



HAL
open science

Quantitative spectral inequalities for the anisotropic Shubin operators and the Grushin operator

Paul Alphonse, Albrecht Seelmann

► **To cite this version:**

Paul Alphonse, Albrecht Seelmann. Quantitative spectral inequalities for the anisotropic Shubin operators and the Grushin operator. 2022. hal-03908252v1

HAL Id: hal-03908252

<https://hal.science/hal-03908252v1>

Preprint submitted on 20 Dec 2022 (v1), last revised 18 Dec 2023 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

QUANTITATIVE SPECTRAL INEQUALITIES FOR THE ANISOTROPIC SHUBIN OPERATORS AND THE GRUSHIN OPERATOR

PAUL ALPHONSE AND ALBRECHT SEELMANN

ABSTRACT. We prove quantitative spectral inequalities for the (anisotropic) Shubin operators and the Grushin operator on the whole Euclidean space, thus relating for functions from spectral subspaces associated to finite energy intervals their L^2 -norm on the whole space to the L^2 -norm on a suitable subset. A particular feature of our estimates is that the constant relating these L^2 -norms is very explicit in geometric parameters of the corresponding subset of the whole space. While these subsets may be thick (i.e. relatively dense) in the classical sense in the case of the Grushin operator, they may become sparse at infinity for the Shubin operators and may even have finite measure. The latter extends results obtained recently by J. Martin and, in the particular case of the harmonic oscillator, by A. Dicke, I. Veselić, and the second author. We apply our results towards null-controllability of the associated (degenerate) parabolic equations, as well as to the ones associated to the Baouendi-Grushin operators acting on $\mathbb{R}^d \times \mathbb{T}^d$.

1. INTRODUCTION

Quantitative spectral inequalities are instances of so-called *uncertainty relations* that, in the context of the present paper, take the form

$$\|f\|_{L^2(\Omega)}^2 \leq d_0 e^{d_1 \lambda^\zeta} \|f\|_{L^2(\omega)}^2, \quad f \in \mathcal{E}_A(\lambda), \quad \lambda \geq 0,$$

where ω is a measurable subset of a domain $\Omega \subset \mathbb{R}^d$, $\mathcal{E}_A(\lambda) = 1_{(-\infty, \lambda]}(A)$ denotes the spectral subspace for a non-negative selfadjoint operator A in $L^2(\Omega)$ associated with the interval $(-\infty, \lambda]$, and $d_0, d_1, \zeta > 0$ are constants. Such inequalities can be viewed as quantitative variants of an identity theorem (in the sense that $f = 0$ on ω implies $f = 0$ on Ω) and are often considered under different names, depending on the context, such as (*quantitative*) *unique continuation estimates*, see e.g. [28, 31], or *uncertainty principles*, see e.g. [41]. The notion *spectral inequality* we adopt is common in the context of control theory, see e.g. [27, 28]. They are also closely related to the so-called *vanishing order*, see, e.g., [18, 27], and *annihilating pairs* in Fourier analysis, see e.g. [7, 22].

In the present work, we prove spectral inequalities from sparse sensor sets ω with an explicit form of the constants when A is either the (anisotropic) Shubin operator in $L^2(\mathbb{R}^d)$,

$$H_{k,m} = (-\Delta)^m + |x|^{2k}, \quad x \in \mathbb{R}^d,$$

where $k, m \geq 1$ are positive integers, or the negative of the Grushin operator in $L^2(\mathbb{R}^{2d})$,

$$\Delta_G = \Delta_x + |x|^2 \Delta_y, \quad (x, y) \in \mathbb{R}^{2d}.$$

While this is, to the best of the authors knowledge, entirely new for the case of the Grushin operator, our inequalities for the Shubin operators complement recent results from [33] and, in the particular case of the harmonic oscillator, from [16]. For instance, very general spectral inequalities have been obtained in [33, Theorem 2.1 (ii)] for every measurable set $\omega \subset \mathbb{R}^d$ with merely positive measure. These inequalities take the form

$$(1.1) \quad \forall \lambda > 0, \forall f \in \mathcal{E}_{\lambda,k,m}, \quad \|f\|_{L^2(\mathbb{R}^d)}^2 \leq K e^{K \lambda^{\frac{1}{2k} + \frac{1}{2m}} |\log \lambda|} \|f\|_{L^2(\omega)}^2,$$

2020 *Mathematics Subject Classification.* 35P05, 93B05, 35P10.

Key words and phrases. Spectral inequalities; null-controllability; Agmon estimates; anisotropic Shubin operators; Grushin operator.

where $K > 0$ is a positive constant depending on the dimension d , k , m , and the set ω . The dependence of K on the set ω , however, is not explicit, even if more information on ω is available. Our inequalities mainly address this dependence if ω is sparse in a sense made precise below. The technique of proof used in the present paper follows the approach by Kovrijkine [25, 26] and builds upon recent developments in this field of research [7, 16, 21, 33, 35]. We apply our results in the context of exact null-controllability for the abstract Cauchy problems associated to both $H_{k,m}$ and Δ_G , as well as to the Baouendi-Grushin operator in $L^2(\mathbb{R}^d \times \mathbb{T}^d)$,

$$\Delta_k = \Delta_x + |x|^{2k} \Delta_y, \quad (x, y) \in \mathbb{R}^d \times \mathbb{T}^d,$$

with $k \geq 1$ a positive integer.

Outline of the work. In Section 2, we present in detail the main results contained in this work. Section 3 is then devoted to the proof of the spectral inequalities for the anisotropic Shubin operators, while the ones for the Grushin operator are derived in Section 4. Both types of spectral inequalities are used in Section 5 to prove null-controllability results for the evolution equations associated with these two classes of operators.

Notations. The following notations and conventions will be used all over this work:

1. \mathbb{N} denotes the set of natural numbers starting from zero.
2. The canonical Euclidean scalar product of \mathbb{R}^d is denoted by \cdot and $|\cdot|$ stands for the associated canonical Euclidean norm. We will also use the Japanese bracket notation $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.
3. The length of any multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ is denoted $|\alpha|$ and defined by
$$|\alpha| = \alpha_1 + \dots + \alpha_d.$$
4. The Lebesgue measure of a measurable set $\omega \subset \mathbb{R}^d$ is denoted $|\omega|$.
5. 1_ω denotes the characteristic function of any subset $\omega \subset \mathbb{R}^d$.
6. For all measurable subsets $\omega \subset \mathbb{R}^d$, the inner product of $L^2(\omega)$ is denoted $\langle \cdot, \cdot \rangle_{L^2(\omega)}$, while $\|\cdot\|_{L^2(\omega)}$ stands for the associated norm.
7. For a nonnegative selfadjoint operator A on $L^2(\mathbb{R}^d)$, $\mathcal{E}_\lambda(A) = 1_{(-\infty, \lambda]}(A)$ with $\lambda \geq 0$ denotes the spectral subspace for A associated with the interval $(-\infty, \lambda]$.

Acknowledgments. The first author thanks J. Martin for many enthusiastic discussions during the preparation of this work and for pointing out some relevant reference. The second author has been partially supported by the DFG grant VE 253/10-1 entitled *Quantitative unique continuation properties of elliptic PDEs with variable 2nd order coefficients and applications in control theory, Anderson localization, and photonics*.

2. STATEMENT OF THE MAIN RESULTS

This section is devoted to present in detail the main results contained in this work.

2.1. Spectral inequalities. In this first section, we present spectral inequalities for the anisotropic Shubin operators and the Grushin operator.

2.1.1. Anisotropic Shubin operators. Given two positive integers $k, m \geq 1$, we consider in $L^2(\mathbb{R}^d)$ the (anisotropic) Shubin operator

$$H_{k,m} = (-\Delta)^m + |x|^{2k}, \quad x \in \mathbb{R}^d,$$

which is a non-negative and selfadjoint operator with purely discrete spectrum when equipped with its maximal domain

$$D(H_{k,m}) = \{g \in L^2(\mathbb{R}^d) : H_{k,m}g \in L^2(\mathbb{R}^d)\}.$$

Moreover, for $\lambda \geq 0$, let $\mathcal{E}_{\lambda,k,m} = \mathcal{E}_{H_{k,m}}(\lambda) = \text{Ran } 1_{(-\infty, \lambda]}(H_{k,m})$ denote the spectral subspace for the operator $H_{k,m}$ associated with the interval $(-\infty, \lambda]$.

For easier comparison, let us first state a result for the harmonic oscillator, corresponding to the case where $k = m = 1$, which covers and extends previous results from [7, 16, 21, 35], see Remark 2.2 below.

Theorem 2.1. *Let $\rho: \mathbb{R}^d \rightarrow (0, +\infty)$ and $\sigma: \mathbb{R}^d \rightarrow (0, 1]$ be functions such that ρ and $1/\sigma$ are locally bounded, and let $\omega \subset \mathbb{R}^d$ be a measurable set satisfying*

$$(2.1) \quad \forall x \in \mathbb{R}^d, \quad |\omega \cap B(x, \rho(x))| \geq \sigma(x)|B(x, \rho(x))|.$$

Then, there exists a positive constant $K > 0$, depending only on the dimension d , such that for all $\lambda \geq 0$ and $f \in \mathcal{E}_{\lambda,1,1}$ we have

$$(2.2) \quad \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \left(\frac{K}{\gamma_\lambda}\right)^{K(1+(L_\lambda)^2+L_\lambda\sqrt{\lambda})} \|f\|_{L^2(\omega)}^2,$$

where

$$\gamma_\lambda := \inf_{|x| < \sqrt{2\lambda}} \sigma(x) \quad \text{and} \quad L_\lambda := \sup_{|x| < \sqrt{2\lambda}} \rho(x).$$

Remark 2.2. Suppose that the functions σ and ρ satisfy the bounds

$$(2.3) \quad \forall x \in \mathbb{R}^d, \quad \sigma(x) \geq \gamma \langle x \rangle^a \quad \text{and} \quad \rho(x) \leq L \langle x \rangle^\delta$$

with some fixed $\gamma \in (0, 1]$, $a \geq 0$, $L > 0$, and $\delta \geq 0$. In this case, we have

$$\gamma_\lambda \geq \gamma^{(1+2\lambda)^{a/2}} \quad \text{and} \quad L_\lambda \leq L(1+2\lambda)^{\delta/2}.$$

It is then straightforward to verify that (2.2) takes the form

$$(2.4) \quad \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \left(\frac{K}{\gamma}\right)^{K^{1+a+\delta}(1+L^2\lambda^{\delta+a/2}+L\lambda^{(1+a+\delta)/2})} \|f\|_{L^2(\omega)}^2, \quad f \in \mathcal{E}_{\lambda,1,1},$$

with a possibly different constant $K \geq 1$. This covers [16, Theorem 2.7], while the particular case of $a = 0$ has also previously been considered in [35, Theorem 2.1] under the additional assumption that ρ is 1/2-Lipschitz continuous.

The case where the functions σ and ρ are constant, and thus the parameters a and δ above can be chosen equal to zero, that is,

$$(2.5) \quad \forall x \in \mathbb{R}^d, \quad |\omega \cap B(x, L)| \geq \gamma|B(x, L)|,$$

corresponds to so-called (γ, L) -thick sets. Such sets have been getting considerable attention in the past and have been previously discussed in this context in [7, Theorem 2.1 (iii)] and [21, Corollary 1.9]. In fact, [21, Corollary 1.9] also makes in this case the dependence on the dimension in (2.4) explicit. This could have been done in (2.2) with our technique as well, but we refrained from doing so for the sake of simplicity.

The spectral inequality in (2.2) is very explicit in terms of σ and ρ . The fact that only the uniform bounds of σ and ρ on the ball $B(0, \sqrt{2\lambda})$ enter the estimate (2.2) is due to the strong decay that the potential enforces on the eigenfunctions of the harmonic oscillator (and finite linear combinations thereof). This is an instance of a much more general phenomenon that also takes place in case of general (anisotropic) Shubin operators and eventually leads to a variant of Theorem 2.1 for these operators that, in particular, gives a positive answer to [17, Conjecture 1.6]. Our corresponding main result considers exactly the same geometry for $\omega \subset \mathbb{R}^d$ as in Theorem 2.1 and reads as follows.

Theorem 2.3. *There exists a constant $K > 0$, depending only on k , m , and the dimension d , such that for all measurable sets $\omega \subset \mathbb{R}^d$ satisfying the geometric condition (2.1), and all $\lambda \geq 0$ and $f \in \mathcal{E}_{\lambda,k,m}$ we have*

$$(2.6) \quad \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \left(\frac{K}{\gamma_{\lambda,k}}\right)^{K(1+(L_{\lambda,k})^{1+\frac{k}{m}}+L_{\lambda,k}\lambda^{\frac{1}{2m}}+\log(1+\lambda))} \|f\|_{L^2(\omega)}^2,$$

where

$$(2.7) \quad \gamma_{\lambda,k} := \inf_{|x| < (2\lambda)^{1/2k}} \sigma(x) \quad \text{and} \quad L_{\lambda,k} := \sup_{|x| < (2\lambda)^{1/2k}} \rho(x).$$

Remark 2.4. Similarly as for the harmonic oscillator, the potential $|x|^{2k}$ enforces a strong decay of (finite linear combinations of) eigenfunctions of the operator $H_{k,m}$, so that such functions are localized around the origin. More precisely, Corollary 3.6 below states that for all $\lambda \geq 0$ and $f \in \mathcal{E}_{\lambda,k,m}$,

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq 2\|f\|_{L^2(B(0,(2\lambda)^{1/2k}))}^2.$$

It is therefore sufficient to prove for functions in $\mathcal{E}_{\lambda,k,m}$ estimates on the ball $B(0, (2\lambda)^{1/2k})$ in order to obtain similar estimates on the whole space \mathbb{R}^d . This also explains why in (2.6) only the bounds of σ and ρ on the ball $B(0, (2\lambda)^{1/2k})$ enter.

While the just mentioned localization behaviour is completely consistent with the case of the harmonic oscillator in Theorem 2.1, it is worth to note that the term $\log(1 + \lambda)$ on the right-hand side of (2.6) does not appear in (2.2). This term turns out to be quite unfavourable (see Remark 2.5 below), and we conjecture that it can indeed be just skipped. The reason why it comes into play within our framework is related to obtaining Agmon estimates for spectral subspaces as explained in Remark 3.3 in Section 3.2 below.

Remark 2.5. Suppose again that the functions σ and ρ satisfy (2.3), so that

$$\gamma_{\lambda,k} \geq \gamma^{(1+(2\lambda)^{1/k})^{a/2}} \quad \text{and} \quad L_{\lambda,k} \leq L(1 + (2\lambda)^{1/k})^{\delta/2}.$$

In this case, it is easy to check that the spectral inequality (2.6) can be written as

$$(2.8) \quad \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \left(\frac{K}{\gamma}\right)^{K^{1+a+\delta(1+\lambda^{\frac{a}{2k}})(1+L^{1+\frac{k}{m}}\lambda^{\delta(\frac{1}{2k}+\frac{1}{2m})+L\lambda^{\frac{\delta}{2k}+\frac{1}{2m}}+\log(1+\lambda)})} \|f\|_{L^2(\omega)}^2$$

with a possibly different constant $K \geq 1$. This extends [33, Theorem 2.1 (i)], where only the case $a = 0$ and $\delta \in [0, 1]$ is considered. At the same time, our bound in (2.8) is much more explicit in the model parameters, which is very useful in the context of control theory as explained below. It should be mentioned, however, that in (2.8) with $a = 0$ the formal homogenization limit as $L \rightarrow 0$ results in a right-hand side where the constant still depends on λ . This is due to the $\log(1 + \lambda)$ -term in (2.8) (resp. (2.6)) but is highly unintuitive and not consistent with the known behaviour for the free Laplacian and the harmonic oscillator. This is one reason why this term is considered unfavourable and should be removed in future research if possible, cf. Remark 3.3 below. Another, more specific, reason in the context of Baouendi-Grushin operators is elaborated in Remark 2.18 below.

It is also worth to note that for $a = 0$ (for simplicity) and $\delta \in [0, 1]$ the estimate (2.8) can for $\lambda \geq 1$ be written as

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq K e^{K\lambda^{\frac{\delta}{2k}+\frac{1}{2m}}} \|f\|_{L^2(\omega)}^2$$

with yet another constant $K > 0$, now also depending on L , γ , and δ . This is stronger than the general estimate (1.1). By contrast, if $a = 0$ and $\delta > 1$, estimate (2.8) writes for $\lambda \geq 1$ as

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq K e^{K\lambda^{\delta(\frac{1}{2k}+\frac{1}{2m})}} \|f\|_{L^2(\omega)}^2$$

and is therefore worse than the general estimate (1.1), although the latter only uses that ω has positive measure. It is not yet clear how to reconcile this different behavior in the two regimes $\delta \leq 1$ and $\delta > 1$.

In the end of this subsection, let us present an example of a measurable set satisfying the geometric condition (2.1).

Example 2.6. Suppose that the local scale $\rho \equiv L > 0$ is constant and that $\sigma = w/(\sqrt{d}+1)^d$ with a radially symmetric function $w: \mathbb{R}^d \rightarrow (0, 1]$ that is nonincreasing with respect to the modulus and for which $1/w$ is locally bounded. Inspired by [16, Example 2.3] and [14, Example 4.17], with $l = L/(\sqrt{d} + 1)$ and $r_j = lw(j)^{1/d}$ consider the set

$$\omega = \bigcup_{j \in l\mathbb{Z}^d} B(j, r_j).$$

This set ω satisfies the geometric condition (2.1). Indeed, given $x \in \mathbb{R}^d$, there is $j \in l\mathbb{Z}^d$ with $|j| \leq |x|$ and $|x - j| < l\sqrt{d}$, so that $|x - j| + r_j < l(\sqrt{d} + 1) = L$. Hence, the ball $B(x, L)$ contains the ball $B(j, r_j)$, so that

$$\frac{|\omega \cap B(x, L)|}{|B(x, L)|} \geq \frac{|B(j, r_j)|}{|B(x, L)|} = \left(\frac{r_j}{L}\right)^d = \sigma(j) \geq \sigma(x).$$

It is worth to note that under the condition $\sum_{j \in l\mathbb{Z}^d} w(j) < \infty$, the above set ω has finite measure.

2.1.2. Grushin operator. Let us now consider the subelliptic model of the Grushin operator in $L^2(\mathbb{R}^{2d}) = L^2(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\Delta_G = \Delta_x + |x|^2 \Delta_y, \quad (x, y) \in \mathbb{R}^{2d}.$$

Equipped with its maximal domain

$$D(\Delta_G) = \{g \in L^2(\mathbb{R}^{2d}) : \Delta_G g \in L^2(\mathbb{R}^{2d})\},$$

this operator is non-positive and selfadjoint.

Theorem 2.7. *There exists a constant $K > 0$, depending only on the dimension d , such that for all $\lambda \geq 0$ and all (γ, L) -thick sets $\omega \subset \mathbb{R}^{2d}$, with $\gamma \in (0, 1]$ and $L > 0$, we have*

$$\forall f \in \mathcal{E}_\lambda(-\Delta_G), \quad \|f\|_{L^2(\mathbb{R}^{2d})}^2 \leq \left(\frac{K}{\gamma}\right)^{K(1+L\lambda+L^2\lambda)} \|f\|_{L^2(\omega)}^2.$$

Remark 2.8. To the best of the authors knowledge, the above spectral inequality is the first one obtained for the Grushin operator on \mathbb{R}^{2d} . However, let us refer to [23, Section 1.3.2, p.16], where spectral inequalities for the Grushin operator considered on $(-1, 1) \times \mathbb{T}$ with Dirichlet boundary conditions are stated. For comparison purposes, let us recall that a quantitative spectral inequality for the free Laplacian on \mathbb{R}^d from (γ, L) -thick sets $\omega \subset \mathbb{R}^d$, announced in [26, Theorem 3] and proved in [25, Theorem 1], takes the form

$$\forall f \in \mathcal{E}_\lambda(-\Delta), \quad \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \left(\frac{K}{\gamma}\right)^{K(1+L\sqrt{\lambda})} \|f\|_{L^2(\omega)}^2.$$

Observe here that the constant on the right-hand side of the estimate for the Laplacian exhibits a sublinear dependence on λ in the exponent, namely via $\sqrt{\lambda}$, whereas the constant in case of the Grushin operator shows a linear dependence on λ in the exponent. This corresponds to the fact that the free Laplacian is elliptic, whereas the Grushin operator is subelliptic. This is also reflected in the proof of Theorem 2.7 when Bernstein inequalities for functions in $\mathcal{E}_\lambda(-\Delta_G)$ are obtained, cf. Section 4.1 below.

2.2. Exact null-controllability. As application of the spectral inequalities from Theorems 2.1, 2.3 and 2.7, we study the exact null-controllability for two classes of diffusive equations, being elliptic and hypoelliptic, respectively.

Definition 2.9 (Exact null-controllability). Let $\Omega \subset \mathbb{R}^d$ be a domain, and let P be a non-negative selfadjoint operator in $L^2(\Omega)$. Given a measurable set $\omega \subset \Omega$, the evolution equation

$$(2.9) \quad \begin{cases} \partial_t f(t, x) + Pf(t, x) = h(t, x)1_\omega(x), & t > 0, x \in \Omega, \\ f(0, \cdot) = f_0 \in L^2(\Omega), \end{cases}$$

is said to be *exactly null-controllable from the control support ω in time $T > 0$* if for every initial datum $f_0 \in L^2(\Omega)$ there exists a control function $h \in L^2((0, T) \times \Omega)$ such that the mild solution to (2.9) satisfies $f(T, \cdot) = 0$.

2.2.1. The fractional anisotropic Shubin evolution equations. Let us first consider the evolution equations of the form (2.9) associated to the elliptic operators $P = H_{k,m}^s$ with $s > 0$, that is,

$$(E_{s,k,m}) \quad \begin{cases} \partial_t f(t, x) + H_{k,m}^s f(t, x) = h(t, x)1_\omega(x), & t > 0, x \in \mathbb{R}^d, \\ f(0, \cdot) = f_0 \in L^2(\mathbb{R}^d). \end{cases}$$

Here, the fractional powers of the operator $H_{k,m}$ are understood via standard functional calculus.

The spectral inequalities in Theorems 2.1 and 2.3 allow us to derive many exact null-controllability results for the equation $(E_{s,k,m})$, and we choose to present only three statements. We first give two general results closely related to Remark 2.5.

Corollary 2.10. *Let $\omega \subset \mathbb{R}^d$ be a measurable set as in (2.1), and suppose that the two functions $\sigma: \mathbb{R}^d \rightarrow (0, 1]$ and $\rho: \mathbb{R}^d \rightarrow (0, +\infty)$ satisfy*

$$\sigma(x) \geq \gamma \langle x \rangle^a \quad \text{and} \quad \rho(x) \leq L \langle x \rangle^\delta, \quad x \in \mathbb{R}^d,$$

with some fixed $L > 0$, $\delta \in [0, 1]$, $\gamma \in (0, 1]$, and $a \geq 0$. Then, for all $s > 0$ satisfying

$$\frac{\delta + a}{2k} + \frac{1}{2m} < s,$$

the equation $(E_{s,k,m})$ is exactly null-controllable from ω in every positive time $T > 0$.

Remark 2.11. Corollary 2.10 extends [34, Corollary 2.12] (cf. also [15, Corollary 1.2]), which only deals with the case $a = 0$. Moreover, recall from [33, Theorem 2.5] (whose proof is based on the general spectral inequalities (1.1)) that whenever $s > 1/(2k) + 1/(2m)$, the equation $(E_{s,k,m})$ is exactly null-controllable from every measurable control support $\omega \subset \mathbb{R}^d$ with positive measure and in every positive time $T > 0$. Corollary 2.10 therefore provides a new result only in the case $0 \leq \delta + a < 1$.

Corollary 2.12. *Let $\omega \subset \mathbb{R}^d$ be a measurable set as in (2.1), where the function σ satisfies*

$$\sigma(x) \geq \gamma \langle x \rangle^a, \quad x \in \mathbb{R}^d,$$

with some fixed $\gamma \in (0, 1]$ and $a \geq 0$, and the function ρ exhibits a growth at infinity that is slower than any power, that is,

$$\forall \delta > 0, \quad \rho(x) = o(|x|^\delta) \quad \text{as } |x| \rightarrow +\infty.$$

Then, for all $s > a/2k + 1/2m$, the equation $(E_{s,k,m})$ is exactly null-controllable from the control support ω in every positive time $T > 0$.

Remark 2.13. Corollary 2.12 is a quite straightforward consequence of Corollary 2.10, see Section 5.1 below. Nevertheless, it should be mentioned that the particular case of $a = 0$, although not explicitly stated in the literature, could have been proven also by using the results from [32, Chapter 6, Section 3].

It is well known from [38, Theorem 1.10] that the equation $(E_{1,1,1})$ is not null-controllable in any positive time whenever the control support $\omega \subset \mathbb{R}^d$ is contained in a half space. In fact, it can be readily checked that a half space satisfies a geometric condition of the form (2.1) with a constant function σ and a function ρ taking the form

$$\rho(x) = L \langle x \rangle, \quad x \in \mathbb{R}^d,$$

with some $L > 0$. Note that the latter exhibits a linear growth and is thus indeed excluded in Corollaries 2.10 and 2.12 above. This, however, raises the question whether local scales ρ can be allowed that exhibit an arbitrary sublinear growth. A first step in this direction is taken by the following last result of this subsection.

Corollary 2.14. *Let $\omega \subset \mathbb{R}^d$ be a measurable set as in (2.1), and suppose that the function σ is constant and that ρ satisfies*

$$(2.10) \quad \rho(x) \leq \frac{L\langle x \rangle}{(g \circ g)^\alpha(|x|)g(|x|)} \quad \text{where} \quad g(r) = \log(e + r), \quad r \geq 0,$$

with some $L > 0$ and $\alpha > 2$. Then, the equation $(E_{1,1,1})$ is exactly null-controllable from the control support ω in every positive time $T > 0$.

2.2.2. The Grushin heat equation. Let us now consider the fractional heat-like hypoelliptic evolution equation associated with the Grushin operator,

$$(E_{G,s}) \quad \begin{cases} \partial_t f(t, x, y) + (-\Delta_G)^s f(t, x, y) = h(t, x, y)1_\omega(x, y), & t > 0, (x, y) \in \mathbb{R}^{2d}, \\ f(0, \cdot, \cdot) = f_0 \in L^2(\mathbb{R}^{2d}). \end{cases}$$

From Theorem 2.7, we immediately obtain the following positive exact null-controllability result.

Corollary 2.15. *For all $s > 1$, the equation $(E_{G,s})$ is exactly null-controllable from every thick control support $\omega \subset \mathbb{R}^{2d}$ and in every positive time $T > 0$.*

While the above corollary allows general thick control supports in \mathbb{R}^{2d} at the restriction of $s > 1$, a different approach based on (2.4) allows to consider also $s = 1$, but with more restrictive control supports and times.

Corollary 2.16. *There exists a constant $K > 0$, only depending on the dimension d , such that for every (γ, L) -thick set $\omega \subset \mathbb{R}^d$ with $L > 0$ and $\gamma \in (0, 1]$, the equation $(E_{G,1})$ is exactly null-controllable from the control support $\omega \times \mathbb{R}^d$ in every positive time $T > 0$ satisfying $T \geq KL^2 \log(K/\gamma)$.*

Remark 2.17. A list of references concerning Grushin models is presented in Remark 2.22 below. Let us now mention that there are only few results concerning the equation $(E_{G,s})$. One of them is [30, Theorem 1.2], which states that in dimension $d = 1$, the equation $(E_{G,1})$ is never exactly null-controllable from any control support of the form $\mathbb{R} \times \omega$ whenever $\bar{\omega} \neq \mathbb{R}$. Corollaries 2.15 and 2.16 are, to the best of our knowledge, the first positive exact null-controllability results for the equation $(E_{G,s})$. Let us finally mention that we could also have studied the Grushin equation $(E_{G,s})$ posed on $\mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$, where $d_x, d_y \geq 1$ are two (possibly different) dimensions, but we refrained from doing so for simplicity.

Remark 2.18. As indicated above, Corollary 2.16 is strictly speaking not an application of our new spectral inequalities from Theorem 2.3, but rather of the already existing more precise spectral inequality for the harmonic oscillator from (2.4) for thick sets, first established in [21, Corollary 1.9]. Indeed, the proof of Corollary 2.16 does not work with the unfavourable $\log(1 + \lambda)$ term from Theorem 2.3, which prohibits us from obtaining a more general variant of Corollary 2.16 for Baouendi-Grushin operators of the form

$$\Delta_x + |x|^{2k} \Delta_y, \quad (x, y) \in \mathbb{R}^{2d},$$

see also Remark 5.6 below.

In contrast to the case of the whole space \mathbb{R}^{2d} , our new spectral inequalities from Theorem 2.3 allow to obtain positive null-controllability results for the fractional Baouendi-Grushin equation posed on $\mathbb{R}^d \times \mathbb{T}^d$,

$$(E_{k,s}) \quad \begin{cases} \partial_t f(t, x, y) + (-\Delta_k)^s f(t, x, y) = h(t, x, y)1_\omega(x, y), & t > 0, (x, y) \in \mathbb{R}^d \times \mathbb{T}^d, \\ f(0, \cdot, \cdot) = f_0 \in L^2(\mathbb{R}^d \times \mathbb{T}^d), \end{cases}$$

where $s > 0$ and $k \geq 1$ is a positive integer. Here, the Baouendi-Grushin operator Δ_k acting on $\mathbb{R}^d \times \mathbb{T}^d$,

$$\Delta_k = \Delta_x + |x|^{2k} \Delta_y, \quad (x, y) \in \mathbb{R}^d \times \mathbb{T}^d,$$

is equipped with its maximal domain, which makes it a positive selfadjoint operator. Our results for the equation $(E_{k,s})$ now read as follows.

Corollary 2.19.

- (i) When $s > (1+k)/2$, the equation $(E_{k,s})$ is exactly null-controllable in every positive time $T > 0$ and from every control support of the form $\omega \times \mathbb{T}^d$ with a thick set $\omega \subset \mathbb{R}^d$.
- (ii) Let $s = (1+k)/2$. Then, there exists a positive constant $K > 0$, depending only on k and the dimension d , such that for all (γ, L) -thick sets $\omega \subset \mathbb{R}^d$, the equation $(E_{k,s})$ is exactly null-controllable from the control support $\omega \times \mathbb{T}^d$ in every positive time $T \geq K(L/\sqrt{\lambda_k})^{k+1} \log(K/\gamma)$, where λ_k denotes the smallest eigenvalue of the anharmonic oscillator $H_{k,1}$.

Remark 2.20. (1) In view of Corollary 2.15, part (i) of the above corollary seems not to be optimal with respect to the type of control supports that can be considered. In fact, in the ongoing work [2], the first author proves that in the high diffusion setting $s > (1+k)/2$, the equation $(E_{k,s})$ is exactly null-controllable from the control support $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ at some positive time $T > 0$ if and only if ω is thick in the sense that

$$\exists \gamma \in (0, 1], \exists L > 0, \forall x \in \mathbb{R}^d, \quad |\omega \cap (B(x, L) \times \mathbb{T}^d)| \geq \gamma |B(x, L)|.$$

Just as for Theorem 2.7, the strategy adopted in [2] is to obtain spectral inequalities for the Baouendi-Grushin operator Δ_k from thick subsets of $\mathbb{R}^d \times \mathbb{T}^d$, but with a different technique as the one used in the present paper. Such estimates could in fact be obtained in the particular case $k = 1$, that is, for the Grushin operator Δ_G acting on $\mathbb{R}^d \times \mathbb{T}^d$, by mimicking the proof of Theorem 2.7 and using Fourier series instead of the Fourier transform. One would then obtain the spectral inequality

$$\forall f \in \mathcal{E}_\lambda(-\Delta_G), \quad \|f\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2 \leq \left(\frac{K}{\gamma}\right)^{K(1+L\lambda+L^2\lambda)} \|f\|_{L^2(\omega)}^2.$$

Corollaries 2.15 and 2.16 therefore also hold for the equation $(E_{G,s})$ considered on $\mathbb{R}^d \times \mathbb{T}^d$. The strategy developed in the present paper seems however not be well adapted to treat also the case $k \geq 2$.

(2) Without the unfavourable log term in Theorem 2.3, part (ii) of Corollary 2.19 would also hold for the equation $(E_{k,s})$ posed on the whole space \mathbb{R}^{2d} , as explained in Remark 5.6 below.

Let us finish this section with an example.

Example 2.21. For some fixed length $L > 0$, we consider the control support

$$\omega_L = B(0, L)^c \times \mathbb{T}^d \subset \mathbb{R}^d \times \mathbb{T}^d,$$

and the associated control time

$$T_{k,s,L} = \inf \{T > 0: (E_{k,s}) \text{ is exactly null-controllable from } \omega_L \text{ at time } T\}.$$

It is easy to see that for all $\varepsilon > 0$ the set $B(0, L)^c$ is $(\gamma_\varepsilon, L_\varepsilon)$ -thick in \mathbb{R}^d , with

$$L_\varepsilon = L + \varepsilon \quad \text{and} \quad \gamma_\varepsilon = 1 - \frac{L^d}{(L + \varepsilon)^d}.$$

One can then show by using part (i) of Corollary 2.19, part (ii) of Corollary 2.19 and part (ii) of Lemma 5.7 below, and part (i) of Lemma 5.7, respectively, that

$$\begin{cases} T_{k,s,L} = 0 & \text{when } s > (1+k)/2, \\ 0 < T_{k,s,L} < +\infty & \text{when } s = (1+k)/2, \\ T_{k,s,L} = +\infty & \text{when } s < (1+k)/2. \end{cases}$$

In the critical diffusion regime $s = (1+k)/2$, we actually have for all $\varepsilon > 0$ the more precise two-sided estimate

$$K \left(\frac{L}{\sqrt{\lambda_k}} \right)^{k+1} \leq T_{k,(1+k)/2,L} \leq K \left(\frac{L + \varepsilon}{\sqrt{\lambda_k}} \right)^{k+1} \log \left(\frac{K(L + \varepsilon)^n}{(L + \varepsilon)^n - L^n} \right),$$

where $\lambda_k > 0$ denotes the smallest eigenvalue of the anharmonic oscillator $H_{k,1}$; the upper bound is a consequence of part (ii) of Corollary 2.19, and the lower bound follows from part (ii) of Lemma 5.7.

Incidentally, the analogous proof as the one for the above lower bound on $T_{k,(1+k)/2,L}$ yields that $(E_{k,s})$ with $(-\Delta_k)^s$ replaced by $i(-\Delta_k)^s$, $s > 0$, is never exactly null-controllable from the control support ω_L , see Remark 5.6 below. This contrasts the behaviour known for the heat and the corresponding Schrödinger equation, see, e.g., [36, Section 2.2].

Remark 2.22. The results presented in Section 2.2.2 above are in line with articles devoted to the study of the null-controllability of Grushin-type heat equations. A pioneering article in this theory is [5], which paved the way for a numerous series of articles of which we can cite [1, 6, 8, 13, 19, 24]. All these works illustrate the fact that the null-controllability of Grushin-type heat equations is governed by minimal times, as in Example 2.21, and some of these works are even devoted to the computation of these minimal times. Let us also mention that the null-controllability of the Schrödinger-Grushin equation is also studied in the papers [11, 29].

3. SPECTRAL INEQUALITIES FOR THE ANISOTROPIC SHUBIN OPERATORS

The objective of this section is to prove Theorems 2.1 and 2.3. To this end, we mainly focus on proving the latter result and then explain briefly how its proof can be adapted in order to obtain the stronger spectral inequality for the harmonic oscillator in Theorem 2.1.

3.1. An abstract uncertainty relation. Let us begin with recalling from [21] the abstract result that plays an essential role in obtaining our spectral inequalities. In order to give its statement, we need to introduce the following definition: given a domain $\Omega \subset \mathbb{R}^d$, a constant $\kappa \geq 1$, and a length $l > 0$, we call a finite or countably infinite family $\{Q_j\}_j$ of non-empty bounded convex open subsets $Q_j \subset \Omega$ a (κ, l) -covering of Ω if

- (i) the set $\Omega \setminus \bigcup_j Q_j$ has Lebesgue measure zero;
- (ii) each Q_j is contained in a hypercube with sides of length l parallel to coordinate axes;
- (iii) the estimate $\sum_j \|g\|_{L^2(Q_j)}^2 \leq \kappa \|g\|_{L^2(\Omega)}^2$ holds for all $g \in L^2(\Omega)$.

We now have the following particular case of an uncertainty relation from [21].

Proposition 3.1 ([21, Proposition 3.1]). *Let $\{Q_j\}_j$ be a (κ, l) -covering of a given domain $\Omega \subset \mathbb{R}^d$, and suppose that $f \in \bigcap_{n \in \mathbb{N}} W^{n,2}(\Omega)$ satisfies*

$$\forall n \in \mathbb{N}, \quad \sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|_{L^2(\Omega)}^2 \leq \frac{C_B(n)}{n!} \|f\|_{L^2(\Omega)}^2,$$

with constants $C_B(n) > 0$ such that

$$h := \sum_{n \in \mathbb{N}} \sqrt{C_B(n)} \frac{(10dl)^n}{n!} < \infty.$$

Then, for every measurable subset $\omega \subset \Omega$ satisfying $\tau := \inf_j |Q_j \cap \omega| / \text{diam}(Q_j)^d > 0$, we have

$$\|f\|_{L^2(\Omega)}^2 \leq \frac{\kappa}{6} \left(\frac{24d|B(0,1)|}{\tau} \right)^{2\frac{\log \kappa}{\log 2} + 4\frac{\log h}{\log 2} + 5} \|f\|_{L^2(\omega)}^2.$$

In view of Proposition 3.1, we therefore need in the following to prove so-called Bernstein inequalities of the form

$$(3.1) \quad \sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|_{L^2(\Omega)}^2 \leq \frac{C_B(n, \lambda)}{n!} \|f\|_{L^2(\Omega)}^2, \quad n \in \mathbb{N}, f \in \mathcal{E}_{\lambda,k,m},$$

with a properly chosen domain $\Omega \subset \mathbb{R}^d$. In order to alleviate the writing, we use throughout this section the abbreviations

$$\mu := \frac{k}{k+m}, \quad \nu := \frac{m}{k+m}, \quad \theta := \frac{1}{2k} + \frac{1}{2m}.$$

3.2. Agmon estimates for spectral subspaces. A key ingredient in obtaining inequalities of the form (3.1) is given by the following variant of Agmon estimates from [3] for spectral subspaces associated with the (anisotropic) Shubin operators $H_{k,m}$.

Proposition 3.2. *There exist positive constants $c_1, c_2, c_3 > 0$ and $t_0 \in (0, 1]$, depending only on k, m , and the dimension d , such that for all $t \in [0, t_0]$, $\lambda \geq 0$, and $f \in \mathcal{E}_{\lambda, k, m}$ we have*

$$(3.2) \quad \|e^{c_1 t \langle x \rangle^{1/\nu}} f\|_{L^2(\mathbb{R}^d)}^2 + \|e^{c_1 t \langle D_x \rangle^{1/\mu}} f\|_{L^2(\mathbb{R}^d)}^2 \leq c_2 \lambda^{d\theta} e^{c_3 t \lambda^\theta} \|f\|_{L^2(\mathbb{R}^d)}^2.$$

Proof. We know from [3, Theorem 2.1] that there exist some positive constants $c_1, \tilde{c} > 0$, and $t_0 \in (0, 1]$ such that for every normalized eigenfunction $\psi \in L^2(\mathbb{R}^d)$ of the operator $H_{k,m}$ and all $t \in [0, t_0]$ we have

$$\|e^{c_1 t \langle x \rangle^{1/\nu}} \psi\|_{L^2(\mathbb{R}^d)} + \|e^{c_1 t \langle D_x \rangle^{1/\mu}} \psi\|_{L^2(\mathbb{R}^d)} \leq \tilde{c} e^{\tilde{c} t \lambda^\theta},$$

where $\lambda > 0$ is the eigenvalue associated with the eigenfunction ψ ; recall that $H_{k,m}$ has purely discrete spectrum. Expanding $f \in \mathcal{E}_{\lambda, k, m}$ for $\lambda \geq 0$ as a linear combination of eigenfunctions, we therefore deduce that for all $t \in [0, t_0]$ we have

$$\|e^{c_1 t \langle x \rangle^{1/\nu}} f\|_{L^2(\mathbb{R}^d)}^2 + \|e^{c_1 t \langle D_x \rangle^{1/\mu}} f\|_{L^2(\mathbb{R}^d)}^2 \leq N(\lambda) \tilde{c}^2 e^{2\tilde{c} t \lambda^\theta} \|f\|_{L^2(\mathbb{R}^d)}^2,$$

where $N(\lambda)$ is chosen as the number of distinct eigenvalues of $H_{k,m}$ less or equal to λ . Using the Weyl law asymptotics from [12, Remark 5.7] for the eigenvalue counting function associated to $H_{k,m}$, cf. also [10, Theorem 2.3.2], we then observe that

$$N(\lambda) \leq c' \lambda^{d\theta},$$

with some constant $c' > 0$ depending only on k, m , and d . The proof is then ended upon choosing $c_2 = \tilde{c}^2 c'$ and $c_3 = 2\tilde{c}$. \square

Remark 3.3. The term $\lambda^{d\theta}$ on the right-hand side of (3.2) is unexpected, and we indeed conjecture that (3.2) holds without this term, that is,

$$(3.3) \quad \|e^{c_1 t \langle x \rangle^{1/\nu}} f\|_{L^2(\mathbb{R}^d)}^2 + \|e^{c_1 t \langle D_x \rangle^{1/\mu}} f\|_{L^2(\mathbb{R}^d)}^2 \leq c_2 e^{c_3 t \lambda^\theta} \|f\|_{L^2(\mathbb{R}^d)}^2.$$

The reason the term $\lambda^{d\theta}$ appears in (3.2) lies in the way we carry quantitative sharp Agmon estimates for single eigenfunctions of the operator $H_{m,k}$ over to finite linear combinations of eigenfunctions. To the best of our knowledge, there are very few results in the literature stating Agmon estimates for spectral subspaces which are sharp with respect to possible parameters involved ($t \in [0, t_0]$ in this case for us), the rare exception being the case of the harmonic oscillator, see [7, Proposition 3.3]. Proving the stronger estimates (3.3) would immediately allow us to remove the unfavorable term $\log(1 + \lambda)$ in the spectral inequalities (2.6).

3.3. Bernstein inequalities. Proposition 3.2 now allows us to prove a global Bernstein inequality, that is, an inequality of the form (3.1) with $\Omega = \mathbb{R}^d$.

Proposition 3.4. *There exist positive constants $c, C > 0$, depending only on k, m , and the dimension d , such that for all $n \geq 0$, $\delta > 0$, $\lambda \geq 0$ and $f \in \mathcal{E}_{\lambda, k, m}$ we have*

$$\sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{C_B(n, \lambda, \delta)/2}{n!} \|f\|_{L^2(\mathbb{R}^d)}^2$$

with

$$(3.4) \quad C_B(n, \lambda, \delta) = 2C^{2(1+n)} \delta^{2n} (n!)^2 (1 + \lambda^{d\theta}) e^{(c+d)\delta^{-1/\nu}} e^{c\delta^{-1}\lambda^{\frac{1}{2m}}}.$$

Proof. Using integration by parts (see also Lemma 2.1 and Remark 2.2 in [21]) and Plancherel's theorem, we have

$$\sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^d)}^2 = \frac{1}{n!} \langle (-\Delta)^n f, f \rangle_{L^2(\mathbb{R}^d)} = \frac{1}{n!} \langle |\xi|^{2n} \hat{f}, \hat{f} \rangle_{L^2(\mathbb{R}^d)} = \frac{1}{n!} \| |\xi|^n \hat{f} \|_{L^2(\mathbb{R}^d)}^2,$$

where \hat{f} denotes the Fourier transform of the function f . We therefore have to estimate the quantity $\| |\xi|^n \hat{f} \|_{L^2(\mathbb{R}^d)}$. Note here that \hat{f} belongs to $\mathcal{E}_{\lambda,m,k}$ since $H_{k,m}$ is similar to $H_{m,k}$ by Fourier transform.

With $c_1, c_2, c_3 > 0$ and $t_0 \in (0, 1]$ as in Proposition 3.2 and $t \in (0, t_0]$, we write

$$|\xi|^n = |\xi|^n e^{-c_1 t \langle \xi \rangle^{1/\mu}} e^{c_1 t \langle \xi \rangle^{1/\mu}},$$

and estimate

$$\| |\xi|^n \hat{f} \|_{L^2(\mathbb{R}^d)} \leq \| \langle \xi \rangle^n e^{-c_1 t \langle \xi \rangle^{1/\mu}} \|_{L^\infty(\mathbb{R}^d)} \| e^{c_1 t \langle \xi \rangle^{1/\mu}} \hat{f} \|_{L^2(\mathbb{R}^d)},$$

with, moreover,

$$\| \langle \xi \rangle^n e^{-c_1 t \langle \xi \rangle^{1/\mu}} \|_{L^\infty(\mathbb{R}^d)} = \sup_{r \geq 0} r^n e^{-c_1 t r^{1/\mu}} \leq \left(\frac{\mu}{c_1 t} \right)^{n\mu} n^{n\mu} \leq \left(\frac{\mu}{c_1 t} \right)^{n\mu} (n!)^\mu.$$

Applying Proposition 3.2 to $\hat{f} \in \mathcal{E}_{\lambda,m,k}$ and taking into account that $\| \hat{f} \|_{L^2(\mathbb{R}^d)} = \| f \|_{L^2(\mathbb{R}^d)}$, we thus obtain from the above that

$$\sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{c_2}{n!} \left(\frac{\mu}{c_1 t} \right)^{2n\mu} (n!)^{2\mu} \lambda^{d\theta} e^{c_3 t \lambda^\theta} \| f \|_{L^2(\mathbb{R}^d)}^2.$$

Suppose that $\lambda \geq (1/\delta)^{2k}$. With the particular choice $t = t_0 \mu \delta^{-1} \lambda^{-1/(2k)} \leq t_0 \mu \leq t_0$, we then have

$$\sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{c_2}{n!} \left(\frac{1}{c_1 t_0} \right)^{2n\mu} \delta^{2n\mu} \lambda^{\frac{n\mu}{k}} (n!)^{2\mu} \lambda^{d\theta} e^{c_3 \mu \delta^{-1} \lambda^{\frac{1}{2m}}} \| f \|_{L^2(\mathbb{R}^d)}^2.$$

We further estimate

$$\left(\lambda^{\frac{1}{2k}} \right)^{\mu n} = \left(\lambda^{\frac{1}{2m}} \right)^{\nu n} = \delta^{\nu n} \left(\frac{\lambda^{\frac{1}{2m}}}{\delta} \right)^{\nu n} \leq \delta^{\nu n} (n!)^\nu e^{\nu \delta^{-1} \lambda^{\frac{1}{2m}}}.$$

Combining the last two inequalities, and taking into account that $\mu + \nu = 1$, we conclude that

$$(3.5) \quad \sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{C^{2(1+n)}}{n!} \delta^{2n} (n!)^2 \lambda^{d\theta} e^{c\delta^{-1} \lambda^{\frac{1}{2m}}} \| f \|_{L^2(\mathbb{R}^d)}^2$$

with $C^2 = \max\{c_2, (c_1 t_0)^{-\mu}\}$ and $c = \max\{1, c_3\}$.

It remains to consider the case $\lambda < (1/\delta)^{2k}$. Since then $\mathcal{E}_{\lambda,k,m} \subset \mathcal{E}_{(1/\delta)^{2k},k,m}$, we obtain from (3.5) with λ replaced by $(1/\delta)^{2k}$ that

$$(3.6) \quad \begin{aligned} \sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^d)}^2 &\leq \frac{C^{2(1+n)}}{n!} \delta^{2n} (n!)^2 \delta^{-2kd\theta} e^{c\delta^{-1} \delta^{-\frac{k}{m}}} \| f \|_{L^2(\mathbb{R}^d)}^2 \\ &\leq \frac{C^{2(1+n)}}{n!} \delta^{2n} (n!)^2 e^{(c+d)\delta^{-1/\nu}} \| f \|_{L^2(\mathbb{R}^d)}^2, \end{aligned}$$

where for the last inequality we used $\delta^{-2kd\theta} = \delta^{-d/\nu} \leq e^{d\delta^{-1/\nu}}$. In light of $e^{(c+d)\delta^{-1/\nu}} \geq 1$ and $e^{\delta^{-1} \lambda^{\frac{1}{2m}}} \geq 1$, the claim now follows from (3.5) and (3.6). \square

Remark 3.5. (1) Bernstein inequalities closely related to Proposition 3.4 have recently been obtained in [33, Eq. (4.5)] using smoothing properties of the semigroup associated to (fractional powers of) $H_{k,m}$ established in [3]. These smoothing properties also rely on the Agmon estimates for eigenfunctions, so that our proof above is more direct. Moreover, our constant in (3.4) incorporates the parameter δ , which may be used to force convergence of

an associated series, see (3.8) below, and thus makes our inequality more suitable for our purposes.

(2) In the particular case of the harmonic oscillator, that is, $k = m = 1$, Bernstein inequalities without the unfavorable term $1 + \lambda^{d\theta}$ have already been obtained in the literature. More precisely, [21, Proposition B.1] (cf. also [7, Proposition 3.3 (i)]) states that for all $n \geq 0$, $\delta > 0$, $\lambda \geq 0$, and $f \in \mathcal{E}_{\lambda,1,1}$ we have

$$\sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{C_B(n, \lambda, \delta)/2}{n!} \|f\|_{L^2(\mathbb{R}^d)}^2$$

with

$$(3.7) \quad C_B(n, \lambda, \delta) = 2(2\delta)^{2n} (n!)^2 e^{e\delta^{-2}} e^{2\delta^{-1}\sqrt{\lambda}}.$$

We are finally able to derive the local Bernstein inequalities of the desired form.

Corollary 3.6. *Let $\lambda > 0$, and let $\Omega \subset \mathbb{R}^d$ be an open set containing the ball $B(0, (2\lambda)^{1/2k})$. Then, for all $n \geq 0$, $\delta > 0$, and $f \in \mathcal{E}_{\lambda,k,m}$ we have*

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq 2\|f\|_{L^2(\Omega)}^2,$$

and

$$\sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|_{L^2(\Omega)}^2 \leq \frac{C_B(n, \lambda, \delta)}{n!} \|f\|_{L^2(\Omega)}^2,$$

with $C_B(n, \lambda, \delta)$ as in (3.4).

Proof. We have

$$\| |x|^k f \|_{L^2(\mathbb{R}^d)}^2 = \langle |x|^{2k} f, f \rangle_{L^2(\mathbb{R}^d)} \leq \langle H_{k,m} f, f \rangle_{L^2(\mathbb{R}^d)} \leq \lambda \|f\|_{L^2(\mathbb{R}^d)}^2,$$

where the last inequality follows by functional calculus. Hence,

$$\|f\|_{L^2(\mathbb{R}^d \setminus B(0, (2\lambda)^{1/2k}))}^2 = \| |x|^{-k} |x|^k f \|_{L^2(\mathbb{R}^d \setminus B(0, (2\lambda)^{1/2k}))}^2 \leq \frac{1}{2\lambda} \| |x|^k f \|_{L^2(\mathbb{R}^d)}^2 \leq \frac{1}{2} \|f\|_{L^2(\mathbb{R}^d)}^2.$$

Since Ω contains the ball $B(0, (2\lambda)^{1/2k})$ by hypothesis, this implies that

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq 2\|f\|_{L^2(B(0, (2\lambda)^{1/2k}))}^2 \leq 2\|f\|_{L^2(\Omega)}^2.$$

Moreover, we deduce from Proposition 3.4 that

$$\sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|_{L^2(\Omega)}^2 \leq \sum_{|\alpha|=n} \frac{1}{\alpha!} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{C_B(n, \lambda, \delta)/2}{n!} \|f\|_{L^2(\mathbb{R}^d)}^2,$$

which, together with the former inequality, proves the claim. \square

For future reference and in light of Proposition 3.1, we now consider for $\delta, l > 0$ and $\lambda > 0$ the quantity

$$\begin{aligned} h(l, \lambda, \delta) &:= \sum_{n \geq 0} \sqrt{C_B(n, \lambda, \delta)} \frac{(10dl)^n}{n!} \\ &= \sqrt{2} C (1 + \lambda^{d\theta})^{1/2} e^{(c+d)2^{-1}\delta^{-1/\nu}} e^{c(2\delta)^{-1}\lambda^{1/2m}} \sum_{n \geq 0} (10dlC\delta)^n. \end{aligned}$$

With the particular choice $\delta^{-1} = 20dlC$, we deduce that there is a constant $C' > 0$, depending only on k, m , and the dimension d , such that

$$(3.8) \quad h(l, \lambda) := h(l, \lambda, (20dlC)^{-1}) \leq C' (1 + \lambda^{d\theta})^{1/2} e^{C'l^{1/\nu}} e^{C'l\lambda^{1/2m}}.$$

3.4. Conclusion of Theorem 2.3. Let $\omega \subset \mathbb{R}^d$ be a measurable set as in (2.1), and let $f \in \mathcal{E}_{\lambda,k,m}$ with $\lambda \geq 0$. Consider $\lambda_0 := \min \sigma(H_{k,m}) > 0$. Then, if $\lambda \in [0, \lambda_0)$, we have $\mathcal{E}_{\lambda,k,m} = \{0\}$ and there is nothing to prove. It therefore suffices to consider $\lambda \geq \lambda_0 > 0$.

The key step is to use the well-known Besicovitch covering theorem in the following formulation taken from [16, Proposition 7.1]; see also [37, Theorem 2.7].

Proposition 3.7 (Besicovitch). *Let $A \subset \mathbb{R}^d$ be bounded, and let \mathcal{B} be a family of open balls such that each point in A is the center of some ball from \mathcal{B} . Then there are at most countably many balls $(B_j)_j \subset \mathcal{B}$ such that*

$$(3.9) \quad 1_A \leq \sum_j 1_{\overline{B_j}} \leq K_{\text{Bes}}^d,$$

where $K_{\text{Bes}} \geq 1$ is a universal constant.

We are finally in position to prove Theorem 2.3.

Proof of Theorem 2.3. Suppose that $\lambda \geq \lambda_0 > 0$, and let

$$A := B(0, (2\lambda)^{1/2k}) \quad \text{and} \quad \mathcal{B} := \{B(x, \rho(x)) : x \in A\}.$$

Besicovitch's covering theorem then implies that there is a finite or countably infinite collection of points $x_j \in A$ such that (3.9) holds with $B_j = B(x_j, \rho(x_j))$. In particular, A is contained in the union $\bigcup_j \overline{B_j}$. Let Ω be the interior of $\bigcup_j \overline{B_j}$. Then, Ω is open and contains the open set A by definition. Moreover, it is easy to see that Ω is a domain.

With $\gamma_{\lambda,k}$ and $L_{\lambda,k}$ from (2.7), for each j we clearly have $\sigma(x_j) \geq \gamma_{\lambda,k}$ and $\rho(x_j) \leq L_{\lambda,k}$. Hence, the family $\{B_j\}_j$ gives a $(K_{\text{Bes}}^d, L_{\lambda,k})$ -covering of Ω in the sense of Section 3.1, and from (2.1) we have

$$\inf_j \frac{|\omega \cap B_j|}{\text{diam}(B_j)^d} = \frac{|B(0,1)|}{2^d} \inf_j \frac{|\omega \cap B(x_j, \rho(x_j))|}{|B(x_j, \rho(x_j))|} \geq \frac{|B(0,1)|}{2^d} \inf_j \sigma(x_j) \geq \frac{|B(0,1)|}{2^d} \gamma_{\lambda,k}.$$

Taking into account Corollary 3.6 and (3.8), applying Proposition 3.1 with $\{Q_j\}_j = \{B_j\}_j$, $l = L_{\lambda,k}$, and $h(\lambda) = h(L_{\lambda,k}, \lambda)$ therefore yields

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq 2\|f\|_{L^2(\Omega)}^2 \leq \frac{K_{\text{Bes}}^d}{3} \left(\frac{24d2^d}{\gamma_{\lambda,k}} \right)^{2 \frac{\log K_{\text{Bes}}^d}{\log 2} + 4 \frac{\log h(\lambda)}{\log 2} + 5} \|f\|_{L^2(\omega \cap \Omega)}^2.$$

Here, we observe that for all $r \geq 0$, we have

$$1 + \lambda^r \leq (1 + \lambda_0^{-r}) \lambda^r,$$

so that

$$\log h(\lambda) \leq \frac{1}{2} \log((C')^2 (1 + \lambda_0^{-d\theta})) + \frac{d\theta}{2} \log(1 + \lambda) + C'(L_{\lambda,k})^{1/\nu} + C' L_{\lambda,k} \lambda^{\frac{1}{2m}}.$$

The claim therefore follows from the above upon an appropriate choice of the constant K , depending on d , C' , λ_0 , ν , θ , and K_{Bes} , that is, effectively only on d , k , and m . \square

We close this section by briefly discussing how the proof of Theorem 2.3 can be adapted to obtain Theorem 2.1.

Proof of Theorem 2.1. As mentioned in Remark 3.5 (2), in the particular case of the harmonic oscillator, that is, $k = m = 1$, there are Bernstein inequalities available that do not contain the unfavorable term $1 + \lambda^{d\theta}$. Upon replacing the constant (3.4) by (3.7), one can then follow the proof of Theorem 2.3 verbatim towards a proof of Theorem 2.1, thereby avoiding the term $\log(1 + \lambda)$ in the final estimate. \square

4. SPECTRAL INEQUALITIES FOR THE GRUSHIN OPERATOR

In this section, we prove Theorem 2.7. The strategy is exactly the same as the one used in the previous section, that is, we use the abstract result stated in Proposition 3.1. However, in the case of the Grushin operator, it suffices to obtain Bernstein inequalities of the form (3.1) only on $\Omega = \mathbb{R}^{2d}$ since, to the best of our knowledge, the elements from the spectral subspaces $\mathcal{E}_\lambda(-\Delta_G)$ do not exhibit any uniform decay properties.

4.1. Bernstein inequalities. Unlike the case of the (anisotropic) Shubin operators, we are not able to use Agmon estimates to derive Bernstein inequalities for functions in $\mathcal{E}_\lambda(-\Delta_G)$. In this section, inspired by the work [33], we instead rely on the smoothing estimates of the evolution operators $e^{t\Delta_G}$ generated by Δ_G . Notice that after applying the partial Fourier transform with respect to the $y \in \mathbb{R}^d$ variable, the Grushin operator Δ_G is transformed as

$$\Delta_G = \Delta_x + |x|^2 \Delta_y \rightsquigarrow -H_\eta := \Delta_x - |\eta|^2 |x|^2,$$

where $\eta \in \mathbb{R}^d$ denotes the dual variable of $y \in \mathbb{R}^d$. Consequently, we have

$$(4.1) \quad (e^{t\Delta_G} g)(x, y) = \int_{\mathbb{R}^d} e^{iy \cdot \eta} (e^{-tH_\eta} g_\eta)(x) d\eta, \quad g \in L^2(\mathbb{R}^{2d}), (x, y) \in \mathbb{R}^{2d},$$

where

$$g_\eta = \int_{\mathbb{R}^d} e^{-iy \cdot \eta} g(\cdot, y) dy.$$

Studying the semigroup generated by the Grushin operator Δ_G can therefore be reduced to studying the semigroup generated by the (negative of the) harmonic oscillator H_η with variably scaled potential.

Let us introduce for $\eta \in \mathbb{R}^d \setminus \{0\}$ the isometry M_η on $L^2(\mathbb{R}^d)$ by

$$M_\eta g = |\eta|^{d/4} g(\sqrt{|\eta|} \cdot), \quad g \in L^2(\mathbb{R}^d).$$

With $H = H_{(1, \dots, 1)}$, a straightforward computation shows that

$$(4.2) \quad (M_\eta)^* H_\eta M_\eta = |\eta| H, \quad \eta \neq 0.$$

In particular, since the smallest eigenvalue of the harmonic oscillator H is 1, we have by functional calculus

$$(4.3) \quad \|e^{-tH_\eta} g\|_{L^2(\mathbb{R}^d)} \leq e^{-t|\eta|} \|g\|_{L^2(\mathbb{R}^d)}, \quad g \in L^2(\mathbb{R}^d), t \geq 0, \eta \in \mathbb{R}^d.$$

Let us first focus on the smoothing properties in the $y \in \mathbb{R}^d$ variable.

Lemma 4.1. *For all $\alpha \in \mathbb{N}^d$, $t > 0$ and $g \in L^2(\mathbb{R}^{2d})$, we have*

$$(4.4) \quad \left\| \partial_y^\alpha (e^{t\Delta_G} g) \right\|_{L^2(\mathbb{R}^{2d})} \leq \frac{1}{t^{|\alpha|}} \alpha! \|g\|_{L^2(\mathbb{R}^{2d})}.$$

Proof. Let $t \geq 0$ and $g \in L^2(\mathbb{R}^{2d})$ be fixed. Let us first check that

$$(4.5) \quad \|e^{t|D_y|} (e^{t\Delta_G} g)\|_{L^2(\mathbb{R}^{2d})} \leq \|g\|_{L^2(\mathbb{R}^{2d})}.$$

From (4.1) we get by Parseval's theorem that

$$\|e^{t|D_y|} (e^{t\Delta_G} g)\|_{L^2(\mathbb{R}^{2d})}^2 = \int_{\mathbb{R}^d} e^{2t|\eta|} \|e^{-tH_\eta} g_\eta\|_{L^2(\mathbb{R}^d)}^2 d\eta.$$

Here, for every $\eta \in \mathbb{R}^d$, we have by (4.3) that

$$e^{2t|\eta|} \|e^{-tH_\eta} g_\eta\|_{L^2(\mathbb{R}^d)}^2 \leq e^{2t|\eta|} e^{-2t|\eta|} \|g_\eta\|_{L^2(\mathbb{R}^d)}^2 = \|g_\eta\|_{L^2(\mathbb{R}^d)}^2,$$

which proves (4.5) by Parseval's theorem. The following estimate, which follows from a straightforward study of function and holds for any $p, q > 0$, $c > 0$ and $x \geq 0$,

$$(4.6) \quad x^p e^{-cx^q} \leq \left(\frac{p}{ecq} \right)^{\frac{p}{q}},$$

and the estimate $|\alpha|^{|\alpha|} \leq e^{|\alpha|} |\alpha|!$ then imply that for all $\alpha \in \mathbb{N}^d$, $t > 0$ and $g \in L^2(\mathbb{R}^{2d})$, we have

$$\begin{aligned} \|\partial_y^\alpha (e^{t\Delta_G} g)\|_{L^2(\mathbb{R}^{2d})} &\leq \| |D_y|^{|\alpha|} e^{-t|D_y|} e^{t|D_y|} (e^{t\Delta_G} g) \|_{L^2(\mathbb{R}^{2d})} \\ &\leq \left(\frac{|\alpha|}{et} \right)^{|\alpha|} \|g\|_{L^2(\mathbb{R}^{2d})} \\ &\leq \frac{1}{t^{|\alpha|}} \alpha! \|g\|_{L^2(\mathbb{R}^{2d})}. \end{aligned}$$

This ends the proof of Lemma 4.1. \square

We are now interested in the smoothing properties in the $x \in \mathbb{R}^d$ variable, for which we obtain a different time behavior compared to the $y \in \mathbb{R}^d$ variable in Lemma 4.1 above.

Lemma 4.2. *There exists a positive constant $c > 0$ such that for all $\alpha \in \mathbb{N}^d$, $t > 0$ and $g \in L^2(\mathbb{R}^{2d})$,*

$$\|\partial_x^\alpha (e^{t\Delta_G} g)\|_{L^2(\mathbb{R}^{2d})} \leq \frac{c^{1+|\alpha|}}{t^{\frac{|\alpha|}{2}}} \alpha! \|g\|_{L^2(\mathbb{R}^{2d})}.$$

Proof. Let $t > 0$ and $g \in L^2(\mathbb{R}^{2d})$ be fixed. We first deduce from Plancherel's theorem and the relation (4.2) that

$$(4.7) \quad \|\partial_x^\alpha (e^{t\Delta_G} g)\|_{L^2(\mathbb{R}^{2d})}^2 = \int_{\mathbb{R}^d} \|\partial_x^\alpha (e^{-tH_\eta} g_\eta)\|_{L^2(\mathbb{R}^d)}^2 d\eta.$$

Let us consider $\eta \in \mathbb{R}^d \setminus \{0\}$. We deduce from (4.2) that

$$\begin{aligned} \|\partial_x^\alpha (e^{-tH_\eta} g_\eta)\|_{L^2(\mathbb{R}^d)} &= \|\partial_x^\alpha (M_\eta e^{-|\eta|tH} (M_\eta)^* g_\eta)\|_{L^2(\mathbb{R}^d)} \\ &= |\eta|^{\frac{|\alpha|}{2}} \|\partial_x^\alpha (e^{-|\eta|tH} (M_\eta)^* g_\eta)\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

In order to control the above term, we will use Theorem 2.3 in [3] which states that there exist positive constants $c_1 > 1$ and $t_0 > 0$ such that for all $0 < t \leq t_0$, $\alpha \in \mathbb{N}^d$ and $g \in L^2(\mathbb{R}^d)$,

$$(4.8) \quad \|\partial_x^\alpha (e^{-tH} g)\|_{L^2(\mathbb{R}^d)} \leq \frac{c_1^{|\alpha|}}{t^{\frac{|\alpha|}{2}}} \sqrt{\alpha!} \|g\|_{L^2(\mathbb{R}^d)}.$$

By using anew that the smallest eigenvalue of the harmonic oscillator H is 1, cf. (4.3), we also deduce that there exists another positive constant $c_2 > 1$ such that for all $t > t_0$, $\alpha \in \mathbb{N}^d$ and $g \in L^2(\mathbb{R}^d)$,

$$(4.9) \quad \|\partial_x^\alpha (e^{-tH} g)\|_{L^2(\mathbb{R}^d)} \leq c_2^{1+|\alpha|} e^{-t} \sqrt{\alpha!} \|g\|_{L^2(\mathbb{R}^d)}.$$

Let us now consider $\eta \in \mathbb{R}^d \setminus \{0\}$ not being equal to zero. There are two cases to consider.

▷ *Case 1:* $|\eta|t \leq t_0$. In this situation, we deduce from the estimate (4.8) that

$$|\eta|^{\frac{|\alpha|}{2}} \|\partial_x^\alpha (e^{-|\eta|tH} (M_\eta)^* g_\eta)\|_{L^2(\mathbb{R}^d)} \leq \frac{c_1^{|\alpha|} |\eta|^{\frac{|\alpha|}{2}}}{(|\eta|t)^{\frac{|\alpha|}{2}}} \sqrt{\alpha!} \|g_\eta\|_{L^2(\mathbb{R}^d)}.$$

As a consequence,

$$(4.10) \quad |\eta|^{\frac{|\alpha|}{2}} \|\partial_x^\alpha (e^{-|\eta|tH} (M_\eta)^* g_\eta)\|_{L^2(\mathbb{R}^d)} \leq \frac{c_1^{|\alpha|}}{t^{\frac{|\alpha|}{2}}} \sqrt{\alpha!} \|g_\eta\|_{L^2(\mathbb{R}^d)}.$$

▷ *Case 2:* $|\eta|t > t_0$. In this case, we use (4.9) to obtain that

$$|\eta|^{\frac{|\alpha|}{2}} \|\partial_x^\alpha (e^{-|\eta|tH} (M_\eta)^* g_\eta)\|_{L^2(\mathbb{R}^d)} \leq c_2^{1+|\alpha|} |\eta|^{\frac{|\alpha|}{2}} e^{-|\eta|t} \sqrt{\alpha!} \|g_\eta\|_{L^2(\mathbb{R}^d)}.$$

The estimates (4.6), $|\alpha|^{|\alpha|} \leq e^{|\alpha|} |\alpha|!$ and $|\alpha|! \leq d^{|\alpha|} \alpha!$ then imply that

$$|\eta|^{\frac{|\alpha|}{2}} e^{-|\eta|t} \leq \left(\frac{|\alpha|}{2et} \right)^{\frac{|\alpha|}{2}} \leq \frac{c_3^{|\alpha|}}{t^{\frac{|\alpha|}{2}}} \sqrt{\alpha!}.$$

In a nutshell, we proved that

$$(4.11) \quad |\eta|^{\frac{|\alpha|}{2}} \left\| \partial_x^\alpha (e^{-|\eta|tH} (M_\eta)^* g_\eta) \right\|_{L^2(\mathbb{R}^d)} \leq \frac{(c_2 c_3)^{1+|\alpha|}}{t^{\frac{|\alpha|}{2}}} \alpha! \|g_\eta\|_{L^2(\mathbb{R}^d)}.$$

Gathering (4.7) and the estimates (4.10), (4.11), we get that

$$\left\| \partial_x^\alpha (e^{t\Delta_G} g) \right\|_{L^2(\mathbb{R}^{2d})} \leq \frac{(c_1 c_2 c_3)^{1+|\alpha|}}{t^{\frac{|\alpha|}{2}}} \alpha! \|g\|_{L^2(\mathbb{R}^{2d})}.$$

This ends the proof of Lemma 4.2. \square

We are now in position to prove a global Bernstein inequality of the desired form (3.1) for functions in $\mathcal{E}_\lambda(-\Delta_G)$.

Proposition 4.3. *There exists a positive constant $c > 0$, depending only the dimension d , such that for all $n \geq 0$, $t > 0$, $\lambda \geq 0$, and $f \in \mathcal{E}_\lambda(-\Delta_G)$ we have*

$$\sum_{|\alpha|+|\beta|=n} \frac{1}{\alpha! \beta!} \|\partial_x^\alpha \partial_y^\beta f\|_{L^2(\mathbb{R}^{2d})}^2 \leq \frac{C_B(n, \lambda, t)}{n!} \|f\|_{L^2(\mathbb{R}^{2d})}^2,$$

with

$$C_B(n, \lambda, t) = \frac{c^{2(1+n)}}{t^{2n}} (n!)^2 e^{2(t+t^2)\lambda}.$$

Proof. First notice that since the operators Δ_G and ∂_y^β commute, we have that for all $(\alpha, \beta) \in \mathbb{N}^{2d}$ and $t, s \geq 0$,

$$\partial_x^\alpha \partial_y^\beta (e^{(s+t)\Delta_G}) = (\partial_x^\alpha e^{s\Delta_G}) (\partial_y^\beta e^{t\Delta_G}).$$

Lemmas 4.1 and 4.2 therefore imply that there exists a positive constant $c > 0$ such that for all $(\alpha, \beta) \in \mathbb{N}^{2d}$, $s, t > 0$ and $g \in L^2(\mathbb{R}^{2d})$,

$$\left\| \partial_x^\alpha \partial_y^\beta (e^{(s+t)\Delta_G} g) \right\|_{L^2(\mathbb{R}^{2d})} \leq \frac{c^{1+|\alpha|+|\beta|}}{s^{\frac{|\alpha|}{2}} t^{|\beta|}} \alpha! \beta! \|g\|_{L^2(\mathbb{R}^{2d})}.$$

By choosing $s = t^2$ and the functional calculus, we therefore deduce that for all $t > 0$, $\lambda \geq 0$ and $f \in \mathcal{E}_\lambda(-\Delta_G)$,

$$\begin{aligned} \left\| \partial_x^\alpha \partial_y^\beta f \right\|_{L^2(\mathbb{R}^{2d})} &= \left\| \partial_x^\alpha \partial_y^\beta (e^{(t+t^2)\Delta_G} e^{-(t+t^2)\Delta_G} f) \right\|_{L^2(\mathbb{R}^{2d})} \\ &\leq \frac{c^{1+|\alpha|+|\beta|}}{t^{|\alpha|+|\beta|}} \alpha! \beta! \left\| e^{-(t+t^2)\Delta_G} f \right\|_{L^2(\mathbb{R}^{2d})} \\ &\leq \frac{c^{1+|\alpha|+|\beta|}}{t^{|\alpha|+|\beta|}} \alpha! \beta! e^{(t+t^2)\lambda} \|f\|_{L^2(\mathbb{R}^{2d})}. \end{aligned}$$

Finally, we get that for all $n \geq 0$, $t > 0$, $\lambda \geq 0$, and $f \in \mathcal{E}_\lambda(-\Delta_G)$,

$$\begin{aligned} \sum_{|\alpha|+|\beta|=n} \frac{1}{\alpha! \beta!} \|\partial_x^\alpha \partial_y^\beta f\|_{L^2(\mathbb{R}^{2d})}^2 &\leq \frac{c^{2(1+n)}}{t^{2n}} \left(\sum_{|\alpha|+|\beta|=n} \alpha! \beta! \right) e^{2(t+t^2)\lambda} \|f\|_{L^2(\mathbb{R}^{2d})}^2 \\ &\leq \frac{c^{2(1+n)}}{t^{2n}} \left(\sum_{|\alpha|+|\beta|=n} |\alpha + \beta|! \right) e^{2(t+t^2)\lambda} \|f\|_{L^2(\mathbb{R}^{2d})}^2 \\ &\leq \frac{c^{2(1+n)}}{t^{2n}} \binom{n+2d-1}{2d-1} n! e^{2(t+t^2)\lambda} \|f\|_{L^2(\mathbb{R}^{2d})}^2 \\ &\leq \frac{c^{2(1+n)} 2^{n+2d-1}}{t^{2n}} n! e^{2(t+t^2)\lambda} \|f\|_{L^2(\mathbb{R}^{2d})}^2 \end{aligned}$$

which is the expected estimate. \square

4.2. Conclusion of Theorem 2.7. Let $\omega \subset \mathbb{R}^{2d}$ be (γ, L) -thick with $\gamma \in (0, 1]$ and $L > 0$. In view of Propositions 3.1 and 4.3, we now consider for $t > 0$ and $\lambda > 0$ the quantity

$$\begin{aligned} h(\lambda, t) &:= \sum_{n \geq 0} \sqrt{C_B(n, \lambda, t)} \frac{(10dL)^n}{n!} \\ &= ce^{(t+t^2)\lambda} \sum_{n \geq 0} \left(\frac{10cdL}{t} \right)^n. \end{aligned}$$

With the particular choice $t = 20cdL$, we deduce that there is a constant $c' > 0$, depending only on the dimension d , such that

$$h(\lambda) := h(\lambda, 20cdL) \leq 2ce^{c'(L+L^2)\lambda}.$$

Theorem 2.7 is then a straightforward application of Proposition 3.1 on $\Omega = \mathbb{R}^d$.

5. PROOF OF THE EXACT NULL-CONTROLLABILITY RESULTS

In this last section we use the spectral inequalities given by Theorems 2.1, 2.3, and 2.7 in order to prove the exact null-controllability results from Section 2.2 for the evolution equations $(E_{s,k,m})$, $(E_{G,s})$ and $(E_{k,s})$.

Since the operators $H_{k,m}^s$, $-\Delta_G$, and $(-\Delta_k)^s$ are selfadjoint in $L^2(\mathbb{R}^d)$, $L^2(\mathbb{R}^{2d})$, and $L^2(\mathbb{R}^d \times \mathbb{T}^d)$, respectively, the Hilbert Uniqueness Method implies that the exact null-controllability of these equations is equivalent to the exact observability of the associated semigroups $(e^{-tH_{k,m}^s})_{t \geq 0}$, $(e^{t\Delta_G})_{t \geq 0}$ and $(e^{-t(-\Delta_k)^s})_{t \geq 0}$. The latter is defined as follows.

Definition 5.1 (Exact observability). Let $\tau > 0$, and let $\Omega \subset \mathbb{R}^d$ and $\omega \subset \Omega$ be measurable. A strongly continuous semigroup $(T(t))_{t \geq 0}$ on $L^2(\Omega)$ is said to be exactly observable from the set ω in time τ if there exists a positive constant $C_{\omega, \tau} > 0$ such that for all $g \in L^2(\Omega)$, we have

$$\|T(\tau)g\|_{L^2(\Omega)}^2 \leq C_{\omega, \tau} \int_0^\tau \|T(t)g\|_{L^2(\omega)}^2 dt.$$

In order to prove exact observability estimates, with an explicit observability constant $C_{\omega, \tau}$, we use the following quantitative result that is based on the well-known Lebeau-Robbiano strategy and is particularly well adapted to the equations we are studying.

Theorem 5.2 ([39, Theorem 2.8]). *Let A be a non-negative selfadjoint operator on $L^2(\mathbb{R}^d)$, and let $\omega \subset \mathbb{R}^d$ be measurable. Suppose that there are $d_0 > 0$, $d_1 \geq 0$, and $\zeta \in (0, 1)$ such that for all $\lambda \geq 0$ and $f \in \mathcal{E}_\lambda(A)$,*

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq d_0 e^{d_1 \lambda^\zeta} \|f\|_{L^2(\omega)}^2.$$

Then, there exist positive constants $c_1, c_2, c_3 > 0$, only depending on ζ , such that for all $T > 0$ and $g \in L^2(\mathbb{R}^d)$ we have the observability estimate

$$\|e^{-TA}g\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{C_{\text{obs}}}{T} \int_0^T \|e^{-tA}g\|_{L^2(\omega)}^2 dt,$$

where the positive constant $C_{\text{obs}} > 0$ is given by

$$(5.1) \quad C_{\text{obs}} = c_1 d_0 (2d_0 + 1)^{c_2} \exp\left(c_3 \left(\frac{d_1}{T^\zeta}\right)^{\frac{1}{1-\zeta}}\right).$$

While the Lebeau-Robbiano strategy in Theorem 5.2 requires that the constant in the spectral inequality exhibits a sublinear power growth in the exponent in terms of the spectral parameter λ , the following statement allows a more general subexponential growth in λ , but does not provide a quantitative observability estimate.

Theorem 5.3 ([20, Theorem 5]). *Let A be a non-negative selfadjoint operator on $L^2(\mathbb{R}^d)$, and let $\omega \subset \mathbb{R}^d$ be measurable. Suppose that the spectral inequality*

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq ce^{c\lambda/((\log \log \lambda)^\alpha \log \lambda)} \|f\|_{L^2(\omega)}^2, \quad f \in \mathcal{E}_\lambda(A), \lambda > e,$$

holds with some $\alpha > 2$ and $c > 0$. Then, for all $T > 0$, there exists a positive constant $C_T > 0$ such that for all $g \in L^2(\mathbb{R}^d)$ we have

$$\|e^{-TA}g\|_{L^2(\mathbb{R}^d)}^2 \leq C_T \int_0^T \|e^{-tA}g\|_{L^2(\omega)}^2 dt.$$

5.1. Proof of Corollaries 2.10, 2.12, and 2.14. Let us first focus on the results regarding the equation $(E_{s,k,m})$. Here, in order to deal with the fractional powers of $H_{k,m}$, we use the fact that by the transformation formula for spectral measures, see, e.g., [40, Proposition 4.24], for all $s > 0$ and $\lambda \geq 0$ we have

$$(5.2) \quad \mathcal{E}_{\lambda,s,k,m} := 1_{(-\infty, \lambda]}(H_{k,m}^s) = 1_{(-\infty, \lambda^{1/s}]}(H_{k,m}) = \mathcal{E}_{\lambda^{1/s}, k, m}.$$

In essence, this implies that a spectral inequality for $H_{k,m}$ yields a spectral inequality for $H_{k,m}^s$ by just replacing λ by $\lambda^{1/s}$ in the corresponding constant.

We are now in position to prove Corollaries 2.10 and 2.12.

Proof of Corollary 2.10. Under the hypotheses on σ and ρ , we are in the situation of Remark 2.5 with $\delta \leq 1$. It therefore immediately follows from (2.8) and (5.2) that for some constants $d_0 > 0$ and $d_1 \geq 0$ we have

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq d_0 e^{d_1 \lambda^\zeta} \|f\|_{L^2(\omega)}^2, \quad f \in \mathcal{E}_{\lambda,s,k,m} = \mathcal{E}_{\lambda^{1/s}, k, m},$$

with

$$\zeta = \frac{\delta + a}{2sk} + \frac{1}{2sm} < 1.$$

The claim then immediately follows by applying Theorem 5.2. \square

Proof of Corollary 2.12. Given $s > a/2k + 1/2m$, we pick a $\delta \in (0, 1)$ such that

$$\frac{a}{2k} + \frac{1}{2m} < \frac{\delta + a}{2k} + \frac{1}{2m} < s.$$

The hypothesis on ρ , namely $\rho(x) = o(|x|^\delta)$ as $|x| \rightarrow +\infty$, then implies that there is $L > 0$ such that

$$\rho(x) \leq L \langle x \rangle^\delta, \quad x \in \mathbb{R}^d.$$

We are thus in the situation of Corollary 2.10 and the claim is just an instance of that result. \square

While the two corollaries above rely on the Lebeau-Robbiano strategy from Theorem 5.2, Corollary 2.14 has to revert to the more general statement in Theorem 5.3.

Proof of Corollary 2.14. Under the hypothesis (2.10), it is easy to see that for, say, $\lambda \geq e+1$ we have

$$L_\lambda = \sup_{|x| < \sqrt{2\lambda}} \rho(x) \leq c' \frac{L\sqrt{\lambda}}{(\log \log \lambda)^\alpha \log \lambda}$$

with a suitably chosen constant $c' > 0$ depending on α , but not on L or λ . It then follows from Theorem 2.1 with a constant function σ that

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq ce^{c\lambda/((\log \log \lambda)^\alpha \log \lambda)} \|f\|_{L^2(\omega)}^2, \quad f \in \mathcal{E}_{\lambda,1,1}, \lambda \geq e+1,$$

where $c > 0$ is another constant, depending on L , c' , and the dimension d . Taking into account that $\mathcal{E}_{\lambda,1,1} \subset \mathcal{E}_{e+1,1,1}$ for $e < \lambda < e+1$, the latter even holds for all $\lambda > e$ after suitably adapting the constant c . The claim then immediately follows from Theorem 5.3. \square

5.2. Proof of Corollaries 2.15 and 2.16. We now consider the Grushin heat equation $(E_{G,s})$. Here, the proof of Corollary 2.15 follows essentially the same lines as the one of Corollary 2.10.

Proof of Corollary 2.15. Analogously as for the Shubin operators in (5.2), the transformation formula for spectral measures yields

$$\mathcal{E}_\lambda((-\Delta_G)^s) = \mathcal{E}_{\lambda^{1/s}}(-\Delta_G), \quad s > 0, \lambda \geq 0.$$

The claim is then an immediate consequence of the spectral inequality for Δ_G in Theorem 2.7 and Theorem 5.2. \square

The proof of Corollary 2.16 requires some preparation. Here, we rely on the notations introduced in Section 4.1, especially for the harmonic oscillators H_η with variably scaled potential and the isometries M_η . We also write $H_\eta = H_{|\eta|}$ and $M_\eta = M_{|\eta|}$ for simplicity.

We first prove an exact observability estimate for the semigroups generated by the harmonic oscillators H_r , $r \geq 0$, with an explicit quantitative observability constant with respect to the parameter r .

Proposition 5.4. *There exists a constant $K > 0$, depending only on the dimension d , such that for all (γ, L) -thick sets $\omega \subset \mathbb{R}^d$, $r > 0$, $T > 0$, and $g \in L^2(\mathbb{R}^d)$ we have*

$$(5.3) \quad \|e^{-TH_r}g\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{C_{\text{obs}}}{T} \int_0^T \|e^{-tH_r}g\|_{L^2(\omega)}^2 dt,$$

where the positive constant $C_{\text{obs}} > 0$ is given by

$$(5.4) \quad C_{\text{obs}} = K \left(\frac{K}{\gamma} \right)^{K(1+rL^2)} \exp\left(\frac{KL^2 \log^2(K/\gamma)}{T} \right).$$

Proof. Recall from Remark 2.2 that for the harmonic oscillator a precise spectral inequality for thick sets has been established in [21, Corollary 1.9]. Following (2.4), for every (γ, L) -thick set ω , this spectral inequality can be written as

$$(5.5) \quad \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \left(\frac{C}{\gamma} \right)^{C(1+L^2+L\sqrt{\lambda})} \|f\|_{L^2(\omega)}^2, \quad f \in \mathcal{E}_{\lambda,1,1}, \lambda \geq 0,$$

with a constant $C > 0$ depending only on the dimension d .

Let us now fix some $r > 0$ and a (γ, L) -thick set $\omega \subset \mathbb{R}^d$. In light of the similarity relation $(M_r)^*H_rM_r = rH$ from (4.2), we clearly have

$$(M_r)^*\mathcal{E}_\lambda(H_r) \subset \mathcal{E}_\lambda(rH) = \mathcal{E}_{\lambda/r}(H) = \mathcal{E}_{\lambda/r,1,1}.$$

Moreover, it is easy to see that the set $\sqrt{r}\omega$ is $(\gamma, \sqrt{r}L)$ -thick. We therefore deduce from (5.5) that for all $\lambda \geq 0$ and $f \in \mathcal{E}_\lambda(H_r)$, we have

$$\begin{aligned} \|f\|_{L^2(\mathbb{R}^d)}^2 &= \|(M_r)^* f\|_{L^2(\mathbb{R}^d)}^2 \leq \left(\frac{C}{\gamma}\right)^{C(1+rL^2+L\sqrt{\lambda})} \|(M_r)^* f\|_{L^2(\sqrt{r}\omega)}^2 \\ &= \left(\frac{C}{\gamma}\right)^{C(1+rL^2+L\sqrt{\lambda})} \|f\|_{L^2(\omega)}^2. \end{aligned}$$

The latter can be rewritten as

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \leq d_0 e^{d_1 \sqrt{\lambda}} \|f\|_{L^2(\omega)}^2, \quad f \in \mathcal{E}_\lambda(H_r),$$

with

$$d_0 = \left(\frac{C}{\gamma}\right)^{C(1+rL^2)} \quad \text{and} \quad d_1 = CL \log\left(\frac{C}{\gamma}\right).$$

Theorem 5.2 then implies that there exist universal positive constants $c_1, c_2, c_3 > 0$ such that for all $T > 0$ and $g \in L^2(\mathbb{R}^d)$, we have

$$\|e^{-TH_r} g\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{C_{\text{obs}}}{T} \int_0^T \|e^{-tH_r} g\|_{L^2(\omega)}^2 dt$$

with $C_{\text{obs}} = C_{\text{obs}}(\omega, T, r)$ given by

$$C_{\text{obs}} = c_1 d_0 (2d_0 + 1)^{c_2} \exp\left(\frac{c_3 d_1^2}{T}\right).$$

It only remains to observe that there exists another positive constant $c_4 > 0$, depending only on the dimension d , such that

$$d_0 (2d_0 + 1)^{c_2} \leq \left(\frac{c_4}{\gamma}\right)^{c_4(1+rL^2)}.$$

This ends the proof of Proposition 5.4 upon a suitable choice of the constant K . \square

We are now able to tackle the proof of Corollary 2.16.

Proof of Corollary 2.16. Let $\omega \subset \mathbb{R}^d$ be a (γ, L) -thick set. We have to show that whenever $T \geq T^*$ with some time $T^* > 0$ depending on γ and L that is to be determined, there exists a constant $C_{\omega, T} > 0$ such that for all $g \in L^2(\mathbb{R}^{2d})$ we have

$$(5.6) \quad \|e^{T\Delta_G} g\|_{L^2(\mathbb{R}^{2d})}^2 \leq C_{\omega, T} \int_0^T \|e^{t\Delta_G} g\|_{L^2(\omega \times \mathbb{R}^d)}^2 dt.$$

To this end, we first observe from (4.1) and Fubini's theorem that for every measurable set $\Omega \subset \mathbb{R}^d$ and all $t > 0$ and $g \in L^2(\mathbb{R}^{2d})$ we have

$$\|e^{t\Delta_G} g\|_{L^2(\Omega \times \mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \|e^{-tH_\eta} g_\eta\|_{L^2(\Omega)}^2 d\eta.$$

Inserting the latter into both sides of (5.6), once with $\Omega = \mathbb{R}^d$ and $t = T$ and once with $\Omega = \omega$, we immediately infer by Fubini's theorem that it suffices to show that

$$(5.7) \quad \|e^{-TH_r} g\|_{L^2(\mathbb{R}^d)}^2 \leq C_{\omega, T} \int_0^T \|e^{-tH_r} g\|_{L^2(\omega)}^2 dt, \quad g \in L^2(\mathbb{R}^d), \quad r > 0,$$

with the constant $C_{\omega, T} > 0$ not depending on r .

On the one hand, we have by (4.3) that

$$\|e^{-TH_r} g\|_{L^2(\mathbb{R}^d)}^2 \leq e^{-Tr} \|e^{-(T/2)H_r} g\|_{L^2(\mathbb{R}^d)}^2, \quad g \in L^2(\mathbb{R}^d), \quad r > 0.$$

On the other hand, it follows from Proposition 5.4 that for all $r > 0$ and $g \in L^2(\mathbb{R}^d)$ we have

$$\|e^{-(T/2)H_r} g\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{2C_{\text{obs}}}{T} \int_0^{T/2} \|e^{-tH_r} g\|_{L^2(\omega)}^2 dt,$$

where $C_{\text{obs}} = C_{\text{obs}}(\omega, T/2, r)$ is given by (5.4) with T replaced by $T/2$. Combining these two estimates, we therefore obtain that for all $r > 0$ and $g \in L^2(\mathbb{R}^d)$,

$$\begin{aligned} \|e^{-TH_r}g\|_{L^2(\mathbb{R}^d)}^2 &\leq \exp(KrL^2 \log(K/\gamma) - rT) \\ &\quad \times K \left(\frac{K}{\gamma}\right)^K \exp\left(\frac{KL^2 \log^2(K/\gamma)}{T/2}\right) \frac{2}{T} \int_0^{T/2} \|e^{-tH_r}g\|_{L^2(\omega)}^2 dt. \end{aligned}$$

Moreover, notice that we have

$$\sup_{r>0} \exp\left(KrL^2 \log(K/\gamma) - rT\right) < +\infty \iff T \geq T^* := KL^2 \log(K/\gamma).$$

The proof of the estimate (5.7) is therefore ended, and so is the one of Corollary 2.16. As a byproduct, we obtain the quantitative estimate

$$\|e^{T\Delta_G}g\|_{L^2(\mathbb{R}^{2d})}^2 \leq K \left(\frac{K}{\gamma}\right)^K \exp\left(\frac{KL^2 \log^2(K/\gamma)}{T/2}\right) \frac{2}{T} \int_0^{T/2} \|e^{t\Delta_G}g\|_{L^2(\omega \times \mathbb{R}^d)}^2 dt. \quad \square$$

5.3. Proof of Corollary 2.19. Let us now explain how the proof of Corollary 2.16 can be adapted to prove Corollary 2.19. We first observe that after passing to the Fourier side with respect to $y \in \mathbb{T}^d$, the Baouendi-Grushin operator is transformed as

$$\Delta_x + |x|^{2k} \Delta_y \rightsquigarrow \Delta_x - |n|^2 |x|^{2k},$$

where $n \in \mathbb{Z}^d$ is the dual variable of $y \in \mathbb{T}^d$. Analogously to Section 4.1, this motivates to introduce the anharmonic oscillator $H_{k,r}$ in $L^2(\mathbb{R}^d)$ with variably scaled potential,

$$H_{k,r} := -\Delta_x - r^2 |x|^{2k}, \quad r \geq 0.$$

A corresponding variant of (4.2) reads

$$(5.8) \quad (M_{k,r})^* (H_{k,r})^s M_{k,r} = r^{\frac{2s}{k+1}} (H_k)^s, \quad r, s > 0,$$

with $H_k = H_{k,1}$ and where the unitary transformation $M_{k,r}$ in $L^2(\mathbb{R}^d)$ is given by

$$(5.9) \quad M_{k,r}g = r^{\frac{d}{2(k+1)}} g(r^{\frac{1}{k+1}} \cdot), \quad g \in L^2(\mathbb{R}^d).$$

We therefore need to prove observability estimates for the operators $H_{k,r}$ similarly to Proposition 5.4.

Proposition 5.5. *Let $s > 1/2$. Then, there exists a constant $K > 0$, depending only on k, s , and the dimension d , such that for all (γ, L) -thick sets $\omega \subset \mathbb{R}^d$, $r \geq 1$, $T > 0$, and $g \in L^2(\mathbb{R}^d)$, we have*

$$\|e^{-T(H_{k,r})^s}g\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{C_{\text{obs}}}{T} \int_0^T \|e^{-t(H_{k,r})^s}g\|_{L^2(\omega)}^2 dt,$$

where the positive constant $C_{\text{obs}} > 0$ is given by

$$C_{\text{obs}} = K \left(\frac{K}{\gamma}\right)^{K(1+rL^{1+k})} \exp\left(\frac{K((1+L) \log(K/\gamma))^{\frac{2s}{2s-1}}}{T^{\frac{1}{2s-1}}}\right).$$

Proof. It follows from (2.8) with $a = 0$ and $\delta = 0$ that for every (γ, L) -thick set $\omega \subset \mathbb{R}^d$, we have

$$(5.10) \quad \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \left(\frac{C}{\gamma}\right)^{C(1+L^{1+k}+L\sqrt{\lambda}+\log(1+\lambda))} \|f\|_{L^2(\omega)}^2, \quad f \in \mathcal{E}_{\lambda,k,1}, \quad \lambda \geq 0,$$

with a constant $C > 0$ depending only on k and the dimension d .

Let us now fix some $r \geq 1$ and a (γ, L) -thick set $\omega \subset \mathbb{R}^d$. In light of the similarity relation (5.8), we clearly have

$$(M_{k,r})^* \mathcal{E}_\lambda(H_{k,r}^s) \subset \mathcal{E}_\lambda(r^{\frac{2s}{k+1}} H_k^s) = \mathcal{E}_{\lambda/r^{2s/(k+1)}}(H_k^s) = \mathcal{E}_{\lambda^{1/s}/r^{2/(k+1),k,1}}.$$

Moreover, one easily checks that the set $\tilde{\omega} := r^{1/(k+1)}\omega$ is $(\gamma, r^{1/(k+1)}L)$ -thick. We therefore deduce from (5.10) that for all $\lambda \geq 0$ and $f \in \mathcal{E}_\lambda(H_{k,r}^s)$, we have

$$\begin{aligned} \|f\|_{L^2(\mathbb{R}^d)}^2 &= \|(M_{k,r})^* f\|_{L^2(\mathbb{R}^d)}^2 \leq \left(\frac{C}{\gamma}\right)^{C(1+rL^{1+k}+L\lambda^{\frac{1}{2s}}+\log(1+r^{-\frac{2}{k+1}}\lambda^{\frac{1}{s}}))} \|(M_{k,r})^* f\|_{L^2(\tilde{\omega})}^2 \\ &\leq \left(\frac{C}{\gamma}\right)^{C(1+rL^{1+k}+(1+L)\lambda^{\frac{1}{2s}})} \|f\|_{L^2(\omega)}^2, \end{aligned}$$

since $r \geq 1$ and, thus, $\log(1+r^{-\frac{2}{k+1}}\lambda^{\frac{1}{s}}) \leq \log(1+\lambda^{\frac{1}{s}}) \leq \lambda^{\frac{1}{2s}}$. The rest of the proof is then analogous to the one of Proposition 5.4. \square

Proof of Corollary 2.19. By mimicking the proof of Corollary 2.16, while using Fourier series instead of the Fourier transform and writing $H_{k,n} = H_{k,|n|}$, $n \in \mathbb{Z}^d$, we see that we need to derive observability estimates of the form

$$\|e^{-T(H_{k,n})^s} g\|_{L^2(\mathbb{R}^d)}^2 \leq C_{\omega,T} \int_0^T \|e^{-t(H_{k,n})^s} g\|_{L^2(\omega)}^2 dt, \quad g \in L^2(\mathbb{R}^d), \quad n \in \mathbb{Z}^d,$$

with a constant $C_{\omega,T} > 0$ not depending on n . Here, for $n = 0$, the operator $(H_{k,0})^s$ reduces to the fractional Laplacian $(-\Delta_x)^s$ on \mathbb{R}^d . Corresponding observability estimates from thick sets are well known in the literature, see, e.g., [39, Theorem 4.10] or [4, Theorem 1.12]. It is therefore sufficient to focus on the case $|n| \geq 1$. In light of (5.8), we have

$$\|e^{-tH_{k,n}^s} g\|_{L^2(\mathbb{R}^d)} \leq e^{-t\lambda_k^s |n|^{\frac{2s}{k+1}}} \|g\|_{L^2(\mathbb{R}^d)}, \quad g \in L^2(\mathbb{R}^d), \quad t \geq 0,$$

where $\lambda_k > 0$ is again the smallest eigenvalue of the anharmonic oscillator H_k . Using the latter together with Proposition 5.5, the rest of the reasoning of the proof of Corollary 2.16 carries over, provided that

$$\sup_{|n| \geq 1} \exp(K|n|L^{1+k} \log(K/\gamma) - |n|^{\frac{2s}{1+k}} \lambda_k^s T) < +\infty.$$

The latter is the case for every $T > 0$ if $s > (k+1)/2$, which proves part (i), and if $s = (k+1)/2$, it requires

$$T \geq K\lambda_k^{-s} L^{k+1} \log(K/\gamma),$$

as claimed in part (ii). \square

Remark 5.6. It is interesting to note that the term $\log(1+\lambda^{1/s}r^{-\frac{2s}{k+1}})$ in the proof of Proposition 5.5 explodes as $r \rightarrow 0^+$. While this does not pose a problem in the context of Corollary 2.19 since it then suffices to consider only the discrete values $r = |n|$, $n \in \mathbb{Z}^d \setminus \{0\}$, it is the very reason why we can not prove the statement (ii) in Corollary 2.19 for the Baouendi-Grushin equation on \mathbb{R}^{2d} when $k \geq 2$. However, if this log term was skipped, then a proof analogous to the one of Corollary 2.19 would lead to a corresponding result for the Baouendi-Grushin equation on \mathbb{R}^{2d} with $k \geq 2$.

5.4. Null-controllability in low diffusion. We finish this section with some calculations regarding Example 2.21.

Lemma 5.7. *Let $\omega_L = B(0, L)^c \times \mathbb{T}^d \subset \mathbb{R}^d \times \mathbb{T}^d$ with some $L > 0$.*

- (i) *If $0 < s < (1+k)/2$, then the equation $(E_{k,s})$ is at no time $T > 0$ exactly null-controllable from the control support ω_L .*
- (ii) *Let $s = (1+k)/2$. If the equation $(E_{k,s})$ is exactly null-controllable from the control support ω_L at some time $T > 0$, then there exists a constant $K > 0$, independent of L , such that*

$$T \geq K \left(\frac{L}{\sqrt{\lambda_k}} \right)^{k+1},$$

where λ_k is the smallest eigenvalue of the anharmonic oscillator $H_{k,1}$.

Proof. Let $T > 0$ be a fixed time. Assume that there exists a positive constant $C_{\omega_L, T} > 0$ such that for all functions $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$, we have the observability estimate

$$(5.11) \quad \|e^{-T(-\Delta_k)^s} g\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2 \leq C_{\omega_L, T} \int_0^T \|e^{-t(-\Delta_k)^s} g\|_{L^2(\omega_L)}^2 dt.$$

Let $\psi_k \in L^2(\mathbb{R}^d)$ be a normalized eigenfunction for the anharmonic oscillator $H_{k,1}$ corresponding to the smallest eigenvalue $\lambda_k > 0$. For each $n \in \mathbb{Z}^d \setminus \{0\}$, consider the function $g_n \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ given by

$$(5.12) \quad g_n(x, y) = |n|^{\frac{d}{2(k+1)}} e^{in \cdot y} \psi_k(|n|^{\frac{1}{k+1}} x), \quad (x, y) \in \mathbb{R}^d \times \mathbb{T}^d.$$

In light of the similarity relation (5.8) with the isometry $M_{k,n}$ from (5.9), it is clear that

$$(5.13) \quad \|g_n\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} = 1 \quad \text{and} \quad \|g_n\|_{L^2(\omega_L)} = \|\psi_k\|_{L^2(|n|^{1/(k+1)}\omega_L)}$$

and also that $(-\Delta_k)g_n = |n|^{\frac{2}{k+1}} \lambda_k g_n$. The latter and the observability estimate (5.11) with $g = g_n$ then imply that

$$(5.14) \quad e^{-2|n|^{\frac{2s}{1+k}} \lambda_k^s T} \leq C_{\omega_L, T} \int_0^T e^{-2|n|^{\frac{2s}{1+k}} \lambda_k^s t} \|\psi_k\|_{L^2(|n|^{1/(k+1)}\omega_L)}^2 dt \\ \leq TC_{\omega_L, T} \|\psi_k\|_{L^2(|n|^{1/(k+1)}\omega_L)}^2.$$

Recall from [3, Theorem 2.1], cf. also the proof of Proposition 3.2, that there are constants $c, c_1, c_2 > 0$, only depending on k and the dimension d , such that

$$\|e^{c|x|^{1+k}} \psi_k\|_{L^2(\mathbb{R}^d)} \leq c_1 e^{c_2 \lambda_k^{\frac{1}{2} + \frac{1}{2k}}}.$$

Thus, for each $n \in \mathbb{Z}^d \setminus \{0\}$ we have

$$(5.15) \quad \|\psi_k\|_{L^2(|n|^{1/(k+1)}\omega_L)} = \|e^{-c|x|^{1+k}} e^{c|x|^{1+k}} \psi_k\|_{L^2(|n|^{1/(k+1)}\omega_L)} \leq c_1 e^{-c|n|L^{1+k}} e^{c_2 \lambda_k^{\frac{1}{2} + \frac{1}{2k}}}.$$

Inserting the latter into (5.14), we deduce for each $n \in \mathbb{Z}^d \setminus \{0\}$ that

$$(5.16) \quad 1 \leq c_1^2 TC_{\omega_L, T} e^{2c_2 \lambda_k^{\frac{1}{2} + \frac{1}{2k}}} \exp(2|n|^{\frac{2s}{1+k}} \lambda_k^s T - 2c|n|L^{1+k}).$$

Now, if $0 < s < (1+k)/2$ or if $s = (1+k)/2$ and $0 < T < c(L/\sqrt{\lambda_k})^{k+1}$, then

$$\exp(2|n|^{\frac{2s}{1+k}} \lambda_k^s T - 2c|n|L^{1+k}) \xrightarrow{|n| \rightarrow +\infty} 0,$$

which contradicts the estimate (5.16). This ends the proof of Lemma 5.7. \square

Remark 5.8. It is worth to note that the Schrödinger-type equation corresponding to the fractional Baouendi-Grushin operator, that is, $(E_{k,s})$ with $(-\Delta_k)^s$ replaced by $i(-\Delta_k)^s$, is for no $s > 0$ and at no time $T > 0$ exactly null-controllable from the control support ω_L . Indeed, assume to the contrary that there exists a positive constant $C_{\omega_L, T} > 0$ such that for all $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$, we have the observability estimate

$$(5.17) \quad \|g\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2 \leq C_{\omega_L, T} \int_0^T \|e^{it(-\Delta_k)^s} g\|_{L^2(\omega_L)}^2 dt.$$

Inserting again the function g_n defined in (5.12) and using the estimate (5.15), we deduce that for all $n \in \mathbb{Z}^d \setminus \{0\}$, we have

$$1 \leq C_{\omega_L, T} T c_1 e^{-c|n|L^{1+k}} e^{c_2 \lambda_k^{\frac{1}{2} + \frac{1}{2k}}} \xrightarrow{|n| \rightarrow +\infty} 0.$$

Hence, the estimate (5.17) can never hold for all $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ simultaneously.

REFERENCES

- [1] D. ALLONSIUS, F. BOYER & M. MORANCEY, *Analysis of the null controllability of degenerate parabolic systems of Grushin type via the moments method*, J. Evol. Equ. **21** (2021), 4799–4843.
- [2] P. ALPHONSE, *Unique continuation estimates for the Baouendi-Grushin equation on unbounded domains*, work in progress.
- [3] P. ALPHONSE, *Null-controllability of evolution equations associated with fractional Shubin operators through quantitative Agmon estimates*, accepted for publication in Annales de l’Institut Fourier (2022), [arXiv:2012.04374](https://arxiv.org/abs/2012.04374).
- [4] P. ALPHONSE & J. BERNIER, *Smoothing properties of fractional Ornstein-Uhlenbeck semigroups and null-controllability*, Bull. Sci. Math. **165** (2020), 102914, 52 pp.
- [5] K. BEAUCHARD, P. CANNARSA & R. GUGLIELMI, *Null controllability of Grushin-type operators in dimension two*, J. Eur. Math. Soc. (JEMS) **16** (2014), 67–101.
- [6] K. BEAUCHARD, J. DARDÉ & S. ERVEDOZA, *Minimal time issues for the observability of Grushin-type equations*, Ann. Inst. Fourier **70** (2020), 247–312.
- [7] K. BEAUCHARD, P. JAMING & K. PRAVDA-STAROV, *Spectral estimates for finite combinations of Hermite functions and null-controllability of hypoelliptic quadratic equations*, Studia Math. **260** (2021), 1–43.
- [8] K. BEAUCHARD, L. MILLER & M. MORANCEY, *2D Grushin-type equations: Minimal time and null controllable data*, J. Differ. Equations **259** (2015), 5813–5845.
- [9] K. BEAUCHARD & K. PRAVDA-STAROV, *Null-controllability of hypoelliptic quadratic differential equations*, J. Éc. polytech. Math. **5** (2018), 1–43.
- [10] P. BOGGIATTO, E. BUZANO & L. RODINO, *Global Hypoellipticity and Spectral Theory*, Mathematical Research, Vol. 92, Akademie Verlag, Berlin (1996).
- [11] N. BURQ & C. SUN, *Time optimal controllability and observability for Grushin Schrödinger equation*, Anal. PDE **15** (2022), 1487–1530.
- [12] M. CHATZAKOU, J. DELGADO & M. RUZHANSKY, *On a class of anharmonic oscillators*, J. Math. Pures Appl. **153** (2021), 1–29.
- [13] J. DARDÉ, A. KOENIG & J. ROYER, *Null-controllability properties of the generalized two-dimensional Baouendi-Grushin equation with non-rectangular control sets*, preprint (2022), [arXiv:2207.03313](https://arxiv.org/abs/2207.03313).
- [14] A. DICKE, *Spectral Inequalities for Schrödinger Operators and Parabolic Observability*, Dissertation, Technische Universität Dortmund (2022).
- [15] A. DICKE & A. SEELMANN, *Uncertainty principles with error term in Gelfand-Shilov spaces*, Arch. Math. **119** (2022), 413–425.
- [16] A. DICKE, A. SEELMANN & I. VESELIĆ, *Uncertainty principle for Hermite functions and null-controllability with sensor sets of decaying density*, accepted for publication in J. Fourier Anal. Appl., preprint (2022), [arXiv:2201.11703](https://arxiv.org/abs/2201.11703).
- [17] A. DICKE, A. SEELMANN & I. VESELIĆ, *Spectral inequality with sensor sets of decaying density for Schrödinger operators with power growth potentials*, preprint (2022), [arXiv:2206.08682](https://arxiv.org/abs/2206.08682).
- [18] H. DONNELLY & C. FEFFERMAN, *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math. **93** (1988), 161–183.
- [19] M. DUPREZ & A. KOENIG, *Control of the Grushin equation: non-rectangular control region and minimal time*, ESAIM Control Optim. Calc. Var. **26** (2020), 3, 18pp.
- [20] T. DUYSKAERTS & L. MILLER, *Resolvent conditions for the control of parabolic equations*, J. Funct. Anal. **11** (2012), 3641–3673.
- [21] M. EGIDI & A. SEELMANN, *An abstract Logvinenko-Sereda type theorem for spectral subspaces*, J. Math. Anal. Appl. **500** (2021), 125149.
- [22] V. HAVIN & B. JÖRICKE, *The Uncertainty Principle in Harmonic Analysis*, Ergeb. Math. Grenzgeb. (3), Vol. 28, Springer, Berlin, 1994.
- [23] A. KOENIG, *Contrôlabilité de quelques équations aux dérivées partielles paraboliques peu diffusives*, Ph.D thesis, Université Côte d’Azur (2019).
- [24] A. KOENIG, *Non-null-controllability of the Grushin operator in 2D*, C. R. Math. Acad. Sci. Paris **355** (2017), 1215–1235.
- [25] O. KOVRIJKINE, *Some estimates of Fourier transforms*, ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)—California Institute of Technology.
- [26] O. KOVRIJKINE, *Some results related to the Logvinenko-Sereda Theorem*, Proc. Amer. Math. Soc. **129** (2001) 3037–3047.
- [27] C. LAURENT & M. LÉAUTAUD, *Observability of the heat equation, geometric constants in control theory, and a conjecture of Luc Miller*, Anal. PDE **14** (2021), 355–423.
- [28] J. LE ROUSSEAU & G. LEBEAU, *On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations*, ESAIM Control Optim. Calc. Var. **18** (2012), 712–747.
- [29] C. LETROUIT & C.M. SUN, *Observability of Baouendi-Grushin type equations through resolvent estimates*, J. Inst. Math. Jussieu (2021).

- [30] P. LISSY, *A non-controllability result for the half-heat equation on the whole line based on the prolate spheroidal wave functions and its application to the Grushin equation*, preprint (2020), [hal-02420212](#).
- [31] A. LOGUNOV & E. MALINNIKOVA, *Lecture notes on quantitative unique continuation for solutions of second order elliptic equations*, In: Harmonic analysis and applications, IAS/Park City Math. Ser., Vol. 27, Amer. Math. Soc., [Providence], RI, 2020, pp. 1–34.
- [32] J. MARTIN, *Conditions géométriques pour la contrôlabilité d'équations aux dérivées partielles linéaires*, Ph.D Thesis, Université de Rennes 1 (2022).
- [33] J. MARTIN, *Spectral inequalities for anisotropic Shubin operators*, preprint (2022), [arXiv:2205.11868](#).
- [34] J. MARTIN, *Uncertainty principles in Gelfand-Shilov spaces and null-controllability*, J. Funct. Anal. **283** (2022), 109619.
- [35] J. MARTIN & K. PRAVDA-STAROV, *Spectral inequalities for combinations of Hermite functions and null-controllability for evolution equations enjoying Gelfand-Shilov smoothing effects*, J. Inst. Math. Jussieu (2022), 1–50.
- [36] J. MARTIN & K. PRAVDA-STAROV, *Geometric conditions for the exact controllability of fractional free and harmonic Schrödinger equations*, J. Evol. Equ. **21** (2021), 1059–1087.
- [37] P. MATTILA, *Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability*, Cambridge Stud. Adv. Math., Vol. 44, Cambridge University Press, Cambridge (1999).
- [38] L. MILLER, *Unique continuation estimates for sums of semiclassical eigenfunctions and null-controllability from cones*, preprint (2008), [hal-00411840](#).
- [39] I. NAKIĆ, M. TÄUFER, M. TAUTENHAN & I. VESELIĆ, *Sharp estimates and homogenization of the control cost of the heat equation on large domains*, ESAIM, Control Optim. Calc. Var. **26** (2020), 26 pp.
- [40] K. SCHMÜDGEN, *Unbounded Self-Adjoint Operators on Hilbert Space*, Grad. Texts in Math., Vol. 265, Springer, Dordrecht (2012).
- [41] P. STOLLMANN & G. STOLZ, *Lower bounds for Dirichlet Laplacians and uncertainty principles*, J. Eur. Math. Soc. (JEMS) **23** (2021), 2337–2360.

(PAUL ALPHONSE) UNIVERSITÉ DE LYON, ENSL, UMPA - UMR 5669, F-69364 LYON
Email address: paul.alphonse@ens-lyon.fr

(ALBRECHT SEELMANN) TECHNISCHE UNIVERSITÄT DORTMUND, GERMANY
Email address: albrecht.seelmann@mathematik.tu-dortmund.de