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CRITICAL POINTS OF ARBITRARY ENERGY FOR THE TRUDINGER-MOSER EMBEDDING IN PLANAR DOMAINS

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Abstract. Given a smoothly bounded non-contractible domain $\Omega \subset \mathbb{R}^2$, we prove the existence of positive critical points of the Trudinger-Moser embedding for arbitrary Dirichlet energies. This is done via degree theory, sharp compactness estimates and a topological argument relying on the Poincaré-Hopf theorem.

1. Introduction

Given a smoothly bounded domain $\Omega \subset \mathbb{R}^2$, i.e. a bounded domain with smooth boundary, let $H_0^1 = H_0^1(\Omega)$ be the usual Sobolev space of functions with zero trace on $\partial\Omega$ endowed with the norm $\|\cdot\|_{H_0^1}$ given by

$$\|u\|_{H_0^1}^2 = \int_\Omega |\nabla u|^2 dx.$$

Given any positive real number $\beta > 0$, our main purpose is to discuss the existence of a nonnegative function, critical point of the classical [5, 19, 20, 24, 32, 34, 37, 39] Moser-Trudinger functional

$$F(u) := \int_\Omega \left(e^{u^2} - 1\right) dx\,, \tag{1.1}$$

constrained to the submanifold

$$\mathcal{M}_\beta := \left\{ v \in H_0^1 \ : \ \|v\|_{H_0^1}^2 = \beta \right\} \tag{1.2}$$

of H_0^1. This is equivalent to finding a solution of the following elliptic problem

$$\left\{ \begin{array}{l} \Delta u = 2\lambda u e^{u^2}\,,\ u > 0 \ \text{ in } \Omega\,, \\ u = 0 \text{ on } \partial\Omega\,, \\ \int_\Omega |\nabla u|^2 dx = \beta\,, \end{array} \right. \tag{$\mathcal{E}_\beta$}$$

for some real number λ, where $\Delta = -\partial_{xx} - \partial_{yy}$ is the (nonnegative) Laplacian. Our main result reads as follows (see also Remark 2.5 below):

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded non-contractible domain. Then, given any positive real number $\beta > 0$, there exists a nonnegative function u, critical point of F constrained to $\mathcal{M}_\beta \subset H_0^1$. In particular, u is smooth and solves $(\mathcal{E}_\beta)$.*

Theorem 1.1 eventually gives a fairly complete answer to a long-standing open problem. Indeed the existence of critical points of $F|_{\mathcal{M}_\beta}$ in the supercritical regime $\beta > 4\pi$ is quite subtle, due to the critical nature of the nonlinearity which, in addition to the failure of the Palais-Smale condition, makes the blow-up estimates very delicate. For $\beta \leq 4\pi$ maximizers are known to exist thanks to the celebrated results of Carleson-Chang [5], Struwe [37] and Flucher [19]. Moreover, still in [37],

Struwe proved the existence of $\varepsilon_0 > 0$ such that $F|_{\mathcal{M}_\beta}$ has a positive local maximizer for every $\beta \in (4\pi, 4\pi+\varepsilon_0)$ and a mountain pass-type positive critical point for *almost every* $\beta \in (4\pi, 4\pi + \varepsilon_0)$. The existence of a second positive critical point for *every* $\beta \in (4\pi, 4\pi + \varepsilon_0)$ was proven by Lamm-Robert-Struwe [25] using a flow approach. In these results, there is no condition on the smoothly bounded domain Ω, but on the other hand, the requirement that β is close to 4π is essential. More recently, via a Lyapunov-Schmidt construction, Del Pino-Musso-Ruf [13] studied the case in which Ω is not simply connected and β is close to 8π. That the topology of Ω plays an important role in the existence of critical points of $F|_{\mathcal{M}_\beta}$ is also suggested by the result of the first and second authors [28], showing that when Ω is a disk $F|_{\mathcal{M}_\beta}$ has no positive critical point for β sufficiently large.

The case in which the domain Ω is replaced by a two-dimensional closed surface was recently studied in [11] using the *barycenter technique* of Djadli-Malchiodi [16] (see also Bahri [2]) taking advantage of the variational structure, a minmax argument together with Struwe's monotonicity trick to obtain solutions of a *subcritical* approximate problem for almost every $\beta > 0$, and blow-up analysis together with energy estimates at critical energy levels to conclude. Here instead, in order to prove Theorem 1.1, we will use the Leray-Schauder degree, together with a deformation argument based on compactness to link our problem to the *Mean-Field* equation

$$\begin{cases} \Delta u = \beta \frac{2e^u}{\int_\Omega e^u dx}\,, \\ u = 0 \text{ on } \partial\Omega\,, \end{cases} \tag{$\mathcal{E}^{MF}_\beta$}$$

studied e.g. in Ding-Jost-Li-Wang [15], and for which a degree theory approach has been proposed by Chen and Lin [7, 8]. In particular, the purpose of Section 3 is to give an argument to get a degree formula for a gradient vector field, proven by [8] to encode the so-called *finite dimensional reduction* of ($\mathcal{E}^{MF}_\beta$) (see Proposition 2.3 below).

In order to deform Problem ($\mathcal{E}_\beta$) into a mean-field type equation, given $\beta > 0$, we consider the following p-dependent family of problems:

$$\begin{cases} \Delta u = p\lambda u^{p-1} e^{u^p}\,,\ u > 0 \ \text{ in } \Omega\,, \\ u = 0 \text{ on } \partial\Omega\,, \\ \frac{\lambda p^2}{2} \left(\int_\Omega \left(e^{u^p} - 1\right) dx\right)^{\frac{2-p}{p}} \left(\int_\Omega u^p e^{u^p} dx\right)^{\frac{2(p-1)}{p}} = \beta\,, \end{cases} \tag{$\mathcal{E}_{p,\beta}$}$$

parametrized by $p \in [1, 2]$. First, as seen when integrating by parts, solving ($\mathcal{E}_{p,\beta}$) for $p = 2$ is equivalent to solving ($\mathcal{E}_\beta$). Moreover, ($\mathcal{E}_{p,\beta}$) still has a nice variational structure for $p \in [1, 2)$ and $\beta > 0$ (see [11]). At last, any nonnegative nontrivial weak H^1_0-solution of the PDE in ($\mathcal{E}_{p,\beta}$) is eventually smooth and positive in Ω, as proven from Trudinger [39] with standard elliptic theory (see for instance Gilbarg-Trudinger [21]).

Let $\mathbb{N}^\star$ be the set of the positive integers and $\mathbb{N}$ the set of the nonnegative ones. The Leray-Schauder degree computation, which is a by-product of our strategy to solve ($\mathcal{E}_\beta$), is given in the following result:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain. Let $p \in (1, 2]$, $N \in \mathbb{N}$ and $\beta \in (4\pi N, 4\pi(N + 1))$ be given. Then the total Leray-Schauder degree $d_{p,\beta}(\Omega)$ of the solutions of ($\mathcal{E}_{p,\beta}$) is well-defined and equals the binomial number $\binom{N-\chi(\Omega)}{N}$,*

where $\chi(\Omega)$ stands for the Euler characteristic of Ω. In particular, if either Ω is not simply connected or if $N = 0$, we have $d_{p,\beta}(\Omega) \neq 0$, so that the set $\mathcal{C}_{p,\beta}(\Omega)$ of the solutions of $(\mathcal{E}_{p,\beta})$ is not empty.

We adopt here the usual convention that $\binom{-1}{0} = 1$ and $\binom{N-1}{N} = 0$ for $N \in \mathbb{N}^\star$. We recall that the genus $g(\Omega) := 1 - \chi(\Omega)$ is basically the *"number of holes"* inside Ω, so that Ω is simply connected if and only if $\chi(\Omega) = 1$.

Interestingly enough, given $p \in [1, 2)$, our topological assumption on Ω is sharp in the following sense: $(\mathcal{E}_{p,\beta})$ does not have any solution for β sufficiently large, as soon as Ω is *simply connected* (see [3, 4]). Yet, for $p = 2$, we state the following delicate question which looks open, except in the aforementioned case of the disk:

Open problem. *Let $\Omega \subset \mathbb{R}^2$ be a* simply connected *domain. Does there exists $\beta^\sharp > 4\pi$ such that $(\mathcal{E}_\beta)$ does not have any solution for $\beta > \beta^\sharp$?*

To conclude this introductory part, we mention that the degree formula in Theorem 1.2 also implies the existence of blowing-up solutions for $(\mathcal{E}_{p,\beta})$ (see Remark 2.4).

2. Proof of Theorems 1.1 and 1.2

As a first ingredient, since the nonlinearities in the right-hand of the PDE in $(\mathcal{E}_{p,\beta})$ are autonomous, we may use the following by-product of the moving plane technique, already used for instance by Adimurthi and Druet [1] in a similar context:

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain. Then there exists $\delta, \delta' > 0$ depending only on Ω such that, for any nonnegative, nondecreasing C^1-function $f : [0, +\infty) \to \mathbb{R}$, for all v solving*

$$\begin{cases} \Delta v = f(v), v > 0 \text{ in } \Omega\,, \\ v = 0 \text{ on } \partial\Omega\,, \end{cases}$$

and all $x \in \partial\Omega$, the function $t \mapsto u(x - t\nu(x))$ has positive derivative in $[0, \delta]$, where $\nu(x)$ is the unit outward normal to $\partial\Omega$ at x, and we have

$$\nabla v(x) = 0 \quad \Longrightarrow \quad d(x, \partial\Omega) \geq \delta' > 0\,. \tag{2.1}$$

Proof of Proposition 2.1 (completed). The existence of $\delta > 0$ as in the statement of the proposition is a direct consequence of the classical moving plane argument in de Figueiredo-Lions-Nussbaum [10] if Ω is convex, while it has to be combined with Kelvin's transform in our general two-dimensional case, as observed by Han [22]. Then, since the map $(t, x) \mapsto x - t\nu(x)$ is a diffeomorphism from $[0, \delta] \times \partial\Omega$ onto a neighborhood of $\partial\Omega$ inside $\bar{\Omega}$, up to reducing $\delta > 0$, (2.1) follows by compactness of $\partial\Omega$. □

Proposition 2.1 is an important ingredient to take advantage of the blow-up analysis developed to get [11, Theorem 4.1] on a closed surface in our present setting of a bounded domain Ω, ensuring that no new phenomena arise close to the boundary. Hence we have the following theorem:

Theorem 2.1. *Let $(\lambda_\varepsilon)_\varepsilon$ be any sequence of positive real numbers and $(p_\varepsilon)_\varepsilon$ be any sequence of numbers in $[1, 2]$. Let $(u_\varepsilon)_\varepsilon$ be a sequence of smooth functions solving*

$$\begin{cases} \Delta u_\varepsilon = p_\varepsilon \lambda_\varepsilon u_\varepsilon^{p_\varepsilon - 1} e^{u_\varepsilon^{p_\varepsilon}}\,, \quad u_\varepsilon > 0 \text{ in } \Omega\,, \\ u_\varepsilon = 0 \quad \text{ on } \partial\Omega\,, \end{cases} \tag{2.2}$$

for all ε. Let β_ε be given by

$$\beta_\varepsilon = \frac{\lambda_\varepsilon p_\varepsilon^2}{2}\left(\int_\Omega \left(e^{u_\varepsilon^{p_\varepsilon}}-1\right)\,dx\right)^{\frac{2-p_\varepsilon}{p_\varepsilon}}\left(\int_\Omega u_\varepsilon^{p_\varepsilon}e^{u_\varepsilon^{p_\varepsilon}}\,dx\right)^{\frac{2(p_\varepsilon-1)}{p_\varepsilon}}$$

for all ε. If we assume the energy bound

$$\lim_{\varepsilon\to 0}\beta_\varepsilon = \beta \in [0,+\infty)\,,$$

but the pointwise blow-up of the u_ε's, namely $\lim_{\varepsilon\to 0}\max_\Omega u_\varepsilon = +\infty$, then $\lambda_\varepsilon \to 0^+$ and there exists an integer $N\in\mathbb{N}^\star$ such that $\beta = 4\pi N$.

Independently, multiplying the equation in $(\mathcal{E}_{p,\beta})$ by a first Dirichlet eigenfunction v_1 of the Laplacian, integrating by parts and using $pt^{p-1}e^{t^p}\geq 2t$ for all $t\geq 0$ and $p\in[1,2]$, we get

$$0<2\lambda\leq\lambda_1 \tag{2.3}$$

as soon as a (nontrivial) solution u of $(\mathcal{E}_{p,\beta})$ exists.

Outline of the proof of Theorem 2.1. Overall, the proof follows closely [11, Sections 2-4]. It is even simpler in the present framework where Dirichlet boundary conditions are imposed, since we no longer need to handle the additional coercivity term hu_ε in the left-hand $\Delta_g u_\varepsilon + hu_\varepsilon$ of the equation in [11, equation (4.1)] analogue to (2.2) here: indeed, all along the argument of [11], this additional term is just estimated by error terms which can be taken now to be zero.

More precisely, as pioneered by Druet [17, Sections 3-4] when $p_\varepsilon = 2$ for all ε (see also [18, Section 3]), we start with the following proposition giving a first pointwise exhaustion of the concentration points, as well as weak gradient estimates under the assumptions of Theorem 2.1:

Proposition 2.2. *Up to a subsequence, there exist an integer $n\in\mathbb{N}^\star$ and sequences $(x_{i,\varepsilon})_\varepsilon$ of points in Ω such that $\nabla u_\varepsilon(x_{i,\varepsilon})=0$, such that, setting $\gamma_{i,\varepsilon}:=u_\varepsilon(x_{i,\varepsilon})$,*

$$\mu_{i,\varepsilon} := \left(\frac{8}{\lambda_\varepsilon p_\varepsilon^2\gamma_{i,\varepsilon}^{2(p_\varepsilon-1)}e^{\gamma_{i,\varepsilon}^{p_\varepsilon}}}\right)^{\frac{1}{2}}\to 0\,, \tag{2.4}$$

and such that

$$\frac{p_\varepsilon}{2}\gamma_{i,\varepsilon}^{p_\varepsilon-1}(\gamma_{i,\varepsilon}-u_\varepsilon(x_{i,\varepsilon}+\mu_{i,\varepsilon}\cdot))\to T_0:=\ln\left(1+|\cdot|^2\right)\ \text{ in } C^1_{loc}(\mathbb{R}^2)\,, \tag{2.5}$$

as $\varepsilon\to 0$, for all $i\in\{1,...,n\}$. Moreover, there exist $C>0$ such that we have

$$\min_{i\in\{1,...,n\}} u_\varepsilon^{p_\varepsilon-1}|x_{i,\varepsilon}-\cdot||\nabla u_\varepsilon|\leq C \text{ in } \Omega$$

for all ε. We also have that $\lim_{\varepsilon\to 0}x_{i,\varepsilon}=x_i$ for all i, and that there exists $u_0\in C^2(\bar\Omega\backslash\mathcal{S})$ such that

$$\lim_{\varepsilon\to 0}u_\varepsilon = u_0 \text{ in } C^1_{loc}(\bar\Omega\backslash\mathcal{S})\,,$$

where $\mathcal{S}\subset\Omega$ consists of all the x_i's.

About the concentration points $x_{i,\varepsilon}$ of this proposition, being critical points of the u_ε's by construction, (2.1) yields that they *cannot collapse to the boundary* so that proving Proposition 2.2 does not require new arguments with respect to the ones to get [11, Proposition 4.1]. In addition, (2.3) and (2.4) yield that $\gamma_{i,\varepsilon}\to+\infty$, so that (2.5) really points out a concentration profile at first order for the u_ε's.

But at that stage, as in [11, Section 2], the strategy is to consider a radially symmetric much more precise ansatz solving the same equation (i.e. set now $h_0 \equiv 0$ in [11, (2.6)]), whose relevance in the non-radial setting can be quantified (see [11, Section 3] picking now $h \equiv 0$ in [11, (3.8)]). Thus assumption [11, (2.5)] and the related [11, (3.6), (3.17)-(3.19), second requirement in (3.21), (4.26), Lemma 4.1] can be now ignored. The terms $w_{l,\varepsilon}$ in [11, (4.47)] and $\tilde{w}_\varepsilon$ in [11, (4.85)] used to control this additional linear term are also now useless and we eventually do get Theorem 2.1 by following the lines of [11, Sections 2-4], since no specific new difficulty may arise close to the boundary thanks to Propositions 2.1 and 2.2. As a last important remark, the lower bound on $d(x_{i,\varepsilon}, \partial\Omega)$ from Proposition 2.2 is also used to get the key property $\lambda_\varepsilon \to 0^+$, arguing as in [11, Step 4.2]. □

Remark 2.1. *As pointed out in* [31]*, the positivity of the u_ε's is essential to have energy quantization (i.e. $\beta \in 4\pi\mathbb{N}^\star$) in Theorem 2.1. Moreover, even restricting to nonnegative functions, we stress that there exist Palais-Smale sequences associated to $(\mathcal{E}_\beta)$ with arbitrary limiting energy $\beta > 4\pi$ and converging weakly to* 0 *in H_0^1 (see Costa-Tintarev* [9]*), so that not only the quantization, but even the quantification proved for solutions by Druet* [17] *fail for general Palais-Smale sequences (see also* [30] *for examples with non-zero weak limit entering in the framework of* [17]*, but where* quantization fails*). This is in striking constrast with a large class of otherwise closely related critical problems for which quantification already holds for Palais-Smale sequences (see for instance the pioneering work* [36] *by Struwe). This is maybe the clearest evidence of the huge difficulty to run directly standard variational techniques generating Palais-Smale sequences to solve $(\mathcal{E}_\beta)$, and motivates our compactness techniques considering only* exact solutions *instead of* Palais-Smale sequences*.*

In the specific case of the mean-field equation, i.e. when $p_\varepsilon = 1$ for all ε, with the same notation and assumptions as in Theorem 2.1, we obviously get

$$\beta_\varepsilon^{MF} := \frac{\lambda_\varepsilon}{2}\int_\Omega e^u dx = \beta_\varepsilon + \frac{\lambda_\varepsilon|\Omega|}{2} \to \beta = 4\pi N \in 4\pi\mathbb{N}^\star \tag{2.6}$$

as $\varepsilon \to 0$. At that stage, another additional ingredient for our proof is the following result stated here with the same notation and conventions as in Theorem 1.2:

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain and $h \in C^1(\bar{\Omega})$ be positive. Let $N \in \mathbb{N}$ and $\beta \in (4\pi N, 4\pi(N+1))$ be given. Then the total Leray-Schauder degree $d_{\beta,h}^{MF}(\Omega)$ of all the solutions of the mean-field equation*

$$\begin{cases} \Delta u = \beta \frac{2he^u}{\int_\Omega he^u dx}\,, \\ u = 0 \text{ on } \partial\Omega\,, \end{cases} \tag{$\mathcal{E}_{\beta,h}^{MF}$}$$

is well-defined and equals $\binom{N-\chi(\Omega)}{N}$.

As an obvious remark, a solution of $(\mathcal{E}_{\beta,h}^{MF})$ has to be positive in Ω. Theorem 2.2, which is only needed here for a constant function $h > 0$, has been stated first in Chen-Lin [8]. Indeed, these authors show in their beautiful work:

Proposition 2.3 (Chen-Lin [8])**.** *For all $N \in \mathbb{N}$ and all $\beta \in (4\pi N, 4\pi(N+1))$, $d_{\beta,h}^{MF}(\Omega)$ is equal to a constant $d_N^{MF}(\Omega)$, which is* 1 *for $N = 0$. Moreover, we have*

the following degree jump formula:

$$d^{MF}_{N+1}(\Omega) - d^{MF}_N(\Omega) = \frac{(-1)^N}{N!} \deg(\nabla\Phi_{N,h}, \Omega^N \setminus D_N, 0), \tag{2.7}$$

where

$$\Phi_{N,h}(x_1, \dots, x_N) = 8\pi \sum_{i\neq j} \mathcal{G}(x_i, x_j) + 4\pi \sum_{1\leq i\leq N} \mathcal{H}(x_i, x_i) + \sum_{1\leq i\leq N} \ln h(x_i) \tag{2.8}$$

is defined in $\Omega^N \setminus D_N$ *with* $D_N := \{(x_1, \dots, x_N) \in \Omega^N : \exists i \neq j\,,\ x_i = x_j\}$*, where* $\mathcal{G}$ *is the Green's function of* Δ *with zero Dirichlet boundary conditions on* Ω*, where* $\mathcal{H}$ *is the regular part of* $\mathcal{G}$ *normalized here as*

$$\mathcal{G}(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|} + \mathcal{H}(x, y), \tag{2.9}$$

and where $\deg(\nabla\Phi_{N,h}, \Omega^N \setminus D_N, 0)$ *is the Brouwer degree (or total index) of the vector field* $\nabla\Phi_{N,h}$*.*

Contrary to what occurs on a closed manifold, it turns out that, in order to compute the degree of $\nabla\Phi_{N,h}$, the classical Poincaré-Hopf formula does *not* directly apply in the present setting of a bounded domain since $\nabla\Phi_{N,h}$ points outwards only on a strict subset of the boundary of its domain, non-empty as soon as $N > 1$. Another goal of this paper is to provide an argument showing that, for topological reasons relying strongly on the fact that every connected component of $\partial\Omega$ has zero Euler characteristic in our two-dimensional case (crucial point in Lemma 3.5, see also Remark 2.2 just below), the following formula for the degree of $\nabla\Phi_{N,h}$ holds in the present setting of a smoothly bounded domain Ω, so that the degree formula in Theorem 2.2 follows from Propositions 2.3 and 2.4.

Remark 2.2. *It is known that any closed odd-dimensional manifold has zero Euler-characteristic, so that this aforementioned property for every connected boundary component of* $\partial\Omega$ *holds more generally for any smoothly bounded* $\Omega \subset \mathbb{R}^n$ *with* n even*. Indeed, with minimal changes, our proof extends to this case which is relevant for instance for the fourth-order mean-field equation in Lin-Wei-Wang* [27].

Proposition 2.4. *The degree of* $\nabla\Phi_{N,h}$ *is well-defined on* $\Omega^N \setminus D_N$ *and one has*

$$\deg(\nabla\Phi_{N,h}, \Omega^N \setminus D_N, 0) = \chi(\Omega)(\chi(\Omega) - 1)...(\chi(\Omega) - N + 1). \tag{2.10}$$

Proving Proposition 2.4 is the purpose of Section 3.

We are now in position to prove Theorems 1.1 and 1.2, starting with the specific case where $\beta > 0$ is out of the set $4\pi\mathbb{N}^\star$ of the critical energy levels.

Proof of the degree formula of Theorem 1.2 for $\beta \notin 4\pi\mathbb{N}^\star$*.* During this whole first part, we fix $N \in \mathbb{N}$ and $\beta \in (4\pi N, 4\pi(N + 1))$. For $t \in [0, 1]$ and $p \in [1, 2]$, let $T_{p,\beta}$ and $T^{MF}_{t,\beta}$ be the Fredholm-type nonlinear operators given for any non-negative non-zero function $v \in C^1(\bar\Omega)$ by

$$T_{p,\beta}(v) = v - \Delta^{-1}\left(\frac{2\beta v^{p-1}e^{v^p}}{p\left(\int_\Omega (e^{v^p} - 1)\,dx\right)^{\frac{2-p}{p}}\left(\int_\Omega v^p e^{v^p} dx\right)^{\frac{2(p-1)}{p}}}\right)$$

and

$$T^{MF}_{t,\beta}(v) = v - \Delta^{-1}\left(\frac{2\beta e^v}{\int_\Omega (e^u - t)dx}\right),$$

where $\Delta^{-1}f$ is the solution w of $\Delta w = f$ with zero Dirichlet boundary conditions. By Theorem 2.1 and formula (2.6) with standard elliptic theory and (2.3), there exist constants $C > c > 0$ such that

$$\|u\|_{C^1(\bar{\Omega})} < C \text{ and } u > cv_1 \text{ in } \Omega$$

for all u positive in Ω solving $T^{MF}_{t,\beta}(u) = 0$ for some $t \in [0,1]$, and all u solving $T_{p,\beta}(u) = 0$, i.e. $(\mathcal{E}_{p,\beta})$, for some $p \in [1,2]$, where v_1 is still a given first Dirichlet eigenfunction of Δ chosen positive in Ω. Let $C^1_0(\bar{\Omega})$ be the closed subspace of $C^1(\bar{\Omega})$ consisting of the functions vanishing on $\partial\Omega$. Let $\mathcal{V}$ be the open subset of $C^1_0(\bar{\Omega})$ given by

$$\mathcal{V} = \left\{ \quad v \in C^1_0(\bar{\Omega}) \quad \text{s.t.} \quad \begin{array}{l} v > cv_1 \text{ in } \Omega\,, \\ \|v\|_{C^1(\bar{\Omega})} < C \end{array} \right\}.$$

Using Hopf's lemma, observe that only positive functions in Ω are in the closure of $\mathcal{V}$ for the $C^1(\bar{\Omega})$-topology, in particular neither the zero nor any sign-changing function is in this closure. Then,

$$(p,v) \mapsto T_{p,\beta}(v) \text{ and } (t,v) \mapsto T^{MF}_{t,\beta}(v)$$

are continuous from $[1,2] \times \mathcal{V}$ and $[0,1] \times \mathcal{V}$ respectively into $C^1_0(\bar{\Omega})$, by standard elliptic theory. Observe in particular that this continuity does hold for $T_{p,\beta}$ up to $p = 1$ despite the jump of $(p,t) \in [1,2] \times (0+\infty) \mapsto pt^{p-1}e^{t^p}$ at $t = 0^+$ as $p \to 1$. Using also that these operators do not vanish on the boundary of $\mathcal{V}$ by construction, the total Leray-Schauder degree $d_{p,\beta}(\Omega) := \deg_{LS}(T_{p,\beta}, \mathcal{V}, 0)$ (resp. $d^{MF}_{t,\beta}(\Omega) := \deg_{LS}(T^{MF}_{t,\beta}, \mathcal{V}, 0)$) of all the solutions of $(\mathcal{E}_{p,\beta})$ (resp. of all the functions $v > 0$ in Ω such that $T^{MF}_{t,\beta}(v) = 0$) is well-defined, see for instance Nirenberg [33].

Remark 2.3. *It is convenient to work here within the space $C^1_0(\bar{\Omega})$, but one could ask whether the degree thus defined coincides with the one that one could define within the natural variational space H^1_0. The arguments in Li* [26, Appendix B] *slightly transposed to the present situation show that it is actually the case.*

Now, for $t = 0$, the formula for the total degree $d^{MF}_{0,\beta}(\Omega)$ of all the solutions of $(\mathcal{E}^{MF}_{\beta})$ is given by Theorem 2.2. Moreover, we clearly have $T_{1,\beta} = T^{MF}_{1,\beta}$, so that $d_{1,\beta}(\Omega) = d^{MF}_{1,\beta}(\Omega)$. Using then our definition of $\mathcal{V}$, Theorem 2.2 and the homotopy invariance of the total Leray-Schauder degree as far as compactness holds, we have $d_{p,\beta}(\Omega) = d_{1,\beta}(\Omega)$ for all $p \in [1,2]$ on the one hand, while $d^{MF}_{t,\beta}(\Omega) = d^{MF}_{0,\beta}(\Omega) = d^{MF}_{1,\beta}(\Omega)$ for all $t \in [0,1]$ on the other hand, which concludes the proof of Theorem 1.2, and thus that of Theorem 1.1, in this first case where $\beta \notin 4\pi\mathbb{N}^\star$. □

Remark 2.4. *Let $p \in [1,2]$ be fixed. A posteriori, by contraposing the homotopy invariance of the Leray-Schauder degree as far as compactness holds, observe that the effective degree jump of $d_{p,\beta}(\Omega)$ at the levels $\beta \in 4\pi\mathbb{N}^\star$, as soon as the genus $g(\Omega)$ is greater than 1, imposes the existence of a sequence of blowing-up solutions with limiting energy $\beta = 4\pi N$ for all $N \in \mathbb{N}^\star$ as in Theorem 2.1 with $p_\varepsilon = p$ (see also Deng-Musso* [14] *or del Pino-Musso-Ruf* [12, 13] *for specific constructions of such blowing-up solutions for $(\mathcal{E}_{p,\beta})$).*

In order to complete the proof of Theorems 1.1 and 1.2, it remains to treat the case $\beta \in 4\pi\mathbb{N}^\star$. This will be based on the estimate in the following theorem, saying roughly that for $p \in (1,2]$ the amount of Dirichlet energy near each blow-up point

approaches 4π *from above*. This estimate was first proven in the radial case and for $p = 2$ by Mancini and the second author [29], then extended to the case of a bounded domain in $\mathbb{R}^2$ by the third author [38] (see also Ibrahim-Masmoudi-Nakanishi-Sani [23] for related results), finally on a closed two-dimensional surface [11, Theorem 5.1]. Taking Proposition 2.1 into account, the proof of [11, Theorem 5.1] can be easily adapted to yield:

Theorem 2.3. *Assume $p_\varepsilon = p \in (1, 2]$ for all ε and let u_ε, λ_ε, $\beta_\varepsilon \to \beta = 4\pi N \in 4\pi\mathbb{N}^\star$ be as in Theorem 2.1. Then*

$$\beta_\varepsilon \geq 4\pi \left(N + \frac{4(p-1)(1+o(1))}{p^2} \sum_{1 \leq i \leq N} \frac{1}{\gamma_{i,\varepsilon}^{2p}} \right) \quad \text{as } \varepsilon \to 0 \tag{2.11}$$

where the sequence $(\gamma_{i,\varepsilon})_\varepsilon$ is given by Proposition 2.2 for all i.

Conclusion of the proof of Theorems 1.1 and 1.2. We fix now $p \in (1, 2]$. Then, from Theorem 2.3 we get that $\beta_\varepsilon > \beta$ for all ε small enough in Theorem 2.1. As a consequence, for any given $N \in \mathbb{N}$ and small $\eta \in (0, 1]$, we have compactness of all the solutions of $(\mathcal{E}_{p,\beta})$ for all $\beta \in [4\pi(N + \eta), 4\pi(N + 1)]$. In particular, we may define as above an open neighborhood $\mathcal{V}' \subset C^1_0(\bar\Omega)$ containing all these solutions and whose closure contains only positive functions in Ω. Then, the map $(\beta, v) \mapsto T_{p,\beta}(v)$ is continuous from $[4\pi(N + \eta), 4\pi(N + 1)] \times \mathcal{V}'$ to $C^1_0(\bar\Omega)$ and, by a deformation argument as above, we get that the total Leray-Schauder degree $\deg_{LS}(T_{p,\beta}, \mathcal{V}', 0)$ of all the solutions of $(\mathcal{E}_{p,\beta})$ is well-defined and does not depend on $\beta \in [4\pi(N + \eta), 4\pi(N + 1)]$: Theorems 1.1 and 1.2 are proven. □

Remark 2.5. Relying now on [18], we may get a more general non-homogeneous version of Theorem 1.1, which can be stated as follows:

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain and h be a smooth positive C^2-function in $\bar\Omega$. Let F_h be given in H^1_0 by*

$$F_h(v) := \int_\Omega \left(e^{v^2} - 1 \right) h \, dx$$

and $\mathcal{M}_\beta \subset H^1_0$ be as in (1.2). Assume that Ω is not contractible. Then, given any positive real number $\beta > 0$, there exists a nonnegative function u, critical point of F_h constrained to $\mathcal{M}_\beta$. In particular, u is smooth and solves $(\mathcal{E}_{p,\beta,h})$ below for $p = 2$.

Proof of Theorem 2.4. We fix $p = 2$. Given $\beta \in (4\pi N, 4\pi(N + 1)]$, we claim that the total Leray-Schauder degree $d_{p,\beta,h}(\Omega)$ of the solutions of

$$\begin{cases} \Delta u = p\lambda h u^{p-1} e^{u^p} \, , \, u > 0 \text{ in } \Omega \, , \\ u = 0 \text{ on } \partial\Omega \, , \\ \frac{\lambda p^2}{2} \left(\int_\Omega \left(e^{u^p} - 1 \right) h \, dx \right)^{\frac{2-p}{p}} \left(\int_\Omega u^p e^{u^p} h \, dx \right)^{\frac{2(p-1)}{p}} = \beta \, , \end{cases} \tag{$\mathcal{E}_{p,\beta,h}$}$$

is well-defined and does not depend of h, positive C^2-function in $\bar\Omega$, so that it always equals $\binom{N-\chi(\Omega)}{N}$, formula given by Theorem 1.2 for $h \equiv 1$. Indeed, first given $\beta > 0$ out of $4\pi\mathbb{N}^\star$, it follows from the quantization and the associated compactness result in [18] that $d_{p,\beta,h_r}(\Omega)$ is well-defined and does not depend of $r \in [0, 1]$, where $h_r = (1 - r) + rh$. Fixing now such an h, $N \in \mathbb{N}$ and $\eta \in (0, 1]$ and using that boundary blow-up cannot either occur for the solutions of $(\mathcal{E}_{p,\beta,h})$

(see Remark 2.6 just below), the energy estimate in Theorem 2.3 from [11, Section 5] still holds here and gives the compactness of all the solutions of $(\mathcal{E}_{p,\beta,h})$ for all $\beta \in [4\pi(N+\eta), 4\pi(N+1)]$, so that $d_{p,\beta,h}(\Omega)$ does not depend on β in this range by following the second part of the above proof: Theorem 2.4 is proven. □

Remark 2.6. *Excluding boundary blow-up is actually the most delicate part in* [18] *where p is fixed equal to 2. The analogue property appears to be open for $p \in (1,2)$.*

3. Degree formula for the vector field associated to $\mathcal{E}^{MF}_{\beta,h}$ on a bounded domain Ω

The purpose of this section is to prove Proposition 2.4. Let $\mathcal{G}$ be the Green's function of Δ with zero Dirichlet boundary conditions and $\mathcal{H}$ be its regular part with the convention in (2.9). Given $y \in \Omega$, notice that

$$\mathcal{G}(x,y) \to +\infty \quad \text{as } x \to y \qquad \text{and that} \qquad \mathcal{H}(x,x) \to -\infty \quad \text{as } x \to \partial\Omega.$$

Given a positive function $h \in C^1(\overline{\Omega})$, let $\Phi_{N,h}$ be given by (2.8). To fix the notation, we will write

$$\nabla\Phi_{N,h}(x_1,\dots,x_N) = (\nabla_{x_1}\Phi_{N,h}(x_1,\dots,x_N),\dots,\nabla_{x_N}\Phi_{N,h}(x_1,\dots,x_N)),$$

with

$$\nabla_{x_j}\Phi_{N,h}(x_1,\dots,x_N) = 16\pi \sum_{\substack{1\le \ell\le N\\ \ell\neq j}} \nabla\mathcal{G}(x_j,x_\ell) + 8\pi\nabla\mathcal{H}(x_j,x_j) + \frac{\nabla h(x_j)}{h(x_j)},$$

where we denote by $\nabla\mathcal{G}(x,y)$ and $\nabla\mathcal{H}(x,y)$ the gradient with respect to the first variable.

In order to get Proposition 2.4, we first need to analyse the compactness of the critical points of $\Phi_{N,h}$.

3.1. Compactness of critical points of $\Phi_{N,h}$.

In this subsection we prove that, uniformly with respect to $t \in [0,1]$, critical the points of

$$\Phi^t_{N,h}(x_1,\dots,x_N) := 8\pi \sum_{\substack{1\le i,j\le N\\ i\neq j}} \mathcal{G}(x_i,x_j) + 4\pi \sum_{1\le i\le N} \mathcal{H}(x_i,x_i) + t \sum_{1\le i\le N} \ln h(x_i)$$

cannot be arbitrarily close to the boundary of $\Omega^N \setminus D_N$, implying that the degree in Proposition 2.4 is well-defined. More precisely, define

$$\begin{aligned}(\Omega^N \setminus D_N)_\delta := \{(x_1,\dots,x_N) \in \Omega^N \setminus D_N : |x_i - x_j| > \delta \text{ for } 1 \le i < j \le N,\\ d(x_i,\partial\Omega) > \delta \text{ for } 1 \le i \le N\}.\end{aligned} \tag{3.1}$$

Lemma 3.1. *For every $M > 0$, there exists $\delta > 0$ such that*

$$|\nabla\Phi^t_{N,h}(x_1,\dots,x_N)| \ge M, \quad \text{for every } (x_1,\dots,x_N) \in (\Omega^N\setminus D_N)\setminus(\Omega^N\setminus D_N)_\delta,\ t\in[0,1].$$

As a consequence, the degree of $\nabla\Phi^t_{N,h}$ is well-defined on $\Omega^N \setminus D_N$ as

$$\deg(\nabla\Phi_{N,h}, \Omega^N\setminus D_N, 0) := \deg(\nabla\Phi_{N,h}, A, 0),$$

for any open set A with $(\Omega^N \setminus D_N)_\delta \subset A \subset \Omega^N \setminus D_N$.

Proof. Consider sequences

$$(t_k)_{k\in\mathbb{N}} \subset [0,1], \quad (x_{1,k},\dots,x_{N,k})_{k\in\mathbb{N}} \subset \Omega^N \setminus D_N$$

such that

$$\nabla\Phi^{t_k}_{N,h}(x_{1,k},\dots,x_{N,k}) = O(1), \quad \text{as } k\to\infty. \tag{3.2}$$

Set

$$r_{i,k} := \min\left\{d_{i,k}, \min_{j\neq i}|x_{i,k}-x_{j,k}|\right\}, \quad d_{i,k} := d(x_{i,k},\partial\Omega). \tag{3.3}$$

Up to extracting a subsequence and reordering the indices we can assume that

$$r_{1,k} \le r_{j,k}, \quad \text{for } 1\le j\le N, \tag{3.4}$$

and set

$$r_k := r_{1,k}, \quad x_k := x_{1,k}, \quad d_k := d_{1,k}. \tag{3.5}$$

It suffices to prove that $\liminf_{k\to\infty} r_k > 0$. Then we assume, by contradiction, that

$$\lim_{k\to\infty} r_k = 0. \tag{3.6}$$

We now scale the domain Ω and the functional $\Phi^{t_k}_{N,h}$ at x_k by a factor r_k. More precisely, we set $\Omega_k := r_k^{-1}(\Omega - x_k)$ and, for $(z_1,\dots,z_N)\in\Omega_k^N\setminus D_N$, we define

$$\begin{aligned}\Xi_k(z_1,\dots,z_N) &= \Phi^{t_k}_{N,h}(x_k+r_kz_1,\dots,x_k+r_kz_N)\\ &= 8\pi\sum_{\substack{1\le i,j\le N\\ i\neq j}}\mathcal{G}_k(z_i,z_j) + 4\pi\sum_{1\le i\le N}\mathcal{H}_k(z_i,z_i) + t_k\sum_{1\le i\le N}\ln h_k(z_i),\end{aligned}$$

where $\mathcal{G}_k$ and $\mathcal{H}_k$ are the Green's function of Δ on Ω_k and its regular part, and $h_k(z) := h(x_k+r_kz)$. Then (3.2) gives

$$\nabla\Xi_k(z_{1,k},\dots,z_{N,k}) = O(r_k), \quad \text{as } k\to\infty, \tag{3.7}$$

where

$$(z_{1,k},\dots,z_{N,k}) := (r_k^{-1}(x_{1,k}-x_k),\dots,r_k^{-1}(x_{N,k}-x_k)).$$

Up to a further subsequence we set

$$\mathcal{D} := \{i\in\{1,\dots,N\} : |x_{i,k}-x_k| = O(r_k) \text{ as } k\to\infty\}, \tag{3.8}$$

and have

$$\lim_{k\to\infty}\frac{|x_{i,k}-x_k|}{r_k} = +\infty \quad \text{for } i\in\{1,\dots,N\}\setminus\mathcal{D}. \tag{3.9}$$

Clearly $\mathcal{D}\neq\emptyset$, since $1\in\mathcal{D}$. Define also, up to a subsequence,

$$Z' = (z_i)_{i\in\mathcal{D}}, \quad Z'_k = (z_{i,k})_{i\in\mathcal{D}} \to Z'_\infty = (z_{i,\infty})_{i\in\mathcal{D}}, \quad \text{as } k\to\infty.$$

Notice that (3.3) and (3.8) imply

$$\frac{1}{C}r_k \le |x_{i,k}-x_{j,k}| \le Cr_k \quad \text{for } i,j\in\mathcal{D}, \quad i\neq j, \tag{3.10}$$

and in particular $z_{i,\infty}\neq z_{j,\infty}$ for $i,j\in\mathcal{D}$, $i\neq j$. We now define the "reduced" functional

$$\begin{aligned}\Xi_{\mathcal{D},k}(z_1,\dots,z_N) :=& 8\pi\sum_{\substack{i,j\in\mathcal{D}\\ i\neq j}}\mathcal{G}_k(z_i,z_j) + 4\pi\sum_{i\in\mathcal{D}}\mathcal{H}_k(z_i,z_i) + t_k\sum_{i\in\mathcal{D}}\ln h_k(z_i)\\ &+ 16\pi\sum_{i\in\mathcal{D}}\sum_{\substack{1\le j\le N\\ j\notin\mathcal{D}}}\mathcal{G}_k(z_i,z_j).\end{aligned} \tag{3.11}$$

From (3.7) we get

$$\nabla_{Z'}\Xi_{\mathcal{D},k}(z_{1,k},\dots,z_{N,k}) = O(r_k). \tag{3.12}$$

Taking (3.6) into account, we now consider 2 cases.

Case 1. Assume that, up to a subsequence,

$$r_k = o(d_k) \quad \text{as } k\to\infty. \tag{3.13}$$

Then Ω_k invades $\mathbb{R}^2$ as $k\to\infty$ and, choosing $K\subset\mathbb{R}^2$ compact containing $z_{i,k}$ for $i\in\mathcal{D}$ and k large, we get from Lemma A.1

$$\begin{aligned}
\nabla\mathcal{G}_k(z_{i,k},z_{j,k}) &= \frac{1}{2\pi}\nabla\ln\frac{1}{|z_{i,k}-z_{j,k}|} + O\left(\frac{r_k}{d_k}\right), \quad \text{for } i,j\in\mathcal{D},\\
\nabla\mathcal{G}_k(z_{i,k},z_{j,k}) &= \frac{1}{2\pi}\nabla\ln\frac{1}{|z_{i,k}-z_{j,k}|} + O\left(\frac{r_k}{d_k}\right) = o(1), \quad \text{for } i\in\mathcal{D},\ j\notin\mathcal{D},\\
\nabla\mathcal{H}_k(z_{i,k},z_{i,k}) &= O\left(\frac{r_k}{d_k}\right) \quad \text{for } i\in\mathcal{D},
\end{aligned}$$

as $k\to\infty$. Moreover $\nabla\ln h_k(z_{i,k}) = O(r_k)$ as $k\to\infty$ for $i\in\mathcal{D}$. Then from (3.11) and (3.12) we infer

$$\nabla\Xi_{\mathcal{D},\mathbb{R}^2}(Z'_k) = o(1), \quad \text{as } k\to\infty, \tag{3.14}$$

where

$$\Xi_{\mathcal{D},\mathbb{R}^2}(Z') := 4\sum_{\substack{i,j\in\mathcal{D}\\ i\neq j}}\ln\frac{1}{|z_i-z_j|}. \tag{3.15}$$

Notice that, thanks to (3.3), (3.8) and (3.13), we have $\mathcal{D}\cap\{2,\dots,N\}\neq\emptyset$. Since also $1\in\mathcal{D}$, the sum in (3.15) is non-empty. Letting now $k\to\infty$ in (3.14) it follows that $\nabla\Xi_{\mathcal{D},\mathbb{R}^2}(Z'_\infty)=0$. This contradicts Lemma A.3.

Case 2. Assume that we are not in Case 1. Then since $r_k\le d_k$, we have up to a subsequence

$$\frac{d_k}{r_k}\to L\ge 1, \quad \text{as } k\to\infty.$$

Then, up to a rotation, Ω_k converges to the half-space $H=\{(x,y)\in\mathbb{R}^2 : y<L\}$. Let $\mathcal{G}_H$ and $\mathcal{H}_H$ be the Green's function on H and its regular part, as given explicitly in the appendix. Choosing $K\subset H$ compact containing $z_{i,k}$ for $i\in\mathcal{D}$ and k large, by Lemma A.2 we estimate

$$\begin{aligned}
\nabla\mathcal{G}_k(z_{i,k},z_{j,k}) &= \nabla\mathcal{G}_H(z_{i,k},z_{j,k}) + O\left(r_k\right), \quad \text{for } i,j\in\mathcal{D}\\
\nabla\mathcal{G}_k(z_{i,k},z_{j,k}) &= o(1), \quad \text{for } i\in\mathcal{D},\ j\notin\mathcal{D}\\
\nabla\mathcal{H}_k(z_{i,k},z_{i,k}) &= \nabla\mathcal{H}_H(z_{i,k},z_{i,k}) + O\left(r_k\right) \quad \text{for } i\in\mathcal{D},
\end{aligned}$$

and $\nabla\ln h_k(z_{i,k}) = O(r_k)$ as $k\to\infty$ for $i\in\mathcal{D}$, hence, from (3.11) and (3.12) we obtain

$$\nabla\Xi_{\mathcal{D},H}(Z'_k) = o(1), \quad \text{as } k\to\infty, \tag{3.16}$$

where

$$\Xi_{\mathcal{D},H}(Z') := 8\pi\sum_{\substack{i,j\in\mathcal{D}\\ i\neq j}}\mathcal{G}_H(z_i,z_j) + 4\pi\sum_{i\in\mathcal{D}}\mathcal{H}_H(z_i,z_i).$$

Letting $k\to\infty$ in (3.16) we get $\nabla\Xi_{\mathcal{D},H}(Z'_\infty)=0$, contradicting Lemma A.5.

These contradictions, arising from assumptions (3.2) and (3.6), complete the proof of the lemma. □

A first consequence of Lemma 3.1 is that the degree of $\Phi_{N,h}$ does not depend on h, so that in the following we will work with $h \equiv 1$.

Corollary 3.1. *We have*

$$\deg(\nabla\Phi_{N,h}, \Omega^N \setminus D_N, 0) = \deg(\nabla\Phi_N, \Omega^N \setminus D_N, 0),$$

where, for $(x_1, \dots, x_N) \in \Omega^N \setminus D_N$,

$$\Phi_N(x_1, \dots, x_N) := 8\pi \sum_{\substack{1 \le i,j \le N \\ i \ne j}} \mathcal{G}(x_i, x_j) + 4\pi \sum_{1 \le i \le N} \mathcal{H}(x_i, x_i).$$

Proof. We can deform $\Phi_{N,h} = \Phi^1_{N,h}$ into $\Phi_N = \Phi^0_{N,h}$ and use the invariance of the degree under homotopy, since for δ sufficiently small and $t \in [0,1]$, $\Phi^t_{N,h}$ has no critical points on $\partial(\Omega^N \setminus D_N)_\delta$. □

3.2. **Bending the Robin function to apply the Poincaré-Hopf theorem.** Since we will not be able to apply the theorem of Poincaré-Hopf directly to $\nabla\Phi_N$ to compute $\deg(\nabla\Phi_N, \Omega^N \setminus D_N, 0)$, as $\nabla\Phi_N$ does not always point outwards near $\partial(\Omega^N \setminus D_N)$, we now introduce a modified function $\tilde{\Phi}_N$ by *bending* the function $\mathcal{H}(y,y)$ upwards near the boundary of Ω, in order to obtain a function that tends to $+\infty$ on $\partial(\Omega^N \setminus D_N)$. More precisely, let $\tau \in C^\infty([0,\infty))$ be a non-decreasing function such that

$$\tau(t) = 1 \quad \text{for } t \ge \frac{1}{2}, \quad \tau(t) = -1 \quad \text{for } t \le \frac{1}{4},$$

and set

$$\tilde{\mathcal{H}}(x) := \mathcal{H}(x,x)\tau\left(\frac{d(y,\partial\Omega)}{\tilde{\delta}}\right), \tag{3.17}$$

see Figure 1. Let then $\tilde{\Phi}_N$ be given in $\Omega^N \setminus D_N$ by

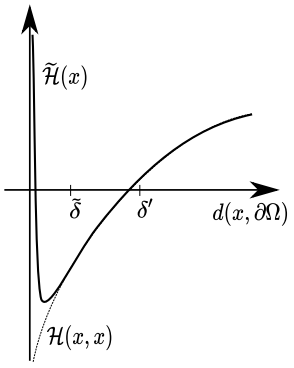


FIGURE 1. Heuristic sketch of $\tilde{\mathcal{H}}$ near $\partial\Omega$

$$\tilde{\Phi}_N(x_1, \dots, x_N) = 8\pi \sum_{\substack{1 \le i,j \le N \\ i \ne j}} \mathcal{G}(x_i, x_j) + 4\pi \sum_{1 \le i \le N} \tilde{\mathcal{H}}(x_i). \tag{3.18}$$

We next show that for $0 < \delta' < \delta/2$ and for $\tilde{\delta}$ sufficiently small (depending on δ'), the topological strips determined by $\delta' < d(y,\partial\Omega) < \delta$ is a *forbidden zone* for critical points of $\tilde{\Phi}$, as made precise in the following lemma.

Lemma 3.2. *For every $M' > 0$ there exist $\delta > 0$, such that for any $\delta' \in (0, \delta/2)$ there exists $\tilde{\delta} \in (0, \delta'/2)$ such that, setting $\tilde{\Phi}_N$ depending on $\tilde{\delta}$ as in (3.17)-(3.18), we have*

$$|\nabla \tilde{\Phi}_N(x_1, \dots, x_N)| \geq M'$$

if $(x_1, \dots, x_N) \in \Omega^N \setminus D_N$ is such that $\delta' \leq d(x_i, \partial\Omega) \leq \delta$ for some $1 \leq i \leq N$. In particular, if $(x_1, \dots, x_N)$ is a critical point of $\tilde{\Phi}_N$, then either $d(x_i, \partial\Omega) > \delta$ or $d(x_i, \partial\Omega) < \delta'$ for every $i = 1, \dots, N$.

Proof. Fix $M' > 0$ and let $\delta := \min_{1 \leq I \leq N} \delta_I$, where δ_I is the constant δ appearing in Lemma 3.1 applied with N replaced by I and with $M = M' + 1 > 0$.

Assuming by contradiction that the lemma is false, we can find $\delta' \in (0, \delta/2)$, and sequences $\tilde{\delta}_k \to 0$ and $(x_{1,k}, \dots, x_{N,k}) \subset \Omega^N \setminus D_N$ such that, for $\tilde{\Phi}_{N,k}$ defined using $\tilde{\delta}_k$, we have

$$|\nabla \tilde{\Phi}_{N,k}(x_{1,k}, \dots, x_{N,k})| < M' \quad \text{as } k \to \infty, \tag{3.19}$$

and such that, up to relabelling,

$$\delta' \leq d(x_{1,k}, \partial\Omega) \leq \delta. \tag{3.20}$$

Up to passing to subsequence and a further relabelling, let $I \in \{1, \dots, N\}$ such that

$$\lim_{k \to \infty} d(x_{i,k}, \partial\Omega) > 0 \quad \text{for } 1 \leq i \leq I, \tag{3.21}$$

and

$$\lim_{k \to \infty} d(x_{j,k}, \partial\Omega) = 0, \quad I + 1 \leq j \leq N.$$

By (3.20) and Lemma 3.1 we have

$$|\nabla \Phi_I(x_{1,k}, \dots, x_{I,k})| \geq M' + 1, \tag{3.22}$$

where

$$\Phi_I(x_1, \dots, x_I) := 8\pi \sum_{\substack{1 \leq i,j \leq I \\ i \neq j}} \mathcal{G}(x_i, x_j) + 4\pi \sum_{1 \leq i \leq I} \mathcal{H}(x_i, x_i).$$

Observe, on the other, hand that

$$\nabla \mathcal{G}(x_{i,k}, x_{j,k}) = o(1), \quad 1 \leq i \leq I, \quad I + 1 \leq j \leq N, \tag{3.23}$$

with $o(1) \to 0$ as $k \to \infty$ (since $x_{j,k}$ is approaching $\partial\Omega$, while $x_{i,k}$ is not as $k \to \infty$). Then, by (3.21), for k large enough (hence $\tilde{\delta}_k$ sufficiently small), we have

$$\begin{aligned} \tilde{\Phi}_{N,k}(x_{1,k}, \dots, x_{N,k}) =& \Phi_I(x_{1,k}, \dots, x_{I,k}) + 16\pi \sum_{\substack{1 \leq i \leq I \\ I+1 \leq j \leq N}} \mathcal{G}(x_{i,k}, x_{j,k}) \\ &+ 8\pi \sum_{I+1 \leq i,j \leq N} \mathcal{G}(x_{i,k}, x_{j,k}) + 4\pi \sum_{I+1 \leq J \leq N} \tilde{\mathcal{H}}_k(x_{j,k}). \end{aligned}$$

Then, setting $X' := (x_1, \dots, x_I)$, from (3.19), (3.22) and (3.23) we get

$$\begin{aligned} M' >& |\nabla_{X'} \tilde{\Phi}_{N,k}(x_{1,k}, \dots, x_{N,k})| = |\nabla \Phi_I(x_{1,k}, \dots, x_{I,k})| + o(1) \\ \geq& M' + 1 + o(1), \end{aligned}$$

giving a contradiction for k large enough. □

We now have the analog of Lemma 3.1 for $\tilde{\Phi}_N$.

Lemma 3.3. *For every $M'' > 0$ there exists $\delta'' \leq \frac{\tilde{\delta}}{2}$ depending on $\tilde{\delta}$ and M'' such that*

$$|\nabla \tilde{\Phi}_N(x_1, \dots, x_N)| \geq M''$$

for every $(x_1, \dots, x_N) \in (\Omega^N \setminus D_N) \setminus (\Omega^N \setminus D_N)_{\delta''}$. In particular, all the critical points of $\nabla \tilde{\Phi}_N$ are in $(\Omega^N \setminus D_N)_{\delta''}$ and we can define

$$\deg(\nabla \tilde{\Phi}_N, \Omega^N \setminus D_N, 0) := \deg(\nabla \tilde{\Phi}_N, A, 0),$$

for any open set A such that $(\Omega^N \setminus D_N)_{\delta''} \subset A \subset \Omega^N \setminus D_N$.

Proof. The proof by contradiction is essentially identical to the proof of Lemma 3.1, upon noticing the following facts. In Case 1, namely $r_k = o(d_k)$, if (up to a subsequence) $d_k \to 0$ as $k \to \infty$, then

$$\tilde{\mathcal{H}}(x_{i,k}) = -\mathcal{H}(x_{i,k}, x_{i,k}) \quad \text{for } i \in \mathcal{D}, \quad k \text{ large},$$

hence

$$\nabla \tilde{\mathcal{H}}_k(z_{i,k}) = -\nabla \mathcal{H}_k(z_{i,k}, z_{i,k}) = O(r_k), \quad \text{for } i \in \mathcal{D}, \quad k \text{ large}.$$

If $d_k \not\to 0$, from (3.17), we bound

$$|\nabla \tilde{\mathcal{H}}(x_{i,k})| = O(\nabla \mathcal{H}(x_{i,k}, x_{i,k}) + C_{\tilde{\delta}} |\mathcal{H}(x_{i,k}, x_{i,k})|) = O(1), \quad \text{as } k \to \infty, \quad i \in \mathcal{D},$$

hence again $\nabla \tilde{\mathcal{H}}_k(z_{i,k}) = O(r_k)$ as $k \to \infty$ for $i \in \mathcal{D}$, where $\tilde{\mathcal{H}}_k(z) := \tilde{\mathcal{H}}(x_k + r_k z)$. Then, up to a subsequence, Z'_k converges to Z'_∞, a critical point of $\Xi_{\mathcal{D},\mathbb{R}^2}$, contradicting Lemma A.3.

In Case 2, since $d(x_{1,k}, \partial\Omega) \to 0$, we also have $d(x_{i,k}, \partial\Omega) \to 0$ as $k \to \infty$ for $i \in \mathcal{D}$, hence $\tilde{\mathcal{H}}(x_{i,k}) = -\mathcal{H}(x_{i,k}, x_{i,k})$ for $i \in \mathcal{D}$ and k large enough. Then, setting $\Xi^-_{\mathcal{D},k}$ as in (3.11) with $h_k \equiv 1$ and $\mathcal{H}_k(z_i, z_i)$ replaced by $-\mathcal{H}_k(z_i, z_i)$, we obtain $\nabla \Xi^-_{\mathcal{D},H}(Z'_\infty) = 0$, where

$$\Xi^-_{\mathcal{D},H}(Z') := 8\pi \sum_{\substack{i,j \in \mathcal{D} \\ i \neq j}} \mathcal{G}_H(z_i, z_j) - 4\pi \sum_{i \in \mathcal{D}} \mathcal{H}_H(z_i, z_i),$$

contradicting Lemma A.5. □

We recall next a variant of the classical Poincaré-Hopf index theorem, which can be found e.g. in [6], pages 99-104 (here we adapt the statement to our purposes).

Proposition 3.1. *Let $U \subset \mathbb{R}^n$ be an open set and consider $f \in C^2(U, \mathbb{R})$. Assume also that, for some $b \in \mathbb{R}$, $f^b := \{x \in U : f(x) < b\}$ has compact closure, and that f has no critical points in $f^{-1}(b)$. Then we have*

$$\deg(\nabla f, f^b, 0) = \chi(f^b). \tag{3.24}$$

Lemma 3.4. *We have*

$$\deg(\nabla \tilde{\Phi}_N, \Omega^N \setminus D_N, 0) = \chi(\Omega)(\chi(\Omega) - 1)...(\chi(\Omega) - N + 1). \tag{3.25}$$

Proof. Since

$$\lim_{(x_1,\dots,x_N) \to \partial(\Omega^N \setminus D_N)} \tilde{\Phi}_N(x_1, \dots, x_N) = +\infty,$$

we can take $b > 0$ large enough such that

$$(\Omega^N \setminus D_N)_{\delta''} \subset (\tilde{\Phi}_N)^b,$$

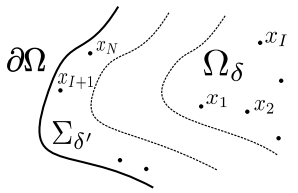


FIGURE 2. $(x_1,\dots,x_N)\in\Theta^*_{\delta,\delta',I}=((\Omega_\delta)^I\setminus D_I)\times((\Sigma_{\delta'})^{N-I}\setminus D_{N-I})$

where δ'' is as in Lemma 3.3 with $M''=1$. Then, since $(\Omega^N\setminus D_N)\setminus(\tilde\Phi_N)^b$ contains no critical points of $\tilde\Phi_N$ by Lemma 3.3, we can apply Proposition 3.1 to get

$$\deg(\nabla\tilde\Phi_N,\Omega^N\setminus D_N,0)=\deg(\nabla\tilde\Phi_N,(\tilde\Phi_N)^b,0)=\chi((\tilde\Phi_N)^b).$$

Since $(\Omega^N\setminus D_N)\setminus(\tilde\Phi_N)^b$ contains no critical points of $\tilde\Phi_N$, we can use a gradient flow to retract $(\Omega^N\setminus D_N)$ to $(\tilde\Phi_N)^b$. Therefore $\chi((\tilde\Phi_N)^b)=\chi(\Omega^N\setminus D_N)$, and we conclude with Proposition B.1 below. □

3.3. **Counting the new critical points.** By bending the Robin function we have in general created new critical points near the boundary. In this section we will show that these new critical points do not change the total degree. To this purpose, we now define the following sets (see Figure 2):

$$\Omega_\delta:=\{y\in\Omega\ :\ d(y,\partial\Omega)>\delta\}\,;\quad \Sigma_{\delta'}:=\Omega\setminus\overline{\Omega}_{\delta'},$$

and for $0\le I\le N$

$$\begin{aligned}\Theta^*_{\delta,\delta',I}&=\big\{(x_1,\dots,x_N)\in\Omega^N\setminus D_N\ :\ x_1,\dots,x_I\in\Omega_\delta,\ x_{I+1},\dots,x_N\in\Sigma_{\delta'}\big\}\\&=((\Omega_\delta)^I\setminus D_I)\times((\Sigma_{\delta'})^{N-I}\setminus D_{N-I});\\\Theta_{\delta,\delta',I}&=\big\{(x_1,\dots,x_N)\in\Omega^N\setminus D_N\ :\ (x_{\sigma(1)},\dots,x_{\sigma(N)})\in\Theta^*_{\delta,\delta',I}\\&\qquad\text{for a permutation }\sigma\in S_N\big\};\\\Theta^*_{\delta,\delta',\delta'',I}&=\Theta^*_{\delta,\delta',I}\cap(\Omega^N\setminus D_N)_{\delta''},\quad \Theta_{\delta,\delta',\delta'',I}=\Theta_{\delta,\delta',I}\cap(\Omega^N\setminus D_N)_{\delta''}.\end{aligned}$$

By Lemmata 3.2 and 3.3, we can define for $0\le I\le N-1$

$$\deg(\nabla\tilde\Phi_N,\Theta_{\delta,\delta',I},0):=\deg(\nabla\tilde\Phi_N,A,0)$$

for any open set A such that $\Theta_{\delta,\delta',\delta'',I}\subset A\subset\Theta_{\delta,\delta',I}$. Moreover, there are no critical points of $\tilde\Phi_N$ in $(\Omega^N\setminus D_N)\setminus\cup_{I=0}^N\Theta_{\delta,\delta',I}$ by Lemma 3.2; therefore by the excision property and noting that $\Theta_{\delta,\delta',N}=(\Omega_\delta)^N\setminus D_N$ and that $\tilde\Phi_N=\Phi_N$ inside $(\Omega_\delta)^N\setminus D_N$, we have

$$\begin{aligned}\deg(\nabla\tilde\Phi_N,\Omega^N\setminus D_N,0)&=\deg(\nabla\Phi_N,(\Omega_\delta)^N\setminus D_N,0)+\sum_{0\le I\le N-1}\deg(\nabla\tilde\Phi_N,\Theta_{\delta,\delta',I},0)\\&=\deg(\nabla\Phi_N,(\Omega_\delta)^N\setminus D_N,0)+\sum_{0\le I\le N-1}\binom{N}{I}\deg(\nabla\tilde\Phi_N,\Theta^*_{\delta,\delta',I},0).\end{aligned}\tag{3.26}$$

Lemma 3.5. *Up to choosing δ' sufficiently small in Lemma 3.2 (and $\tilde\delta$ and δ'' correspondingly), we have that*

$$\deg(\nabla\tilde\Phi_N,\Theta^*_{\delta,\delta',I},0)=\deg(\nabla_{X'}\Phi_I,(\Omega_\delta)^I\setminus D_I,0)\times\deg(\nabla_{X''}\tilde\Phi_{N-I},(\Sigma_{\delta'})^{N-I}\setminus D_{N-I},0),$$

for $0 \le I \le N-1$, *where*

$$X' = (x_1, \dots, x_I), \quad X'' = (x_{I+1}, \dots, x_N), \quad X = (X', X''),$$

and the degrees are well-defined intersecting the domains with $(\Omega^I \setminus D_I)_{\delta''}$ *and* $(\Omega^{N-I} \setminus D_{N-I})_{\delta''}$.

Proof. For $X \in \Theta^*_{\delta,\delta',I}$ we have

$$x_i \in \Omega_\delta \quad \text{for } 1 \le i \le I, \qquad x_a \in \Sigma_{\delta'} \quad \text{for } I+1 \le a \le N. \tag{3.27}$$

Interestingly, the gradient almost *decouples* into the first I components and the last $N-I$. In fact, since $\tilde{\mathcal{H}}(x)$ coincides with $\mathcal{H}(x,x)$ for $x \in \Omega_\delta$, we have

$$\tilde{\Phi}_N(X) = \Phi_I(X') + \tilde{\Phi}_{N-I}(X'') + \Xi_{I,N}(X),$$

where

$$\tilde{\Phi}_{N-I}(X'') = 8\pi \sum_{\substack{I+1 \le a,b \le N \\ a \ne b}} \mathcal{G}(x_a, x_b) + 4\pi \sum_{I+1 \le a \le N} \tilde{\mathcal{H}}(x_a);$$

$$\Xi_{I,N}(X) = 16\pi \sum_{1 \le i \le I < a \le N} \mathcal{G}(x_i, x_a).$$

Notice that

$$\mathcal{G}(x_i, x_a) = o_{\delta'}(1), \quad \nabla \mathcal{G}(x_i, x_a) = o_{\delta'}(1),$$

with $o_{\delta'}(1) \to 0$ as $\delta' \to 0$ (for δ fixed), uniformly with respect to x_i and x_a as in (3.27). Now observe that $X = (x_1, \dots, x_N) \in \partial\Theta^*_{\delta,\delta',\delta'',I}$ (with $\delta'' < \frac{\tilde{\delta}}{2}$ to be fixed) implies that one of the following holds

(1) $|x_i - x_j| = \delta''$ for some $1 \le i < j \le I$;
(2) $|x_a - x_b| = \delta''$ for some $I+1 \le a < b \le N$;
(3) $d(x_i, \partial\Omega) = \delta$ for some $1 \le i \le I$;
(4) $d(x_a, \partial\Omega) = \delta'$ for some $I+1 \le a \le N$;
(5) $d(x_a, \partial\Omega) = \delta''$ for some $I+1 \le a \le N$.

If we choose $M = M' = M'' = 1$ in Lemmata 3.1, 3.2 and 3.3 and $\delta, \delta', \delta''$ sufficiently small, we obtain that in all the above cases

$$\max\{|\nabla_{X'} \Phi_I(X')|, |\nabla_{X''} \tilde{\Phi}_{N-I}(X'')|\} \ge 1, \quad \text{for } X \in \partial\Theta^*_{\delta,\delta',\delta'',I}. \tag{3.28}$$

Then, by the analog of (3.23), we can choose δ' even smaller (changing also $\tilde{\delta}$ accordingly) so that in addition to (3.28) we also have

$$|\nabla \Xi_{I,N}(X)| \le \frac{1}{2} \quad \text{for } X \in \partial\Theta^*_{\delta,\delta',\delta'',I}. \tag{3.29}$$

Consider now for $t \in [0,1]$

$$\tilde{\Phi}^t_N(X) := \Phi_I(X') + \tilde{\Phi}_{N-I}(X'') + t\Xi_{I,N}(X).$$

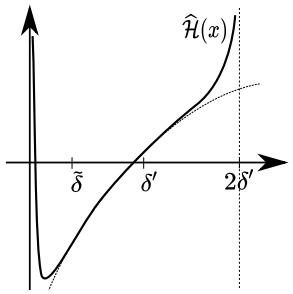
Then (3.28)-(3.29) imply at once

$$|\nabla \tilde{\Phi}^t_N(X)| \ge |(\nabla_{X'} \Phi_I(X'), \nabla_{X''} \tilde{\Phi}_{N-I}(X''))| - |\nabla \Xi_{I,N}(X)| > 0 \text{ for} X \in \partial\Theta^*_{\delta,\delta',\delta'',I}.$$

By the invariance of the degree under homotopy, we obtain

$$\begin{aligned} \deg(\nabla \tilde{\Phi}_N, \Theta^*_{\delta,\delta',\delta'',I}, 0) &= \deg(\nabla \tilde{\Phi}^0_N, \Theta^*_{\delta,\delta',\delta'',I}, 0) \\ &= \deg((\nabla_{X'} \Phi_I(X'), \nabla_{X''} \tilde{\Phi}_{N-I}(X'')), \Theta^*_{\delta,\delta',\delta'',I}, 0), \end{aligned} \tag{3.30}$$

and we conclude by the product formula of the degree. □



Lemma 3.6. *Up to choosing δ' sufficiently small (reducing $\tilde{\delta}, \delta''$ accordingly), for $1 \leq J \leq N$ we have*

$$\deg(\nabla\tilde{\Phi}_J, (\Sigma_{\delta'})^J \setminus D_J, 0) = 0.$$

Proof. We will work on the the larger domain $\Sigma_{2\delta'}$, and modify $\tilde{\mathcal{H}}$ in the strip $\Sigma_{2\delta'} \setminus \Sigma_{\delta'} = \bar{\Omega}_{\delta'} \setminus \bar{\Omega}_{2\delta'}$ by defining

$$\hat{\mathcal{H}}(x) := \tilde{\mathcal{H}}(x) + \sigma\left(\frac{d(x, \partial\Omega)}{\delta'}\right),$$

where $\sigma \in C^\infty([0,2))$ is non-negative, non-decreasing, and

$$\sigma(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq 1, \\ \ln\left(\frac{1}{2-t}\right) & \text{for } \frac{3}{2} \leq t < 2. \end{cases}$$

Clearly $\hat{\mathcal{H}} = \tilde{\mathcal{H}}$ in $\Sigma_{\delta'}$, so that

$$\deg(\nabla\tilde{\Phi}_J, (\Sigma_{\delta'})^J \setminus D_J, 0) = \deg(\nabla\hat{\Phi}_J, (\Sigma_{\delta'})^J \setminus D_J, 0), \tag{3.31}$$

where, for $(x_1, \dots, x_J) \in (\Sigma_{2\delta'})^J \setminus D_J$,

$$\begin{aligned} \hat{\Phi}_J(x_1, \dots, x_J) &= \tilde{\Phi}_J(x_1, \dots, x_J) + \sum_{1 \leq i \leq J} \sigma\left(\frac{d(x_i, \partial\Omega)}{\delta'}\right) \\ &= 8\pi \sum_{\substack{1 \leq i,j \leq J \\ i \neq j}} \mathcal{G}(x_i, x_j) + 4\pi \sum_{1 \leq i \leq J} \hat{\mathcal{H}}(x_i). \end{aligned} \tag{3.32}$$

Thanks to Lemmata 3.7 and 3.8 below, we can choose $\delta' > 0$ even smaller than before, and $\tilde{\delta}$ accordingly small (without affecting the previous results, which are valid for δ', $\tilde{\delta}$ and δ'' sufficiently small), so that $\hat{\Phi}_J$ has no critical points in $\left((\Sigma_{2\delta'})^J \setminus (\Sigma_{\delta'})^J\right) \setminus D_J$. Then

$$\deg(\nabla\hat{\Phi}_J, (\Sigma_{\delta'})^J \setminus D_J, 0) = \deg(\nabla\hat{\Phi}_J, (\Sigma_{2\delta'})^J \setminus D_J, 0). \tag{3.33}$$

Since now

$$\lim_{(x_1,\dots,x_J) \to \partial((\Sigma_{2\delta'})^J \setminus D_J)} \hat{\Phi}_J(x_1, \dots, x_J) = +\infty,$$

we can apply Proposition 3.1, and get

$$\deg(\nabla\hat{\Phi}_J, (\Sigma_{2\delta'})^J \setminus D_J, 0) = \chi((\Sigma_{2\delta'})^J \setminus D_J). \tag{3.34}$$

On the other hand, $\chi(\Sigma_{2\delta'}) = 0$, since $\Sigma_{2\delta'}$ retracts to a union of circles (up to choosing δ' sufficiently small) and $\chi(\mathbb{S}^1) = 0$. Then it follows from Proposition B.1

$$\chi((\Sigma_{2\delta'})^J \setminus D_J) = 0.$$

This, together with (3.31), (3.33) and (3.34) yields $\deg(\nabla\tilde{\Phi}_J, (\Sigma_{\delta'})^J \setminus D_J, 0) = \chi((\Sigma_{2\delta'})^J \setminus D_J) = 0$. □

Lemma 3.7. *For $1 \le J \le N$ and $M' > 0$ and $\delta > 0$ there exists $\delta' \in (0, \delta/2)$ such that*

$$|\nabla\Psi_J| \ge M' \quad \text{in } (\Sigma_{2\delta'})^J \setminus D_J,$$

where, for $x_1, \dots, x_J \in (\Omega \setminus \overline{\Omega}_{2\delta'})^J \setminus D_J$,

$$\begin{aligned}\Psi_J(x_1,\dots,x_J) &:= \Phi_J(x_1,\dots,x_J) + \sum_{1\le i\le J} \sigma\left(\frac{d(x_i,\partial\Omega)}{\delta'}\right)\\ &= 8\pi \sum_{\substack{1\le i,j\le J\\ i\ne j}} \mathcal{G}(x_i,x_j) + 4\pi \sum_{1\le i\le J} \mathcal{H}(x_i) + \sum_{1\le i\le J} \sigma\left(\frac{d(x_i,\partial\Omega)}{\delta'}\right),\end{aligned} \tag{3.35}$$

and $\mathcal{G}$ and $\mathcal{H}$ are still the Green's function of Δ in Ω (not in $\Omega \setminus \overline{\Omega}_{2\delta'}$) and its regular part.

Proof. We assume by contradiction that there exist sequences $\delta'_k \to 0^+$ and

$$(x_{1,k}, \dots, x_{J,k}) \in (\Sigma_{2\delta'_k})^J \setminus D_J$$

such that

$$\nabla\Psi^k_J(x_{1,k},\dots,x_{J,k}) = O(1) \quad \text{as } k \to \infty, \tag{3.36}$$

where the notation Ψ^k_J is meant to emphasize that Ψ_J depends on δ'_k, due to the last term in (3.35). We proceed as in the proof of Lemma 3.1. Set

$$r_{i,k} = \min\left\{d_{i,k}, \hat{d}_{i,k}, \min_{j\ne i} |x_{i,k} - x_{j,k}|\right\},$$

where

$$d_{i,k} := d(x_{i,k}, \partial\Omega), \quad \hat{d}_{i,k} := d(x_{i,k}, \partial\Omega_{2\delta'_k}) = 2\delta'_k - d(x_{i,k}, \partial\Omega).$$

Up to reordering we assume $r_{1,k} \le r_{i,k}$ for every $1 \le i \le J$, and set

$$r_k := r_{1,k}, \quad d_k := d_{1,k}, \quad \hat{d}_k := \hat{d}_{1,k}, \quad x_k := x_{1,k}.$$

Define $\mathcal{D}$, Z_k, $Z'_k \to Z'_\infty$, Ω_k, as in the proof of Lemma 3.1, and similarly

$$(\Sigma_{2\delta'_k})_k := r_k^{-1}(\Sigma_{2\delta'_k} - x_k).$$

Then we set, for $Z = (z_1, \dots, z_J) \in ((\Sigma_{2\delta'_k})_k)^J \setminus D_J$,

$$\begin{aligned}\hat{\Xi}_{\mathcal{D},k}(z_1,\dots,z_N) :=& 8\pi \sum_{\substack{i,j\in\mathcal{D}\\ i\ne j}} \mathcal{G}_k(z_i,z_j) + 16\pi \sum_{i\in\mathcal{D}} \sum_{\substack{1\le j\le N\\ j\notin\mathcal{D}}} \mathcal{G}_k(z_i,z_j)\\ &+ 4\pi \sum_{i\in\mathcal{D}} \mathcal{H}_k(z_i,z_i) + \sum_{i\in\mathcal{D}} \sigma\left(\frac{r_k d(z_i,\partial\Omega_k)}{\delta'_k}\right),\end{aligned} \tag{3.37}$$

where $\mathcal{G}_k$ and $\mathcal{H}_k$ are the Green's function of Δ on Ω_k and its regular part, respectively. Observe that $r_k \le \delta'_k \to 0$ as $k \to \infty$. We consider several cases.

Case 1 Assume that, up to a subsequence, $r_k = o(d_k)$, $r_k = o(\hat{d}_k)$ as $k \to \infty$. Since

$$\nabla_z \sigma\left(\frac{r_k d(z,\partial\Omega_k)}{\delta'_k}\right) = \nabla_z \sigma\left(\frac{d(x_k + r_k z, \partial\Omega)}{\delta'_k}\right) = r_k \nabla_x \sigma\left(\frac{d(x,\partial\Omega)}{\delta'_k}\right)\Big|_{x = x_k + r_k z},$$

and

$$\sigma'\left(\frac{d(x_{i,k},\partial\Omega)}{\delta'_k}\right)=O\left(\frac{\delta'_k}{\hat{d}_k}\right),\quad |\nabla d(\cdot,\partial\Omega)|=1,\quad \text{for } i\in\mathcal{D}$$

we infer

$$\nabla_z\sigma\left(\frac{r_k d(z_{i,k},\partial\Omega_k)}{\delta'_k}\right)=O\left(\frac{r_k}{\hat{d}_k}\right)=o(1)\quad \text{as } k\to\infty,\quad \text{for } i\in\mathcal{D}.$$

Then we can proceed as in Case 1 of the proof of Lemma 3.1 to get that Z'_∞ is critical point of $\Xi_{\mathcal{D},\mathbb{R}^2}$, as defined in (3.15), contradicting Lemma A.3.

Case 2 Assume that, up to a subsequence, $r_k=o(\hat{d}_k)$, $\frac{d_k}{r_k}\to L\geq 1$ as $k\to\infty$. This implies that $r_k=o(\delta'_k)$ as $k\to\infty$ and, up to a rotation, $(\Sigma_{\delta'_k})_k$ converges to a half-space $H=\{(x,y)\in\mathbb{R}^2: y<L\}$. We proceed as in Case 1, except that now

$$\sigma\left(\frac{r_k d(z_{i,k},\partial\Omega_k)}{\delta'_k}\right)=0,\quad \text{for } i\in\mathcal{D},\quad k \text{ large enough.}$$

Then $\hat{\Xi}_{\mathcal{D},k}(Z_k)=\Xi_{\mathcal{D},k}(Z_k)$ for k large enough, where $\Xi_{\mathcal{D},k}$ is as in (3.11) (with $t_k=0$). Then we are in the same situation as Case 2 of the proof of Lemma 3.1, leading to the same contradiction, i.e. Z'_∞ is a critical point of $\Xi_{\mathcal{D},H}$, violating Lemma A.5.

Case 3 Assume that, up to a subsequence, $r_k=o(d_k)$, $\frac{\hat{d}_k}{r_k}\to\hat{L}_1\geq 1$. This implies in particular that, up to a subsequence,

$$r_k=o(\delta'_k),\quad \frac{\hat{d}_{i,k}}{r_k}\to\hat{L}_i\geq 1\quad \text{as } k\to\infty,\ \text{for } i\in\mathcal{D}. \tag{3.38}$$

Then, up to a rotation $(\Sigma_{2\delta'_k})_k$ converges to $\hat{H}=\{(x,y)\in\mathbb{R}^2: y>-\hat{L}\}$. As before, $r_k=o(d_k)$ implies $r_k=o(d_{i,k})$ for $i\in\mathcal{D}$ and it follows $\nabla\mathcal{H}_k(z_{i,k},z_{i,k})=O(r_k)$ as $k\to\infty$ for $i\in\mathcal{D}$. Considering that

$$\frac{r_k d(z_{i,k},\partial\Omega_k)}{\delta'_k}=\frac{d(x_{i,k},\partial\Omega)}{\delta'_k}\to 2,\quad \text{as } k\to\infty,\quad \text{for } i\in\mathcal{D},$$

and $\sigma(t)=-\ln(2-t)$ for $\frac{3}{2}\leq t<2$, we have that

$$\begin{aligned}\nabla_z\sigma\left(\frac{r_k d(z_{i,k},\partial\Omega_k)}{\delta'_k}\right)&=\frac{r_k\nabla d(z_{i,k},\partial\Omega_k)}{2\delta'_k-r_k d(z_{i,k},\partial\Omega_k)}=\frac{-r_k\nabla d(z_{i,k},\partial(\Omega\setminus\Omega_{2\delta'_k})_k)}{\hat{d}_{i,k}}\\&\to-\frac{\nabla_z d(z_{i,\infty},\partial\hat{H})}{\hat{L}_i}=\nabla_z\ln\frac{1}{d(z_{i,\infty},\partial\hat{H})},\end{aligned} \tag{3.39}$$

since $\hat{L}_i=d(z_{i,\infty},\partial\hat{H})$. Then Z'_∞ is a critical point of

$$\hat{\Xi}_{\mathcal{D},\hat{H}}(Z)=4\sum_{\substack{i,j\in\mathcal{D}\\ i\neq j}}\ln\frac{1}{|z_i-z_j|}+\sum_{i\in\mathcal{D}}\ln\frac{1}{d(z_i,\partial\hat{H})},$$

contradicting Lemma A.4.

Case 4 If we are not in any of the above cases, then, up to a subsequence,

$$\frac{d_k}{r_k}\to L_1\geq 1,\quad \frac{\hat{d}_k}{r_k}\to\hat{L}_1\geq 1,\quad \text{as } k\to\infty.$$

Then,

$$\frac{r_k}{\delta'_k} \to \gamma > 0, \quad \text{as } k \to \infty,$$

and,

$$\frac{d_{i,k}}{r_k} \to L_i \geq 1, \quad \frac{\hat{d}_{i,k}}{r_k} \to \hat{L}_i \geq 1, \quad \frac{\hat{d}_{i,k}}{\delta'_k} \to \rho_i \in (0,2), \quad \text{as } k \to \infty, \quad i \in \mathcal{D}.$$

Up to a rotation $(\Sigma_{2\delta'_k})_k$ converges to $H \cap \hat{H}$, where

$$H = \{(x,y) \in \mathbb{R}^2 : y < L_1\}, \quad \hat{H} = \{(x,y) \in \mathbb{R}^2 : y > -\hat{L}_1\}.$$

Considering now that

$$L_i + o(1) = \frac{d_{i,k}}{r_k} = d(z_{i,k}, \partial\Omega_k) = d(z_{i,\infty}, \partial H) + o(1), \quad \text{as } k \to \infty, \quad \text{for } i \in \mathcal{D},$$

we compute

$$\begin{aligned}\nabla_z \sigma\left(\frac{r_k d(z_{i,k}, \partial\Omega_k)}{\delta'_k}\right) &= \frac{r_k}{\delta'_k}\sigma'\left(\frac{d_{i,k}}{\delta'_k}\right)\nabla_z d(z_{i,k}, \partial\Omega_k) \\ &\to \gamma\sigma'(\gamma L_i)\nabla_z d(z_{i,\infty}, \partial H) \\ &= \nabla_z \sigma(\gamma d(z_{i,\infty}, \partial H)), \quad \text{as } k \to \infty, \quad \text{for } i \in \mathcal{D}.\end{aligned}$$

Then Z'_∞ is a critical point of

$$\hat{\Xi}_{\mathcal{D},\hat{H}}(Z') = 8\pi \sum_{\substack{i,j \in \mathcal{D} \\ i \neq j}} \mathcal{G}_H(z_i, z_j) + 4\pi \sum_{i \in \mathcal{D}} \mathcal{H}_H(z_i) + \sum_{i \in \mathcal{D}} \sigma(\gamma d(z_i, H)).$$

This contradicts Lemma A.6.

These contradictions, arising from (3.36), complete the proof of the lemma. □

Lemma 3.8. *Given $1 \leq J \leq N$ and δ' as in Lemma 3.7 (or smaller), there exists $\tilde{\delta} \in (0, \frac{\delta'}{2})$ sufficiently small such that $\hat{\Phi}_J$ (defined as in (3.32), hence depending on $\tilde{\delta}$) has no critical points in $((\Sigma_{2\delta'})^J \setminus (\Sigma_{\delta'})^J) \setminus D_J$.*

Proof. This follows as in the proof of Lemma 3.2 by a contradiction argument, using Lemma 3.7 instead of Lemma 3.1. Here $2\delta'$ plays the role of δ in Lemma 3.7. □

3.4. **Proof of Proposition 2.4 (completed).** By Corollary 2.3 we can assume $h \equiv 1$ and work with Φ_N instead of $\Phi_{N,h}$. Using Lemma 3.1, considering that Φ_N coincides with $\tilde{\Phi}_N$ on $(\Omega_\delta)^N \setminus D_N$, using formula (3.26) and Lemmata 3.5 and 3.6 we now get

$$\begin{aligned}\deg(\nabla\Phi_N, \Omega^N \setminus D_N, 0) &= \deg(\nabla\Phi_N, (\Omega_\delta)^N \setminus D_N, 0) \\ &= \deg(\nabla\tilde{\Phi}_N, (\Omega_\delta)^N \setminus D_N, 0) \\ &= \deg(\nabla\tilde{\Phi}_N, \Omega^N \setminus D_N, 0) - \sum_{0 \leq I \leq N-1} \deg(\nabla\tilde{\Phi}_N, \Theta_{\delta,\delta',I}, 0) \\ &= \deg(\nabla\tilde{\Phi}_N, \Omega^N \setminus D_N, 0).\end{aligned}$$

By Lemma 3.4 we finally obtain $\deg(\nabla\Phi_N, \Omega^N \setminus D_N, 0) = \chi(\Omega)(\chi(\Omega)-1)...(\chi(\Omega)-N+1)$. □

Appendix A. Properties of the Green's function

For some fixed $L, \hat{L} > 0$, let H and $\hat{H}$ denote the half spaces

$$H = \{(x,y) \in \mathbb{R}^2 : y < L\}, \quad \hat{H} = \{(x,y) \in \mathbb{R}^2 : y > -\hat{L}\}. \tag{A.1}$$

On the half-space H, the Green's function and its regular part are given explicitly by

$$\mathcal{G}_H(z,\eta) = \frac{1}{2\pi} \ln \frac{|z-\eta^*|}{|z-\eta|}, \quad \mathcal{H}_H(z,\eta) = \frac{1}{2\pi} \ln |z-\eta^*|, \tag{A.2}$$

where η^* is the reflection of η across ∂H.

Consider now $\Omega \subset \mathbb{R}^2$ smoothly bounded. For $x_k \in \Omega$, $d_k := d(x_k, \partial\Omega)$, $r_k > 0$, let $\mathcal{G}_k(z,\eta)$ and $\mathcal{H}_k(z,\eta)$ denote the Green's function of Δ in $\Omega_k := r_k^{-1}(\Omega - x_k)$ and its regular part.

Lemma A.1. *With the above notation, assume that, up to a subsequence, $r_k = o(d_k)$, so that Ω_k invades $\mathbb{R}^2$ as $k \to \infty$. Then, for every compact set $K \subset \mathbb{R}^2$, there exists C_K such that*

$$\left| \nabla_z \left(\mathcal{G}_k(z,\eta) - \frac{1}{2\pi} \ln \frac{1}{|z-\eta|} \right) \right| \leq C_K \frac{r_k}{d_k}$$

and

$$|\nabla_z \mathcal{H}_k(z,z)| \leq C_K \frac{r_k}{d_k},$$

uniformly for $z \in K$, $\eta \in \Omega_k$.

Proof. This follows from Appendix B in [18]. □

Lemma A.2. *With the above notation, assume that, up to a subsequence and a rotation, $r_k \to 0$, $d_k/r_k \to L \in (0,\infty)$, and Ω_k converges to the half-space H as $k \to \infty$. Then for every compact set $K \subset H$ and for k large enough*

$$\|\nabla(\mathcal{H}_k - \mathcal{H}_H)\|_{L^\infty(K \times \Omega_{k,\rho})} \leq C_K r_k \max\{1, \rho^{-1}\},$$

and

$$\|\nabla(\mathcal{G}_k - \mathcal{G}_H)\|_{L^\infty(K \times \Omega_{k,\rho})} \leq C_K r_k \max\{1, \rho^{-1}\},$$

where $\Omega_{k,\rho} := \{z \in \Omega_k : d(z, \partial\Omega_k) > \rho\}$.

Proof. We have

$$\begin{cases} \Delta_z \mathcal{H}_k(z,\eta) = 0 & \text{in } \Omega_k \times \Omega_k \\ \mathcal{H}_k(z,\eta) = \frac{1}{2\pi} \ln |z-\eta| & \text{for } z \in \partial\Omega_k,\ y \in \Omega_k. \end{cases} \tag{A.3}$$

We apply the maximum principle to the function

$$\mathcal{H}_k(z,\cdot) - \mathcal{H}_H(z,\cdot) : \Omega_k \to \mathbb{R},$$

where $z \in K$ and $K \subset H$ is a fixed compact set. We want to prove that

$$\sup_{z \in K, \eta \in \partial\Omega_k} |\mathcal{H}_k(z,\eta) - \mathcal{H}_H(z,\eta)| \leq C(K) r_k, \tag{A.4}$$

for k sufficiently large. By definition we have

$$\mathcal{H}_k(z,\eta) = \frac{1}{2\pi} \ln |z-\eta|, \quad \mathcal{H}_H(z,\eta) = \frac{1}{2\pi} \ln |z-\eta^*| \quad \text{for } \eta \in \partial\Omega_k.$$

Let $\delta > 0$ be so small that $\partial\Omega \cap B_\delta(p(x_k))$ can be written as the graph over $[-\delta, \delta]$ of a function f, where $p(x_k)$ is the nearest point projection of x_k onto $\partial\Omega$. Let

$\mathcal{L} \subset \mathbb{R}^2$ be the tangent line to $\partial\Omega$ at $p(x_k)$ and $f : \mathcal{L} \cap B_\delta(x_k) \to \mathbb{R}$. Up to a rotation we can assume that $f'(p(x_k)) = 0$ and by a Taylor expansion we obtain

$$|f(t)| \leq \frac{\sup_{[-\delta,\delta]} f''}{2} t^2, \quad t \in [-\delta, \delta].$$

In particular for each $x \in \partial\Omega \cap B_\delta(x_k)$ we have $d(x, L) \leq C_{\Omega,\delta}|x - x_k|^2$. Scaling to Ω_k we then obtain

$$\eta \in \partial\Omega_k, \quad |\eta| \leq \frac{\delta}{r_k} \quad \Rightarrow \quad d(\eta, \partial H) = O(r_k).$$

Then, for such η and for $z \in K$, we have $|z - \eta| \leq |z - \eta^*| + O(r_k)$ and

$$|\mathcal{H}_k - \mathcal{H}_H| = \left| \ln \frac{|z-\eta|}{|z-\eta^*|} \right| \leq \ln \left(1 + \frac{O(r_k)}{|z-\eta^*|} \right) = C(K)O(r_k),$$

since $\frac{1}{|z-\eta^*|} \geq \varepsilon(K) > 0$ uniformly for $z \in K$. On the other hand

$$\eta \in \partial\Omega_k, \quad |\eta| \geq \frac{\delta}{r_k} \quad \Rightarrow \quad |z - \eta^*| - |z - \eta| \leq C(K).$$

Then

$$|\mathcal{H}_k - \mathcal{H}_H| = \left| \ln \frac{|z-\eta|}{|z-\eta^*|} \right| \leq \ln \left(1 + \frac{r_k C(K)}{\delta} \right) = C(K)O(r_k),$$

so that (A.4) is proven. By the maximum principle we obtain

$$\sup_{z \in K, \eta \in \Omega_k} |\mathcal{H}_k(z, \eta) - \mathcal{H}_H(z, \eta)| = C(K)O(r_k), \tag{A.5}$$

and the bound on $\nabla(\mathcal{H}_k - \mathcal{H}_H)$ follows by elliptic estimates, since both $\mathcal{H}_H(z, \cdot)$ and $\mathcal{H}_k(z, \cdot)$ are harmonic. The estimate of $\nabla(\mathcal{G}_k - \mathcal{G}_H)$ follows at once since $\mathcal{G}_k - \mathcal{G}_H = \mathcal{H}_k - \mathcal{H}_H$. □

We state Lemmata A.3, A.4 and A.5 below with general positive coefficients $\alpha_{i,j}$ and β_i for future reference, and also to remark that the coefficients 8π and 4π appearing in the definitions of $\Phi_{N,h}$, Φ_N, etc..., can be replaced by any other positive constants.

Lemma A.3. *Consider for some $J \geq 2$ the functional $\Xi_{J,\mathbb{R}^2} : (\mathbb{R}^2)^J \setminus D_J \to \mathbb{R}$*

$$\Xi_{J,\mathbb{R}^2}(z_1, \ldots, z_J) := \sum_{\substack{1 \leq i,j \leq J \\ i \neq j}} \alpha_{i,j} \ln \frac{1}{|z_i - z_j|},$$

with $\alpha_{i,j} > 0$ for $1 \leq i, j \leq J$, $i \neq j$. Then $\Xi_{J,\mathbb{R}^2}$ has no critical points.

Proof. Consider $1 \leq i \leq J$ such that z_i lies on the boundary of the convex hull of $\{z_1, \ldots, z_J\}$, and observe that the derivative with respect to z_i in a direction pointing outwards of the convex hull is negative, as $|z_i - z_j|$ is increasing for every $1 \leq i \neq j \leq J$. □

Lemma A.4. *Consider for some $J \geq 1$ the functional $\hat{\Xi}_{J,\hat{H}} : \hat{H}^J \setminus D_J \to \mathbb{R}$*

$$\hat{\Xi}_{J,\hat{H}}(z_1, \ldots, z_J) := \sum_{\substack{1 \leq i,j \leq J \\ i \neq j}} \alpha_{i,j} \ln \frac{1}{|z_i - z_j|} + \sum_{1 \leq i \leq J} \gamma_i \ln \frac{1}{d(z_i, \partial\hat{H})},$$

with $\alpha_{i,j} > 0$ for $1 \leq i < j \leq J$ and $\gamma_i > 0$ for $1 \leq i \leq J$. Then $\hat{\Xi}_{J,\hat{H}}$ has no critical point.

Proof. Given $(z_1,\dots,z_J)$, write $z_i=(a_i,b_i)$ and up to reordering, assume that $b_1\le\cdots\le b_J$, i.e. z_J is one of the points farthest from $\partial\hat{H}$. Then

$$\frac{\partial}{\partial b_J}\ln\frac{1}{|z_i-z_J|}\le 0,\quad \text{for } 1\le i<J,\quad \frac{\partial}{\partial b_J}\ln\frac{1}{d(z_J,\partial\hat{H})}<0,\quad \text{hence } \frac{\partial\hat{\Xi}_{J,\hat{H}}(z_1,\dots,z_J)}{\partial b_J}<0.$$

□

Lemma A.5. *Consider for some $J\ge 1$ the functionals $\Xi^{\pm}_{J,H}:H^J\setminus D_J\to\mathbb{R}$ given as*

$$\Xi^{\pm}_{J,H}(z_1,\dots,z_J):=\sum_{\substack{1\le i,j\le J\\ i\neq j}}\alpha_{i,j}\mathcal{G}_H(z_i,z_j)\pm\sum_{1\le i\le J}\beta_j\mathcal{H}_H(z_i,z_i),$$

with $\alpha_{i,j}>0$ for $1\le i,j\le J$, $i\neq j$, $\beta_j>0$ for $1\le i\le J$. Then $\Xi^{\pm}_{J,H}$ has no critical points.

Proof. Given $(z_1,\dots,z_J)$, write in coordinates $z_i=(a_i,b_i)\in H$ for $1\le i\le J$ and, up to reordering, assume first that $J>1$, $a_1\le a_2\le\cdots\le a_J$ and $a_1<a_J$. Then, it follows from (A.2)

$$\frac{\partial\mathcal{G}_H(z_1,z_J)}{\partial a_1}>0,\quad \frac{\partial\mathcal{G}_H(z_1,z_j)}{\partial a_1}\ge 0,\ \text{for } 2\le j\le J-1,\quad \frac{\partial\mathcal{H}_H(z_1,z_1)}{\partial a_1}=0,$$

hence $\nabla_{z_1}\Xi^{\pm}_{J,H}(z_1,\dots,z_J)\neq 0$.

If $J=1$ or all the $z_i's$ are vertically, i.e. $a_1=\cdots=a_J$, then, up to reordering, assume that $b_1<\cdots<b_J$. Then, again from (A.2), we get

$$\frac{\partial\mathcal{G}_H(z_1,z_j)}{\partial b_1}<0,\quad \text{for } 2\le j\le J,\quad \frac{\partial\mathcal{H}_H(z_1,z_1)}{\partial b_1}<0,\tag{A.6}$$

hence $\nabla_{z_1}\Xi^{+}_{J,H}(z_1,\dots,z_J)\neq 0$, and

$$\frac{\partial\mathcal{G}_H(z_J,z_j)}{\partial b_J}>0,\quad \text{for } 2\le j\le J,\quad \frac{\partial\mathcal{H}_H(z_J,z_J)}{\partial b_J}<0,$$

hence $\nabla_{z_N}\Xi^{-}_{J,H}(z_1,\dots,z_J)\neq 0$. □

Lemma A.6. *Consider for some $J\ge 1$ the functional $\Xi^{\varphi}_{J,H,\hat{H}}:(H\cap\hat{H})^J\setminus D_J\to\mathbb{R}$ given as*

$$\Xi^{\varphi}_{J,H,\hat{H}}(z_1,\dots,z_J):=\Xi^{+}_{J,H}(z_1,\dots,z_J)+\sum_{1\le i\le J}\gamma_i\varphi(z_i),$$

where H and $\hat{H}$ are as in (A.1) for some $L,\hat{L}>0$, $\Xi^{+}_{J,H}$ is as in Lemma A.5, $\gamma_i\ge 0$, for $1\le i\le J$, $\varphi\in C^1(H\cap\hat{H})$ and $\nabla\varphi(z)\cdot(0,1)\le 0$ for every $z\in H\cap\hat{H}$. Then $\Xi^{\varphi}_{J,H,\hat{H}}$ has no critical points.

Proof. The proof is identical to the proof of Lemma A.5, using (A.6) together with

$$\frac{\partial\varphi(z_1)}{\partial b_1}=\nabla\varphi(z_1)\cdot(0,1)\le 0.$$

□

Appendix B. The Euler characteristic of configuration spaces

We will need the following general fact about *configuration spaces*.

Proposition B.1. *Let Ω be a smoothly bounded domain of $\mathbb{R}^2$ and call for any $N \in \mathbb{N}^\star$*

$$F(\Omega, N) := \Omega^N \setminus D_N = \left\{(x_1, \dots, x_N) \in \Omega^N \ : \ x_i \neq x_j \text{ for } i \neq j\right\}.$$

Then

$$\chi(F(\Omega, N)) = \chi(\Omega)(\chi(\Omega) - 1)...(\chi(\Omega) - N + 1).$$

Proof. This result follows from the fact that $F(\Omega, N)$ fibers on $F(\Omega, N-1)$ with fiber Ω_{N-1} (projecting on the last of the N points), where we denote by Ω_j the set Ω with j points removed, see [35, Chapter 2].

Claim The above fibration is *orientable*, in the sense that $\pi_1(F(\Omega, N-1))$ acts trivially on the homology $H_*(\Omega_{N-1}, \mathbb{R})$ of the fiber.

Proof of the claim. Since Ω_{N-1} is homotopically equivalent to a (connected) finite union of circles, the only relevant homology group of Ω_{N-1} is $H_1(\Omega_{N-1}, \mathbb{R})$. Consider now a closed path $\gamma \in C^0([0,1], F(\Omega, N-1))$ representing a homotopy class $[\gamma] \in \pi_1(F(\Omega, N-1))$. Set

$$(x_1(t), \dots, x_{N-1}(t)) := \gamma(t).$$

For the point $\gamma(0) =: (x_1, \dots, x_{N-1}) \in F(\Omega, N-1)$ consider the fiber

$$\Omega \setminus \{x_1, \dots, x_{N-1}\} \simeq \Omega_{N-1}$$

and choose generators of its first homology group as follows. Since Ω is a two-dimensional domain, it is homotopic to a disk with r points removed, for $r = 0, 1, 2, \dots$. Given the i-th hole of Ω, $i = 1, \dots, r$, choose an element σ_i of $H_1(\Omega, \mathbb{R})$ in the form of a smooth Jordan curve that does not contain any of the points $x_1(t), \dots, x_{N-1}(t)$ with $t \in [0,1]$. Then it is clear that the action of γ on $\sigma_1, \dots, \sigma_r$ is trivial.

Now, for each x_j, $j = 1, \dots, k-1$, choose a small and smooth Jordan curve η_j around x_j not intersecting $\sigma_1, \dots, \sigma_r$, for instance the oriented boundary of $B_\varepsilon(x_j)$. By compactness, if we choose ε small enough we get

$$\overline{B_\varepsilon(x_j(t))} \cap \overline{B_\varepsilon(x_i(t))} = \emptyset, \quad \text{for } 1 \leq i < j \leq N-1, \ t \in [0,1]$$

and

$$\overline{B_\varepsilon(x_j(t))} \cap \sigma_i = \emptyset, \quad \text{for } 1 \leq j \leq N-1, \ 1 \leq i \leq r, \ t \in [0,1].$$

Again it follows immediately that the action of $[\gamma]$ of η_j is trivial. Since

$$H_1(\Omega_{N-1}, \mathbb{R}) = \operatorname{span}\{\sigma_i, \eta_j, 1 \leq i \leq r, 1 \leq j \leq N-1\}$$

the claim is proven. □

By the product formula for orientable fibrations, see Spanier [35, p.481], we infer

$$\chi(F(\Omega, N)) = \chi(F(\Omega, N-1))\chi(\Omega_{N-1}) = \chi(F(\Omega, N-1))(\chi(\Omega) - N + 1).$$

Now the proposition follows at once by induction in N. □

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