



Single Point Particle Motion: Mass, Force, Momenta, Impulse, Work

Luciano da Fontoura Costa

► To cite this version:

Luciano da Fontoura Costa. Single Point Particle Motion: Mass, Force, Momenta, Impulse, Work. 2022. ⟨hal-03907279⟩

HAL Id: hal-03907279

<https://hal.science/hal-03907279v1>

Preprint submitted on 19 Dec 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Single Point Particle Motion: Mass, Force, Momenta, Impulse, Work...

Luciano da F. Costa
luciano@ifsc.usp.br

São Carlos Institute of Physics – DFCM/USP

15th Nov. 2022

Abstract

In this work, which mostly constitutes a sequence of a previous work (Point Motion in Flat Spaces: An Ample Starting Point), we approach the interesting problem of representing and studying the position, velocity, acceleration and forces on point particles with mass appearing in respective trajectories unfolded in geometrical space, which is considered from the classic perspective. While the previous work addressed the motion of a massless point from an almost exclusive mathematical point of view, in the present work we study how the incorporation of physical principles, especially Newton’s law of motion, effectively restrict the otherwise ampler mathematical possibilities related to massless point motion. It is hoped that this type of study provides an interesting prospect from which to better understand several basic physical concepts including mass, force (especially gravitational interactions and the elastic force, though electromagnetic forces are also briefly discussed), free fall, conservative fields, linear and angular momentum, impulse, torque, the relationship between forces and curvature, as well as the linear and non-linear pendula. The impulse-momentum theorem, as well as the principles of conservation of linear momentum are also briefly addressed. The presented study is complemented by the consideration of composition (or coupling) of forces, a simple informal derivation of the wave equation, as well as by basic related concepts including density, pressure, flow and flux. The important concept of work is then briefly covered as a last topic, providing a bridge to complementary studies involving the concept of energy.

“ [A body in vacuum] will either stay at rest or move forever,
or until colliding with some other object.”

Aristotle (Physics, Book IV).

1 Introduction

In a *previous* work, we addressed the motion of massless point particles specified in purely mathematical terms and then obtained the respective velocity and acceleration in terms of the first and second derivatives of respective parametric curves (trajectories). That approach had several motivations, including focusing on the mathematical aspects of motion, which also provided a broader perspective since no physical constraints were taken into account. In a sense, that approach tried to go as far into physics while relying only on mathematics. Among other aspects, we were then able to consider a relatively wider range of possible manners in which a scalar or vector field could interact, from the mathematical point of view, regarding

massless point particle motion.

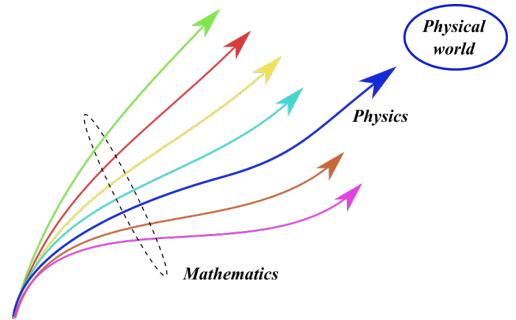


Figure 1: Abstract illustration of the concept that Physics only takes one (or a few) of the many paths provided by Mathematics.

One of the main objectives of the *present* work is to incorporate into the previous approach several related physical principles ([1, 2, 3]), which therefore imply choices among the several previously discussed mathematical possibilities, an idea illustrated in Figure 1.

For instance, as indicated by Newton’s second law of motion, the interaction between the properties

(e.g. derivatives) of a point particle with mass m involves a directly proportional relationship between the resulting applied force and the *second* derivative of motion, with the mass acting as the proportionality constant. However, we shall also consider some of the consequences of having the gravitational interaction between masses to depend reciprocally of distinct powers of the distance between them, which is then found to define completely distinct types of motion and orbits.

However, the above endeavor constitutes only one of the objectives of the present work. Additional aims include illustrating of how several related mathematical aspects from linear algebra (e.g. [4]), multivariate calculus (e.g. [5, 6, 7, 8]), differential equations (e.g. [9]), and differential geometry (e.g. [10]) can be effectively applied to model an ample range of physical phenomena.

It is hoped that the present work also illustrates how just a few physical laws, especially Newton’s laws of motion, can be generally applied to better understand, through modeling, a wide range of interesting problems including weightlessness, composite mechanical systems, linear and non-linear oscillations, as well as transversal and longitudinal wave equations. In addition, the many presented and illustrated concepts and methods should also provide some preliminary rudiments for subsequent studies in several related areas not only in Physics (e.g. classical mechanics, electromagnetism, relativity, quantum mechanics, thermodynamics, and statistical physics), as well as from other disciplines (e.g. electric engineering, computer graphics, image analysis, pattern recognition, transportation systems, biophysics, etc.).

Here, we shall be happy to incorporate mass and force in our previous study [11], obtaining the motion of a point particle with mass m by integrating over force fields while considering Newton’s laws, which establish (for non-relativistic velocities) a relationship between acceleration of an infinitesimal particle with mass m and the instantaneous force applied along time on that particle.

As it could be expected, a great deal of our attention in the present work focuses on gravitation and elastic forces. Though the important electromagnetic force is also briefly addressed, for the sake of completeness and comparison, this subject is worth a further, more comprehensive study.

An interesting aspect of studying particle motion that is not so often realized is that, in addition to its intrinsic interest from the Physics perspective, these studies also provide an excellent opportunity for consolidating and better understanding mathematical concepts and methods. Even more important, the study of point particle motion provides a prototypical example of mathematical modeling (e.g. [12]).

Though it is hoped that the present work may contribute regarding all the above aspects, it is also inter-

esting to observe that, as with [11], it has been mostly conceived as a resource that could contribute with part of the basic concepts and methods that can potentially help researchers with mathematics, engineering and computer science backgrounds to probe further into areas related to complex systems, pattern recognition, data sciences, deep learning, scientific visualization, image analysis, computer vision, non-linear systems, statistical physics, variational calculus, as well as tensor algebra and calculus. That is basically why a more effective understanding of the concepts and methods presented here could benefit from preliminary familiarization with basic concepts from linear algebra, calculus, and differential geometry.

Several of the concepts and methods addressed here are illustrated by respective case examples, which have been chosen/devised in order to hopefully minimize the cost/benefit ratio. Efforts have been spent also in trying to follow a logical order characterized by successive integration of concepts and following, whenever possible, from simpler to progressively more elaborate subjects.

It should be observed that the present work is not intended to be a first course on particle motion. Not only concepts from multivariate calculus, differential equations, and linear algebra are required, but also emphasis is specifically placed on point particles and Newton’s second law. In addition, some specific approaches are adopted, including the concepts of *artificial forces*, *conjugated pairs of forces* and *ghost mechanical elements*, that are not orthodoxly adopted in basic physics courses.

2 Space and Time from the Human Perspective

Possibly the most intuitive and basic concepts humans have about physics concerns *space* and *time*, into which we are immersed from the very onset of our lives. Indeed, as our daily experiences seem to attest, we live in a 3-dimensional Cartesian space (i.e. \mathbb{R}^3), where time is independently measured from space, progressing inexorably from past to future. Though these notions have been proved not to be fully accurate (e.g for speeds close to the velocity of light in relativity theory), the present work will focus on classical physics, limited to substantially smaller speeds, where the the 3-dimensional Cartesian space and the linear time hypotheses largely hold with remarkable accuracy.

One of the not so straightforward classical aspects of space and time that are not so intuitive to humans concerns the choice of an absolute frame of reference for observing motion. Basically, each human being carries a personal reference frame, relative to our own position. Though this frame works fine in most situations, we start

perceiving its limitations when we are in motion, especially accelerated motion. For instance, it becomes difficult to discern the types of surrounding motions when one is in rotation, such as when one is on a carousel or going around an accentuated curve.

Our difficulties while figuring out surrounding space and motion (e.g. [13, 14, 15, 16]) seem to be related not only to the mathematical fact that motion changes when expressed respectively to an accelerating frame, but also from some intrinsic cognitive difficulties related to the increased demands required for integrating several clues about velocity and distance (e.g. parallax, perspective, relative motion, occlusion, inner ear action, etc.). Be that as it may, a sound conceptual understanding and familiarity of relative motion constitutes a particular important aspect regarding human understanding and modeling of the surrounding 3D space.

The present work will be limited to frames of reference that are still or undergoing uniform linear motion, which are known in classic physics as *inertial frames of reference*. Still, it is often necessary to choose a suitable origin for our frames.

Though inherently simple, given its 1D nature, time also poses some perceptual difficulties in the sense that the way in which it is perceived by humans tends to vary in terms of distinct contexts and situations. For instance, we tend to perceive time as running slower when we focus attention on it (e.g. [17, 18]).

3 Mass and Force

While the above discussed approach concerning classic time and space is justifiable from the mathematical perspective, it is only by introducing the concepts of *mass* and *force* that time and space can be start being appreciated from the physical manner, even if only from the classic perspective. The introduction of these two concepts has a major impact in restricting the otherwise ample mathematical possibilities regarding how motion could, in principle, interact with a field. As a consequence, as illustrated in Figure 1, only a narrower range of otherwise viable mathematical models becomes feasible from the real-world physical perspective.

We start our approach to mass by considering a single particle with mass m , measured in kilograms (kg), moving in an otherwise completely empty universe, and therefore subjected to no field or other types of possible influences. Quite interestingly, as figured out by Galileu Galilei (1564–1642), this particle will necessarily move with constant velocity (magnitude and direction) respectively to any reference frame also moving with constant speed, which are called *inertial frames of reference*. Ob-

serve that this includes the situation where the particle remains stopped, because this constitutes but a special type of motion, with null constant speed. Interestingly, the fact that linear motion is not qualitatively altered when expressed from inertial frames of reference has been shown in our previous discussion [11] to be a purely mathematical concept, not necessarily related to physics or mass.

It is interesting to observe that Galileu’s discovery already strongly suggests that, given that velocity is preserved in absence of external effects, interactions between the particle motion and external fields probably involve the second time-derivative of the particle position. We already have already seen in [11] that a possible way in which the motion of a point can be influenced by a vector field $\vec{\phi}$ consists in having one of the time-derivatives of the motion to be proportional and parallel to the vector field, i.e.:

$$b \frac{d^n \vec{u}(t)}{dt^n} = c \vec{\phi}(x, y, z, t) \quad (1)$$

where n is the order of the only time derivative of the point position to be influenced by the field, and b, c are real-valued proportionality constants.

It was Isaac Newton (1642–1726) who discovered that the above mathematical possible interaction between particle motion and a vector field was valid (respectively to the non-relativistic velocities), taking pace for $n = 2$, while the constant b was found to correspond to the mass m of the particle. In the case of the time-constant gravitational forces addressed by Newton, we have that the constant c also corresponds to the mass m of the particle, with the force vector being then expressed as:

$$c \vec{\phi}(x, y, z) = m \vec{g}(x, y, z) \quad (2)$$

where $\vec{g}(x, y, z)$ is the *gravitational acceleration*, point towards the center of the earth, and is therefore normal to its surface. These essential findings regarding gravitation can then be summarized in terms of the following relationship, known as *Newton’s second law of motion*:

$$\begin{aligned} m \frac{d^2 \vec{u}(t)}{dt^2} &= \vec{a}(x, y, z, t) = m \vec{g}(x, y, z, t) \implies \\ \implies \boxed{\frac{d^2 \vec{u}(t)}{dt^2} &= \vec{a}(x, y, z, t) = \vec{g}(x, y, z, t)} \end{aligned} \quad (3)$$

where $\vec{a}(t)$ is the acceleration of the particle. Though consisting of an approximation, oftentimes the gravitational force / acceleration are taken as being constant along time, which is assumed henceforth unless contrariwise specified.

There are three Newton’s law of motion, which are summarized as follows, concerning point particles with mass m :

- I. Every point particle undergoes uniform linear motion (including rest) whenever there is no resulting force acting on them;
- II. $\vec{f} = m \vec{a}$, where \vec{f} is the resulting force on the particle at each time instant t ;
- III. Every applied force necessarily implies a *reaction* force.

The two first laws are closely interrelated, with the former being a consequence of the second when the resulting force is zero. The third law actually seems to have been more concerned with conservation of linear momentum. Though seemingly simple, it has often been misunderstood, so it will receive additional attention here, especially in Section 14, in the context of coupled forces. Actually, Newton's third law has an immediate important implication that is not always realized or considered: every force occurs in a paired manner or, in other words, *there is no such a thing as a single, isolated force* in nature.

While it does not follow that every problem involving particle motion should consider all action-reaction pairs, it is a good practice always at least to be aware about which are the involved pairs and how not considering some of the reactions could impact the understanding and solution of a given problem. The consideration of all involved pairs is all the most relevant when conservations of linear momentum, angular momentum, for of energy are involved.

An interesting additional discussion of Newton's law of motion can be found in [19].

The gravitational action on mass is a special type of force, being governed by *Newton's law of universal gravitation*, which states that the *gravitational force* between two point particles with respective masses m_1 and m_2 that are distant one another by r can be expressed in terms of magnitude values as:

$$f = G \frac{m_1 m_2}{r^2} \quad (4)$$

where r is the distance between the two particles and $G = 6.67430 \cdot 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1}$, recommended for the year of 2018 (interestingly, the gravitational acceleration of the Earth varies along time).

A more complete version of the previous law, in terms of vector quantities, can be expressed as follows (please refer to Figure 2):

$$\vec{f}_{1 \rightarrow 2} = G \frac{m_1 m_2}{\|\vec{r}_{1 \rightarrow 2}\|^2} \hat{r}_{1 \rightarrow 2} = -G \frac{m_1 m_2}{\|\vec{r}_{2 \rightarrow 1}\|^2} \hat{r}_{2 \rightarrow 1} \quad (5)$$

$$\vec{f}_{2 \rightarrow 1} = G \frac{m_1 m_2}{\|\vec{r}_{2 \rightarrow 1}\|^2} \hat{r}_{2 \rightarrow 1} = -G \frac{m_1 m_2}{\|\vec{r}_{1 \rightarrow 2}\|^2} \hat{r}_{1 \rightarrow 2} \quad (6)$$

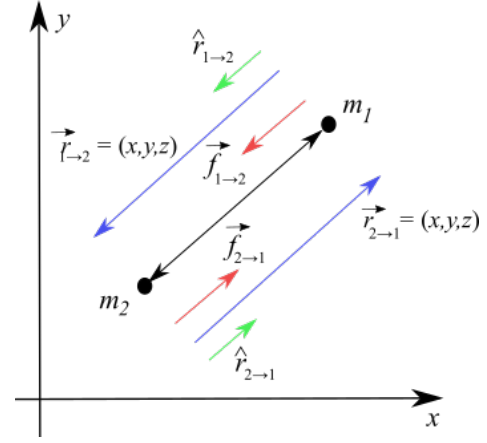


Figure 2: Basic configuration underlying the 2D version of *Newton's law of universal gravitation*. The two point particles have infinitesimal dimensions. The motion of the particles will not change qualitatively when expressed respectively to any inertial frame of reference (i.e. moving with constant speed).

Observe that:

$$\begin{aligned} \vec{r}_{1 \rightarrow 2} = -\vec{r}_{2 \rightarrow 1} &\implies \\ \implies \|\vec{r}_{1 \rightarrow 2}\| = \|\vec{r}_{2 \rightarrow 1}\| &\implies \|\vec{f}_{1 \rightarrow 2}\| = \|\vec{f}_{2 \rightarrow 1}\| \end{aligned} \quad (7)$$

Figure 2 depicts the situation commonly assumed regarding Newton's law of universal gravitation. The shown $x \times y$ coordinate system is assumed to correspond to a valid inertial frame of reference.

As can be realized from the previous equations and Figure 2, Newton's law of universal gravitation involves a pair of forces with identical magnitudes, but pointing one towards the other, as in a compression, in full agreement with Newton's third law. Actually, as we will see later, the linear motion of this system is conserved because the two opposite forces cancel one another, yielding no net impulse.

Because there are forces applied to the particles with mass m , respective accelerations take place along the same direction of the forces, according to Newton's second law. Thou simple, this example illustrates how Newton's three laws nicely complement one another while completely describing the interactions between the two particles.

In the case of earth gravitation, the, particle corresponding to the earth is understood to remain mostly fixed at the origin of the frame of reference. There are two possible justifications for this approximation: (i) being much larger than the masses of objects at human scale, the earth hardly moves in reaction to free falls; and (ii) the frame is understood to be attached to earth. The problem with the latter approach is that the earth undergoes acceleration as a reaction to the free fall, and therefore does not correspond, strictly speaking, to a classic iner-

tial frame of reference.

In the case of Figure 2, we have that placing the frame at particle 2 (assumed to correspond to the earth) will no longer ensure conservation of linear moment (see Section 5), which should nevertheless be expected in this specific situation. Indeed, when the frame is attached to particle 2, only particle 1 will have non-null velocity, which will actually be larger than that expressed respectively in an inertial frame of reference (it will correspond to the sum of the otherwise implied velocities of the two particles along time). As a consequence, given that there are no velocities in opposite orientation, even though the total impulse results zero, the linear momentum will be mistakenly understood as not being conserved. In other words the impulse-momentum theorem simply does not hold for non-inertial frames of reference.

It is interesting to observe that it is not only that particle 1 is “falling” towards particle 2, but the opposite is also verified. So, when an object is falling towards the surface of earth, the earth is also falling towards the object, but with a much smaller velocity because of its substantially larger mass.

It follows from the above discussion that, though just one of the forces implied by gravitation may be eventually taken as isolated, it should be kept in mind that the impulse momentum theorem considering the whole system (i.e. the two particles from which the force originates) will be undermined.

Now, by assuming that particle 2 undergoes negligible motion as a consequence of the reaction to the motion of particle 1, we can understand particle 2 as “fixed” use on the origin of the adopted frame of reference. Newton’s second can then be brought in to obtain:

$$\begin{aligned}\vec{f} &= m_1 \vec{g}(x, y, z) = -G \frac{m_1 m_2}{\|\vec{r}\|^2} \hat{r} \implies \\ \implies \vec{g}(x, y, z) &= -G \frac{m_2}{\|\vec{r}\|^2} \frac{\vec{r}}{\sqrt{x^2 + y^2 + z^2}}\end{aligned}$$

where $\vec{r} = (x, y, z)$ is the position of particle 1 and \hat{r} is the respective versor pointing from $(0, 0, 0)$ towards particle 1. By making $m_2 = m$ for simplicity, the previous equation can be rewritten as:

$$\vec{g}(x, y, z) = -\frac{G m}{\|\vec{r}\|^2} \hat{r} = -\frac{G m}{\|\vec{r}\|^3} \vec{r} \quad (8)$$

which is known as the *gravitational field*, having $N kg^{-1}$ as unit or $m s^{-2}$, and can then be understood as acceleration.

Now, if we plug in the earth’s mass $m_2 \approx 5.972 \cdot 10^{24} kg$ and radius $r \approx 6.371 \cdot 10^6 m$, we get:

$$\|g\| = \frac{G m}{r^2} \approx 6.67430 \frac{5.98 \cdot 10^{24}}{[6.38 \cdot 10^6]^2} \approx 9.819973 m s^{-2} \quad (9)$$

It should be observed that the gravitational force implied by a non-infinitesimal sphere can in several situations, including the above calculation, be modeled by replacing the sphere by a point particle with the same original mass, as it is adopted throughout the present work.

Let us consider another situation, involving two point masses of $1 kg$ each, separated by a distance of $1 m$. The gravitational force in this case will be:

$$f = G \frac{m_1 m_2}{r^2} = G = 6.67430 \cdot 10^{-11} N \quad (10)$$

Now, this is a proverbially diminutive force, which can by no means be compared to other common forces such as those produced by elastic materials or magnets. The reason the gravitational field of the earth is capable of exerting an appreciable force on us (our weights), is because the earth has such a big mass. In addition, observe that in this case it would be by no means a good approximation to understand one of the particles as being nearly at rest, as it was the case with the earth.

Being a conservative field, the *gravitational* field is the gradient of a respective potential. More specifically, the *gravitational potential energy* between two masses m and M distant one another by r can be given as:

$$U(r) = -G \frac{m M}{r} \quad (11)$$

which has Joules (J) as unit.

The *gravitational potential*, which is a different but related concept, corresponds to:

$$V(r) = \frac{U(r)}{m} = -G \frac{M}{r} \quad (12)$$

which has energy per mass ($J kg^{-1}$) as unit. This potential at a given point (x, y, z) can be understood as the total work necessary to bring a mass of $1 kg$ from the infinite to that position.

An interesting property of Newton’s law of motion, as well as all other physical principles that we shall adopt in the present work, is that all the obtained motions are time-reversible, in the sense that if we make $\tilde{t} = -t$, all the obtained trajectories would still satisfy all considered laws. The irreversibility of the physical world turns out to be related to statistical mechanics aspects, in particular the second law of thermodynamics.

4 Trajectories and Orbits

Having discussed gravitation to some extent, it is now interesting to consider the types of trajectories that can be respectively obtained. Figure 3 illustrates several trajectories obtained under a gravitational field by a point

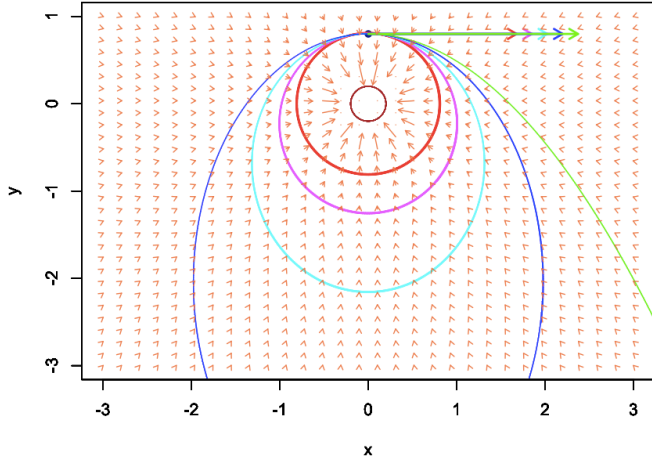


Figure 3: Examples of possible trajectories of a particle initiating its motion from the point marked as a blue dot with five equally spaced velocities (shown by arrows), increasing from the orbital speed to the escape speed. The gravitational field, shown in coral, is understood to be generated by the brown disk, whose center coincides with the frame origin. A circular orbit is obtained for the smallest speed (orbital speed). The four successive trajectories correspond to elliptic orbits, while the final parabolic trajectory is characterized by the particle escaping from being captured into a closed orbit. The velocities and accelerations are shown in relative scales among each of these two types of vectors, but not across them. Results obtained numerically by Verlet integration (e.g. [20]).

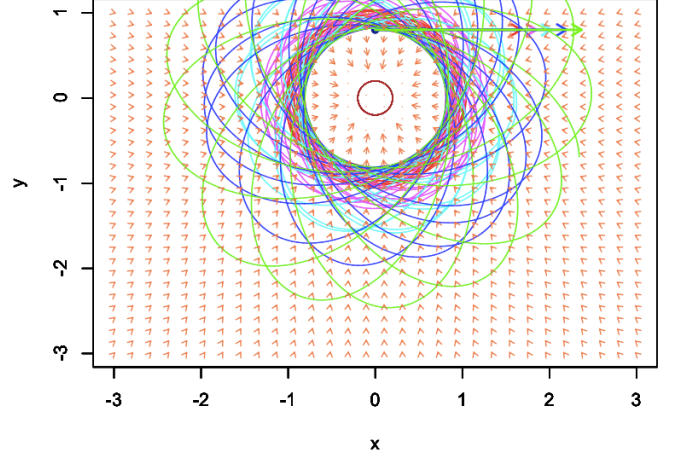


Figure 4: The trajectories, assuming the same setting as for the results shown in Fig. 3 that would be obtained in case Newton’s law of universal gravitation hypothetically followed $f = G(m_1 m_2)/r$ instead of $f = G(m_1 m_2)/r^2$. All obtained trajectories are orbits corresponding to Lissajous curves. This type of universe would be characterized by intricate ‘precessional’ orbits, which would at the same time remain closer to the reference center than in our real world (compare with Fig. 3).

particle, assuming diverse initial conditions shown in the figure.

All initial velocities are tangent to a circle, at a given distance, around the earth. A circular orbit is obtained for the smallest considered initial velocity magnitude. Larger magnitudes can be found to yield ellipses of increasing sizes, up to a scape point where the particle cannot be captured by the considered gravitational field.

In the spirit of considering how physics could be different, but still mathematically well-defined, Figure 4 shows the trajectories that would be obtained for the same situations as in the previous example, but now with the gravitational force varying inversely with the first power (instead of the second power, as is the case in reality) of the distance between the moving particle and the center of the larger mass. Interestingly, “precessional” trajectories are obtained in all cases, with sizes that increase with the magnitude of the initial velocity.

Another interesting hypothetical situation would concern the gravitational field magnitude being inversely proportional to the third power of the distance between the masses. Figure 5 presents the trajectories that are obtained for the same configurations as in the previous examples. Now, all the obtained motions are little affected by the gravitational field, escaping quickly from its action.

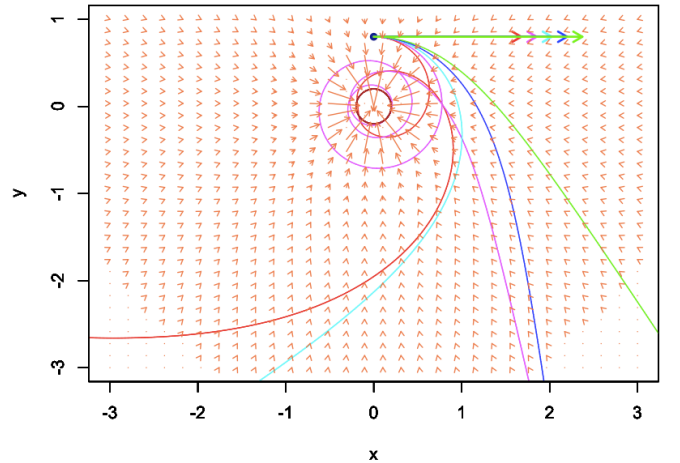


Figure 5: Trajectories of concerning the same initial situation as in Fig. 3, but with $f = G(m_1 m_2)/r^3$. No stable, closed orbit is now obtained, with the force field casting the moving mass away.

5 Linear Momentum and Impulse

Moving particles have several properties in addition to their position and respective derivatives. One particularly important addition feature of a point particle with mass m and velocity $\vec{v}(t)$ concerns its *linear momentum*, a vector quantity that is defined as follows:

$$\boxed{\vec{p}(t) = m \vec{v}(t) = m \dot{\vec{u}}(t)} \quad (13)$$

Observe that the above definition holds not only for the whole trajectory, but also for each specific time instant t . In addition, the linear momentum is always parallel to the velocity of the particle, with the mass m acting as a proportionality constant between velocity and linear momentum magnitude.

The linear momentum of a particle is particularly interesting as it is directly related to the particle kinetic energy, and also because it can be related to the applied forces by using the concept of impulse, as discussed in this section. In addition, the linear momentum plays a central role in Hamiltonian mechanics.

It follows from the above definition that the force is *identical* to the time derivative of the particle momentum:

$$\vec{f}(t) = \frac{d\vec{p}(t)}{dt} = \dot{\vec{p}}(t) \quad (14)$$

We also have that:

$$\vec{p}(t) = \int_{t_i}^{t_f} \vec{f} dt = \vec{p}(t_f) - \vec{p}(t_i) \quad (15)$$

where the integral term is called the *impulse* on the particle taking place from time t_i to time t_f :

$$\vec{J}(t_i, t_f) = \int_{t_i}^{t_f} \vec{f} dt = \vec{p}(t_f) - \vec{p}(t_i) \quad (16)$$

which can be summarized as:

$$\boxed{\vec{J} = \Delta \vec{p}} \quad (17)$$

The above identity is called the *impulse-momentum* theorem. One of its immediate implications is that the linear momentum of a particle remains constant in absence of resulting applied forces, which is directly related to Newton's third law of motion.

The fact that the linear momentum of a particle does not change in the absence of resulting force is referred to as the principle of *linear momentum conservation*, one of the several conservation laws in physics.

It is important to realize that, in Equation 16, the force is a vector involving three components that can vary in time, i.e.:

$$\vec{f}(t) = (f_x(t), f_y(t), f_z(t))$$

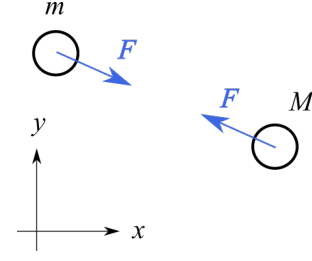


Figure 6: Two free particles in a frame of reference are acted upon by respective gravitational forces with the same magnitude but opposite directions, during a time interval ΔT . The sum of the impulse received by each particle has necessarily identical magnitude, but opposite directions, from which we have that the net impulse is the zero vector. The impulse-momentum theorem then indicates that the total linear momentum of the two particles is conserved during Δt .

so that the respective impulse is obtained by integrating each of this components with respect to time along the interval t_i to t_f , i.e.:

$$\begin{aligned} J_x &= \int_{t_0}^{t_f} f_x(t) dt \\ J_y &= \int_{t_0}^{t_f} f_y(t) dt \\ J_z &= \int_{t_0}^{t_f} f_z(t) dt \end{aligned}$$

Leading to the following resulting impulse:

$$\vec{J} = (J_x, J_y, J_z) \quad (18)$$

Oftentimes, it becomes interesting to consider the linear momentum and/or impulse not only on a single particle, but on two or more particles. Figure 6 illustrates two particles with masses m and M onto which forces with the same magnitude F , but opposite orientations (which we shall call a conjugated pair of forces) are respectively applied. No additional forces are imposed, and the motion is considered with respect to the inertial frame of reference shown as $x \times y$ in the figure. The application of the forces is assumed to last for a time period Δt .

Because the impulse received by each mass necessarily has the same magnitude, but opposite orientations, when they are vectorially added the zero vector during Δt , indicating no net impulse on the system of two particles. Then, by considering the impulse-momentum theorem, it can be concluded that the total linear momentum of the system of two particles is not modified, being therefore conserved. The same result is obtained for several pair or particles under similar opposite forces, and also for situations where the resulting force is null.

Given that the same particle motion will yield varying velocities along time when considered from distinct inertial frames of reference, it becomes an interesting issue

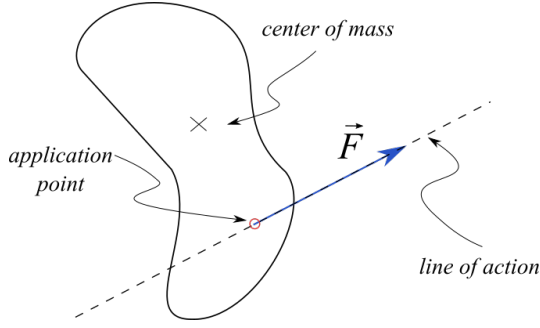


Figure 7: Illustration of the concepts of *application point* and *line of action* of a given force \vec{F} , relative to a non-infinitesimal object whose center of mass is also identified.

to consider what are the respective effects for the linear momentum of the motion. We now immediately that it will very likely be distinct in quantitative different frames. However, interestingly, the impulse-momentum theorem, and therefore the conservation of linear momentum in absence of resulting forces *is preserved* independently of the inertial frame adopted, even though the involved quantities are different.

6 Angular Momentum

Given a particle of mass m moving with velocity $\vec{v}(t)$ under possible influence of a resulting force $\vec{f}(t)$, as well as a reference point $P = (x_P, y_P)$ in the same space, the *angular momentum* of that particle is a vector quantity defined as:

$$\vec{L}(t) = \vec{r}(t) \times \vec{p}(t) = \vec{r}(t) \times [m \vec{v}(t)] \quad (19)$$

Observe that the resulting force does not appear directly in the angular momentum.

The above definition evidentiates that angular momentum, as well as torque to be soon presented, mainly arise when we the respective systems involve pivots or forces with their line of action not passing through the center of mass of non-infinitesimal objects.

Figure 7 illustrates the concepts of *application point* and *line of action* of a given force \vec{F} . Observe that these concepts only apply to objects with non-infinitesimal sides, otherwise the application point will coincide with the particle and the line of action will pass necessarily through the particle position.

The consideration of non-infinitesimal mechanical objects, which are systematically studied in *Classic Mechanics*, typically require the application of volume and surface integrals, and therefore will not be considered in the present work. The non-infinitesimal objects here concern situations in which the symmetry and/or mechanical situation allow them to be summarized in terms of a point

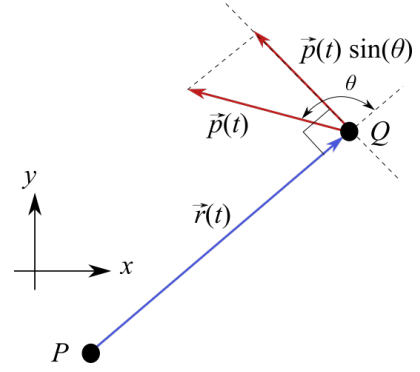


Figure 8: The angular momentum of a particle of mass m moving with velocity $\vec{v}(t)$ under possible influence of a resulting force $\vec{f}(t)$ respectively to a reference point P is defined as the cross product between the vector corresponding to the relative position of the particle with respect to P and the respective linear momentum. The resulting angular momentum is a vector that points out of the plane of this page toward the reader.

particle with mass equal to the mass of the object.

Figure 8 illustrates the concept of angular momentum.

The magnitude of the cross product can be expressed as:

$$||\vec{L}(t)|| = ||\vec{r}(t)|| ||\vec{p}(t)|| \sin(\theta) \quad (20)$$

where θ is the smallest angle between the vectors $\vec{r}(t)$ and $\vec{v}(t)$.

Thus, a closer analysis of the definition of the angular momentum of a particle from the perspective of reveals that the magnitude of the angular momentum can be understood as effectively corresponding to the product of the magnitude of the relative position vector $\vec{r}(t)$ and the magnitude of the linear momentum of the particle projected so as to be orthogonal to $\vec{r}(t)$, namely $||\vec{p}(t)|| \sin(\theta)$.

This implies that, if the magnitudes of $\vec{r}(t)$ and $\vec{p}(t)$ are kept constant, the maximum magnitude of angular momentum will be obtained for $\theta = \pi/2$, i.e. when those two vectors are orthogonal. Minimum torque magnitude will be obtained for $\theta = 0$.

The magnitude of the cross product between two vectors is also known to be equal to the area A of the parallelogram respectively defined by the two vectors, as illustrated in Figure 9.

The concept of angular momentum is directly related to the concepts of moment of inertia and torque. As a linguistic aside, it is interesting to observe that typically one refers to “*moment of inertia*”, differently from “*linear momentum*” and “*angular momentum*”.

The *moment of inertia* of a particle with mass m respectively to a point P at distance r from the particle

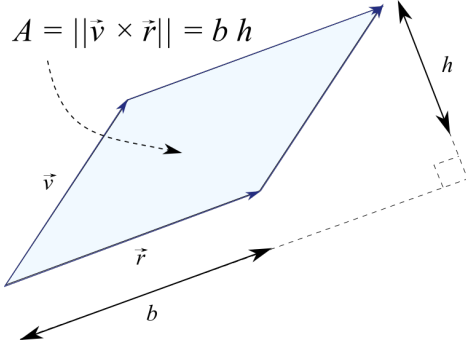


Figure 9: The cross-product between two vectors \vec{v} and \vec{r} is equal to the area A of the respectively defined parallelogram. Geometrically, the area of a parallelogram of base b and height h is given as $A = b h$.

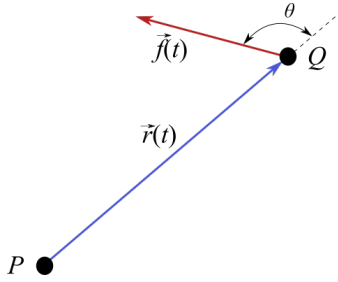


Figure 10: The *torque* of a force $\vec{f}(t)$ applied at a point Q respectively to a point P is the vector that has magnitude equal to $\vec{r}(t) \times \vec{f}(t)$ and goes out from the page toward the reader. The relative position vector of Q from P is indicated as $\vec{r}(t)$.

corresponds to:

$$I = r^2 m \quad (21)$$

The *torque* of a force $\vec{f}(t)$ applied at a point Q , respectively to a point P , can be expressed as:

$$\vec{\tau}(t) = \vec{r}(t) \times \vec{f}(t) = \|\vec{r}(t)\| \|\vec{f}(t)\| \sin(\theta) \quad (22)$$

where $\vec{r}(t)$ is the position of the point Q of application of the force respectively to the point P . Observe that the definition of torque is independent of mass. Figure 10 depicts the concept of *torque* of a force $\vec{f}(t)$ respectively to a point P .

The *lever principle*, one of the earliest mechanical concepts, is based on torque. Consider the situation shown in Figure 11, which includes a beam pivoted to a fulcrum, with distinct lengths at each of the two arms. Forces f and g are applied orthogonally to each extremity, as shown.

Provided the lever is at rest (or that there is no change of angular momentum), the two torques with respect to P will cancel one another, i.e:

$$r f - s g = 0 \implies f = \frac{s}{r} g \quad (23)$$

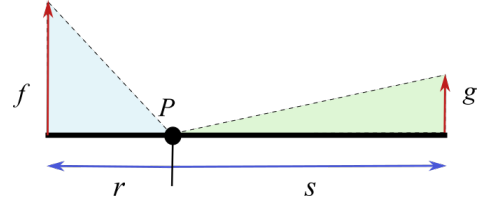


Figure 11: The *lever principle*: A beam pivoted so as to have different side lengths can greatly amplify one of the forces (in this case g). The beam is assumed not to undergo changes of angular momentum. As a consequence of the equilibrium of the torques, the areas A shown in blue and green are necessarily identical to half the value of $r f = s g = 2A$.

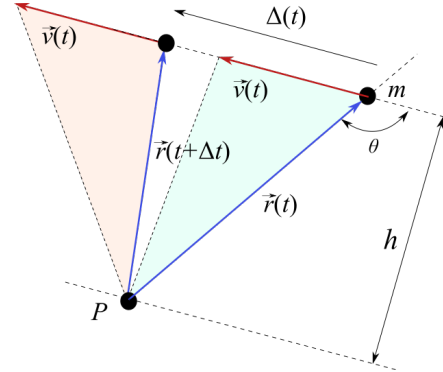


Figure 12: Geometrical verification of conservation of angular momentum at times t and $t + \Delta t$. The velocity vector is constant as a consequence of absence of forces acting on the particle. The areas of the two triangles, which are equal to half the respective angular momenta, are identical because the two triangles share the same basis (velocity magnitude $\|\vec{v}\|$) and height h .

Therefore, the force g can be greatly increased provided $s/r \gg 1$.

It can be shown that the torque and angular momentum, both taken respectively to the same reference point P , are related as:

$$\vec{\tau}(t) = \frac{d\vec{L}(t)}{dt} \quad (24)$$

Therefore, the torque can be understood as being related to the variation of angular momentum of the particle, establishing an interesting analogy with the relationship between force and linear momentum, expressed as:

$$\vec{\tau}(t) = \frac{d\vec{L}(t)}{dt} \quad \text{analogous to} \quad \vec{f}(t) = \frac{d\vec{p}(t)}{dt}$$

In an analogy with the linear momentum, the angular momentum of a particle moving in absence of torque remains constant, which is known as the principle of *angular momentum conservation*. Figure 12 presents an intuitive geometrical verification of the conservation of angular momentum in absence of resulting force on the particle.

The concepts of linear and angular momenta turn out to be frequently of great practical usefulness, especially when analyzing linear and curved motion, respectively, provided there is absence of resulting force/torque on the particle.

7 First Case-Example: Throwing a Ball

As a first complete analytic example of how forces can influence the motion of a point particle, we now consider the situation in which a small ball, approximated by a particle with the same mass m , moves when thrown from a specific initial position \vec{u}_0 with a given initial velocity \vec{v}_0 given as:

$$\begin{aligned}\vec{u}_0 &= \vec{u}(t=0) = (x(t=0), y(t=0)) = (x_0, y_0) \\ \vec{v}_0 &= \vec{v}(t=0) = (\dot{x}(t=0), \dot{y}(t=0)) = (v_{x,0}, v_{y,0})\end{aligned}$$

In this case, we know that the only acting force is a consequence of the gravitational acceleration, which is time-constant and always vertical to the motion, which allows us to instantiate Newton's second law of motion as:

$$\begin{aligned}\vec{f}(x(t), y(t)) &= m\vec{a}(t) = m(\ddot{x}(t), \ddot{y}(t)) = (0, -mg) \implies \\ \implies \begin{cases} \ddot{x}(t) &= 0 \\ \ddot{y}(t) &= -g \end{cases} \quad (25)\end{aligned}$$

where $\vec{r}(t) = (x(t), y(t))$ is a parametric curve, with time as parameter, describing the motion of the particle in this specific experiment. Therefore $(x(t), y(t))$ constitutes the sought solution of the problem.

By integrating the two Equation in 25 we obtain:

$$\begin{cases} \dot{x} = c_x \\ \dot{y} = -gt + c_y \end{cases}$$

We can now impose the initial conditions on the velocity:

$$\begin{cases} \dot{x}(t=0) = c_x = v_{x,0} \\ \dot{y}(t=0) = c_y = v_{y,0} \end{cases}$$

which leads to:

$$\begin{cases} \dot{x} = v_{x,0} \\ \dot{y} = -gt + v_{y,0} \end{cases}$$

A subsequent integration yields:

$$\begin{cases} x = v_{x,0}t + c_{xx} \\ y = -\frac{1}{2}gt^2 + v_{y,0}t + c_{yy} \end{cases}$$

Now, considering the initial position:

$$\begin{cases} x(t=0) = c_{xx} = x_0 \\ y(t=0) = c_{yy} = y_0 \end{cases}$$

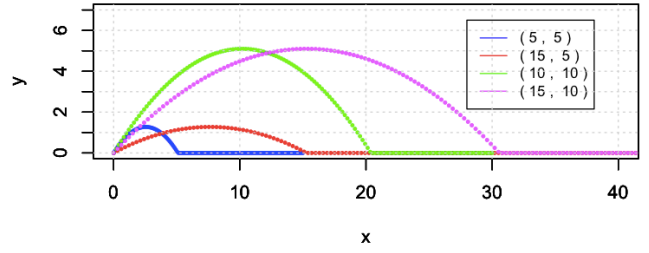


Figure 13: Several trajectories defined by throwing a point particle from the same initial position but with distinct initial $(v_{x,0}, v_{y,0})$ shown in the respective legend.

we obtain the sought solution as:

$$\begin{cases} x(t) = v_{x,0}t + x_0 \\ y(t) = -\frac{1}{2}gt^2 + v_{y,0}t + y_0 \end{cases}$$

The situation considered in this section provides an interesting example of what is often called *motion composition*, in the sense that throwing a ball involves the vertical movement in which the ball rises, stops, and the falls, combined with an horizontal linear displacement with constant speed (i.e. uniform linear motion).

While the vertical displacement $x(t)$ is simply a uniform linear motion, the vertical motion $y(t)$ is uniformly accelerated (constant acceleration), being relatively more elaborated. This type of motion of a point particle under the action of gravity is frequently called a *free fall*, which assumes complete absence of dissipative forces (e.g. drag, also called air resistance).

It is also interesting to observe that the two involved parametric curves described in the equation above are completely independent one another, since the gravitational force is always normal to the horizontal motion of the particle and therefore can have no effect on this velocity. Another interesting aspect of the above equations, characteristic of gravitational interactions, is that they do not involve the mass m of the particle, so that both the horizontal and vertical components of the motion are completely independent of the mass of the particle.

Figure 13 illustrates several trajectories obtained respectively to the indicated initial conditions.

The time when the ball reaches its maximum height can now be obtained by using standard calculus-based optimization of $y(t)$:

$$\frac{dy(t)}{dt} = -gt + v_{y,0} = 0 \implies t = \frac{v_{y,0}}{g}$$

The sign of the second derivative can be obtained from:

$$\frac{d^2y(t)}{dt^2} = -g < 0$$

indicating that the obtained extremum is a maximum.

At the above obtained time, the ball height is:

$$\begin{aligned} y(t) &= -\frac{1}{2}g \left(\frac{v_{y,0}}{g} \right)^2 + v_{y,0} \left(\frac{v_{y,0}}{g} \right) + y_0 = \\ &= -\frac{(v_{y,0})^2}{2g} + \frac{(v_{y,0})^2}{g} + y_0 = \frac{(v_{y,0})^2}{2g} + y_0 \end{aligned} \quad (26)$$

It is also possible to find the time t_G at which the ball hits the ground by solving:

$$y(t) = -\frac{1}{2}gt^2 + v_{y,0}t + y_0 = 0$$

which leads to:

$$t_G = \frac{v_{y,0} + \sqrt{v_{y,0}^2 + 2gy_0}}{g} \quad (27)$$

There is, however, the following additional root, which turns out to be negative for $y_0 > 0$:

$$\tilde{t}_G = \frac{v_{y,0} - \sqrt{v_{y,0}^2 + 2gy_0}}{g} \quad (28)$$

This can be understood as the time the ball would hit the ground in case the motion were time-reversed, which would also imply $x(\tilde{t}_G) < x_0$.

At the positive obtained time, the distance from the throwing point can be obtained as:

$$x_G = v_{x,0} \frac{v_{y,0} + \sqrt{v_{y,0}^2 + 2gy_0}}{g} + x_0 \quad (29)$$

We are now in position to address the particularly interesting problem in which one wants to identify which initial orientation of the velocity would lead to the maximum distance x_G . In other words, how the ball should be cast with constant magnitude but adjustable orientation so as to get the as far as possible.

An analysis of Equation 29 indicates that it is enough to maximize the quantity:

$$\begin{aligned} z(\theta) &= v_{x,0} \left[v_{y,0} + \sqrt{v_{y,0}^2 + c} \right] = \\ &= \cos(\theta) \left[\sin(\theta) + \sqrt{\cos^2(\theta) + c} \right] = \\ &= \cos(\theta) \sin(\theta) + \cos(\theta) \sqrt{\cos^2(\theta) + c} \end{aligned} \quad (30)$$

where $c = 2gy_0$.

Let us find the solution assuming that $y_0 = 0$, which implies $c = 0$. In this particular case, we have:

$$z(\theta) = \cos^2(\theta) + \sin(\theta) \cos(\theta) \quad (31)$$

By maximizing respectively to θ , we obtain:

$$\begin{aligned} \frac{dz(\theta)}{d\theta} &= -2\cos(\theta) \sin(\theta) + \cos^2(\theta) - \sin^2(\theta) = 0 \implies \\ \implies [\cos(\theta) - \sin(\theta)]^2 &= 0 \implies \\ \implies \cos(\theta) &= \sin(\theta) \implies \boxed{\theta = \frac{\pi}{4}} \end{aligned} \quad (32)$$

It is also interesting to illustrate the concept of linear momentum respectively to the above throwing examples.

Let us consider the following situation regarding a point particle with mass $m = 1\text{kg}$:

$$\begin{aligned} \vec{u}_0 &= \vec{u}(t=0) = (x(t=0), y(t=0)) = (0, 0) \\ \vec{v}_0 &= \vec{v}(t=0) = (\dot{x}(t=0), \dot{y}(t=0)) = (10, 10) \end{aligned}$$

What will be the total impulse exerted on the particle during its whole trajectory, and what will be its influence on the linear momentum?

Given that the gravitational force is constant and equal to mg , all we additionally need in order to determine the total impulse is the total time of the trajectory. This can be obtained from Equation 27 as:

$$t_G = \frac{v_{y,0} + \sqrt{v_{y,0}^2 + 2gy_0}}{g} = \frac{10 + \sqrt{10^2}}{g} = \frac{20}{g} \text{ s} \quad (33)$$

Therefore, the total impulse can be determined as:

$$\vec{J} = -mg t_G = -20 \hat{i} \text{ N s} \quad (34)$$

where \hat{i} is the *versor* of \mathbb{R}^3 pointing vertically from the floor toward the sky.

By applying the impulse momentum theorem, we obtain:

$$\Delta \vec{p} = \vec{J} = -20 \hat{i} \text{ kg m s}^{-1} \quad (35)$$

where, as before, we have assumed the positive orientation of the vertical speed as corresponding to the x -axis direction.

This results is fully compatible with the fact that, given the symmetry of this motion and absence of dissipative forces, the final velocity horizontal will have the same magnitude than the initial velocity, i.e. 10 m s^{-1} , but reversed orientation, i.e. -10 m s^{-1} , yielding the total variation of linear momentum $\Delta \vec{p} = -20 \hat{i} \text{ kg m s}^{-1}$.

Before proceeding with our discussion of mass and force, it is interesting to consider how the initial vertical velocity $v_{y,0}$ in the just considered ball throwing example influences the resulting dynamics along time. In order to quantitatively address this interesting question, let us express the relative influence of the original velocity on the current velocity in terms of the following fraction:

$$\nu(t) = \frac{\|v_{y,0}\|}{\|v(t)\|} = \frac{\|v_{y,0}\|}{\| -gt + v_{y,0} \|} \quad (36)$$

In the limit when time tends to infinity, we have:

$$\nu = \lim_{t \rightarrow \infty} \frac{\|v_{y,0}\|}{\| -gt + v_{y,0} \|} = 0 \quad (37)$$

As also illustrated in Figure 14, the influence of $v_{y,0}$ tends to fade away in an asymptotical manner, therefore never disappearing. The figure assumes $v_{y,0} = 10$.

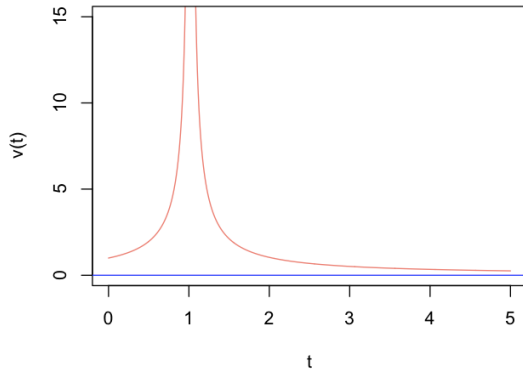


Figure 14: The relative influence of the initial vertical speed $v_{y,0}$ tends to vanish asymptotically to zero. However, a more substantial influence can be observed from the instant the ball was released ($t = 0$) to about $t = 2$ s. The large clipped positive peak, followed by a negative counterpart, correspond to the divergence of the ration that takes place when the velocity $v_y(t)$ changes sign, as it passes through zero at this instant.

In addition, it is also interesting to note that the shorter the time t elapsed from the moment the ball was thrown, the larger the relative influence of the initial vertical speed. Informally speaking, the observed *asymptotic* behavior can be understood as the effect of $v_{y,0}$ getting closer and closer to zero, but never reaching it.

8 Free Fall, Weightlessness, and Singularities

In the previous case-example, the vertical motion corresponded to a free fall, involving the respective motion equation:

$$y(t) = -\frac{1}{2}gt^2 + v_{y,0}t + y_0 \quad (38)$$

which does not depend on the mass of the particle. Therefore, particles with any mass, when released under identical initial conditions, will all take the same time to reach the floor, which is known since Galileo. This important aspect of gravity also implies that all objects released proximate one another and at the same time, when of absence of additional forces other than that implied by gravity, will keep their relative positions as they undergo a free fall.

In particular, if one object is directly below and initially in contact another, they will remain so while there can be no force, and therefore no pressure, between them. Thus, the weight of a mass undergoing free fall will be zero when measured by contact with a scale (see Fig. 15). This situation, called *weightlessness*, is ultimately achieved because in a free fall all objects will necessarily undergo the same acceleration, equal to the gravitational acceleration. All

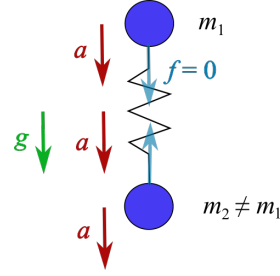


Figure 15: If two objects with possibly distinct respective masses m_1 and m_2 , linked by a spring initially under no tension, are released into a free fall under action of the gravity g , the spring will keep its length, implying the appearance of no relative forces between the two masses as a consequence of their motion. That is because all the three involved objects will undergo exactly the same constant acceleration $a = g$, therefore keeping their relative positions as long as the free fall lasts.

in all, objects in free fall behave exactly as they would in total absence of external forces.

This is a direct consequence of the fact that the gravitational force is a special type of force that depends linearly of the mass, which implies Newton's second law of motion to become mass-independent, i.e.:

$$\begin{aligned} f &= f(m, g) = m g = m a_y(y) \implies \\ \implies a_y(t) &= g = \text{constant} \end{aligned} \quad (39)$$

However, it should be kept in mind that taking g constant is but an approximation considering that the amplitude of the free fall is much smaller than the radius of the earth. Actually, we have from Newton's law of universal gravitation that g varies with the distance of the mass to the center of the Earth.

However, were the objects released in a parallel gravitational field during a long period of time in which the magnitude of g changed along time, objects release in proximity and at the same time would remain weightless while of the duration of the free fall because any changes of the gravitational acceleration would also be equally shared among all involved objects.

Interestingly, the weightlessness state is by no means observed for other types of force that are not proportional to the mass, such as the elastic or electromagnetic forces. In these cases, when a same force is applied to two point particles with distinct masses, they will undergo different motions. In particular, if $m_1 > m_2$, it follows that:

$$\begin{cases} f = m_1 a_1 \\ f = m_2 a_2 \end{cases} \implies \begin{cases} a_1 = \frac{f}{m_1} \\ a_2 = \frac{f}{m_2} \end{cases} \implies a_1 < a_2 \quad (40)$$

From which it immediately follows that particle 2, which has less mass, will move faster than particle 1. The obtained different accelerations will imply in possible forces appearing between accelerating objects that are in contact, as well as relative motion between them.

Though gravitation does not influence the relative motion of free falling objects, the understanding of this situation as a possible frame of reference drifts us away from the classical context that has been assumed from the outset of this work into the world of general relativity, so that this subject is not addressed further here.

The concepts of free fall and weightlessness are sometimes mistakenly understood from the perspective of that, when objects undergo long falls in the real-world, therefore with present of air drag, their speed will tend to become constant, which could suggest an inertial frame of reference which, however, has no direct relationship on weightlessness or free fall. The interesting tendency of falling objects to reach a constant speed is a consequence of the force implied by the air resistance on the falling object, whose magnitude increases with the velocity of the object, but in inverse direction, until near null resulting force is obtained on the object, therefore implying no respective acceleration.

There is another aside to Newton's law of universal gravitation, also common to the electric field, that deserves particular attention. This has to do with *singularities*. The point here is that as the radius separating the two particles becomes smaller and smaller, the implied interaction force grows enormously. Actually, the force will diverge to infinity when the separation between the two point particles completely vanishes. At this point, we would have what is called a *singularity*. Observe that these singularities only occur as a consequence of two point particles interaction, because the otherwise implied non-infinitesimal size of the objects avoids the two masses to get too close. Actually, in these cases (such as the Earth), the total mass m is distributed according to a possibly non uniform density (see Section 19) throughout their respective volumes.

9 Second Case-Example: Moving Helix

Another interesting situation involving the composition of two types of motions, depicted in Figure 16, is addressed next in this section. External forces are applied as required to maintain the *uniform circular motion* (constant speed magnitude while rotation in a circular orbit) of the particle, as it undergoes uniform linear motion along the horizontal. The addressed situation *does not incorporate gravitational interactions*.

As presented in Figure 16, the situation concerns a particle with mass m rotating with uniform angular speed $\dot{\theta} = 2\pi = \omega$, with $f_0 = 1\text{Hz} \Rightarrow T_0 = 1\text{s}$ around a reference point, all of which moving with constant linear speed equal to the perimeter $2\pi\rho$ of one complete circular tra-

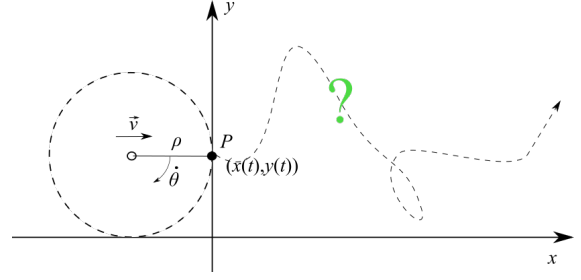


Figure 16: A point particle P undergoes a motion composed by: (i) a uniform circular motion around a pivot with angular velocity $\dot{\theta}$ and radius ρ ; and (ii) a linear horizontal motion with constant velocity $\vec{v} = 2\pi\rho$. How will be the trajectory of this particle, expressed in terms of its positions $(x(t), y(t))$?

jectory, so that the two motions are synchronized and the particle actually moves as if the massless circle defined by its trajectory were a wheel touching the ground.

In this case, we can write:

$$P : \begin{cases} x(t) = \rho \cos(\omega t) + 2\pi\rho t + x_0 \\ y(t) = \rho \sin(\omega t) + y_0 \end{cases} \quad (41)$$

Implying the following scalar velocities:

$$\vec{v}_P(t) : \begin{cases} v_x(t) = -\rho\omega \sin(\omega t) + 2\pi\rho \\ v_y(t) = \rho\omega \cos(\omega t) \end{cases} \quad (42)$$

and accelerations:

$$\vec{a}_P(t) : \begin{cases} a_x(t) = -\rho\omega^2 \cos(\omega t) \\ a_y(t) = -\rho\omega^2 \sin(\omega t) \end{cases} \quad (43)$$

The resulting motion $x(t)$ and $y(t)$, as well as scalar velocities, are presented in Figure 17. As the horizontal speed is constant in both magnitude and orientation, it does not influence the centripetal force that is applied onto the particle in order to ensure its uniform circular motion. Actually, it would be possible to attach an inertial frame of reference to the center of the circular motion, it wanted, as that point undergoes uniform linear motion.

Interestingly, the resulting trajectory $(x(t), y(t))$ shown in Figure 17(a) seems to suggest discontinuities of the orientation of the velocity at the point where the particle almost touches the ground, which happens at time intervals of 1s. However, an analysis of the involved scalar velocities, shown in (d) and (e), reveal that, actually, what happens is that both components of the scalar velocity, namely $v_x(t)$ and $v_y(t)$, reach value zero in those situations, implying $\|\vec{v}_P(t)\| = 0$.

10 Artificial Forces and Linear Superimposition

Another interesting situation regarding the trajectories of point particles in presence of forces regards the situation

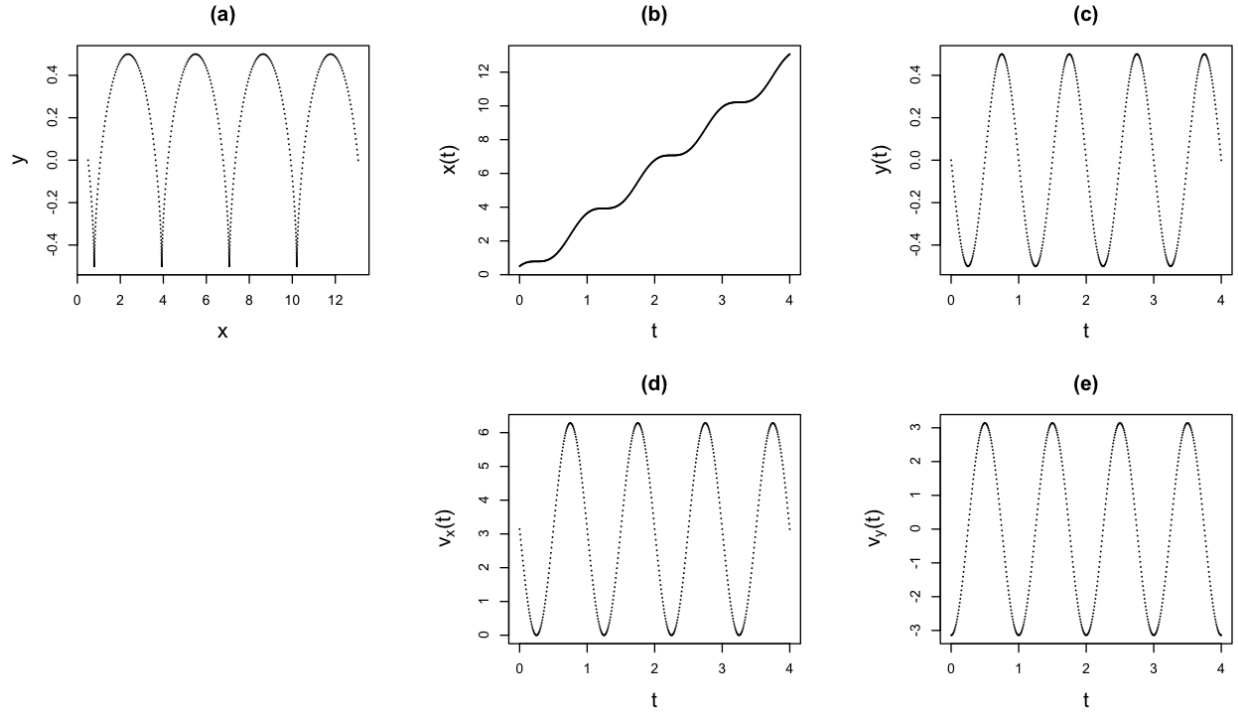


Figure 17: The motion of the point particle in the considered situation, shown in (a), seems to suggest discontinuities of the orientation of the scalar velocity of the particle. Also shown are the positions $x(t)$ and $y(t)$ along time (b,c), as well as the respective scalar velocities (d,e). An analysis of the obtained velocities however indicate that, instead of velocity discontinuities, what happens is that the scalar velocity magnitude becomes zero at those positions, which happens at intervals of 1 second, i.e. each whole rotation.

in which an “artificially” imposed force is exerted onto the particle so as to modify the trajectory it would have otherwise undergone in consequence of other “natural” forces, such as gravitation. Here, we employ the terms “natural” and “artificial” in order to designate forces that are typically found in nature, such as a consequence of gravity, and forces that are applied under control of some agent such as humans with the objective of “shaping” the trajectory in specific ways.

Though very common, the artificial control of motion is not always realized in the context of particle motion. Figure 18 presents a possible computationally supervised system capable of controlling the position, along time, of a base, where a “particle” or some other object can be placed. Though there are other possible approaches, such as controlling an electromagnetic field or using magnets, the system shown in the figure relies on mechanical contact with the object to be displaced according to a pre-established trajectory, possibly calculated from some mathematical model, and imposed by the processor P via the control unit C and respective electromechanical actuators implementing orthogonal displacements according to a respective Cartesian frame.

Needless to say, another particularly interesting possibility to impose completely arbitrary forces and motions is through *numerical simulation* in computational physics

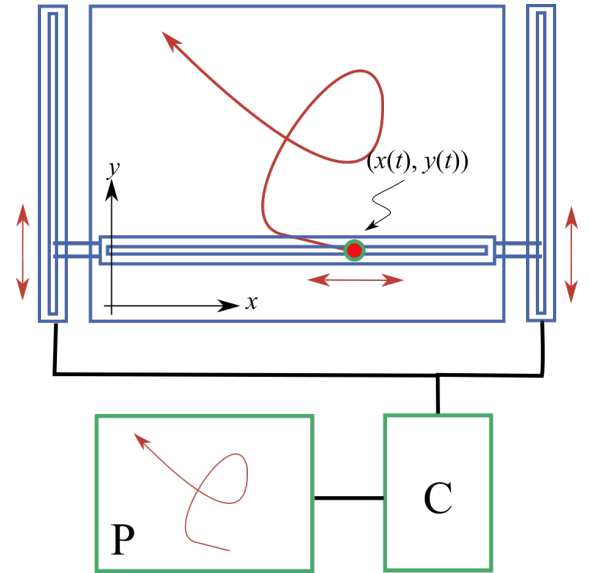


Figure 18: A possible computational-electro-mechanical system for imposing a generic motion of a “particle” or some other object placed on a respective base. Typically, the motion to be imposed has been previously determined mathematically, so that it can be realized by the electromechanical actuators under control of the processor P . Except for inertial, speed, and maximum force limitations, virtually any motion, including trajectories with intersecting points, can be realized by using a device similar to that shown in this figure.

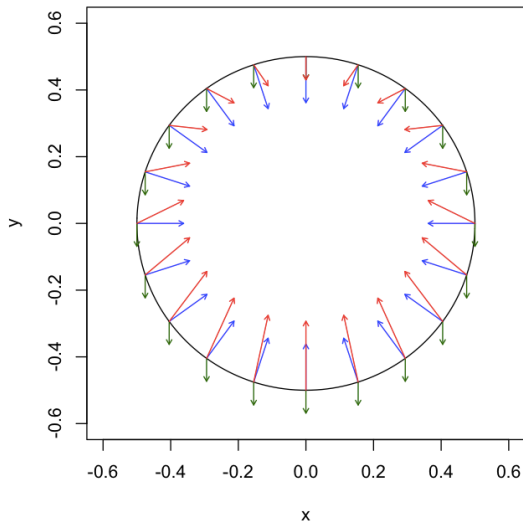


Figure 19: A uniform circular particle motion is to be performed under the action of gravitation by artificially incorporating an additional force $\vec{f}(t)$ (shown in red) acting continuously on the particle so as to compensate the gravitational force (shown in green), ensuring a resulting force (shown in blue) required for maintaining the constant normal acceleration necessary for changing the orientation of the particle velocity (not shown for simplicity), while the respective magnitude remains constant. Observe the parallelogram composition of the three involved forces.

(e.g. [21, 22]), where Newton's laws and other physical principles can be brought in as resources for simulating an endless variety of particle motion and mechanical systems. An important feature of numerical simulations is that it allows any differential equation to be solved in terms of finite differences or elements. Numerical methods have been applied frequently along the present work.

In several cases, such as in robot arm and motion control, the “artificial” force is necessary to obtain a specific type of motion that would otherwise be completely different if allowed to proceed only under the “natural” forces.

For instance, one may want to make a particle with mass m to undergo uniform circular motion in presence of gravity. It is understood that the plan of circular motion is orthogonal to the earth's surface. If left alone, the particle would imply undergo a free fall under action of the gravity. Therefore, it is required to continuously apply, at each time instant t , a force which, composed with that implied by the gravitational acceleration, ensures constant angular speed, therefore implementing a respective uniform circular motion. Interestingly, this imposition of a pathway to an otherwise free moving particle corresponds directly to the consideration of parametric curves in our previous study [11].

Figure 19 presents the diagram of forces involved in this situation. The gravitational force $\vec{f}_g(t)$ is shown in green, the forces $\vec{f}_c(t)$ related to sought centripetal acceleration

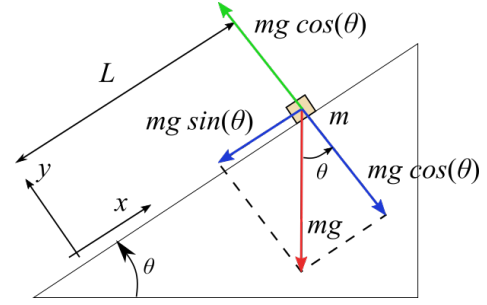


Figure 20: A point particle with mass m moving along a tilted plane. The presence of the plane implies the particle to undergo a motion that it would not have otherwise undergone. Indeed, in absence of the plane, the particle would undergo a free fall, which corresponds to the variational trajectory of minimum energy.

are shown in blue (observe their constant magnitude), while the “artificial” forces $\vec{f}_a(t)$ required for obtaining the latter are shown in red. Observe that, at each time instant, we necessarily have:

$$\vec{f}_c(t) = \vec{f}_g(t) + \vec{f}_a(t) \quad (44)$$

as can be readily verified in the parallelograms additions respectively implied in the diagram in Figure 19.

It is interesting to observe that the “artificial” force field tends to have its magnitude increased when the particle gets at its lower positions, while almost vanishing at its highest point, because at that position the gravitational force is more similar to the required centripetal force.

As a matter of fact, because the imposed force field depends on the particle mass, rotation radius, and desired angular speed, it becomes an interesting problem to find for which of these parameters no “artificial” force will be required at the precise moment when the particle reaches at its highest position. This is left as an exercise.

This example also illustrates the possibility to *linearly superimpose* (add, with possible scalings) several fields of force, which is a property intrinsically considered in Newton's laws of motion. In a sense, this property reveals that fields do not interact one another.

A simpler, but also interesting situation involves a point particle progressing through an imposed pathway, such as illustrated in Figure 20, involving a point particle (we assume that point masses can have arbitrary shapes) with mass m moving along an inclined plane.

This situation, known as the *inclined*, or *tilted* plane, can be solved by first identifying that the only force imposed on the particle corresponds to that implied by gravitation, which can be readily decomposed as shown in the figure. Given that the particle does not move along the orientation normal to the plane, we have that the component with magnitude $m g \cos(\theta)$ is completely counterbal-

anced by the force (shown in green) exerted by the plane on the particle, so that these two forces lead to zero resulting force along the orientation normal to the plane. However, the component $m g \sin \theta$ of the gravitational interaction is not counterbalanced by any other force (we assume no drag along the surface), so that it will necessarily imply the particle to be accelerated.

We also impose an arbitrary frame of reference $x \times y$ as shown in the figure, so that the initial position of the particle is $x(t=0) = L$. It is assumed that the initial velocity is $v_{x,0}$ and $v_{y,0} = 0$. Newton's second law implies:

$$m g \sin(\theta) = -m \ddot{x}(t) \quad (45)$$

from which:

$$\begin{aligned} \ddot{x}(t) &= -g \sin(\theta) \implies \\ \dot{x}(t) &= -g \sin(\theta) t + c \implies \\ x(t) &= -\frac{1}{2} g \sin(\theta) t^2 + c t + d \implies \end{aligned} \quad (46)$$

Now, by imposing the initial position:

$$x(t=0) = L = d \implies \quad (47)$$

The consideration of the initial velocity yields:

$$\dot{x}(t=0) = v_{x,0} = c \implies \quad (48)$$

and the complete solution of the particle position can be expressed as:

$$\boxed{x(t) = -\frac{1}{2} g \sin(\theta) t^2 + v_{x,0} t + L} \quad (49)$$

As could be expected, the motion resembles a free fall, but with an attenuated acceleration implied by the presence of the plane, which constrains the block particle to maintain contact with the tilted plane.

As an interesting aside, there is an alternative and effective manner of looking at motion (as well as all other physical phenomena), which relates to the fact that the trajectories defined by “natural” motion correspond to those that minimize the overall energy, which is known as the *Principle of Least Action*, or *Stationary-Action Principle*. These approaches are completely compatible with the trajectories obtained by using Newton's law of motion, leading to the same results though they consider the whole trajectory (global) instead of the otherwise differential (local) approach to which we shall be constrained in the present work. This interesting area is related to the areas known as Calculus of Variations, and Lagrangian and Hamiltonian Mechanics.

While the “natural” motion of a point particle when left free in the vacuum is a respective free fall, ensuring least action (energy), we have just seen that additional constraints, such as an inclined plane or more elaborated schemes, can be imposed so as to shape the trajectory in specific ways. Thus, it becomes an interesting issue to search for the imposed constraints to a point particle under the action of gravity that would lead to the *shortest time of motion* between the initial position of the particle and another distinct point beneath it. This interesting problem is known as the *brachistochrone trajectory* (e.g. [23, 24]), whose solution involves the application of calculus of variations (e.g. [25, 26, 27]) and therefore will be addressed in an eventual future related work.

11 Force and Curvature

We have already seen from the point of view of differential geometry (e.g. [10]) that, given a parametric curve on a generic parameter such as time t , it is often possible to transform this curve to *arc-length* parametrization, characterized by velocity with constant magnitude equal to 1, though with possible orientation changes. It is now interesting to discuss the implications of this type of trajectories, typically obtained artificially by imposing additional force fields, from the perspective of the physics of particle motion, which we do as follows.

Let $r(s) = (r_x(s), r_y(s))$ be a parametric curve on the arc-length parameter s . Then, it necessarily follows that:

$$\vec{T}(s) = \frac{d\vec{r}(s)}{ds} \quad (50)$$

$$||\vec{T}(s)|| = \sqrt{\left[\frac{dr_x(s)}{ds}\right]^2 + \left[\frac{dr_y(s)}{ds}\right]^2} = 1 \quad (51)$$

meaning that the first derivative of $\vec{r}(s)$, which is analogous to the respective “velocity” in case the parameter corresponded to time, has constant magnitude equal to one, implying that:

$$S = \int_{s_i}^{s_f} ds = \text{arc-length from } s_i \text{ to } s_f \quad (52)$$

However, though the magnitude of the first derivative, which is tangent to the curve $\vec{r}(s)$ at each value of s , is constant and equal to one, this does not necessarily mean that the velocity does not change its orientation. The remainder of this section assumes that all considered curves are in arc-length parametrization.

A more complete description of the changes undergone by the first derivative of $\vec{r}(s)$ as s changes can be provided by considering the second derivative (“acceleration”) of

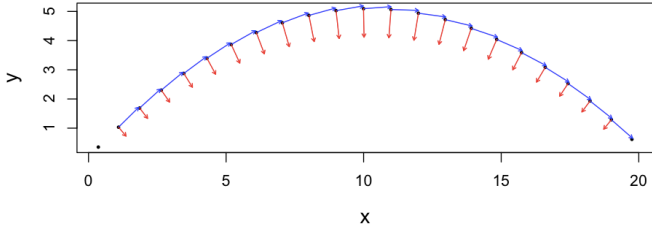


Figure 21: Parabolic motion parametrized by arc-length, implying constant unit speed magnitude (in blue), also called the *tangent field* of the curve, shown along equally spaced time intervals. The acceleration (in red) is implied by the forces needed to ensure constant velocity magnitude under continuous orientation changes. The magnitude of the tangent field corresponds, by definition, to the *curvature* of the motion. Interestingly, the pair of vectors defined by the first and second derivative of this type of motion provide an orthogonal basis associated to each of the points along the trajectory which, however, does not correspond to an inertial frame of reference because of the continuous changes undergone by the velocity orientation.

the curve $\vec{r}(s)$:

$$\vec{a}(s) = \vec{N}(s) = \frac{d\vec{T}(s)}{ds} = \frac{d^2\vec{r}(s)}{ds^2} \quad (53)$$

$$\kappa(s) = \|\vec{N}(s)\| = \|\vec{a}(s)\| \quad (54)$$

where $\kappa(s)$ is the *curvature* of $g(s)$ at s .

Interestingly, in case the particle has mass 1, the curvature can be understood as being identical to the magnitude of the force at any arc-length s . In the case of non-unit mass, the magnitude of the force will still be proportional, through the mass m , to the curvature at each point along the trajectory. These normal forces are required to change the orientation of the velocity vectors, even though they have constant unit magnitude.

The vector $\vec{a}(s)$ can be shown to be either zero (e.g. straight portions of the curve) or normal to the curve for each value of s , being therefore orthogonal to the first derivative of $\vec{r}(s)$. In the cases of non-zero acceleration, these two vectors can be understood as providing a moving orthogonal frame that adapts to the parametric curve for each value of s , as illustrated in Figure 21 respectively to a parabolic motion with arc-length parametrization.

12 Other Types of Forces

It turns out that Newton's second law of motion holds not only for the gravitational action $m\vec{g}(x, y, z)$, but actually for *any other force* $\vec{f}(x, y, z, t)$ applied to a point particle with mass m :

$$\vec{f}(x, y, z, t) = m \frac{d^2\vec{u}(t)}{dt^2} \quad (55)$$

Examples of other types of forces found in nature include the *electromagnetic force* and the *elastic force*. However, as discussed in Section 10, we can also have arbitrary forces imposed along time on an object through some controlled mechanical device such as linear or angular motors. In this way, an infinite number of force functions can be obtained. Interestingly, Newton's second law of motion makes no distinction whatsoever between these forces.

Together with the strong and weak forces found at the scale of fundamental particles, the gravitational and electromagnetic forces — the so called *fundamental forces* — are the *only way* in which force fields, implying action from distance without involving contact with the objects, can be obtained in the physical world. However, there is in principle no limitation to the types of localized “artificial” forces that can be imposed, typically by contact with some device, onto a given mass.

Given that we can have several types of forces, it becomes an interesting issue to *define* a force in a more general manner. A reasonable approach is to understand that a force is any effect that can change the velocity of a particle with mass m , according to Newton's second law of motion. Therefore, the identification of any change in the velocity of a particle, which means a respective acceleration, necessarily reveals that a force has been respectively applied.

However, the other way round does not hold. More specifically, the identification that a given force is applied to a particle does not necessarily means that the particle is undergoing acceleration, or even motion. This situation happens when the sum of forces applied at the particle is zero. Therefore, it is important to bear in mind that the force appearing in Newton's second law corresponds to the *resulting sum of every force acting on that particle*.

Let us now briefly discuss the electromagnetic force. For simplicity, we start with the magnitude of the *electric force* established between on a point particle with electrical charges Q_1 as a consequence of its interaction with another point particle with charge Q_2 (all charges having C =Coulomb as unit):

$$|f(t)| = k_e \frac{|Q_1||Q_2|}{r^2} \quad (56)$$

where the two particles are at distance r one another, k_e is the Coulomb constant $k_e \approx 8.988 \cdot 10^9 N m C^{-2}$ and \hat{r} is the versor pointing from charge Q toward charge q . The above equation is known as *Coulomb's law*.

Electric fields are measured in $N kg^{-1}$.

As with the gravitational interaction, more complete version of the previous law expressed in terms of vectors (please refer to Figure 2) ,considering all charges being

positive, is as follows:

$$\vec{f}_{1 \rightarrow 2} = k_e \frac{Q_1 Q_2}{\|\vec{r}_{1 \rightarrow 2}\|^2} \hat{r}_{1 \rightarrow 2} = -k_e \frac{Q_1 Q_2}{\|\vec{r}_{2 \rightarrow 1}\|^2} \hat{r}_{2 \rightarrow 1} \quad (57)$$

$$\vec{f}_{2 \rightarrow 1} = k_e \frac{Q_1 Q_2}{\|\vec{r}_{2 \rightarrow 1}\|^2} \hat{r}_{2 \rightarrow 1} = -k_e \frac{Q_1 Q_2}{\|\vec{r}_{1 \rightarrow 2}\|^2} \hat{r}_{1 \rightarrow 2} \quad (58)$$

In direct analogy with the gravitational force, we can define a vector field, namely the *electric field* (with unit $V m^{-1}$), corresponding to the forces implied by one of the particles, let us say $Q_2 > 0$, leading to:

$$\vec{E}(x, y, z, t) = \frac{1}{4\pi\epsilon_0} \frac{Q_2}{r^2} \hat{r} \quad (59)$$

where ϵ_0 is the *vacuum permittivity* ($\epsilon_0 = 8.854187 \cdot 10^{-12} F m$, $F = \text{Faraday}$). The versor \hat{r} points from the central particle $Q_2 > 0$ towards particle $q > 0$.

The electric force on a particle with charge q can now be expressed respectively to the electric field as:

$$\vec{f}(t) = q \vec{E}(t) \quad (60)$$

where the direction of the obtained force depends on the charge of q .

Again in direct analogy to the gravitational interaction, we have that the electric field is a vector field corresponding to minus the gradient of a respective scalar field known as *electric potential* (with unit $V = \text{Volt} = J/C$):

$$U(x, y, z, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad (61)$$

Therefore, we again have that the electric field is *conservative*.

The strong analogy between the gravitational and electric field mostly ends here, as there are also several differences. To start with, in the case of the electric field, we have both positive and negative charges, which can form *dipoles* as those shown in Figure 22(b) assuming a positive probing charge (i.e. the forces are shown as if implied on a point particle with positive charge). There is a third possibility (not shown), corresponding to two negative charges. How would be the respectively defined electric field look?

While negative charges are very common in nature, the concept of negative masses in nature is often understood as ‘exotic’, being mostly related to regions of negative pressure density following from the Casimir effect [28, 29] (Hendrik B. G. Casimir, 1909–2000).

Observe the singularities implied at the exact position of the two point charges, which is analogous to that observed for particles with mass in Newton’s law of universal gravitation. In addition, except at these singularities, the divergent of such electric fields, including the case of single charges, at any other point will be zero in these respective

cases. Non-zero divergent will however be obtained in the case of spatial charge densities (Maxwell’s first law).

Another important difference between the electric and gravitational forces and fields is that the electric field potentially coexists with a dual field, the *magnetic field*, while there is no known counterpart in the case of the gravitational interaction.

The magnetic field can originate from magnets (always dipoles) or from time variations of the electric field (e.g. a moving electric charge), the latter being also possible to be obtained as a consequence of time variations of the former field. The lines of force of the magnetic field in a dipole such as a magnet will always extend from one of its *poles* toward the other pole with opposite type (i.e. N/S). Interestingly, it is not known for sure why there are no magnetic monopoles.

There are two closely related types of magnetic field, represented as \vec{B} (with unit $T = \text{Tesla}$) and \vec{H} (with unit $A m$), but here we will be constrained to the former type.

The *magnetic force* on a point particle with charge q traveling with velocity $\vec{v}(t)$, in the presence of a magnetic field $\vec{B}(t)$, can be expressed as:

$$\vec{f}(t) = q \vec{v}(t) \times \vec{B}(t) \quad (62)$$

The electric and magnetic fields can be linearly superimposed, leading to the following respectively linearly combined force, which is known as *Lorenz force law*:

$$\vec{f}(t) = q \left[\vec{E}(t) + \vec{v}(t) \times \vec{B}(t) \right] \quad (63)$$

We can now very informally and briefly present and discuss the following *Maxwell’s equations*:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \text{Gauss’s law} \quad (64)$$

$$\nabla \cdot \vec{B} = 0 \quad \text{Gauss’s law of magnetism} \quad (65)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{Faraday’s induction law} \quad (66)$$

$$\nabla \times \vec{B} = \mu_0 \left[\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right] \quad \text{Ampère’s circuital law} \quad (67)$$

where ρ is the *charge density*, μ_0 is the *vacuum permeability* ($\mu_0 \approx 1.256637 \cdot 10^{-6} N A^{-2}$, $A = \text{Ampère}$).

Observe an intrinsic pairwise duality between the four equations above, in the sense that the divergent of the electric field is treated in the first law, while the second rule relates to the divergent of the magnetic field. A similar dual pair is observed, between the third and fourth laws, regarding the curl of the electric and magnetic field,

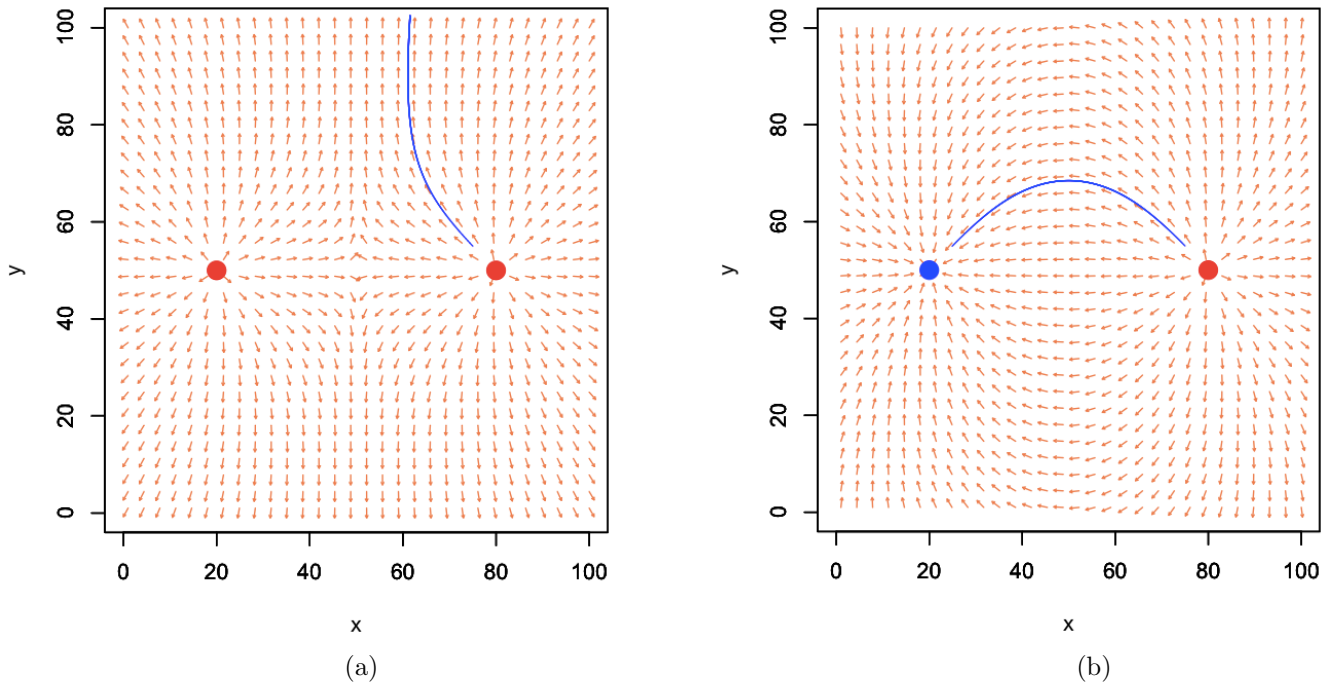


Figure 22: Electric fields originating from: (a) two positive (red) charges; (b) a negative (blue) and a positive charge. The vectors are shown normalized to unit magnitude for the sake of enhanced visualization. Situation (b) is understood as constituting a *electric dipole*. The *lines of force* defined by these fields (one of them illustrated in blue), obtained by considering positive probing charges, are tangent to the field vectors at every point (x, y) and do not cross one another. For generality's sake, and in particular to allow compatibility with Gauss's law, the 2D vector field shown in this figure could be thought of as a slice of a 3D counterpart, where Coulomb's law is verified.

respectively. However the symmetry is not perfect because the net values resulting in each paired law are not the same.

We have seen [11] that the curl of any conservative field is null. So, Faraday's induction law mostly concerns situations in which the electric field is not constant along time, as would be the case of originating from a static charge. Indeed, it is only when the magnetic field \vec{B} varies along time that the curl of an electric field will be non-zero. This fact evidentiates that the electric field can be generated not only from static charges, but also from a co-existing magnetic field.

It should be taken into account that all Maxwell laws, as stated above, rely on multivariate differential operations, so any analogy with current induction as in the case of the induction law should be considered in terms of infinitesimal closed trajectories around the point of interest.

As should have become plain, a better understanding of Maxwell's laws depends strongly on our familiarity with the involved differential operators and their properties. At the same time, these physical laws provide an interesting perspective from which to start better understanding the differential operators.

Electromagnetism constitutes an important enough area, with almost unending basic and applied implica-

tions, deserving a more comprehensive study to be addressed in a possible subsequent work, as presently we focus our attention on the gravitational and elastic forces, which are simpler than the electromagnetic force. Be that as it may, it is worth recalling that, taken separately, the electric field bears a strong analogy with the gravitational field, though it does not depend on the mass of the involved particles, but only on their charges. In this sense, the translation of an electric field to the motion of a point particle involves the action of the electrical force on that particle with charge $q \neq 0$, which will then have its dynamics specified by Newton's second law of motion in terms of its mass m .

Table 1 presents a summary of some of the main properties of the gravitational, electric, magnetic, and elastic force fields. Of particular interest are the several symmetries (and asymmetries) between many of the considered properties.

13 Elastic Force

Let us now turn our attention to *elastic forces* that are characteristic of springs and rubber bands, among several other devices and materials. A linear elastic force f acting on an elastic device that can go horizontal dis-

<i>property/field</i>	gravitational	electric	magnetic	elastic (analogy)
<i>conservative</i>	yes	yes	<i>no</i>	yes
<i>unit</i>	$N\,kg^{-1}$	$V\,m^{-1} = N\,C^{-1}$	$N\,m^{-1}\,A^{-1}$	$N\,m^{-1}$
<i>force equation</i>	$-G \frac{mM}{r^2}$	$\frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r^2}$	$q \vec{v}(t) \times \vec{B}(t)$	$k \Delta x$
<i>potential field</i>	$-G \frac{M}{r}$	$\frac{1}{4\pi\epsilon_0} \frac{Q}{r}$	—	$\frac{1}{2} k [\Delta x]^2$
<i>dual field</i>	<i>no</i>	yes (magnetic)	yes (electric)	<i>no</i>
<i>dist./contact action</i>	distance	distance	distance	contact
<i>multiple particles</i>	yes	yes	yes	<i>no</i>
<i>monopole</i>	yes	yes	<i>no</i>	yes
<i>dipole</i>	yes	yes	yes	yes
<i>opposite sources</i>	<i>no</i>	yes (pos./neg.)	yes (north/south)	yes (compr./tension)

Table 1: Comparison of some of the main properties of the four types of force fields discussed in the present work: gravitational, electric, magnetic, and elastic.

placements Δx and sustain both tension and compression is characterized by the following relationship:

$$f(\Delta x) = k \Delta x \quad (68)$$

where $k \in \mathbb{R}^+$ is a proportionality constant. The displacement Δx corresponds to the change of length of the spring respectively to its free length L . In the above equation, the convention of positive sign of the force f means that the spring is under tension, so that the force at the extremity where it is measured has orientation equal to that of the spring while pointing towards the spring center.

As with almost everything else in nature, the above linear relationship is only an approximation, because every real-world elastic material will present some level of non-linearity, especially for relatively large displacements Δx .

In addition, observe that rubber bands act differently from springs, in the sense that they can only exert forces when under traction, and not compression, which may implies discontinuity of respective forces.

Though elastic forces take place by *contact*, rather than at distance, they also have a respective potential “field” (see Section 15), corresponding to the following *mechanical potential energy*:

$$U(\Delta x) = \frac{1}{2} k [\Delta x]^2 \quad (69)$$

Interesting, in a similar manner to the gravitational and electric field, the “elastic field” defined Equation 68 concerning the elastic force at each point of space can be obtained as the gradient of the mechanical potential:

$$f(\Delta x) = k \Delta x = \frac{\partial U(\Delta x)}{\partial \Delta x} \quad (70)$$

where we used the partial derivative in order to emphasize the relationship with the gradient. Actually, two and

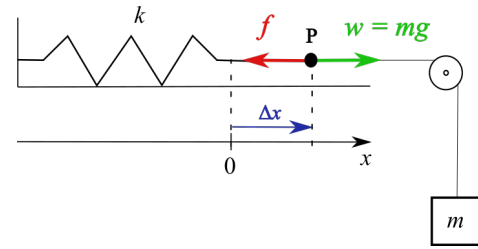


Figure 23: A simple scale involving a spring, a thread and a massless pulley, can be used to measure the weight (or mass) of a given object attached to the thread.

three-dimensional analogs of vector fields can be defined for spring forces (see Section 15). The unit of the elastic potential (or energy) is Joules (J).

One interesting manner to better understanding weight stems directly from the main approach that has been traditionally used for its measurements, namely the devices known as *scales*. One of the simplest types of scales is depicted in Figure 23, involving an ideal (massless, so that no torque appears) pulley and spring, as well as total absence of friction or any other type of dissipation. The weight to be measured, the *force* implied by the object mass m under the action of the gravitational acceleration g , is attached to the thread connected to the spring.

It is interesting to realize that *massless pulleys* can be more generally used to change the orientation of given forces while preserving their respective magnitude.

Once the weight has been attached and eventual oscillations implied by its attachment have vanished, point P will be at rest. The force $w = mg$ is a force acting on point P , as a consequence of the weight of the attached mass. This force will be matched by an opposite force f of equal magnitude (Newton’s third law of motion, considering that point P is at rest), which corresponds to the

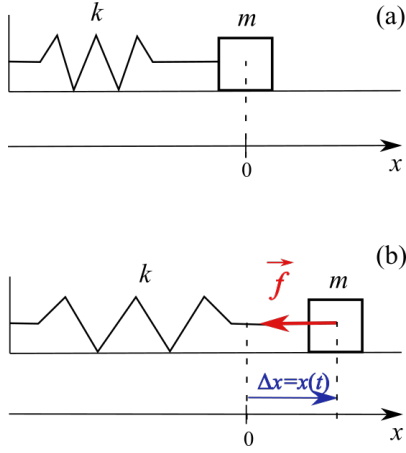


Figure 24: An object of mass m is attached to one of the extremities of a string with constant k , while its other extremity remains fixed. The initial position and velocity of the particle are A and zero, respectively. The spring has free length L , but by placing the origin of the x -axis as shown in the figure, this length does not appear in the equations. What will the motion of this mass be?

force which the spring. So, we can write:

$$f + w = 0 \implies f = -mg = -k \Delta x \implies m = \frac{k \Delta x}{g} \quad (71)$$

where k is the *spring constant* with unit N/m . Observe that for $x = x_0$ it follows that $f = 0 N$.

The above equation allows us to measure either the weight force $w = k \Delta x$, or the mass $m = (k \Delta x)/g$ of the object. The scale of the scale then needs to be calibrated by using the respective values of k and g .

Let us now consider a situation involving a spring and a point mass in which it is possible to have respective displacements, i.e. dynamics. The addressed situation is illustrated in Figure 24.

The only force to be considered corresponds to the force implied by the spring on the particle. Thus, we can write:

$$f = m \frac{d^2 x(t)}{dt^2} = -k x(t) \implies \quad (72)$$

$$\frac{d^2 x(t)}{dt^2} = -\frac{k}{m} x(t) \quad (73)$$

The solution of the above ordinary differential equation (ODE) on the time variable requires finding a function $x(t)$ that is proportional, through the proportionality constant $\frac{k}{m}$, to the negative of its second derivative. In real spaces (as contrasted to complex spaces), there are two functions that satisfy this condition, namely the *sine* and the *cosine*.

As a consequence of the linearity of the respective differential equation, any linear combination of these two functions will also be a solution of the system, actually corresponding to its most general solution. As allowed by the interesting (and not so often employed) property 136,

here we will consider $a \cos(x + \phi)$, with $a, \phi \in \mathbb{R}$ because this function is identical to a linear combination of the sine and cosine functions.

Therefore, the general solution involving the parameters a, b and ϕ still to be determined, is as follows:

$$x(t) = a \cos(bt + \phi) \quad (74)$$

Let us determine the respective first and second derivatives:

$$\begin{aligned} \frac{dx(t)}{dt} &= -ab \sin(bt + \phi) \implies \\ \implies \frac{d^2 x(t)}{dt^2} &= -ab^2 \cos(bt + \phi) \end{aligned} \quad (75)$$

This result should be identical to Equation 73, which implies:

$$\begin{aligned} \frac{d^2 x(t)}{dt^2} &= -\frac{k}{m} x(t) = -\frac{k}{m} a \cos(bt + \phi) = \\ &= -ab^2 \cos(bt + \phi) \implies b = \sqrt{\frac{k}{m}} \end{aligned} \quad (76)$$

Let us introduce the angular frequency ω as:

$$\omega = 2\pi f = \sqrt{\frac{k}{m}} \implies f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad (77)$$

implying a respective period of:

$$T = 2\pi \sqrt{\frac{m}{k}} \quad (78)$$

Thus, we can write:

$$x(t) = a \cos(\omega t + \phi) \quad (79)$$

Now, by applying the initial position condition $x(t = 0) = A$:

$$\begin{aligned} x(t = 0) &= A = a \cos(\omega \cdot 0 + \phi) = a \cos(\phi) \implies \\ \implies \cos(\phi) &= \frac{A}{a} \end{aligned} \quad (80)$$

Now, we also have to consider the initial velocity $\dot{x}(t = 0) = v_0$:

$$\begin{aligned} \dot{x}(t = 0) &= v_0 = -a\omega \sin(\omega \cdot 0 + \phi) \implies \\ \implies \sin(\phi) &= -\frac{v_0}{a\omega} \end{aligned} \quad (81)$$

Combining the two previous equations, we have:

$$\begin{aligned} \frac{\sin(\phi)}{\cos(\phi)} &= \tan(\phi) = -\frac{v_0}{A\omega} \implies \\ \implies \phi &= \arctan\left(-\frac{v_0}{A\omega}\right) \end{aligned} \quad (82)$$

from which it follows that:

$$\begin{aligned}
 \cos(\phi) = \frac{A}{a} &\implies a = \frac{A}{\cos(\phi)} = \\
 &= \frac{A}{\cos(\arctan(-\frac{v_0}{A\omega}))} = \\
 &= \frac{A}{\frac{1}{\sqrt{(-\frac{v_0}{A\omega})^2 + 1}}} = A \sqrt{\left(\frac{v_0}{A\omega}\right)^2 + 1} \quad (83)
 \end{aligned}$$

Therefore, we can now express the solution to the considered spring-mass system under generic initial conditions as:

$$x(t) = A \sqrt{\left(\frac{v_0}{A\omega}\right)^2 + 1} \cos\left(\omega t + \arctan\left(\frac{v_0}{A\omega}\right)\right) \quad (84)$$

In case we have $v_0 = 0$, it follows that:

$$x(t) = A \cos(\omega t) \quad (85)$$

It is interesting to observe from the previous example that it was precisely the fact that Newton's second law of motion establishes a link between the force and the second derivative of the position of the particle that ultimately led to the harmonic (frequency independent of the oscillation amplitude) oscillations characterizing the considered spring-mass system. In a sense, the fact that our world and universe has plentiful of harmonic or near-harmonic oscillations is to a great extent related to this intrinsic relationship underlying Newton's second law of motion.

14 Coupled Forces

The existence of forces mediated by contact, such as elastic forces, motivates further attention to be given to the problem of how forces interact among themselves along objects. This issue bring us back to a more careful consideration of Newton's third law of motion.

One first important point about this third law is that it is often misunderstood. In this section we will try to develop an intuitive approach to its application, which is based on the concept of understanding action and reaction in terms of a conceptual approach that we shall call *conjugated forces*.

Consider the situation depicted in Figure 25(a). Here, we have a point particle with mass m under the application of the respective gravitational force $f = mg$. This force appears alone, which seems to violate Newton's third law, because there is no respective reaction to be accounted for.

A little reflection, however, reveals that there is indeed a paired reaction as predicted by Newton's third law, and this reaction happens to correspond to the force implied

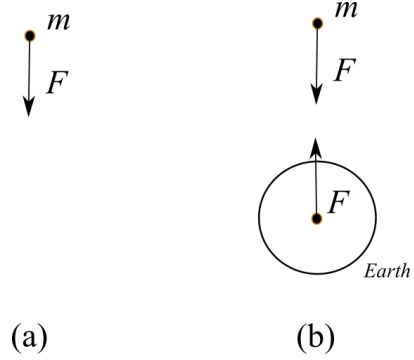


Figure 25: Understanding Newton's third law of motion in terms of the concept of *conjugated forces*. A point particle appears under the force of gravity in (a), missing a respective reaction. The reaction to this force is actually to be found at the center of the earth, which corresponds to the second mass involved in Newton's law of universal gravitation. These pair of action-reaction opposite forces is henceforth called *conjugated forces*. Observe that no additional mass or force is required, as could be the case in coupled mechanical systems involving other types of force (e.g. elastic), because gravity takes place from distance.

by the point particle m onto the Earth, as illustrated in Figure 25(b). These pairs of force-reaction are henceforth called *conjugated forces*. They can be better understood by considering a rubber band, which can only exert forces provided it is tensioned with opposite forces at each of its extremity. Another interesting analogy is that of a car jack, which can only work provided one of its extremities is in contact with the floor. In a sense, these situations are somewhat analogous to the lever principle, whose action requires a pivot.

Henceforth, it is assumed that not torques are present in the considered mechanical systems, and that no possible rotations are therefore observed.

In the case of the gravitational and electric fields, these pairs of forces do not involve any additional masses, as they take place through distance effects. For simplicity's sake, we shall assume henceforth that even mechanically coupled systems will have conjugated forces, mediated by no additional mass (e.g. massless springs or action from distance).

The main reason why the reaction to the gravitational force is often not considered is that frequently the gravitational potential is assumed from the beginning, therefore not explicitly involving the mass of the earth in the analyses.

Interestingly, when all involved forces are taken into account, the net sum of forces acting on all parts of the system becomes necessarily zero ($F - F$ in the case of Fig. 25), meaning null resulting applied forces, and therefore, as a consequence of the impulse-momentum theorem, conservation of linear moment.

As it happens, the analysis of mechanical systems in-

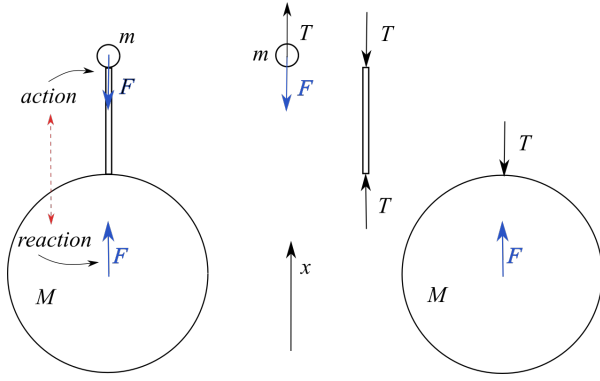


Figure 26: A simple example of decomposing a system involving coupled masses. For simplicity's sake, and given that the orientation of the forces are indicated by the respective vectors, the shown force magnitudes refer to absolute values. The objects are not shown to geometric scale.

volving several solid masses coupled, or in contact one another, can often be more effectively approached with the help of the above concept.

Let us start with the simple situation shown in Figure 26. A solid object of mass m , placed over a massless bar onto the surface of the earth, which is also considered to be solid (rigid).

Observe that it is important to adopt a reference axis x from which the signs of the forces and derivatives can be determined. The sense of this axis is completely immaterial, as the signs of the involved quantities adapt respectively.

$$\begin{aligned} +T - F &= m(0) \implies T = F \\ -T + T &= (0)(0) \\ -T + F &= M(0) \end{aligned}$$

Therefore, we have that $T = F$. Interestingly, in systems where all parts are at rest, the composition of the several forces, including conjugated pairs, can be seen as “closing a circuit”, in the sense that the respective vector composition of all involved forces results in the zero vector, therefore “closing the circuit.”

One particular important point regarding the previous construction regards the fact that the elements that have no mass cannot contribute to the motion of the system other than by transmitting forces. As the mass is null, they undergo no acceleration so that the exerted forces necessarily correspond to a conjugate pair whose two components necessarily cancel one another.

The proper construction of a diagram of forces modeling a mechanical system can be helped by paying special attention to the eventual massless components, henceforth called *ghosts*, since they will necessarily be under tension, compression, or free, while having equal forces at

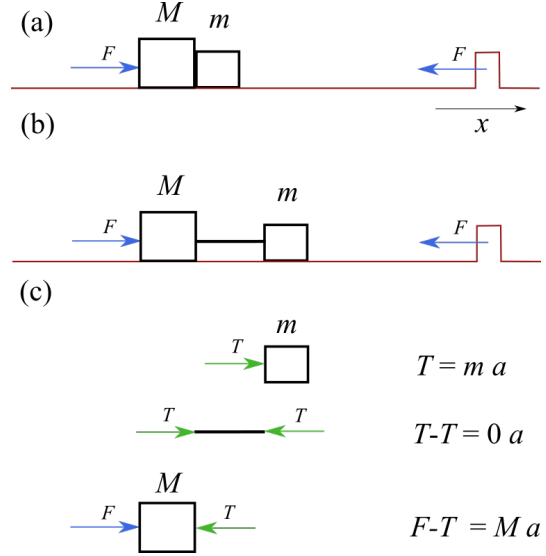


Figure 27: A mechanical system involves two masses m and M in contact one another, while a force F is applied only onto the former (a). The decomposition of forces can be helped by introducing a massless rod between the two objects (b), which *should be considered as having length zero*. Newton's second law of motion can now be applied respectively to the identified forces, leading to the solution (c).

both sides. It is also important to identify all involved actions and reactions of the forces, even if some of them are eventually not taken into account. In case a system incorporates no ghosts, they may be artificially temporarily added as a resource for better understanding and defining the forces, provided they do not introduce torque (null-length ghost elements can also be used).

For instance, consider the situation in Figure 27, which illustrates two masses m and M that contact one another while an external force F applied to the mass at the left-hand side of the system. In the case of this example, force F is understood to take place at distance, as could be the practical case with electric or magnetic interactions.

A ghost rod can be incorporated between the two masses, as also illustrated in Figure 26(a). In situations involving torque and/or angular momentum, these additional ghost rods need to be considered as *having zero length*.

Provided the rod is solid, has null mass, and it is applied so as not to introduce additional rotation or torque, the rod can have no possible effect in changing the dynamics of the system. At the same time, it makes it evident that the contact between the two masses involves a conjugated pair of forces, which therefore have identical magnitude but opposite directions, as shown in the figure. The obtained decomposition of forces so that a respective force diagram is assigned to each of the components of interest (b), allowing the determination of the resulting force

and respective consideration in Newton's second law of motion, which yields:

$$\begin{aligned} T &= m a \\ T - T &= (0) (0) \\ F - T &= M a \implies F = M a + m a = a (M + m) \end{aligned}$$

As the acceleration is shared by all involved bodies, the above equations can be immediately solved in order to reveal the involved forces, velocities, and accelerations.

Let us now proceed to the slightly more elaborate system shown in Figure 28(a), which involves several components as well as an external force F .

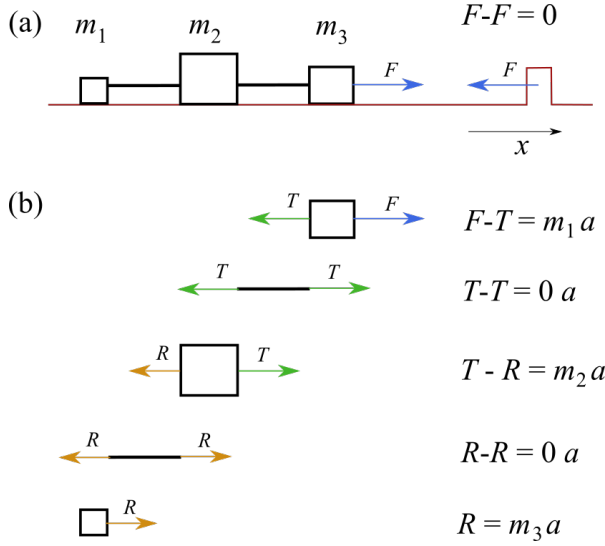


Figure 28: Motion involving several components under action of an external source. The objective is to determine all the involved forces, as well as accelerations and velocities.

The decomposition of the objects and forces is shown in (b), involving two ghost rods. Similarly to the previous example, all masses undergo the same acceleration since they are stably linked one another so that there is no possible relative motion between them,

$$\begin{aligned} F - T &= m_1 a \implies 12 - T = 2 a \\ T - R &= m_2 a \implies T - R = 3 a \\ R &= m_3 a \implies R = 1 a \end{aligned}$$

which allows the immediate solution of the involved accelerations and forces as follows:

$$\begin{aligned} (F - T) + (T - R) + R &= F = a(m_1 + m_2 + m_3) \implies \\ \implies 12 &= 6 a \implies a = 2 \text{ ms}^{-2} \\ R &= 1 a \implies R = (1) (2) = 2 \text{ N} \\ T - R &= 3 a \implies T = 6 + 2 = 8 \text{ N} \\ F - T &= 2 a \implies 12 - 8 = (2) (2) = 4 \text{ N} \blacksquare \end{aligned}$$

Interestingly, the tension forces through the rods are progressively reduced as one moves away from mass m_3 .

These examples evidenciate that Newton's third law applies only regarding conjugate pairs, not necessarily being verified for every decomposed elements. For instance, we have that the component with mass m_3 has a resulting force $F - T = m a$, which can by no means understood as an action-reaction pair. Actually, the nullification of forces at every component will only take place in case all components are at rest or equilibrium of forces (conservation of linear motion).

Figure 29 illustrates the decomposition of forces in a system where a particle with mass m is attached to a massless frame of rods and beams. It is assumed that there is no resulting force or torque in the system, so that it is at equilibrium. The gravitational force (in magenta) acts on the particle, which is counterbalanced by force resulting (in cyan) from the the tensions in the two rods (in green).

The action reaction pairs (Newton's third law of motion) corresponds to the gravitational force on the particle \times resulting force implied by the tensions in the frame; the tensions if the frame resulting at its upper and lower points; and the action of the frame on the earth \times the reaction of the gravitational force on the center of the earth. In these equilibrium cases, each individual force in each component will be necessarily counterbalanced by other forces applied to that object, so as to result zero. However, in non-equilibrium situations including systems moving with constant vector velocity, only the action-reaction pairs are guaranteed to lead to zero resulting force.

Interestingly, a torque diagram similar to the above discussed force decomposition, can also be obtained for the considered systems of coupled objects. For instance, the frame in the previous example is under torque as implied by the pairs of forces acting upon it which have disjoint lines of action, but no rotation is obtained as the frame is rigidly attached to the soil.

Though the consideration of the ghost elements can substantially help with the construction of the force diagrams, it is also possible, especially after some familiarization, to proceed more directly to the decomposition of the system, as illustrated in Figure 30.

The last example suggests a way to infer the distribution of longitudinal forces along a rectangular object with total mass M and length L , which can be approached as illustrated in Figure 31.

Our formulation assumes that the system is analyzed at a fixed instant time t as shown in the figure, but the obtained results then hold for any duration of the force. We immediately find that the acceleration of the particle

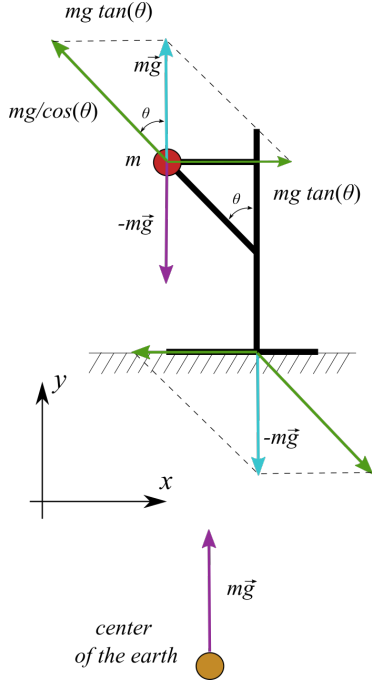


Figure 29: Example of coupled forces in a system at equilibrium, implying that the resulting force is zero. The involved beams and rods are all massless, and the particle (in red) is placed at the junction of two rods. Since there is no resulting force or torque, the massless framework of beams can be understood as a single point, allowing the decomposition of forces corresponding to the reactions of the frame in the upper portion to be mirrored at its lower portion, where the frame attaches to the floor. Though not necessary, the earth, represented by its center of mass, has been incorporated in order to complete the “circuit” of forces, leading to zero resulting force. The objects are not shown to scale.

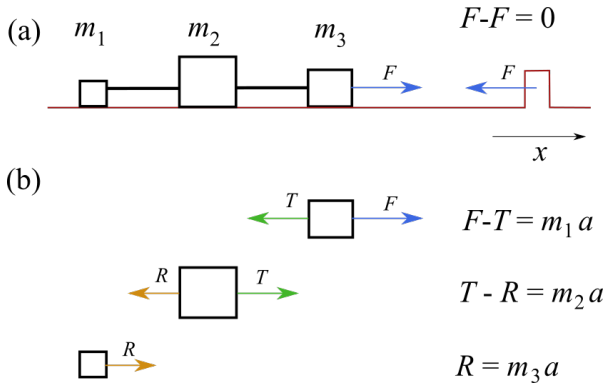


Figure 30: Simplified analysis of the system in Fig. 28, obtaining by not considering explicitly the ghost rod components.

is:

$$a = \frac{F}{M} \quad (86)$$

As the mass contained in the left-hand portion of the divided object can be expressed as:

$$m(s) = \frac{s}{L} M \quad (87)$$

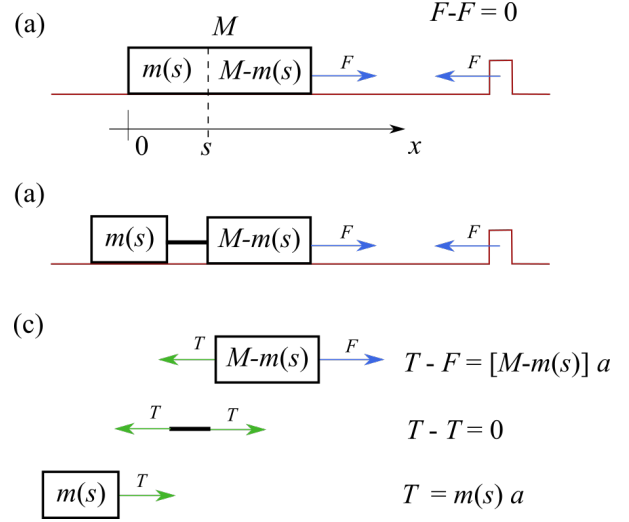


Figure 31: Construction adopted for determining the progressive distribution of the tension along a rectangular block with mass M and length L , under the influence of the force F .

by applying Newton’s second law, it follows that:

$$T = m(s) a = \frac{s}{L} M a = s \frac{M a}{L} = s \frac{F}{L} \quad (88)$$

which indicates, as could be expected, that the tension T along the rectangular block increases linearly from 0 at its left-hand extremity, up to F at its right-hand side, as illustrated in Figure 32.

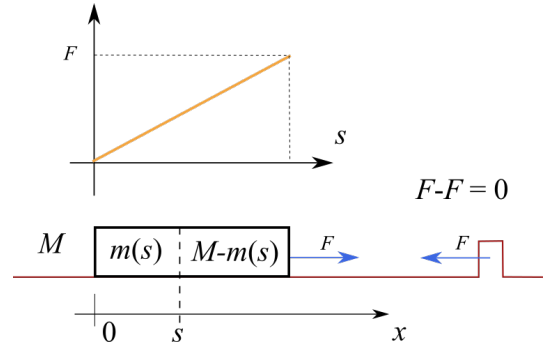


Figure 32: The solution of the problem of finding the distribution of tensions within the rectangular block. The tension force varies linearly from 0, at the left-hand extremity of the block, up to F at the other extremity.

The just presented procedure can be used as a subsidy when addressing coupled systems incorporating force-generating devices with intrinsic mass distribution, such as real-world spring. In particular, it can be applied to better understand the sensation of our own weight. To do so, we consider the result in Figure 32 applied vertically, so that the force F can be associated to the gravitational force.

Thus, while we are lying down on the ground, the greatest force (and therefore pressure, see Section 20) will be

felt along the portions of our body that are in contact with the ground, while the upper portions will receive decreasing forces until no force results at the upper surface of the body.

15 Oscillating Spring-Mass Systems

We now consider the particularly interesting spring-mass system shown in Figure 33, involving two masses bound by a spring moving along the x -axis under the effect of an external constant force F . The decomposition of the system is also shown in the figure.

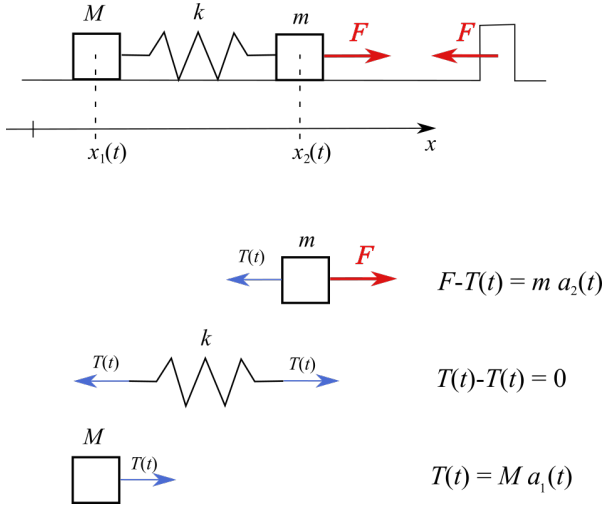


Figure 33: Two masses M and m linked by a spring of constant k move along the x -axis under the influence of the external constant force F . How can we characterize this motion?

Therefore, we have that:

$$\begin{aligned} M \ddot{x}_1(t) &= T(t) \\ m \ddot{x}_2(t) &= F - T(t) \\ T(t) &= k (x_2(t) - x_1(t) - L) \end{aligned}$$

which can be rewritten by using the equation of elastic force as:

$$\begin{aligned} M \ddot{x}_1(t) &= k (x_2(t) - x_1(t) - L) \\ m \ddot{x}_2(t) &= F - k (x_2(t) - x_1(t) - L) \end{aligned} \quad (89)$$

The motion can then be differentially expressed in terms of the following system of second order ODEs:

$$\begin{cases} \ddot{x}_1(t) = \frac{1}{M} [k x_2(t) - k x_1(t) - k L] \\ \ddot{x}_2(t) = \frac{1}{m} [F - k x_2(t) - k x_1(t) - k L] \end{cases}$$

Now, we aim at transforming this system of two second order ODEs into a system of four first order ODEs,

which can be done by introducing the following auxiliary variables:

$$\begin{aligned} u_1 &= x_1 \\ u_2 &= \dot{x}_1 \\ u_3 &= x_2 \\ u_4 &= \dot{x}_2 \end{aligned}$$

which leads to the following differential model of the original system:

$$\begin{cases} \dot{u}_1 = u_2 \\ \dot{u}_2 = \frac{1}{M} [k u_3 - k u_1 - k L] \\ \dot{u}_3 = u_4 \\ \dot{u}_4 = \frac{1}{m} [F - k u_3 - k u_1 - k L] \end{cases}$$

Figure 34 illustrates the obtained solution for the parametric configuration specified by spring free length $L = 5m$, $M = 0.5kg$, $m = 0.1kg$, $F = 1N$, $k = 1Nm^{-1}$. The positions of the center of mass of the two bodies, namely $x_1(t)$ and $x_2(t)$ are shown in (a) and (c), respectively, while the scalar velocities $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are presented in (b) and (d). The result indicates that the velocities undergo an overall linear increase, but in presence of continuing oscillations which are larger in the case of the smaller mass. The positions present a mostly parabolic form, also incorporating respective oscillations.

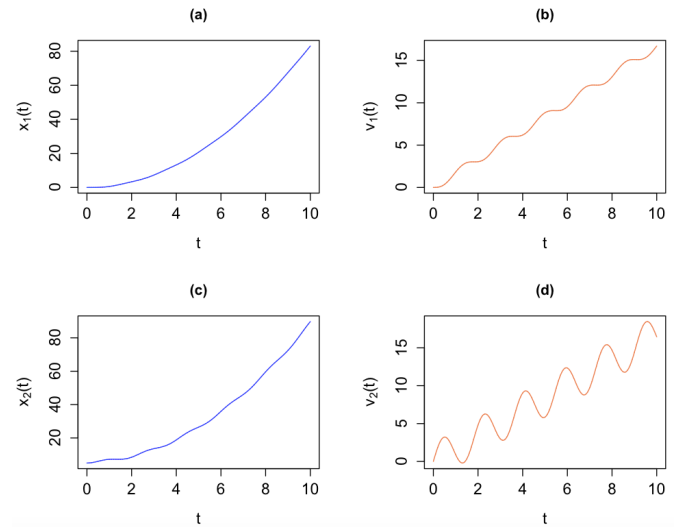


Figure 34: The numeric solution (e.g. [22, 21]) of the spring-mass system in Fig. 33 assuming $L = 5m$, $M = 0.5kg$, $m = 0.1kg$, $F = 1N$, $k = 1Nm^{-1}$.

Figure 35 illustrates another interesting situation that will be now addressed, which will turn out to be surprising.

We start by considering a generic configuration of the system involving the mass displaced by (x, y) , yielding the

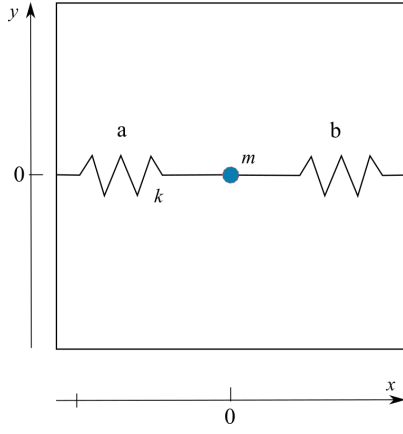


Figure 35: A point particle with mass m is attached to two massless springs — identified as a, b — with identical constants k . There are no energy dissipation, gravity or external forces. All springs are not tensioned or compressed when the mass is at the origin $(0,0)$ of the reference frame. What kind of motion can this particle undergo?

diagram shown in Figure 36. Observe that the coordinates of the adopted frame are identified as $X \times Y$.

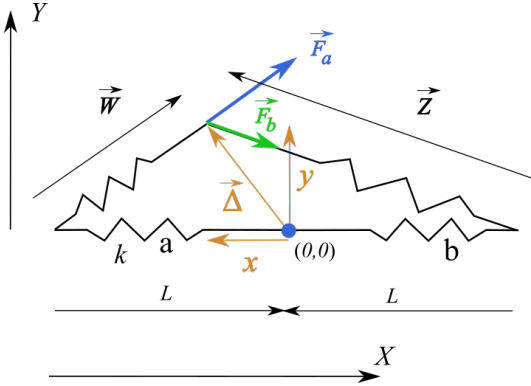


Figure 36: Forces imposed on the particle as a consequence of displacing the particle by (x, y) from its equilibrium position $(0,0)$.

Now, we can obtain the vectors \vec{W} and \vec{Z} corresponding to the relative size and orientation of the springs a and b as a consequence of the displacement (x, y) of the particle:

$$\begin{aligned}\vec{W} &= (L, 0) + (x, y) = (L + x, y) \\ \vec{Z} &= (-L, 0) + (x, y) = (-L + x, y)\end{aligned}$$

These vectors have respective magnitudes corresponding to:

$$\begin{aligned}\|\vec{W}\| &= \sqrt{(L + x)^2 + y^2} = \sqrt{L^2 + 2Lx + x^2 + y^2} \\ \|\vec{Z}\| &= \sqrt{(-L + x)^2 + y^2} = \sqrt{L^2 - 2Lx + x^2 + y^2}\end{aligned}$$

Now, the forces \vec{F}_a and \vec{F}_b imposed on the particle as a consequence of its relative displacement can be expressed

as:

$$\begin{aligned}\vec{F}_a &= k \left[L - \|\vec{W}\| \right] \frac{\vec{W}}{\|\vec{W}\|} \\ \vec{F}_b &= k \left[L - \|\vec{Z}\| \right] \frac{\vec{Z}}{\|\vec{Z}\|}\end{aligned}$$

which allows us to obtain the following resulting force:

$$\vec{F} = \vec{F}_a + \vec{F}_b$$

At this point, it is interesting to visualize the forces implied onto the particle for several possible relative displacements, which is shown in Figure 37.

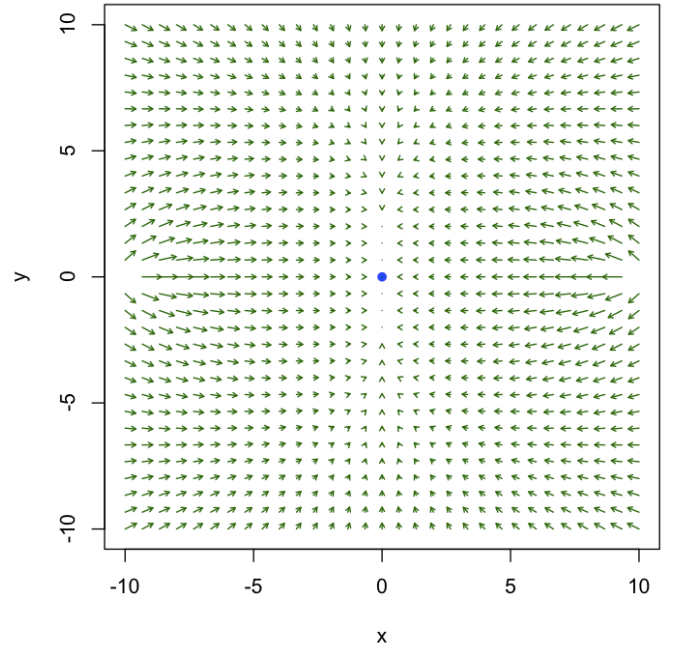


Figure 37: “Vector field” defined by the two massless springs onto the particle when it is displaced to the positions corresponding to the point of application of the shown force vectors. The quotes in “vector field” are in order to emphasize that, strictly speaking, only one particle and respective force exist at any time, so that there is no possibility to have simultaneous forces implied by more than one particle as is the case with traditional vector fields. However, in the case of a single particle, the obtained motion will be indistinguishable between the cases considering the adopted spring system and a respective, full parallel, vector field characterized by the same force interaction. In which aspects does the vector field shown in this picture resemble or differ from the electric field defined by two positive charges separated horizontally by a distance $2L$?

Observe the null forces obtained along the middle vertical line, which is a consequence of the symmetry of the system. Also, observe that the forces along the horizontal line extending through the fixed extremities of the springs are exclusively horizontal, as it should be. In addition, the forces become stronger as the position of the displaced particles approaches the fixed extremities of any of the two springs, as well as when it reaches the corners

of the region shown in the figure. All these properties are coherent with what could be expected regarding the action of the springs on the particle.

Newton's second law of motion can now be applied on the two components of the obtained resulting force:

$$\begin{cases} F_x = m \ddot{x}(t) \\ F_y = m \ddot{y}(t) \end{cases}$$

leading to the following pair of second order ODEs:

$$\begin{cases} \ddot{x}(t) = \frac{1}{m} F_x \\ \ddot{y}(t) = \frac{1}{m} F_y \end{cases}$$

By introducing the following auxiliary variables:

$$\begin{aligned} u_1 &= x \\ u_2 &= \dot{x} \\ u_3 &= y \\ u_4 &= \dot{y} \end{aligned}$$

We can rewrite the previous system of two ODEs as:

$$\begin{cases} \dot{u}_1 = u_2 \\ \dot{u}_2 = \frac{1}{m} F_x \\ \dot{u}_3 = u_4 \\ \dot{u}_4 = \frac{1}{m} F_y \end{cases} \quad (90)$$

Figure 38 depicts some of the many interesting possible trajectories obtained respectively to the two spring system considering diverse initial conditions indicated in the figure caption. Numerical integration has been adopted given that the two spring system is highly non-linear.

The previous two spring system can be modified in a particular interesting manner by placing a single spring with null free length with one of its extremities attached to the point $(0, 0)$. The equations obtained for the two spring system can be readily adapted to this new configuration, being left as an exercise. Figure 39 illustrates some of the trajectories obtained for this system, which closely resemble the gravitational orbits observed in Figure 3.

16 The Linear Pendulum

The linear pendulum, shown in Figure 40, consists in one of the most traditional examples of harmonic oscillation. Here, we have a point particle with mass m attached to a rod, left to undergo oscillations, which are assumed to be small enough so that $\sin(\theta) \approx \theta$.

By decomposing the gravitational force along the tangent to the motion and the orientation of the rod, we have that the only resulting force is the former, with magnitude $m g \sin(\theta(t))$, so that Newton's second law yields:

$$m g \sin(\theta(t)) = -m L \frac{d^2 \theta(t)}{dt^2} \quad (91)$$

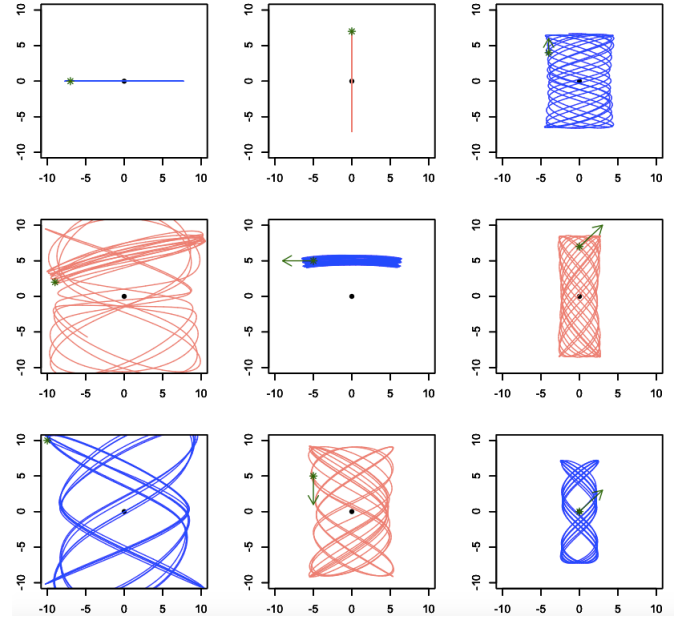


Figure 38: Several trajectories obtained for the two spring system considering several initial conditions and $m - k = 1$ and $L = 10$. The trajectories are shown along their first 100 s. Observe that the trajectories are always tangent to the initial velocity, unless in the case of zero initial velocity vector. Completely distinct trajectories can be obtained depending on the initial conditions. For instance, in the first two cases we will simply have an oscillation with constant amplitude continuously taking place along time (there is no dissipation). Initial positions close to the point $(0, 0)$ tend to yield bilaterally symmetric trajectories, but those near the fixed extremity of the springs may lead to highly irregular motion. What are the equilibrium points of this system? Can it undergo chaotic dynamics?

This is a *non-linear* ordinary differential equation (ODE) on the variable $\theta(t)$.

Because we have assumed small amplitudes of oscillations, we can make the approximation:

$$\frac{d^2 \theta(t)}{dt^2} = -\frac{g}{L} \theta(t) \quad (92)$$

Except for the proportionality constant $\frac{g}{L}$, this ODE is identical to that we obtained respectively to the spring-mass system, i.e. Equation 73. Thus, by making:

$$\omega = \sqrt{\frac{g}{L}} \quad (93)$$

So, the sought solution is the same as in Equation 84.

In the particular case that $\theta_0 = 0$ and $\dot{\theta}_0 = 0$, the solution simplifies to:

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{L}} t\right)$$

Given that $\omega = 2\pi$ is the angular frequency of the mo-

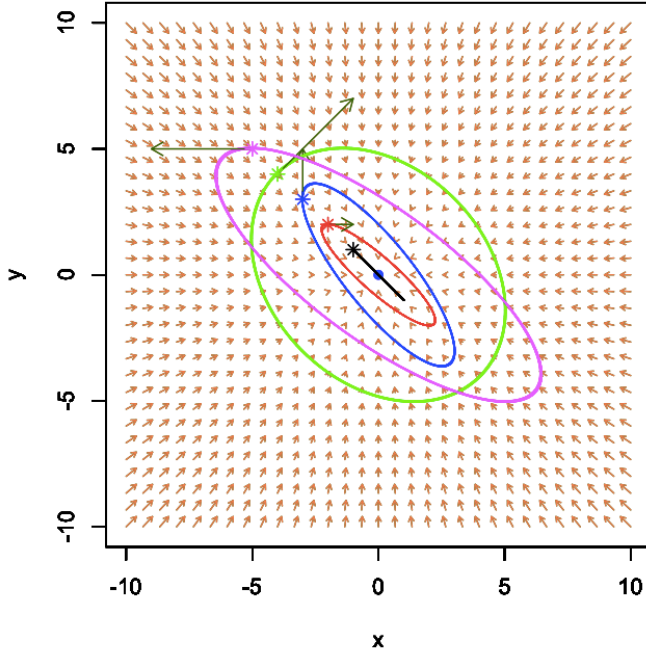


Figure 39: Examples of trajectories, shown in diverse colors, obtained for the one spring system considering diverse initial conditions (shown in dark green) and assuming $m = k = 1$ and absence of other forces than that exerted by the null free length spring attached to the central point. The respectively defined “force vector field” is also shown for reference. As before, every obtained trajectory is tangent to the initial velocity vector. Are these trajectories ellipses? Is it possible to identify initial conditions leading to circular trajectories?

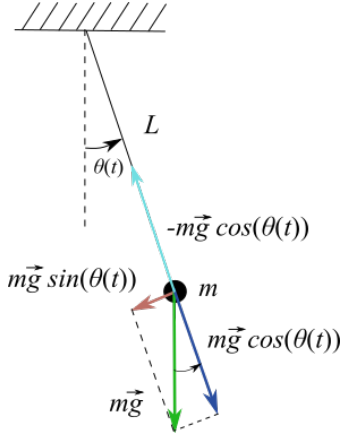


Figure 40: The *linear pendulum* consists of a point particle with mass m attached to a rod, or arm, of length L . The gravitational force $m g$ applied to the particle can be decomposed into two components, one tangent to the motion (shown in brown) and another in the direction parallel to the rod (shown in blue). The latter is canceled by the tension of the rod, and the former is left to modify the tangent velocity of the particle. The pendulum is understood to undergo small oscillations, allowing the approximation $\sin(\theta) \approx \theta$, which is required in order to obtain a respective linear differential equation of motion.

tion, we have that:

$$\omega_0 = 2\pi f = \sqrt{\frac{g}{L}} t \Rightarrow f = \frac{1}{2\pi} \sqrt{\frac{g}{L}} \quad (94)$$

implying a period equal to:

$$T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{L}{g}} \quad (95)$$

Though we obtained an equation of motion that is completely analogous to that obtained for the spring-mass system, the solution obtained for the pendulum is not completely accurate because it assumed quite small oscillations. In this sense, the spring-mass oscillation is of particular interest for being intrinsically linear (though this turns out to be, in practice, also an assumption), requiring no approximations.

17 The Non-Linear Pendulum

Though we assumed small amplitudes of oscillations while obtaining in the previous section an equation of motion for a simple pendulum (linear), most pendulum motions involve larger amplitudes yielding, as a consequence, the following *non-linear* ODE:

$$\frac{d^2 \theta(t)}{dt^2} = -\frac{g}{L} \sin(\theta(t)) \quad (96)$$

As it is typical with non-linear systems, obtaining the respective solution constitutes not a particularly simple task that will not be developed here (please refer to, e.g. [30, 31]). The exact solution of this equation (e.g. [30]), is as follows:

$$\theta(t) = 2 \arcsin \left\{ \sin(\theta_2) \operatorname{sn} \left[K(\sin^2(\theta_2)) - \omega_0 t; \sin^2(\theta_2) \right] \right\} \quad (97)$$

$$\text{where: } \theta_2 = \frac{\theta}{2},$$

$$\omega_0 = \sqrt{\frac{g}{L}},$$

and the function:

$$K(m) = \int_0^1 \frac{1}{\sqrt{(1-z^2)(1-mz^2)}} dz \quad (98)$$

is the *complete elliptic integral of the first kind*, and $\operatorname{sn}(x; m)$ is one of the *Jacobi elliptic functions*, being analogous to the traditional *sine* function. These functions are analogous to the trigonometric functions, but while the latter are defined respectively to a circle, the Jacobi elliptic functions assume conic sections, in particular ellipses, in their definition.

The exact angular frequency of this motion is given as:

$$\omega(\theta_0) = \frac{\pi \omega_0}{2 K(\sin^2(\theta_0/2))} \quad (99)$$

from which we realize that the frequency and period of the non-linear oscillation *are actually a function of the initial angular position* θ_0 .

Approximate solutions of the non-linear pendulum can also be obtained numerically (by using an effective scheme and high resolution) by considering the respective system of first order ODEs:

$$\begin{cases} \dot{u}_1 = u_2 \\ \dot{u}_2 = -\frac{g}{L} \sin(u_1) \end{cases} \quad (100)$$

where:

$$\begin{aligned} u_1 &= \theta \\ u_2 &= \dot{\theta} \end{aligned}$$

Figure 41(a) depicts the solution of the linear and non-linear pendulum for $L = 0.2m$, $m = 0.2kg$, $\theta_0 = 0.97\pi$, and $\dot{\theta}_0 = 0ms^{-1}$. The linear solution, shown in cyan, has a completely distinct frequency from the exact solution for the non-linear case, shown in blue. Observe the distinct shape of the result obtained for the non-linear motion. Also shown (in red) is the solution for the linear case, but using the exact angular frequency in Equation 99, i.e.:

$$\theta(t) = \theta_0 \cos(\omega(\theta_0) t) \quad (101)$$

Interestingly, it has been observed [30] that this approximation, consisting of a combination of the linear and non-linear aspects, is reasonably accurate for $|\theta_0| < 0.75\pi$.

The obtained angular velocity of the non-linear is shown in Figure 41(b), which indicates that the angular motion at the lowest point is maximum and relatively more accelerated in magnitude than when the mass is at its highest points, around which case the pendulum moves much slower and with less angular acceleration.

18 The Wave Equation

The wave equation constitutes a markedly important concept in several areas in the Physical sciences as it appears in several distinct areas and contexts, not only in more classical areas such as electromagnetism and acoustics, but also in more recent areas including quantum mechanics.

Interestingly, the concepts and methods we have discussed so far in this work allow us to develop a simplified derivation of the wave equation which, though not completely formal, provide insights about the most important related aspects.

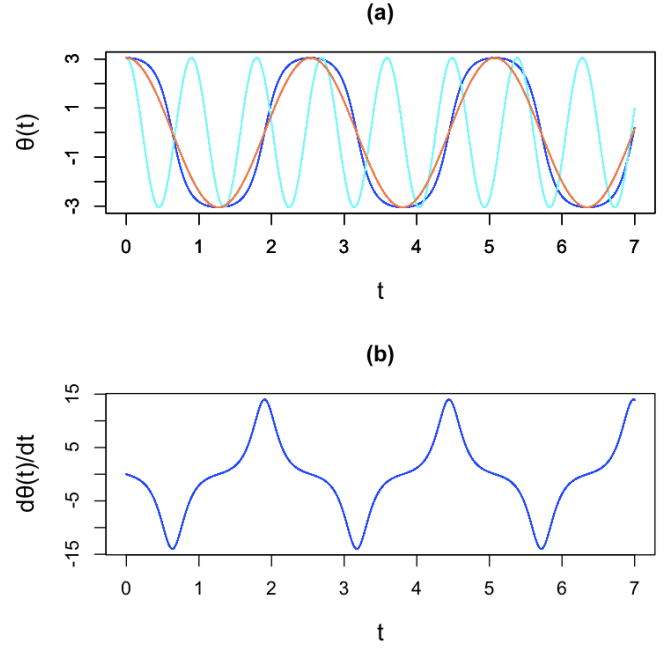


Figure 41: (a): Comparison between the angular position $\theta(t)$ obtained for linear and non-linear pendula for $L = 0.2m$, $m = 0.2kg$, $\theta_0 = 0.97\pi$, and $\dot{\theta}_0 = 0ms^{-1}$: the exact solution for the non-linear pendulum is shown in blue, while the solution obtained the linear pendulum equation is shown in cyan. The results are markedly distinct regarding both the shape of the functions as well as their frequency, which is markedly smaller for the non-linear case. A third solution, obtained by using the exact angular frequency in Equation 99 in the linear solution is shown in red. The angular velocity of the non-linear motion is shown in (b).

In order to do so, we will consider the spring-mass system shown in Figure 42, which is assumed not to be influenced by external forces (no gravity) and devoid of dissipation (i.e. no drag or friction).

The considered system is constituted by a sequence of vertically moveable rods to which masses and springs are attached. All masses have the same value m , and all springs have the same respective constant k . The initial condition consists of having all masses aligned to the dotted line at the middle of the vertical height, which coincides with the origin of the axis adopted for representing the position $y(t, x)$ of the masses.

All springs are at their rest or equilibrium position when all masses are at the middle line, as illustrated for the first spring from left to right in Figure 42. Both the rods and the springs are massless, and in this case the length of the rods can be as long as one desires.

The possible forces acting on mass i include the elastic force from the springs at masses $i - 1$ and $i + 1$, with

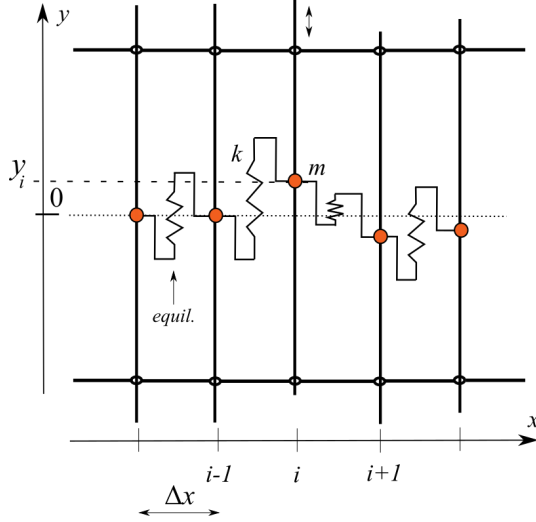


Figure 42: Simple spring-mass system used in the present work to infer the *wave equation*. Individual masses m are attached to respective rods that are free to move vertically, in complete absence of external forces including gravitational ones, as well as dissipative forces (e.g. drag, friction, etc.). Massless springs with constant k provide coupling between each adjacent mass along the system. All springs are free of tension when the vertical positions of their left and right arms coincide. Though extremely simple, the model incorporates the minimal requirements that lead to the wave equation, a partial differential equation on two variables, t and x . The solution of this equation is expressed as $y(t, x)$. What will happen is the arms of a given spring go across each another?

respective resulting force F_i being expressed as:

$$\begin{aligned} \vec{F}_i &= \vec{f}_{i-1} + \vec{f}_{i+1} = \\ &= -k(y_i - y_{i-1}) - k(y_i - y_{i+1}) = \\ &= k[y_{i-1} - 2y_i + y_{i+1}] = m \frac{d^2 y_i}{dt^2} \end{aligned} \quad (102)$$

Now, let us remember that the second derivative (partial or not) can be understood in terms of the following limit:

$$\frac{\partial^2 y}{\partial x^2} = \lim_{\Delta x \rightarrow 0} \frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} \quad (103)$$

By applying Newton's second law on the i -th mass, and considering the limit situation regarding the separation Δx along the x -axis, we obtain the following approximation:

$$\begin{aligned} \vec{F}_i &= m \frac{\partial^2 y}{\partial t^2} \implies \\ \implies k \lim_{\Delta x \rightarrow 0} \frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} &= m \frac{\partial^2 y}{\partial t^2} \implies \\ \implies \boxed{\frac{\partial^2 y(t, x)}{\partial t^2} = \frac{k}{m} \frac{\partial^2 y(t, x)}{\partial x^2} = \omega_0^2 \frac{\partial^2 y(t, x)}{\partial x^2}} \end{aligned} \quad (104)$$

Which corresponds to the partial differential equation (PDE) describing a *transversal wave equation*. Observe

that in the above developments, which are neither completely formal nor mathematically rigorous, we took into account that the form in Equation 102 corresponds to a finite difference operator for the second derivative, assuming spacing Δx as turns out to be the case in the system in Figure 42. Interestingly, the adopted construction corresponds to possibly the simplest manner to informally derive the transversal wave equation.

Observe that the resulting PDE in Equation 104 involves two variables, namely time t and horizontal position x . Solutions of this equation are scalar fields $y(t, x)$, which are called one-dimensional transversal mechanical waves.

In both cases, i.e. as a system of ODEs or as a single PDE, the solution of these equations require two additional information: (i) *initial condition*, consisting of the position and velocity of each mass at $t = 0$; and (ii) *boundary conditions*, involving how the two extremities of the system are set. Two possibilities are to let them free or to fix the masses at the left and right extremities at $y = 0$, the latter being characterized by implying reflections of the wave.

Interestingly, the system of Equations 102, considering all $i = 1, 2, \dots, N$, is not a PDE, but actually a system of second order differential equations, which can be simulated numerically without particular difficulty.

In order to do so, we rewrite each second order ODE as two respective first order ODEs.

First, we make:

$$\begin{aligned} u_i &= y_i \\ v_i &= \dot{y}_i \end{aligned}$$

which allows us to rewrite the original system of ODEs as:

$$\begin{aligned} \frac{d^2 y_i}{dt^2} &= \frac{k}{m} [y_{i-1} - 2y_i + y_{i+1}] \implies \\ \begin{cases} \dot{u}_i &= v_i \\ \dot{v}_i &= u_{i-1} - 2u_i + u_{i+1} \end{cases} \end{aligned}$$

Figure 43 illustrates the solution of the wave equation above for $N = 51$ masses with $k = 1Nm^{-1}$ and $m = 0.2kg$, considering an impulse with width 3 as initial condition and assuming fixed boundary conditions. The result incorporates several interesting features, such as the displacement of the initial impulse towards both sides, as well as their later reflections. Interferences between the components of the initial impulse can also be observed.

Though the above developments considered transversal waves, they can be immediately adapted to model *longitudinal waves*. All that is needed to do is to re-organize the mass-spring system linearly, as illustrated in Figure 44.

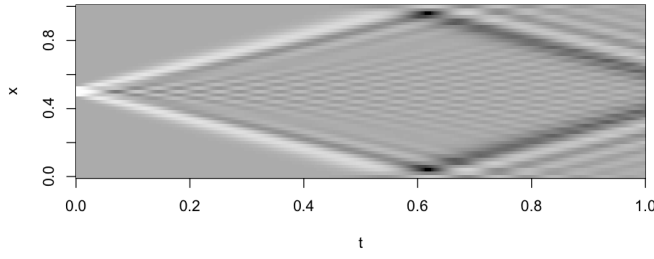


Figure 43: The numeric solution of the wave equation in terms of a system of first order ODES considering $N = 51$. The initial condition corresponds to an impulse of width 3 placed at the center of the system of masses.

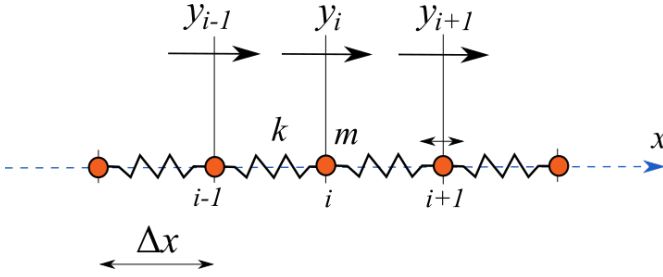


Figure 44: The model of a transversal wave equation described above can be immediately adapted to model *longitudinal waves*, as illustrated in the depicted diagram.

While longitudinal waves can be considered in analogy (reduced dimensionality) to acoustic waves, where the propagation takes place through longitudinal pressure variations, waves on water surfaces can be understood in analogy to transversal waves.

19 Density

The concept of *density* turns out to be related to a normalized counting procedure. Let us approach this concept by considering that N particles of infinitesimal size are contained within a closed region of \mathbb{R}^2 of area A , as illustrated in Figure 45.

The concept of density can now be mathematically expressed as:

$$\rho = \frac{N}{A} \quad (105)$$

Though the above definition is well in accordance with our intuitive understanding of density, it refers to a non-infinitesimal area A . However, in case we understand that the density is constant and equal to ρ within the region R , we can rewrite the previous equation as:

$$\rho(x, y) = \lim_{A \rightarrow 0} \frac{N}{A} \quad (106)$$

where the area A is made to contract around one of its original points $(x, y) \in A$. It is implicitly assumed that

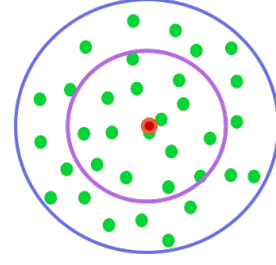


Figure 45: The density of elements, shown as green particles, constitutes a simple intuitive concept considering the number of elements comprised within a region with a given area (2D spaces, as shown in this figure) or volume (3D spaces). A constant density is such that it will not depend on infinitesimal areas (or volumes) taken around the point of interest (shown in red), as is the case in this figure (circular regions shown in blue and violet). The concept of density can be immediately extended in the case of continuous materials (scalar density field), instead of the discrete situation considered in this figure.

there are enough particles so that the numerator in the above ratio remains always non-null. This is an idealized abstraction not guaranteed to take place in the real world, as a consequence of the possibly discrete nature of matter.

Now, there is no reason why the local (or point) density $\rho(x, y)$ (or $\rho(x)$ and $\rho(x, y, z)$ in 1D and 3D spaces, respectively), should not vary with the position (x, y) . For instance, we could have:

$$\rho(x, y) = \rho_x x + \rho_y y \quad (107)$$

which corresponds to a scalar field on \mathbb{R}^2 providing an example of a *density field*.

The total mass within any respective region can then be obtained by integrating $\rho()$ along a specific region. As a simple example, let's consider the total mass M contained in a 2D mass density function $\rho(x, y) = \frac{1}{2}x + \frac{1}{3}y^2 \text{ kg m}^{-2}$ in the domain region $0 \leq x \leq 1, 0 \leq y \leq 1$:

$$\begin{aligned} M &= \int_0^1 \int_0^1 \left(\frac{1}{2}x + \frac{1}{3}y^2 \right) dx dy = \\ &= \int_0^1 \left(x^2 \Big|_0^1 + \frac{1}{3}y^2 \right) dy = \left(1 + y^3 \Big|_0^1 \right) = 2 \text{ kg} \end{aligned} \quad (108)$$

20 Pressure

Given a force resulting on a surface as a consequence of possibly several internal and external effects, as illustrated in Figure 46, it is often interesting to define the respectively exerted *pressure*, which can be done as follows:

$$P = \frac{\vec{F} \cdot \hat{i}_A}{A} = \frac{\|\vec{F}\| \cos(\theta)}{A} \quad (109)$$

where “.” refers to the dot product (see Appendix C).

As such, pressure is proportional to the spatial integration along the area A , at a given time instant t , of all

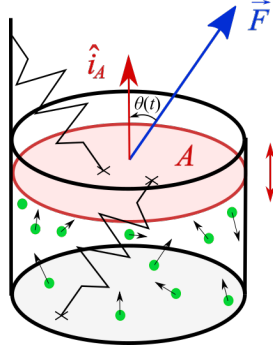


Figure 46: The mechanical pressure on a plane surface with area A , originating from various effects including elastic forces, motion of particles, vacuum, etc., is defined as the dot product between the resulting force $\vec{F}(t)$ on the surface and the versor normal to that surface, divided by the area A . Observe that, in the present diagram, the surface is restricted to undertake vertical motions without any rotation.

forces originating from possibly several components and causes (e.g. the force from each colliding particle).

It is often interesting to consider a differential version of the pressure, which can be done by taking the area A as being infinitesimal, which leads to:

$$dP(x, y, z) = \vec{f}(x, y, z) \cdot d\hat{A} = \vec{f}(x, y, z) \cdot [\hat{n} dA] \quad (110)$$

where $\vec{f}(x, y, z, t)$ is the force at each of the area points and $d\hat{A}$ is the normalized oriented differential area, so that $||d\hat{A}|| = 1$, and \vec{n} is the versor orthogonal to the surface.

Now, the net pressure along a given surface can be obtained by surface integration of the above defined differential pressure.

21 Flow or Flux?

The concept of flux, not to be confounded with flow, plays an important role in several areas, especially fluid dynamics, electricity, quantum mechanics, thermal sciences, transportation systems, and even biology and medicine.

Despite their intrinsic simplicity, the terms *flow* and *flux* often cause some confusion.

By flow it is colloquially meant a steady, continuous, movement of something towards a given direction, which does not directly correspond to a quantitative measurement. A more specific name, *flow rate*, is typically employed in a more informal way in order to quantify the number or quantity of some entity that passes along time through some delimited boundary, which can be expressed as:

$$f(t) = \frac{dq}{dt} \quad (111)$$

This concept is illustrated in Figure 47. Here, we have some discrete elements, represented as green balls, that can move through the circular section, shown in red. These elements can be of any nature, shape, etc. The unit of flow rate therefore is (quantity of entities) s^{-1} .

Observe that the flow rate can refer to either countable or continuous material, the latter situation leading to flow rates such as $kg s^{-1}$, *liters s⁻¹*, etc.

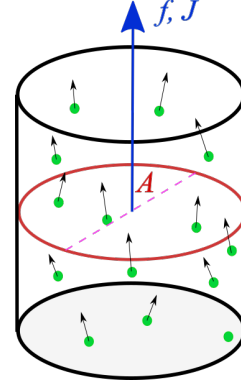


Figure 47: Basic construction underlying the concepts of both *flow rate* and *flux*.

While flow rate will suffice in several situations, *it does not take into account the area* of the section through with the entities pass along time. This has the important implication that the flow rate will depend on the assumed area (it will be actually proportional to it), which makes this concept more specific and restrict.

The incorporation of the area into the idea of flow leads to the concept of *flux*. The natural manner to consider area is as follows:

$$J(t) = \frac{f(t)}{A} = \frac{\frac{dq}{dt}}{A} = \frac{1}{A} \frac{dq}{dt} \quad (112)$$

which becomes independent of the area of the section crossed by the considered quantity.

Consider that we have divided the circular cross-section into two identical halves, as shown by the dashed magenta dividing line. We shall also assume that the flow is uniform across the section. While the flow rate through any of the two halves will be halved respectively to the flow through the whole section, at the same time, the flux will remain identical in whatever portion of the cross-section that is eventually considered.

Similarly to pressure, we can consider the flux as a differential quantity, i.e.:

$$d\vec{J}(x, y, z) = \vec{j}(x, y, z) \cdot d\hat{A} \quad (113)$$

in which we also incorporated a vector nature to the flux $\vec{j}(x, y, z)$ taking place infinitesimally at each point of the surface.

Now, the flow through a given surface can be obtained by respective integration of the above differential quantity along the surface in a manner analogous to that developed in Section 20.

22 Work

And, so, we have reached the penultimate topic in the present work, namely *work*. This special position in the sequence of covered topics did not happen by chance, but because not only it is very important from the physical perspective, but also because it provides the link to a subsequent topic that plays a special role in the physical word: *energy*.

A good news is that the concept of work turns out to be actually simple, taking place as a direct consequence of a force applied onto a point particle along a displacement (or trajectory), not depending directly of its mass. More specifically, we have that:

$$W(T) = \int_0^T \langle \vec{f}(t), \vec{v}(t) dt \rangle = \int_0^T \vec{f}(t) \cdot \vec{v}(t) dt \quad (114)$$

where $\vec{v}(t) = \dot{\vec{r}}(t)$ is the velocity of the particle along its trajectory, as represented by a parametric curve $\vec{r}(t)$ on t . In the SI, work (and energy) is measured in Joules ($J = 10^7 \text{ ergs}$).

The concept of work respectively to a trajectory of a point particle with mass m in a force vector field $\vec{\phi}(x, y, z, y)$ can be similarly expressed in terms of the following *line integral* (see [11]):

$$W(T) = \int_0^T \vec{\phi}(x, y, z, t) \cdot \vec{v}(t) dt \quad (115)$$

Observe that neither of the above expression depend directly of the initial position (x_0, y_0) .

In the case of $\vec{\phi}$ being a *conservative vector field*, we have that $W(T)$ will be equal to the difference of the values of the potential scalar field $\psi(x, y, z, t)$ from which the conservative field was derived, taken at times $t = T$ and $t = 0$, i.e.:

$$\begin{aligned} W(T) &= \\ &= \psi(r_x(T), r_y(T), r_z(T), T) - \psi(r_x(0), r_y(0), r_z(0), 0) \end{aligned} \quad (116)$$

Or, in the case the force vector field is constant along time:

$$W(T) = \psi(r_x(T), r_y(T), r_z(T)) - \psi(r_x(0), r_y(0), r_z(0)) \quad (117)$$

By recalling that the dot product between two vectors can be understood as a measurement of orientation similarity, or alignment, between those two vectors (see Appendix C), it follows that the work performed by a force along a given trajectory $\vec{r}(t)$ reflects the net agreement between the orientations of the force and the velocity of the particle during the considered motion.

The concept of work can be better appreciated by considering some progressively more elaborated examples.

We start by one of the simplest situations, concerning a point particle with mass m undergoing horizontal uniform linear motion with velocity v_x , as shown in Figure 48.

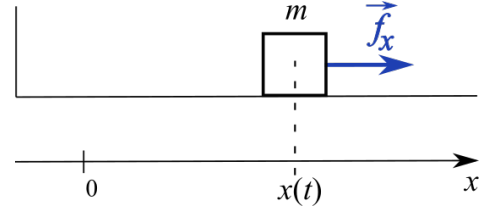


Figure 48: A point particle with mass m undergoes uniform linear motion with speed v_x along the horizontal direction along a time period T . What is the work performed on this particle?.

More specifically, we want to calculate the work on the particle during its T initial seconds of motion. By applying the definition of work, we have:

$$\begin{aligned} W(T) &= \int_0^T \vec{f}(t) \cdot \vec{v}(t) dt = \\ &= \int_0^T \vec{f}(t)(0, 0) \cdot (1, 0) dt = \int_0^T 0 dt = 0 \end{aligned} \quad (118)$$

which is null as could be expected, given that there are no forces applied to the particle, implying no similarity at all between the force orientation and the velocity of the particle.

The incorporation of a constant horizontal force $f_x > 0$, with initial conditions $x_0 = 0$ and $\dot{x}(t) = v_{x,0} > 0$ leads to the following respective trajectory:

$$\vec{r}(t) = (x(t), y(t)) \implies \begin{cases} x(t) = \frac{1}{2m} f_x t^2 + v_{x,0} t + x_0 \\ y(t) = 0 \end{cases} \quad (119)$$

from which:

$$\dot{\vec{r}}(t) = (\dot{x}(t), \dot{y}(t)) \implies \begin{cases} \dot{x}(t) = \frac{1}{m} f_x t + v_{x,0} \\ \dot{y}(t) = 0 \end{cases} \quad (120)$$

So that we can now apply the definition of work to

obtain:

$$\begin{aligned}
W(T) &= \int_0^T \vec{f}(t) \cdot \vec{v}(t) dt = \\
&= \int_0^T \vec{f}(t)(f_x, 0) \cdot \left(\frac{1}{m} f_x t + v_{x,0}, 0 \right) dt = \\
&= \int_0^T \left[\frac{1}{m} f_x^2 t + f_x v_{x,0} \right] dt = \\
&= \left[\frac{1}{2} \frac{1}{m} f_x^2 t^2 + f_x v_{x,0} t \right] \Big|_0^T = \\
&= \frac{1}{2} \frac{1}{m} f_x^2 T^2 + f_x v_{x,0} T > 0
\end{aligned} \tag{121}$$

The positive result indicates that the force orientation had a net agreement with the orientation of the motion. In this particular case, the agreement is total, because the force remains parallel to the velocity during the whole considered motion. Moreover, the resulting work performed by force f_x during the considered trajectory is a function of the particle mass m , the magnitude of the applied force f_x , as well as of the length of the period of time along which the displacement took place.

The next example relates to the situation shown in Figure 49, which concerns a point particle with mass m attached to a spring with free length L and constant k . The spring is initially compressed by a respective relative displacement Δx from its free length, therefore implying a force $f_x(t) = k \Delta x$. We want to determine the work performed by this force as the particle goes from its depicted initial position until the spring reaches its free length L .

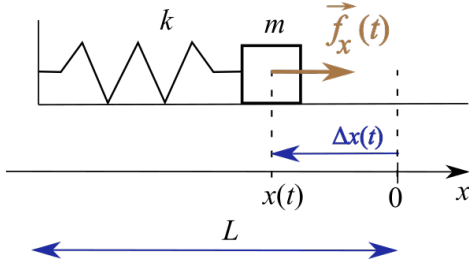


Figure 49: A point particle with mass m undergoes uniform linear motion with null initial speed and initial position $x_0 = 0$ along the horizontal direction under the influence of the force, which is a function of time in this case, implied by the spring with free length L and constant k . What is the work performed by the spring on this particle?

The spring force is:

$$f_x(t) = \begin{cases} f_x(t) = -k \Delta x = -k x(t) \\ f_y(t) = 0 \end{cases} \tag{122}$$

Let us henceforth assume that $x(t=0) = -A$.

By applying Newton's second law of motion:

$$\begin{cases} \ddot{x}(t) = -\frac{k}{m} x(t) \\ \ddot{y}(t) = 0 \end{cases} \tag{123}$$

Assuming $v_{x,0} = 0$, it follows that:

$$\vec{r}(t) : \begin{cases} x(t) = -A \cos \left(\sqrt{\frac{k}{m}} t \right) = -A \cos(\omega t) \\ \dot{y}(t) = 0 \end{cases} \tag{124}$$

where we have considered that $\omega = \sqrt{\frac{k}{m}}$.

Therefore:

$$\dot{\vec{r}}(t) : \begin{cases} \dot{x}(t) = A \sqrt{\frac{k}{m}} \sin \left(\sqrt{\frac{k}{m}} t \right) = A \omega \sin(\omega t) \\ \dot{y}(t) = 0 \end{cases} \tag{125}$$

The work performed by the spring under the considered initial conditions can now be calculated as:

$$\begin{aligned}
W(T) &= \int_0^T \vec{f}(t) \cdot d\vec{v} = \\
&= \int_0^T (k A \cos \omega t, 0) \cdot (A \omega \sin \omega t, 0) dt = \\
&= \int_0^T k A^2 \omega \sin(\omega t) \cos(\omega t) dt = \\
&= \left[-\frac{1}{2} k A^2 \omega \cos^2(\omega t) \right] \Big|_0^T = \\
&= -\frac{1}{2} k A^2 \omega \cos^2(\omega T) + \frac{1}{2} k A^2 \omega
\end{aligned} \tag{126}$$

Figure 50 illustrates the work performed by the spring on the particle under the adopted initial conditions, assuming $k = m = A = 1$, and $x(t=0) = -A$. Interestingly, as the particle motion $x(t)$ itself, the work is also a periodic function, but with double the frequency. At each half-cycle of the particle motion, the accumulated work becomes zero, changing its sign as the force and velocity change their relative orientations.

The obtained result shown in Figure 50 evidentiates that the work along time in a system can oscillate, being produced and consumed. In this case, the work has an oscillation that is twice as fast as that of the particle. In addition, observe that no negative work is observed respectively to this case example.

It is important to keep in mind that it is also possible to calculate the work of the force that the particle exerts on the spring. In the case of the above example, this work corresponds to the negative of the work implemented by the spring force on the point mass.

Our last considered situation regards throwing a ball of mass m as discussed in Section 7, assuming $0 \leq t \leq T$. We have already obtained the following equation of motion (defining the particle trajectory):

$$\vec{r}(t) = (x(t), y(t)) \implies \begin{cases} x(t) = v_{x,0} t + x_0 \\ y(t) = -\frac{1}{2} g t^2 + v_{y,0} t + y_0 \end{cases} \tag{127}$$

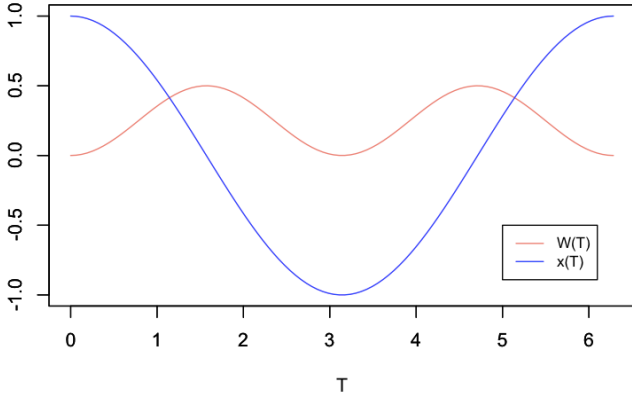


Figure 50: The work performed by the spring force on the particle, as in Fig. 49, shown in salmon, as well as the position of the particle, shown in blue, along time $T \in [0, 2\pi]$, corresponding to a complete cycle of the motion. It has been assumed that $k = m = A = 1$, $x(t = 0) = -A$ and $\dot{x}(t = 0) = 0$. Observe that positive work is observed whenever the speed orientation aligns with force orientation, being negative otherwise. As a consequence, the overall work becomes zero at each half-cycle of the particle motion.

which we need in order to obtain $d\vec{r}$ as

$$\dot{\vec{r}}(t) = (\dot{x}(t), \dot{y}(t)) \implies \begin{cases} \dot{x}(t) = v_{x,0} \\ \dot{y}(t) = -gt + v_{y,0} \end{cases} \quad (128)$$

We also know that:

$$\vec{f}(t) = (f_x(t), f_y(t)) \implies \begin{cases} f_x(t) = 0 \\ f_y(t) = -mg \end{cases} \quad (129)$$

We can now apply Equation 114 as:

$$\begin{aligned} W(T) &= \int_0^T \vec{f}(t) \cdot d\vec{v} = \\ &= \int_0^T (0, -mg) \cdot (v_{x,0}, -gt + v_{y,0}) dt = \\ &= \int_0^T (mg^2 t - mg v_{y,0}) dt = \\ &= \left[\frac{1}{2} mg^2 t^2 - mg v_{y,0} t \right] \Big|_0^T = \\ &= \frac{1}{2} mg^2 T^2 - mg v_{y,0} T \end{aligned}$$

The negative term concerns the fact that $v_{y,0}$ is oriented along the orientation of the y -axis, while the gravitational force proceeds in the directly opposite orientation.

Let us consider the initial velocity vector $(\dot{x}_0 = v_{x,0}, \dot{y}_0 = v_{y,0}) = (0, 0)$, which yields:

$$W(T) = \frac{1}{2} (mgT)^2 \quad (130)$$

If $m = 1 \text{ kg}$, $T = 2 \text{ s}$ and $g = 9.81 \text{ ms}^{-2}$, we obtain:

$$W = \frac{1}{2} (gT)^2 = \frac{1}{2} (9.81)^2 (2)^2 = 192.4722 \text{ J} \quad (131)$$

The fact that a positive work has been obtained indicate that the orientation of the resulting force on the particle mostly agreed with the orientation of the velocity vector along the considered displacement. Actually, since there was no horizontal motion, the orientation of the gravitational force agreed completely with the that of the velocity along the whole considered trajectory in this particular case.

Now, let us repeat the determination of the work of the gravitational force in the previous example, but considering that $v_{x,0} > 0$. In this case, we have:

$$W(T) = \frac{1}{2} (mgT)^2 \quad (132)$$

Therefore, despite the longer length of the displacement of the particle in this case, the horizontal velocity did not contribute whatsoever to changing the work of the force respectively to the previous example. That is because the considered force had no horizontal component, so that the orientation of the vector force was not parallel to the orientation of the vector velocity.

It is important to observe that we have considered only the work of *one* of the two forces involved in the particle motion. Indeed, as the gravitational force unfolds its respective work as calculated above, its reaction force on earth will also lead to a respective work. What would be the total work obtained by summing the works implemented individually by the two involved forces?

The concept of work has central importance in physics, especially because it is directly related to the concept of *energy*, with which it shares its unit, in the sense that the energy of a system can be changed as a consequence of the system performing or receiving work. As such, this present section on work (as well as the subsequent, on power) provides a link to a possible further text focusing on energy as well as its conservation and dissipation.

23 Power

Power can be simply understood as the time derivative of work:

$$P = \frac{dW(t)}{dt} \quad (133)$$

In SI, power is measured in Watts (W), being equal to Js^{-1} .

The power of a particle undergoing a trajectory $\vec{r}(t)$ under a force $\vec{f}(t)$ can be simply obtained as:

$$P(t) = \vec{f}(t) \cdot \vec{v}(t) = \vec{f}(t) \cdot \dot{\vec{r}}(t) \quad (134)$$

where $\dot{\vec{r}}(t)$ is the *vector velocity* of the motion at instant t .

In case the force originates from a respective force vector field $\vec{\phi}(x, y, z, t)$, we have:

$$P(t) = \vec{\phi}(x, y, z, t) \cdot \dot{\vec{r}}(t) \quad (135)$$

Interestingly, as it follows from the definition of dot product, the value of the power at each position (or time) along the trajectory is proportional to the magnitudes of the velocity and force vectors, as well as the cosine of the smallest angle between these two vectors (reflecting the orientation similarity between them). Therefore, the larger and more aligned the force and the velocity are, the larger the resulting power at that position.

24 Concluding Remarks

Though the present work can be read independently, it represents mostly a continuation of a previous article [11], in which the interesting problem of point motion had been approached almost exclusively from the mathematical perspective, without the direct or explicit consideration of physical aspects, including masses or forces. One of the main objectives of the present work was to incorporate important physical rules, especially Newton's laws of motion, into the previously discussed mathematical point motion framework, therefore implying in a respective restriction required for compatibility while describing and modeling the real world.

It has been hoped that, once acquainted with an ampler context of possible solutions as allows by the mathematically focused approach, the subsequent incorporation of physical constraints can represent an interesting experience that motivates one to better appreciate the specific aspects of the physical reality, in the light of the classical perspective that has been followed here.

Be that as it may, the present work also aims at providing a brief journey into the impressive range of implications of Newton's law of motion, hopefully providing a compelling illustration of the generality that is intrinsic to the physical laws. In addition, the relatively ample range of covered topics also make the present work as an interesting resource for those who are reasonably acquainted with the required basic mathematical concepts to approach several concepts and methods related to the motion of single particles, providing a preliminary bridge to several other areas in physics and other disciplines in the physical sciences. Another possible use of the present work is as a refresher for those who are already familiarized with the involved concepts, but have not exercised them more recently. It is also believed that the range of topics covered here can provide an interesting occasion for illustrating the use of several mathematical concepts and methods from multivariate calculus, linear algebra,

differential geometry, and differential equations.

After a relatively broad tour of several concepts and methods related to the central topics of mass and force, it concluded by presenting the concepts of work and power, which are directly related to one of the most important concepts in the physical world, namely *energy*. A subsequent work resuming the discussion from this point is planned, to be developed with emphasis on the relationship between work and energy, as well as conservation and dissipation of the later.

Acknowledgments

Luciano da F. Costa thanks CNPq (grant no. 307085/2018-0) and FAPESP (grant 15/22308-2).

Observations

As all other preprints by the author, the present work contains preliminary work subject to further revision and validation. Respective modification, commercial use, or distribution of any of its parts are not possible, as this work has author copyright. Many of the preprints by the author are also available in Hal and/or arXiv. This work can also be cited by using the DOI number or article identification link. Thanks for reading.

A Some Interesting Trigonometric Properties

A linear combination of a cosine and a sine functions with the same argument satisfies the following property:

$$a \cos(\alpha) + b \sin(\alpha) = A \cos(\alpha + \phi) \quad (136)$$

$$\text{where: } \begin{cases} A = \sqrt{a^2 + b^2} \\ \phi = -\arctan\left(\frac{b}{a}\right) \end{cases} \quad (137)$$

which holds $\forall \alpha, a, b, \in \mathbb{R}, a \neq 0$. The quantity ϕ is often called *phase shift*, *relative phase*, *phase lag*, and sometimes simply *phase*, though actually speaking the latter is not strict because the term phase typically corresponds to the whole argument $\alpha + \phi$.

This interesting, and not so widely known property, indicates that the function $A \cos(\alpha + \phi)$ is as general as the function $a \cos(\alpha) + b \sin(\alpha)$ because they turn out to be, actually, identical.

We also have that:

$$a \sin(\alpha + \phi_a) + b \sin(\alpha + \phi_b) = A \sin(\alpha + \phi)$$

where: $\begin{cases} A = \sqrt{a^2 + b^2} + 2ab \cos(\phi_a - \phi_b) \\ \phi = \arctan\left(\frac{a \sin(\phi_a) + b \sin(\phi_b)}{a \cos(\phi_a) + b \cos(\phi_b)}\right) \end{cases}$ (138)

Some other trigonometric properties that are often useful while addressing the motion of a particle include, but are by no means limited to:

$$\sin(-\alpha) = -\sin(\alpha) \quad (139)$$

$$\cos(-\alpha) = \cos(\alpha) \quad (140)$$

$$\sin(\alpha) = \cos(\pi/2 - \alpha) \quad (141)$$

$$\cos(\alpha) = \sin(\pi/2 - \alpha) \quad (142)$$

$$\sin(\alpha) = -\cos(\pi/2 + \alpha) \quad (143)$$

$$\cos(\alpha) = \sin(\pi/2 + \alpha) \quad (144)$$

$$\tan(-\alpha) = -\tan(\alpha) \quad (145)$$

$$\sin^2(\alpha) + \cos^2(\alpha) = 1 \quad (146)$$

$$\cos^2(\alpha) - \sin^2(\alpha) = \cos(2\alpha) \quad (147)$$

$$\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \sin(\beta) \cos(\alpha) \quad (148)$$

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\beta) \sin(\alpha) \quad (149)$$

$$\sin(\alpha) \cos(\beta) = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2} \quad (150)$$

$$\sin(\alpha) \sin(\beta) = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2} \quad (151)$$

$$\cos(\alpha) \cos(\beta) = \frac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2} \quad (152)$$

$$\sin(\alpha) + \sin(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (153)$$

$$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (154)$$

$$\sin(\alpha) \approx \tan(\alpha) \approx \alpha \text{ for } \alpha \text{ very small.} \quad (155)$$

$$\cos(\alpha) \approx 1 \text{ for } \alpha \text{ very small.} \quad (156)$$

$$\sin(\arctan(x)) = \frac{x}{\sqrt{x^2 + 1}} \quad (157)$$

$$\cos(\arctan(x)) = \frac{1}{\sqrt{x^2 + 1}} \quad (158)$$

$$\arctan(x) = 2 \arctan\left(\frac{x}{1 + \sqrt{x^2 + 1}}\right) \quad (159)$$

B The Ellipse

The ellipse, which can be thought of as a *generalization of the circle*, is illustrated in Figure 51(a), being understood

as being specified by points x_1 and x_2 , which are called the *foci* (plural of *focus*) F_1 and F_2 of the ellipse. The average position $(x_1 + x_2)/2$ of the foci x -coordinate corresponds to the *center* of the ellipse.

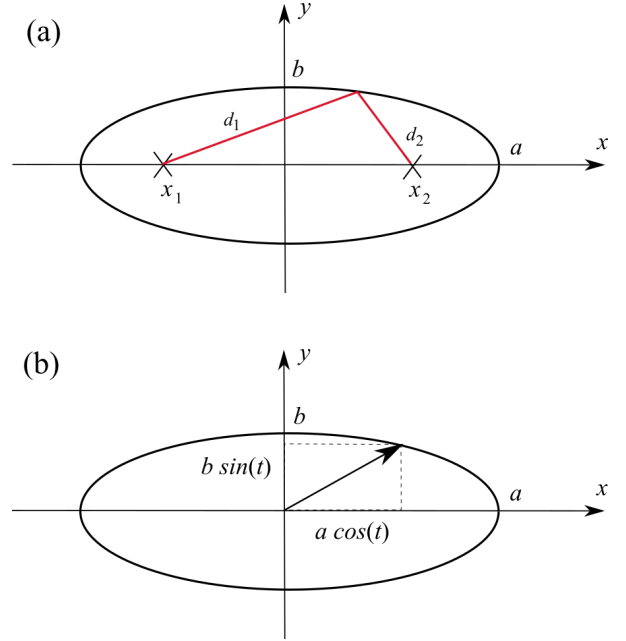


Figure 51: An ellipse as defined from its two foci (a), and as a parametric curve (b) on parameter t , which does *not* correspond to the angle of the points along the ellipse.

The ellipse is then defined as the geometric position (*locus*) of all points that have equal sum of their distances, i.e. $d_1 + d_2 = 2a$, where $a > x_2 - x_1$, to the two foci. That the sum of the distances equals $2a$ can be easily inferred by considering the right-most point of the ellipse, namely $(a, 0)$. The distance between this point and the focus F_2 is $d_1 = a - x_1$, and the distance from this same point to F_1 is $d_2 = a - x_2$, which immediately implies in $d_1 + d_2 = 2a - x_1 - x_2$. However, because the coordinate origin is at the midpoint between the foci, we know that $x_1 = -x_2$, implying $d_1 + d_2 = 2a$. Observe that an ellipse therefore becomes a circle when $x_1 = x_2$.

It can also be shown that:

$$\boxed{\frac{x^2(t)}{a^2} + \frac{y^2(t)}{b^2} = 1} \quad (160)$$

The parametric representation of an ellipse can be obtained as:

$$\boxed{\vec{u}(t) = (x(t), y(t)) = (a \cos(t), b \sin(t))}, \quad (161)$$

where: $t \in [0, 2\pi]$

C The Scalar Product and Orientation Similarity

Given two vectors $\vec{v} = (v_x, v_y, v_z)$ and $\vec{r} = (r_x, r_y, r_z)$, with θ corresponding to the smallest angle between them, the *dot product* or *scalar product* between these vectors can be expressed as:

$$\begin{aligned}\langle \vec{v}, \vec{r} \rangle &= \vec{v} \cdot \vec{r} = \\ &= v_x r_x + v_y r_y + v_z r_z = \|\vec{v}\| \|\vec{r}\| \cos(\theta) \quad (162)\end{aligned}$$

So, if we keep the vector magnitudes $\|\vec{v}\|$ and $\|\vec{r}\|$ fixed, the dot product can be understood as providing an indication of the *similarity* between the orientation of the two vectors.

For instance, in case the two vectors are fully aligned, and therefore with identical orientations, we have $\cos(\theta = 0) = 1$, which implies maximum possible dot product between the two vectors. If $\theta = \pi/2$, it follows that $\vec{v} \cdot \vec{r} = 0$, indicating null similarity of orientations of the two vectors. When $\theta = \pi$, the two vectors are counter-aligned, leading to the minimum possible absolute similarity between their respective orientations.

References

- [1] R. A. Serway and J. W. Jewett. *Physics for scientists and engineers*. Cengage learning, 2018.
- [2] R. P. Feynman, R. B. Leighton, and M. Sands. The Feynman lectures on physics; vol. i. *American Journal of Physics*, 33(9):750–752, 1965.
- [3] R. Shankar. Fundamentals of physics. In *Fundamentals of Physics*. Yale University Press, 2014.
- [4] K. Hoffman. *Linear Algebra*. Pearson India, 2015.
- [5] J. Stewart. *Multivariable Calculus*. Cengage Learning, 2015.
- [6] C. H. Edwards and D. E. Penney. *Calculus with geometry analytic*, 2002.
- [7] L. da F. Costa. A mosaic of multivariate calculus. Researchgate, 2020. https://www.researchgate.net/publication/340570370_A_Mosaic_of_Multivariate_Calculus_CDT-27. [Online; accessed 18-Apr-2020].
- [8] E. Kreyszig. *Advanced Engineering Mathematics*. Wiley and Sons, 2015.
- [9] R. K. Nagle, E. B. Saff, and A. D. Snider. *Fundamentals of Differential Equations*. Pearson, 2017.
- [10] M. P. do Carmo. *Differential geometry of curves and surfaces*. Prentice Hall, 1976.
- [11] L. da F. Costa. Point motion in flat spaces: An ample starting point. https://www.researchgate.net/publication/365806206_Point_Motion_in_Flat_Spaces_An_Ample_Starting_Point, 2022.
- [12] L. da F. Costa. Modeling: The human approach to science. Researchgate, 2019. https://www.researchgate.net/publication/333389500_Modeling_The_Human_Approach_to_Science_CDT-8. [Online; accessed 1-Oct-2020].
- [13] E. B. Goldstein. *Sensation and Perception*. Cengage Learning, 2012.
- [14] D. Burr and C. Morrone. Time perception: space-time in the brain. *Current Biology*, 16(5):R171–R173, 2006.
- [15] B. Pavard and A. Berthoz. Linear acceleration modifies the perceived velocity of a moving visual scene. *Perception*, 6(5):529–540, 1977.
- [16] S. Runeson. Constant velocity?not perceived as such. *Psychological Research*, 37(1):3–23, 1974.
- [17] L. G. Allan. The perception of time. *Perception & Psychophysics*, 26(5):340–354, 1979.
- [18] M. Wittmann and S. Lehnhoff. Age effects in perception of time. *Psychological Reports*, 97(3):921–935, 2005.
- [19] G. Smith. Newton’s Philosophiae Naturalis Principia Mathematica. In E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Winter 2008 edition, 2008.
- [20] E. Hairer, C. Lubich, and G. Wanner. Geometric numerical integration illustrated by the störmer–verlet method. *Acta Numerica*, 12:399–450, 2003.
- [21] P. L. DeVries and R. P. Wolf. A first course in computational physics. *Computers in Physics*, 8(2):178–179, 1994.
- [22] N. J. Giordano, M. L. De Jong, S. R. McKay, and W. Christian. Computational physics. *Computers in Physics*, 11(4):351–351, 1997.
- [23] L. Haws and T. Kiser. Exploring the brachistochrone problem. *The American Mathematical Monthly*, 102(4):328–336, 1995.
- [24] R. T. Boute. The brachistochrone problem solved geometrically: A very elementary approach. *Mathematics Magazine*, 85(3):193–199, 2012.

- [25] S. Hildebrandt and A. Tromba. *The parsimonious universe: shape and form in the natural world*. Springer Science & Business Media, 1996.
- [26] M. J. Forray. *Variational Calculus in Science and Engineering*. McGraw-Hill, 1968.
- [27] H. Sagan. *Boundary and eigenvalue problems in mathematical Physics*. Dover, 1989.
- [28] G. Plunien, B. Müller, and W. Greiner. The Casimir effect. *Physics Reports*, 134(2-3):87–193, 1986.
- [29] H. B. G. Casimir and D. Polder. Influence of retardation on the london–van der waals forces. *Nature*, 158(4022):787–788, 1946.
- [30] A. Beléndez, C. Pascual, D. I. Méndez, T. Beléndez, and C. Neipp. Exact solution for the nonlinear pendulum. *Revista brasileira de ensino de física*, 29:645–648, 2007.
- [31] K. Ochs. A comprehensive analytical solution of the nonlinear pendulum. *European Journal of Physics*, 32(2):479, 2011.