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MEAN DIMENSION OF NATURAL EXTENSION OF ALGEBRAIC SYSTEMS

BINGBING LIANG AND RUXI SHI

ABSTRACT. Mean dimension may decrease after taking natural extension. In this paper we show that mean dimension stays the same after taking natural extension for an endomorphism on a compact metrizable abelian group. As an application, we obtain that the mean dimension of algebraic cellular automaton coincides with its natural extension, which strengthens a result of Burguet and Shi [1] with a different proof.

1. INTRODUCTION

By a dynamical system, we mean a pair (X, φ) for a compact metrizable space X and $\varphi: X \rightarrow X$ a continuous map. Denote by \widetilde{X} the natural extension of (X, φ) with the extension map $\widetilde{\varphi}$ (see Definition 2.5). It is well known that the topological entropy of (X, φ) coincides with the topological entropy of $(\widetilde{X}, \widetilde{\varphi})$.

Mean (topological) dimension is a dynamical invariant which can distinguish the dynamical systems of infinite entropy. Gromov introduced this invariant when measuring the size of some holomorphic function spaces [6]. Mean dimension takes a crucial role in the dynamical embedding problems [13, 7, 9] and has close relations with other areas like information theory [12] and operator algebras [11, 5]. As is pointed out by Burguet and the second author, the mean dimension of the natural extension $(\widetilde{X}, \widetilde{\varphi})$ is no greater than the mean dimension of (X, φ) and the inequality can be strict [1, Section 3.2]. In fact, it roughly follows from the facts that the natural extension is defined as the inverse limit (see Definition 2.5) and that the inverse limit of compact metrizable spaces can decrease the (covering) dimension [10, Example 183]. This motivates the following question:

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Question 1.1. *For which dynamical system (X, φ) do we have the equality $\text{mdim}(X, \varphi) = \text{mdim}(\widetilde{X}, \widetilde{\varphi})$?*

In other words, we concern which dynamical properties would guarantee the non-decreasing of mean dimension of its natural extension. In this paper, we show that the above equality holds when the dynamical system admits an algebraic structure. That is,

Theorem 1.2. *For a compact metrizable abelian group X and a continuous endomorphism $\varphi: X \rightarrow X$, we have $\text{mdim}(X, \varphi) = \text{mdim}(\widetilde{X}, \widetilde{\varphi})$.*

To show this theorem, taking advantage of the abelian group structure of X , we can translate this question into its dual action of Pontryagin dual space. We first study the surjective endomorphism φ on X . By introducing a proper notion of mean rank, we will see that mean dimension of (X, φ) coincides with the mean rank of its Pontryagin dual. With the help of a result of Gutman [8] regarding the mean dimension of nonwandering subsystems, we can finally deal with the nonsurjective case.

It is worth-mentioning that the approach of interplay between an action on a compact abelian group and its dual action on Pontryagin dual space already appears in [11], where the addition formula for mean dimension of algebraic actions is hard to prove directly. However, when transforming the problem into the algebraic setting, the discreteness of algebraic objects makes the proof accessible.

On the other hand, Burgeut and the second author [1, Theorem 6.4] showed that the mean dimension of a unit cellular automaton coincides with that of its natural extension. In general, they [1, Question 3.6] asked that

Question 1.3. *Does the mean dimension of a general cellular automaton coincide with the mean dimension of its natural extension?*

Based on a computation result concerning cellular automaton [1, Lemma 7.3], it is implicitly proved that the mean dimension of an algebraic permutative cellular automaton (see Definition 4.3) stays the same as the mean dimension of its natural extension.

Our result improves the result of [1] with a different proof and give a partial answer to Question 1.3.

Corollary 1.4. *For an algebraic cellular automaton $F: X^{\mathbb{Z}} \rightarrow X^{\mathbb{Z}}$, we have $\text{mdim}(X^{\mathbb{Z}}, F) = \text{mdim}(\widetilde{X^{\mathbb{Z}}}, \widetilde{F})$.*

The paper is organized as follows. In Section 2 we recall the definitions of mean dimension and natural extension. As a preparation,

in Section 3 we introduce the notion of mean rank for a general group homomorphism, show some nice properties, and establishes an equality relating mean dimension and mean rank. In Section 4 we finish the proof of Theorems 1.2 and Corollary 1.4.

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2. BACKGROUND

2.1. Mean dimension. Let X be a compact metrizable space with a compatible metric d . Fix $\varepsilon > 0$. Recall that a continuous map $f: (X, d) \rightarrow Y$ for another topological space Y is an (ε, d) -embedding if

$$\text{diam}(f^{-1}(y), d) < \varepsilon$$

for every $y \in Y$ where diam denotes the diameter of set. Denote by $\text{Wdim}_\varepsilon(X, d)$ the minimal dimension of a polyhedron P bearing an (ε, d) -embedding $f: (X, d) \rightarrow P$.

The following important lemma is proved first by Gromov [6, Section 1.11 and 1.12] and admits a proof via Brouwer's fixed point theorem [2, Proposition 4.6.5].

Lemma 2.1. *For every $0 < \varepsilon < a$ and $d \in \mathbb{N}$, we have*

$$\text{Wdim}_\varepsilon([0, a]^d, |\cdot|_\infty) = d.$$

Let $\varphi: X \rightarrow X$ be a continuous map. For a compatible metric d on X and $n \in \mathbb{N}$, we obtain a larger compatible metric

$$d_n(x, x') := \max_{0 \leq i \leq n-1} d(\varphi^i(x), \varphi^i(x')).$$

It is easy to check that the sequence $a_n := \text{Wdim}_\varepsilon(X, d_n)$ is *subadditive* in the sense that $a_{n+m} \leq a_n + a_m$ for any $n, m \in \mathbb{N}$. Thus we obtain a well-defined notation:

$$\text{mdim}_\varepsilon(X, d) := \lim_{n \rightarrow \infty} \frac{\text{Wdim}_\varepsilon(X, d_n)}{n}.$$

Definition 2.2. The *mean (topological) dimension* of (X, φ) is

$$\text{mdim}(X, \varphi) := \sup_{\varepsilon > 0} \text{mdim}_\varepsilon(X, d).$$

It is easy to see that the definition is independent of choices of compatible metrics.

Remark 2.3. Lindenstrauss and Weiss introduced another equivalent definition of mean (topological) dimension in terms of the overlapping number of finite open covers [13]. This definition works for actions on

general compact Hausdorff spaces. See a friendly introduction of mean dimension theory in [2].

2.2. Inverse limit. Let $\{X_n\}_{n=1}^\infty$ be a sequence of compact metrizable spaces with surjective continuous maps $\psi_n: X_{n+1} \rightarrow X_n$ for all $n \in \mathbb{N}$. The *inverse limit of ψ_n 's*, denoted as

$$\varprojlim X_n,$$

is defined as the subset of $\prod_{n \geq 1} X_n$ consisting of all elements $(x_n)_{n \in \mathbb{N}}$ satisfying $\psi_n(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$.

Now let $\varphi_n: X_n \rightarrow X_n$ be a surjective continuous map such that $\psi_n \circ \varphi_{n+1} = \varphi_n \circ \psi_n$ for every $n \in \mathbb{N}$. In other words, $\psi_n: (X_{n+1}, \varphi_{n+1}) \rightarrow (X_n, \varphi_n)$'s is a sequence of factor maps of dynamical systems (X_n, φ_n) 's. Define $\Phi: \varprojlim X_n \rightarrow \varprojlim X_n$ by sending $(x_n)_{n \in \mathbb{N}}$ to $(\varphi_n(x_n))_{n \in \mathbb{N}}$.

Definition 2.4. The dynamical system given by $(\varprojlim X_n, \Phi)$ is called the *inverse limit of dynamical systems (X_n, φ_n) 's*.

For a (not necessarily surjective) dynamical system (X, φ) , it is easy to check that the continuous map φ restricted on $\cap_{k \geq 1} \varphi^k(X)$ is surjective. Notice that if φ is surjective, then $\cap_{k \geq 1} \varphi^k(X) = X$.

Definition 2.5. For a dynamical system (X, φ) , setting $X_n = \cap_{k \geq 1} \varphi^k(X)$ and $\psi_n = \varphi_n = \varphi$, the induced inverse limit is called the *natural extension* of (X, φ) , which we denote by $(\widetilde{X}, \widetilde{\varphi})$.

The following proposition shows that the mean dimension of the natural extension $(\widetilde{X}, \widetilde{\varphi})$ is no greater than the mean dimension of (X, φ) .

Proposition 2.6. [15, Proposition 5.8] *For an inverse limit of dynamical systems (X_n, φ_n) 's, we have*

$$\text{mdim}(\varprojlim X_n, \varprojlim \varphi_n) \leq \varliminf_{n \rightarrow \infty} \text{mdim}(X_n, \varphi_n).$$

In particular, for a dynamical system (X, φ) , we have

$$\text{mdim}(\widetilde{X}, \widetilde{\varphi}) \leq \text{mdim}(X, \varphi).$$

3. MEAN RANK AND MEAN DIMENSION

In this section, similar to [11], we introduce the notion of mean rank, discuss its nice properties, and establish its connection with mean dimension.

Let A be a discrete abelian group and $\varphi: A \rightarrow A$ a group homomorphism. For a finite subset E of A , consider the subgroup of A generated by $\{\varphi^i(E) : i = 0, \dots, n-1\}$, which we denote by $T_n(E, \varphi)$.

We call $T_n(E, \varphi)$ the n -trajectory of E . For a subgroup B of A , we denote by $\text{rk}(B)$ the rank of B . Since the sequence $\{\text{rk}(T_n(E, \varphi))\}_{n \in \mathbb{N}}$ is subadditive, the limit $\lim_{n \rightarrow \infty} \frac{\text{rk}(T_n(E, \varphi))}{n}$ exists, which we denote by $\text{mrk}(E, \varphi)$.

Definition 3.1. For a discrete abelian group A and $\varphi: A \rightarrow A$ a group homomorphism, the *mean rank* of A is defined as

$$\text{mrk}(A, \varphi) := \sup_E \text{mrk}(E, \varphi)$$

for E ranging over all finite nonempty subsets of A .

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of discrete abelian groups and $\varphi_n \in \text{End}(A_n)$ space of endomorphisms for every $n \in \mathbb{N}$. Suppose that $\{\psi_{n,m}: A_n \rightarrow A_m\}$ is a directed system of discrete abelian groups A_n 's, i.e. $\psi_{n,m} \circ \varphi_n = \varphi_m \circ \psi_{n,m}$ for any $n, m \in \mathbb{N}$ with $n \leq m$. Denote by λ_n the natural embedding $A_n \rightarrow \bigoplus_{n \geq 1} A_n$. Then an explicit construction of the colimit of $\{\psi_{nm}: A_n \rightarrow A_m\}$ is

$$\varinjlim A_n := (\bigoplus_{n \geq 1} A_n) / S$$

where S is the subgroup of $\bigoplus_{n \geq 1} A_n$ generated by all elements

$$\lambda_m \circ \psi_{n,m}(a_n) - \lambda_n(a_n)$$

with $n \leq m$ and $a_n \in A_n$ (see [14, Proposition B-7.7]).

The compatible φ_n 's then induce a homomorphism $\Phi: \varinjlim A_n \rightarrow \varinjlim A_n$ sending $\overline{(a_n)_n}$ to $\overline{(\varphi_n(a_n))_n}$. In particular, as $A_n = A$, $\psi_{n,m} = \text{id}$, and $\varphi_n = \varphi$ for all $n, m \in \mathbb{N}$, we denote by \overrightarrow{A} the colimit and $\overrightarrow{\varphi}$ the corresponding map on \overrightarrow{A} .

The following proposition is in the same line of [3, Proposition 3.5].

Proposition 3.2. *Let $\{\psi_{n,m}: A_n \rightarrow A_m\}$ be a directed system of discrete abelian groups and $\varphi_n \in \text{End}(A_n)$ such that $\psi_{n,m} \circ \varphi_n = \varphi_m \circ \psi_{n,m}$ for any $n, m \in \mathbb{N}$ with $n \leq m$. Denote by A'_n the quotient group $A_n / \bigcup_{m \geq n} \ker(A_n \xrightarrow{\psi_{n,m}} A_m)$ and φ'_n the map $A'_n \rightarrow A'_n$ sending \bar{x} to $\overline{\varphi_n(x)}$. Then we have*

$$\text{mrk}(\varinjlim A_n, \Phi) = \sup_{n \geq 1} \text{mrk}(A'_n, \varphi'_n).$$

In particular, if A is a discrete abelian group and a group homomorphism $\varphi: A \rightarrow A$ is injective, then $\text{mrk}(A, \varphi) = \text{mrk}(\overrightarrow{A}, \overrightarrow{\varphi})$.

Proof. Consider the injective homomorphism $\iota_n: A'_n \rightarrow \varinjlim A_n$ by sending \bar{x} to $\overline{\lambda_n(x)}$. We have that ι_n intertwines φ'_n with Φ and $\iota_n(A'_n)$

increases with $\cup_{n \geq 1} \iota_n(A'_n) = \varinjlim A_n$. Since mrk is upper continuous, we see that $\text{mrk}(A'_n, \varphi'_n)$ increases to $\text{mrk}(\varinjlim A_n, \Phi)$.

On the other hand, for a single injective homomorphism $\varphi: A \rightarrow A$, we have $A'_n = A$ and $\varphi'_n = \varphi$. Thus

$$\text{mrk}(\vec{A}, \vec{\varphi}) = \sup_{n \geq 1} \text{mrk}(A, \varphi) = \text{mrk}(A, \varphi).$$

□

Now we consider a compact metrizable abelian group X . The *Pontryagin dual* of X , denoted as \widehat{X} , is defined as the collection of continuous group homomorphisms from X to the unit circle \mathbb{T} . Under the pointwise multiplication \widehat{X} becomes an abelian group and under the compact-open topology \widehat{X} makes a discrete group. The classical Pontryagin duality says that the Pontryagin dual of \widehat{X} is isomorphic to X as topological groups.

Let $\varphi: X \rightarrow X$ be a group homomorphism. Then φ induces a group homomorphism $\widehat{\varphi}: \widehat{X} \rightarrow \widehat{X}$ sending χ to $\chi \circ \varphi$. For convenience, we may write

$$\langle \widehat{\varphi}(\chi), x \rangle = \langle \chi, \varphi(x) \rangle$$

for every $\chi \in \widehat{X}$ and $x \in X$.

The following theorem establishes the connection between mean dimension and mean rank, which already appears in [11, Theorem 4.1]. For readers' convenience, we present a proof here.

Theorem 3.3. *Let X be a compact metrizable group and $\varphi: X \rightarrow X$ a continuous endomorphism. Then $\text{mdim}(X, \varphi) = \text{mrk}(\widehat{X}, \widehat{\varphi})$.*

Proof. Let us first estimate the upper bound of mean dimension. Fix a compatible metric d on X . We may consider X as the Pontryagin dual of \widehat{X} . Then for any $\varepsilon > 0$, there exists $A \subseteq \widehat{X}$ finite such that whenever $\chi|_A = \chi'|_A$ for $\chi, \chi' \in X$ we have $d(\chi, \chi') < \varepsilon$. Fix $n \in \mathbb{N}$ and pick a maximal independent subset \mathcal{E} of $\cup_{i=0}^{n-1} \widehat{\varphi}^i(A)$ and hence $|\mathcal{E}| = \text{rk}(\langle \widehat{\varphi}^i(A) : i = 0, \dots, n-1 \rangle)$. Consider the continuous map $\pi_{\mathcal{E}}: X \rightarrow \mathbb{T}^{\mathcal{E}}$ sending χ to the restriction map $\chi|_{\mathcal{E}}$. Assume that $\chi|_{\mathcal{E}} = \chi'|_{\mathcal{E}}$ for some $\chi, \chi' \in X$. By our selection of \mathcal{E} , for every $a \in A$ and $i = 0, 1, \dots, n-1$, we have

$$\langle \varphi^i(\chi), a \rangle = \langle \chi, \widehat{\varphi}^i(a) \rangle = \langle \chi', \widehat{\varphi}^i(a) \rangle = \langle \varphi^i(\chi'), a \rangle.$$

It follows that $\pi_{\mathcal{E}}$ is an (ε, d_n) -embedding and hence

$$\text{Wdim}_{\varepsilon}(X, d_n) \leq \dim(\mathbb{T}^{\mathcal{E}}) = |\mathcal{E}| = \text{rk}(T_n(A, \widehat{\varphi})).$$

Let $n \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, we obtain $\text{mdim}(X, \varphi) \leq \text{mrk}(\widehat{X}, \widehat{\varphi})$.

Now we estimate the lower bound of mean dimension. Fix a finite subset $A \in \widehat{X}$ and $n \in \mathbb{N}$. Let $\mathcal{E} \subseteq \cup_{i=0}^{n-1} \widehat{\varphi}^i(A)$ be a maximal linearly independent subset and hence $|\mathcal{E}| = \text{rk}(T_n(A, \widehat{\varphi}))$. We then desire to embed $[0, 1/2]^\mathcal{E}$ into X topologically.

Consider the collection $\{H \leq \widehat{X} : H \cap \langle \mathcal{E} \rangle = 0\}$ with the partial order of set containment. By Zorn's Lemma, there exists a maximal subgroup $H_\mathcal{E}$ of \widehat{X} such that $H_\mathcal{E} \cap \langle \mathcal{E} \rangle = 0$. Then for any $u \in \widehat{X}$ there exists $k \in \mathbb{Z}$ such that $ku \leq H_\mathcal{E} \oplus \langle \mathcal{E} \rangle$ (If not we will get a strictly larger subgroup $H_\mathcal{E} \oplus \mathbb{Z}u!$). Write ku as

$$ku = \sum_{e \in \mathcal{E}} \lambda_e e + v$$

with $v \in H_\mathcal{E}$ and $\lambda_e \in \mathbb{Z}$. Define the map $f: ([0, 1/2]^\mathcal{E}, \|\cdot\|_\infty) \rightarrow X$ sending $(x_e)_{e \in \mathcal{E}}$ to $(\sum_{e \in \mathcal{E}} \frac{\lambda_e}{k} x_e + \mathbb{Z})_u$. Clearly f is well defined, continuous, and injective. What's left is to introduce a compatible metric d on X such that f is an (ε, d_n) -embedding. Enumerate the elements of \widehat{X} as $a_1, \dots, a_{|A|}, \dots$ such that $A = \{a_1, \dots, a_{|A|}\}$. Define a compatible metric θ on \mathbb{T} by

$$\theta(x + n, y + m) = \min_{k \in \mathbb{Z}} |x - y + k|.$$

Then the desired metric d can be defined via

$$d(\chi, \chi') := \max_{a \in A} \theta(\chi(a), \chi'(a)) + \max_{j > |A|} \frac{1}{2^j} \theta(\chi(a_j), \chi'(a_j)).$$

This completes the proof. \square

Remark 3.4. In light of [4, Definition 4.2], we can extend the notion of mean rank to the actions by amenable semigroups and expect the above connection holds.

4. MEAN DIMENSION OF THE NATURAL EXTENSION

In this section we give the proof of Theorem 1.2 and Corollary 1.4.

For a dynamical system (X, φ) , recall that a point $x \in X$ is called *non-wandering* if for every open set U containing x and $N \geq 1$, there is $k \geq N$ such that $U \cap \varphi^{-k}U \neq \emptyset$. Denote by $\Omega(X)$ the collection of non-wandering points of (X, φ) .

Lemma 4.1. *For a dynamical system (X, φ) , we have $\text{mdim}(X, \varphi) = \text{mdim}(\cap_{n \geq 1} \varphi^n(X), \varphi)$.*

Proof. Following the same argument of [8, Lemma 7.2] for the non-invertible case, we still have $\text{mdim}(X, \varphi) = \text{mdim}(\Omega(X), \varphi)$. Since

$X \supseteq \bigcap_{n \geq 1} \varphi^n(X) \supseteq \Omega(X)$, it follows that

$$\text{mdim}(X, \varphi) \geq \text{mdim}(\bigcap_{n \geq 1} \varphi^n(X), \varphi) \geq \text{mdim}(\Omega(X), \varphi) = \text{mdim}(X, \varphi).$$

□

Now we are ready to complete the proof of the main theorem. We repeat our main result as follows.

Theorem 4.2 (Theorem 1.2). *For any compact metrizable abelian group X and a continuous homomorphism $\varphi: X \rightarrow X$, we have the equality $\text{mdim}(X, \varphi) = \text{mdim}(\widetilde{X}, \widetilde{\varphi})$.*

Proof. First we deal with the case that φ is surjective. From Proposition 2.6, it suffices to show $\text{mdim}(X, \varphi) \leq \text{mdim}(\widetilde{X}, \widetilde{\varphi})$.

By definition, for each element $\overline{(\chi_n)_{n \in \mathbb{N}}} \in \overrightarrow{\widetilde{X}}$ we have $\chi_{n+1} = \chi_n \circ \varphi$ for all $n \in \mathbb{N}$. $\overline{(\chi_n)_{n \in \mathbb{N}}}$ induces an element of $\widehat{\widetilde{X}}$ by sending $(x_n)_n \in \widetilde{X}$ to $\chi_1(x_1)$, which we denote by $\pi(\overline{(\chi_n)_n})$. Since $\varphi(x_{n+1}) = x_n$ for every $n \in \mathbb{N}$, we have

$$\chi_{n+1}(x_{n+1}) = \chi_n \circ \varphi(x_{n+1}) = \chi_n(x_n).$$

Thus $\chi_1(x_1) = \chi_n(x_n)$ for all $n \in \mathbb{N}$. Moreover we see that the map π is well defined and injective. Note that $\overrightarrow{\widetilde{\varphi}}$ sends $\overline{(\chi_n)_{n \in \mathbb{N}}}$ to $\overline{(\chi_n \circ \varphi)_{n \in \mathbb{N}}}$ and for each $\xi \in \widehat{\widetilde{X}}$, $\widehat{\widetilde{\varphi}}(\xi)$ sends each $(x_n)_{n \in \mathbb{N}} \in \widetilde{X}$ to $\xi((\varphi(x_n))_{n \in \mathbb{N}})$. It follows that $\pi \circ \overrightarrow{\widetilde{\varphi}} = \widehat{\widetilde{\varphi}} \circ \pi$, i.e. $\pi: \overrightarrow{\widetilde{X}} \rightarrow \widehat{\widetilde{X}}$ is an injective group homomorphism that intertwines $\overrightarrow{\widetilde{\varphi}}$ with $\widehat{\widetilde{\varphi}}$. Thus we have $\text{mrk}(\overrightarrow{\widetilde{X}}, \overrightarrow{\widetilde{\varphi}}) \leq \text{mrk}(\widehat{\widetilde{X}}, \widehat{\widetilde{\varphi}})$.

Since φ is surjective, we get that the map $\widehat{\varphi}$ is injective. By Proposition 3.2, we obtain $\text{mrk}(\widehat{\widetilde{X}}, \widehat{\widetilde{\varphi}}) = \text{mrk}(\overrightarrow{\widetilde{X}}, \overrightarrow{\widetilde{\varphi}})$. Applying Theorem 3.3 we have

$$\text{mdim}(X, \varphi) = \text{mrk}(\widehat{\widetilde{X}}, \widehat{\widetilde{\varphi}}) = \text{mrk}(\overrightarrow{\widetilde{X}}, \overrightarrow{\widetilde{\varphi}}) \leq \text{mrk}(\widehat{\widetilde{X}}, \widehat{\widetilde{\varphi}}) = \text{mdim}(\widetilde{X}, \widetilde{\varphi}).$$

This completes the proof when the map φ is surjective.

Now we consider the general case that φ is not necessarily surjective. Denote by Y the subspace $\bigcap_{n \geq 1} \varphi^n(X)$. Since $\varphi|_Y$ is surjective, we have $\text{mdim}(\widetilde{Y}, \widetilde{\varphi}) = \text{mdim}(Y, \varphi)$. By Lemma 4.1 and the definition of $(\widetilde{X}, \widetilde{\varphi})$, we have

$$\text{mdim}(X, \varphi) = \text{mdim}(Y, \varphi) = \text{mdim}(\widetilde{Y}, \widetilde{\varphi}) = \text{mdim}(\widetilde{X}, \widetilde{\varphi}).$$

□

In the rest of this section, we will prove Corollary 1.4. Firstly, we recall the definition of algebraic cellular automata.

Definition 4.3. For a compact metric space X and a continuous map $f: X^I \rightarrow X$ with a finite set $I \subseteq \mathbb{Z}$, the *cellular automaton* on $X^{\mathbb{Z}}$ with local rule f is the continuous map $F: X^{\mathbb{Z}} \rightarrow X^{\mathbb{Z}}$ sending $(x_n)_{n \in \mathbb{Z}}$ to $(f((x_{n+j})_{j \in I}))_n$. We say F is *algebraic* if X is a compact metric abelian group and f is a group homomorphism.

Next, we give a proof of Corollary 1.4.

Corollary 4.4 (Corollary 1.4). *For an algebraic cellular automaton, we have $\text{mdim}(X^{\mathbb{Z}}, F) = \text{mdim}(\widetilde{X^{\mathbb{Z}}}, \widetilde{F})$.*

Proof. Since f is a continuous homomorphism between compact metric abelian groups, so is F . Applying Theorem 1.2 to the induced dynamical systems $(X^{\mathbb{Z}}, F)$, we see that the mean dimension of the natural extension $(\widetilde{X^{\mathbb{Z}}}, \widetilde{F})$ stays the same as $\text{mdim}(X^{\mathbb{Z}}, F)$. \square

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