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# A MULTISCALE HYBRID METHOD\*

GABRIEL R. BARRENECHEA <sup>†</sup>, ANTONIO TADEU A. GOMES <sup>‡</sup>, AND DIEGO  
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**Abstract.** A new multiscale finite element method, the Multiscale Hybrid (MH) method, is proposed, analyzed, and tested in this work. The method is built as a close relative to the multiscale hybrid mixed (MHM) method, but with the fundamental difference that a novel definition of the Lagrange multiplier is introduced. The practical implication of this is that both the local problems to compute the basis functions, as well as the global problem, are elliptic, as opposed to the MHM method (and also other previous methods) that is needed to solve a mixed global problem, as well as constrained local problems to compute the basis functions. The error analysis of the method is based on a hybrid formulation, and a static condensation process is done at the discrete level, so the final global system only involves the Lagrange multipliers. The performance of the method is tested by numerical experiments for problems with multiscale coefficients, and comparisons with the MHM method are carried out in terms of performance, accuracy, and memory requirements.

**Key words.** Multiscale finite element; hybrid problem; multiscale diffusion equation.

**MSC codes.** 65N30; 65M12; 65Y20.

**1. Introduction.** The numerical analysis of multiscale problems, and the proposal of effective multiscale finite element methods, has been a very active area of research in the last few decades. Starting with the pioneering work by Babuska and Osborn [4], a myriad of multiscale finite element methods have come to life, see, e.g., [11, 3, 24], and for further references [10, 23]. One particular example of such a method, developed in the last decade, is the Multiscale Hybrid Method (MHM), originally proposed in [20, 2] for multiscale problems in porous media, and further developed for different partial differential equations (see, e.g., [21, 9] for its application for different models, and [6, 15] for its extension to polytopic meshes). The starting point of the MHM method is the proposal of a hybrid formulation for the partial differential equation at hand. Then, static condensation is performed at the continuous level leading to a problem with two unknowns, namely, the Lagrange multipliers in the inter-element facets, and one unknown per element that belongs to a local kernel (piecewise constants in the case of scalar problems [2], rigid modes in the case of linear elasticity [19]). As such, the resulting system is a *mixed* problem. The reason for the mixed character of the problem can be explained as follows: the Lagrange multipliers are the normal fluxes, that is, the normal derivative of the primal unknown in the case of the Poisson equation. So, the local problems defining the basis functions include Neumann conditions, and thus to get a well-posed problem the solution needs to be sought in the space of functions with zero mean value in each element.

The issue of local problems with Neumann boundary conditions has been encountered in several different contexts, probably the most noticeable being domain

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decomposition methods. In fact, this is one of the main reasons behind the use of Robin boundary conditions in the Optimized Schwarz Method (OSM) proposed as far back as [22]. Since then, several different proposals to use Robin boundary conditions have been used to remove local singularities. For example, and just restricting ourselves to the context of recent multiscale finite element methods, in [17] the Multiscale Robin Coupled Method (MRCM) starts from a mixed problem in porous media, and generates basis functions that are solutions of local problems containing Robin boundary conditions; both the local and the global problems are written in mixed form, and comparisons are drawn (see also [18, 30] for the application of the same idea to different models, mostly linked to flow through porous media). Also, in [25] the multiscale Latin method is analyzed, and the optimal choice of the Robin parameter is studied. This last point is a common thread in the domain decomposition community. In fact, looking for optimal Robin parameters has been the topic of several works in domain decomposition, see [14] as an example.

The purpose of this work is to build up on the idea of using Robin boundary conditions on the local problems to propose a new Multiscale Hybrid (MH) finite element method. After partitioning the domain into elements (of general shape) a new hybrid formulation is proposed. In it, the Lagrange multiplier (denoted  $\lambda$ ) is not equal to the normal flux (as in standard hybrid methods), but it is perturbed by adding a term that depends on the primal variable (denoted  $u$ ). This process presents the advantage that the bilinear form in the bulk is elliptic in the whole broken space (thus removing the kernel that would have arisen otherwise), and that its well-posedness can be analyzed using standard arguments for variational problems with constraints. We then discretize this hybrid problem using discontinuous approximations for  $\lambda$ , and globally discontinuous (but continuous in each polyhedral element) finite element spaces for  $u$ . Due to the particular structure of the new formulation, error estimates that are independent of the Robin parameter can be proven.

The final step in the building of the method is the static condensation process. For this, the discontinuous character of the approximation space for  $u$  is exploited to write its degrees of freedom in terms of those for  $\lambda$ . This is where the fact that the bilinear form in the bulk is not only elliptic in the discrete kernel, but in fact *elliptic in the whole broken space, and locally in each element of the partition*, can be fully exploited. In fact, this leads to two main advantages of this method: first, the local problems are well-posed in the whole local space (without any extra constraints); in addition, and as a consequence of this, the global problem is driven by an elliptic bilinear form involving only the unknowns for the Lagrange multiplier (thus avoiding the solution of a mixed problem). To keep the presentation simple, we have focused in this work to the diffusion equation with multiscale coefficients, but the strategy proposed herein can be extended naturally to more complex situations.

The rest of the paper is organized as follows. We finish this introduction by presenting the model problem and the main notations, assumptions, and preliminary results used in the rest of the manuscript. In [section 2](#) we present the new hybrid formulation, and prove its well-posedness and equivalence with the original primal weak form. We also present its discrete counterpart, and prove stability and optimal-order error estimates. The MH method is then introduced in [section 3](#) as a result of a static condensation procedure. We detail the rewriting of the hybrid problem as one whose only unknowns are the discrete Lagrange multipliers, and prove its ellipticity. In addition, we present a short section on the implementation and main features of the computational algorithm. The performance of the method, and comparisons with previously existing methods, are presented in [section 4](#) using four series of numerical

experiments. Finally, we outline some conclusions and open questions.

**1.1. The model problem.** Let  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$ , be an open, bounded, polygonal, domain with Lipschitz boundary  $\partial\Omega$ . We consider the problem of finding  $u : \Omega \rightarrow \mathbb{R}$  such that:

$$(1.1) \quad \begin{cases} \operatorname{div}(-\mathcal{K} \nabla u) = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases},$$

where  $f$  is a given datum and the diffusive coefficient  $\mathcal{K} = (\mathcal{K}_{ij})_{i,j=1}^d \in L^\infty(\Omega)^{d \times d}$  is a symmetric tensor supposed to be uniformly elliptic, i.e., there exist positive constants  $\mathcal{K}_{\min}$  and  $\mathcal{K}_{\max}$ , such that

$$(1.2) \quad \mathcal{K}_{\min} |\boldsymbol{\xi}|^2 \leq \mathcal{K}_{ij}(\mathbf{x}) \xi_i \xi_j \leq \mathcal{K}_{\max} |\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{\xi} = \{\xi_i\} \in \mathbb{R}^d, \text{ and almost all } \mathbf{x} \in \bar{\Omega}.$$

The functions  $f$  and  $\mathcal{K}$  may involve multi-scale features.

**1.2. Partitions and triangulations.** Following closely the definitions initially proposed in [6], we introduce two partitions which do not necessarily coincide but are not independent. To avoid technical diversions, we will restrict the presentation, and all the proofs, to the two-dimensional situation. Nevertheless, the results presented below can be extended to the three-dimensional case by proceeding as in [15]. First, we introduce a family  $\{\mathcal{P}_{\mathcal{H}}\}_{\mathcal{H}>0}$ , of partitions of  $\bar{\Omega}$ , composed by closed, bounded, disjoint polygons. (We make more precise hypotheses on the polygons later on.) The diameter of a polygonal element  $K \in \mathcal{P}_{\mathcal{H}}$  is denoted by  $\mathcal{H}_K$ , and the radius of its inscribed circle is denoted by  $\rho_K$ . We set  $\mathcal{H} := \max_{K \in \mathcal{P}_{\mathcal{H}}} \mathcal{H}_K$  and assume the existence of  $c > 0$  such that

$$(1.3) \quad \frac{\mathcal{H}_K}{\rho_K} \leq c, \quad \text{for all } K \in \mathcal{P}_{\mathcal{H}}, \text{ for all } \mathcal{H} > 0.$$

Each polygon  $K$  has a boundary  $\partial K$  consisting of edges  $E$ . The set of boundaries of the elements in  $\mathcal{P}_{\mathcal{H}}$  is denoted by  $\partial\mathcal{P}_{\mathcal{H}}$  and the set of its edges by  $\mathcal{E}$ ; that is  $\partial\mathcal{P}_{\mathcal{H}} = \{\partial K : K \in \mathcal{P}_{\mathcal{H}}\}$ , and,

$$\mathcal{E} = \left\{ \begin{array}{l|l} E = \partial K \cap \partial K', \text{ or} & K, K' \in \mathcal{P}_{\mathcal{H}}, K \neq K', \text{ and} \\ E = \partial K \cap \partial\Omega & E \text{ is not a single point} \end{array} \right\}.$$

To each  $E \in \mathcal{E}$  we associate a normal  $\mathbf{n}^E$ ; its orientation is not important, but is fixed. For each  $K \in \mathcal{P}_{\mathcal{H}}$  we further denote by  $\mathbf{n}^K$  the outward normal vector on  $\partial K$ , and let  $\mathbf{n}_E^K := \mathbf{n}^K|_E$  for each  $E \subset \partial K$ . The first step toward defining a computational mesh is to introduce  $\mathcal{E}_H$ , a partition of  $\mathcal{E}$  into segments  $F$  of length  $H_F \leq H := \max_{F \in \mathcal{E}_H} H_F$ , and such that each  $F \in \mathcal{E}_H$  is a subset of only one  $E \in \mathcal{E}$ . We will not assume that the segments are of equal length, but we will require that they are not too different. More precisely, we impose the following Assumption.

**Assumption (A1)** The mesh  $\mathcal{E}_H$  is such that in every  $K \in \mathcal{P}_{\mathcal{H}}$  a shape-regular simplicial triangulation  $\Xi_H(K)$  of  $K$  can be built such that its trace on  $\partial K$  coincides with  $\mathcal{E}_H$ .

The last ingredient in the definition of the discrete scheme is the mesh that will be used to approximate the local basis functions: for each  $K \in \mathcal{P}_{\mathcal{H}}$ , we introduce a shape regular family of simplicial triangulations  $\{\mathcal{T}_h^K\}_{h>0}$  built in the following way:

1. first, on each  $K \in \mathcal{P}$ , the triangulation  $\Xi_H(K)$  is refined once using a *red refinement* [5]. The resulting triangulation is called *minimal triangulation*;

2. then, for each  $K$ , the family  $\{\mathcal{T}_h^K\}_{h>0}$  is formed by regular refinements of the minimal triangulation.

The diameter of  $\mathfrak{T} \in \mathcal{T}_h^K$  is denoted by  $h_{\mathfrak{T}}$ , and  $h := \max_{K \in \mathcal{P}_{\mathcal{H}}} \max_{\mathfrak{T} \in \mathcal{T}_h^K} h_{\mathfrak{T}}$ , and  $\mathcal{T}_h := \cup_{K \in \mathcal{P}_{\mathcal{H}}} \mathcal{T}_h^K$ . It is important to remark that, if  $E = K \cap K' \in \mathcal{E}$ , then the traces of the two neighbouring triangulations  $\mathcal{T}_h^K$  and  $\mathcal{T}_h^{K'}$  do not necessarily coincide. In [Figure 1](#) we show one example of partitions that satisfy the above assumptions.

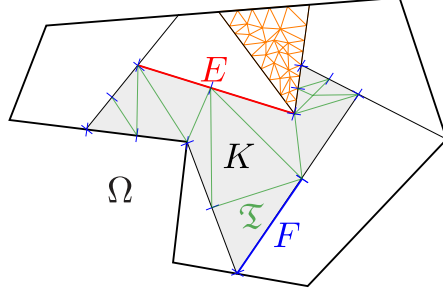


FIG. 1. A domain  $\Omega$  partitioned by  $\mathcal{P}_{\mathcal{H}}$  with non-conforming polygonal elements. Observe the sub-meshes (green and orange) discretizing two different elements of  $\mathcal{P}_{\mathcal{H}}$  with different granularity, where  $\mathfrak{T} \in \mathcal{T}_h^K$  is an element of the discretization of  $K \in \mathcal{P}_{\mathcal{H}}$ . The red line represents an edge  $E \in \mathcal{E}$  and the blue line represents an element  $F \in \mathcal{E}_H$  of the skeleton mesh.

**1.3. Spaces, inner products, and norms.** We follow the standard notation for Lebesgue and Sobolev spaces, in line with, e.g., [12]. For  $K \in \mathcal{P}_{\mathcal{H}}$  and  $m \geq 1$ , we consider the local space  $H^m(K)$  equipped with the semi-norm  $|\cdot|_{m,K}$  and the norm  $\|\cdot\|_{m,K}$ , with their usual definitions. We then define the broken space

$$(1.4) \quad H^m(\mathcal{P}_{\mathcal{H}}) := \{v \in L^2(\Omega) : v|_K \in H^m(K) \text{ for all } K \in \mathcal{P}_{\mathcal{H}}\},$$

equipped with the semi-norm and norm,

$$|v|_{m,\mathcal{P}_{\mathcal{H}}} := \left( \sum_{K \in \mathcal{P}_{\mathcal{H}}} |v|_{m,K}^2 \right)^{1/2} \quad \text{and} \quad \|v\|_{m,\mathcal{P}_{\mathcal{H}}} := \left( \sum_{K \in \mathcal{P}_{\mathcal{H}}} \|v\|_{m,K}^2 \right)^{1/2},$$

and set  $V := H^1(\mathcal{P}_{\mathcal{H}})$  equipped with the norm

$$(1.5) \quad \|v\|_V := \left( \|\nabla v\|_{0,\mathcal{P}_{\mathcal{H}}}^2 + \frac{1}{d_{\Omega}^2} \|v\|_{0,\mathcal{P}_{\mathcal{H}}}^2 \right)^{1/2},$$

for all  $v \in V$ . Here,  $d_{\Omega} > 0$  is a constant that depends on the domain  $\Omega$ , and that will be specified later. In addition, we define the inner product

$$(u, v)_{\mathcal{P}_{\mathcal{H}}} := \sum_{K \in \mathcal{P}_{\mathcal{H}}} (u, v)_K, \quad \text{for all } u, v \in V.$$

Since the aim of this work is to introduce a new hybrid formulation, we shall need an appropriate space for the Lagrange multipliers. The natural space in our case consists of functions that are normal traces of  $H(\text{div}; \Omega)$  functions; more precisely, we define the space

$$(1.6) \quad \Lambda := \{\mathbf{q} \cdot \mathbf{n}^K|_{\partial K}, \text{ for all } K \in \mathcal{P}_{\mathcal{H}} : \mathbf{q} \in H(\text{div}; \Omega)\},$$

equipped with the norm

$$(1.7) \quad \|\mu\|_\Lambda := \inf \{ \|\mathbf{q}\|_{\text{div},\Omega} : \mathbf{q} \in H(\text{div};\Omega) \text{ and } \mathbf{q} \cdot \mathbf{n}^K|_{\partial K} = \mu, \forall K \in \mathcal{P}_h \},$$

with,

$$\|\mathbf{p}\|_{\text{div},\Omega} := \left( \|\mathbf{p}\|_{0,\Omega}^2 + d_\Omega^2 \|\nabla \cdot \mathbf{p}\|_{0,\Omega}^2 \right)^{1/2}, \text{ for all } \mathbf{p} \in H(\text{div};\Omega).$$

Denoting by  $\langle \cdot, \cdot \rangle_{\partial K}$  the duality pairing between  $H^{-\frac{1}{2}}(\partial K)$  and  $H^{\frac{1}{2}}(\partial K)$ , the Trace Theorem (see, e.g., [12, Theorem 3.10]) ensures that the following *duality pairing* between  $\prod_{K \in \mathcal{P}_h} H^{-\frac{1}{2}}(\partial K)$  and  $V$  is well-defined:

$$(1.8) \quad \langle \mu, v \rangle_{\partial \mathcal{P}_h} := \sum_{K \in \mathcal{P}_h} \langle \mu, v \rangle_{\partial K}, \text{ for all } (\mu, v) \in \prod_{K \in \mathcal{P}_h} H^{-\frac{1}{2}}(\partial K) \times V.$$

Moreover, the following identity can be proven as in [15, pp. 5]

$$(1.9) \quad \|\mu\|_\Lambda = \sup_{v \in V} \frac{\langle \mu, v \rangle_{\partial \mathcal{P}_h}}{\|v\|_V}, \text{ for all } \mu \in \Lambda.$$

Here, and thereafter, we lighten the notation by writing  $\sup_{v \in V}$  instead of  $\sup_{v \in V \setminus \{\mathbf{0}\}}$ .

Finally, associated to  $\mathcal{E}_H$  and  $\mathcal{T}_h^K$ , for  $\ell \geq 0$ , we introduce the following finite element spaces:

$$(1.10) \quad V_h := \prod_{K \in \mathcal{P}_h} V_h(K),$$

$$(1.11) \quad V_h(K) := \{v_h \in C^0(K) : v_h|_{\mathfrak{T}} \in \mathbb{P}_k(\mathfrak{T}), \forall \mathfrak{T} \in \mathcal{T}_h^K\},$$

$$(1.12) \quad \Lambda_H := \{\mu_H \in \Lambda : \mu_H|_F \in \mathbb{P}_\ell(F), \forall F \in \mathcal{E}_H\}.$$

**2. A new Hybrid Formulation.** As it was mentioned in the introduction, the starting point in building the method is to derive an equivalent weak formulation of (1.1) based on a hybridization procedure. Our presentation below borrows ideas from [20, 2, 6], which in turn are inspired by [29]. First, the standard weak formulation of (1.1) is given by: *Find*  $u \in H_0^1(\Omega)$  *such that*

$$(2.1) \quad (\mathcal{K} \nabla u, \nabla v)_\Omega = (f, v)_\Omega, \text{ for all } v \in H_0^1(\Omega).$$

We shall propose a hybrid weak formulation, equivalent to (2.1). The first ingredient is a linear mapping  $\varsigma : V \rightarrow H^{-\frac{1}{2}}(\partial \mathcal{P}_h)$  that, for now, will only be required to be continuous, and satisfy

$$(2.2) \quad v \in H_0^1(\Omega) \Rightarrow \varsigma(v) \in \Lambda.$$

Using this mapping, we propose the following hybrid variational formulation: *Find*  $(u, \lambda) \in V \times \Lambda$  *such that*

$$(2.3) \quad \begin{cases} a(u, v) + \langle \lambda, v \rangle_{\partial \mathcal{P}_h} = (f, v)_{\mathcal{P}_h}, \text{ for all } v \in V \\ \langle \mu, u \rangle_{\partial \mathcal{P}_h} = 0, \text{ for all } \mu \in \Lambda \end{cases},$$

where the bilinear form  $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$  is defined by,

$$(2.4) \quad a(w, v) := (\mathcal{K} \nabla w, \nabla v)_{\mathcal{P}_h} + \langle \varsigma(w), v \rangle_{\partial \mathcal{P}_h}.$$

Notice that if  $\varsigma$  is identically zero, then (2.3) is the classical hybrid formulation proposed in [29] and the rest of the derivation would coincide with what was presented in [2]. So, following ideas from [2, 6] we now prove that (2.1) and (2.3) are equivalent.

**THEOREM 2.1.** *The function  $u \in H_0^1(\Omega)$  solves (2.1) if and only if  $(u, \lambda) \in V \times \Lambda$  solves (2.3). Moreover, the following relation between  $u$  and  $\lambda$  holds*

$$(2.5) \quad \lambda = (-\mathcal{K} \nabla u \cdot \mathbf{n}^K - \varsigma(u))|_{\partial K}, \text{ for all } K \in \mathcal{P}_{\mathcal{H}}.$$

*Proof.* Let  $(u, \lambda) \in V \times \Lambda$  be a solution of (2.3). Using [28, Lemma 1] the following characterization holds

$$(2.6) \quad H_0^1(\Omega) = \{v \in V : \langle \mu, v \rangle_{\partial \mathcal{P}_{\mathcal{H}}} = 0, \text{ for all } \mu \in \Lambda\}.$$

So, using the second equation in (2.3) we conclude that  $u \in H_0^1(\Omega)$  and  $\varsigma(u) \in \Lambda$ . Hence  $\langle \varsigma(u), v \rangle_{\partial \mathcal{P}_{\mathcal{H}}} = 0$ , for all  $v \in H_0^1(\Omega)$ , and the first equation in (2.3) implies that

$$(\mathcal{K} \nabla u, \nabla v)_{\Omega} = (f, v)_{\Omega}, \text{ for all } v \in H_0^1(\Omega),$$

and thus  $u$  solves (2.1).

Conversely, let  $u \in H_0^1(\Omega)$  be the solution of (2.1) and let us consider the following continuous linear functional on  $V$

$$(2.7) \quad L(v) = (f, v)_{\partial \mathcal{P}_{\mathcal{H}}} - (\mathcal{K} \nabla u, \nabla v)_{\partial \mathcal{P}_{\mathcal{H}}} - \langle \varsigma(u), v \rangle_{\partial \mathcal{P}_{\mathcal{H}}}.$$

This functional vanishes on  $H_0^1(\Omega)$ ; so, thanks to the inf-sup condition (1.9) there exists a unique  $\lambda \in \Lambda$  such that

$$\langle \lambda, v \rangle_{\partial \mathcal{P}_{\mathcal{H}}} = L(v), \text{ for all } v \in V.$$

Hence, the pair  $(u, \lambda) \in V \times \Lambda$  solves (2.3).

Finally, since  $f = \operatorname{div}(-\mathcal{K} \nabla u)$ , replacing and integrating by parts in (2.7), we arrive at

$$L(v) = \sum_{K \in \mathcal{P}_{\mathcal{H}}} -\langle \mathcal{K} \nabla u \cdot \mathbf{n}^K + \varsigma(u), v \rangle_{\partial K} = \langle \lambda, v \rangle_{\partial \mathcal{P}_{\mathcal{H}}},$$

which proves (2.5).  $\square$

**2.1. Explicit definition of  $\varsigma(\cdot)$ .** The result given by Theorem 2.1 provides existence and uniqueness of solution for (2.3), as long as  $\varsigma(\cdot)$  satisfies (2.2). So, this condition needs to be kept in mind when defining a concrete mapping  $\varsigma(\cdot)$ . Let us consider  $\boldsymbol{\sigma} \in W^{1,\infty}(\Omega)^d$  and  $v \in V$ . It is easy to notice that  $v \boldsymbol{\sigma}|_K \in H(\operatorname{div}; K)$  for all  $K \in \mathcal{P}_{\mathcal{H}}$ . Moreover, if  $v \in H_0^1(\Omega)$  then  $v \boldsymbol{\sigma} \in H(\operatorname{div}; \Omega)$ . Thus, the following is a valid definition for  $\varsigma$ :

$$(2.8) \quad \begin{aligned} \varsigma : V &\rightarrow H^{-\frac{1}{2}}(\partial \mathcal{P}_{\mathcal{H}}), \\ v &\mapsto \varsigma(v)|_{\partial K} := (v \boldsymbol{\sigma})|_{\partial K} \cdot \mathbf{n}^K, \text{ in each } K \in \mathcal{P}_{\mathcal{H}}. \end{aligned}$$

Now, we give a concrete definition of  $\boldsymbol{\sigma} \in W^{1,\infty}(\Omega)^d$  that will ensure that the bilinear form  $a(\cdot, \cdot)$  is elliptic. Let us define an open and bounded rectangular parallelepiped

$$\tilde{\Omega} = \prod_{i=1}^d (a_i, b_i) \subset \mathbb{R}^d,$$

such that  $\bar{\Omega} \subset \tilde{\Omega}$ . In what follows we denote  $\boldsymbol{\sigma}(\mathbf{x}) = (\sigma_1(\mathbf{x}), \dots, \sigma_d(\mathbf{x}))$  where  $\mathbf{x} = (x_1, \dots, x_d) \in \tilde{\Omega}$ . Then, for  $j \in \{1, \dots, d\}$  we define:

$$(2.9) \quad \sigma_j(\mathbf{x}) := \frac{\nu}{d} (x_j - a_j), \text{ for all } \mathbf{x} \in \tilde{\Omega},$$

where  $\nu > 0$  will be chosen later. The function  $\boldsymbol{\sigma}$  just defined satisfies the following important properties

$$(2.10) \quad \operatorname{div} \boldsymbol{\sigma}(\mathbf{x}) = \sum_{j=1}^d \partial_{x_j} \sigma_j(\mathbf{x}) = \sum_{j=1}^d \frac{\nu}{d} = \nu,$$

and

$$(2.11) \quad \|\boldsymbol{\sigma}\|_{0,\infty,\Omega} = \operatorname{ess\,sup}_{\mathbf{x} \in \Omega} \left( \sum_{j=1}^d |\sigma_j(\mathbf{x})|^2 \right)^{\frac{1}{2}} \leq \nu \max_{j \in \{1, \dots, d\}} |b_j - a_j| \leq d_{\tilde{\Omega}} \nu,$$

where  $d_{\tilde{\Omega}} > 0$  is the diameter of  $\tilde{\Omega}$ . Then, using the above definitions the bilinear form  $a(\cdot, \cdot)$  can be expressed in the following equivalent ways:

$$(2.12) \quad a(u, v) = (\mathcal{K} \nabla u, \nabla v)_{\mathcal{P}_{\mathcal{H}}} + \langle (u \boldsymbol{\sigma}) \cdot \mathbf{n}^K, v \rangle_{\partial \mathcal{P}_{\mathcal{H}}}$$

$$(2.13) \quad = (\mathcal{K} \nabla u, \nabla v)_{\mathcal{P}_{\mathcal{H}}} + (\boldsymbol{\sigma} \cdot \nabla u, v)_{\mathcal{P}_{\mathcal{H}}} + (u, \boldsymbol{\sigma} \cdot \nabla v)_{\mathcal{P}_{\mathcal{H}}} + \nu (u, v)_{\mathcal{P}_{\mathcal{H}}}.$$

The aforementioned ellipticity of  $a(\cdot, \cdot)$  as well as its continuity are established in the following result.

**THEOREM 2.2.** *Let us suppose that, in the definition of (2.9),  $\nu$  is chosen such that*

$$(2.14) \quad 0 \leq \nu \leq \frac{\mathcal{K}_{\min}}{4 d_{\tilde{\Omega}}^2}.$$

*Then, the bilinear form  $a(\cdot, \cdot)$  defined in (2.12) is continuous and elliptic. More precisely, for all  $v, w \in V$  the following holds*

$$\begin{aligned} a(w, v) &\leq 2 \left( \|\mathcal{K}^{\frac{1}{2}} \nabla w\|_{0, \mathcal{P}_{\mathcal{H}}}^2 + \nu \|w\|_{0, \mathcal{P}_{\mathcal{H}}}^2 \right)^{\frac{1}{2}} \left( \|\mathcal{K}^{\frac{1}{2}} \nabla v\|_{0, \mathcal{P}_{\mathcal{H}}}^2 + \nu \|v\|_{0, \mathcal{P}_{\mathcal{H}}}^2 \right)^{\frac{1}{2}}, \\ a(v, v) &\geq \frac{1}{2} \left( \|\mathcal{K}^{\frac{1}{2}} \nabla v\|_{0, \mathcal{P}_{\mathcal{H}}}^2 + \nu \|v\|_{0, \mathcal{P}_{\mathcal{H}}}^2 \right). \end{aligned}$$

*Proof.* Let  $w, v \in V$ . Using the writing (2.13) for  $a(\cdot, \cdot)$ , Cauchy-Schwarz's inequality, (2.10), and (2.11) we arrive at

$$\begin{aligned} a(w, v) &= (\mathcal{K}^{\frac{1}{2}} \nabla w, \mathcal{K}^{\frac{1}{2}} \nabla v)_{\mathcal{P}_{\mathcal{H}}} + (\boldsymbol{\sigma} \cdot \nabla w, v)_{\mathcal{P}_{\mathcal{H}}} + (w, \boldsymbol{\sigma} \cdot \nabla v)_{\mathcal{P}_{\mathcal{H}}} + \nu (w, v)_{\mathcal{P}_{\mathcal{H}}} \\ &\leq \|\mathcal{K}^{\frac{1}{2}} \nabla w\|_{0, \mathcal{P}_{\mathcal{H}}} \|\mathcal{K}^{\frac{1}{2}} \nabla v\|_{0, \mathcal{P}_{\mathcal{H}}} + \nu \|w\|_{0, \mathcal{P}_{\mathcal{H}}} \|v\|_{0, \mathcal{P}_{\mathcal{H}}} \\ &\quad + \frac{1}{2} \nu^{\frac{1}{2}} (\|\mathcal{K}^{\frac{1}{2}} \nabla w\|_{0, \mathcal{P}_{\mathcal{H}}} \|v\|_{0, \mathcal{P}_{\mathcal{H}}} + \|w\|_{0, \mathcal{P}_{\mathcal{H}}} \|\mathcal{K}^{\frac{1}{2}} \nabla v\|_{0, \mathcal{P}_{\mathcal{H}}}) \\ &\leq \left( \|\mathcal{K}^{\frac{1}{2}} \nabla w\|_{0, \mathcal{P}_{\mathcal{H}}} + \nu^{\frac{1}{2}} \|w\|_{0, \mathcal{P}_{\mathcal{H}}} \right) \left( \|\mathcal{K}^{\frac{1}{2}} \nabla v\|_{0, \mathcal{P}_{\mathcal{H}}} + \nu^{\frac{1}{2}} \|v\|_{0, \mathcal{P}_{\mathcal{H}}} \right) \\ &\leq 2 \left( \|\mathcal{K}^{\frac{1}{2}} \nabla w\|_{0, \mathcal{P}_{\mathcal{H}}}^2 + \nu \|w\|_{0, \mathcal{P}_{\mathcal{H}}}^2 \right)^{\frac{1}{2}} \left( \|\mathcal{K}^{\frac{1}{2}} \nabla v\|_{0, \mathcal{P}_{\mathcal{H}}}^2 + \nu \|v\|_{0, \mathcal{P}_{\mathcal{H}}}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

and,

$$\begin{aligned} a(v, v) &= (\mathcal{K} \nabla v, \nabla v)_{\mathcal{P}_{\mathcal{H}}} + 2 (\boldsymbol{\sigma} \cdot \nabla v, v)_{\mathcal{P}_{\mathcal{H}}} + \nu (v, v)_{\mathcal{P}_{\mathcal{H}}} \\ &\geq \|\mathcal{K}^{\frac{1}{2}} \nabla v\|_{0, \mathcal{P}_{\mathcal{H}}}^2 - 2 d_{\tilde{\Omega}} \nu |v|_{1, \mathcal{P}_{\mathcal{H}}} \|v\|_{0, \mathcal{P}_{\mathcal{H}}} + \nu \|v\|_{0, \mathcal{P}_{\mathcal{H}}}^2 \\ &\geq \frac{1}{2} \left( \|\mathcal{K}^{\frac{1}{2}} \nabla v\|_{0, \mathcal{P}_{\mathcal{H}}}^2 + \nu \|v\|_{0, \mathcal{P}_{\mathcal{H}}}^2 \right) + \nu \left( \sqrt{2} d_{\tilde{\Omega}} |v|_{1, \mathcal{P}_{\mathcal{H}}} - \frac{1}{\sqrt{2}} \|v\|_{0, \mathcal{P}_{\mathcal{H}}} \right)^2 \\ &\geq \frac{1}{2} \left( \|\mathcal{K}^{\frac{1}{2}} \nabla v\|_{0, \mathcal{P}_{\mathcal{H}}}^2 + \nu \|v\|_{0, \mathcal{P}_{\mathcal{H}}}^2 \right), \end{aligned}$$



which proves the Theorem.  $\square$

To link the ellipticity and continuity of  $a(\cdot, \cdot)$  just proved with the norm  $\|\cdot\|_V$ , we note that using (2.14) we obtain

$$(2.15) \quad a(w, v) \leq 2\mathcal{K}_{\max} \|w\|_V \|v\|_V, \text{ and, } a(v, v) \geq \frac{1}{2}\nu d_\Omega^2 \|v\|_V^2, \text{ for all } w, v \in V.$$

Interestingly, the bilinear form  $a(\cdot, \cdot)$  is elliptic in  $H_0^1(\Omega)$  even if  $\nu = 0$ . It is therefore sufficient to use the standard theory for variational problems with constraints, which is precisely what the next theorem shows.

**THEOREM 2.3.** *Suppose that, in the definition of (2.9),  $\nu$  is chosen such that (2.14) holds. Then, problem (2.3) has a unique solution  $(u, \lambda) \in V \times \Lambda$ . Moreover, the following stability holds*

$$(2.16) \quad \|\lambda\|_\Lambda + \mathcal{K}_{\max} \|u\|_V \leq C \frac{\mathcal{K}_{\max}}{\mathcal{K}_{\min}} \|f\|_{0,\Omega},$$

where,  $C > 0$  does not depend on any mesh size, data of the problem, or  $\nu$ .

*Proof.* The existence and uniqueness of the solution to (2.3) is a consequence of its equivalence with (2.1) and the fact that the latter has a unique solution. The remaining part of the proof is related to giving the bound (2.16). We first notice that for all  $v \in H_0^1(\Omega)$  we have  $\|v\|_{0,\Omega} \leq c_P \|\nabla v\|_{0,\Omega}$ , where the Poincaré constant  $c_P > 0$  depends only on  $\Omega$ . Thus, from the ellipticity given in Theorem 2.2 and the fact that  $c_P \leq d_\Omega$  we arrive at

$$a(v, v) \geq \frac{1}{2} \mathcal{K}_{\min} \|v\|_V^2, \text{ for all } v \in H_0^1(\Omega).$$

Then,

$$\frac{1}{2} \mathcal{K}_{\min} \|u\|_V^2 \leq a(u, u) = (f, u)_{0,\mathcal{F}_\mathcal{H}} \leq c_P \|f\|_{0,\Omega} \|u\|_V.$$

So, using (2.15) and the inf-sup condition (1.9) we get

$$\|\lambda\|_\Lambda = \sup_{v \in V} \frac{\langle \lambda, v \rangle_{\partial \mathcal{F}_\mathcal{H}}}{\|v\|_V} = \sup_{v \in V} \frac{(f, v)_{0,\mathcal{F}_\mathcal{H}} - a(u, v)}{\|v\|_V} \leq c_P \|f\|_{0,\Omega} + 2\mathcal{K}_{\max} \|u\|_V.$$

The proof is finished by combining the above bounds.  $\square$

*Remark 2.4.* Although the equivalence result presented in Theorem 2.1 holds for any choice of mapping  $\zeta(\cdot)$ , the uniform stability (with respect to  $\nu$ ) on the solution of the hybrid problem (2.3) is due to the precise choice of  $\sigma$  done in this work. More general definitions of  $\zeta(\cdot)$  could be considered leading to related stability results, based this time in inf-sup conditions for  $a(\cdot, \cdot)$ , but in such a case the uniform stability with respect to  $\zeta(\cdot)$  would no longer be guaranteed.

**2.2. The discrete hybrid problem.** We start by recalling the definition of the spaces  $V_h$  and  $\Lambda_H$  given in (1.10) and (1.12). The Galerkin discretization of (2.3) using these spaces reads as follows: Find  $(u_h, \lambda_H) \in V_h \times \Lambda_H$  such that

$$(2.17) \quad \begin{cases} a(u_h, v_h) + \langle \lambda_H, v_h \rangle_{\partial \mathcal{F}_\mathcal{H}} = (f, v_h)_{\mathcal{F}_\mathcal{H}}, \text{ for all } v_h \in V_h \\ \langle \mu_H, u_h \rangle_{\partial \mathcal{F}_\mathcal{H}} = 0, \text{ for all } \mu_H \in \Lambda_H \end{cases}.$$

The first step in analyzing the discrete problem is to define the discrete kernel

$$(2.18) \quad \mathfrak{N}_h := \{v_h \in V_h : \langle \mu_H, v_h \rangle_{\partial \mathcal{P}_h} = 0, \forall \mu_H \in \Lambda_H\}.$$

Then, the Poincaré–Friedrichs inequality for piecewise  $H^1$  functions given in [7, Eq. (1.3)] implies the existence of  $c_B > 0$  that depends only on the shape of the polygons of  $\mathcal{P}_h$  (and not on  $h$ ,  $H$ , or  $\mathcal{H}$ ) such that

$$(2.19) \quad \|v_h\|_{0,\Omega} \leq c_B \|\nabla v_h\|_{0,\mathcal{P}_h}, \text{ for all } v_h \in \mathfrak{N}_h.$$

So,  $v_h \mapsto \|\mathcal{K}^{\frac{1}{2}} \nabla v_h\|_{0,\mathcal{P}_h}$  is a norm on  $\mathfrak{N}_h$ . This observation also allows us to make the ellipticity in [Theorem 2.2](#) more precise. In fact, the ellipticity in [Theorem 2.2](#) implies:

$$(2.20) \quad a(v_h, v_h) \geq \frac{1}{2} \left( \|\mathcal{K}^{\frac{1}{2}} \nabla v_h\|_{0,\mathcal{P}_h}^2 + \nu \|v_h\|_{0,\mathcal{P}_h}^2 \right) \geq \frac{1}{2} \|\mathcal{K}^{\frac{1}{2}} \nabla v_h\|_{0,\mathcal{P}_h}^2,$$

for all  $v_h \in \mathfrak{N}_h$ , which is indeed an ellipticity in the discrete kernel  $\mathfrak{N}_h$ . Moreover, thanks to the Poincaré inequality [\(2.19\)](#) we have

$$(2.21) \quad a(v_h, v_h) \geq \frac{1}{2} \left( \frac{d_\Omega^2}{d_\Omega^2 + c_B^2} \right) \mathcal{K}_{\min} \|v_h\|_V^2.$$

In addition, the hypotheses made in [subsection 1.3](#) on the partitions  $\mathcal{P}_h$  and  $\mathcal{T}_h$  allow us to use the results from [6] and [15] to prove the following discrete inf-sup condition:

$$(2.22) \quad \frac{1}{c_F} \|\mu_H\|_\Lambda \leq \sup_{v_h \in V_h} \frac{\langle \mu_H, v_h \rangle_{\partial \mathcal{P}_h}}{\|v_h\|_V},$$

where  $c_F > 0$  is independent of  $h$ ,  $H$ , and  $\mathcal{H}$ . As a consequence, the finite element method [\(2.17\)](#) is well-posed, and has stability and error constants independent of  $\nu$ . This is stated in the following result.

**THEOREM 2.5.** *There exists a unique  $(u_h, \lambda_H) \in V_h \times \Lambda_H$  that is the solution to [\(2.17\)](#). In addition, the discrete solution satisfies the following stability*

$$(2.23) \quad \frac{1}{c_F} \|\lambda_H\|_\Lambda + \mathcal{K}_{\max} \|u_h\|_V \leq C \frac{\mathcal{K}_{\max}}{\mathcal{K}_{\min}} \|f\|_{0,\Omega},$$

where,  $C > 0$  does not depend on any mesh size, data, or  $\nu$ . Moreover we have the following error estimate

$$(2.24) \quad \|u - u_h\|_V \leq c_1 \inf_{v_h \in V_h} \|u - v_h\|_V + c_2 \inf_{\rho_H \in \Lambda_H} \|\lambda - \rho_H\|_\Lambda,$$

$$(2.25) \quad \|\lambda - \lambda_H\|_\Lambda \leq c_3 \inf_{v_h \in V_h} \|u - v_h\|_V + c_4 \inf_{\rho_H \in \Lambda_H} \|\lambda - \rho_H\|_\Lambda,$$

where  $c_1, c_2, c_3$ , and  $c_4$  are positive constants independent of mesh sizes, and  $\nu$ .

*Proof.* The proof of the existence and uniqueness of solution to [\(2.17\)](#) follows from the ellipticity [\(2.20\)](#) of  $a(\cdot, \cdot)$  on  $\mathfrak{N}_h$ , the discrete inf-sup condition [\(2.22\)](#), and standard results on saddle-point problems (see, e.g., [31, 13]). The stability [\(2.23\)](#) can be obtained following closely the proof of [Theorem 2.3](#). Finally, the error bounds [\(2.24\)](#) and [\(2.25\)](#) are obtained by applying [13, Lemma 50.2].  $\square$

The Cea-like estimates (2.24) and (2.25) lead to optimal-order error estimates. For this, we first state the following result, whose proof is a slight variation of that from [6, Lemma 3].

LEMMA 2.6. *Suppose  $w \in H^{\ell+2}(\mathcal{P}_{\mathcal{H}}) \cap H_0^1(\Omega)$ ,  $\mathcal{K} \nabla w \in H^{\ell+1}(\mathcal{P}_{\mathcal{H}})$ , with  $\ell \geq 0$ , and  $\mathcal{K} \nabla w \in H(\text{div}; \Omega)$ . Let  $\mu \in \Lambda$  defined by*

$$\mu|_E = -(\mathcal{K} \nabla w \cdot \mathbf{n}^K + w(\boldsymbol{\sigma} \cdot \mathbf{n}^K))|_E$$

for each  $E \in \mathcal{E}$ . Then, there exists a positive constant  $C > 0$ , independent of  $h$ ,  $H$ ,  $\mathcal{H}$ , and  $\mathcal{K}$ , such that

$$(2.26) \quad \inf_{\mu_H \in \Lambda_H} \|\mu - \mu_H\|_{\Lambda} \leq C H^{\ell+1} |\mathcal{K} \nabla w + w \boldsymbol{\sigma}|_{\ell+1, \mathcal{P}_{\mathcal{H}}},$$

where  $\Lambda_H$  and  $\boldsymbol{\sigma} \in [C^\infty(\Omega)]^d$  are given in (1.12) and (2.9), respectively.

The last approximation result, along with standard interpolation estimates (see, e.g., [12]) lead, in a natural way, to the following error estimate for the discrete hybrid problem (2.17). We stress once again the robustness of the error constants with respect to  $\nu$ .

COROLLARY 2.7. *Suppose that the solution  $(u, \lambda)$  of (2.3) satisfies the hypotheses of Lemma 2.6. Then, there exist positive constants  $C_1, C_2$ , independent of any mesh size and  $\nu$ , such that*

$$(2.27) \quad \|u - u_h\|_V \leq C_1 (H^{\ell+1} |\mathcal{K} \nabla u + u \boldsymbol{\sigma}|_{\ell+1, \mathcal{P}_{\mathcal{H}}} + h^k |u|_{k+1, \mathcal{P}_{\mathcal{H}}}),$$

$$(2.28) \quad \|\lambda - \lambda_H\|_{\Lambda} \leq C_2 (H^{\ell+1} |\mathcal{K} \nabla u + u \boldsymbol{\sigma}|_{\ell+1, \mathcal{P}_{\mathcal{H}}} + h^k |u|_{k+1, \mathcal{P}_{\mathcal{H}}}).$$

The fact that the stability and error results just proven are robust with respect to  $\nu$  prompts the question of what happens in the limit  $\nu \rightarrow 0$ . It is important to notice that for  $\nu = 0$  the discrete problem (2.17) is, in fact, equivalent to the MHM method proposed in [6], and so the limit problem when  $\nu = 0$  has a unique solution as well. To explain the transition between the two cases ( $\nu \neq 0$  and  $\nu = 0$ ), in the next result we study the convergence of  $u_h$  as  $\nu \rightarrow 0$ , and provide an upper bound for the error.

THEOREM 2.8. *Let  $(u_h, \lambda_H) \in V_h \times \Lambda_H$  be the solution of (2.17), and let us denote the solution of (2.17) when  $\nu = 0$  by  $(u_h^0, \lambda_H^0)$ . Then, there exists a constant  $C$ , independent of  $\nu$ , and every physical and mesh parameters, such that*

$$(2.29) \quad \|\mathcal{K}^{\frac{1}{2}} \nabla(u_h - u_h^0)\|_{0, \mathcal{P}_{\mathcal{H}}} \leq C \frac{1}{\mathcal{K}_{\min}^{\frac{3}{2}}} \nu \|f\|_{0, \Omega}.$$

*Proof.* Let  $e_h := u_h - u_h^0$ . Since both  $u_h$  and  $u_h^0$  belong to  $\mathfrak{N}_h$ , using (2.17) we get

$$a(u_h, e_h) = (f, e_h)_{\mathcal{P}_{\mathcal{H}}} = (\mathcal{K} \nabla u_h^0, \nabla e_h)_{\mathcal{P}_{\mathcal{H}}} = a(u_h^0, e_h) - \langle (u_h^0 \boldsymbol{\sigma}) \cdot \mathbf{n}^K, e_h \rangle_{\partial \mathcal{P}_{\mathcal{H}}}.$$

Rearranging terms, using (2.13), (2.10), (2.11), (2.23), and (2.19), we arrive at

$$\begin{aligned}
\frac{1}{2} \|\mathcal{K}^{\frac{1}{2}} \nabla e_h\|_{0, \mathcal{T}_h}^2 &\leq a(e_h, e_h) \\
&= -\langle (u_h^0 \boldsymbol{\sigma}) \cdot \mathbf{n}^K, e_h \rangle_{\partial \mathcal{T}_h} \\
&= -\left[ (\boldsymbol{\sigma} \cdot \nabla u_h^0, e_h)_{\mathcal{T}_h} + (u_h^0, \boldsymbol{\sigma} \cdot \nabla e_h)_{\mathcal{T}_h} + ((\nabla \cdot \boldsymbol{\sigma}) u_h^0, e_h)_{\mathcal{T}_h} \right] \\
&\leq \sqrt{2} (d_{\tilde{\Omega}}^2 + d_{\tilde{\Omega}} c_B) \frac{1}{\mathcal{K}_{\min}^{\frac{1}{2}}} \nu \|u_h^0\|_V \|\mathcal{K}^{\frac{1}{2}} \nabla e_h\|_{0, \mathcal{T}_h},
\end{aligned}$$

yielding

$$(2.30) \quad \|\mathcal{K}^{\frac{1}{2}} \nabla e_h\|_{0, \mathcal{T}_h} \leq 2\sqrt{2} (d_{\tilde{\Omega}}^2 + d_{\tilde{\Omega}} c_B) \frac{1}{\mathcal{K}_{\min}^{\frac{1}{2}}} \nu \|u_h^0\|_V.$$

Finally, from the definition of  $u_h^0$  and (2.21) we arrive at:

$$(2.31) \quad \|u_h^0\|_V \leq 2 \left( 1 + \frac{c_B^2}{d_{\tilde{\Omega}}^2} \right) d_{\tilde{\Omega}} \frac{1}{\mathcal{K}_{\min}} \|f\|_{0, \Omega}.$$

Therefore, inserting (2.31) in (2.30) we get

$$\|\mathcal{K}^{\frac{1}{2}} \nabla e_h\|_{0, \mathcal{T}_h} \leq C \frac{1}{\mathcal{K}_{\min}^{\frac{3}{2}}} \nu \|f\|_{0, \Omega},$$

which finishes the proof.  $\square$

**3. The Multiscale Hybrid Method.** A common feature in all the works related to the MHM method is a process of static condensation. The purpose of this feature is twofold; first, it allows for a natural upscaling to be carried out in parallel, and second, it leads in a natural way to a formulation whose only unknowns are the approximation of the fluxes in the interelement facets, that is, a global linear system with  $\lambda_H$  as the sole unknown. So, we now describe that process in the context of the present approach. We start by observing that the first equation of (2.17) can be rewritten as follows:

$$(3.1) \quad a(u_h, v_h) = -\langle \lambda_H, v_h \rangle_{\partial \mathcal{T}_h} + (f, v_h)_{\mathcal{T}_h}.$$

Due to the linearity of the above problem and the fact that  $a(\cdot, \cdot)$  is an elliptic bilinear form, the solution  $u_h \in V_h$  of (2.17) can be written in the following way

$$(3.2) \quad u_h = T_h \lambda_H + \hat{T}_h f,$$

where  $\hat{T}_h f \in V_h$  and  $T_h \rho \in V_h$  (for any  $\rho \in \Lambda$ ) are the unique solutions to the following discrete problems:

$$(3.3) \quad a(T_h \rho, v_h) = -\langle \rho, v_h \rangle_{\partial \mathcal{T}_h}, \text{ for all } v_h \in V_h,$$

$$(3.4) \quad a(\hat{T}_h f, v_h) = (f, v_h)_{\mathcal{T}_h}, \text{ for all } v_h \in V_h.$$

The ellipticity of  $a(\cdot, \cdot)$  in  $V_h$  yields the fact that the mappings  $T_h$  and  $\hat{T}_h$  are well-defined. The writing (3.2) constitutes a static condensation procedure. In fact, as

it will be detailed in [Remark 3.3](#), the mappings  $T_h$  and  $\hat{T}_h$  can be computed locally. These local computations allow the elimination of the degrees of freedom internal to each element, so that we can write the global problem only in terms of  $\lambda_H$ . The final step in the derivation of the Multiscale Hybrid (MH) method is to replace [\(3.2\)](#) in the second equation of [\(2.17\)](#) to arrive at the following equation involving only  $\lambda_H$ : Find  $\lambda_H \in \Lambda_H$  such that

$$(3.5) \quad \mathfrak{B}(\lambda_H, \mu_H) = \langle \mu_H, \hat{T}_h f \rangle_{\partial \mathcal{D}_{\mathcal{H}}}, \text{ for all } \mu_H \in \Lambda_H,$$

where the bilinear form  $\mathfrak{B}$  is given by

$$(3.6) \quad \mathfrak{B}(\rho_H, \mu_H) = - \langle \mu_H, T_h \rho_H \rangle_{\partial \mathcal{D}_{\mathcal{H}}}, \text{ for all } (\rho_H, \mu_H) \in \Lambda_H \times \Lambda_H.$$

The problems [\(3.5\)](#) and [\(2.17\)](#) are equivalent. Thus, Problem [\(3.5\)](#) has a unique solution  $\lambda_H \in \Lambda_H$ . Once  $\lambda_H$  is computed, then  $u_h$  is built using [\(3.2\)](#). As it will be detailed in the next section, this does not involve any extra computation (local or global). In addition, the error estimates proven in [subsection 2.2](#) for the solution of [\(2.17\)](#) are, of course, still valid for  $u_h$  built using [\(3.2\)](#).

The MH's global problem [\(3.5\)](#) can be proven well-posed directly. This is stated in the next result.

**THEOREM 3.1.** *Suppose Assumption (A1) and [\(2.14\)](#) to hold. Then, for all  $\rho_H, \mu_H \in \Lambda_H$  the following holds*

$$(3.7) \quad \mathfrak{B}(\rho_H, \mu_H) \leq \frac{2}{d_\Omega^2 \nu} \|\mu_H\|_\Lambda \|\rho_H\|_\Lambda, \text{ for all } (\rho_H, \mu_H) \in \Lambda_H \times \Lambda_H,$$

$$(3.8) \quad \mathfrak{B}(\mu_H, \mu_H) \geq \frac{1}{2 c_F^2 \mathcal{K}_{max}} \|\mu_H\|_\Lambda^2, \text{ for all } \mu_H \in \Lambda_H.$$

*Proof.* Let  $\mu_H \in \Lambda_H$ . Using the inf-sup condition [\(2.22\)](#) (which follows from (A1)) followed by [\(3.3\)](#), the continuity of  $a(\cdot, \cdot)$ , and [\(2.14\)](#) we get to:

$$(3.9) \quad \frac{1}{c_F} \|\mu_H\|_\Lambda \leq \sup_{v_h \in V_h} \frac{\langle \mu_H, v_h \rangle_{\partial \mathcal{D}_{\mathcal{H}}}}{\|v_h\|_V} = \sup_{v_h \in V_h} \frac{a(T_h \mu_H, v_h)}{\|v_h\|_V}$$

$$(3.10) \quad \leq \mathcal{K}_{\max}^{\frac{1}{2}} \left( \|\mathcal{K}^{\frac{1}{2}} \nabla T_h \mu_H\|_{0, \mathcal{D}_{\mathcal{H}}}^2 + \nu \|T_h \mu_H\|_{0, \mathcal{D}_{\mathcal{H}}}^2 \right)^{\frac{1}{2}}.$$

Then, using the definition of  $\mathfrak{B}$ , followed by [\(3.3\)](#), and the ellipticity of  $a(\cdot, \cdot)$  in [Theorem 2.2](#) we arrive at

$$(3.11) \quad \begin{aligned} \mathfrak{B}(\mu_H, \mu_H) &= - \langle \mu_H, T_h \mu_H \rangle_{\partial \mathcal{D}_{\mathcal{H}}} = a(T_h \mu_H, T_h \mu_H) \\ &\geq \frac{1}{2} \left( \|\mathcal{K}^{\frac{1}{2}} \nabla T_h \mu_H\|_{0, \mathcal{D}_{\mathcal{H}}}^2 + \nu \|T_h \mu_H\|_{0, \mathcal{D}_{\mathcal{H}}}^2 \right) \geq \frac{1}{2 c_F^2 \mathcal{K}_{\max}} \|\mu_H\|_\Lambda^2, \end{aligned}$$

which proves [\(3.8\)](#). Next, to obtain [\(3.7\)](#) we use again the ellipticity of  $a(\cdot, \cdot)$  followed by its continuity to obtain

$$\frac{1}{2} d_\Omega^2 \nu \|T_h \mu_H\|_V^2 \leq a(T_h \mu_H, T_h \mu_H) = - \langle \mu_H, T_h \mu_H \rangle_{\partial \mathcal{D}_{\mathcal{H}}} \leq \|\mu_H\|_\Lambda \|T_h \mu_H\|_V,$$

which proves the following continuity of  $T_h$

$$\|T_h \mu_H\|_V \leq \frac{2}{d_\Omega^2 \nu} \|\mu_H\|_\Lambda.$$

Hence,

$$\mathfrak{B}(\rho_H, \mu_H) = \langle \mu_H, T_h \rho_H \rangle_{\partial \mathcal{P}_h} \leq \|\mu_H\|_\Lambda \|T_h \lambda_H\|_V \leq \frac{2}{d_\Omega^2 \nu} \|\mu_H\|_\Lambda \|\rho_H\|_\Lambda,$$

thus finishing the proof.  $\square$

*Remark 3.2.* The fact that the matrix associated to  $\mathfrak{B}$  is symmetric positive-definite has multiple advantages from a computational point of view, the possibility of using faster Krylov solvers for its solution being one of them. In addition, it is important to remark that the ellipticity of the bilinear form  $\mathfrak{B}$  is independent of  $\nu$ . Of course,  $\mathfrak{B}$  is not defined for  $\nu = 0$ , but as long as  $\nu > 0$ , its ellipticity is uniform with respect to it. On the other hand, the lack of uniform stability of  $T_h$  with respect to  $\nu$  is represented by a continuity constant that blows up as  $\nu \rightarrow 0$ . This does not affect the error estimates proven before, as (3.5) is a rewriting of the hybrid formulation (2.17). Nevertheless, for small values of  $\nu$  the matrix associated to (3.5) may become ill-conditioned as  $\nu \rightarrow 0$ . We will study the extent to which this holds in the numerical experiments in section 4.

*Remark 3.3.* We finish this section by showing how to implement the MH method solving local problems, thus avoiding a global solution to compute the mappings  $T_h$  and  $\hat{T}_h$ . From (3.3) and (3.4), and thanks to the local definition of the space  $V_h$ , we can see that, in each  $K \in \mathcal{P}_h$ ,  $T_h \rho|_K \in V_h(K)$  and  $\hat{T}_h f|_K \in V_h(K)$  are the unique solutions of the following local problems:

$$(3.12) \quad a_K(T_h \rho, v_h) = -\langle \rho, v_h \rangle_{\partial K}, \quad \text{for all } v_h \in V_h(K),$$

$$(3.13) \quad a_K(\hat{T}_h f, v_h) = (f, v_h)_K, \quad \text{for all } v_h \in V_h(K),$$

where the local bilinear form  $a_K(\cdot, \cdot)$  is given by

$$(3.14) \quad a_K(u, v) := (\mathcal{K} \nabla u, \nabla v)_K + \langle (u \boldsymbol{\sigma}) \cdot \mathbf{n}^K, v \rangle_{\partial K}.$$

Problems (3.12) and (3.13) are well-posed. In fact, following exactly the same steps as in the proof of Theorem 2.2, we can see that the following holds

$$(3.15) \quad a_K(v, v) \geq \frac{1}{2} \left( \|\mathcal{K}^{\frac{1}{2}} \nabla v\|_{0,K}^2 + \nu \|v\|_{0,K}^2 \right), \quad \text{for all } v \in H^1(K).$$

**3.1. The MH-Algorithm.** The decomposition of (3.3) into the local problems (3.12), coupled with an appropriate selection of basis for  $\Lambda_H$ , provides an *embarrassingly parallel* process to compute the basis functions in the MH method (3.5). More precisely, if  $\{\psi_1, \dots, \psi_N\}$  is a basis for  $\Lambda_H$ , then we start by rewriting it in the following way

$$(3.16) \quad \{\psi_1, \dots, \psi_N\} = \bigcup_{F \in \mathcal{E}_H} \{\phi_1^F, \dots, \phi_{\ell+1}^F\},$$

where each  $\phi_1^F, \dots, \phi_{\ell+1}^F$  has support in each  $F \in \mathcal{E}$ . To give a concrete definition of  $\phi_j^F$  for each  $j = 1, \dots, \ell + 1$ , we shall fix a basis  $\{\varphi_1, \dots, \varphi_{\ell+1}\}$  for  $\mathbb{P}_\ell(F)$ , then we define  $\phi_j^F = (\mathbf{n}^F \cdot \mathbf{n}_F^K) \varphi_j$  for all  $j = 1, \dots, \ell + 1$ , i.e. if  $K_1^F, K_2^F \in \mathcal{P}_h$  are such that  $F = K_1^F \cap K_2^F$  and  $K_1^F \neq K_2^F$ , then  $\phi_j^F|_{\partial K'} = -\phi_j^F|_{\partial K}$ . Definition (3.16) induces the existence of a global-local bijection  $\psi_i \leftrightarrow \phi_k^F$ , i.e., it is possible to build a transformation from local to global indexes, denoted by

$$(3.17) \quad (k, F) \mapsto i(k, F),$$

where we have abused the notation using  $F$  instead of the index of  $F$  in some data structure associated to the mesh  $\mathcal{E}_H$ . Using (3.16) we rewrite the global problem (3.5) as the linear system: *Find*  $(c_1, \dots, c_N) \in \mathbb{R}^N$  *such that*

$$(3.18) \quad - \begin{pmatrix} \langle \psi_1, T_h \psi_1 \rangle_{\partial \mathcal{P}_H} & \cdots & \langle \psi_1, T_h \psi_N \rangle_{\partial \mathcal{P}_H} \\ \vdots & \ddots & \vdots \\ \langle \psi_N, T_h \psi_1 \rangle_{\partial \mathcal{P}_H} & \cdots & \langle \psi_N, T_h \psi_N \rangle_{\partial \mathcal{P}_H} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} \langle \psi_1, \hat{T}_h f \rangle_{\partial \mathcal{P}_H} \\ \vdots \\ \langle \psi_N, \hat{T}_h f \rangle_{\partial \mathcal{P}_H} \end{pmatrix}.$$

So, (3.2) can be rewritten as

$$(3.19) \quad u_h = \sum_{i=1}^N c_i T_h \psi_i + \hat{T}_h f.$$

Now, we will briefly analyse how to compute the entries  $\langle \psi_j, T_h \psi_i \rangle_{\partial \mathcal{P}_H}$ . As  $\psi_j$  is an element of the basis (3.16), its support is a subset of exactly one  $F \in \mathcal{E}_H$ . This element  $F$  is shared by at most two polygons  $\{K_1^F, K_2^F\} := \mathcal{P}_H^F \subseteq \mathcal{P}_H$ . Following a close construction we associate to  $\psi_i$  two polygons  $\{K_1^{F'}, K_2^{F'}\} := \mathcal{P}_H^{E'} \subseteq \mathcal{P}_H$  with  $F' \in \mathcal{E}_H$ . Using this writing we obtain

$$(3.20) \quad \langle \psi_j, T_h \psi_i \rangle_{\partial \mathcal{P}_H} = \sum_{K \in \mathcal{P}_H^F \cap \mathcal{P}_H^{E'}} \langle \psi_j, T_h \psi_i \rangle_{\partial K} = \sum_{K \in \mathcal{P}_H^F \cap \mathcal{P}_H^{E'}} \int_F \psi_j T_h \psi_i.$$

So, the sum above may be zero (when  $\mathcal{P}_H^F \cap \mathcal{P}_H^{E'} = \emptyset$ ), have only one term (when the edge  $F$  is on the boundary of  $\Omega$ ) or be composed by two terms. Just as in a classical finite element method implementation, we propose to approach (3.20) through contributions element-by-element with respect to the partition  $\mathcal{P}_H$ , i.e., for each  $K \in \mathcal{P}_H$ ,  $F, F' \subset \partial K$ ,  $k, m \in \{1, \dots, \ell + 1\}$ :

$$(3.21) \quad \int_F \phi_k T_h \phi_m \xrightarrow{\text{contributes to}} \langle \psi_{i(k,F)}, T_h \psi_{i(m,F')} \rangle_{\partial \mathcal{P}_H}$$

where the mapping  $i : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  was introduced in (3.17). Similarly, we notice that

$$(3.22) \quad \int_F \phi_k \hat{T}_h f \xrightarrow{\text{contributes to}} \langle \psi_{i(k,F)}, \hat{T}_h f \rangle_{\partial \mathcal{P}_H}.$$

The only part left to be explained in the MH algorithm is how the terms  $T_h \phi_m$  and  $\hat{T}_h f$  are computed on each  $K \in \mathcal{P}_H$ . Let  $\{\xi_1, \dots, \xi_{N_K}\}$  be a basis for  $V_h(K)$ . From the discrete variational problems (3.3) we see that  $T_h \psi_i$  can be written as  $T_h \psi_i = \sum_{k=1}^{N_K} t_k^{(i)} \xi_k$ , where  $(t_1^{(i)}, \dots, t_{N_K}^{(i)}) \in \mathbb{R}^{N_K}$  solves the following linear system:

$$(3.23) \quad \begin{pmatrix} a_K(\xi_1, \xi_1) & \cdots & a_K(\xi_{N_K}, \xi_1) \\ \vdots & \ddots & \vdots \\ a_K(\xi_1, \xi_{N_K}) & \cdots & a_K(\xi_{N_K}, \xi_{N_K}) \end{pmatrix} \begin{pmatrix} t_1^{(i)} \\ \vdots \\ t_{N_K}^{(i)} \end{pmatrix} = - \begin{pmatrix} \int_F \psi_i \xi_1 \\ \vdots \\ \int_F \psi_i \xi_{N_K} \end{pmatrix},$$

where  $F \in \mathcal{E}_H$  is the only element containing the support of  $\psi_i$ . Similarly, from (3.4) we obtain  $\hat{T}_h f|_{K_n} = \sum_{k=1}^{N_K} f_k \xi_k$  by solving

$$(3.24) \quad \begin{pmatrix} a_K(\xi_1, \xi_1) & \cdots & a_K(\xi_{N_K}, \xi_1) \\ \vdots & \ddots & \vdots \\ a_K(\xi_1, \xi_{N_K}) & \cdots & a_K(\xi_{N_K}, \xi_{N_K}) \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_{N_K} \end{pmatrix} = \begin{pmatrix} (f, \xi_1)_K \\ \vdots \\ (f, \xi_{N_K})_K \end{pmatrix}.$$

Finally, we notice that after solving the linear systems (3.23) and (3.24), the right hand side of (3.23) provides an easy way to compute the contributions (3.21) and (3.22) to assemble (3.18) through the following matrix-vector products

$$(3.25) \quad \int_F \psi_i T_h \psi_j = \sum_{k=1}^{N_K} t_k^j \int_F \psi_i \xi_k, \text{ and, } \int_F \psi_i \hat{T}_h f = \sum_{k=1}^{N_K} f_k \int_F \psi_i \xi_k.$$

Algorithm 3.1 summarizes the main steps explained above to compute an MH solution.

---

**Algorithm 3.1** Compute an MH solution.

---

**parfor**  $K \in \mathcal{P}_H$  **do**

Assemble the left-hand side  $\mathbb{A}$  of (3.23);

Assemble the right-hand side  $\mathbf{b}_f$  of (3.24);

**parfor**  $F \in \partial K$  **do**

**parfor**  $j = 1, \dots, \ell + 1$  **do**

Assemble the  $j$ -th column of the right-hand side  $\mathbb{B}_K(:, j)$  of (3.23);

**end parfor**

**end parfor**

Solve  $\mathbb{A} * [\mathbb{E}_K \mathbf{f}_K] = [\mathbb{B}_K \mathbf{b}_f]$ ; (see (3.23) and (3.24))

$-\mathbb{E}_K^T * \mathbb{B}_K \xrightarrow{\text{contributes to}} \mathbb{M}$ ; (left-hand side of (3.18), see also (3.21) and (3.25))

$\mathbf{f}_K^T * \mathbb{B}_K \xrightarrow{\text{contributes to}} \mathbf{L}$ ; (right-hand side of (3.18), see also (3.22) and (3.25))

**end parfor**

Solve  $\mathbb{M} * \mathbf{V} = \mathbf{L}$ ;

**parfor**  $K \in \mathcal{P}_H$  **do**

Extract  $\mathbf{V}_K$  from  $\mathbf{V}$ ; (coefficients related to  $K$ )

$\mathbf{u}_K = \mathbf{V}_K^T * \mathbb{E}_K + \mathbf{f}_K$ ; (see (3.19))

**end parfor**

---

**4. Computational Experiments.** In this section we set the unit square,  $\Omega = (0, 1)^2$ , as the domain of problem (1.1). We will present four series of numerical experiments showcasing the performance of the method from different points of view. To avoid any pollution from the solution of the local problems, we have always taken  $k = \ell + 2$ . So, the convergence will always be measured with respect to the order of approximation in  $\Lambda_H$ , namely  $\ell$ .

**4.1. Convergence Assessment.** We first test the convergence of the method for a problem with smooth data and solutions. We consider  $\mathcal{K} = \mathcal{I}$  and

$$f(x, y) = 4(n^2 + m^2)\pi^2 \sin(2n\pi x) \sin(2m\pi y), \text{ in } \Omega,$$

with  $n, m \in \mathbb{N}$ . So, the exact solution of (1.1) is given by

$$(4.1) \quad u(x, y) = \sin(2n\pi x) \sin(2m\pi y).$$

In the convergence results presented below we set  $n = 3$  and  $m = 7$  in the exact solution (4.1).

The first convergence test is carried out on a family of partitions composed by L-shaped polygons. In Figure 2 we depict the discrete solution for one particular mesh in the family, using  $H = \frac{1}{64}$  and  $h = \frac{1}{1024}$ . Thanks to our choice  $k = \ell + 2$  and



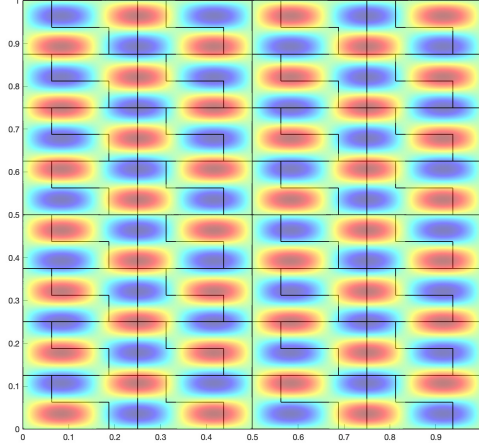


FIG. 2. Example of approximated solution obtained with the MH method on a mesh composed by L-shaped polygons. The shortest edge in each polygon has size  $\frac{1}{16}$  and the longest has size  $\frac{1}{8}$ . The computational edges in  $\mathcal{E}_H$  have size  $H = \frac{1}{64}$ . In addition,  $h = \frac{1}{1024}$ ,  $\ell = 1$ , and  $k = 3$ .

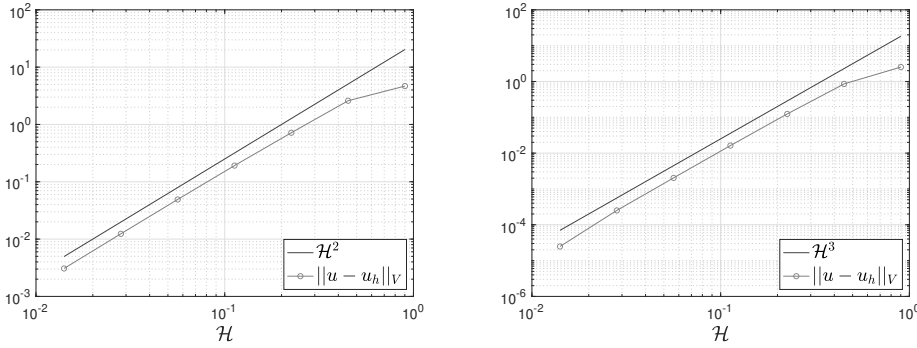


FIG. 3. Mesh based convergence for  $\nu = \frac{1}{4}$  and  $H = \mathcal{H}$ . Here it was used  $\ell = 1$  and  $k = 3$  (left) and  $\ell = 2$  and  $k = 4$  (right). The second level was computed with  $h = \frac{1}{256}$ .

the small value  $h = \frac{1}{256}$ , both chosen to avoid any pollution arising from the second level calculations on the convergence, [Corollary 2.7](#) predicts an  $O(\mathcal{H}^{\ell+1})$  convergence rate. The results shown in [Figure 3](#) reflect this optimal convergence rate both for  $\ell = 1, k = 3$  (left panel) and  $\ell = 2, k = 4$  (right panel). For this first experiment, the value  $\nu$  has been chosen equal to  $\frac{1}{4}$ , and the skeleton mesh  $\mathcal{E}_H$  coincides with  $\mathcal{E}$ , i.e.  $H = \mathcal{H}$ .

We next study the convergence behavior on  $H$  by fixing the partition  $\mathcal{P}_H$  as the mesh presented in [Figure 2](#) and making successive refinements on  $\mathcal{E}_H$  (referred to as *space-based* convergence in [\[6\]](#)). The value  $h = \frac{1}{256}$  is sufficiently small as to ensure that the second level calculations do not pollute the errors. The results presented in [Figure 4](#) show a *super-convergence* with respect to the theoretical estimate given

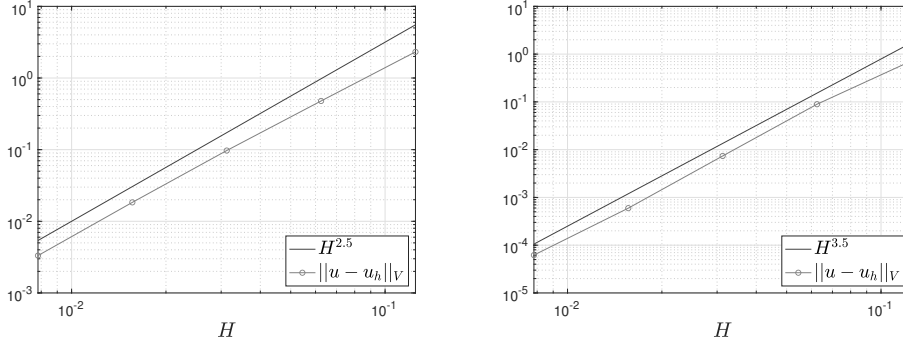


FIG. 4. Space based convergence for  $\mathcal{H} = \frac{1}{4}\sqrt{13}$  and  $\nu = \frac{1}{4}$ . The second level was computed with  $h = \frac{1}{256}$ . The error curves on the left side were made for  $\ell = 1$  and  $k = 3$ , whereas on the right side we present curves for  $\ell = 2$  and  $k = 4$ .

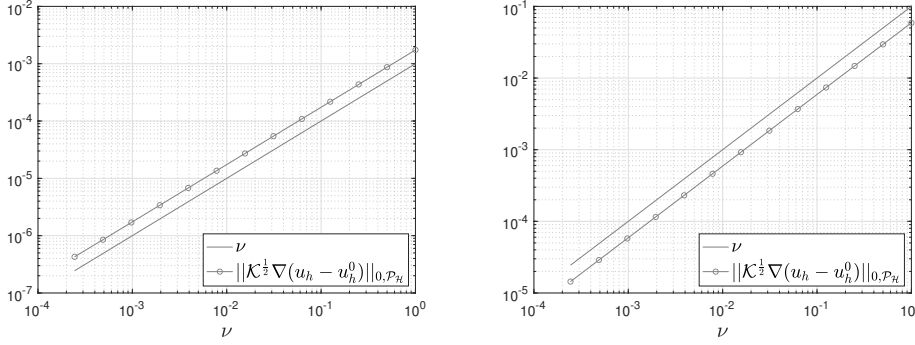


FIG. 5. Difference between the MH and MHM solutions as  $\nu \rightarrow 0$ . On the left panel we use a mesh using squares of size  $\mathcal{H} = \frac{1}{8}$ , with  $\ell = 1$ ,  $k = 3$ ,  $h = \frac{1}{8}\mathcal{H}$ ,  $H = \frac{1}{4}\mathcal{H}$ . On the right panel we used a criss-cross mesh composed by triangles with  $\mathcal{H} = \frac{1}{4}$ ,  $\ell = 2$ ,  $k = 4$ ,  $h = \frac{1}{16}\mathcal{H}$ ,  $H = \mathcal{H}$ .

by [Corollary 2.7](#). More precisely, both rates are one half higher than expected. Such behavior was observed in the MHM method in [\[6\]](#) and has been recently analyzed in [\[8\]](#).

Finally, we test the convergences presented in [Theorem 2.8](#). For this, we use two types of meshes. More precisely, we consider two fixed partitions  $\mathcal{P}_H$ , one composed of triangles and the other one of squares (see [Figure 5](#) for details). In [Figure 5](#) we report the evolution of the error  $\|\mathcal{K}^{\frac{1}{2}}\nabla(u_h - u_h^0)\|_{0, \mathcal{P}_H}$ , where we see that, as predicted by [Theorem 2.8](#), the decrease in the error is linear, thus confirming that the solution of the MH method does approach that of the MHM method as  $\nu \rightarrow 0$ .

#### 4.2. Conditioning with respect to $\nu$ and its numerical consequences.

In [section 3](#) we analyzed the stability of local and global problems, [\(3.3\)](#)-[\(3.4\)](#) and [\(3.5\)](#), respectively. This analysis reveals that the ellipticity of the local problems depends strongly on  $\nu$ , and that the norm of the bilinear form associated to the global problem grows up as  $\nu^{-1}$  when  $\nu \rightarrow 0$ . In this section we validate those claims numer-

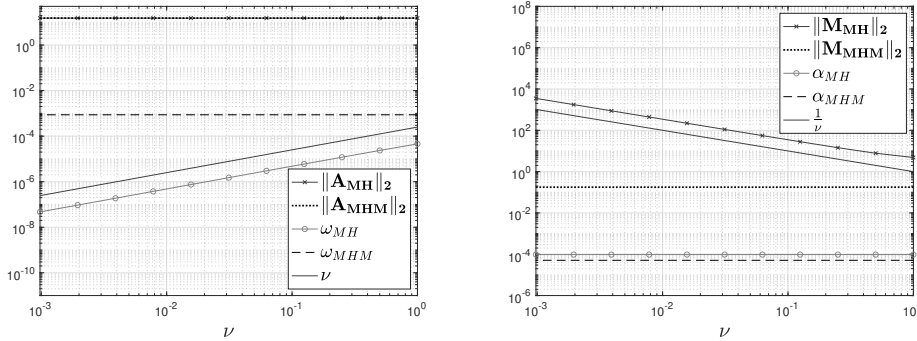


FIG. 6. Behaviour of ellipticity constants and norms for local (left) and global (right) systems for the presented method compared with same quantities obtained for the MHM method. The used configuration was  $\ell = 1$ ,  $H = \frac{\mathcal{H}}{4}$ ,  $h = \frac{\mathcal{H}}{8}$ , and,  $k = 3$ , with  $\mathcal{H} = \frac{1}{8}$ , on a criss-cross mesh.

ically by computing the largest and smallest singular values of the respective matrices obtained by approximating the solution to the problem described in subsection 4.1 with  $n = 3$  and  $m = 7$  and using the space settings detailed in Figure 6. We depict the obtained results in Figure 6, along with the same quantities obtained for the MHM method (the latter is added for reference, as the MHM method does not involve  $\nu$ ).

The results depicted in Figure 6 show that the MH method indeed produces matrices whose condition number is worse than those of the MHM method (the ellipticity constants are slightly better than those of the MHM method, but the continuity ones clearly blow up with  $\nu^{-1}$ , reproducing the fact that the MH method is not defined for  $\nu = 0$ ). These results raise the question about the influence of the value of  $\nu$  in the quality of the results provided by the MH method. From a theoretical point of view, the answer is clear. As a matter of fact, the convergence rates presented in Corollary 2.7 are not affected by the value of  $\nu$ . This implies, in particular, that the solution obtained by the MH algorithm (i.e., the function  $u_h$  computed using (3.2)) converges to the exact solution with rates (and values) that are independent of  $\nu$ . However, from a practical point of view, the degeneration of the condition number can affect the quality of the approximate solution to a discrete problem. To make a finer assessment on this situation, in Figure 8 we present the convergence curves for different choices of  $\nu$ . The results depicted in that figure show that the rates (and values) of the error are virtually unaffected with values of  $\nu$  larger than, or equal to,  $6 \times 10^{-8}$ . When  $\nu$  reaches that value, the condition number of the local and global problems start getting affected by the value of  $\nu$ , and this can affect the overall quality of the solution, as can be concluded from the error curve depicted using  $\nu = 3.7 \times 10^{-9}$ .

To complement the above discussion, in Figure 9 we depict the error of the MH method for a range of values of  $\nu$ , and also report the error obtained when solving the exact same problem with the MHM method. From those results we observe that for a moderate range of  $\nu$ , that is, for  $\nu$  ranging from  $10^{-8}$  to 1, the error of the MH method is similar or slightly smaller than that of the MHM method, while for smaller values of  $\nu$  the quality of the solution of the MH method deteriorates.

To summarise the above discussion, in our experience it is safe to use the MH method for values of  $\nu$  that are not smaller than  $10^{-6}$ .

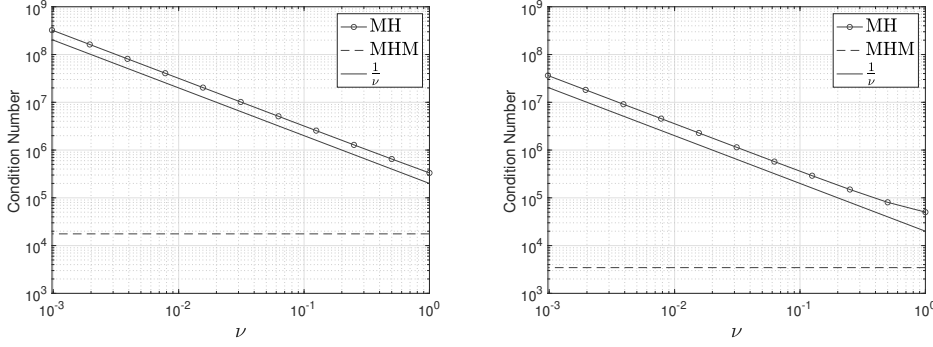


FIG. 7. Condition Numbers of the local (left) and global (right) linear varying with respect to  $\nu$  for the presented method compared with same quantities obtained for the MHM method. The used configuration was  $\ell = 1$ ,  $H = \frac{\mathcal{H}}{4}$ ,  $h = \frac{\mathcal{H}}{8}$ , and,  $k = 3$ , with  $\mathcal{H} = \frac{1}{8}$ , on a criss-cross mesh.

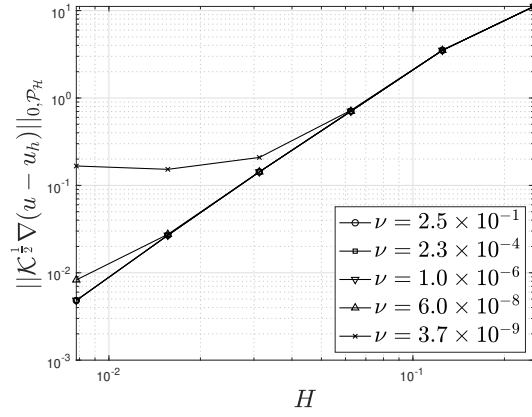


FIG. 8. Influence of  $\nu$  on the convergence rates with respect to  $H$ . Here it was used  $\ell = 1$ ,  $k = 3$ ,  $h = \frac{1}{64}\mathcal{H}$ , and,  $\mathcal{H} = \frac{1}{4}$ , on a criss-cross mesh.

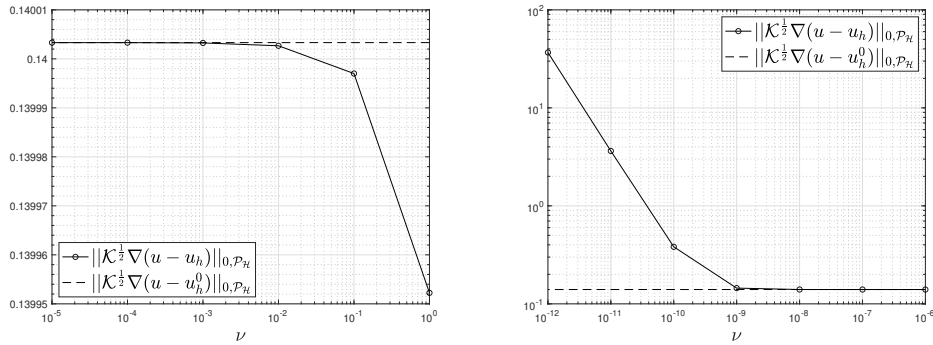


FIG. 9. Two regimes for the behaviour of the error obtained for the presented method compared with the error of the MHM method: Big (left) and small (right) values of  $\nu$ . The used configuration was  $\ell = 1$ ,  $k = 3$ ,  $h = \frac{\mathcal{H}}{16}$ , and,  $H = \frac{\mathcal{H}}{8}$ , with  $\mathcal{H} = \frac{1}{4}$ , on a criss-cross mesh.

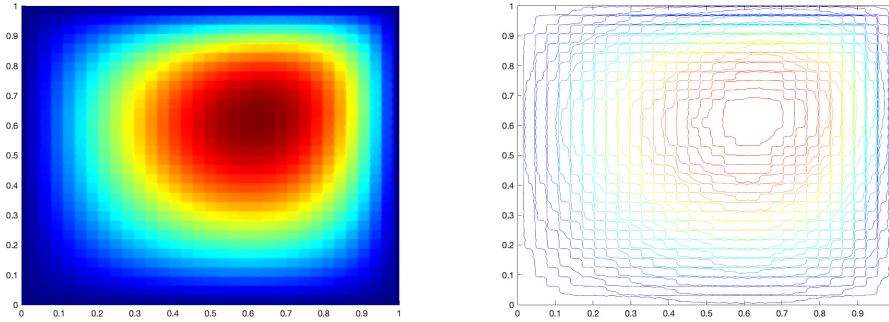


FIG. 10. Reference solution computed with the classical Galerkin method using  $\mathbb{Q}_3$  elements on a square mesh with 4,194,304 elements and 37,736,449 degrees of freedom. Isovalues (left) and levelset curves (right).

**4.3. Capture of Multiscale Features.** To showcase the capabilities of the presented method to capture the multiscale features of the problem we will approximate the solution to (1.1) under highly-oscillatory coefficients of periodicity  $\varepsilon > 0$  using an example proposed in [26, 27]. Specifically, we consider  $f(x, y) = \sin(x) \sin(y)$  and

$$\mathcal{K}(x, y) = 1 + 100 \cos^2\left(\frac{\pi x}{\varepsilon}\right) \sin^2\left(\frac{\pi y}{\varepsilon}\right),$$

both, for all  $(x, y) \in \Omega$ .

We first consider  $\varepsilon = \frac{1}{32}$ , and compute a reference solution using the Galerkin method with cubic polynomials in a highly refined mesh (containing 4,194,304 uniform quadrilateral elements). Figure 10 depicts the reference solution, where we can observe its highly oscillating character due to definition of  $\mathcal{K}$ . Then, we compute the MH approximation in a mesh containing 64 square elements, with spaces  $\Lambda_H$  and  $V_h$  using  $\ell = 1$ ,  $k = 3$ ,  $h = \frac{1}{64} \mathcal{H}$ ,  $H = \frac{1}{16} \mathcal{H}$ , and  $\mathcal{H} = \frac{1}{8}$ . The MH solution is depicted in Figure 11 where we can observe that it matches very well the oscillating character of the reference solution. (See also a zoom containing its isovalues and isolines in Figure 12.)

In order to test the convergence of the scheme in this case we consider  $\varepsilon = \frac{1}{16}$ . In Figure 13 we depict the error between the MH solution and the reference one as  $H \rightarrow 0$ . For this we have used a space-based approach (so, keeping  $\mathcal{H}$  fixed). We can observe in the left panel that the error decreases with the expected rate  $O(H^{2.5})$ . In addition, on the right panel of Figure 13 we depict the error for the largest value of  $H$  and a range of values of  $\nu$ . In there, we can observe that the error is very robust with respect to the value of  $\nu$ .

**4.4. Performance Analysis.** In this section we compare the computational performance of the MH method with that of the MHM method presented in [6]. Both methods were used for solving problem (1.1) with the same  $\Omega$ ,  $\mathcal{K}$  and  $f$  used in subsection 4.1 (see Figure 14 for details about the meshes and polynomial degrees used). The simulations were conducted using a C++ framework developed in-house for implementing finite-element solvers.<sup>1</sup> The framework implements the MH and MHM

<sup>1</sup>The code used for this performance analysis and some other experiments presented in this paper is freely available under request at [https://gitlab.com/ipes/msl\\_kernelfreediffusion](https://gitlab.com/ipes/msl_kernelfreediffusion).

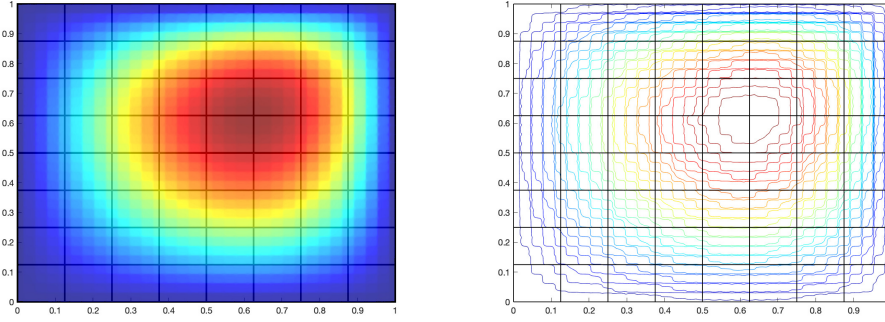


FIG. 11. *Approximated solution computed by the MH method for  $\varepsilon = \frac{1}{32}$ . Here,  $\ell = 1$ ,  $k = 3$ ,  $h = \frac{1}{64} \mathcal{H}$ ,  $H = \frac{1}{16} \mathcal{H}$ , with  $\mathcal{H} = \frac{\sqrt{2}}{8}$ , on a mesh composed by 64 squares.*

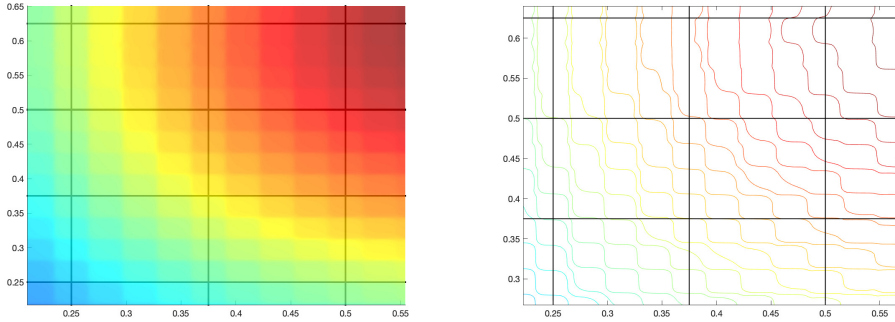


FIG. 12. *Zoom on the isovales (left) and levelset curves (right) presented in Figure 11.*

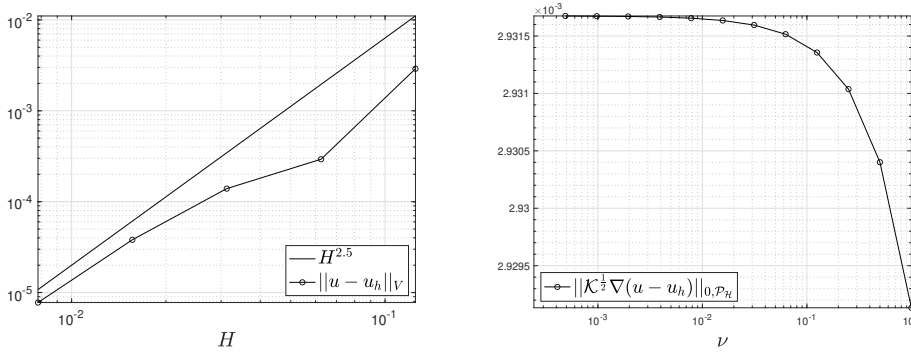


FIG. 13. *Convergence on  $H$  with respect to the reference (left) and influence of  $\nu$  on the approximation error for the MH method (right). Here it was used  $\ell = 1$ ,  $k = 3$ ,  $h = \frac{1}{64} \mathcal{H}$ , with  $\mathcal{H} = \frac{1}{8}$  on a mesh composed by squares.*

methods, and also the classical Galerkin method used for solving the local problems of both multiscale methods. The framework allows for multithreaded, shared memory parallelism using OpenMP, and multiprocessing, distributed memory parallelism using MPI. The framework is linked to two main libraries: (i) Eigen [16], for general matricial operations and also for solving the linear systems generated by the local problems; and (ii) MUMPS [1], for solving the linear system generated by the global problem.

For the experiments conducted in this paper, we used a high-performance cluster. Each node in this cluster has 2 CPUs of 24 cores each and 768 Gb of RAM memory. We considered three simulation configurations: running on a single node, on two nodes, and on four nodes. The computing system is configured so that each MPI process occupies a single CPU; therefore, our simulations run with 2, 4 and 8 MPI processes.

For each simulation, the local problems are uniformly partitioned between the MPI processes. Within each of these processes, the “**parfor**” loops shown in [Algorithm 3.1](#) are implemented with OpenMP so that the local problems pertaining to an MPI process are solved in an embarrassingly parallel way using all the available 24 cores in its CPU. Only the assembly of  $\mathbb{M}$  and  $\mathbf{L}$ , and the solution of  $\mathbb{M} * \mathbf{V} = \mathbf{L}$  in [Algorithm 3.1](#) employ tightly-coupled parallelism through MPI messages within the MUMPS library.

In the simulations with MHM, the LU factorization implementations of Eigen (for local problems) and MUMPS (for the global problem) are used. For the implementation of the current MH method, the symmetric positive-definite character of the local and global problems allows us to use the Choleski factorization implementations of Eigen (for local problems) and MUMPS (for the global problem).

The performance metrics used for the comparison between MH and MHM are: (i) the time to assemble and solve all local problems; (ii) the time to assemble and solve the global problem; and (iii) the maximum memory (resident set size – RSS) consumed by the most demanding MPI process in each simulation.

We start describing the case  $\ell = 0$ . The results depicted in [Figure 14](#) show that, as  $H$  decreases, the performance difference in terms of time needed to solve the global problem increases in favour of the MH method. Regarding memory consumption, the MHM method presents better performance for smaller values of  $H$ , but this situation changes very early in the refinement process. In fact, when  $H \approx 2.44 \times 10^{-4}$  the MHM method is already more memory consuming than the MH method.

To perform the same comparison for higher order elements, in [Figure 15](#) we depict the memory consumption for both the present MH and the MHM methods. From that figure it is clear that the MH method requires considerable less memory than the MHM method, both for  $\ell = 1$  and  $\ell = 2$ . In fact, for  $H \approx 2.44 \times 10^{-4}$  the MHM method could not be solved in 4 cluster nodes due to the lack of memory resources.

The last results were related to the solution of the whole problem, including both the online solution of the global problem, as well as the offline solution of the local ones. We now will take a closer look at the second ingredient. More precisely, we place ourselves in the setting described in [Figure 16](#), and measure the impact of the degree  $k$  used to solving the local problems both in the MH and the MHM methods. To achieve that, we have measured the time needed to solve all local problems in both methods. Interestingly, for lower values of  $k$  the local problems in the MHM method take slightly less time than those associated to the MH method. This can be explained as follows: the local problems for the MH method are symmetric and elliptic (as opposed to those of the MHM method that are indefinite), but they

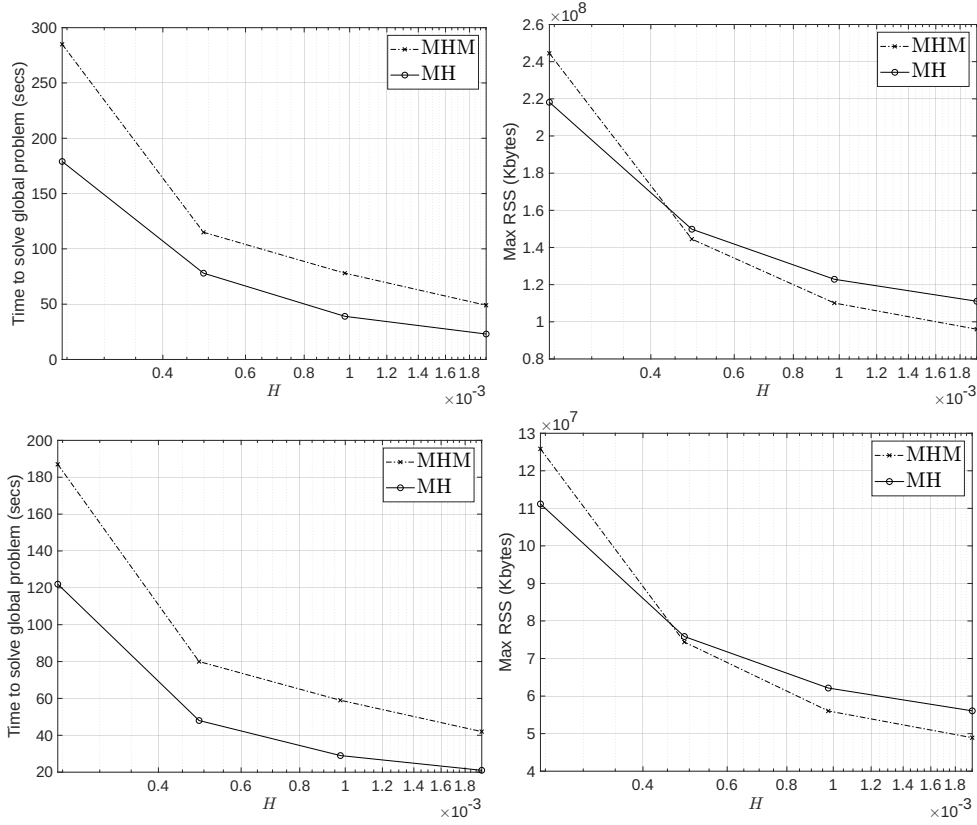


FIG. 14. Performance results with respect to  $H$ . Both the MH and MHM methods used a criss-cross mesh with  $\mathcal{H} = \frac{1}{512}$ ,  $\ell = 0$ ,  $k = 2$ , and  $h = \frac{H}{16}$ . For the MH method,  $\nu = \frac{1}{4}$ . Left side: Time to compute the global problem. Right side: Maximum RSS of the simulation. Two system configurations were used: 2 MPI processes (top); 4 MPI processes (bottom).

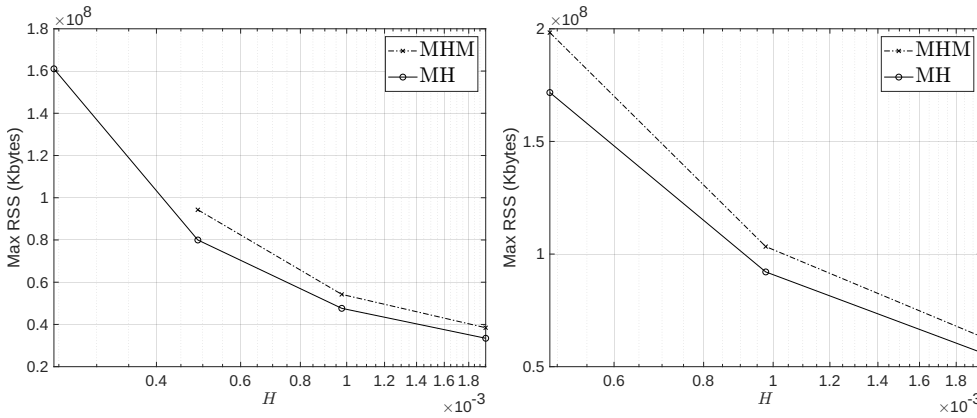


FIG. 15. Maximum RSS of the simulation with respect to  $H$ . Both the MH and MHM methods used a criss-cross mesh with  $\mathcal{H} = \frac{1}{512}$  in 8 MPI processes, and were configured with  $h = \frac{H}{16}$ . For the MH method,  $\nu = \frac{1}{4}$ . Left side:  $\ell = 1$ ,  $k = 3$ . Right side:  $\ell = 2$ ,  $k = 4$ .



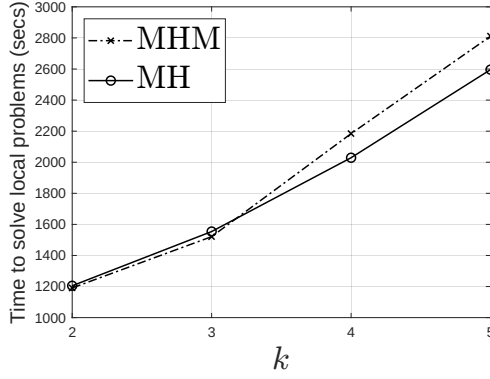


FIG. 16. Time to solve local problems with respect to  $k$ . Both the MH and MHM methods used a criss-cross mesh with  $\mathcal{H} = \frac{1}{512}$  in 4 MPI processes, and were configured with  $H = \mathcal{H}$ ,  $\ell = k - 2$  and  $h = \frac{\mathcal{H}}{16}$ . For the MH method,  $\nu = \frac{1}{4}$ .

require to assemble the term  $\langle (\boldsymbol{\sigma} \cdot \mathbf{n}^K) u_h, v_h \rangle_{\partial K}$  as well, which takes some more time. Now, once the polynomial degree  $k$  gets high, the symmetry and ellipticity of the local problems for the MH method starts paying off and as a consequence the total time computing the local problems for the MH method becomes lower than the corresponding time for the MHM method. These results can be seen in detail in Figure 16.

These performance results are indicative that the MH method is particularly adequate for challenging problems that demand more space-based refinement and/or higher-order polynomials. To support this argument, we conducted an additional set of experiments using  $n = 128$  and  $m = 256$  in the model problem presented in subsection 4.1, thus we consider the case of a highly oscillatory solution. Figure 17 shows that the MH method considerably reduces the amount of memory needed for reaching the same approximation error as the MHM method in this problem. Again, the MHM simulation could not complete in the most demanding configuration because of the lack of memory resources.

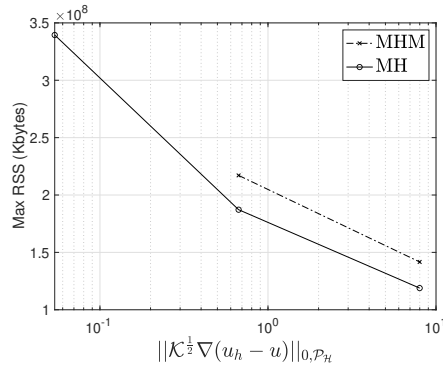


FIG. 17. Maximum RSS of the simulation with respect to  $\|\mathcal{K}^{\frac{1}{2}} \nabla(u_h - u)\|_{0, \mathcal{P}_H}$ . Both the MH and MHM methods used a criss-cross mesh with  $\mathcal{H} = \frac{1}{512}$  in 4 MPI processes, and were configured with  $\ell = 2$ ,  $k = 4$  and  $h = \frac{\mathcal{H}}{16}$ . In both methods,  $H$  varied from  $\mathcal{H}$  to  $\frac{\mathcal{H}}{4}$  to achieve the intended approximation errors. For the MH method,  $\nu = \frac{1}{4}$ .

**5. Conclusion and outlook.** In this work we presented an alternative way to build multiscale methods based on a hybrid formulation. The main advantage of this approach lies on the fact that both the global problem for the discrete Lagrange multipliers, as well as the local problems defining the basis functions, are elliptic. In addition, the approach followed to analyze the error, that is, interpreting the MH method as a reformulation of a hybrid formulation, is, up to our best knowledge, new (and could also be applied to analyze the MHM method itself). Interestingly, the error constants for the MH method are independent of the parameter  $\nu$ , and also the solution of the MH method was proven to converge to that of the MHM method when  $\nu \rightarrow 0$ . When exploring the performance of the method in this regime, one particular phenomenon emerged, namely, the deterioration of the condition number of the local and global problems for very small values of  $\nu$ . Thus, our advice to use this scheme with values of  $\nu$  that are not excessively small.

Several problems remain open at this point. From a theoretical point of view, the fact that both the local and global kernels are removed from the formulations can help to adapt the present framework to more involved models, such as the one considered in [9], where geometrical conditions had to be imposed on the mesh and coefficients to be able to deal with the presence of a local kernel. In addition, it is interesting to remark that, even if in practice we use a Robin boundary condition (depending on  $\nu$ ) for the local problems, the control provided by it is not the one we would expect from the use of a Robin condition (cf. the ellipticity in [Theorem 2.2](#)). So, a more detailed comparison between the present approach and others including a more *traditional* Robin condition is also of interest. Finally, from a numerical perspective, the relation between the parameter  $\nu$  and the condition number of both the local and global problems needs to be studied more in detail, and alternatives to overcome this possible issue need to be clarified. These and other topics are currently underway and will be the subject of future research.

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