



**HAL**  
open science

# Standing waves for semilinear Schrödinger equations with discontinuous dispersion

Olivier Goubet, Imen Manoubi

► **To cite this version:**

Olivier Goubet, Imen Manoubi. Standing waves for semilinear Schrödinger equations with discontinuous dispersion. *Rendiconti del Circolo Matematico di Palermo*, 2022, 71 (3), pp.1159-1171. 10.1007/s12215-022-00782-3 . hal-03906178

**HAL Id: hal-03906178**

**<https://hal.science/hal-03906178v1>**

Submitted on 19 Dec 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# STANDING WAVES FOR SEMILINEAR SCHRÖDINGER EQUATIONS WITH DISCONTINUOUS DISPERSION

Olivier Goubet and Imen Manoubi

O. Goubet Laboratoire Paul Painlevé CNRS UMR 8524, équipe projet INRIA PARADYSE,  
Université de Lille, 59 655 Villeneuve d'Ascq cedex. olivier.goubet@univ-lille.fr

I. Manoubi Laboratoire Multifractales et Ondelettes, Av. de l'environnement, 5000 Monastir, Tunisie ISSAT-Université de Gabès, Avenue Omar Ibn El Khattab, 6029 Gabès, Tunisie. imen.manoubi@yahoo.fr

ABSTRACT. We consider here semilinear Schrödinger equations with a non standard dispersion that is discontinuous at  $x = 0$ . We first establish the existence and uniqueness of standing wave solutions for these equations. We then study the orbital stability of these standing wave into a subspace of the energy space that where classical methods as the concentration-compactness method of P.L. Lions can be used.

## 1. INTRODUCTION

We consider here a class of Schrödinger equations that read

$$(1) \quad u_t - i\partial_x(\nu(x)\partial_x u) - ig(|u|^2)u = 0,$$

where either  $g(\xi) = |\xi|^{\frac{p-1}{2}}$  for  $p > 1$  (pure power case) or  $g(\xi) = \log(\xi)$  (logarithmic case; we do not address here the case  $g(\xi) = -\log(\xi)$  as in [7]). The unknown  $u(t, x)$  maps  $\mathbb{R} \times \mathbb{R}$  into  $\mathbb{C}$ .

These semilinear Schrödinger equations are models in nonlinear optics (see [9] and the references therein). The classical case is when  $\nu(x) = 1$ . Here we are interested in the case of some impurity in the material that affects the propagation of the wave; here we model the impurity as follows: we consider a discontinuity at  $x = 0$  that reads  $\nu(x) = \nu_+ > 0$  if  $x > 0$  and  $\nu_- > 0$  if  $x < 0$ . To fix ideas we assume in the sequel that  $0 < \nu_- < \nu_+$ .

Various models of impurity has been studied in the literature. Let us first point out the case where the classical dispersion operator  $i\partial_x^2 u$  is perturbed by a Dirac mass at 0 and replaced by  $i\partial_x^2 u + iZu\delta_0$  for some constants  $Z$  (see [4], [11], [12], [14], [15], [16], [18] and the references therein). In the present article we have a singularity at the origin that involves the first space derivative of  $u$  at  $x = 0$ . The problem of the study of the corresponding linear operator (with  $\nu(x)$  non constant) has been addressed in [5] and [6]. Let us point out that our problem differs from the one where we have rough

coefficients, that are time dependent but space independent, in front of the dispersion operator (see [2], [10] and the references therein).

We are interested here in *standing wave* solutions to Schrödinger equations (1) that read  $u(t, x) = \exp(i\omega t)\varphi(x)$  where  $\varphi(x)$  is the profile of the solution. In the classical case where  $\nu_+ = \nu_-$ , solutions are unique up to a multiplication by a complex number of modulus 1 and up to space translations; actually these solutions read  $u(t, x) = \exp(i\omega t + i\theta)\varphi(x - y)$  for any  $y, \theta$  in  $\mathbb{R}$  where the profile  $\varphi$  is uniquely determined (see [9] and the references therein). Moreover these standing wave solutions are *orbitally stable* in the  $H^1$  subcritical case ( $p < 5$ ) in the pure power case and in the logarithmic case, i.e. stable up to the transformations that leave the equation invariant (see the definition below in our case or in [9] in the classical case).

Here we study the existence and uniqueness of standing wave solutions for (1), and the issue of orbital stability for these particular solutions. Since our problem is not invariant by space translations our case differs from the classical one.

Before stating the main result of the article we need to introduce some notations. Consider the bilinear form defined as

$$(2) \quad (u, v) \mapsto \operatorname{Re} \int_{\mathbb{R}} \nu(x) u_x(x) \bar{v}_x(x) dx.$$

This bilinear form is symmetric continuous coercive in  $H^1(\mathbb{R})$ . Setting

$$(Au, v)_{L^2(\mathbb{R})} = \operatorname{Re} \int_{\mathbb{R}} \nu(x) u_x(x) \bar{v}_x(x) dx,$$

then the domain of  $A$  is

$$\{u \in H^1(\mathbb{R}); v \mapsto (Au, v)_{L^2(\mathbb{R})} \text{ is continuous in } L^2(\mathbb{R})\}.$$

Then the associated unbounded  $A = -\partial_x(\nu(x)\partial_x \cdot)$  operator satisfies  $D(A^{\frac{1}{2}}) = H^1(\mathbb{R})$  and it is standard to prove that its domain is

$$(3) \quad D(A) = \{u \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{0\}); \nu_+ u_x(0^+) = \nu_- u_x(0^-)\}.$$

**Remark 1.1.** *Our problem can be understood as follows: we solve two equations respectively for  $x < 0$  and  $x > 0$  that read respectively*

$$(4) \quad u_t - i\nu_{\mp} \partial_x^2 u - ig(|u|^2)u = 0,$$

*and we couple these equations by transmission condition of Neumann type  $\nu_+ u_x(0^+) = \nu_- u_x(0^-)$ .*

We seek a standing wave solution whose profile  $\varphi$  belongs to  $D(A)$ . Our first result reads as follows

**Theorem 1.2.** *Up to a multiplication by a complex number of modulus 1, there exists a unique solution of (1). Moreover this solution reads  $\varphi(x) = \psi(\sqrt{\nu(x)}x)$  where  $\psi$  is the classical (even) solution for  $\nu(x) = 1$ .*

We then address the issue of orbital stability for the equation. Orbital stability reads as follows for the standing wave in some Banach space  $E$ .

**Definition 1.3.** *The standing wave is orbitally stable in  $E$  if for any  $\varepsilon > 0$  there exists  $\alpha > 0$  such that for  $u_0$  in  $E$  that satisfies  $\|u_0 - \varphi\|_E < \alpha$  we have that  $\inf_{\theta \in \mathbb{R}} \|u(t)e^{i\theta} - \varphi\|_E < \varepsilon$ , where  $u(t)$  is the solution of (1) that starts at  $t = 0$  from initial data  $u_0$ .*

This definition means that up to the invariance of the equation (here the multiplication by a complex number of modulus 1) if the initial data are close the orbits  $\{e^{i\omega t}\varphi\}$  and  $\{u(t)\}$  remain close to each other. Proving the orbital stability requires to establish that the profile is solution to some minimization problem. We will see in the sequel that we cannot establish such a result in the energy space  $H^1(\mathbb{R})$  (or in the suitable Orlicz space in the logarithmic case); actually our problem is not translation invariant in space. Therefore we have to restrict the energy space to a closed subspace  $H$  that contains the profile. We define  $H = \{u \in H^1(\mathbb{R}); x \mapsto u(\sqrt{\nu(x)}x) \text{ is even}\}$ . Our second main result states as follows

**Theorem 1.4.** *In the pure power case if  $p < 5$  or in the logarithmic case, the standing wave solution of (1) is orbitally stable (in  $H$  in the pure power case or in a suitable Orlicz space in the logarithmic case).*

This article is organized as follows. The next section is devoted to prove Theorem 1.2 either in the pure power case or in the logarithmic case. In a last section we introduce the new mathematical framework, i.e. the space of functions that have some symmetries, and we prove Theorem 1.4 following the guidelines in [9] for the pure power case and in [3] for the logarithmic case.

We complete this introduction by some notations. We set  $\log_+ x = \max(0, \log x)$  and  $\log_- x = -\min(0, \log x)$ . We set  $o(1)$  for any sequence of real numbers that converges to 0. Since we are dealing we complex valued functions the scalar product on  $L^2(\mathbb{R})$  reads

$$(5) \quad (u, v)_{L^2(\mathbb{R})} = \operatorname{Re} \int_{\mathbb{R}} u(x)\bar{v}(x)dx.$$

For a non negative function  $\mu(x)$  the weighted space  $L^2(\mu)$  is endowed with the scalar product

$$(6) \quad (u, v)_{L^2(\mu)} = \operatorname{Re} \int_{\mathbb{R}} u(x)\bar{v}(x)\mu(x)dx.$$

For any function  $u$ , we write either  $\partial_x u$  or  $u'$  for the space derivative of  $u$ .

## 2. EXISTENCE OF STANDING WAVE SOLUTIONS

**2.1. Mathematical framework.** Let us first recall that  $D(A^{\frac{1}{2}}) = H^1(\mathbb{R})$  and that  $(Au, u)_{L^2(\mathbb{R})}$  defines a seminorm that is equivalent to the Poincaré seminorm, i.e. since  $\nu_- < \nu_+$

$$(7) \quad \nu_- \int_{\mathbb{R}} |u'(x)|^2 dx \leq (Au, u)_{L^2(\mathbb{R})} \leq \nu_+ \int_{\mathbb{R}} |u'(x)|^2 dx.$$

We now recall some results about the initial value problem associated with (1). For the pure power case, the classical method for  $\nu(x) = 1$  as described in [9] works straightforwardly. For  $p < 5$  (the so-called  $H^1$ -subcritical case) for any initial data in  $H^1(\mathbb{R})$  there exists a unique global solution  $u(t)$  in  $C(\mathbb{R}; H^1(\mathbb{R}))$ . Moreover the *mass*  $\|u(t)\|_{L^2(\mathbb{R})}$  and, setting  $G(\xi) = \int_0^\xi g(s) ds$ , the *energy*

$$(8) \quad E(u) = \int_{\mathbb{R}} \nu(x) |u'(x)|^2 dx - \int_{\mathbb{R}} G(|u(x)|^2) dx,$$

are conserved along the flow of solutions. For the logarithmic case, the method introduced in [8] gives that the initial value problem is well posed for initial data in  $H^1(\mathbb{R}) \cap W$  where  $W$  is the Orlicz space

$$W = \{u \in L^1_{loc}(\mathbb{R}); \int_{\mathbb{R}} |u(x)|^2 |\log(|u(x)|)| dx < +\infty\}.$$

See [1] for details. Once again the mass and the energy are conserved along the flow.

**2.2. The pure power case.** We prove Theorem 1.2 in the case  $g(\xi) = \xi^{\frac{p-1}{2}}$ . Setting  $u(t, x) = \exp(i\omega t)\varphi(x)$  in (1) with polynomial nonlinearity leads to the following equation for the profile  $\varphi$

$$(9) \quad -\omega\varphi + (\nu\varphi')' + |\varphi|^{p-1}\varphi = 0.$$

For  $x > 0$ , we multiply by  $\bar{\varphi}'$  and integrate the real part of the resulting equation between  $x$  and  $+\infty$ . Since  $\varphi$  and  $\varphi'$  vanish at  $+\infty$  we obtain

$$(10) \quad -\omega|\varphi(x)|^2 + \nu_+|\varphi'(x)|^2 + \frac{2}{p+1}|\varphi(x)|^{p+1} = 0.$$

Due to the embedding  $H^1(\mathbb{R}) \subset C^{\frac{1}{2}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$  then  $\varphi$  is continuous at  $x = 0$ . Thus we infer from (10) and from its analogous for  $x < 0$  that

$$(11) \quad \nu_-|\varphi'(0^-)|^2 = \nu_+|\varphi'(0^+)|^2.$$

Since  $\varphi$  belongs to  $D(A)$  then the following transmission condition

$$(12) \quad \nu_- \varphi'(0^-) = \nu_+ \varphi'(0^+)$$

is valid. Using the fact that  $\nu_- < \nu_+$ , then from (11) and (12) we deduce that  $\varphi'(0^-) = \varphi'(0^+) = 0$ . Therefore  $\varphi$  is continuously differentiable on  $\mathbb{R}$ . Let us observe that setting  $x = 0$  in (10) leads to  $\omega > 0$ .

We now set  $\psi(x) = \varphi(\sqrt{\nu(x)}x)$ . Then  $\psi$  is solution to the ODE, defined for  $x \in \mathbb{R}$ , such that  $\psi'(0) = 0$ ,

$$(13) \quad -\omega\psi + \psi'' + |\psi|^{p-1}\psi = 0.$$

Appealing the results for the classical case  $\nu(x) = 1$  (see [9] and the references therein) we have that  $\psi$  (up to a multiplication by a complex number of modulus 1 and a space translation) is uniquely determined. Since  $\psi'(0) = 0$  then there is no translation and the proof of the theorem is completed.

**2.3. The logarithmic case.** We prove Theorem 1.2 in the case  $g(\xi) = \log \xi$ . Setting  $u(t, x) = \exp(i\omega t)\varphi(x)$  in (1) with logarithmic nonlinearity leads to the following equation for the profile  $\varphi$

$$(14) \quad -\omega\varphi + (\nu\varphi')' + \log(|\varphi|^2)\varphi = 0.$$

For  $x > 0$ , we multiply by  $\bar{\varphi}'$  and integrate the real part of the resulting equation between  $x$  and  $+\infty$ . Since  $\varphi$  and  $\varphi'$  vanish at  $+\infty$  we obtain

$$(15) \quad -\omega|\varphi(x)|^2 + \nu_+|\varphi'(x)|^2 + |\varphi(x)|^2(\log(|\varphi(x)|^2) - 1) = 0.$$

Due to the embedding  $H^1(\mathbb{R}) \subset C^{\frac{1}{2}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$  then  $\varphi$  is continuous at  $x = 0$ . Thus we infer from (15) and from its analogous for  $x < 0$  that  $\nu_-|\varphi'(0^-)|^2 = \nu_+|\varphi'(0^+)|^2$ . Since  $\varphi$  belongs to  $D(A)$  then  $\nu_- \varphi'(0^-) = \nu_+ \varphi'(0^+)$  and then  $\varphi'(0^-) = \varphi'(0^+) = 0$ . Therefore  $\varphi$  is continuously differentiable on  $\mathbb{R}$ .

We now set  $\psi(x) = \varphi(\sqrt{\nu(x)}x)$ . Then  $\psi$  is solution to the ODE, defined for  $x \in \mathbb{R}$ , such that  $\psi'(0) = 0$ .

$$(16) \quad -\omega\psi + \psi'' + \psi \log(|\psi|^2) = 0.$$

Appealing the results for the classical case  $\nu(x) = 1$  (see [9] and the references therein) we have that  $\psi$  (up to a multiplication by a complex number of modulus 1 and a space translation) is uniquely determined. Since  $\psi'(0) = 0$  then there is no translation then  $\psi(x) = \exp(i\theta + \frac{1+\omega}{2}x - \frac{x^2}{2})$  and the proof is completed.

### 3. STABILITY OF STANDING WAVES

**3.1. A new mathematical framework.** The classical method to prove the orbital stability is to prove that the profile is solution to a minimization problem. We first have the following negative result

**Proposition 3.1.** *There is no function  $u$  that achieves the infimum of the energy function  $E(u)$  under the constraint  $\|u\|_{L^2} = 1$ .*

*Proof.* Let us argue by contradiction. Consider  $u$  that achieves the infimum. Consider for  $y$  in  $\mathbb{R}$  the translate of  $u$  that reads  $u_y(x) = u(x + y)$ . Then  $u_y$  has the same mass than  $u$  but

$$(17) \quad E(u_y) = E(u) + (\nu_+ - \nu_-) \int_0^y |u'(x)|^2 dx.$$

We then can chose  $y$  such that  $E(u_y) < E(u)$  and then the contradiction occurs.  $\square$ .

Therefore it is difficult to prove the orbital stability of the standing wave among all functions in the energy space. For this purpose we change the framework as follows. We introduce the change of variable  $u(t, x) = v(t, \frac{x}{\sqrt{\nu(x)}})$ . Hence (1) yields

$$(18) \quad iv_t + \frac{1}{\sqrt{\nu}} \partial(\sqrt{\nu} \partial v) + g(|v|^2)v = 0,$$

associated to the Hilbert space  $L^2(\sqrt{\nu})$ . It is worth to point out that the unbounded operator  $v \mapsto \frac{1}{\sqrt{\nu}} \partial(\sqrt{\nu} \partial v)$  is self-adjoint in  $L^2(\sqrt{\nu})$ . We then introduce

$$(19) \quad H_0 = \{v \in L^2(\sqrt{\nu}); v(x) = v(-x)\}.$$

The space  $H_0$  is an Hilbert space that contains the profile associated to the standing wave. Moreover if the initial data  $v_0$  belongs to  $H = H_0 \cap H^1(\mathbb{R})$  (or to  $H \cap W$  in the logarithmic case) the solution  $v(t)$  of (18) belongs also to  $H$  (or to  $H \cap W$  in the logarithmic case). Then we shall prove the orbital stability in the space  $H$ . For the new equation (18) the new energy that reads

$$E(v) = \int_{\mathbb{R}} |v'(x)|^2 \sqrt{\nu(x)} dx - \int_{\mathbb{R}} G(|v(x)|^2) \sqrt{\nu(x)} dx,$$

and the mass  $\|v\|_{H_0} = \|v\|_{L^2(\sqrt{\nu})}$  are invariant along the flow of solutions.

**3.2. Proof of Theorem 1.4 in the pure power case.** We assume in this section that  $p < 5$ , i.e. that we are in the  $H^1(\mathbb{R})$  subcritical case. To begin with we state and prove

**Proposition 3.2.** *The profile  $\varphi$  achieves the minimum of the energy  $E(v)$  among all functions in  $H$  that satisfies  $\|v\|_{H_0} = \|\varphi\|_{H_0}$ . The set of all minimizers reads  $\{\varphi(x)e^{i\theta}\}$  for  $\theta$  a real number.*

*Proof.* Introduce  $C = \{v \in H; \|v\|_{H_0} = \|\varphi\|_{H_0}\}$ . Let us observe that if  $v$  belongs to  $H$

$$(20) \quad E(v) = \frac{\sqrt{\nu_-} + \sqrt{\nu_+}}{2} \left( \int_{\mathbb{R}} |v'(x)|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}} |v(x)|^{p+1} dx \right).$$

Since

$$(21) \quad \int_{\mathbb{R}} |v(x)|^{p+1} \leq \|v\|_{L^2(\mathbb{R})}^{\frac{p+3}{2}} \|v'\|_{L^2(\mathbb{R})}^{\frac{p-1}{2}},$$

and  $\frac{p-1}{2} < 2$  we have that for  $\|v\|_{H_0} = \|\varphi\|_{H_0}$  the energy  $E(v)$  is bounded by below and  $\mu = \inf_C E(v) > -\infty$ .

**Remark 3.3.** *The estimate (21) above is instrumental to prove that if  $p < 5$  the solutions cannot blow up in  $H^1(\mathbb{R})$  and then that the solutions are global, i.e. defined for any  $t \in \mathbb{R}$ . This is the reason why this is called the  $H^1$  subcritical case.*

Let us prove that  $\mu < 0$ . The set  $C$  is invariant by dilation  $\varepsilon \mapsto \sqrt{\varepsilon}v(\varepsilon x) = v_\varepsilon(x)$ . Since, for a given  $v$  in  $C$

$$(22) \quad E(v_\varepsilon) = \frac{\sqrt{\nu_-} + \sqrt{\nu_+}}{2} \left( \varepsilon^2 \int_{\mathbb{R}} |v'(x)|^2 dx - \frac{2}{p+1} \varepsilon^{\frac{p-1}{2}} \int_{\mathbb{R}} |v(x)|^{p+1} dx \right)$$

we have that for  $\varepsilon > 0$  small enough  $E(v_\varepsilon) < 0$  and then  $\mu < 0$ . Consider now  $\{v_k\}$  a minimizing sequence, i.e. such that  $v_k$  belongs to  $C$  and satisfies

$$(23) \quad \mu \leq E(v_k) < \mu + o(1)$$

when  $k$  diverges to  $\infty$ . Let us observe that since  $\mu < 0$  we have that

$$(24) \quad \liminf \int_{\mathbb{R}} |v_k|^{p+1} = \alpha > 0.$$

Observing that  $|v_k|$  is also a minimizing sequence we may assume that  $v_k$  takes values in  $[0, +\infty)$ . We now apply the concentration-compactness lemma of P.L. Lions (see section 1.7 in [9]). Up to a subsequence extraction the following alternative occurs

- (1) Vanishing:  $v_k \rightarrow 0$  in  $L^{p+1}(\mathbb{R})$ .
- (2) Dichotomy:  $v_k$  splits into two parts  $a_k, b_k$  ( $|v_k| \geq |a_k| + |b_k|$ ) whose supports are disjoint, such that  $\liminf_k (E(a_k) + E(b_k) - E(v_k)) \leq 0$ , and such that there exists  $\theta$  in  $(0, 1)$  such that  $\|a_k\|_{H_0}^2 = \|\varphi\|_{H_0}^2 \theta + o(1)$  and  $\|b_k\|_{H_0}^2 = \|\varphi\|_{H_0}^2 (1 - \theta) + o(1)$ .
- (3) Compactness: there exists  $y_k$  in  $\mathbb{R}$  such that  $v_k(x - y_k)$  converges in  $L^2(\mathbb{R})$  strongly to a function  $v$  that is non zero.

Vanishing is impossible due to (24). Let us prove that Dichotomy is impossible. Actually writing  $E\left(\frac{\|\varphi\|_{H_0}}{\|a_k\|_{H_0}} a_k\right) \geq \mu$  we have

$$(25) \quad \mu \frac{\|a_k\|_{H_0}^2}{\|\varphi\|_{H_0}^2} \leq \frac{\nu_- + \nu_+}{2} \int_{\mathbb{R}} |a'_k(x)|^2 dx - \frac{2}{p+1} \frac{\|a_k\|_{H_0}}{\|\varphi\|_{H_0}}^{1-p} \int_{\mathbb{R}} |a_k|^{p+1}(x) dx.$$

Gathering with the analogous estimate for  $b_k$  we have



$$(26) \quad \mu \left( \frac{\|a_k\|_{H_0}^2 + \|b_k\|_{H_0}^2}{\|\varphi\|_{H_0}^2} \right) \leq \frac{\nu_- + \nu_+}{2} \int_{\mathbb{R}} (|a'_k(x)|^2 + |b'_k(x)|^2) dx - \frac{2}{p+1} \min \left( \left( \frac{\|a_k\|_{H_0}}{\|\varphi\|_{H_0}} \right)^{1-p}, \left( \frac{\|b_k\|_{H_0}}{\|\varphi\|_{H_0}} \right)^{1-p} \right) \int_{\mathbb{R}} (|a_k|^{p+1}(x) + |b_k|^{p+1}(x)) dx.$$

Therefore passing to the limit when  $k$  diverges towards  $\infty$  we have the following contradiction

$$(27) \quad \mu + o(1) \leq \mu + \frac{2}{p+1} (1 - \min(\theta^{p-1}, (1-\theta)^{1-p})) \alpha < \mu.$$

Therefore we have compactness and the minimum is achieved; to check this assertion let us define the symmetric rearrangement  $v_k^*$  of  $x \mapsto v_k(x-y_k)$  that is an even function that satisfies, since the rearrangement map conserves the  $L^2$  norm and decays the Poincaré seminorm (see [17])

$$\mu \leq E(v_k^*) \leq E(v_k) \leq \mu + o(1),$$

and  $v_k^*$  converges towards  $v^*$  strongly in  $L^2(\mathbb{R})$ , since the rearrangement map is a contraction in  $L^2$  (see [17]). Then  $v^*$  belongs to  $H$  and achieves the minimum of the energy. This proves that the set of minimizers is non empty.

For any minimizer  $v$ , there exists then a Lagrange multiplier  $\omega$  such that

$$(28) \quad E'(v) = \omega v.$$

Then  $v$  solves the equation in section 2.2. Then  $v$  is equal to  $e^{i\theta}\varphi$  that has the same energy than  $\varphi$ . The proof of the proposition is then completed.  $\square$

We now complete the proof of the orbital stability. Let us argue by contradiction. Assume that there exists a sequence  $v_k$  in  $H^1(\mathbb{R})$ , two sequences of real numbers  $t_k, \theta_k$  such that for  $v_k(t)$  the solution of (18) starting from  $v_k$  at  $t = 0$

$$(29) \quad \begin{aligned} \|\varphi - v_k\|_{H^1(\mathbb{R})} &= o(1), \\ \|\varphi - v_k(t_k)e^{i\theta_k}\|_{H^1(\mathbb{R})} &\geq \varepsilon > 0. \end{aligned}$$

Let us observe that due to the first equation in (29) the sequence  $\tilde{v}_k = \frac{\|\varphi\|_{H_0}}{\|v_k\|_{H_0}} v_k$  is a minimizing sequence for the energy under the constraint  $C$ . Set  $w_k = v_k(t_k)$ . Due to the conservation of the energy and of the mass we have that  $E(v_k) = E(w_k)$  and  $\|v_k\|_{H_0} = \|w_k\|_{H_0}$ . Therefore  $\tilde{w}_k = \frac{\|\varphi\|_{H_0}}{\|w_k\|_{H_0}} w_k$  is also a minimizing sequence for the energy under the constraint  $C$ . Applying the method in the proof of Proposition 3.2 we have that up to a subsequence extraction  $w_k$  converges towards a solution of (28). This is in contradiction with the second equation in (29). The proof of the theorem is then completed.

**3.3. An instability result if  $p \geq 5$  in the pure power case.** We recall the following classical result essentially due to Weinstein [19] and that relies on the viriel method of Glassey [13]. For the sake of simplicity we give a complete proof only in the  $H^1$  critical case  $p = 5$ .

**Proposition 3.4.** *Assume here  $p = 5$ . Assume  $u_0$  be in  $H^1(\mathbb{R}) \cap L^2(x^2)$  and  $E(u_0) < +\infty$ . Then the trajectory starting from  $u_0$  blow ups in finite time.*

*Proof.* First we prove that as long as the solution remains in  $H^1(\mathbb{R})$  then the solution remains in the weighted space  $L^2(x^2)$ . Introducing

$$q(t) = \int_{\mathbb{R}} x^2 |u(t, x)|^2 \frac{dx}{\nu(x)},$$

we have that

$$q_t = 2\operatorname{Re} \int_{\mathbb{R}} x^2 u(t, x) \bar{u}_t(t, x) \frac{dx}{\nu(x)} = 2\operatorname{Im} \int_{\mathbb{R}} u(t, x) \partial_x(\nu(x) \partial_x \bar{u}(t, x)) \frac{dx}{\nu(x)}.$$

Integrating by parts and using that  $(\frac{x^2}{\nu(x)})' = \frac{2x}{\nu(x)}$  we obtain that, with  $u' = \partial_x u$

$$(30) \quad q_t = 4V = 4\operatorname{Im} \int_{\mathbb{R}} x u'(t, x) \overline{u(t, x)} dx.$$

Therefore by Cauchy-Schwarz inequality

$$(31) \quad q_t \leq 4 \|u'\|_{L^2(\mathbb{R})} \sqrt{q},$$

and  $q$  remains bounded in any compact subinterval  $[-T, T]$  of  $\mathbb{R}$  and the  $L^2(x^2)$  norm of the solution cannot blow up in finite time, as long as the solution remains in  $H^1(\mathbb{R})$ .

We now differentiate  $V$  with respect to  $t$ . We have

$$(32) \quad V_t = \operatorname{Im} \int_{\mathbb{R}} x u'_t(t, x) \overline{u(t, x)} dx + \operatorname{Im} \int_{\mathbb{R}} x u'(t, x) \overline{u_t(t, x)} dx$$

Integrating by parts the first integral in the right hand side of (32) leads to

$$(33) \quad V_t = \operatorname{Im} \int_{\mathbb{R}} u(t, x) \overline{u_t(t, x)} dx + 2\operatorname{Im} \int_{\mathbb{R}} u(t, x) \overline{u_t(t, x)} dx.$$

Integrating by parts

$$(34) \quad \operatorname{Im} \int_{\mathbb{R}} u(-i(\nu \bar{u}')' - i|u|^4 u) dx = \int_{\mathbb{R}} \nu |u'|^2 dx - \int_{\mathbb{R}} |u|^6 dx.$$

Integrating by parts twice

$$(35) \quad 2\text{Im} \int_{\mathbb{R}} xu'(-i(\nu\bar{u}')' - i|u|^4u)dx = \int_{\mathbb{R}} \nu|u'|^2dx + \frac{1}{3} \int_{\mathbb{R}} |u|^6dx.$$

Therefore  $V_t = 2E(u_0)$  and  $q(t) = 4E(u_0)t^2 + 4V(u_0)t + q(u_0)$  cannot remain non negative for all  $t$ .  $\square$

**Proposition 3.5.** *The standing wave  $\varphi$  is not orbitally stable if  $p = 5$ .*

*Proof.* As pointed out in [9] in the critical case  $E(\varphi) = 0$ . Since for  $\lambda > 1$ ,  $\lambda$  close to 1.

$$(36) \quad E(\lambda\varphi) = \lambda^2(E(\varphi) - \frac{1 - \lambda^4}{3} \int_{\mathbb{R}} \varphi^6 dx) < 0.$$

We then have a trajectory issue from  $\lambda\varphi$  which is close to the standing wave at  $t = 0$  and that blows up in finite time.  $\square$

**3.4. Proof of Theorem 1.4 in the logarithmic case.** We follow here [3] and the Nehari's manifold method. Here we deal with the Orlicz space  $V = H \cap W$ . Introduce for  $\omega > 0$  the modified energy

$$(37) \quad E_{\omega}(v) = E(v) + \omega\|v\|_{H_0}^2,$$

where

$$E(v) = \int_{\mathbb{R}} \sqrt{\nu(x)}|v'(x)|^2dx - \int_{\mathbb{R}} \sqrt{\nu(x)}(\log(|v(x)|^2) - 1)|v(x)|^2dx.$$

The Nehari's manifold reads

$$(38) \quad \mathcal{N} = \{v \neq 0, N(v) = \langle E'_{\omega}(v), v \rangle = 0\},$$

where

$$\langle E'_{\omega}(v), v \rangle = \int_{\mathbb{R}} \sqrt{\nu(x)} (|v'(x)|^2 + \omega|v(x)|^2 - \log(|v(x)|^2)|v(x)|^2) dx.$$

Let us observe that since  $v$  is even the Nehari's manifold reads also  $v \neq 0$  and

$$(39) \quad \int_{\mathbb{R}} (|v'(x)|^2 + \omega|v(x)|^2 - \log(|v(x)|^2)|v(x)|^2) dx = 0.$$

We then state

**Proposition 3.6.** *The infimum of the modified energy equation is achieved under the constraint that  $v$  belongs to  $\mathcal{N}$ .*

**Remark 3.7.** *If the infimum is achieved there exists a Lagrange multiplier  $\lambda$  such that for any  $w$  in the energy space  $V$*

$$(40) \quad \langle E'_\omega(v), w \rangle = \lambda \langle E'_\omega(v), w \rangle + \lambda E''_\omega(v)(v, w).$$

*Setting  $w = v$  in (40) we have that  $\lambda E''_\omega(v)(v, v) = 0$ . Besides since  $E''_\omega(v)(v, v) = 2\|v\|_{H_0}^2 \neq 0$  then  $\lambda = 0$  and  $v$  is solution to the equation in section 2.3.*

*Proof.* For  $v$  in  $\mathcal{N}$ , we have that  $E_\omega(v) = 2\|v\|_{H_0}^2$ . We begin with the following statement

**Lemma 3.8.** *We have  $\mu = \inf_{v \in \mathcal{N}} \|v\|_{H_0}^2 > 0$ .*

*Proof.* The key is the logarithmic Sobolev inequality (see [17]) that reads for any function  $u$  in  $H^1(\mathbb{R})$

$$(41) \quad \int_{\mathbb{R}} |u(x)|^2 \log |u(x)|^2 dx \leq \|u'\|_{L^2(\mathbb{R})}^2 + \|u\|_{L^2(\mathbb{R})}^2 (\log \|u\|_{L^2(\mathbb{R})}^2 - 1 - \log \sqrt{\pi}).$$

Therefore if  $v$  belongs to the Nehari's manifold we have that

$$(42) \quad ((1 + \log \sqrt{\pi}) + \omega) \|v\|_{L^2(\mathbb{R})}^2 \leq \|v\|_{L^2(\mathbb{R})}^2 \log \|v\|_{L^2(\mathbb{R})}^2,$$

and then  $\|v\|_{H_0}^2 = \frac{\sqrt{\nu_-} + \sqrt{\nu_+}}{2} \|v\|_{L^2(\mathbb{R})}^2 \geq \frac{\sqrt{\nu_-} + \sqrt{\nu_+}}{2} \exp(1 + \omega) \sqrt{\pi}$ .  $\square$

Consider now a minimizing sequence  $v_k$  that belongs to the Nehari's manifold and such that  $\|v_k\|_{H_0}^2 = \mu + o(1)$ . We have that

$$(43) \quad \|v'_k\|_{H_0}^2 + \omega \|v_k\|_{H_0}^2 + \int_{\mathbb{R}} |v_k|^2 \log_- |v_k|^2 \sqrt{\nu(x)} dx \leq \int_{\mathbb{R}} |v_k|^2 \log_+ |v_k|^2 \sqrt{\nu(x)} dx.$$

Due to Sobolev embedding the right hand side of (42) is bounded by above by

$$\|v_k\|_{H_0}^2 \log_+ \|v_k\|_{L^\infty(\mathbb{R})}^2 \leq C \|v_k\|_{H_0}^2 \log_+ \|v_k\|_{H^1(\mathbb{R})}^2.$$

Then the sequence  $v_k$  remains in a bounded set of  $H^1(\mathbb{R})$ . Then we may extract a subsequence (still denoted  $v_k$ ) that converges weakly in  $H^1(\mathbb{R})$  towards  $v$  and almost everywhere. We follow once again [3]. We have the following alternative that is related to the Concentration-Compactness lemma of P.L. Lions: if

$$(44) \quad \lim_{k \rightarrow +\infty} \sup_{a \in \mathbb{R}} \int_a^{a+1} |v_k(x)|^2 dx = 0,$$

then  $v_k \rightarrow 0$  in  $L^4(\mathbb{R})$ . Let us prove

**Lemma 3.9.** *There exists  $\alpha > 0$  such that  $\inf_k \|v_k\|_{L^4(\mathbb{R})} \geq \alpha$ .*

*Proof.* Consider  $\delta$  in  $(0, 1)$  to be chosen later. Then

$$(45) \quad \mu \leq \|v_k\|_{H_0}^2 \leq \delta^{-2} \|v_k\|_{L^2(\sqrt{\nu})}^4 + \int_{|v_k| \leq \delta} |v_k|^2 \frac{\log(|v_k|^2)}{\log(\delta^2)} \sqrt{\nu(x)} dx.$$

Since  $v_k$  belongs to the Nehari's manifold we have that

$$(46) \quad \int_{|v_k| \leq \delta} |v_k|^2 \frac{\log(|v_k|^2)}{\log(\delta^2)} \sqrt{\nu(x)} dx \leq - \int_{|v_k| \geq \delta} |v_k|^2 \frac{\log(|v_k|^2)}{\log(\delta^2)} \sqrt{\nu(x)} dx.$$

Therefore since  $\xi^2 \log \xi^2 \leq \xi^4 - \xi^2$  we have that

$$(47) \quad \mu \leq \left( \delta^{-2} - \frac{1}{\log(\delta^2)} \right) \|v_k\|_{L^2(\sqrt{\nu})}^4 + \frac{\mu + o(1)}{\log(\delta^2)}.$$

This concludes the proof of the lemma choosing for instance  $\log \delta^2 = -1$ .  $\square$

We now complete the proof of the proposition. Since (44) is false there exists a sequence  $y_k$  such that  $w_k(x) = v_k(x - y_k)$  converges to a non zero function  $v$  weakly in  $V$  and uniformly in any compact subset of  $\mathbb{R}$ . Repeating the rearrangement argument as above we may assume that  $w_k$  and  $v$  belong to  $H \cap W$ . Let us prove now that  $v$  belongs to  $\mathcal{N}$ . We argue by contradiction. For  $\lambda > 0$  a positive real number we have that

$$(48) \quad N(\lambda v) = \lambda^2 (N(v) - \log \lambda^2 \|v\|_{H_0}^2).$$

If  $N(v) < 0$  then there exists  $\lambda < 1$  such that  $N(\lambda v) = 0$ . Then by weak convergence we have the following contradiction

$$(49) \quad \mu \leq \|\lambda v\|_{H_0}^2 < \|v\|_{H_0}^2 \leq \liminf_{k \rightarrow \infty} \|v_k\|_{H_0}^2 = \mu.$$

If  $N(v) > 0$ , by Brezis-Lieb lemma (see Lemma 2.3 in [3]) we have that for  $k$  large enough  $N(w_k - v) < 0$ . Therefore proceeding as in (49) we have the following contradiction (for some  $\lambda$  in  $(0, 1)$ )

$$(50) \quad \mu \leq \|\lambda(w_k - v)\|_{H_0}^2 < \|w_k - v\|_{H_0}^2 = \mu + o(1) - \|v\|_{H_0}^2.$$

Therefore  $v$  belongs to  $\mathcal{N}$  and since by weak convergence we have that

$$\mu \leq \|v\|_{H_0}^2 \leq \liminf_{k \rightarrow \infty} \|v_k\|_{H_0}^2 = \mu$$

the proof of the proposition is completed.  $\square$

We now complete the proof of the orbital stability. Let us argue by contradiction. Assume that there exists a sequence  $v_k$  in  $V$ , two sequences of real numbers  $t_k, \theta_k$  such that for  $v_k(t)$  the solution of (18) starting from  $v_k$  at  $t = 0$

$$(51) \quad \begin{aligned} \|\varphi - v_k\|_V &= o(1), \\ \|\varphi - v_k(t_k)e^{i\theta_k}\|_V &\geq \varepsilon > 0. \end{aligned}$$

Set  $w_k = v_k(t_k)$ . Appealing the first equation in (51) and due to the conservation of the energy and of the mass by the flow of the equation we have that

$$(52) \quad N(v_k) = N(w_k) = o(1) \text{ and } \|v_k\|_{H_0}^2 = \|w_k\|_{H_0}^2 = \mu + o(1).$$

We may exhibit a sequence of real numbers  $\lambda_k = 1 + o(1)$  such that  $\lambda_k w_k$  is a minimizing sequence belonging to the Nehari's manifold. Then up to a subsequence extraction  $\lambda_k w_k$  converges towards a solution of the equation in section 2.3 that contradicts the second equality in (51).

acknowledgements

This work was partially supported by a grant from the Simons Foundation when the second author I.M. was visiting Laboratoire Paul Painlevé in Lille. O.G. is supported by Labex CEMPI (ANR-11-LABX-0007-01).

#### REFERENCES

- [1] M. Abidi PhD dissertation, to appear.
- [2] P. Antonelli, J.-C. Saut and C. Sparber, *Well-posedness and averaging of NLS with time periodic dispersion management*, Adv. Di. Eq. , 18, 12, 49-68, (2013).
- [3] A. Ardila *Orbital stability of Gausson solutions to logarithmic Schrödinger equations*, Electronic Journal of Differential Equations, 2016(335):1–9, (2016).
- [4] A. Ardila and J. Pava, *Stability of standing waves for logarithmic Schrödinger equation with attractive delta potential*, Indiana University Mathematics Journal 2 (67), 471–494, (2018).
- [5] V. Banica, *Dispersion and Strichartz inequalities for Schrödinger equations with singular coefficients*, SIAM J. Math. Analysis, vol 35, N 4, 868-883, (2003).
- [6] N. Burq and F. Planchon, *Smoothing and dispersive estimates for 1D Schrödinger equations with BV coefficients and applications*, J. Funct. Anal. 236 , no. 1, 265-298, (2006).
- [7] R. Carles and I. Gallagher *Universal dynamics for the defocusing logarithmic Schrödinger equation*, Duke Math. J. 167 (9), 1761-1801, (2018).
- [8] T. Cazenave, A. Haraux *Equation d'évolution avec nonlinéarité logarithmique*, A. Fac. des Sciences de Toulouse, 2 , 21-51 (1980).
- [9] T. Cazenave, *Semilinear Schrödinger equations*, vol 10, Courant Lectures Notes in Mathematics, NYU Courant Institute of Mathematical Sciences, New York (2003).
- [10] A. de Bouard and A. Debussche, *The nonlinear Schrödinger equation with white noise dispersion*, J. Func. Anal. 259, 1300-1321, (2006).
- [11] R. Fukuizumi, L. Jeanjean, *Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential*, Discrete Contin. Dyn. Syst. 21, , no. 1, 121-136, (2008).
- [12] F. Genoud, B. Alomed and R. Weishaupt *Stable NLS solitons in a cubic-quintic medium with a delta-function potential*, Nonlinear Anal Theory Methods Appl., ser A, Theory Methods 133, 28-50 (2016).
- [13] R. Glassey, *On the blowing up to the solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. 18, 1794-1797, (1977).

- [14] J. Holmer and C. Liu, *Blow-up for the 1D nonlinear Schrödinger equation with point nonlinearity I, basic theory*, Math. AP., (2015).
- [15] M. Ikeda and T. Inui, *Global dynamics below the standing waves for the focusing semilinear Schrödinger equation with a repulsive Dirac delta potential*, Analysis & PDE, (2017).
- [16] S. Le Coz, R. Fukuizumi, G. Fibich, B. Ksherim and Y. Sivan, *Instability of bound states of a nonlinear Schrödinger equation with a Dirac potential*, Physica D, 237, pp 1103-1128, (2008).
- [17] E. Lieb and M. Loss. *Analysis*, volume 14 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2 edition, (2001).
- [18] S. Masaki, J. Murphy and J-I. Segata, *Stability of small solitary waves for the 1D NLS equation with an attractive delta potential*, Anal. PDE. 13 , no. 4, 1099-1128. (2020)
- [19] M. Weinstein *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phy. 87, 567-576. (1983)