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Testing copula-based dependence hypotheses: a proofreading based on functional decompositions

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Abstract

Tests of multivariate independence may rely on asymptotically independent Cramér-von Mises statistics derived from a Möbius decomposition of the empirical copula process. We generalize this approach to some other copula-based assumptions, with the help of a functional decomposition based on commuting idempotent maps. As soon as the null hypothesis reflects the stability of the copula under the action of the composition of such operators, the methodology applies. The empirical testing process, which depends on the decomposition, allows the derivation of a new family of test statistics. The asymptotic distributions are obtained. Since the latter depend on the unknown copula being tested, we adapt parametric bootstrap or subsampling procedure to our setting to approximate p-values. The benefits in deriving test statistics from a functional decomposition are illustrated and discussed through simulations.

Keywords: Copula models, Functional decomposition, Idempotent maps, Rank-based inference

MSC Classification: 62H15 , 62E10 , 41A63

1 Introduction

The nature and strength of cross-sectional dependence is of crucial importance to understand economic or environmental systems. One possible measure relies on copulas, which have become popular over the last decades. In this paper, we review and provide a new light on the literature for some testing problems.

Consider $\mathbf{X}_1, \dots, \mathbf{X}_n$ a sample of d -variate observations where \mathbf{X}_j stands for $(X_{j1}, \dots, X_{jd})^T$. At first, one may think that this n -sample consists of independent copies of a d -dimensional random vector $\mathbf{X} = (X_1, \dots, X_d)^T$. However, most of the results hold true for some strictly stationary time series. We assume that the cumulative distribution function (c.d.f.) F of the representative vector \mathbf{X} has continuous univariate margins denoted by F_1, \dots, F_d . There exists then a unique copula $C : [0, 1]^d \rightarrow [0, 1]$, that is a d -dimensional c.d.f. with standard uniform margins such that $F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d))$ for all $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$. This representation, due to [1], illustrates that the copula C characterizes the dependence between the components of \mathbf{X} .

The present work is concerned with testing structural hypotheses for the copula. There exists indeed a large number of copula families and testing procedures help guide the choice of the most appropriate one. Tests based on empirical copula processes have been successfully proposed in the literature. Let us cite for instance [2], [3], [4], [5] or [6] that handled the independence, serial independence, independence by blocks, or broader classes such as extreme value copulas. Whereas these references focus on one hypothesis at once, the aim of this paper is to demonstrate that several structural hypotheses for dependence share a common pattern. Our contribution is to unify part of the theory as well as to provide new combined statistics.

Our procedure could be roughly illustrated by the pioneering idea of [2] which reveals the independence through the Möbius decomposition of the empirical process. The null hypothesis is thus equivalent to the intersection of a finite set of hypotheses since all secondary terms of the decomposition vanish. We generalize this method by applying another functional decomposition, chosen in accordance with the structural assumption being tested. Again, a collection of sub-hypotheses holds true under the null hypothesis. In consequence, new test statistics are defined by extracting and combining all the associated information.

The remainder of this paper is organized as follows. Section 2 recalls the functional decomposition based on operators and makes it explicit in the context of dependence structures. After a first asymptotic statement, Section 3 explains how to construct independent copies of some limiting processes. Section 4 is devoted to the practical implementation of the theoretical results: definition of the test statistics, associated asymptotics, practical computation and approximation of the p-value are discussed. Two numerical experiments based on simulation end this section. Concluding remarks are given in Section 5 and proofs are postponed to Section 6.

2 Dependence structures and associated operators

Our aim in this section is to explain how a general functional decomposition allows to handle various null hypotheses of dependence with a common mechanism. Such a null hypothesis is defined as the stability of the copula under a composition of some

operators. A collection of sub-hypotheses arises naturally. The last part of this section provides a large list of examples.

2.1 A general functional decomposition

Let \mathcal{F} be the linear space of real-valued functions acting on $[0, 1]^d$. Let $\mathbf{I} : \mathcal{F} \rightarrow \mathcal{F}$ denote the identity map. For $i \in \{1, \dots, d\}$, let $\mathbf{P}_i : \mathcal{F} \rightarrow \mathcal{F}$ be an operator. We assume that the collection of functionals $\mathbf{P}_1, \dots, \mathbf{P}_d$ commutes. The composition of the maps \mathbf{P}_i for $i \in \mathcal{A}$ will be denoted as $\prod_{i \in \mathcal{A}} \mathbf{P}_i$ and equals \mathbf{I} in the case where $\mathcal{A} = \emptyset$.

Every $f \in \mathcal{F}$ can be decomposed as

$$f = \prod_{i=1}^d (\mathbf{I} - \mathbf{P}_i + \mathbf{P}_i)(f) = \sum_{\mathcal{A} \in \mathcal{P}_d} \mathbf{M}_{\mathcal{A}}(f), \quad (1)$$

where \mathcal{P}_d stands for the superset of $\{1, \dots, d\}$ and where $\mathbf{M}_{\mathcal{A}}$ is defined as

$$\mathbf{M}_{\mathcal{A}} = \prod_{i \in \mathcal{A}} (\mathbf{I} - \mathbf{P}_i) \prod_{i \notin \mathcal{A}} \mathbf{P}_i. \quad (2)$$

Another way of writing the equation (1) is

$$f - \mathbf{M}_{\emptyset}(f) = \sum_{\mathcal{A} \in \mathcal{P}_d^*} \mathbf{M}_{\mathcal{A}}(f), \quad (3)$$

where $\mathcal{P}_d^* = \mathcal{P}_d \setminus \emptyset$. From (2) $\mathbf{M}_{\emptyset} = \prod_{i=1}^d \mathbf{P}_i$, the composition of all functionals.

2.2 The null hypothesis and a list of examples

Consider now copula functions C associated with continuous random vectors \mathbf{X} . The main objective of this section is to identify, for some copula-based structural dependence, their associated set of operators $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$ that allows to write the dependence null hypothesis as $(\mathcal{H}) C = \mathbf{M}_{\emptyset}(C) = \prod_{i=1}^d \mathbf{P}_i(C)$.

- $(\mathcal{H}) C(\mathbf{x}) = x_1 \times \dots \times x_d$ the complete independence among all components of \mathbf{X} is obtained with $\mathbf{P}_i(C)(\mathbf{x}) = x_i \times C(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d)$.
- Let us structure \mathbf{X} as $(\mathbf{X}_{\{1\}}, \dots, \mathbf{X}_{\{p\}})$ the concatenation of p subvectors of dimension d_1, \dots, d_p . Therefore $d = d_1 + \dots + d_p$. The assertion $(\mathcal{H}) \mathbf{X}_{\{1\}}, \dots, \mathbf{X}_{\{p\}}$ are independent, or equivalently

$$(\mathcal{H}) C(\mathbf{x}) = C(\mathbf{x}_{\{1\}}, \mathbf{1}_{-\{1\}}) \cdots C(\mathbf{x}_{\{i\}}, \mathbf{1}_{-\{i\}}) \cdots C(\mathbf{x}_{\{p\}}, \mathbf{1}_{-\{p\}})$$

is associated with $\mathbf{P}_i(C)(\mathbf{x}) = C(\mathbf{x}_{\{i\}}, \mathbf{1}_{-\{i\}}) \times C(\mathbf{1}_{\{i\}}, \mathbf{x}_{-\{i\}})$.

- According to [7], a function $f : [0, 1]^d \rightarrow [0, 1]$ is said associative whenever for $(x_1, \dots, x_d, \dots, x_{2d-1}) \in [0, 1]^{2d-1}$ it holds

$$f(f(x_1, \dots, x_d), x_{d+1}, \dots, x_{2d-1}) = \dots = f(x_1, \dots, x_{d-1}, f(x_d, \dots, x_{2d-1})) .$$

We thus consider $\mathbf{P}_i(C)(\mathbf{x}) = C(x_1, \dots, x_i, C(1, \dots, 1, x_{i+1}, \dots, x_d), 1, \dots, 1)$ to describe a weak form of associativity.

- Fix φ the generator of interest that is a non-negative, continuous, strictly decreasing and convex function defined on $[0, 1]$ satisfying $\varphi(1) = 0$. Its pseudo-inverse, denoted $\varphi^{[-1]}$, is defined as the usual inverse on $[0, \varphi(0)]$ and equals 0 elsewhere. Writing the specific Archimedean copula generated by φ

$$(\mathcal{H}) C(\mathbf{x}) = \varphi^{[-1]} [\varphi(x_1) + \dots + \varphi(x_d)]$$

corresponds to the choice

$$\mathbf{P}_i(C)(\mathbf{x}) = \varphi^{[-1]} [\varphi(C(x_i, \mathbf{1}_{-i})) + \varphi(C(\mathbf{1}_i, \mathbf{x}_{-i}))] .$$

- The symmetric logistic extreme value copula is a particular case of the latter example. Let $\ell : [0, \infty]^d \rightarrow [0, \infty]$ be a stable tail dependence function. Recall that an extreme value copula can be written as $C(\mathbf{x}) = \exp[-\ell\{-\ln(x_1), \dots, -\ln(x_d)\}]$. See for instance Chapter 7 of [8] for more details on the ℓ function. It is called the symmetric logistic extreme value copula model when there exists a real $\theta \in [1, \infty[$ such that $\ell(x_1, \dots, x_d) = (x_1^\theta + \dots + x_d^\theta)^{1/\theta}$. Testing the symmetric logistic extreme value model

$$(\mathcal{H}) C(\mathbf{x}) = \exp\left[-\{(-\ln(x_1))^\theta + \dots + (-\ln(x_d))^\theta\}^{1/\theta}\right]$$

corresponds to the last expression of \mathbf{P}_i with $\varphi(t) = (-\ln(t))^\theta$.

- A possible extension consists in mixing previous examples. Recall that the random vector \mathbf{X} might be seen as the concatenation of p subvectors $\mathbf{X}_{\{1\}}, \dots, \mathbf{X}_{\{p\}}$. The independence by blocks could be replaced by an Archimedean structure by blocks associated with φ . Then the null hypothesis

$$(\mathcal{H}) C(\mathbf{x}) = \varphi^{-1} [\varphi(C(\mathbf{x}_{\{1\}}, \mathbf{1}_{-\{1\}})) + \dots + \varphi(C(\mathbf{1}_{-\{p\}}, \mathbf{x}_{\{p\}}))]]$$

could be obtained using the following functionals

$$\mathbf{P}_i(C)(\mathbf{x}) = \varphi^{-1} [\varphi(C(\mathbf{x}_{\{i\}}, \mathbf{1}_{-\{i\}})) + \varphi(C(\mathbf{x}_{-\{i\}}, \mathbf{1}_{\{i\}}))] .$$

Copulas which satisfy the corresponding null hypothesis have an easy interpretation. Only p -uplets of variables, each belonging to one of the p blocks, are completely specified: Their dependence structure follows the Archimedean copula

generated by φ . The dependence within any groups of variables belonging partially to the same block is not fixed. This differs from the notion of nested or hierarchical copulas.

- As before, consider φ a generator associated with an Archimedean structure. And consider $\ell : [0, \infty]^d \rightarrow [0, \infty]$ a stable tail dependence function. Recall from [9, 10] that

$$C(\mathbf{x}) = \varphi^{-1} [\ell(\varphi(x_1), \dots, \varphi(x_d))]$$

is called an Archimax copula. We restrict here the form of ℓ as following

$$\ell(x_1, \dots, x_d) = g^{-1} [g\{\ell(\mathbf{x}_{\{1\}}, \mathbf{0}_{-\{1\}})\} + \dots + g\{\ell(\mathbf{0}_{-\{p\}}, \mathbf{x}_{\{p\}})\}], \quad (4)$$

where g is a continuous bijection from \mathbb{R}_+ to \mathbb{R}_+ satisfying $g(1) = 1$. From Theorem 6 in [11], one knows that $g(x) = x^\theta$ for some $\theta \geq 1$. For the sake of simplicity, set $\varphi \mathbf{x}_{\{i\}} = \sum_{j \in \{i\}} \varphi(x_j) \mathbf{e}_j$. For $i \in \{1, \dots, p\}$, let define \mathbf{P}_i by

$$\mathbf{P}_i(C)(\mathbf{x}) = \varphi^{-1} \left[\left\{ \left(\ell(\varphi \mathbf{x}_{\{i\}}, \mathbf{0}_{-\{i\}}) \right)^\theta + \left(\varphi \circ C(\mathbf{1}_{\{i\}}, \mathbf{x}_{-\{i\}}) \right)^\theta \right\}^{1/\theta} \right]$$

completed by $\mathbf{P}_{p+1} = \dots = \mathbf{P}_d = \mathbf{I}$, to describe the null hypothesis

$$(\mathcal{H}) C(\mathbf{x}) = \varphi^{-1} \left[\left\{ \left(\ell(\varphi \mathbf{x}_{\{1\}}, \mathbf{0}_{-\{1\}}) \right)^\theta + \dots + \left(\ell(\mathbf{0}_{-\{p\}}, \varphi \mathbf{x}_{\{p\}}) \right)^\theta \right\}^{1/\theta} \right].$$

- For a given positive integer r , let us consider the null hypothesis

$$(\mathcal{H}_r) C(\mathbf{x}) = C^r(\mathbf{x}^{1/r}) \quad \forall \mathbf{x} \in [0, 1]^d.$$

The max-stability assumption, which is the intersection of any such null hypothesis $(\mathcal{H}) = \bigcap_{r \in \mathbb{N}^*} (\mathcal{H}_r)$, corresponds to the choice of functional $\mathbf{P}_i(C)(\mathbf{x}) = C^{r_i}(\mathbf{x}^{1/r_i})$.

- Let \mathfrak{S}_d be the set of all permutations of $\{1, \dots, d\}$ and set $\mathbf{x}_\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(d)})$ for any $\sigma \in \mathfrak{S}_d$. Testing symmetry of the copula

$$(\mathcal{H}) C(\mathbf{x}) = C(\mathbf{x}_\sigma) \quad \forall \mathbf{x} \in [0, 1]^d \text{ and } \forall \sigma \in \mathfrak{S}_d$$

can be handled in a very similar way to the previous one. Let $T_{1,d}$ denote the set consisting of the $d - 1$ transpositions $\tau_i = (1i)$ for $i = 2, \dots, d$. Noting that $T_{1,d}$ generates \mathfrak{S}_d , it is also possible to write here that $(\mathcal{H}) = \bigcap_{i=2}^d (\mathcal{H}_{\tau_i})$. It is thus sufficient to consider $\mathbf{P}_i(C)(\mathbf{x}) = C(\mathbf{x}_{\tau_i})$.

2.3 An associated collection of sub-hypotheses

Recall that the null hypothesis $(\mathcal{H}) C = \mathbf{M}_\emptyset(C)$ can also be written as following

$$(\mathcal{H}) C - \mathbf{M}_\emptyset(C) = 0.$$

From (3),

$$C - \mathbf{M}_\emptyset(C) = \sum_{\mathcal{A} \in \mathcal{P}_d^*} \mathbf{M}_\mathcal{A}(C)$$

so that the summation $\sum_{\mathcal{A} \in \mathcal{P}_d^*} \mathbf{M}_\mathcal{A}(C)$ vanishes when (\mathcal{H}) holds true.

It is thus interesting to consider for any $\mathcal{A} \in \mathcal{P}_d^*$ the sub-hypothesis

$$(\mathcal{H}^\mathcal{A}) \mathbf{M}_\mathcal{A}(C) = 0. \quad (5)$$

An immediate property is $\cap_{\mathcal{A}} (\mathcal{H}^\mathcal{A}) \subseteq (\mathcal{H})$. So that a relevant question is to analyze whether any $(\mathcal{H}^\mathcal{A})$ holds true under the null hypothesis (\mathcal{H}) . What is its link exactly with the intersection? In the next proposition, we answer part of the question.

Proposition 1. *Let $\mathbf{P}_1, \dots, \mathbf{P}_d$ be a commuting collection of idempotent operators on \mathcal{F} . Then, the null hypothesis satisfies the equality*

$$(\mathcal{H}) = \bigcap_{\mathcal{A} \in \mathcal{P}_d^*} (\mathcal{H}^\mathcal{A}).$$

Naturally, one may wonder whether the list of examples satisfies these conditions or not. More precisely, the question is as follows: under (\mathcal{H}) , are the associated operators $\mathbf{P}_1, \dots, \mathbf{P}_d$ commutative and idempotent? The answer is affirmative in each case presented in Section 2.2. Their commutativity property is evident. As for idempotence, we regularly need to use the following: $C(\mathbf{1}) = 1$ and $\varphi(1) = 0$.

3 Behavior and approximation of the testing process

The purpose of this section is to introduce the empirical testing processes. Consider a structural dependence hypothesis for copulas expressed as $(\mathcal{H}) C = \mathbf{M}_\emptyset(C)$. Recall that $\{\mathbf{M}_\mathcal{A}\}_{\mathcal{A} \in \mathcal{P}_d^*}$ is the set depending, through Formula (2), on a collection of operators $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$ defined on \mathcal{F} . It is assumed that $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$ are commuting and idempotent maps, at least when (\mathcal{H}) holds true. Starting from a copula estimator C_n , it is natural to construct the testing process as $(\sqrt{n}(C_n - \mathbf{M}_\emptyset(C_n)))(\mathbf{x})$, $\mathbf{x} \in [0, 1]^d$ when considering (\mathcal{H}) . This is precisely what is done in the literature. Nevertheless, since (\mathcal{H}) implies any sub-hypothesis $(\mathcal{H}^\mathcal{A}) \mathbf{M}_\mathcal{A}(C) = 0$, another choice is possible.

3.1 Weak convergence of the empirical processes

Consider $\mathbf{X}_1, \dots, \mathbf{X}_n$ a sample of d -variate observations of \mathbf{X} where \mathbf{X}_j stands for $(X_{j1}, \dots, X_{jd})^T$. Set $\mathbf{U}_j = (F_1(X_{j1}), \dots, F_d(X_{jd}))$ for $j \in \{1, \dots, n\}$. The empirical cumulative distribution function based on $\mathbf{U}_1, \dots, \mathbf{U}_n$ is denoted by G_n and we set $\mathbb{G}_n = \sqrt{n}(G_n - C)$. Under regular conditions, the empirical process \mathbb{G}_n converges weakly in $\ell^\infty([0, 1]^d)$ to a tight centered Gaussian process \mathbb{G}_C concentrated on

$$\mathcal{C}_0 = \{h : [0, 1]^d \rightarrow \mathbb{R} \text{ continuous such that } h(\mathbf{1}) = 0 \text{ and}$$

$$h(\mathbf{x}) = 0 \text{ if some components of } \mathbf{x} \text{ are equal to } 0\}. \quad (6)$$

Throughout the paper, we assume the existence and the paths continuity of

$$\mathbb{W}_C(\mathbf{x}) = \mathbb{G}_C(\mathbf{x}) - \sum_{i=1}^d \partial C_i(\mathbf{x}) \mathbb{G}_C(x_i, \mathbf{1}_{-i}), \quad \mathbf{x} \in [0, 1]^d. \quad (7)$$

We introduce and study in this section the concatenated empirical testing process

$$\left(\sqrt{n}(C_n - \mathbf{M}_\emptyset(C_n)), \{ \sqrt{n} \mathbf{M}_A(C_n) \}_{A \in \mathcal{P}_d^*} \right).$$

Theorem 2. *Assume, at least when $(\mathcal{H}) C = \mathbf{M}_\emptyset(C) = \prod_{i=1}^d \mathbf{P}_i(C)$ holds true, that*

- *The operators $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$ are commuting and idempotent maps.*
- *The maps $\{\mathbf{M}_A\}_{A \in \mathcal{P}_d}$ derived from (2) are Hadamard-differentiable at C tangentially to \mathcal{C}_0 .*

Consider an empirical copula C_n such that, as n tends to infinity, the empirical copula process $\sqrt{n}(C_n - C)$ converges weakly in $\ell^\infty([0, 1]^d)$ to \mathbb{W}_C given in (7).

Then, under (\mathcal{H}) and as n tends to infinity, the joint empirical processes converge weakly in $\{\ell^\infty([0, 1]^d)\}^{2^d}$ as following

$$\left(\sqrt{n}(C_n - \mathbf{M}_\emptyset(C_n)), \{ \sqrt{n} \mathbf{M}_A(C_n) \}_{A \in \mathcal{P}_d^*} \right) \xrightarrow[n \rightarrow \infty]{w} \left(\mathbb{W}_C - \mathbf{M}'_\emptyset(C; \mathbb{W}_C), \{ \mathbf{M}'_A(C; \mathbb{W}_C) \}_{A \in \mathcal{P}_d^*} \right). \quad (8)$$

Several copula estimates C_n satisfy the required convergence. The last lines of Section 2 in [12] list carefully the conditions under which $\sqrt{n}(C_n - C) \xrightarrow[n \rightarrow \infty]{w} \mathbb{W}_C$ in $\ell^\infty([0, 1]^d)$ for the following list of well-known empirical copulas

- the non-parametric estimators $\tilde{C}_n(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \prod_{i=1}^d \mathbf{1}_{\{F_{n_j}(X_{ji}) \leq x_i\}}$ and $\hat{C}_n(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \prod_{i=1}^d \mathbf{1}_{\{R_{ji,n}/(n+1) \leq x_i\}}$ where $R_{ji,n}$ = rank of X_{ji} among X_{1i}, \dots, X_{ni} ,
- the checkerboard version $C_n^\#(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \prod_{i=1}^d \min\{\max\{nx_i - R_{ji,n}, 0\}, 1\}$,
- and the empirical beta copula, $C_n^\beta(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \prod_{i=1}^d F_{n, R_{ji,n}}(x_i)$ where $F_{n,r}$ stands for the probability distribution function of the Beta distribution $\mathcal{B}(r, n+1-r)$.

3.2 Subsampling empirical testing processes

The weak convergence of the empirical processes has just been proved but the covariance structures of the limiting processes $\mathbb{D}_\emptyset := \mathbb{W}_C - \mathbf{M}'_\emptyset(C; \mathbb{W}_C)$ and $\mathbb{D}_A := \mathbf{M}'_A(C; \mathbb{W}_C)$ depend on the unknown copula C . For this reason it is not always directly applicable for statistical testing. To reproduce independently the asymptotic behavior of such processes, resampling techniques have to be applied.

Parametric bootstrap can be used under some specific null hypotheses. Subsampling can be performed otherwise, as discussed in the following lines. To simplify notation, set $D_{\emptyset,n} := \sqrt{n}(C_n - \mathbf{M}_{\emptyset}(C_n))$ and $D_{\mathcal{A},n} := \sqrt{n}\mathbf{M}_{\mathcal{A}}(C_n)$ for $\mathcal{A} \in \mathcal{P}_d^*$, as well as $\mathbb{C}_n := \sqrt{n}(C_n - C)$. Theorem 2 has stated the weak convergence of $\{D_{\mathcal{A},n}\}_{\mathcal{A} \in \mathcal{P}_d}$ to $\{\mathbb{D}_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$ under some conditions, in particular the weak convergence of \mathbb{C}_n . The subsampling method is a substitute to approximate the limiting distribution. See [13] and its adaptation in [12] that inspired this development.

Let $b < n$ denote the size of the samples extracted from $\mathbf{X}_1, \dots, \mathbf{X}_n$ and let $B_{b,n}$ be the corresponding number of possible subsamples. Under the i.i.d. setting, $B_{b,n} = \binom{n}{b}$ since the subsamples may be obtained without replacement. Since $B_{b,n}$ might be too large, the complete enumeration could not be possible. The practical solution is to obtain its stochastic approximation through a N -sample of integers $I_{1,n}, \dots, I_{N,n}$ independently extracted with replacement from $\{1, \dots, B_{b,n}\}$. The quantities of interest would be then computed for these N values of the index m . In the serial context, the sampling should preserve the dependence so that the subsamples must have consecutive index, and $B_{b,n} = n - b + 1$. Denote by $C_b^{[m]}$ the replicates of the estimator C_n evaluated on such b -subsamples. Set now $\mathbb{C}_b^{[m]} = \sqrt{b}(C_b^{[m]} - C_n)$ the associated replicates of the empirical copula processes. Set finally $D_{\emptyset,b}^{[m]} = \sqrt{b}(C_b^{[m]} - \mathbf{M}_{\emptyset}(C_b^{[m]}))$ and $D_{\mathcal{A},b}^{[m]} = \sqrt{b}\mathbf{M}_{\mathcal{A}}(C_b^{[m]})$.

Theorem 3. *Assume, at least when $(\mathcal{H}) C = \mathbf{M}_{\emptyset}(C) = \prod_{i=1}^d \mathbf{P}_i(C)$ holds true, that*

- *The operators $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$ are commuting and idempotent maps.*
- *The maps $\{\mathbf{M}_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$ derived from (2) are Hadamard-differentiable at C tangentially to \mathcal{C}_0 .*
- *The derivatives $\{\mathbf{M}'_{\mathcal{A}}(C; \cdot)\}_{\mathcal{A} \in \mathcal{P}_d}$ are continuous on $\ell^\infty([0, 1]^d)$.*

Let $v_n \rightarrow 1$ and $b_n \rightarrow \infty$ such that $b_n = o(n)$. Consider an empirical copula C_n such that $(\mathbb{C}_n, v_n \mathbb{C}_{b_n}^{[I_{1,n}]}, v_n \mathbb{C}_{b_n}^{[I_{2,n}]})$ converges weakly in $\{\ell^\infty([0, 1]^d)\}^3$, as n tends to infinity, to $(\mathbb{W}_C, \mathbb{W}_C^{[1]}, \mathbb{W}_C^{[2]})$ where $\mathbb{W}_C^{[1]}$ and $\mathbb{W}_C^{[2]}$ are independent copies of \mathbb{W}_C . Then, under (\mathcal{H}) and as n tends to infinity,

$$\left(\{D_{\mathcal{A},n}\}_{\mathcal{A} \in \mathcal{P}_d}, \left\{ v_n D_{\mathcal{A},b_n}^{[I_{1,n}]} \right\}_{\mathcal{A} \in \mathcal{P}_d}, \left\{ v_n D_{\mathcal{A},b_n}^{[I_{2,n}]} \right\}_{\mathcal{A} \in \mathcal{P}_d} \right)$$

converges weakly in $\{\ell^\infty([0, 1]^d)\}^{3 \times 2^d}$ to $(\{\mathbb{D}_{\mathcal{A}}\}_{\mathcal{A}}, \{\mathbb{D}_{\mathcal{A}}^{[1]}\}_{\mathcal{A}}, \{\mathbb{D}_{\mathcal{A}}^{[2]}\}_{\mathcal{A}})$, where $\{\mathbb{D}_{\mathcal{A}}^{[2]}\}_{\mathcal{A} \in \mathcal{P}_d}$ and $\{\mathbb{D}_{\mathcal{A}}^{[1]}\}_{\mathcal{A} \in \mathcal{P}_d}$ are independent copies of $\{\mathbb{D}_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$.

The previous result assumes the subsampling of the empirical copula process \mathbb{C}_n . Theorem 3.3 in [12] states such a convergence

- under i.i.d. observations, $b/n \rightarrow \alpha \in [0, 1]$, $v_n = (1 - b/n)^{-1/2}$ when C_n stands for $\hat{C}_n, C_n^\#$ or C_n^β ,
- under strictly stationary alpha-mixing sequence with $\alpha(k) = O(k^{-a})$ for some $a > 1$, $b/n \rightarrow 0$, $v_n = 1$ when C_n stands for \tilde{C}_n ,
- under strictly stationary alpha-mixing sequence with $\alpha(k) = O(a^k)$ for some $a \in (0, 1)$, $b/n \rightarrow 0$, $v_n = 1$, when C_n stands for $\hat{C}_n, C_n^\#$ or C_n^β ,

assuming additionally classical regularity properties on the true copula C .

3.3 Weighted version of the subsampling methodology

A weight is used to emphasize the region where the copula differs from the tested copula. See [14] or more recently [15]. Following [16] and [15] for instance we provide, under stronger assumptions, the weak convergence with respect to stronger metrics. It also includes the validity of the subsampling methodology for the empirical testing process, as it has been done in [12].

Theorem 4. *Assume, at least when $(\mathcal{H})C = \mathbf{M}_\emptyset(C) = \prod_{i=1}^d \mathbf{P}_i(C)$ holds true, that*

- *The operators $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$ are commuting and idempotent maps.*
- *The maps $\{\mathbf{M}_A\}_{A \in \mathcal{P}_d}$ derived from (2) are Hadamard-differentiable at C tangentially to \mathcal{C}_0 .*
- *The derivatives $\{\mathbf{M}'_A(C; \cdot)\}_{A \in \mathcal{P}_d}$ are continuous on $\ell^\infty([0, 1]^d)$.*

Let $v_n \rightarrow 1$ and $b_n \rightarrow \infty$ such that $b_n = o(n)$. Let C_n be an empirical copula and q a weight function such that, as n tends to infinity, $\left(\frac{C_n}{q}, v_n \frac{\mathbb{C}_{b_n}^{[I_1, n]}}{q}, v_n \frac{\mathbb{C}_{b_n}^{[I_2, n]}}{q}\right)$ converges weakly in $\{\ell^\infty([0, 1]^d)\}^3$ to $\left(\frac{\mathbb{W}_C}{q}, \frac{\mathbb{W}_C^{[1]}}{q}, \frac{\mathbb{W}_C^{[2]}}{q}\right)$, where $\mathbb{W}_C^{[1]}$ and $\mathbb{W}_C^{[2]}$ are independent copies of \mathbb{W}_C . Then, under (\mathcal{H}) and as n tends to infinity,

$$\left(\{D_{A,n}/q\}_{A \in \mathcal{P}_d}, \{v_n D_{A,b_n}^{[I_1, n]}/q\}_{A \in \mathcal{P}_d}, \{v_n D_{A,b_n}^{[I_2, n]}/q\}_{A \in \mathcal{P}_d}\right)$$

converges weakly in $\{\ell^\infty([0, 1]^d)\}^{3 \times 2^d}$ to

$$\left(\{\mathbb{D}_A/q\}_{A \in \mathcal{P}_d}, \{\mathbb{D}_A^{[1]}/q\}_{A \in \mathcal{P}_d}, \{\mathbb{D}_A^{[2]}/q\}_{A \in \mathcal{P}_d}\right),$$

where $\{\mathbb{D}_A^{[2]}\}_{A \in \mathcal{P}_d}$ and $\{\mathbb{D}_A^{[1]}\}_{A \in \mathcal{P}_d}$ are independent copies of $\{\mathbb{D}_A\}_{A \in \mathcal{P}_d}$.

The subsampling of the process \mathbb{C}_n/q assumed in Theorem 4 has been proved in [12]. Let \vee and \wedge stand respectively for the maximum and the minimum. On $[0, 1]^d$ let g be the weight function $g(\mathbf{x}) = \wedge_{i=1}^d \left\{x_i \wedge \vee_{k=1, k \neq i}^d (1 - x_k)\right\}$. With $w \in [0, 1/2]$ and $q = g^w$, see [12, Theorem 4.3] for the statement concerning both \hat{C}_n and \tilde{C}_n and [12, Theorem 4.5] for that involving $C_n^\#$ and C_n^β .

4 The testing procedure

Natural measures of departure from the null hypothesis are Kolmogorov–Smirnov or Cramér–von Mises statistics. We give their expressions, and a new combination derived from the concatenated testing processes is introduced and studied. As these statistics are not distribution-free, the subsampling methodology or the parametric bootstrap will be needed to approximate p-values.

4.1 Well-known families of test statistics

Consider $(\mathcal{H})C = \mathbf{M}_\emptyset(C)$, depending through (2) on the operators $\mathbf{P}_1, \dots, \mathbf{P}_d$. It is assumed, at least when (\mathcal{H}) holds true, that $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$ are commuting and idempotent maps. Recall that the definition of $D_{\mathcal{A},n}$ is specific when $\mathcal{A} = \emptyset$ since $D_{\emptyset,n} = C_n - \mathbf{M}_\emptyset(C_n) = C_n - (\prod_{i=1}^d \mathbf{P}_i)(C_n)$ whereas, if $\mathcal{A} \neq \emptyset$, the definition is $D_{\mathcal{A},n} = \mathbf{M}_{\mathcal{A}}(C_n) = (\prod_{i \in \mathcal{A}} (\mathbf{I} - \mathbf{P}_i) \prod_{i \notin \mathcal{A}} \mathbf{P}_i)(C_n)$.

We introduce the 2^d Cramér–von Mises (CvM) statistics, derived from the functional decomposition, and associated limits as

$$I_{\mathcal{A},n,q} = \int_{[0,1]^d} \left\{ \frac{D_{\mathcal{A},n}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 d\mathbf{x} \quad \text{and} \quad \mathbb{I}_{\mathcal{A},q} = \int_{[0,1]^d} \left\{ \frac{\mathbb{D}_{\mathcal{A}}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 d\mathbf{x}$$

for any $\mathcal{A} \in \mathcal{P}_d$. Therein, the weight function q is the function already introduced in Subsection 3.3. Replacing, in the CvM statistics $I_{\mathcal{A},n,q}$, the Lebesgue product measure $d\mathbf{x}$ on $[0,1]^d$ by the empirical $dC_n(\mathbf{x})$ is of greater interest. We thus introduce, for any $\mathcal{A} \in \mathcal{P}_d$, the alternative CvM statistics as

$$S_{\mathcal{A},n,q} = \int_{[0,1]^d} \left\{ \frac{D_{\mathcal{A},n}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 dC_n(\mathbf{x}) \quad \text{and} \quad \mathbb{S}_{\mathcal{A},q} = \int_{[0,1]^d} \left\{ \frac{\mathbb{D}_{\mathcal{A}}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 dC(\mathbf{x}).$$

Similarly, the Kolmogorov–Smirnov statistics and associated limits are defined, for any $\mathcal{A} \in \mathcal{P}_d$, by

$$U_{\mathcal{A},n,q} = \sup_{\mathbf{x} \in [0,1]^d} \left| \frac{D_{\mathcal{A},n}(\mathbf{x})}{q(\mathbf{x})} \right| \quad \text{and} \quad \mathbb{U}_{\mathcal{A},q} = \sup_{\mathbf{x} \in [0,1]^d} \left| \frac{\mathbb{D}_{\mathcal{A}}(\mathbf{x})}{q(\mathbf{x})} \right|.$$

When C_n is specifically defined as \hat{C}_n the notation $D_{\mathcal{A},n}$ becomes $\hat{D}_{\mathcal{A},n}$, and the previous statistics are denoted by $\hat{I}_{\mathcal{A},n,q}$, $\hat{S}_{\mathcal{A},n,q}$ and $\hat{U}_{\mathcal{A},n,q}$. In view of subsampling considerations, $I_{\mathcal{A},n,q}^{[I_m]}$, $S_{\mathcal{A},n,q}^{[I_m]}$ and $U_{\mathcal{A},n,q}^{[I_m]}$ (resp. $\mathbb{I}_{\mathcal{A},q}^{[I_m]}$, $\mathbb{S}_{\mathcal{A},q}^{[I_m]}$ and $\mathbb{U}_{\mathcal{A},q}^{[I_m]}$ for their limits) are obtained while $D_{\mathcal{A},n}$ (resp. $\mathbb{D}_{\mathcal{A}}$) is replaced by $D_{\mathcal{A},n}^{[I_m]}$ (resp. $\mathbb{D}_{\mathcal{A}}^{[I_m]}$).

Corollary 5. (i) *Under the assumptions of Theorem 2, the random vectors $\{I_{\mathcal{A},n,1}\}_{\mathcal{A}}$, $\{S_{\mathcal{A},n,1}\}_{\mathcal{A}}$ and $\{U_{\mathcal{A},n,1}\}_{\mathcal{A}}$ converge in distribution to $\{\mathbb{I}_{\mathcal{A},1}\}_{\mathcal{A}}$, $\{\mathbb{S}_{\mathcal{A},1}\}_{\mathcal{A}}$ and $\{\mathbb{U}_{\mathcal{A},1}\}_{\mathcal{A}}$ respectively.*

(ii) *The assumptions of Theorem 3 imply the convergence in distribution of*

$(\{I_{\mathcal{A},n,1}\}_{\mathcal{A}}, \{v_n^2 I_{\mathcal{A},b_n,1}^{[I_{1,n}]}\}_{\mathcal{A}}, \{v_n^2 I_{\mathcal{A},b_n,1}^{[I_{2,n}]}\}_{\mathcal{A}})$ to $(\{\mathbb{I}_{\mathcal{A},1}\}_{\mathcal{A}}, \{\mathbb{I}_{\mathcal{A},1}^{[1]}\}_{\mathcal{A}}, \{\mathbb{I}_{\mathcal{A},1}^{[2]}\}_{\mathcal{A}})$, where the latter is the concatenation of independent copies. Similar results hold true for the S and U families of test statistics (for U use v_n instead of v_n^2).

(iii) The assumptions of Theorem 4 imply the convergence in distribution of $(\{S_{\mathcal{A},n,q}\}_{\mathcal{A}}, \{v_n^2 S_{\mathcal{A},b_n,q}^{[I_{1,n}]}\}_{\mathcal{A}}, \{v_n^2 S_{\mathcal{A},b_n,q}^{[I_{2,n}]}\}_{\mathcal{A}})$ to $(\{\mathbb{S}_{\mathcal{A},q}\}_{\mathcal{A}}, \{\mathbb{S}_{\mathcal{A},q}^{[1]}\}_{\mathcal{A}}, \{\mathbb{S}_{\mathcal{A},q}^{[2]}\}_{\mathcal{A}})$, where the latter is the concatenation of independent copies. Similar results hold true for I and U families of test statistics (for U use v_n instead of v_n^2).

4.2 New test statistics derived from the functional decomposition

Note that under independence or independence by block, the p-values associated with the above family of test statistics are also asymptotically mutually independent (with respect to \mathcal{A}). As a consequence, individual critical values can be chosen to achieve an asymptotic global significance level. Furthermore, it is possible to combine individual p-values and get a global p-value thanks to the method *à la Fisher* (the resulting statistics are denoted by W_n) as well as *à la Tippett* (the resulting statistics is denoted T_n). For more details, we refer to the discussion in [17] or the paragraph “Combining p-values” in Section 3 of [3].

In general, under other types of null hypotheses, the asymptotic mutual independence is no more true. However, the use of the functional decomposition

$$\mathbf{M}_\emptyset(C) - C = \sum_{\mathcal{A} \in \mathcal{P}_p^*} \mathbf{M}_{\mathcal{A}}(C)$$

allows to improve the well-known statistics (that only considers the left hand member of the above equality) by combining both the left hand member and all the right hand member terms.

Let $w = \{w^{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$ be a vector of positive weights. The latter reflects the importance we put in the test $(\mathcal{H})C = \mathbf{M}_\emptyset(C)$ through w^\emptyset , or in the test $(\mathcal{H}^{\mathcal{A}}) \mathbf{M}_{\mathcal{A}}(C) = 0$ through $w^{\mathcal{A}}$. We introduce the statistic and associated limit by

$$S_{w,n,q} = \sum_{\mathcal{A} \in \mathcal{P}_d} w^{\mathcal{A}} S_{\mathcal{A},n,q} \quad \text{and} \quad \mathbb{S}_{w,q} = \sum_{\mathcal{A} \in \mathcal{P}_d} w^{\mathcal{A}} \mathbb{S}_{\mathcal{A},q}. \quad (9)$$

Similarly, the test statistics $I_{w,n,q}$ and $U_{w,n,q}$ (and respective limits $\mathbb{I}_{w,q}$ and $\mathbb{U}_{w,q}$) are the weighted linear combination constructed from the collections $I_{\mathcal{A},n,q}$ and $U_{\mathcal{A},n,q}$ (and from $\mathbb{I}_{\mathcal{A},q}$ and $\mathbb{U}_{\mathcal{A},q}$ for the limits).

Corollary 6. *The results stated in Corollary 5 are inherited by the weighted combinations $I_{w,n,q}$, $S_{w,n,q}$ and $U_{w,n,q}$. To give an example, under the assumptions of Theorem 2, the test statistic $S_{w,n,1}$ converges in distribution to $\mathbb{S}_{w,1}$.*

4.3 Practical implementation of the tests

In some particular cases, and when C_n is taken as \hat{C}_n it is possible to provide the expression of the statistics in terms of the pseudo-observations only. Let $R_{ji,n}$ denote the rank of X_{ji} among X_{1i}, \dots, X_{ni} and set $\hat{\mathbf{U}}_{j,n} = (R_{j1,n}/n, \dots, R_{jd,n}/n)$. Then,

$$S_{\mathcal{A},n,q} = \frac{1}{n} \sum_{j=1}^n \left\{ \frac{D_{\mathcal{A},n}(\hat{\mathbf{U}}_{j,n})}{q(\hat{\mathbf{U}}_{j,n})} \right\}^2.$$

In general, we proceed by numerical approximation based on a grid. Let K be a large integer and let $\mathbf{u}_1, \dots, \mathbf{u}_K$ be K uniformly spaced points on $(0, 1)^d$. Then $I_{\mathcal{A},n,q} \simeq \frac{1}{K} \sum_{k=1}^K \{D_{\mathcal{A},n}(\mathbf{u}_k)/q(\mathbf{u}_k)\}^2$ and $U_{\mathcal{A},n,q} \simeq \max_{k=1, \dots, K} |D_{\mathcal{A},n}(\mathbf{u}_k)/q(\mathbf{u}_k)|$.

Algorithm 1: Approximating the p-value (Illustrated with the S family)

Compute $S_{w,n,q}^{(0)}$ the value of $S_{w,n,q}$ on the original series

Generate from subsampling $D_{\mathcal{A},n,q}^{[k]}$ for $k = 1, \dots, N$

Compute $S_{w,n,q}^{(1)}, \dots, S_{w,n,q}^{(N)}$ the value of $S_{w,n,q}$ on one of these processes

Define an approximate p-value for the test statistic as following $\frac{1}{N+1} \left(\frac{1}{2} + \sum_{k=1}^N \mathbf{1}\{S_{w,n,q}^{(k)} > S_{w,n,q}^{(0)}\} \right)$

Taking into account Section 3, the p-values are approximately uniform on $[0, 1]$ under the null hypothesis (\mathcal{H}). In the case of goodness-of-fit tests (\mathcal{H}) $C = C_\theta$, samples of reference should not be obtained from subsampling but by parametric bootstrapping instead. The alternative version of the algorithm is given below.

Algorithm 2: Approximating the p-value (Goodness-of-fit tests)

Compute the pseudo-observations $\mathbf{U}_1, \dots, \mathbf{U}_n$ on the original series, $\theta_n = \hat{\theta}(\mathbf{U}_1, \dots, \mathbf{U}_n)$ and $S_{w,n,q, [\theta_n]}$

for $k = 1$ to N **do**

Generate $\mathbf{U}_1^{\{k\}}, \dots, \mathbf{U}_n^{\{k\}}$ from C_{θ_n}

Compute $\theta_n^{\{k\}} = \hat{\theta}(\mathbf{U}_1^{\{k\}}, \dots, \mathbf{U}_n^{\{k\}})$ and $S_{w,n,q, [\theta_n^{\{k\}]}$ on the parametric sample

end for

Define an approximate p-value for the test statistic as following

$$\frac{1}{N+1} \left(\frac{1}{2} + \sum_{k=1}^N \mathbf{1}\{S_{w,n,q, [\theta_n^{\{k\}]} > S_{w,n,q, [\theta_n]}\} \right)$$

4.4 Numerical experiments

In this final section, we shall consider the use of the functional decomposition in two experiments and analyze the results. First, we explore the testing problem of block independence and specifically one of the practical settings imagined in [3]. We investigate then the goodness-of-fit test where two Archimedean copulas, namely Clayton and Gumbel, are opposed.

4.4.1 Independence between three continuous r -dimensional random vectors

We adapt here Section 4 from [3] that implements testing procedures for block independence. Let $\mathbf{X} = (X_1, \dots, X_{12})$ and consider the 3 groups $\{X_1, \dots, X_4\}$, $\{X_5, \dots, X_8\}$ and $\{X_9, \dots, X_{12}\}$ so that $r = 4$, $p = 3$ and $d = 12$. The dependence is described by the normal copula. The $d \times d$ correlation matrices Σ are structured as follows

	X_1	...	X_4	X_5	...	X_8	X_9	...	X_{12}
X_1	1		ρ_{intra}						
\vdots									
X_4	ρ_{intra}		1		ρ_{inter}			ρ_{inter}	
X_5				1		ρ_{intra}			
\vdots									
X_8		ρ_{inter}				1		ρ_{inter}	
X_9				ρ_{intra}			1		ρ_{intra}
\vdots									
X_{12}		ρ_{inter}			ρ_{inter}		ρ_{intra}		1

The quantity ρ_{inter} (resp. ρ_{intra}) controls the amount of dependence among (resp. within) the three random vectors. Under the normal copula, the values $\rho_{\text{inter}} \in \{0.000, 0.025, 0.050, \dots, 0.275, 0.300\}$ for $\rho_{\text{intra}} = 0.5$ are considered. We generate 1000 samples composed of $n = 200$ independent realizations of \mathbf{X} . Note that, in all the simulations, the number of randomized samples is set to 1000.

Table 1 shows the rejection rates of the null hypothesis, the proportion of times that the different tests reject the null hypothesis, with respect to the value of ρ_{inter} . The significance level is arbitrarily set to 5% and measured on the first column of the table. The global Cramér-von-mises statistic $I_n = I_{\emptyset, n, 1}$, as well as W_n the test statistic *à la Fisher*, and T_n the test statistic *à la Tippett* are those studied in Figure 3 of [3], with the difference that $n = 200$ here. Four additional measures of the form $S_{w, n, 1}$ are included. Recall that taking into account the form of the null hypothesis with $p = 3$ blocks, the weights w have the following structure

$$w = (w^{\emptyset}, w^{\{1\}}, w^{\{2\}}, w^{\{3\}}, w^{\{12\}}, w^{\{13\}}, w^{\{23\}}, w^{\{123\}}).$$

More precisely, we consider

- $w_1 = (1, 0, 0, 0, 0, 0, 0, 0)$ that only measures the left hand term of the decomposition,

- $w_2 = (1, 0, 0, 0, 1, 1, 1, 1)$ that combines the left hand term of the decomposition with the right hand terms of order 2 and 3 (recall that the right hand terms associated with singletons all vanish),
- $w_3 = (0, 0, 0, 0, 1, 1, 1, 1)$ that only combines the non-null right hand terms of the decomposition.

		ρ_{inter}						
		0	0.025	0.05	0.075	0.10	0.125	0.15
Test statistics	W_n	0.060	0.146	0.332	0.540	0.770	0.901	0.982
	T_n	0.047	0.110	0.234	0.404	0.584	0.766	0.932
	I_n	0.054	0.128	0.287	0.503	0.711	0.854	0.971
	$S_{w_1, n, 1}$	0.049	0.176	0.376	0.673	0.875	0.964	0.994
	$S_{w_2, n, 1}$	0.046	0.181	0.388	0.691	0.883	0.971	0.995
	$S_{w_3, n, 1}$	0.041	0.124	0.311	0.572	0.819	0.953	0.992

Table 1 Percentage of rejection of the null hypothesis that stipulates the block independence of $\{X_1, \dots, X_4\}$, $\{X_5, \dots, X_8\}$ and $\{X_9, \dots, X_{12}\}$ that comes from Normal copula with correlation matrices Σ whereas the null hypothesis is only true under the value $\rho_{\text{inter}} = 0$. The statistics W_n , T_n and I_n are those presented in [3]. The statistics $S_{w, n, 1}$, defined by (9), are evaluated for several weights w as indicated in the text.

As can be seen in Table 1, the S -type statistics perform best among all the approaches. There is no uniformly better S , even if $S_{w_2, n, 1}$ is roughly speaking the best choice globally.

4.4.2 Goodness-of-fit tests

Turning to the Archimedean Goodness-of-Fit tests, we consider the Clayton or the Gumbel family in a 3-dimensional setting. These classes will both be used as the generator of datasets or as the family being tested. To generate the original samples, three values of Kendall's τ are chosen: $\tau = .1$, $\tau = .2$ and $\tau = .3$. Test statistics $S_{w, n, 1}$ given by (9) and where

$$w = (w^\emptyset, w^{\{1\}}, w^{\{2\}}, w^{\{3\}}, w^{\{12\}}, w^{\{13\}}, w^{\{23\}}, w^{\{123\}})$$

are computed for the weights: $w_1 = (1, 0, 0, 0, 0, 0, 0, 0)$, $w_2 = (1, 0, 0, 0, 1, 1, 1, 1)$ and $w_3 = (0, 0, 0, 0, 1, 1, 1, 1)$. The results are provided in Table 2 for $n = 100$. The first lines are dedicated to test whether the dependence structure is given by the Clayton copula. Similarly, Gumbel copula is tested in the last lines of the table. The parameter associated with the generator φ is estimated at each step as the mean of empirical Kendall's τ . The parametric bootstrap described in Algorithm 2 with $n_{\text{boot}} = 200$ is used to compute the p-value. The rejection rates are estimated through $n_{\text{rep}} = 500$ repetitions of each experiment. Two characteristics are of interest: the empirical level might be close to the nominal level, arbitrarily fixed at 0.05, and the empirical power.

The analysis of Table 2 is not straightforward but the results are interesting. First, on the right upper corner of the table, one can remark that $S_{w_1, n, 1}$ always dominates in discriminating true Gumbel from supposed Clayton. This becomes true for $S_{w_2, n, 1}$

		Simulated copulas						
		Clayton			Gumbel			
		$\tau = .1$	$\tau = .2$	$\tau = .3$	$\tau = .1$	$\tau = .2$	$\tau = .3$	
Tested copulas	Clayton	w_1	0.056	0.044	0.048	0.364	0.690	0.934
		w_2	0.052	0.044	0.038	0.294	0.674	0.924
		w_3	0.044	0.060	0.046	0.160	0.494	0.852
	Gumbel	w_1	0.282	0.720	0.922	0.070	0.048	0.056
		w_2	0.306	0.766	0.948	0.036	0.042	0.052
		w_3	0.254	0.710	0.960	0.024	0.042	0.048

Table 2 Rejection rates of the null hypothesis. In the first lines, Clayton copula is being tested, whereas Gumbel copula is tested in the last lines of the table. The datasets are simulated for three different strengths of dependence, calibrated through the Kendall's τ : .1, .2 and .3. The test statistics $S_{w,n}$ are studied for three different weights: w_1, w_2 and w_3 (definition in the text). Additional parameters of the numerical study: sample size $n = 100$, parametric bootstrap size $n_{\text{boot}} = 200$ and number of repetitions of the experiment $n_{\text{rep}} = 500$.

in the left bottom corner, except when the dependence becomes stronger with $\tau = 0.3$, the more powerful is then $S_{w_3,n,1}$.

4.4.3 Concluding remarks

The numerical section provides two simple studies, one analyzing the independence among $p = 3$ blocks in a ($d =$)12-dimensional setting and one examining particular Archimedean copula families in dimension $d = 3$. Both reveal situations where one can take advantage in applying the functional decomposition associated with the null hypothesis in order to derive powerful weighted test statistics.

5 Summary and perspectives

Identifying and modeling dependencies with copulas remain an important topic, which has become very popular over the last decades since it has been applied in almost every discipline. The aim in this paper is to unify various papers, as [2], [17], [18], [3], [4] among others, that derive copula-based tests of the structure of dependence. The solution here is to dip them in a functional decomposition context in order to reveal a common pattern.

The dimensions d or p are small in our experiments. Nevertheless, the current paper provides an interesting perspective on high dimensional problems. The practical implementation of the tests relies indeed on a trade-off between exhaustivity (all subsets of \mathcal{P}_d) and dimensionality (exponential growth in d). When d becomes larger, it could be interesting to use only part of the decomposition. With the help of the weight w introduced in the definition of the test statistics, one can focus only in a given size of subsets or in all sizes that do not exceed a given size. This way, we can control the underlying complexity of the method. The question will be then: how much this selection affects the corresponding power of the testing procedure?

6 Proofs

Proof of Proposition 1. Suppose $\mathbf{M}_{\mathcal{A}}(C) \equiv 0$ for any non-empty subset \mathcal{A} of $\{1, \dots, d\}$. By application of (1), one obtains $C - \mathbf{M}_{\emptyset}(C) \equiv 0$ which is (\mathcal{H}) . Reciprocally, if (\mathcal{H}) holds true, then $C = \mathbf{M}_{\emptyset}(C) = (\prod_{j=1}^d \mathbf{P}_j)(C)$. Combined with (2), it yields by commutativity,

$$\begin{aligned} \mathbf{M}_{\mathcal{A}}(C) &= \left(\prod_{i \in \mathcal{A}} (\mathbf{I} - \mathbf{P}_i) \prod_{i \notin \mathcal{A}} \mathbf{P}_i \right) (C) = \left(\prod_{i \in \mathcal{A}} (\mathbf{I} - \mathbf{P}_i) \prod_{i \notin \mathcal{A}} \mathbf{P}_i \right) \left(\prod_{j=1}^d \mathbf{P}_j(C) \right) \\ &= \left(\prod_{i \in \mathcal{A}} (\mathbf{P}_i - \mathbf{P}_i^2) \prod_{i \notin \mathcal{A}} \mathbf{P}_i \right) \left(\prod_{j \notin \mathcal{A}} \mathbf{P}_j(C) \right) \end{aligned}$$

which vanishes for $\mathcal{A} \neq \emptyset$, since $\mathbf{P}_i = \mathbf{P}_i^2$ by the idempotence assumption. \square

Proof of Theorem 2. By assumption, $\sqrt{n}(C_n - C) \xrightarrow[n \rightarrow \infty]{w} \mathbb{W}_C$ in $\ell^\infty([0, 1]^d)$ and any $\mathbf{M}_{\mathcal{A}}$ is Hadamard-differentiable at C . The functional version of the Delta method (see Section 3.9 of [19]) applied to $f \mapsto (f, \mathbf{M}_{\emptyset}(f), \{\mathbf{M}_{\mathcal{A}}(f)\}_{\mathcal{A} \in \mathcal{P}_d^*})$ yields

$$\begin{aligned} (\sqrt{n}(C_n - C), \sqrt{n}(\mathbf{M}_{\emptyset}(C_n) - \mathbf{M}_{\emptyset}(C)), \{\sqrt{n}(\mathbf{M}_{\mathcal{A}}(C_n) - \mathbf{M}_{\mathcal{A}}(C))\}_{\mathcal{A} \in \mathcal{P}_d^*}) &\xrightarrow[n \rightarrow \infty]{w} \\ (\mathbb{W}_C, \mathbf{M}'_{\emptyset}(C; \mathbb{W}_C), \{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C)\}_{\mathcal{A} \in \mathcal{P}_d^*}) \end{aligned}$$

in $\{\ell^\infty([0, 1]^d)\}^{2^d+1}$. From the continuous mapping theorem applied to the functional $T(f, g, \{h_{\mathcal{A}}\}_{\mathcal{A}}) = (f - g, \{h_{\mathcal{A}}\}_{\mathcal{A}})$, we obtain the weak convergence of

$$(\sqrt{n}(C_n - \mathbf{M}_{\emptyset}(C_n)) - \sqrt{n}(C - \mathbf{M}_{\emptyset}(C)), \{\sqrt{n}(\mathbf{M}_{\mathcal{A}}(C_n) - \mathbf{M}_{\mathcal{A}}(C))\}_{\mathcal{A} \in \mathcal{P}_d^*})$$

in $(\ell^\infty([0, 1]^d))^{2^d}$ to $(\mathbb{W}_C - \mathbf{M}'_{\emptyset}(C; \mathbb{W}_C), \{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C)\}_{\mathcal{A} \in \mathcal{P}_d^*})$. Now, when (\mathcal{H}) holds true, the collection of maps $\{\mathbf{P}_i\}_{i=1, \dots, d}$ is assumed to form an idempotent and commuting family. Consequently, Proposition 1 applies, $\cap_{\mathcal{A} \in \mathcal{P}_d^*} (\mathcal{H}^{\mathcal{A}})$ holds true so that $C - \mathbf{M}_{\emptyset}(C) = 0$ as well as $\mathbf{M}_{\mathcal{A}}(C) = 0$ for any $\mathcal{A} \in \mathcal{P}_d^*$. Then, the left hand side of the last convergence reduces to the process under study. \square

Proof of Theorem 3. Since by assumption both $\mathbf{M}_{\emptyset}(C) = C$ and $\mathbf{M}_{\mathcal{A}}(C) = 0$ for any subset $\mathcal{A} \in \mathcal{P}_d^*$, one can observe that,

$$D_{\emptyset, n} = \mathbb{C}_n - \sqrt{n} \left(\mathbf{M}_{\emptyset} \left(C + \frac{\mathbb{C}_n}{\sqrt{n}} \right) - \mathbf{M}_{\emptyset}(C) \right) \quad (10)$$

$$D_{\mathcal{A}, n} = \sqrt{n} \left(\mathbf{M}_{\mathcal{A}} \left(C + \frac{\mathbb{C}_n}{\sqrt{n}} \right) - \mathbf{M}_{\mathcal{A}}(C) \right) \quad (11)$$

$$D_{\emptyset, b}^{[m]} = \mathbb{C}_b^{[m]} + \sqrt{\frac{b}{n}} \mathbb{C}_n - \sqrt{b} \left(\mathbf{M}_{\emptyset} \left(C + \frac{\mathbb{C}_b^{[m]}}{\sqrt{b}} + \frac{\mathbb{C}_n}{\sqrt{n}} \right) - \mathbf{M}_{\emptyset}(C) \right) \quad (12)$$

$$D_{\mathcal{A},b}^{[m]} = \sqrt{b} \left(\mathbf{M}_{\mathcal{A}} \left(C + \frac{\mathbb{C}_b^{[m]}}{\sqrt{b}} + \frac{\mathbb{C}_n}{\sqrt{n}} \right) - \mathbf{M}_{\mathcal{A}}(C) \right) \quad (13)$$

for $m = I_{1,n}$ or $I_{2,n}$ in the last two displays. By assumption, the weak convergence of $(\mathbb{C}_n, v_n \mathbb{C}_b^{[I_{1,n}]}, v_n \mathbb{C}_b^{[I_{2,n}]})$ in $\{\ell^\infty([0,1]^d)\}^3$ to $(\mathbb{W}_C, \mathbb{W}_C^{[1]}, \mathbb{W}_C^{[2]})$ holds true. Using the fact that $b = o(n)$, the following

$$\left(\mathbb{C}_n, v_n \mathbb{C}_b^{[I_{1,n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n, v_n \mathbb{C}_b^{[I_{2,n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right)$$

shares the same asymptotic behavior. Continuous mapping theorem with

$$(f_1, f_2, f_3) \mapsto (-\mathbf{M}'_{\emptyset}(C; f_1), -\mathbf{M}'_{\emptyset}(C; f_2), -\mathbf{M}'_{\emptyset}(C; f_3), \\ \{\mathbf{M}'_{\mathcal{A}}(C; f_1)\}_{\mathcal{A}}, \{\mathbf{M}'_{\mathcal{A}}(C; f_2)\}_{\mathcal{A}}, \{\mathbf{M}'_{\mathcal{A}}(C; f_3)\}_{\mathcal{A}})$$

then delivers the weak convergence of

$$\left(-\mathbf{M}'_{\emptyset}(C; \mathbb{C}_n), -\mathbf{M}'_{\emptyset} \left(C; v_n \mathbb{C}_b^{[I_{1,n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right), -\mathbf{M}'_{\emptyset} \left(C; v_n \mathbb{C}_b^{[I_{2,n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right), \right. \\ \left. \{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{C}_n)\}_{\mathcal{A}}, \left\{ \mathbf{M}'_{\mathcal{A}} \left(C; v_n \mathbb{C}_b^{[I_{1,n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right) \right\}_{\mathcal{A}}, \right. \\ \left. \left\{ \mathbf{M}'_{\mathcal{A}} \left(C; v_n \mathbb{C}_b^{[I_{2,n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right) \right\}_{\mathcal{A}} \right)$$

in $\{\ell^\infty([0,1]^d)\}^{3 \times 2^d}$ to

$$\left(-\mathbf{M}'_{\emptyset}(C; \mathbb{W}_C), -\mathbf{M}'_{\emptyset}(C; \mathbb{W}_C^{[1]}), -\mathbf{M}'_{\emptyset}(C; \mathbb{W}_C^{[2]}), \right. \\ \left. \{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C)\}_{\mathcal{A}}, \{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C^{[1]})\}_{\mathcal{A}}, \{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C^{[2]})\}_{\mathcal{A}} \right).$$

It remains to show that, for $m = I_{1,n}$ or $I_{2,n}$,

$$\sup_{[0,1]^d} |D_{\emptyset,n} - \mathbb{C}_n + \mathbf{M}'_{\emptyset}(C; \mathbb{C}_n)| \xrightarrow{\mathbb{P}} 0 \quad ; \quad \sup_{[0,1]^d} \left| \frac{D_{\mathcal{A},n}}{q} - \mathbf{M}'_{\mathcal{A}}(C; \mathbb{C}_n) \right| \xrightarrow{\mathbb{P}} 0, \\ \sup_{[0,1]^d} \left| v_n D_{\emptyset,b}^{[m]} - v_n \mathbb{C}_b^{[m]} - v_n \sqrt{\frac{b}{n}} \mathbb{C}_n + \mathbf{M}'_{\emptyset} \left(C; v_n \mathbb{C}_b^{[m]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right) \right| \xrightarrow{\mathbb{P}} 0, \\ \sup_{[0,1]^d} \left| v_n D_{\mathcal{A},b}^{[m]} - \mathbf{M}'_{\mathcal{A}} \left(C; v_n \mathbb{C}_b^{[m]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right) \right| \xrightarrow{\mathbb{P}} 0,$$

which are, once the equations (10), (11), (12) and (13) taken into account, only the consequences of what precedes combined with the functional Delta Method applied to the maps \mathbf{M}'_{\emptyset} or $\mathbf{M}'_{\mathcal{A}}$. \square

Proof of Theorem 4. By assumption, $\left(\frac{\mathbb{C}_n}{q}, v_n \frac{\mathbb{C}_b^{[I_1, n]}}{q}, v_n \frac{\mathbb{C}_b^{[I_2, n]}}{q}\right)$ converges weakly in $\{\ell^\infty([0, 1]^d)\}^3$ to $\left(\frac{\mathbb{W}_C}{q}, \frac{\mathbb{W}_C^{[1]}}{q}, \frac{\mathbb{W}_C^{[2]}}{q}\right)$. Consequently, using the fact that $b = o(n)$, the following $\left(\frac{\mathbb{C}_n}{q}, v_n \frac{\mathbb{C}_b^{[I_1, n]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q}, v_n \frac{\mathbb{C}_b^{[I_2, n]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q}\right)$ shares the same asymptotic behaviour. Using the continuous mapping theorem with

$$(f_1, f_2, f_3) \mapsto (-\mathbf{M}'_\emptyset(C; f_1), -\mathbf{M}'_\emptyset(C; f_2), -\mathbf{M}'_\emptyset(C; f_3), \\ \{\mathbf{M}'_{\mathcal{A}}(C; f_1)\}_{\mathcal{A}}, \{\mathbf{M}'_{\mathcal{A}}(C; f_2)\}_{\mathcal{A}}, \{\mathbf{M}'_{\mathcal{A}}(C; f_3)\}_{\mathcal{A}})$$

we thus obtain that

$$\left(-\mathbf{M}'_\emptyset\left(C; \frac{\mathbb{C}_n}{q}\right), -\mathbf{M}'_\emptyset\left(C; v_n \frac{\mathbb{C}_b^{[I_1, n]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q}\right), -\mathbf{M}'_\emptyset\left(C; v_n \frac{\mathbb{C}_b^{[I_2, n]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q}\right), \\ \left\{\mathbf{M}'_{\mathcal{A}}\left(C; \frac{\mathbb{C}_n}{q}\right)\right\}_{\mathcal{A}}, \left\{\mathbf{M}'_{\mathcal{A}}\left(C; v_n \frac{\mathbb{C}_b^{[I_1, n]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q}\right)\right\}_{\mathcal{A}}, \\ \left\{\mathbf{M}'_{\mathcal{A}}\left(C; v_n \frac{\mathbb{C}_b^{[I_2, n]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q}\right)\right\}_{\mathcal{A}}\right)$$

converges weakly in $\{\ell^\infty([0, 1]^d)\}^{3 \times 2^d}$ to

$$\left(-\mathbf{M}'_\emptyset\left(C; \frac{\mathbb{W}_C}{q}\right), -\mathbf{M}'_\emptyset\left(C; \frac{\mathbb{W}_C^{[1]}}{q}\right), -\mathbf{M}'_\emptyset\left(C; \frac{\mathbb{W}_C^{[2]}}{q}\right), \\ \left\{\mathbf{M}'_{\mathcal{A}}\left(C; \frac{\mathbb{W}_C}{q}\right)\right\}_{\mathcal{A}}, \left\{\mathbf{M}'_{\mathcal{A}}\left(C; \frac{\mathbb{W}_C^{[1]}}{q}\right)\right\}_{\mathcal{A}}, \left\{\mathbf{M}'_{\mathcal{A}}\left(C; \frac{\mathbb{W}_C^{[2]}}{q}\right)\right\}_{\mathcal{A}}\right)$$

which is equal to

$$\left(-\frac{\mathbf{M}'_\emptyset(C; \mathbb{W}_C)}{q}, -\frac{\mathbf{M}'_\emptyset(C; \mathbb{W}_C^{[1]})}{q}, -\frac{\mathbf{M}'_\emptyset(C; \mathbb{W}_C^{[2]})}{q}, \\ \left\{\frac{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C)}{q}\right\}_{\mathcal{A}}, \left\{\frac{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C^{[1]})}{q}\right\}_{\mathcal{A}}, \left\{\frac{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C^{[2]})}{q}\right\}_{\mathcal{A}}\right)$$

by linearity of the Hadamard derivatives in their second argument. The result follows, when $m = I_{1, n}$ or $I_{2, n}$, from

$$\sup_{[0, 1]^d} \left| \frac{D_{\emptyset, n}}{q} - \frac{\mathbb{C}_n}{q} - \mathbf{M}'_\emptyset\left(C; \frac{\mathbb{C}_n}{q}\right) \right| \xrightarrow{\mathbb{P}} 0 \quad ; \quad \sup_{[0, 1]^d} \left| \frac{D_{\mathcal{A}, n}}{q} - \mathbf{M}'_{\mathcal{A}}\left(C; \frac{\mathbb{C}_n}{q}\right) \right| \xrightarrow{\mathbb{P}} 0, \\ \sup_{[0, 1]^d} \left| v_n \frac{D_{\emptyset, b}^{[m]}}{q} - v_n \frac{\mathbb{C}_b^{[m]}}{q} - v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q} - \mathbf{M}'_\emptyset\left(C; v_n \frac{\mathbb{C}_b^{[m]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q}\right) \right| \xrightarrow{\mathbb{P}} 0,$$

$$\sup_{[0, 1]^d} \left| v_n \frac{D_{\mathcal{A}, b}^{[m]}}{q} - \mathbf{M}'_{\mathcal{A}}\left(C; v_n \frac{\mathbb{C}_b^{[m]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q}\right) \right| \xrightarrow{\mathbb{P}} 0. \quad \square$$

Before going through the rest of the proofs, one should remark that any statement of the current paper assumes that \mathbb{C}_n converges weakly to \mathbb{W}_C given in (7). It is assumed that the latter presents continuous paths. As a consequence,

$$\|\mathbb{C}_n/\sqrt{n}\|_\infty = \|C_n - C\|_\infty \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \quad (14)$$

Proof of Corollary 5. (i) Let us focus on the S family of test statistics. To integrate with respect to C_n , we adapt the proof of [4, Proposition 3]. Applying the continuous mapping theorem, a convergence is obtained while concatenating C with the vector under study in Theorem 2. Since $\|C_n - C\|_\infty$ tends to zero in probability from (14), one can replace C by C_n on the left hand side of the concatenated convergence. Again, the continuous mapping theorem allows to conclude. (ii) - (iii) - (iv) as the proof of Corollary 6 are based on similar arguments as those used throughout the proof section. \square

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