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Antonin Chambolle, Vito Crismale. Phase-field approximation for a class of cohesive fracture energies with an activation threshold. *Advances in Calculus of Variation*, 2021, 14 (4), pp.475-497. 10.1515/acv-2019-0018 . hal-03908450

HAL Id: hal-03908450

<https://hal.science/hal-03908450>

Submitted on 20 Dec 2022

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PHASE-FIELD APPROXIMATION FOR A CLASS OF COHESIVE FRACTURE ENERGIES WITH AN ACTIVATION THRESHOLD

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ABSTRACT. We study the Γ -limit of Ambrosio-Tortorelli-type functionals $D_\varepsilon(u, v)$, whose dependence on the symmetrised gradient $e(u)$ is different in $\mathbb{A}u$ and in $e(u) - \mathbb{A}u$, for a \mathbb{C} -elliptic symmetric operator \mathbb{A} , in terms of the prefactor depending on the phase-field variable v . The limit energy depends both on the opening and on the surface of the crack, and is intermediate between the Griffith brittle fracture energy and the one considered by Focardi and Iurlano in [43]. In particular we prove that $G(S)BD$ functions with bounded \mathbb{A} -variation are $(S)BD$.

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1. INTRODUCTION

The energy functionals in Fracture Mechanics are usually expressed in terms of the *displacement* $u: \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ as the sum of a volume part, accounting for the mechanical properties of the uncracked material in the bulk region, and of a surface part, concentrated on a $(n-1)$ -dimensional discontinuity set of u (the *crack set*) and representing the energy dissipated in the crack process.

The presence of the crack set entails difficulties in the effective computation of minimisers, for instance by numerical simulations. A possible, and by now classical, way out is to approximate the energy in the sense of Γ -convergence, through simpler functionals. These depend on the two variables $u: \Omega \rightarrow \mathbb{R}^n$, which is now a Sobolev function and represents the regularised displacement, and the phase-field $v: \Omega \rightarrow [0, 1]$, whose sublevels $\{v < s\}$, for $s \in (0, 1)$, may be used to approximate the limit discontinuity set. Such approximations are often called of Ambrosio-Tortorelli type, from the breakthrough paper [10] they realised to approximate the Mumford-Shah functional [56] in image reconstruction.

In the context of Fracture Mechanics, Ambrosio-Tortorelli approximations are largely employed since the Francfort-Marigo's work [46] on the variational approach to fracture

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2010 *Mathematics Subject Classification.* 49J45, 26A45, 49Q20, 74R99, 35Q74.

Key words and phrases. free discontinuity problems, Γ -convergence, special functions of bounded deformation, cohesive fracture.

and the first numerical experiments [17] (see e.g. [16, 6, 1] and references therein). The first case that has been considered is the Griffith energy [52]

$$\int_{\Omega} f_p(e(u)) \, dx + \mathcal{H}^{n-1}\left(J_u \cup (\partial_D \Omega \cap \{\text{tr}_{\partial \Omega} u \neq \text{tr}_{\partial \Omega} u_0\})\right) \quad (\text{G})$$

where $u_0 \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^n)$ enforces a Dirichlet boundary condition (by penalising $\text{tr}_{\partial \Omega} u$, the trace on $\partial \Omega$ of u , where different from that of u_0 on the Dirichlet boundary $\partial_D \Omega$), $e(u) := \frac{\nabla u + \nabla u^T}{2} \in \mathbb{M}_{sym}^{n \times n}$ is the *linearised strain* (in the bulk) in *small strain assumptions*, J_u is the *jump set* of u (see Section 2), \mathcal{H}^{n-1} is the $(n-1)$ -dimensional Hausdorff measure, and $f_p: \mathbb{M}_{sym}^{n \times n} \rightarrow [0, +\infty)$ is convex with

$$f_p(0) = 0, \quad C_{f_p}(|\xi|^p - 1) < f_p(\xi) < C'_{f_p}(|\xi|^p + 1), \quad p > 1, \quad (\text{HP1 } f_p)$$

for the Frobenius norm $|\cdot|$ on $\mathbb{M}_{sym}^{n \times n}$. As explained e.g. in [31, Section 1] and [53, Sections 10 and 11], the reference form for f_p is for every $\mu > 0$

$$f_{p,\mu}(\xi) := \frac{1}{p} \left((\Sigma \xi : \xi + \mu)^{\frac{p}{2}} - \mu^{\frac{p}{2}} \right), \quad (1.1)$$

where Σ , such that $\Sigma(\xi - \xi^T) = 0$ and $\Sigma \xi \cdot \xi \geq c_0 |\xi + \xi^T|^2$ for all $\xi \in \mathbb{M}_{sym}^{n \times n}$, is the fourth-order Hooke's tensor: this is a slight generalisation of the original Griffith energy, where the bulk energy is the *linear elastic* energy, that is $p = 2$, $\mu = 0$ and

$$\Sigma \xi \cdot \xi = \frac{1}{4} \lambda_1 |\xi + \xi^T|^2 + \frac{1}{2} \lambda_2 (\text{Tr } \xi)^2,$$

with λ_1, λ_2 the Lamé coefficients. The $f_{p,\mu}$ are quadratic for small ξ and with p -growth for large ξ , and for $p \neq 2$ this may account for plastic deformation at large strain.

The Griffith energy (G) is approximated by the functionals

$$\int_{\Omega} \left((v + \eta_{\varepsilon}) f_p(e(u)) + \frac{(1-v)^2}{4\varepsilon} + \varepsilon^{q-1} |\nabla v|^q \right) dx, \quad \lim_{\varepsilon \rightarrow 0} \frac{\eta_{\varepsilon}}{\varepsilon^{p-1}} = 0, \quad (\text{G}_{\varepsilon})$$

for $u \in W_{u_0}^{1,p}(\Omega; \mathbb{R}^n) := W^{1,p}(\Omega; \mathbb{R}^n) \cap \{u : \text{tr}_{\partial \Omega}(u - u_0) = 0 \text{ on } \partial_D \Omega\}$, $v \in W_1^{1,q}(\Omega; [0, 1]) := W^{1,q}(\Omega; [0, 1]) \cap \{v : \text{tr}_{\partial \Omega} v = 1 \text{ on } \partial_D \Omega\}$, and $+\infty$ otherwise: such approximation has been proven without any *a priori* assumption on u , for any $p > 1$, and in any dimension in [26], together with compactness and regularity for minimisers (see [28] and [27]), assuming that

$$O_{\delta, x_0}(\partial_D \Omega) \subset \Omega \quad \text{for } \delta \in (0, \bar{\delta}), \quad (1.2)$$

for some $\bar{\delta} > 0$ and $x_0 \in \mathbb{R}^n$, where $O_{\delta, x_0}(x) := x_0 + (1 - \delta)(x - x_0)$. This generalises [22, 23, 55], assuming *a priori* $u \in L^2$ and $p = 2$, [30], requiring $u \in L^p$, $p > 1$, and [47], obtained in dimension 2 (see also e.g. [42, 49, 57, 19] for the antiplane shear case and different approximations).

In [43] Focardi and Iurlano studied the limit of the functionals

$$\int_{\Omega} \left((v + \varepsilon) f_{2,0}(e(u)) + \frac{\psi(v)}{\varepsilon} + \varepsilon^{q-1} |\nabla v|^q \right) dx, \quad (\text{C}_{\varepsilon})$$

for $u \in H^1(\Omega; \mathbb{R}^n)$, $v \in W^{1,q}(\Omega; [0, 1])$, and $+\infty$ otherwise (with $\psi \in C([0, 1])$ decreasing, $\psi(1) = 0$) and proved that they Γ -converge to

$$\int_{\Omega} f_{2,0}(e(u)) \, dx + c_1 \mathcal{H}^{n-1}(J_u) + c_2 \int_{J_u} |[u] \odot \nu_u| \, d\mathcal{H}^{n-1}, \quad (\text{C})$$

for suitable $c_1, c_2 > 0$. The energy space for (C) is SBD^2 , a subspace of the Special Bounded Deformation functions SBD (see Section 2). For $v \in SBD$, the distributional gradient $\text{Ev} := \frac{\text{D}v + \text{D}^T v}{2}$ is a bounded Radon measure, J_v is the set of points x at which

v has two different approximate limits $v^+(x)$, $v^-(x)$ with respect to a suitable direction $\nu_v(x)$, and $[v](x) := v^+(x) - v^-(x)$ is the *jump*. We denote by \odot the symmetrised tensor product, and notice that $[u] \odot \nu_u$ is the part of the total strain Eu concentrated on J_u , see (2.2).

The energy (C) depends also on the jump amplitude, reflecting mechanical interaction between the fracture lips. This is typical of *cohesive* fracture energies, in contrast to the *brittle* energy (G). On the other hand, (C) has not the form of the classical cohesive fracture energies in Barenblatt's model [14], which in particular do not depend on $\mathcal{H}^{n-1}(J_u)$. The presence of the measure of the crack surface corresponds to an activation energy which is necessary to nucleate the crack: this is considered also in [2], where it is called “depinning energy”, in [5], that studies a model for quasistatic evolution, and in the approximation result [13]. A few others have succeeded in approximating particular instances of pure cohesive energies, as in [54, 29, 38], see also [3] and [4] (in these works the bulk energy is a function of the full gradient ∇u).

In this work we approximate fracture energies that, as (C), include the measure of J_u , but whose cohesive term now depends only on a part of the strain, for instance on its deviatoric part (for $n \geq 3$). Moreover, we consider general p -growth ($p > 1$) in $e(u)$ of the bulk energy, no integrability assumptions on u , and study the Dirichlet boundary problem. To present the general case we consider a constant-coefficient, linear, first order differential operator

$$\mathbb{A}u = \sum_{j=1}^n A_j \partial_j u, \quad u: \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad (1.3)$$

for $A_j \in \mathcal{L}(\mathbb{R}^n, \mathbb{M}^{n \times n})$ linear mappings. We assume that $(A_j)_i = (A_i)_j$ and that $\mathbb{A}u: \mathbb{R}^n \rightarrow \mathbb{M}_{sym}^{n \times n}$, so that there is an endomorphism A of $\mathbb{M}_{sym}^{n \times n}$ for which

$$\mathbb{A}u = A(e(u)). \quad (1.4)$$

A *Fourier symbol mapping* $\mathbb{A}[z]: \mathbb{R}^n \rightarrow \mathbb{M}_{sym}^{n \times n}$ is introduced for every $z = (z_1, \dots, z_n) \in \mathbb{R}^n$, defined by

$$\mathbb{A}[z]v := v \otimes_{\mathbb{A}} z := \sum_{j=1}^n z_j A_j v = A([v] \odot z) \quad (1.5)$$

for $v \in \mathbb{R}^n$; the operator \mathbb{A} is \mathbb{R} -elliptic if $\mathbb{A}[z]$ is injective for all $z \in \mathbb{R}^n \setminus \{0\}$, and \mathbb{C} -elliptic if (take the extension of $\mathbb{A}[z]v$ on \mathbb{C}^n) $\mathbb{A}[z]: \mathbb{C}^n \rightarrow \mathbb{C}^{n \times n}$ is injective for all $z \in \mathbb{C}^n \setminus \{0\}$. These operators have been recently considered in e.g. [18, 50, 51, 40, 11, 61]. The deviator operator $E_D u := Eu - \frac{1}{n}(\operatorname{div} u) \operatorname{Id}_n$ is \mathbb{C} -elliptic for $n \geq 3$, but not for $n = 2$ (see Remark 2.5).

From a mechanical point of view, the reference problem is to minimise the energy F under a Dirichlet boundary condition on a part of the boundary $\partial_D \Omega \neq \emptyset$ with possibly the presence of volume forces, and surface forces on the remaining part of the boundary $\partial_N \Omega$, with

$$\partial \Omega = \partial_D \Omega \cup \partial_N \Omega \cup N, \quad \partial_D \Omega \cap \partial_N \Omega = \emptyset, \quad \mathcal{H}^{n-1}(N) = 0, \quad \partial(\partial_D \Omega) = \partial(\partial_N \Omega), \quad (1.6)$$

for $\partial_D \Omega$ and $\partial_N \Omega$ relatively open. Here we assume all forces null, (1.2), and that

$$\lim_{s \rightarrow \pm\infty} \frac{f_p(s\xi)}{|s|^p} = \tilde{f}_p(\xi) \quad \text{uniformly as } s \rightarrow \pm\infty, \quad \xi \in \mathbb{M}_{sym}^{n \times n}. \quad (\text{HP2 } f_p)$$

We have that \tilde{f}_p is positively p -homogeneous, and $(\tilde{f}_p)^{\frac{1}{p}}$ is a norm on $\mathbb{M}_{sym}^{n \times n}$ (cf. e.g. [44, Remark 2.7]). Then we prove the following, main result of this work.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be open bounded Lipschitz satisfying (1.2), (1.6), $u_0 \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^n)$, $p, q > 1$, $\gamma > 0$, $\varepsilon > 0$, $\eta_\varepsilon > 0$ such that $\lim_{\varepsilon \rightarrow 0} \frac{\eta_\varepsilon}{\varepsilon^{p-1}} = 0$, f_p and \tilde{f}_p*

satisfying (HP1 f_p), (HP2 f_p), $\psi \in C([0, 1])$ decreasing with $\psi(1) = 0$, and \mathbb{A} be \mathbb{C} -elliptic. Then the functionals $D_\varepsilon(u, v)$ defined on u, v measurable by

$$D_\varepsilon(u, v) := \int_{\Omega} \left[(v + \varepsilon^{p-1}) f_p(\mathbb{A}u) + (v + \eta_\varepsilon) f_p(e(u) - \mathbb{A}u) + \frac{\psi(v)}{\varepsilon} + \gamma \varepsilon^{q-1} |\nabla v|^q \right] dx$$

if $u \in W_{u_0}^{1,p}(\Omega; \mathbb{R}^n)$, $v \in W_1^{1,q}(\Omega; [0, 1])$ and by $+\infty$ otherwise, Γ -converge, as $\varepsilon \rightarrow 0$, to

$$D(u, v) := \int_{\Omega} \left[f_p(A(e(u))) + f_p(e(u) - A(e(u))) \right] dx + \int_{J_u} \left[a + b(\tilde{f}_p)^{\frac{1}{p}}([u] \otimes_{\mathbb{A}} \nu_u) \right] d\mathcal{H}^{n-1} \\ + \int_{\partial_D \Omega \cap \{\text{tr}_{\partial \Omega}(u - u_0) \otimes_{\mathbb{A}} \nu_{\partial \Omega} \neq 0\}} \left[a + b(\tilde{f}_p)^{\frac{1}{p}}(\text{tr}_{\partial \Omega}(u - u_0) \otimes_{\mathbb{A}} \nu_{\partial \Omega}) \right] d\mathcal{H}^{n-1},$$

if

$$u \in SBD^p(\Omega), \quad v = 1 \text{ a.e. in } \Omega,$$

and by $+\infty$ otherwise for u, v measurable, with respect to the topology of convergence in \mathcal{L}^n -measure for u and v . Above A is the operator introduced in (1.4), and $(\frac{1}{p'} + \frac{1}{p} = \frac{1}{q'} + \frac{1}{q} = 1)$

$$a := 2(q')^{1/q'} (\gamma q)^{1/q} \int_0^1 \psi^{1/q'}, \quad b := p^{1/p} (p')^{1/p'} \psi(0)^{1/p}.$$

Moreover, for every $M > 0$ and $\varepsilon < 1$, the sublevel $\{(u, v) : D_\varepsilon(u, v) \leq M\}$ is contained in

$$\left\{ (u, v) : \int_{\Omega} |\mathbb{A}u| dx \leq C_M, \text{tr}_{\partial \Omega} u = \text{tr}_{\partial \Omega} u_0 \text{ on } \partial_D \Omega, \int_{\Omega} \psi(v) dx \leq M\varepsilon \right\}.$$

Then a sequence of quasi-minimisers for D_ε converge, up to a subsequence, to a minimiser of D , with respect to the product of the strong $L^r(\Omega; \mathbb{R}^n)$ topology for u , for any $r \in [1, \frac{n}{n-1}]$, times the topology of convergence in \mathcal{L}^n -measure for v , provided $\mathcal{H}^{n-1}(\partial_D \Omega \cap \partial \Omega_j) > 0$, for each Ω_j connected component of Ω .

The functionals $f_{p,\mu}$ in (1.1) satisfy (HP2 f_p), and $f_{2,\mu}(\mathbb{A}u) + f_{2,\mu}(e(u) - \mathbb{A}u) = f_{2,\mu}(e(u))$ if $\mathbb{A} = E_D$, so, in this case, we recover the linear elastic energy in the bulk.

Our approximating functionals are in some sense intermediate between those in (G_ε) and (C_ε) , since the part corresponding to $e(u) - \mathbb{A}u$ is multiplied by $v + \eta_\varepsilon$ as in (G_ε) , while $f_p(\mathbb{A}u)$ is multiplied by $v + \varepsilon^{p-1}$ as in (C_ε) for $p = 2$. This results in an interaction between $(v + \varepsilon^{p-1})f_p(\mathbb{A}u)$ and $\frac{\psi(v)}{\varepsilon}$ that gives the term in $[u]$ in the limit. As usual, the surface of J_u is approximated by the Ambrosio-Tortorelli part $\frac{\psi(v)}{\varepsilon} + \gamma \varepsilon^{q-1} |\nabla v|^q$.

Since the integrals of $(v + \eta_\varepsilon)f_p(e(u) - \mathbb{A}u)$ and $\frac{\psi(v)}{\varepsilon}$ are not energetically of the same order, we have an *a priori* control only on $\mathbb{A}u$ as a Radon measure, differently from [43], where this control is on the whole Eu . For this reason we initially work in the space $GSBD$ of *generalised SBD* functions, introduced by Dal Maso in [37] to study brittle fracture (in [43] the control on Eu allowed to work directly in SBD). A crucial point is to establish the expression of $\mathbb{A}u$ on the set J_u , in particular to show that

$$\mathbb{A}u \llcorner J_u = [u] \otimes_{\mathbb{A}} \nu_u \mathcal{H}^{n-1} \llcorner J_u$$

($J_u, [u], \nu_u$ are well defined in $GSBD$, see Section 2): we prove this equality employing the tools developed in [18, 50, 51] to show the existence of a trace for functions with bounded \mathbb{A} -variation if and only if \mathbb{A} is \mathbb{C} -elliptic. This is enough to conclude that $GSBD$ functions with bounded \mathbb{A} -variation are in fact in SBD , because we deduce that $[u]$ is integrable on J_u (this is also true for GBD, BD in place of $GSBD, SBD$, see Theorem 2.9). Technical problems arise to prove the same in dimension 2 for $\mathbb{A} = E_D$ (see Remark 2.10).

We remark that in [25] the approximating functionals weight differently $E_D u + \text{div}^+ u$ (multiplied by $(v + \epsilon)$) and $\text{div}^- u$ (without any prefactor), so that $\text{div}^- u$ is equibounded in L^2 and in the limit $[u] \cdot \nu \geq 0$, a linearised non-interpenetration condition. Here we could

also separate the behaviours of $\operatorname{div}^+ u$ and $\operatorname{div}^- u$ (Remarks 3.2 and 4.6), but the meaning of our approach is in some sense opposite to [25], since we do not pay, in the limit part in $[u]$, for the terms multiplied by $v + \eta_\varepsilon$, while in [25] the concentration of terms without prefactor pay infinite energy. One might also consider a non-interpenetration condition in our model, for instance by studying the Γ -limit of (for Id the identity $n \times n$ matrix)

$$\int_{\Omega} \left[(v + \varepsilon^{p-1}) f_p(E_D u) + (v + \eta_\varepsilon) f_p(\operatorname{div}^+ u \operatorname{Id}) + f_p(\operatorname{div}^- u \operatorname{Id}) + \frac{\psi(v)}{\varepsilon} + \gamma \varepsilon^{q-1} |\nabla v|^q \right] dx,$$

but the Γ -lim sup inequality presents hard difficulties. In this respect, we point out that for the Γ -lim sup inequality in Theorem 1.1 we employ the approximation result [34] (see also [33]) for SBD^p functions in BD -norm, which allows us to prove the result without any regularity assumption on the displacement, as the uniform L^∞ bound in [43]. We have also to refine the argument of [34], in order to deal with Dirichlet boundary conditions.

2. NOTATION AND A PRELIMINARY RESULT

We denote by \mathcal{L}^n and \mathcal{H}^k the n -dimensional Lebesgue measure and the k -dimensional Hausdorff measure. For any locally compact subset B of \mathbb{R}^n , the space of bounded \mathbb{R}^m -valued Radon measures on B is indicated as $\mathcal{M}_b(B; \mathbb{R}^m)$. For $m = 1$ we write $\mathcal{M}_b(B)$ for $\mathcal{M}_b(B; \mathbb{R})$ and $\mathcal{M}_b^+(B)$ for the subspace of positive measures of $\mathcal{M}_b(B)$. For every $\mu \in \mathcal{M}_b(B; \mathbb{R}^m)$, $|\mu|(B)$ stands for its total variation. We use the notation: $B_\varrho(x)$ [and $Q_\varrho(x)$] for the open ball [hypercube] with center x and radius [sidelength] ϱ ; $x \cdot y$, $|x|$ for the scalar product and the norm in \mathbb{R}^n ; 1^* for $n/(n-1)$, n being the space dimension; $d(x, A)$ for the distance of x from A ; $A \Subset K$ when A compactly contained in K .

We recall the definition of approximate limit with respect to the convergence in measure and approximate jump set for measurable functions.

Definition 2.1. Let $A \subset \mathbb{R}^n$, $v: A \rightarrow \mathbb{R}^m$ an \mathcal{L}^n -measurable function, $x \in \mathbb{R}^n$ such that

$$\limsup_{\varrho \rightarrow 0^+} \frac{\mathcal{L}^n(A \cap B_\varrho(x))}{\varrho^n} > 0.$$

A vector $a \in \mathbb{R}^n$ is the *approximate limit* of v as y tends to x if for every $\varepsilon > 0$

$$\lim_{\varrho \rightarrow 0^+} \frac{\mathcal{L}^n(A \cap B_\varrho(x) \cap \{|v - a| > \varepsilon\})}{\varrho^n} = 0,$$

and then we write

$$\operatorname{ap} \lim_{y \rightarrow x} v(y) = a. \quad (2.1)$$

Definition 2.2. Let $U \subset \mathbb{R}^n$ open, and $v: U \rightarrow \mathbb{R}^m$ be \mathcal{L}^n -measurable. The *approximate jump set* J_v is the set of points $x \in U$ for which there exist $a, b \in \mathbb{R}^m$, with $a \neq b$, and $\nu \in \mathbb{S}^{n-1}$ such that

$$\operatorname{ap} \lim_{(y-x) \cdot \nu > 0, y \rightarrow x} v(y) = a \quad \text{and} \quad \operatorname{ap} \lim_{(y-x) \cdot \nu < 0, y \rightarrow x} v(y) = b.$$

The triplet (a, b, ν) is uniquely determined up to a permutation of (a, b) and a change of sign of ν , and is denoted by $(v^+(x), v^-(x), \nu_v(x))$. The jump of v is the function defined by $[v](x) := v^+(x) - v^-(x)$ for every $x \in J_v$.

BV and BD functions. For $U \subset \mathbb{R}^n$ open, a function $v \in L^1(U)$ is a *function of bounded variation* on U , denoted by $v \in BV(U)$, if $D_i v \in \mathcal{M}_b(U)$ for $i = 1, \dots, n$, where $Dv = (D_1 v, \dots, D_n v)$ is its distributional gradient. A vector-valued function $v: U \rightarrow \mathbb{R}^m$ is $BV(U; \mathbb{R}^m)$ if $v_j \in BV(U)$ for every $j = 1, \dots, m$.

A \mathcal{L}^n -measurable bounded set $E \subset \mathbb{R}^n$ is a set of *finite perimeter* if χ_E is a function of bounded variation. The *reduced boundary* of E , denoted by $\partial^* E$, is the set of points $x \in \text{supp } |D\chi_E|$ such that the limit $\nu_E(x) := \lim_{\varrho \rightarrow 0^+} \frac{D\chi_E(B_\varrho(x))}{|D\chi_E|(B_\varrho(x))}$ exists and satisfies $|\nu_E(x)| = 1$. The reduced boundary is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable, and the function ν_E is called *generalised inner normal* to E .

The space of *functions of bounded deformation* on U is

$$BD(U) := \{v \in L^1(U; \mathbb{R}^n) : Ev \in \mathcal{M}_b(U; \mathbb{M}_{sym}^{n \times n})\},$$

where Ev is the distributional symmetric gradient of v . It is well known (see [8, 62]) that for $v \in BD(U)$, the *jump set* J_v , defined as the set of points $x \in U$ where v has two different one sided Lebesgue limits $v^+(x)$ and $v^-(x)$ with respect to a suitable direction $\nu_v(x) \in \mathbb{S}^{n-1}$, is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable (see, e.g. [41, 3.2.14]), and that

$$Ev = E^a v + E^c v + E^j v,$$

where $E^a v$ is absolutely continuous with respect to \mathcal{L}^n , $E^c v$ is singular with respect to \mathcal{L}^n and such that $|E^c v|(B) = 0$ if $\mathcal{H}^{n-1}(B) < \infty$, while

$$E^j v = [v] \odot \nu_v \mathcal{H}^{n-1} \llcorner J_v. \quad (2.2)$$

In the above expression of $E^j v$, $[v]$ denotes the *jump* of v at any $x \in J_v$ and is defined by $[v](x) := (v^+ - v^-)(x)$, the symbols \odot and \llcorner stands for the symmetric tensor product and the restriction of a measure to a set, respectively. Since $|a \odot b| \geq |a||b|/\sqrt{2}$ for every a, b in \mathbb{R}^n , it holds $[v] \in L^1(J_v; \mathbb{R}^n)$. The density of $E^a v$ with respect to \mathcal{L}^n is denoted by $e(v)$, and we have that (see [8, Theorem 4.3]) for \mathcal{L}^n -a.e. $x \in U$

$$\text{ap} \lim_{y \rightarrow x} \frac{(v(y) - v(x) - e(v)(x)(y - x)) \cdot (y - x)}{|y - x|^2} = 0. \quad (2.3)$$

The space $SBD(U)$ is the subspace of all functions $v \in BD(U)$ such that $E^c v = 0$, while for $p \in (1, \infty)$

$$SBD^p(U) := \{v \in SBD(U) : e(v) \in L^p(U; \mathbb{M}_{sym}^{n \times n}), \mathcal{H}^{n-1}(J_v) < \infty\}.$$

Analogous properties hold for BV , as the countable rectifiability of the jump set and the decomposition of Dv , and the spaces $SBV(U; \mathbb{R}^m)$ and $SBV^p(U; \mathbb{R}^m)$ are defined similarly, with ∇v , the density of $D^a v$ with respect to \mathcal{L}^n , in place of $e(v)$.

We now recall some slicing properties of SBD that will be useful in Theorem 2.9. As general notation, fixed $\xi \in \mathbb{S}^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$, for any $y \in \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ let

$$\Pi^\xi := \{y \in \mathbb{R}^n : y \cdot \xi = 0\}, \quad B_y^\xi := \{t \in \mathbb{R} : y + t\xi \in B\},$$

and for every function $v: B \rightarrow \mathbb{R}^n$ and $t \in B_y^\xi$ let

$$v_y^\xi(t) := v(y + t\xi), \quad \widehat{v}_y^\xi(t) := v_y^\xi(t) \cdot \xi.$$

The following proposition collects some results from [8] (see Propositions 3.2, 4.7, and Theorem 4.5 therein).

Proposition 2.3. *Let $v \in L^1(U; \mathbb{R}^n)$ and e_1, \dots, e_n be a basis of \mathbb{R}^n . Then $v \in BD(U)$ [resp. $SBD(U)$] if and only if for every $\xi = e_i + e_j$, $1 \leq i, j \leq n$*

$$\begin{aligned} \widehat{v}_y^\xi &\in BV(U_y^\xi) \text{ [resp. } SBV(U_y^\xi)\text{]} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi, \\ \int_{\Pi^\xi} |D\widehat{v}_y^\xi|(U_y^\xi) d\mathcal{H}^{n-1}(y) &< +\infty. \end{aligned} \quad (2.4)$$

Moreover, let $v \in BD(U)$, $\xi \in \mathbb{S}^{n-1}$ and $J_v^\xi := \{x \in J_v : [v] \cdot \xi \neq 0\}$ (it holds that $\mathcal{H}^{n-1}(J_v \setminus J_v^\xi) = 0$ for \mathcal{H}^{n-1} -a.e. $\xi \in \mathbb{S}^{n-1}$). Then

$$E^J v \xi \cdot \xi = \int_{\Pi^\xi} \int_{J_{\widehat{v}_y}^\xi} [\widehat{v}_y^\xi] dt d\mathcal{H}^{n-1}(y), \quad |E^J v \xi \cdot \xi|(U) = \int_{\Pi^\xi} \int_{J_{\widehat{v}_y}^\xi} |[\widehat{v}_y^\xi]| dt d\mathcal{H}^{n-1}(y), \quad (2.5)$$

and for \mathcal{H}^{n-1} -a.e. $y \in \Pi^\xi$

$$e(v)_y^\xi \xi \cdot \xi = \nabla \widehat{v}_y^\xi \quad \mathcal{L}^1\text{-a.e. on } U_y^\xi, \quad (2.6a)$$

$$(J_v^\xi)_y^\xi = J_{\widehat{v}_y}^\xi \quad \text{and} \quad v^\pm(y + t\xi) \cdot \xi = (\widehat{v}_y^\xi)^\pm(t) \quad \text{for } t \in (J_v)_y^\xi, \quad (2.6b)$$

where the normals to J_v and $J_{\widehat{v}_y}^\xi$ are oriented so that $\xi \cdot \nu_v \geq 0$ and $\nu_{\widehat{v}_y}^\xi = 1$.

For more details on the spaces BV , SBV and BD , SBD we refer to [9] and to [8, 15, 12, 62], respectively.

GBD functions. The space GBD of *generalised functions of bounded deformation* has been introduced in [37] (to which we refer for a general treatment) and it is defined by slicing as follows.

Definition 2.4 ([37]). Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and $v: \Omega \rightarrow \mathbb{R}^n$ be \mathcal{L}^n -measurable. Then $v \in GBD(\Omega)$ if there exists $\lambda_v \in \mathcal{M}_b^+(\Omega)$ such that the following equivalent conditions hold for every $\xi \in \mathbb{S}^{n-1}$:

- (a) for every $\tau \in C^1(\mathbb{R})$ with $-\frac{1}{2} \leq \tau \leq \frac{1}{2}$ and $0 \leq \tau' \leq 1$, the partial derivative $D_\xi(\tau(v \cdot \xi)) = D(\tau(v \cdot \xi)) \cdot \xi$ belongs to $\mathcal{M}_b(\Omega)$, and for every Borel set $B \subset \Omega$

$$|D_\xi(\tau(v \cdot \xi))|(B) \leq \lambda_v(B);$$

- (b) $\widehat{v}_y^\xi \in BV_{\text{loc}}(\Omega_y^\xi)$ for \mathcal{H}^{n-1} -a.e. $y \in \Pi^\xi$, and for every Borel set $B \subset \Omega$

$$\int_{\Pi^\xi} \left(|D\widehat{v}_y^\xi|(B_y^\xi \setminus J_{\widehat{v}_y}^1) + \mathcal{H}^0(B_y^\xi \cap J_{\widehat{v}_y}^1) \right) d\mathcal{H}^{n-1}(y) \leq \lambda_v(B), \quad (2.7)$$

$$\text{where } J_{\widehat{v}_y}^1 := \left\{ t \in J_{\widehat{v}_y}^\xi : |[\widehat{v}_y^\xi]|(t) \geq 1 \right\}.$$

The function v belongs to $GSBD(\Omega)$ if $v \in GBD(\Omega)$ and $\widehat{v}_y^\xi \in SBV_{\text{loc}}(\Omega_y^\xi)$ for every $\xi \in \mathbb{S}^{n-1}$ and for \mathcal{H}^{n-1} -a.e. $y \in \Pi^\xi$.

$GBD(\Omega)$ and $GSBD(\Omega)$ are vector spaces, as stated in [37, Remark 4.6], and one has the inclusions $BD(\Omega) \subset GBD(\Omega)$, $SBD(\Omega) \subset GSBD(\Omega)$, which are in general strict (see [37, Remark 4.5 and Example 12.3]). Every $v \in GBD(\Omega)$ has an *approximate symmetric gradient* $e(v) \in L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, still characterised by (2.3) and (2.6a), and the *approximate jump set* J_v is still countably $(\mathcal{H}^{n-1}, n-1)$ -rectifiable (cf. [37, Theorem 6.2]) and can be reconstructed by (2.6b) (see [37, Theorem 8.1]).

First order differential operators \mathbb{A} and functions of bounded \mathbb{A} -variation. In this paragraph we recall recent results from [18, 50, 51], starting from the notions of \mathbb{R} - and \mathbb{C} -ellipticity for operators \mathbb{A} of the form (1.3), introduced in Section 1. Such an operator can be seen as $\mathbb{A}u = A(Eu)$, for A endomorphism on $\mathbb{M}_{\text{sym}}^{n \times n}$, as in (1.4).

First (see [18, Theorem 2.6]) \mathbb{A} is \mathbb{C} -elliptic if and only if the kernel of \mathbb{A} , defined by

$$N(\mathbb{A}) = \{v \in \mathcal{D}'(\mathbb{R}^n; \mathbb{R}^n) : \mathbb{A}v \equiv 0\},$$

is finite dimensional and contained in the space of polynomials of degree less than $l = l(\mathbb{A}) \in \mathbb{N}$.

Remark 2.5. For the symmetrised gradient $\mathbb{A}v = \mathbb{E}v = \frac{1}{2}(\nabla v + \nabla v^T)$, we have $N(\mathbb{E}) = \{x \mapsto Mx + b : M \in \mathbb{M}^{n \times n}, M = -M^T, b \in \mathbb{R}^n\}$. For $\mathbb{A}v = \mathbb{E}_D v = \mathbb{E}v - \frac{1}{n}(\operatorname{div} v) \operatorname{Id}_n$, if $n \geq 3$ this operator is \mathbb{C} -elliptic with

$$N(\mathbb{E}_D) = \{x \mapsto Mx + b + (2(a \cdot x)x - |x|^2 a) : M \in \mathbb{M}^{n \times n}, M = -M^T, a, b \in \mathbb{R}^n\},$$

while, if $n = 2$, \mathbb{E}_D is only \mathbb{R} -elliptic and $N(\mathbb{E}_D)$ consists of the holomorphic functions, with the identification $\mathbb{C} \cong \mathbb{R}^2$. (The elements of $N(\mathbb{E}_D)$ are usually called conformal Killing vectors, see [35, 48].)

By [18, Lemma 2.3] if \mathbb{A} is \mathbb{R} -elliptic there exist $0 < \kappa_1 < \kappa_2 < \infty$ such that

$$\kappa_1 |w| |z| \leq |w \otimes_{\mathbb{A}} z| \leq \kappa_2 |w| |z| \quad \text{for all } w, z \in \mathbb{R}^n. \quad (2.8)$$

For every open domain $U \subset \mathbb{R}^n$, the *total \mathbb{A} -variation* of $v \in L^1_{\operatorname{loc}}(U; \mathbb{R}^n)$ is (notice that \mathbb{A} is symmetric)

$$|\mathbb{A}v|(U) := \sup \left\{ \int_U v \cdot \mathbb{A}\varphi \, dx : \varphi \in C_c^1(U; \mathbb{R}^n), |\varphi| \leq 1 \right\}. \quad (2.9)$$

A function $v \in L^1(U; \mathbb{R}^n)$ is of *bounded \mathbb{A} -variation* if $|\mathbb{A}v|(U) < \infty$ and we denote

$$BV^{\mathbb{A}}(U) := \{v \in L^1(U; \mathbb{R}^n) : \mathbb{A}v \in \mathcal{M}_b(U; \mathbb{M}_{sym}^{n \times n})\}.$$

The following proposition collects [63, Theorem 1.3] (see also [50, Theorem 1.1]), [50, Proposition 4.2 and Lemma 5.8], [51, Proposition 2.5], and [48, Theorem 3].

Proposition 2.6. *Let U be bounded and star-shaped with respect to a ball (that is star-shaped with respect to each point of a ball $B \subset U$). If \mathbb{A} is \mathbb{C} -elliptic then there exist a constant $C > 0$ such that*

$$\|v\|_{L^{1^*}(U; \mathbb{R}^n)} \leq C \|\mathbb{A}v\|_{L^1(U; \mathbb{M}_{sym}^{n \times n})} \quad (2.10)$$

for every $v \in C_c^1(U; \mathbb{R}^n)$, and (denoting by \hookrightarrow and $\hookrightarrow\hookrightarrow$ continuous and compact embeddings, respectively)

$$\begin{aligned} BV^{\mathbb{A}}(U) &\hookrightarrow L^{1^*}(U; \mathbb{R}^n) \\ BV^{\mathbb{A}}(U) &\hookrightarrow\hookrightarrow L^p(U; \mathbb{R}^n) \text{ for every } p \in [1, 1^*). \end{aligned} \quad (2.11)$$

If \mathbb{A} is \mathbb{R} -elliptic then for every $p \in [1, 1^*)$ there exist $C_p > 0$ such that

$$\|v\|_{L^p(U; \mathbb{R}^n)} \leq C_p \|\mathbb{A}v\|_{L^1(U; \mathbb{M}_{sym}^{n \times n})} \quad (2.12)$$

for every $v \in C_c^1(U; \mathbb{R}^n)$. Moreover, if \mathbb{A} is \mathbb{C} -elliptic then there is $C > 0$, depending only on n , such that for every $v \in BV^{\mathbb{A}}(U)$

$$\|v - \pi_U v\|_{L^{1^*}(U)} \leq C |\mathbb{A}v|(U), \quad (2.13)$$

for a suitable $\pi_U v \in N(\mathbb{A})$. If $n = 2$, then for every $p \in [1, 1^*)$ there exists $C > 0$ depending only on p , such that it holds

$$\|v - \pi_U v\|_{L^p(U)} \leq C \operatorname{diam}(U)^{1-n+\frac{n}{p}} |\mathbb{E}_D v|(U), \quad (2.14)$$

for some $\pi_U v \in N(\mathbb{E}_D)$, namely $\pi_U v$ is holomorphic (see Remark 2.7).

Remark 2.7. In [63] it is proven that (2.10) is equivalent to the fact that \mathbb{A} is \mathbb{R} -elliptic and cancelling, a weaker property than \mathbb{C} -ellipticity. For $n = 2$, we have that \mathbb{E}_D is only \mathbb{R} -elliptic but not cancelling, so only (2.12) holds, and $N(\mathbb{E}_D)$ can be identified with the space of holomorphic functions (see [50, Example 2.4 c)).

Remark 2.8. The estimates (2.13) and (2.14) may be extended to any connected set U finite union of sets U_i which are bounded and star-shaped with respect to a ball. Indeed, since $N(\mathbb{A})$ is made of polynomials and due to (2.13), one can find $\pi_U \in N(\mathbb{A})$ such that $\|\pi_{U_i} v - \pi_U v\|_{L^1(U)} \leq C|\mathbb{A}v|(U)$ for any i . This is true, for $\pi_{U_j} v$ in place of $\pi_U v$, for any U_i, U_j with $\mathcal{L}^n(U_i \cap U_j) > 0$, by rigidity of polynomials, and it is extended to a finite union. As for (2.14), see [48, comment before Theorem 3]. In particular, one sees that (2.13) and (2.14) hold if U is a connected Lipschitz domain.

We prove below the main result of the section.

Theorem 2.9. *Let $U \subset \mathbb{R}^n$ be an open bounded domain. If \mathbb{A} as in (1.3) (i.e., \mathbb{A} symmetric) is \mathbb{C} -elliptic, then*

$$GBD(U) \cap BV^{\mathbb{A}}(U) = BD(U) \quad (2.15a)$$

and

$$GSBD(U) \cap BV^{\mathbb{A}}(U) = SBD(U). \quad (2.15b)$$

Proof. By (2.4), we have that from (2.15a) one gets (2.15b). It is also immediate that $BD(U) \subset GBD(U) \cap BV^{\mathbb{A}}(U)$, being \mathbb{A} symmetric. In order to show the opposite inclusion, let us fix $u \in GBD(U) \cap BV^{\mathbb{A}}(U)$ and first prove (in the spirit of the blow up technique [45]) that

$$\frac{d|\mathbb{A}u|}{d\mathcal{H}^{n-1} \llcorner J_u} \geq |[u]_{\otimes \mathbb{A}} \nu_u| \quad \mathcal{H}^{n-1}\text{-a.e. in } J_u. \quad (2.16)$$

Since J_u is countably rectifiable, so that $\mathcal{H}^{n-1} \llcorner J_u$ is σ -finite, and $\mathbb{A}u \in \mathcal{M}_b(U; \mathbb{M}_{sym}^{n \times n})$, the Radon-Nikodym derivative of $|\mathbb{A}u|$ with respect to $\mathcal{H}^{n-1} \llcorner J_u$ exists (more precisely, it is the function $\theta \in L^1(J_u)$ such that $|\mathbb{A}u^a| = \theta \mathcal{H}^{n-1} \llcorner J_u$, where $\mathbb{A}u = \mathbb{A}u^a + \mathbb{A}u^s$, for $\mathbb{A}u^a \ll \mathcal{H}^{n-1} \llcorner J_u$, $\mathbb{A}u^s \perp \mathcal{H}^{n-1} \llcorner J_u$). Moreover, it may be computed explicitly by (see e.g. [9, Theorems 1.28 and 2.83]):

$$\frac{d|\mathbb{A}u|}{d\mathcal{H}^{n-1} \llcorner J_u}(x) = \lim_{\varrho \rightarrow 0} \frac{|\mathbb{A}u|(B_\varrho(x))}{\mathcal{H}^{n-1}(J_u \cap B_\varrho(x))} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_u. \quad (2.17)$$

For \mathcal{H}^{n-1} -a.e. $x \in J_u$, we have also that

$$\lim_{\varrho \rightarrow 0} \frac{\mathcal{H}^{n-1}(J_u \cap B_\varrho(x))}{\omega_{n-1} \varrho^{n-1}} = 1, \quad (2.18)$$

for ω_{n-1} the $n-1$ -dimensional measure of the unit ball in \mathbb{R}^{n-1} , and that, if we introduce $u_{\varrho,x}(y) := u(x + \varrho y) : B_1(0) \rightarrow \mathbb{R}^n$, then (denoting $B := B_1(0)$)

$$\lim_{\varrho \rightarrow 0^+} u_{\varrho,x} \rightarrow u_0 := u^+(x)\chi_{B^+} + u^-(x)\chi_{B^-} \quad \text{in } \mathcal{L}^n\text{-measure in } B, \quad (2.19)$$

where $u^\pm(x) \in \mathbb{R}^n$ are the Lebesgue limits at x on the two sides of J_u with respect to $\nu_u(x)$, and $B^\pm := B \cap \{y \in \mathbb{R}^n : (y-x) \cdot \nu_u(x) \in \mathbb{R}^\pm\}$ (see also e.g. [37, Theorem 6.2, below (6.4)]). Let us fix x such that these three conditions hold, and denote $u_\varrho \equiv u_{\varrho,x}$.

Since the derivative in (2.17) exists finite, by (2.18) and the fact that

$$|\mathbb{A}u_\varrho|(B) = \frac{|\mathbb{A}u|(B_\varrho(x))}{\varrho^{n-1}},$$

we obtain that there exists $C > 0$ independent of ϱ such that

$$|\mathbb{A}u_\varrho|(B) \leq C. \quad (2.20)$$

By the embeddings (2.11) we get that $\|u_\varrho\|_{L^1(B)} = \varrho^{-n} \|u\|_{L^1(B_\varrho(x))} < \infty$, so that $u_\varrho \in BV^{\mathbb{A}}(B)$ for any $\varrho > 0$ and (2.13), (2.20) imply

$$\|u_\varrho - \pi_\varrho\|_{L^1(B)} \leq C, \quad (2.21)$$

where $\pi_\varrho := \pi_B u_\varrho$. This gives that $(u_\varrho - \pi_\varrho)_\varrho$ is bounded in $BV^\mathbb{A}(B)$. Then, by (2.11), up to a (not relabelled) subsequence, $u_\varrho - \pi_\varrho \rightarrow \tilde{v} \in \mathbb{R}^n$ a.e. in B . Recalling (2.19), π_ϱ belong to the finite dimensional space of polynomials $N(\mathbb{A})$ of degree less than $l(\mathbb{A}) \in \mathbb{N}$ (being \mathbb{A} elliptic, cf. before Remark 2.5) and converge \mathcal{L}^n -a.e. in B . Therefore π_ϱ converge uniformly to a suitable polynomial π_0 (indeed, if $\|\pi_\varrho\| \rightarrow \infty$, for any norm on the finite dimensional space of polynomials of degree less than $l(\mathbb{A})$, then $\frac{\pi_\varrho}{\|\pi_\varrho\|}$ converges to a polynomial of degree less than $l(\mathbb{A})$, so $|\pi_\varrho|$ converges to $+\infty$ up to a \mathcal{L}^n -negligible set).

By difference we obtain that the convergence in (2.19) is strong in $L^1(B; \mathbb{R}^n)$, passing to a suitable subsequence ϱ_k . Looking at the definition of $|\mathbb{A}u|$ in (2.9), we deduce immediately the lower semicontinuity with respect to L^1 -convergence of u_ϱ , so by (2.17), (2.18)

$$\begin{aligned} |\mathbb{A}u_0|(B) &\leq \liminf_{k \rightarrow \infty} |\mathbb{A}u_{\varrho_k}|(B) = \liminf_{k \rightarrow \infty} \frac{|\mathbb{A}u|(B_{\varrho_k}(x))}{(\varrho_k)^{n-1}} \\ &= \lim_{\varrho \rightarrow 0} \frac{|\mathbb{A}u|(B_\varrho(x))}{\varrho^{n-1}} = \omega_{n-1} \frac{d|\mathbb{A}u|}{d\mathcal{H}^{n-1} \llcorner J_u}(x). \end{aligned}$$

By the special form of u_0 (see (2.19)), we have directly that

$$|\mathbb{A}u_0|(B) = \omega_{n-1} |[u](x) \otimes_{\mathbb{A}} \nu_u(x)|.$$

This proves the claim (2.16).

Combining (2.16) with (2.8) (recall that $\mathbb{A}u$ has bounded variation) we obtain that

$$[u] \in L^1(J_u; \mathbb{R}^n),$$

for $[u] = u^+ - u^-$, where u^\pm are the Lebesgue limits in the sense of GBD , cf. Definition 2.2.

It is now possible to fill the gap between the slicing conditions (2.7) for $G(S)BD$ and the characterisation of $(S)BD$ functions (2.4), by the area formula for rectifiable sets (see e.g. [60, (12.4) in Section 12]). Since J_u is countably $(\mathcal{H}^{n-1}, n-1)$ -rectifiable and $\nu_u \cdot \xi$ is the Jacobian of the projection $p_\xi: J_u \rightarrow \Pi^\xi$ (we consider $\nu_u \cdot \xi \geq 0$) we obtain for any $\xi \in \mathbb{S}^{n-1}$

$$\int_{J_u^\xi} |[u] \cdot \xi| (\nu_u \cdot \xi) d\mathcal{H}^{n-1} = \int \sum_{\Pi^\xi} \sum_{t \in (J_u^\xi)_y} |[u](y + t\xi) \cdot \xi| d\mathcal{H}^{n-1} = \int \sum_{\Pi^\xi} \sum_{t \in J_{\hat{u}_y^\xi}^\xi} |[\hat{u}_y^\xi](t)| d\mathcal{H}^{n-1},$$

recalling that (2.6b) holds also for $u \in GBD(U)$. Employing (2.5) and (2.7) in Definition 2.4 (now $\hat{u}_y^\xi \in SBV_{\text{loc}}(U_y^\xi)$ for \mathcal{H}^{n-1} -a.e. ξ if $u \in GSBD(U)$), and the fact that (2.6a) holds both in $(S)BD$ and $G(S)BD$, we get (2.4) and then $u \in BD(U)$. This concludes the proof. \square

Remark 2.10. For $n = 2$ and $\mathbb{A} = \mathbb{E}_D$ we are not able to deduce that $GBD \cap BV^{\mathbb{E}_D} = BD$ as above, the issue being property (2.16) (if this was true, then we would conclude by using (2.8)). Indeed, in this case $N(\mathbb{E}_D)$ consists of the holomorphic functions (with the identification $\mathbb{C} \cong \mathbb{R}^2$); employing (2.14) instead of (2.13) we get (2.21) with any fixed $p \in [1, 1^*)$ in place of 1^* , where π_ϱ is holomorphic. Now the problem is that it is not true that the \mathcal{L}^n -a.e. convergence of π_ϱ to $\pi_0 := u_0 - \tilde{v}$ takes place also in L^1 : in general, the convergence is locally uniform just on an open dense subset of $B_1(0)$ (by Osgood's theorem [59]).

Corollary 2.11. *If \mathbb{A} is an operator as in Theorem 2.9 and $u \in GBD(U) \cap BV^\mathbb{A}(U)$, then*

$$\mathbb{A}u \llcorner J_u = [u] \otimes_{\mathbb{A}} \nu_u \mathcal{H}^{n-1} \llcorner J_u,$$

applying the operator \mathbb{A} to (2.2).

3. Γ -lim inf INEQUALITY

Let us fix a sequence ε_k and denote by D_k the functionals D_{ε_k} , with analogous notation for all the quantities depending on ε . We consider an open bounded domain $\Omega' \subset \mathbb{R}^n$ such that $\Omega \subset \Omega'$ and $\Omega' \cap \partial\Omega = \partial_D\Omega$ and set, for each u, v defined in Ω , their extensions

$$\tilde{u} := \begin{cases} u & \text{in } \Omega, \\ u_0 & \text{in } \Omega' \setminus \Omega, \end{cases} \quad \tilde{v} := \begin{cases} v & \text{in } \Omega, \\ 1 & \text{in } \Omega' \setminus \Omega, \end{cases}$$

Then we have that

$$\begin{aligned} D_k^{\Omega'}(\tilde{u}, \tilde{v}) - D_k^{\Omega}(u, v) &= D^{\Omega'}(\tilde{u}, \tilde{v}) - D^{\Omega}(u, v) \\ &= \int_{\Omega' \setminus \Omega} \left[(1 + \varepsilon_k^{p-1}) f_p(\mathbb{A}u_0) + (1 + \eta_{\varepsilon_k}) f_p(e(u_0) - \mathbb{A}u_0) \right] dx, \end{aligned} \quad (3.1)$$

where D_k^{Ω} , D^{Ω} and $D_k^{\Omega'}$, $D^{\Omega'}$ are the functionals D_k and D with the integrals evaluated on Ω and Ω' . Then it is enough to argue in the enlarged domain Ω' . We denote $\tilde{D}_k := D_k^{\Omega'}$, $\tilde{D} := D^{\Omega'}$.

First we prove that for given sequences u_k, v_k converging in \mathcal{L}^n -measure to some $u: \Omega \rightarrow \mathbb{R}^n$, $v: \Omega \rightarrow \mathbb{R}$ measurable, such that $D_k(u_k, v_k) < \infty$, that is

$$\tilde{D}_k(\tilde{u}_k, \tilde{v}_k) < \infty, \quad (3.2)$$

(we may assume without loss of generality that $D(u_k, v_k)$, and then $\tilde{D}(\tilde{u}_k, \tilde{v}_k)$, converges to some finite limit) we have

$$\tilde{u} \in GSB D^p(\Omega') \cap BV^{\mathbb{A}}(\Omega') \quad \text{and} \quad \tilde{v} = 1 \text{ a.e. in } \Omega'. \quad (3.3)$$

Since $\tilde{D}_k(\tilde{u}_k, \tilde{v}_k) \geq \int_{\Omega} \frac{\psi(\tilde{v}_k)}{\varepsilon_k} dx$ and ψ is decreasing with $\psi(1) = 0$, we get readily that $\tilde{v} = 1$ a.e. in Ω' and $\tilde{v}_k \rightarrow 1$ in \mathcal{L}^n -measure. As for u , recalling the assumptions on f_p , we have that for any $\lambda \in [0, 1)$

$$\begin{aligned} \tilde{D}_k(\tilde{u}_k, \tilde{v}_k) &\geq C_{f_p} \int_{\Omega'} \left[\lambda |A(e(\tilde{u}_k))|^p \chi_{\{\tilde{v}_k \geq \lambda\}} + \varepsilon_k^{p-1} |A(e(\tilde{u}_k))|^p \chi_{\{\tilde{v}_k \leq \lambda\}} \right] dx \\ &\quad + \frac{\psi(\lambda)}{\varepsilon_k} \mathcal{L}^n(\{\tilde{v}_k \leq \lambda\}) - C_{f_p} \mathcal{L}^n(\Omega') \\ &\geq \tilde{C}_{f_p, p} \left[\lambda \int_{\{\tilde{v}_k \geq \lambda\}} |A(e(\tilde{u}_k))|^p dx + p^{\frac{1}{p}} (p')^{\frac{1}{p'}} \psi(\lambda)^{\frac{1}{p'}} \int_{\{\tilde{v}_k \leq \lambda\}} |A(e(\tilde{u}_k))| dx \right] - C_{f_p} \mathcal{L}^n(\Omega') \\ &\geq \tilde{C} \int_{\Omega'} |A(e(\tilde{u}_k))| dx - \hat{C}, \end{aligned} \quad (3.4)$$

for suitable \tilde{C}, \hat{C} depending on $f_p, p, \psi, \mathcal{L}^n(\Omega)$, and λ . Notice that we have employed the operator A in (1.4) to underline the dependence on the absolutely continuous part $e(u_k)$ and used the Young inequality for the second estimate above.

Since $\tilde{u}_k \in L^1(\Omega'; \mathbb{R}^n)$, from (2.13) and Remark 2.8 (we may assume here Ω' connected, arguing for each connected component of Ω) we get that there are suitable $\pi_{\Omega'} \tilde{u}_k \in N(\mathbb{A})$ for which

$$\|\tilde{u}_k - \pi_{\Omega'} \tilde{u}_k\|_{L^{1^*}(\Omega')} \leq \|\mathbb{A}(e(\tilde{u}_k))\|_{L^1(\Omega')}.$$

By (3.4) and the compact embedding in (2.11), up to a subsequence the functions $\tilde{u}_k - \pi_{\Omega'} \tilde{u}_k$ converge strongly in $L^1(\Omega'; \mathbb{R}^n)$. Since \tilde{u}_k converge in measure to \tilde{u} , then $\pi_{\Omega'} \tilde{u}_k$ converge uniformly to a polynomial in $N(\mathbb{A})$ (see the proof of Theorem 2.9), and then, for every $p \in [1, 1^*)$,

$$\tilde{u}_k \rightarrow \tilde{u} \in BV^\Delta(\Omega') \quad \text{in } L^p(\Omega'; \mathbb{R}^n), \quad \mathbb{A}\tilde{u}_k \xrightarrow{*} \mathbb{A}\tilde{u} \quad \text{in } \mathcal{M}_b(\Omega'; \mathbb{M}_{sym}^{n \times n}). \quad (3.5)$$

Let us now prove that \tilde{u} is in $GSBD^p(\Omega')$, employing the two terms depending only on v in D_k to estimate the measure of $J_{\tilde{u}}$. Consider the function $\phi(t) := \int_0^t \psi^{\frac{1}{q'}}(s) \, ds$ for $t \in [0, 1]$. By the Young inequality $\frac{\alpha^q}{q} + \frac{\beta^{q'}}{q'} \geq \alpha\beta$ for

$$\alpha = (\gamma q \varepsilon_k^{q-1} |\nabla v_k|^q)^{1/q}, \quad \beta = (q' \psi(v_k) \varepsilon_k^{-1})^{1/q'}, \quad (3.6)$$

we find that for any $\lambda \in (0, 1)$

$$\begin{aligned} \tilde{D}_k(\tilde{u}_k, \tilde{v}_k) &\geq \int_{\{\tilde{v}_k > \lambda\}} \left[\frac{\psi(\tilde{v}_k)}{\varepsilon_k} + \gamma \varepsilon_k^{q-1} |\nabla \tilde{v}_k|^q \right] dx \geq q'^{\frac{1}{q'}} (\gamma q)^{\frac{1}{q}} \int_{\{\tilde{v}_k > \lambda\}} \psi^{\frac{1}{q'}}(\tilde{v}_k) |\nabla \tilde{v}_k| dx \\ &= q'^{\frac{1}{q'}} (\gamma q)^{\frac{1}{q}} \int_{\{\tilde{v}_k > \lambda\}} |\nabla(\phi(\tilde{v}_k))| dx \\ &= q'^{\frac{1}{q'}} (\gamma q)^{\frac{1}{q}} \int_{\phi(\lambda)}^{\phi(1)} \mathcal{H}^{n-1}(\partial^* \{\phi(\tilde{v}_k) > s\}) ds, \end{aligned} \quad (3.7)$$

employing the Coarea formula for $\phi(\tilde{v}_k)$. Therefore, fixed $\lambda \in (0, 1)$, for any $\lambda' \in (\lambda, 1)$ the Mean Value theorem guarantees the existence of $\tilde{\lambda}_k \in (\lambda, \lambda')$ such that (notice that ϕ is strictly increasing)

$$\begin{aligned} \mathcal{H}^{n-1}(\partial^* \{\tilde{v}_k > \lambda_k\}) &= \mathcal{H}^{n-1}(\partial^* \{\phi(\tilde{v}_k) > \phi(\lambda_k)\}) \\ &\leq (\phi(\lambda') - \phi(\lambda))^{-1} \int_{\phi(\lambda)}^{\phi(\lambda')} \mathcal{H}^{n-1}(\partial^* \{\phi(\tilde{v}_k) > s\}) ds < C. \end{aligned}$$

It follows that the functions $\hat{u}_k := \tilde{u}_k \chi_{\{\tilde{v}_k > \lambda_k\}}$ satisfy

$$E\hat{u}_k = e(\tilde{u}_k) \chi_{\{\tilde{v}_k > \lambda_k\}} \mathcal{L}^n + \tilde{u}_k \odot \nu_{\partial^* \{\tilde{v}_k > \lambda_k\}} \mathcal{H}^{n-1} \llcorner \partial^* \{\tilde{v}_k > \lambda_k\}.$$

Since $\lambda_k \geq \lambda > 0$, we get a control for $e(\tilde{u}_k)$ in L^p , and with the estimate above this gives

$$\int_{\Omega} |e(\hat{u}_k)|^p dx + \mathcal{H}^{n-1}(J_{\hat{u}_k}) \leq C, \quad (3.8)$$

and

$$\mathcal{L}^n(\{\tilde{u}_k \neq \hat{u}_k\}) = \mathcal{L}^n(\{\tilde{v}_k \leq \lambda_k\}) \leq \mathcal{L}^n(\{\tilde{v}_k \leq \lambda'\}) \leq \varepsilon_k \frac{\tilde{D}_k(\tilde{u}_k, \tilde{v}_k)}{\psi(\lambda')} \rightarrow 0,$$

so that

$$\hat{u}_k \rightarrow \tilde{u} \quad \text{in } \mathcal{L}^n\text{-measure in } \Omega'. \quad (3.9)$$

By (3.8) and (3.9) (this latter condition implies that there exists a continuous function $\tilde{\psi}$ diverging to $+\infty$ such that $\int_{\Omega'} \tilde{\psi}(\tilde{u}_k) dx < C < +\infty$), we may apply [37, Theorem 11.3] (or we may use the compactness theorem for $GSBD$ [28, Theorem 1.1], since the exceptional set A therein is empty by (3.9)) to get

$$\tilde{u} \in GSBD^p(\Omega'), \quad e(\hat{u}_k) \rightarrow e(\tilde{u}) \text{ in } L^p(\Omega'; \mathbb{M}_{sym}^{n \times n}). \quad (3.10)$$

Together with (3.5) this proves the claim (3.3). At this stage Theorem 2.9 implies that

$$\tilde{u} \in SBD^p(\Omega').$$

By the weak convergences (3.10), the fact that \tilde{v}_k converge to 1 uniformly up to a set of vanishing measure, and the Ioffe-Olech semicontinuity theorem, see e.g. [20, Theorem 2.3.1],

we get that for any $\lambda \in (0, 1)$ (cf. [43, (4.4) in proof of Theorem 3.3] and [26, (5.4a) in proof of Theorem 5.1])

$$\begin{aligned} & \int_{\Omega'} \left[f_p(A(e(\tilde{u}))) + f_p(e(\tilde{u}) - A(e(\tilde{u}))) \right] dx \\ & \leq \liminf_{k \rightarrow \infty} \int_{\{\tilde{v}_k > \lambda\}} \left[(\tilde{v}_k + \varepsilon_k^{p-1}) f_p(\mathbb{A}\tilde{u}_k) + (\tilde{v}_k + \eta_{\varepsilon_k}) f_p(e(\tilde{u}_k) - \mathbb{A}\tilde{u}_k) \right] dx. \end{aligned} \quad (3.11)$$

As for the Ambrosio-Tortorelli term $\int_{\Omega} \left[\frac{\psi(\tilde{v}_k)}{\varepsilon_k} + \gamma \varepsilon_k^{q-1} |\nabla \tilde{v}_k|^q \right] dx$ in \tilde{D}_k , by a standard argument (see e.g. [43, (4.18)], now we argue in the enlarged domain Ω') we obtain

$$\liminf_{k \rightarrow \infty} \mathcal{H}^{n-1}(\partial^* \{\phi(\tilde{v}_k) > s\}) \geq 2\mathcal{H}^{n-1}(J_{\tilde{u}}) = 2\mathcal{H}^{n-1}(J_u \cup (\partial_D \Omega \cap \{\text{tr}(u - u_0) \neq 0\}))$$

for every $s \in (\phi(\lambda), \phi(1))$. Together with (3.7) this gives

$$\begin{aligned} & 2(q')^{1/q'} (\gamma q)^{1/q} (\phi(1) - \phi(\lambda)) \mathcal{H}^{n-1}(J_u \cup (\partial_D \Omega \cap \{\text{tr}(u - u_0) \neq 0\})) \\ & \leq \liminf_{k \rightarrow \infty} \int_{\{\tilde{v}_k > \lambda\}} \left[\frac{\psi(\tilde{v}_k)}{\varepsilon_k} + \gamma \varepsilon_k^{q-1} |\nabla \tilde{v}_k|^q \right] dx. \end{aligned} \quad (3.12)$$

Let us now estimate the other significant term in the limit by

$$\int_{\{\tilde{v}_k \leq \lambda\}} \left[(\tilde{v}_k + \varepsilon_k^{p-1}) f_p(\mathbb{A}\tilde{u}_k) + \frac{\psi(\tilde{v}_k)}{\varepsilon_k} \right] dx \geq p^{\frac{1}{p}} (p')^{\frac{1}{p'}} \psi(\lambda)^{\frac{1}{p'}} \int_{\{\tilde{v}_k \leq \lambda\}} (f_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u}_k) dx, \quad (3.13)$$

thanks to the Young inequality. We claim that for any $\lambda \in (0, 1)$

$$\int_{J_{\tilde{u}}} (\tilde{f}_p)^{\frac{1}{p}}([\tilde{u}] \otimes_{\mathbb{A}} \nu_u) d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{\{\tilde{v}_k \leq \lambda\}} (f_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u}_k) dx. \quad (3.14)$$

Up to a subsequence, that we do not relabel, we may assume that the \liminf above is a limit, so it is enough to prove (3.14) along any further subsequence. Let us introduce the positive measures defined on any $B \subset \Omega'$ Borel set by

$$\mu_k(B) := \int_{B \cap \{\tilde{v}_k \leq \lambda\}} (f_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u}_k) dx, \quad \hat{\mu}_k(B) := \int_B (\tilde{f}_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u}_k) dx.$$

By (3.5) we get that μ_k and $\hat{\mu}_k$ are equibounded, and then (up to a subsequence)

$$\mu_k \xrightarrow{*} \mu, \quad \hat{\mu}_k \xrightarrow{*} \hat{\mu} \quad \text{in } \mathcal{M}_b^+(\Omega').$$

Therefore we want to prove that the Radon-Nikodym derivatives of μ and $\hat{\mu}$ satisfy

$$\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner J_{\tilde{u}}} = \frac{d\hat{\mu}}{d\mathcal{H}^{n-1} \llcorner J_{\tilde{u}}} \geq (\tilde{f}_p)^{\frac{1}{p}}([\tilde{u}] \otimes_{\mathbb{A}} \nu_{\tilde{u}}) \quad \mathcal{H}^{n-1}\text{-a.e. in } J_{\tilde{u}}. \quad (3.15)$$

With (3.15) at disposal, we conclude (3.14) since then

$$\int_{J_{\tilde{u}} \cap B} (\tilde{f}_p)^{\frac{1}{p}}([\tilde{u}] \otimes_{\mathbb{A}} \nu_{\tilde{u}}) d\mathcal{H}^{n-1} \leq \mu(B)$$

as (positive) measures on Ω' , and

$$\mu(\Omega') \leq \liminf_{k \rightarrow \infty} \mu_k(\Omega') = \liminf_{k \rightarrow \infty} \int_{\{\tilde{v}_k \leq \lambda\}} (f_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u}_k) dx.$$

In order to show (3.15), we argue in the spirit of [43, Proof of (4.6)] (the functions giving the density of elastic energy are there supposed to be quadratic in $e(u)$, we include the

case where these have p -growth and are not p -homogeneous, cf. (HP2 f_p)). Let us define the measures $\zeta_k \in \mathcal{M}_b^+(\Omega')$ by

$$\zeta_k(B) := D_k^B(\tilde{u}_k, \tilde{v}_k), \quad \text{for } B \subset \Omega' \text{ Borel}, \quad (3.16)$$

where D_k^B denotes the localisation of D_k to the set B . By (3.2), ζ_k are equibounded in $\mathcal{M}_b^+(\Omega')$, so, up to a subsequence, $\zeta_k \xrightarrow{*} \zeta \in \mathcal{M}_b^+(\Omega')$. Recalling Corollary 2.11 and since $(\tilde{f}_p)^{\frac{1}{p}}$ is a norm, we have that

$$\frac{d(\tilde{f}_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u})}{d\mathcal{H}^{n-1} \llcorner J_{\tilde{u}}} = (\tilde{f}_p)^{\frac{1}{p}}([\tilde{u}] \otimes_{\mathbb{A}} \nu_{\tilde{u}}) \quad \mathcal{H}^{n-1}\text{-a.e. in } J_{\tilde{u}}. \quad (3.17)$$

Let us fix $x \in J_{\tilde{u}}$ such that the derivatives in (3.15) plus $\frac{d\zeta}{d\mathcal{H}^{n-1} \llcorner J_{\tilde{u}}}$ exist finite in x , and (3.17) is verified in x (this holds for \mathcal{H}^{n-1} -a.e. $x \in J_{\tilde{u}}$); let

$$I := \{\varrho \in (0, d(x, \partial\Omega')) : \mu(\partial B_{\varrho}(x)) = \hat{\mu}(\partial B_{\varrho}(x)) = \zeta(\partial B_{\varrho}(x)) = 0\}.$$

For every $\varrho \in (0, d(x, \partial\Omega'))$ consider the three sets (that partition $B_{\varrho}(x)$)

$$\begin{aligned} E_1 &:= B_{\varrho}(x) \cap \{|\mathbb{A}\tilde{u}_k| \leq \varrho^{-\frac{1}{2}}\} \cap \{\tilde{v}_k \leq \lambda\}, \\ E_2 &:= B_{\varrho}(x) \cap \{|\mathbb{A}\tilde{u}_k| > \varrho^{-\frac{1}{2}}\} \cap \{\tilde{v}_k \leq \lambda\}, \\ E_3 &:= B_{\varrho}(x) \cap \{\tilde{v}_k > \lambda\}. \end{aligned}$$

Since, by (HP1 f_p), there exists $C'_{f_p} \geq C_{f_p} > 0$ such that $f_p(\xi) \leq C'_{f_p}(1 + |\xi|^p)$ and $\tilde{f}_p(\xi) \leq C'_{f_p}|\xi|^p$, it holds that

$$\int_{E_1} (f_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u}_k) dx \leq C'_{f_p}(\omega_{n-1}\varrho^n + \varrho^{n-\frac{1}{2}}), \quad \int_{E_1} (\tilde{f}_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u}_k) dx \leq C'_{f_p}\varrho^{n-\frac{1}{2}}. \quad (3.18)$$

By (HP2 f_p) We have that

$$\int_{E_2} |(f_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u}_k) - (\tilde{f}_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u}_k)| dx \leq \delta_{\varrho} \int_{B_{\varrho}(x) \cap \{\tilde{v}_k \leq \lambda\}} |\mathbb{A}\tilde{u}_k| dx \leq C(C_{f_p})\delta_{\varrho}\mu_k(B_{\varrho}(x)), \quad (3.19)$$

for

$$\delta_{\varrho} := \sup_{s > \varrho^{-1/2}, |\xi|=1} \left| \frac{(f_p)^{\frac{1}{p}}(s\xi)}{|s|} - (\tilde{f}_p)^{\frac{1}{p}}(\xi) \right|,$$

using (HP1 f_p) and the fact that

$$\sup_{s > \varrho^{-1/2}} \left| \frac{(f_p)^{\frac{1}{p}}(s \frac{\mathbb{A}\tilde{u}_k}{|\mathbb{A}\tilde{u}_k|})}{|s|} - (\tilde{f}_p)^{\frac{1}{p}}\left(\frac{\mathbb{A}\tilde{u}_k}{|\mathbb{A}\tilde{u}_k|}\right) \right| \leq \delta_{\varrho}.$$

By (HP2 f_p), $\lim_{\varrho \rightarrow 0} \delta_{\varrho} = 0$ (uniformly in k). Thus, the estimate (3.19) and the fact that $\lim_{\varrho} \lim_k \varrho^{-(n-1)} \mu_k(B_{\varrho}(x)) < C$ (by the choice of ϱ and x , in particular $\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner J_{\tilde{u}}}$ exists finite at x) give that

$$\lim_{\substack{\varrho \in I \\ \varrho \rightarrow 0}} \lim_{k \rightarrow \infty} \varrho^{-(n-1)} \int_{E_2} |(f_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u}_k) - (\tilde{f}_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u}_k)| dx = 0. \quad (3.20)$$

On the other hand Hölder's inequality gives

$$\begin{aligned} \int_{E_3} (f_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u}_k) dx &\leq \left(\int_{E_3} f_p(\mathbb{A}\tilde{u}_k) dx \right)^{\frac{1}{p}} (\mathcal{L}^n(E_3))^{\frac{1}{p'}} \\ &\leq \lambda^{-\frac{1}{p}} (\zeta_k(B_{\varrho}(x)))^{\frac{1}{p}} (\mathcal{L}^n(B_{\varrho}(x)))^{\frac{1}{p'}}, \end{aligned} \quad (3.21)$$

since

$$\zeta_k(B_\varrho(x)) = D_k^{B_\varrho(x)}(\tilde{u}_k, \tilde{v}_k) \geq \int_{E_3} (\tilde{v}_k + \varepsilon_k^{p-1}) f_p(\mathbb{A}\tilde{u}_k) \, dx \geq \lambda \int_{E_3} f_p(\mathbb{A}\tilde{u}_k) \, dx.$$

Therefore we obtain

$$\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner J_{\tilde{u}}} = \lim_{\varrho \rightarrow 0} \lim_{\substack{\varrho \in I \\ k \rightarrow \infty}} \frac{\mu_k(B_\varrho(x))}{\omega_{n-1} \varrho^{n-1}} = \lim_{\varrho \rightarrow 0} \lim_{\substack{\varrho \in I \\ k \rightarrow \infty}} \frac{\hat{\mu}_k(B_\varrho(x))}{\omega_{n-1} \varrho^{n-1}} = \frac{d\hat{\mu}}{d\mathcal{H}^{n-1} \llcorner J_{\tilde{u}}}. \quad (3.22)$$

Indeed, the first and the last equalities follow by definition of Radon-Nikodym derivative and the choice of I , while the central equality descends by putting together (3.18), (3.21) (divided by $\omega_{n-1} \varrho^{n-1}$), and (3.20). In order to deal with (3.21), we remark that $\lim_{\varrho} \lim_k \varrho^{-(n-1)} \zeta_k(B_\varrho(x)) < C$ since $\frac{d\zeta}{d\mathcal{H}^{n-1} \llcorner J_{\tilde{u}}}$ exists finite at x .

Since $\hat{\mu}_k$ is defined in terms of the convex positively 1-homogeneous $(\tilde{f}_p)^{\frac{1}{p}}$ and $\mathbb{A}\tilde{u}_k \xrightarrow{*} \mathbb{A}\tilde{u}$ in $\mathcal{M}_b(\Omega'; \mathbb{M}_{sym}^{n \times n})$ by (3.5), Reshetnyak Semicontinuity Theorem (see e.g. [9, Theorem 2.38]) implies that

$$(\tilde{f}_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u})(B_\varrho(x)) \leq \liminf_{k \rightarrow \infty} (\tilde{f}_p)^{\frac{1}{p}}(\mathbb{A}\tilde{u}_k)(B_\varrho(x)) = \liminf_{k \rightarrow \infty} \hat{\mu}_k(B_\varrho(x)) = \hat{\mu}(B_\varrho(x)),$$

if $\varrho \in I$. Taking the Radon-Nikodym derivative of the above inequality with respect to $\mathcal{H}^{n-1} \llcorner J_{\tilde{u}}$ at x , for $I \ni \varrho \rightarrow 0$, and recalling (3.22) and the choice of x (that gives in particular (3.17) at x), we deduce (3.15) and then prove the claim (3.14).

We now collect (3.11), (3.12), (3.13), (3.14) and use the arbitrariness of $\lambda \in (0, 1)$ (indeed we let $\lambda \rightarrow 0$) to conclude the Γ -lim inf inequality

$$\tilde{D}(\tilde{u}, \tilde{v}) \leq \liminf_{k \rightarrow \infty} \tilde{D}_k(\tilde{u}_k, \tilde{v}_k),$$

that gives the desired inequality $D(u, v) \leq \liminf_{k \rightarrow \infty} D_k(u_k, v_k)$, by (3.1).

Moreover, notice that (3.4) gives also the inclusion stated in Theorem 1.1 for the sublevels of D_ε . The corresponding compactness property follows arguing as done for proving (3.5), but now the boundedness of the polynomials $\pi_{\Omega'} \tilde{u}_k$ is a consequence of the fact that $\tilde{u}_k = u_0$ in $\Omega' \setminus \bar{\Omega}$ (we argue separately on each connected component, using Remark 2.8). The convergence of quasi-minimisers for D_ε to a minimiser for D follows by general properties of Γ -convergence (see e.g. [36, Corollary 7.17]).

Remark 3.1. If $n = 2$ and $\mathbb{A} = E_D$, by (3.4) and (2.12) we get still (3.5), as well as (3.10), arguing as done for $n \geq 3$. If we had at disposal the analogous of Theorem 2.9 (and then Corollary 2.11) we could follow the proof of Γ -lim inf inequality as above.

Remark 3.2. We could reproduce the proof of the Γ -lim inf inequality above for $n \geq 3$ and the operator

$$\mathbb{B}u := E_D u + \frac{\operatorname{div}^+ u}{n} \operatorname{Id},$$

employed in [25] for a phase-field approximation of the Griffith brittle fracture energy with a non-interpenetration constraint, which is not \mathbb{C} -elliptic since it does not satisfy (1.3). Indeed $BV^{\mathbb{B}}(\Omega') \subset BV^{E_D}(\Omega')$, so that $GSBD(\Omega') \cap BV^{\mathbb{B}}(\Omega') = SBD(\Omega')$, and then, applying \mathbb{B} to (2.2),

$$\mathbb{B}u \llcorner J_u = \left[([u] \odot \nu_u)_D + \frac{([u] \cdot \nu_u)^+}{n} \operatorname{Id} \right] \mathcal{H}^{n-1} \llcorner J_u =: [u] \otimes_{\mathbb{B}} \nu_u. \quad (3.23)$$

Moreover, it holds $\mathbb{B}\tilde{u}_k \xrightarrow{*} \mathbb{B}\tilde{u}$ in $\mathcal{M}_b(\Omega'; \mathbb{M}_{sym}^{n \times n})$. Thus we get (3.17) for $[u] \otimes_{\mathbb{B}} \nu_u$, and so the corresponding version of (3.14).

4. Γ -lim sup INEQUALITY

As in [43], we construct by hand a recovering sequence starting from a function u with regular jump set and smooth outside J_u . However, since our result is formulated for general SBD functions without requiring *a priori* integrability for u it is not enough now to apply neither the density results for $GSBD$ [55, Theorem 3.1] and [47, 30, 26], nor the approximations [22, 23] for SBD . Indeed all these results do not approximate the jump part of u without assuming $u \in L^\infty(\Omega; \mathbb{R}^n)$: this request is not natural because the functionals, that depend on $e(u)$, are not decreasing by truncation of u .

The analysis is then based on the following approximation for SBD^p functions in BD -norm, recently proven in [34, Theorem 1.1].

Theorem 4.1. *Let Ω be an open bounded Lipschitz subset of \mathbb{R}^n , and $u \in SBD^p(\Omega)$, with $p > 1$. Then there exist $u_k \in SBV^p(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ such that each J_{u_k} is closed and included in a finite union of closed connected pieces of C^1 hypersurfaces, $u_k \in C^\infty(\overline{\Omega} \setminus J_{u_k}; \mathbb{R}^n) \cap W^{m,\infty}(\Omega \setminus J_{u_k}; \mathbb{R}^n)$ for every $m \in \mathbb{N}$, and:*

$$\lim_{k \rightarrow \infty} \left(\|u_k - u\|_{BD(\Omega)} + \|e(u_k) - e(u)\|_{L^p(\Omega; \mathbb{M}_{sym}^{n \times n})} + \mathcal{H}^{n-1}(J_{u_k} \triangle J_u) \right) = 0. \quad (4.1)$$

We combine the previous approximation with a well-known result by Cortesani and Toader, that allows us to work with the so-called “piecewise smooth” SBV -functions, denoted $\mathcal{W}(\Omega; \mathbb{R}^n)$, namely

$$u \in \mathcal{W}(\Omega; \mathbb{R}^n) \text{ if } \begin{cases} u \in SBV(\Omega; \mathbb{R}^n) \cap W^{m,\infty}(\Omega \setminus J_u; \mathbb{R}^n) \text{ for every } m \in \mathbb{N}, \\ \mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0, \\ \overline{J_u} \text{ is the intersection of } \Omega \text{ with a finite union of } (n-1)\text{-dimensional simplexes.} \end{cases}$$

We report below the result by Cortesani and Toader, in a slightly less general version.

Theorem 4.2 ([32], Theorem 3.1). *Let Ω be an open bounded Lipschitz set. For every $u \in SBV^p(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ there exist $u_k \in \mathcal{W}(\Omega; \mathbb{R}^n)$ such that*

$$\begin{aligned} \lim_{k \rightarrow \infty} \left(\|u_k - u\|_{L^1(\Omega; \mathbb{R}^n)} + \|\nabla u_k - \nabla u\|_{L^p(\Omega; \mathbb{M}^{n \times n})} + \mathcal{H}^{n-1}(J_{u_k} \triangle J_u) \right) &= 0, \\ \lim_{k \rightarrow \infty} \int_{J_{u_k} \cap A} \phi(x, u_k^+, u_k^-, \nu_{u_k}) d\mathcal{H}^{n-1} &= \int_{J_u \cap A} \phi(x, u^+, u^-, \nu_u) d\mathcal{H}^{n-1}, \end{aligned}$$

for every $A \subset \Omega$, $\mathcal{H}^{n-1}(\partial A \cap J_u) = 0$, and every ϕ strictly positive, continuous, and BV -elliptic (see e.g. [7] or [32, equation (2.4)] for the notion of BV -ellipticity).

Remark 4.3. In Theorem 4.2 we may assume also $J_{u_k} \Subset \Omega$, by [34, Remark 6.3], in turn using [39]. At this stage, [39, Lemma 5.2] gives that for any $p > 1$ the $n-1$ dimensional simplexes in the decomposition of $\overline{J_u}$ may be taken *pairwise disjoint* and such that also $J_u \cap \Pi_j \cap \Pi_i = \emptyset$ for any two different hyperplanes Π_i, Π_j (if $p \in (1, 2]$ it is enough to employ the capacitary argument in [32, Remark 3.5]). Moreover, we notice that our function $(\tilde{f}_p)^{\frac{1}{p}}$ is BV -elliptic.

The combination of the density results described so far guarantees that for a given $u \in SBD^p(\Omega)$ we can find approximating functions $u_k \in \mathcal{W}(\Omega, \mathbb{R}^n)$ with $J_{u_k} \Subset \Omega$ and $J_{u_k} \cap \Pi_j \cap \Pi_i = \emptyset$ for any two different hyperplanes Π_i, Π_j . The last property we have to ensure is that

$$\text{tr}_{\partial\Omega} u_k = \text{tr}_{\partial\Omega} u_0 \quad \text{on } \partial_D \Omega. \quad (4.2)$$

This is possible in view of the assumption (1.2), arguing as in [26, Theorem 5.5] with tools from [34], as sketched below.

For given $u \in SBD^p(\Omega)$ and $\varepsilon > 0$, one first defines a suitable extension \hat{u}_k of u on $\Omega_t := \Omega + B(0, t)$, for $t < 32k^{-1}$, as follows. This extension will be equal to u_0 in a

neighbourhood of $\partial_D \Omega$ but will have a jump set with the same regularity of J_u . For this reason we will correct it by improving its regularity without changing the values around $\partial_D \Omega$, by an argument using [34, Theorem 4.1].

Let us now construct \widehat{u}_k . We can find pairwise disjoint hypercubes $(Q_h)_{h=1}^{\bar{h}}$ and hypersurfaces $(\Gamma_h)_{h=1}^{\bar{h}}$ with the following properties. Each Q_h is centered at $x_h \in \partial_N \Omega$ with sidelength ϱ_h , $d(Q_h, \partial_D \Omega) > d_\varepsilon > 0$ (recall (1.6)),

$$\int_{\partial_N \Omega \setminus \widehat{Q}} 1 + |\operatorname{tr}_{\partial \Omega}(u - u_0)| d\mathcal{H}^{n-1} < \varepsilon, \quad D^{\Omega \cap \widehat{Q}}(u, 1) < \eta_\varepsilon, \quad \text{for } \widehat{Q} := \bigcup_h \overline{Q}_h, \quad (4.3)$$

for suitable $d_\varepsilon, \eta_\varepsilon$ with $\lim_{\varepsilon \rightarrow 0} d_\varepsilon = \lim_{\varepsilon \rightarrow 0} \eta_\varepsilon = 0$ (D^A denotes the energy D in Theorem 1.1 localised on a set A), $\mathcal{H}^{n-1}(J_u \cap \partial Q_h) = 0$ for each h , and

$$u \in L^1\left(\Omega \cap \bigcup_h \partial Q_h; \mathbb{R}^n\right), \quad u_0 \in L^1\left(\bigcup_h \partial Q_h; \mathbb{R}^n\right). \quad (4.4)$$

Moreover, each Γ_h is a C^1 hypersurface with $x_h \in \Gamma_h \subset \overline{Q}_h$ and (cf. [34, (4.2)])

$$\mathcal{H}^{n-1}((\partial_N \Omega \triangle \Gamma_h) \cap \overline{Q}_h) < \varepsilon (2\varrho_h)^{n-1} < \frac{\varepsilon}{1-\varepsilon} \mathcal{H}^{n-1}(\partial_N \Omega \cap \overline{Q}_h), \quad (4.5)$$

Γ_h is a C^1 graph with respect to $\nu_{\partial \Omega}(x_h)$ with Lipschitz constant less than $\varepsilon/2$.

We can say that $\partial_N \Omega \cap Q_h$ is “almost” the intersection of Q_h with the hyperplane through x_h and normal $\nu_{\partial \Omega}(x_h)$, and it is approximated by the C^1 hypersurface Γ_h . Let

$$\widetilde{u} := u\chi_\Omega + u_0\chi_{\Omega^c \setminus \Omega},$$

and notice that by (4.3) and (4.5) we can say that, up to modify η_ε , which has to be greater than ε , we have

$$\int_{\partial_N \Omega \setminus (\bigcup_h \Gamma_h)} |\widetilde{u}| d\mathcal{H}^{n-1} < \eta_\varepsilon. \quad (4.6)$$

We now approximate \widetilde{u} with respect to the energy D , arguing in each Q_h , by a sequence of functions, depending on a parameter k . We notice that the choice of the finite family of hypercubes Q_h is done before the construction of these approximations, and depends only on ε . Then we can argue, as follows, for a fixed hypercube, denoting $Q \equiv Q_h$, $\Gamma \equiv \Gamma_h$ and assuming, up to a rotation and a translation, $x_h = 0$ and $\nu_{\partial \Omega}(x_h) = e_n$ (notice that all the notation indeed depends on h). Let Q^- denote the almost half hypercube contained in Q which is below Γ (that is Q^- is almost contained in Ω).

We partition Q^- in hyperrectangles with first $n-1$ coordinates in squares of sidelength $(\eta_\varepsilon k)^{-1}$

$$F_{\mathbf{m}} := \{(y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1} : y_i \in (\eta_\varepsilon k)^{-1} m_i + (0, (\eta_\varepsilon k)^{-1})\}$$

(we have $\mathbf{m} = (m_1, \dots, m_{n-1}) \in \{-\eta_\varepsilon k \varrho, -\eta_\varepsilon k \varrho + 1, \dots, 0, \dots, \eta_\varepsilon k \varrho - 1\}^{n-1} \subset \mathbb{N}^{n-1}$, we may assume $\eta_\varepsilon k \varrho \in \mathbb{N}$) so that

$$\Gamma_h \cap (F_{\mathbf{m}} \times \mathbb{R}) \subset F_{\mathbf{m}} \times (m_n, m_n + 1/2)k^{-1},$$

for some $m_n \in \mathbb{N}$ (cf. [34, (4.7), (4.8)], and recall $\varepsilon < \eta_\varepsilon$). As in [34, (4.9)], setting $Q_{\mathbf{m}}^- := Q^- \cap (F_{\mathbf{m}} \times \mathbb{R})$ we use the Nitsche-type extension [34, Lemma 2.1] (see also [58]) to extend $\widetilde{u}|_{Q_{\mathbf{m}}^-}$ along the vertical direction, employing the (part of) hyperplans $F_{\mathbf{m}} \times \{m_n k^{-1}\}$ as the flat interface needed in [34, Lemma 2.1]: we obtain a function $\widetilde{u}_{\mathbf{m}}^-$ defined on

$$\widetilde{Q}_{\mathbf{m}}^- := Q \cap (F_{\mathbf{m}} \times (-\infty, (m_n + 33)k^{-1})) = F_{\mathbf{m}} \times (-\varrho, (m_n + 33)k^{-1}),$$

with $\widetilde{u}_{\mathbf{m}}^- = \widetilde{u}$ on $F_{\mathbf{m}} \times (-\varrho, (m_n - 33)k^{-1})$. We do not create a jump on the common boundary between adjacent $Q_{\mathbf{m}}^-$ except for a region with height of order k^{-1} , and this is true also for the jump created with respect to the original \widetilde{u} on the “boundary hyperrectangles”, namely the $Q_{\mathbf{m}}^-$ with $\partial Q_{\mathbf{m}}^- \cap \partial Q \neq \emptyset$. With the same arguments of [34, Section 4], one

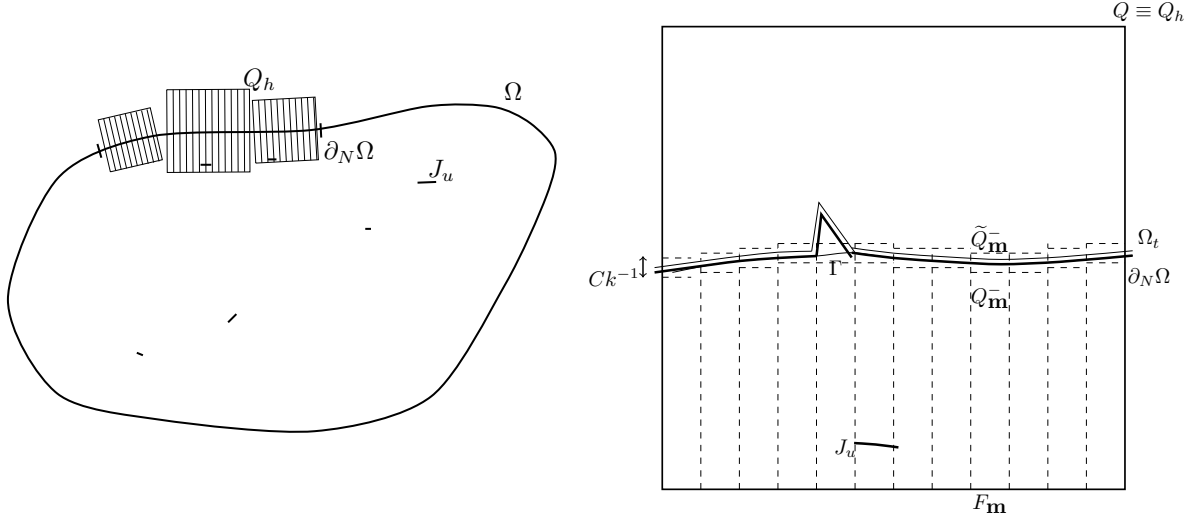


FIGURE 1. In the first figure, the hypercubes Q_h covering almost all $\partial_N \Omega$. In the second one, a single hypercube $Q \equiv Q_h$ with the relative (almost) hyperrectangles $Q_{\mathbf{m}}^-$, their bottom faces $F_{\mathbf{m}}$, and their extensions $\tilde{Q}_{\mathbf{m}}^-$. We see the enlarged domain Ω_t , the C^1 hypersurface Γ almost splitting the hypercube (notice that $\partial_N \Omega$ is only Lipschitz, see the corner in the second figure over the label Γ), and the pieces of hyperplanes $F_{\mathbf{m}} \times \{m_n k^{-1}\}$, below Γ , along which the original function is extended into $\tilde{u}_{\mathbf{m}}^-$. The zones in which we extend *à la* Nitsche have height of order k^{-1} .

can control both the measure of the union of these small interfaces (by $C\eta_\varepsilon \varrho^{n-1}$, see [34, (4.27)]), and the integral of the jump amplitude over this set (cf. [34, (4.32)]).

We obtain that, for a universal $c > 1$, (neglect the boundary contribution in D)

$$D^{\tilde{Q}_{\mathbf{m}}^-}(\tilde{u}_{\mathbf{m}}^-, 1) < cD^{Q_{\mathbf{m}}^-}(\tilde{u}, 1), \quad D^{\tilde{Q}_{\mathbf{m}}^- \cup \tilde{Q}_{\mathbf{m}'}^-}(\tilde{u}_{\mathbf{m}}^- \chi_{\tilde{Q}_{\mathbf{m}}^-} + \tilde{u}_{\mathbf{m}'}^- \chi_{\tilde{Q}_{\mathbf{m}'}^-}, 1) < cD^{Q_{\mathbf{m}}^- \cup Q_{\mathbf{m}'}^-}(\tilde{u}, 1), \quad (4.7)$$

for adjacent \mathbf{m}, \mathbf{m}' . Notice that, since the extension is done with respect to the vertical direction, for the “boundary hyperrectangles” $Q_{\mathbf{m}}^-$ we have that

$$\|\text{tr } \tilde{u}_{\mathbf{m}}^-\|_{L^1(\{\text{d}(\cdot, \partial\Omega) < t\} \cap \partial Q)} \leq c\|u\|_{L^1(\Omega \cap \{\text{d}(\cdot, \partial\Omega) < 2t\} \cap \partial Q)}, \quad (4.8)$$

which vanishes as $k \rightarrow \infty$, for $\varepsilon > 0$, by (4.4) (this is true also taking the union of ∂Q_h , since Q_h are in finite number, independent of k). Eventually, since $u \in BD$, we are able to estimate the trace of $\tilde{u}_{\mathbf{m}}^-$ on $F_{\mathbf{m}} \times \{(m_n + 33)k^{-1}\}$, in terms of the trace of u on Γ (cf. e.g. [34, (4.35)]); then we can say that, if $\bar{\Omega}_t$ intersects $F_{\mathbf{m}} \times \{(m_n + 33)k^{-1}\}$ (as in the corner for $\partial_N \Omega$ in the Figure 1), then

$$\int_{\bar{\Omega}^t \setminus (F_{\mathbf{m}} \times \{(m_n + 33)k^{-1}\})} |\text{tr } \tilde{u}_{\mathbf{m}}^- - u_0| d\mathcal{H}^{n-1} < \int_{\partial_N \Omega \setminus \Gamma_h} \|\tilde{u}\| d\mathcal{H}^{n-1} + o_{k \rightarrow \infty}(1). \quad (4.9)$$

By (4.3)–(4.9) we can see (cf. again [34, (4.16)–(4.34)]) that the extension

$$\hat{u}_k := \begin{cases} u & \text{in } \Omega \setminus \hat{Q}, \\ \tilde{u}_{\mathbf{m}}^- & \text{in } \tilde{Q}_{\mathbf{m}}^-, \text{ for any } \mathbf{m} \text{ and any } h, \\ u_0 & \text{elsewhere in } \Omega_t. \end{cases}$$

satisfies

$$D^{\Omega_t}(\widehat{u}_k, 1) < D^{\Omega}(u, 1) + c D^{\Omega \cap \widehat{Q}}(u, 1) + \int_{\partial_N \Omega \setminus (\cup_h \Gamma_h)} |[\widetilde{u}]| d\mathcal{H}^{n-1} + o_{k \rightarrow \infty}(1).$$

The vanishing term $o_{k \rightarrow \infty}(1)$ accounts also for the jump created on $\{d(\cdot, \partial\Omega) < 2t\} \cap \bigcup_h \partial Q_h$, that is controlled by (4.4) and (4.8), due to the choice of the hypercubes Q_h . At this stage, we can follow the strategy of [26, Theorem 5.5]. For $\delta = t/2 = 16k^{-1}$, the function

$$\widehat{u}'_k := \widehat{u}_k \circ (O_{\delta, x_0})^{-1} + u_0 - u_0 \circ (O_{\delta, x_0})^{-1}$$

is equal to u_0 in a neighbourhood of $\partial_D \Omega$, by (1.2), and satisfies

$$D^{\Omega_{t/2}}(\widehat{u}'_k, 1) < D^{\Omega_t}(\widehat{u}_k, 1) + o_{k \rightarrow \infty}(1).$$

(Notice that \widehat{u}'_k is still in $SBD(\Omega)$, since $\nabla((O_{\delta, x_0})^{-1}) = (1 - \delta)^{-1} \text{Id}$.) At this stage, we ensured the condition (4.2) for \widehat{u}'_k , without increasing the energy except for a vanishing amount. However, the jump of \widehat{u}'_k may be still not smooth. Therefore the last step is to construct a smooth approximation of \widehat{u}'_k without modifying it in a neighbourhood of $\partial_D \Omega$. In this respect, we apply the construction of [34, Theorem 1.1] to \widehat{u}'_k , to get \widetilde{u}_k with

$$D^{\Omega}(\widetilde{u}_k, 1) < D^{\Omega_{t/2}}(\widehat{u}'_k, 1) + o_{k \rightarrow \infty}(1),$$

and $\widetilde{u}_k = u_0 * \varrho_k$ in a neighbourhood of $\partial_D \Omega$, for ϱ_k a convolution kernel at scale k^{-1} . Eventually, we obtain the approximating function u_k , satisfying (4.2) and close in energy to u , by

$$u_k := \widetilde{u}_k + u_0 - u_0 * \varrho_k.$$

We are therefore allowed to start (employing Theorem 4.2, that preserves the boundary condition, in a neighbourhood of $\partial_D \Omega$) from a function $u \in \mathcal{W}(\Omega; \mathbb{R}^n)$ with $u = u_0$ in a neighbourhood of $\partial_D \Omega$, $J_u \Subset \Omega$, and it is not restrictive to consider the case $J_u \subset \Pi$ for a suitable hyperplane Π , say $\Pi = \{x_n = 0\}$ to fix a simple notation. (From now on we regard $x \in \mathbb{R}^n$ as (x', x_n) for $x' \in \mathbb{R}^{n-1}$.)

Remark 4.4. To get (4.2) it is enough to assume (1.2) separately for $\partial_D \Omega \cap \Omega_j$ and suitable $x_0^j \in \mathbb{R}^n$, $\bar{\delta}^j > 0$, for each connected component Ω_j of Ω . Moreover, if $p \leq 1^* = \frac{n}{n-1}$, the hypothesis (1.2) may be dropped by using partitions of the unity to guarantee the condition (4.2). This is possible since $u \in L^{1^*}(\Omega; \mathbb{R}^n)$, and so $e(\varphi u) = \varphi e(u) + u \odot \nabla \varphi$ is well controlled in L^p for any smooth φ . We also refer to [21] for a corresponding treatment of smooth domains.

Remark 4.5. A variant of Theorem 4.1 with a strong approximation of $\mathbb{A}u$ in L^p and $e(u) - \mathbb{A}u$ in L^t for functions in $SBD^{p \wedge t}$ would allow us to prove the result for functionals D_k depending on $e(u) - \mathbb{A}u$ through a function g_t with t -growth, $t \neq p$. Unfortunately, following the proof of Theorem 4.1, this would follow from a refined version of [24, Proposition 3] controlling two different powers of $\mathbb{A}u$ and $e(u) - \mathbb{A}u$: this seems out of reach with the strategy in [24] that relies on slicing properties, useless for $\mathbb{A}u$. For this reason we take p growth both in $\mathbb{A}u$ and in $e(u) - \mathbb{A}u$ (we could consider two different functions f_p and g_p , but it is almost the same, taking f_p that acts very differently in the two cases).

Let us now construct a recovery sequence corresponding to a regular u , in the sense described above, by adapting the argument in [43, Theorem 3.4]. We set

$$\sigma_k(x) := \frac{\varepsilon_k}{2 p' \psi^{\frac{1}{p'}}(0)} p^{\frac{1}{p}}(p')^{\frac{1}{p'}} (\tilde{f}_p)^{\frac{1}{p}}(|[u](x', 0) \otimes_{\mathbb{A}} e_n|) \quad \text{for } x \in J_u = J_u \cap \Pi. \quad (4.10)$$

Since u is Lipschitz up to J_u , then also σ_k is Lipschitz with

$$|\nabla \sigma_k| \leq C \varepsilon_k, \quad (4.11)$$

where $C > 0$ depends on the Lipschitz constant of u , f_p , \mathbb{A} . As in [43], let for any $\varrho < 1$

$$h_1(\varrho) := \psi(1 - \varrho), \quad h_2(\varrho) := \left(\int_0^{1-\varrho} \psi^{-\frac{1}{q}}(s) \, ds \right)^{-1}, \quad h(\varrho) := h_1 h_2(\varrho).$$

Since ψ is positive and vanishing in 1, we have that h is increasing and vanishing in 0 and $\frac{h_1}{h_2}$ also vanishes in 0, so that $\varrho_k := h^{-1}(\varepsilon_k)$ is vanishing and

$$\lim_{k \rightarrow \infty} \frac{h_1(\varrho_k)}{\varepsilon_k} = \lim_{k \rightarrow \infty} \frac{\varepsilon_k}{h_2(\varrho_k)} = 0. \quad (4.12)$$

Let w_k be the unique solution to the Cauchy problem

$$\begin{cases} w'_k = \left(\frac{q'}{\gamma q} \right)^{\frac{1}{q}} \varepsilon_k^{-1} \psi^{\frac{1}{q}}(w_k), \\ w_k(0) = 0, \end{cases}$$

in $[0, T_k)$, where $T_k := \left(\frac{\gamma q}{q'} \right)^{\frac{1}{q}} \varepsilon_k \int_0^1 \psi^{-1/q}(s) \, ds \in (0, \infty]$. We have that w_k is the inverse of the function

$$z \in (\varepsilon_k, 1] \mapsto \left(\frac{\gamma q}{q'} \right)^{\frac{1}{q}} \varepsilon_k \int_0^z \psi^{-1/q}(s) \, ds,$$

in $[0, T_k)$. Let $\tau_k := w_k^{-1}(1 - \varrho_k)$, namely

$$\tau_k = \left(\frac{\gamma q}{q'} \right)^{\frac{1}{q}} \varepsilon_k \int_0^{1-\varrho_k} \psi^{-1/q}(s) \, ds \in (0, T_k),$$

which is infinitesimal in view of (4.12), and define the sets

$$\begin{aligned} A_k &:= \{x \in \mathbb{R}^n : (x', 0) \in J_u, |x_n| < \sigma_k(x')\}, \\ B_k &:= \{x \in \mathbb{R}^n : (x', 0) \in J_u, 0 \leq |x_n| - \sigma_k(x') \leq \tau_k\}, \\ C_k &:= \{x \in \mathbb{R}^n : (x', 0) \notin J_u, d(x, J_u) \leq \tau_k\}. \end{aligned}$$

The candidate recovery sequence (u_k, v_k) is then

$$u_k(x) := \begin{cases} \frac{x_n + \sigma_k(x')}{2\sigma_k(x')} (u(x', \sigma_k(x')) - u(x', -\sigma_k(x'))) + u(x', -\sigma_k(x')), & \text{if } x \in A_k, \\ u(x) & \text{if } x \notin A_k \end{cases}$$

and (recall that in the functional there are $v + \varepsilon_k$ and $v + \eta_{\varepsilon_k}$)

$$v_k(x) := \begin{cases} 0 & \text{if } x \in A_k, \\ w_k(|x_n| - \sigma_k(x')) & \text{if } x \in B_k, \\ w_k(d(x, J_u)) & \text{if } x \in C_k, \\ 1 - \varrho_k & \text{otherwise.} \end{cases}$$

It is immediate that the sequences $(u_k)_k$ and $(v_k)_k$ converge pointwise to u and 1. Moreover, for the components u_k^i of u_k ,

$$\partial_n u_k^i(x) = \frac{u^i(x', \sigma_k(x')) - u^i(x', -\sigma_k(x'))}{2\sigma_k(x')}, \quad i = 1, \dots, n, \quad (4.13)$$

and, by straightforward calculations (see also [43])

$$\begin{aligned} |\partial_j u_k^i(x)| &\leq |\partial_j \sigma_k(x')| \left(\frac{|[u^i](x', 0)|}{2\sigma_k(x')} + 4L \right) + 3L \leq C \quad \text{for } j = 1, \dots, n-1, \\ |\partial_n u_k^i(x)| &\leq L + \frac{|[u^i](x', 0)|}{2\sigma_k(x')} \leq \frac{C}{\varepsilon_k}, \end{aligned} \quad (4.14)$$

in A_k , where L is the Lipschitz constant of u in $\Omega \setminus J_u$ and C depends on L (recall also (4.11)). By the way, u_k is a Lipschitz function. Notice also that

$$\lim_{k \rightarrow \infty} |\partial_n u_k^i(x)| = \infty \quad \text{for } x \in J_u. \quad (4.15)$$

Let us estimate the energy $D_k(u_k, v_k)$. We have that

$$\limsup_{k \rightarrow \infty} \int_{\Omega \setminus A_k} (v_k + \varepsilon_k^{p-1}) f_p(\mathbb{A} u_k) \, dx = \limsup_{k \rightarrow \infty} \int_{\Omega \setminus A_k} (v_k + \varepsilon_k^{p-1}) f_p(\mathbb{A} u) \, dx \leq \int_{\Omega} f_p(\mathbb{A} u) \, dx,$$

and

$$\limsup_{k \rightarrow \infty} \int_{\Omega \setminus A_k} (v_k + \eta_{\varepsilon_k}) f_p(e(u_k) - \mathbb{A} u_k) \, dx \leq \int_{\Omega} f_p(e(u) - \mathbb{A} u) \, dx.$$

Now, recalling (4.14), we get

$$\int_{A_k} (v_k + \eta_{\varepsilon_k}) f_p(e(u_k) - \mathbb{A} u_k) \, dx = \int_{A_k} \eta_{\varepsilon_k} f_p(e(u_k) - \mathbb{A} u_k) \, dx \leq \mathcal{L}^n(A_k) \eta_{\varepsilon_k} \frac{C^p}{(\varepsilon_k)^p} \leq C \frac{\eta_{\varepsilon_k}}{(\varepsilon_k)^{p-1}},$$

and this tends to 0 since $\lim_{\varepsilon \rightarrow 0} \frac{\eta_{\varepsilon}}{\varepsilon^{p-1}} = 0$.

In view of the fact that $\lim_{k \rightarrow \infty} \mathcal{L}^n(A_k) = 0$ and of the estimates for the tangential derivatives (4.14), we do not see the contribution of the tangential derivatives in the limit. Moreover, (4.15) and assumption (HP2 f_p) allows us to replace f_p with \tilde{f}_p for the normal derivatives, in the limit. Then (see (1.5))

$$\limsup_{k \rightarrow \infty} \int_{A_k} (v_k + \varepsilon_k^{p-1}) f_p(\mathbb{A} u_k) \, dx = \limsup_{k \rightarrow \infty} \int_{A_k} \varepsilon_k^{p-1} \tilde{f}_p(\partial_n u_k \otimes_{\mathbb{A}} e_n) \, dx, \quad (4.16)$$

for $\partial_n u_k$ the vector of the normal derivatives in (4.13) and $e_n = (0, \dots, 0, 1) \in \mathbb{R}^n$. By (4.13) and the fact that \tilde{f}_p is positively p -homogeneous we get

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \int_{A_k} (v_k + \varepsilon_k^{p-1}) f_p(\mathbb{A} u_k) \, dx \\ &= \limsup_{k \rightarrow \infty} \int_{A_k} \frac{\varepsilon_k^{p-1}}{(2 \sigma_k(x'))^p} \tilde{f}_p((u(x', \sigma_k(x')) - u(x', -\sigma_k(x'))) \otimes_{\mathbb{A}} e_n) \, d\mathcal{H}^{n-1}(x'). \end{aligned}$$

Recalling the definitions of σ_k and A_k , the pointwise convergence of $u(x', \sigma_k(x')) - u(x', -\sigma_k(x'))$ to $[u](x) \equiv [u](x')$, a change of variables and the Dominated Convergence Theorem give that the terms above are equal to

$$\int_{J_u} \lim_{k \rightarrow \infty} \frac{\varepsilon_k^{p-1}}{(2 \sigma_k(x'))^{p-1}} \tilde{f}_p([u](x') \otimes_{\mathbb{A}} e_n) \, d\mathcal{H}^{n-1}(x')$$

and then to

$$\frac{p^{1/p} (p')^{1/p'} \psi(0)^{1/p}}{p} \int_{J_u} (\tilde{f}_p)^{\frac{1}{p}}([u] \otimes_{\mathbb{A}} e_n) \, d\mathcal{H}^{n-1},$$

since

$$(p')^{(1-1/p')(p-1)} = (p')^{1/p'}, \quad p^{-\frac{p-1}{p}} = p^{1/p}/p.$$

As for the remaining terms of D_k , notice that by the definition of v_k and (4.12) we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} \int_{\Omega} \left[\frac{\psi(v_k)}{\varepsilon_k} + \gamma \varepsilon_k^{q-1} |\nabla v_k|^q \right] dx &\leq \limsup_{k \rightarrow \infty} \int_{B_k \cup C_k} \left[\frac{\psi(v_k)}{\varepsilon_k} + \gamma \varepsilon_k^{q-1} |\nabla v_k|^q \right] dx \\ &\quad + \limsup_{k \rightarrow \infty} \int_{A_k} \frac{\psi(v_k)}{\varepsilon_k} dx. \end{aligned}$$

We deduce now that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \int_{B_k} \left[\frac{\psi(v_k)}{\varepsilon_k} + \gamma \varepsilon_k^{q-1} |\nabla v_k|^q \right] dx &= \limsup_{k \rightarrow \infty} \int_{J_u} \left(\int_0^{\tau_k} \left[\frac{\psi(w_k)}{\varepsilon_k} + \gamma \varepsilon_k^{q-1} (w'_k)^q \right] dx_n \right) d\mathcal{H}^{n-1}(x') \\ &= 2(q')^{1/q'} (\gamma q)^{1/q} \limsup_{k \rightarrow \infty} \int_{J_u} \left(\int_0^{\tau_k} \psi^{1/q'}(w_k) w'_k dx_n \right) d\mathcal{H}^{n-1}(x') \\ &= 2(q')^{1/q'} (\gamma q)^{1/q} \limsup_{k \rightarrow \infty} \left(\int_0^{1-\varrho_k} \psi^{1/q'}(s) ds \right) \mathcal{H}^{n-1}(J_u) \\ &= a \mathcal{H}^{n-1}(J_u). \end{aligned}$$

Indeed in the first equality we have used the estimate (4.11) to neglect the contribution of the tangential derivatives of v_k in the limit, and the second one follows from the definition of w_k (w'_k represents the normal derivative of v_k) that gives $\alpha^q = \beta^{q'}$, that is the condition to have the Young equality $\frac{\alpha^q}{q} + \frac{\beta^{q'}}{q'} = \alpha \beta$, for (recall (3.6))

$$\alpha = (\gamma q \varepsilon_k^{q-1} (w'_k)^q)^{1/q}, \quad \beta = (q' \psi(w_k) \varepsilon_k^{-1})^{1/q'}.$$

Furthermore, arguing similarly and using the Coarea formula (cf. [43, eq. (4.49)]) we get

$$\int_{C_k} \left[\frac{\psi(v_k)}{\varepsilon_k} + \gamma \varepsilon_k^{q-1} |\nabla v_k|^q \right] dx \leq C \mu_k \int_0^{1-\varrho_k} \psi^{1/q'}(s) ds \leq C \mu_k,$$

so that

$$\limsup_{k \rightarrow \infty} \int_{C_k} \left[\frac{\psi(v_k)}{\varepsilon_k} + \gamma \varepsilon_k^{q-1} |\nabla v_k|^q \right] dx = 0.$$

Eventually

$$\int_{A_k} \frac{\psi(v_k)}{\varepsilon_k} dx = \int_{J_u} \frac{2\sigma_k(x')}{\varepsilon_k} \psi(0) d\mathcal{H}^{n-1}(x') = \frac{p^{1/p}(p')^{1/p'} \psi(0)^{1/p}}{p'} \int_{J_u} (\tilde{f}_p)^{\frac{1}{p}}([u] \otimes_{\mathbb{A}} e_n) d\mathcal{H}^{n-1}.$$

Collecting all the estimates below (4.15) we then conclude the Γ -lim sup inequality.

Remark 4.6. With the notation of Remark 3.2, we could reproduce also the proof of the Γ -lim sup inequality for \mathbb{B} in place of \mathbb{A} . Indeed, we define σ_k and u_k in terms of \mathbb{B} , and notice that in (4.16) we see in the limit $(\partial_n u_k \odot e_n)_D$ plus the contribution of $(\partial_n u_k^n)^+$, asymptotically equal to that of $\operatorname{div}^+ u_k$ by (4.14). Now $2\sigma_k(\partial_n u_k \odot e_n)_D$ converge pointwise to $([u] \odot e_n)_D$ and $2\sigma_k(\partial_n u_k^n)^+$ to $[u^n]^+ = ([u] \cdot e_n)^+$, which gives $\mathbb{B}u$ in the limit, according to (3.23).

Acknowledgements. Vito Crismale has been supported by the Marie Skłodowska-Curie Standard European Fellowship No. 793018, and by a public grant as part of the *Investissement d'avenir* project, reference ANR-11-LABX-0056-LMH, LabEx LMH.

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