{I}_\lambda^{(1)} + \hat{I}_\lambda^{(2)}) + \frac{1-\mathbb{I}}{1+p} \hat{I}_\lambda^{(1)} \right), \\ T(\nu) &= \mathcal{L} \left( \frac{\mathbb{I}}{1+p} (\hat{I}_\nu^{(1)} + \hat{I}_\nu^{(2)}) + \frac{1-\mathbb{I}}{1+p} \hat{I}_\nu^{(1)} \right), \end{aligned}$$



$$\begin{aligned}\ell_2^2(T(\lambda), T(\nu)) &\leq 2 \frac{\mathbb{E}[\mathbb{I}^2]}{(1+p)^2} \mathbb{E}[|\hat{I}_\lambda - \hat{I}_\nu|]^2 + \frac{\mathbb{E}[(1-\mathbb{I})^2]}{(1+p)^2} \mathbb{E}[|\hat{I}_\lambda - \hat{I}_\nu|]^2 \\ &= \frac{1}{1+p} \mathbb{E}[|\hat{I}_\lambda - \hat{I}_\nu|]^2.\end{aligned}$$

Therefore, we have  $\ell_2^2(T(\lambda), T(\nu)) \leq \frac{1}{1+p} \ell_2^2(\lambda, \nu)$ . Since  $\frac{1}{1+p} < 1$ , we deduce that  $T$  is a contraction with respect to the  $\ell_2$ -metric. Thus, Banach's fixed point theorem provides existence and uniqueness of solutions of the fixed point equation  $T(\lambda) = \lambda$ . By (19), if  $\hat{I}_n$  converges in distribution to some random variable  $\hat{I}$ , then  $\hat{I}$  satisfies (18). Consequently, the distribution of  $\hat{I}$  will be  $\lambda_0$ , the unique fixed point of  $T$ , i.e.  $\mathcal{L}(\hat{I}) = T(\lambda_0) = \lambda_0$ . Therefore, we have to prove  $\lim_{n \rightarrow \infty} \ell_2(\hat{I}_n, \hat{I}) = 0$  to conclude  $\hat{I}_n \xrightarrow{\mathcal{D}} \hat{I}$ .

Since  $\hat{I}_n^{(i)} \stackrel{d}{=} \hat{I}_n$ ,  $\hat{I}^{(i)} \stackrel{d}{=} \hat{I}$ , for  $i = 1, 2, 3$ , we deduce

$$\begin{aligned}\lim_{n \rightarrow \infty} \ell_2^2(\hat{I}_n, \hat{I}) &\leq \frac{p}{(1+p)^2} \lim_{n \rightarrow \infty} \left( \left\| \frac{n-1}{n} \hat{I}_{n-1}^{(1)} - \hat{I}^{(1)} \right\|_2^2 + \left\| \frac{n-1}{n} \hat{I}_{n-1}^{(2)} - \hat{I}^{(2)} \right\|_2^2 \right) \\ &\quad + \frac{1-p}{(1+p)^2} \lim_{n \rightarrow \infty} \left\| \frac{n-1}{n} \hat{I}_{n-1}^{(3)} - \hat{I}^{(3)} \right\|_2^2 \\ &\leq \frac{1}{1+p} \lim_{n \rightarrow \infty} \|\hat{I}_n - \hat{I}\|_2.\end{aligned}$$

Therefore, we have

$$\lim_{n \rightarrow \infty} \ell_2^2(\hat{I}_n, \hat{I}) \leq \frac{1}{1+p} \lim_{n \rightarrow \infty} \ell_2^2(\hat{I}_n, \hat{I}).$$

This is true only if  $\lim_{n \rightarrow \infty} \ell_2(\hat{I}_n, \hat{I}) = 0$ . □

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