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# Testing copula-based dependence hypotheses: a proofreading based on functional decompositions

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**Abstract:** Tests of multivariate independence may rely on asymptotically independent Cramér-von Mises statistics derived from a Möbius decomposition of the empirical copula process. We generalize this approach to some other copula-based assumptions, with the help of a functional decomposition based on commuting idempotent maps. As soon as the null hypothesis reflects the stability of the copula under the action of the composition of such operators, the methodology applies. The empirical testing process, which depends on the decomposition, allows the derivation of a new family of test statistics. The asymptotic distributions are obtained. Since the latter depend on the unknown copula being tested, we adapt the subsampling procedure and the multiplier bootstrap to our setting and recall that the parametric bootstrap also applies to approximate p-values. The benefits in deriving test statistics from a functional decomposition are illustrated and discussed through simulations.

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## 1. Introduction

The nature and strength of cross-sectional dependence is of crucial importance to understand economic or environmental systems. One possible measure relies on copulas, which have become popular over the last decades. In this paper, we review and provide a new light on the literature for some testing problems.

Consider  $\mathbf{X}_1, \dots, \mathbf{X}_n$  a sample of  $d$ -variate observations where  $\mathbf{X}_j$  stands for  $(X_{j1}, \dots, X_{jd})^T$ . At first, one may think that this  $n$ -sample consists of independent copies of a  $d$ -dimensional random vector  $\mathbf{X} = (X_1, \dots, X_d)^T$ . However, most of the results hold true for some strictly stationary time series. We assume that the cumulative distribution function (c.d.f.)  $F$  of the representative vector  $\mathbf{X}$  has continuous univariate margins denoted by  $F_1, \dots, F_d$ . There exists then a unique copula  $C : [0, 1]^d \rightarrow [0, 1]$ , that is a  $d$ -dimensional c.d.f. with

standard uniform margins such that  $F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d))$  for all  $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ . This representation, due to [51], illustrates that the copula  $C$  characterizes the dependence between the components of  $\mathbf{X}$ .

The present work is concerned with testing structural hypotheses for the copula. There exists indeed a large number of copula families and testing procedures help guide the choice of the most appropriate one. Tests based on empirical copula processes have been successfully proposed in the literature. Let us cite for instance [18], [35], [36], [24] or [9] that handled the independence, serial independence, independence by blocks, or broader classes such as extreme value copulas. Whereas earlier papers focus on one hypothesis at once, the aim of this paper is to demonstrate that several structural hypotheses for dependence share a common pattern. Our contribution is thus to unify part of the theory.

In the literature, some papers already propose a common procedure to a list of copula-based dependence hypotheses. Based on an arbitrary finite set of points in  $[0, 1]^d$ , [39] reduce the weak convergence of the renormalized empirical copula to a multivariate normal convergence. This is strongly different to our tools. The method in [43] consists in rewriting the null hypotheses with quadratic functionals. But it still differs from our methodology. So, as far as we are aware, our generalization is new.

Our procedure could be roughly illustrated by the pioneering idea of [18] which reveals the independence through the Möbius decomposition of the empirical process. The null hypothesis is thus equivalent to the intersection of a finite set of assumptions since all secondary terms of the decomposition vanish. We generalize this method by applying another functional decomposition, chosen in accordance with the structural assumption being tested. Indeed, for a given structural form of dependence, the null hypothesis is *often* characterized by the stability of the copula under the action of a transformation  $\mathbf{M}_\theta$ . It leads to the test of  $(\mathcal{H}) C = \mathbf{M}_\theta(C)$  against its negation. Under various interesting examples, the transformation  $\mathbf{M}_\theta$  can be obtained as the composition of several operators. This common functional pattern offers the possibility of generalizing the writing as unifying the method. The study based on a functional decomposition reveals that a collection of sub-hypotheses  $(\mathcal{H}^A)$  hold true under  $(\mathcal{H})$ . In consequence, new test statistics are defined by combining the information extracted from  $(\mathcal{H})$  with that extracted from any  $(\mathcal{H}^A)$ .

The remainder of this paper is organized as follows. Section 2 recalls the functional decomposition based on operators and makes it explicit in the context of dependence structures. After a first asymptotic statement, Section 3 explains how to construct independent copies of some limiting processes: both subsampling and multiplier bootstrap procedures are presented. Section 4 is devoted to the practical implementation of the theoretical results: definition of the test statistics, associated asymptotics, practical computation and approximation of the p-value are discussed. Two numerical experiments based on simulation end

this section. Concluding remarks are given in Section 5 and proofs are postponed to Section 6.

## 2. Dependence structures and associated operators

Our aim in this section is to outline a functional decomposition based on commuting idempotent maps which will be the basis of our further developments. The link with the null hypothesis is the stability of the copula under a composition of some of these operators. A collection of sub-hypotheses arises naturally. Two lists of examples are gathered in the third and fourth part: one listing the cases where the operators are always commuting and idempotent and one where all the operators reduce to the identity under the null hypothesis.

### 2.1. The functional decomposition of Kuo, Sloan, Wasilkowski, and Woźniakowski

The functional decomposition mentioned here has a long story that is nicely described in [54]. To provide a short presentation, let us start by quoting [30]. His pioneering work uses  $L^2$  projections to decompose and study  $U$ -statistics. But it is in [31] that the author proposes a recursive construction, based on conditional expectations, of what can be called the Hoeffding decomposition. Its first terms, depending on combinations of measurable functions of only one variable, corresponds to the Hoeffding projection. [19] seems to be the first reference with a clear statement and proof of the Hoeffding decomposition. It appears also in [52], with its own proof. This work had a major impact in the field of Global Sensitivity Analysis. This explains why his name is now attached to the first one. The generalization of the Hoeffding-Sobol decomposition, allowing the combination of commuting, idempotent and linear operators instead of conditional expectations, is due to [38] and extended in [41]. In the latter, the authors have relaxed the linear condition of the maps that are no longer projections.

Let  $\mathcal{F}$  be the linear space of real-valued functions acting on  $[0, 1]^d$ . For each  $i \in \{1, \dots, d\}$ , let  $\mathbf{P}_i : \mathcal{F} \rightarrow \mathcal{F}$  be an idempotent operator. We assume that the collection of functionals  $\mathbf{P}_1, \dots, \mathbf{P}_d$  commutes. Let  $\mathbf{I} : \mathcal{F} \rightarrow \mathcal{F}$  denote the identity map. Set  $\mathcal{P}_d$  as the collection of all subsets of  $\{1, \dots, d\}$  and  $\mathcal{P}_d^* = \mathcal{P}_d \setminus \emptyset$ . Fix  $\mathcal{A} \in \mathcal{P}_d$ . Note that  $-\mathcal{A}$  stands for  $\{1, \dots, d\} \setminus \mathcal{A}$ . The composition of the maps  $\mathbf{P}_i$  for  $i \in \mathcal{A}$  will be denoted as  $\prod_{i \in \mathcal{A}} \mathbf{P}_i$  or  $\mathbf{P}_{\mathcal{A}}$ , and equals  $\mathbf{I}$  in the case where  $\mathcal{A} = \emptyset$ . From Proposition 1 of [41], starting with such a collection  $(\mathbf{P}_i)_{i \in \{1, \dots, d\}}$ , every  $f \in \mathcal{F}$  can be decomposed as

$$f = \sum_{\mathcal{A} \in \mathcal{P}_d} \mathbf{M}_{\mathcal{A}}(f), \quad (1)$$

for the operator

$$\mathbf{M}_{\mathcal{A}} = \prod_{i \in \mathcal{A}} (\mathbf{I} - \mathbf{P}_i) \prod_{i \notin \mathcal{A}} \mathbf{P}_i. \quad (2)$$

The operator  $\mathbf{M}_{\mathcal{A}}$  is such that  $\mathbf{P}_i \times \mathbf{M}_{\mathcal{A}} = \mathbf{M}_{\mathcal{A}} \times \mathbf{P}_i = \mathbf{M}_{\mathcal{A}}$  whenever  $i \notin \mathcal{A}$  and vanishes otherwise. Another way of writing the equation (1) is

$$f - \mathbf{M}_{\emptyset}(f) = \sum_{\mathcal{A} \in \mathcal{P}_d^*} \mathbf{M}_{\mathcal{A}}(f), \quad (3)$$

where from (2) one knows that  $\mathbf{M}_{\emptyset} = \prod_{i=1}^d \mathbf{P}_i$ , the composition of all functionals. The main objective of this section is to identify, for some structural dependence null hypotheses  $(\mathcal{H})$ , their associated set of operators  $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$  that allows to write

$$(\mathcal{H}) C = \mathbf{M}_{\emptyset}(C) \quad \text{against} \quad (\mathcal{K}) C \neq \mathbf{M}_{\emptyset}(C).$$

## 2.2. Associated family of null sub-hypotheses

Recall that  $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$  is a collection of commuting idempotent functionals and  $(\mathcal{H}) C = \mathbf{M}_{\emptyset}(C)$  where  $\mathbf{M}_{\mathcal{A}}$  denotes the combination of (2) with this set of operators. From (3),

$$C - \mathbf{M}_{\emptyset}(C) = \sum_{\mathcal{A} \in \mathcal{P}_d^*} \mathbf{M}_{\mathcal{A}}(C).$$

As a consequence, the summation  $\sum_{\mathcal{A} \in \mathcal{P}_d^*} \mathbf{M}_{\mathcal{A}}(C)$  vanishes when  $(\mathcal{H})$  holds true. It is thus interesting to consider for any  $\mathcal{A} \in \mathcal{P}_d^*$  the null sub-hypothesis

$$(\mathcal{H}^{\mathcal{A}}) \mathbf{M}_{\mathcal{A}}(C) = 0. \quad (4)$$

A relevant question is to analyze whether any  $(\mathcal{H}^{\mathcal{A}})$  holds true under the null hypothesis  $(\mathcal{H})$ . What is its link exactly with the intersection? In the next proposition, we answer part of the question.

**Proposition 2.1.** *Let  $\mathbf{P}_1, \dots, \mathbf{P}_d$  be a commuting collection of idempotent operators on  $\mathcal{F}$ . Then, the null hypothesis satisfies the equality*

$$(\mathcal{H}) = \bigcap_{\mathcal{A} \in \mathcal{P}_d^*} (\mathcal{H}^{\mathcal{A}}).$$

**Remark 1.** *When  $(\mathcal{H})$  reflects the independence among subvectors, as it is discussed in Subsection 2.3.1, note that Proposition 7 in [35] (restricted to  $H$  being a copula) can be viewed as a consequence of Proposition 2.1.*

## 2.3. Examples of null hypotheses and associated maps

We discuss a list of null hypotheses, describing a type of dependence or partially characterizing the copula. The independence by blocks (including the complete independence), a weak form of associativity as well as particular Archimedean or Archimax copula are treated. Each associated alternative, denoted by  $(\mathcal{K})$  above, represents the negation of the null hypothesis. In these examples, the associated maps are always both commuting and idempotent, regardless of whether the null hypothesis is true or not.

### 2.3.1. Independence among subvectors

The copula approach, for testing mutual independence of the components of  $\mathbf{X}$ , starts with the very well known contribution of [18]. Subsequent analyses are recalled below in Remark 2.

Set  $\mathcal{F}_1 = \{f \in \mathcal{F}, f(\mathbf{1}) \neq 0\}$ . Note that  $\mathcal{F}_1$  contains the set of copulas. Let us structure the random vector  $\mathbf{X}$  as the concatenation of  $p$  subvectors  $\mathbf{X} = (\mathbf{X}_{\{1\}}, \dots, \mathbf{X}_{\{p\}})$  of dimension  $d_1, \dots, d_p$ . Therefore,  $d = d_1 + \dots + d_p$ . For  $\mathbf{x}, \mathbf{y}$  in  $\mathbb{R}^d$ ,  $(\mathbf{y}_{\{i\}}, \mathbf{x}_{-\{i\}})$  stands for  $(\mathbf{x}_{\{1\}}, \dots, \mathbf{x}_{\{i-1\}}, \mathbf{y}_{\{i\}}, \mathbf{x}_{\{i+1\}}, \dots, \mathbf{x}_{\{p\}})$ . Define, for  $i \in \{1, \dots, p\}$ , the map  $\mathbf{P}_i : \mathcal{F}_1 \rightarrow \mathcal{F}_1$  as following

$$\mathbf{P}_i(f)(\mathbf{x}) = \frac{f(\mathbf{x}_{\{i\}}, \mathbf{1}_{-\{i\}}) \cdot f(\mathbf{1}_{\{i\}}, \mathbf{x}_{-\{i\}})}{f(\mathbf{1}_{\{1\}}, \dots, \mathbf{1}_{\{p\}})}, \quad (5)$$

One could check that  $\{\mathbf{P}_1, \dots, \mathbf{P}_p\}$  defines a collection of commuting and idempotent operators. The assertion  $(\mathcal{H}_1) \mathbf{X}_{\{1\}}, \dots, \mathbf{X}_{\{p\}}$  are independent, or equivalently

$$(\mathcal{H}_1) C(\mathbf{x}) = C(\mathbf{x}_{\{1\}}, \mathbf{1}_{-\{1\}}) \cdots C(\mathbf{x}_{\{i\}}, \mathbf{1}_{-\{i\}}) \cdots C(\mathbf{x}_{\{p\}}, \mathbf{1}_{-\{p\}})$$

can be rewritten as  $(\mathcal{H}_1) C = \mathbf{M}_\emptyset(C)$  where  $\mathbf{M}_A$  stands now for the combination of (2) and (5).

**Remark 2.** Assume  $p = d$  so that  $(\mathcal{H}_1)$  describes the complete independence among all components of  $\mathbf{X}$ . The projection (5), applied to a copula  $C$ , satisfies

$$\mathbf{P}_i(C)(\mathbf{x}) = x_i \cdot C(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d)$$

for  $i = 1, \dots, d$ . As already mentioned in [41], the functional decomposition (1) associated with the map (2) and above projection is known as the Möbius decomposition. As explained and extended in [26] or in the introduction of [25], the pioneering work of [18] makes already use, without naming it, of the Möbius decomposition. Some results for the case where the number of variables is of the same size, or even larger, than the sample size, have recently been obtained by [9]. However, none of these works do a presentation in terms of commuting idempotent maps.

**Remark 3.** In the more general case where  $p$  is possibly strictly lower than  $d$ , the reader should be aware that [35] have successfully handled the question of testing  $(\mathcal{H}_1)$  with the help of the extended Möbius decomposition. The latter is the one which combines (1), (2) and (5). As already mentioned in [41], Lemma 6 in [35] is then a particular case of (1).

### 2.3.2. Associativity

According to [53], a function  $f : [0, 1]^d \rightarrow [0, 1]$  is said associative whenever for  $(x_1, \dots, x_d, \dots, x_{2d-1}) \in [0, 1]^{2d-1}$  it holds

$$f(f(x_1, \dots, x_d), x_{d+1}, \dots, x_{2d-1}) = \dots = f(x_1, \dots, x_{d-1}, f(x_d, \dots, x_{2d-1})).$$

Any Archimedean copula  $C$  is associative. In particular, under archimedeanity

$$C(x_1, \dots, x_i, C(1, \dots, 1, x_{i+1}, \dots, x_d), 1, \dots, 1) = C(x_1, \dots, x_d),$$

for all  $\mathbf{x} = (x_1, \dots, x_d) \in [0, 1]^d$ , and any  $i \in \{1, \dots, d-1\}$ . We thus consider

$$\mathbf{P}_i(f)(\mathbf{x}) = f(x_1, \dots, x_i, f(1, \dots, 1, x_{i+1}, \dots, x_d), 1, \dots, 1), \quad (6)$$

completed by  $\mathbf{P}_d = \mathbf{I}$ . Note that  $\mathbf{P}_{d-1} = \mathbf{I}$ . The definition (6) is an arbitrary choice of maps in comparison with the natural previous example. To be more explicit,  $\mathbf{P}_i(f)(\mathbf{x}) = f(1, \dots, 1, f(x_1, \dots, x_i, 1, \dots, 1), x_{i+1}, \dots, x_d)$  would be convenient, and it is not the only one. Restricted to functions  $f$  satisfying  $f(\mathbf{1}_{-i}, x_i) = x_i$  for any  $i$  in  $\{1, \dots, d\}$  which of course includes copulas, each operator above is idempotent and the family commutes. We do not characterize archimedeanity this way, only a weak version. However, it remains interesting to investigate  $(\mathcal{H}_2)C = \mathbf{M}_\emptyset(C)$  when  $\mathbf{M}_\mathcal{A}$  is given by the combination of (2) and (6). It is not so obvious what  $C = \mathbf{M}_\emptyset(C)$  means here. It can be written  $C(\mathbf{x}) = C(x_1, z_2, 1, \dots, 1)$  with  $z_i = C(1, \dots, 1, x_i, z_{i+1})$  for  $i = 2, \dots, d-1$  and  $z_d = x_d$ .

**Remark 4.** Consider the 3-dimensional copula  $C(x_1, x_2, x_3) = x_1 \cdot \min(x_2, x_3)$ . It satisfies  $(\mathcal{H}_2)$  but it is non-Archimedean. More generally, testing archimedeanity consists in testing associativity as well as an appropriate condition on the behavior of  $C$  on the diagonal. For complete details we refer to [8].

### 2.3.3. Describing a given Archimedean copula

The next null hypotheses need some preliminary remarks. Goodness-of-fit tests arise when  $C$  is unknown but assumed to belong to a particular class  $(\tilde{\mathcal{H}})C \in \{C_\theta, \theta \in \Theta\}$  where  $\Theta$  is an open subset of  $\mathbb{R}^D$  for some integer  $D \geq 1$ . Natural tests consist in measuring a “distance” between the empirical copula and an estimate of  $C$  obtained under  $(\tilde{\mathcal{H}})$ . We skip here the details for the inference on the parameter and we focus below only on the testing part of the procedure. The null hypothesis becomes  $(\mathcal{H})C = C_{\theta_0}$  where the reader should think of  $C_{\theta_0}$  as  $C_{\hat{\theta}_n}$ , where  $\hat{\theta}_n$  estimates  $\theta$  in  $\Theta$ . Of course, convergence of the practical procedure with respect to  $(\tilde{\mathcal{H}})$  needs appropriate regularity conditions on both the parametric family and the sequence of estimators  $\hat{\theta}_n$ . We refer to [28] for a review and discussion on combining both testing steps.

We focus here on testing a given Archimedean copula which differs from testing archimedeanity. Fix  $\varphi$  the generator of interest that is a non-negative, continuous, strictly decreasing and convex function defined on  $[0, 1]$  satisfying  $\varphi(1) = 0$ . Its pseudo-inverse, denoted  $\varphi^{[-1]}$ , is defined as the usual inverse on  $[0, \varphi(0)]$  and equals 0 elsewhere. See McNeil and Neslehova (2009) for a complete characterization of the generator  $\varphi$ . For  $i \in \{1, \dots, d\}$ , consider the functional

$$\mathbf{P}_i(f)(\mathbf{x}) = \varphi^{[-1]}[\varphi(f(x_i, \mathbf{1}_{-i})) + \varphi(f(\mathbf{1}_i, \mathbf{x}_{-i})) - \varphi(f(\mathbf{1}))]. \quad (7)$$

It forms a set of commuting idempotent maps. Testing the specific Archimedean copula generated by  $\varphi$

$$(\mathcal{H}_3) C(\mathbf{x}) = \varphi^{[-1]} [\varphi(x_1) + \cdots + \varphi(x_d)] \quad (8)$$

can be rewritten as  $(\mathcal{H}_3) C = \mathbf{M}_\theta(C)$  where  $\mathbf{M}_\theta$  combines (2) with maps given in (7).

**Remark 5.** *The symmetric logistic extreme value copula is a particular case. Let  $\ell : [0, \infty]^d \rightarrow [0, \infty]$  be a stable tail dependence function. Recall that an extreme value copula can be written as  $C(\mathbf{x}) = \exp[-\ell\{-\ln(x_1), \dots, -\ln(x_d)\}]$ . See for instance Chapter 7 of [14] for more details on the  $\ell$  function. It is called the symmetric logistic extreme value copula model when there exists a real  $\theta \in [1, \infty[$  such that  $\ell(x_1, \dots, x_d) = (x_1^\theta + \cdots + x_d^\theta)^{1/\theta}$ . Testing the symmetric logistic extreme value model*

$$(\mathcal{H}_4) C(\mathbf{x}) = \exp \left[ - \left\{ (-\ln(x_1))^\theta + \cdots + (-\ln(x_d))^\theta \right\}^{1/\theta} \right]$$

corresponds to  $(\mathcal{H}_4) C = \mathbf{M}_\theta(C)$  as soon as  $\mathbf{M}_\theta$  stands for (2) with maps (8) and substituting  $\varphi(t) = (-\ln(t))^\theta$ .

#### 2.3.4. Extension to Archimedean by blocks

A possible extension consists in mixing previous sections. Recall that the random vector  $\mathbf{X}$  might be seen as the concatenation of  $p$  subvectors  $\mathbf{X}_{\{1\}}, \dots, \mathbf{X}_{\{p\}}$  of dimension  $d_1, \dots, d_p$  where  $d = d_1 + \cdots + d_p$ . The independence by blocks of Section 2.3.1 could be replaced by an Archimedean structure by blocks. Let  $\varphi$  be a generator as described in Section 2.3.3. Then the null hypothesis

$$(\mathcal{H}_5) C(\mathbf{x}) = \varphi^{-1} [\varphi(C(\mathbf{x}_{\{1\}}, \mathbf{1}_{-\{1\}})) + \cdots + \varphi(C(\mathbf{1}_{-\{p\}}, \mathbf{x}_{\{p\}}))] ]$$

could be written as  $(\mathcal{H}_5) C = \mathbf{M}_\theta(C)$  when  $\mathbf{M}_\theta$  calls (2) while using the following functionals

$$\mathbf{P}_i(f)(\mathbf{x}) = \varphi^{-1} [\varphi(f(\mathbf{x}_{\{i\}}, \mathbf{1}_{-\{i\}})) - \varphi(f(\mathbf{1})) + \varphi(f(\mathbf{x}_{-\{i\}}, \mathbf{1}_{\{i\}}))] . \quad (9)$$

Copulas which satisfy  $(\mathcal{H}_5)$  have an easy interpretation. Only  $p$ -uplets of variables, each belonging to one of the  $p$  blocks, are completely specified: Their dependence structure follows the Archimedean copula generated by  $\varphi$ . The dependence within any groups of variables belonging partially to the same block is not fixed. This differs from the notion of nested or hierarchical copulas.

#### 2.3.5. Extension to some specific Archimax copulas

As before, consider  $\varphi$  a generator associated with an Archimedean structure. And consider  $\ell : [0, \infty]^d \rightarrow [0, \infty]$  a stable tail dependence function. Recall from [12] or more recently from [13] that

$$C(\mathbf{x}) = \varphi^{-1} [\ell(\varphi(x_1), \dots, \varphi(x_d))] ]$$



is called an Archimax copula. We restrict here the form of  $\ell$  as following

$$\ell(x_1, \dots, x_d) = g^{-1} \left[ g\{\ell(\mathbf{x}_{\{1\}}, \mathbf{0}_{-\{1\}})\} + \dots + g\{\ell(\mathbf{0}_{-\{p\}}, \mathbf{x}_{\{p\}})\} \right], \quad (10)$$

where  $g$  is a continuous bijection from  $\mathbb{R}_+$  to  $\mathbb{R}_+$  satisfying  $g(1) = 1$ . From Theorem 6 in [45], one knows that  $g(x) = x^\theta$  for some  $\theta \geq 1$ . For the sake of simplicity, set  $\varphi_{\mathbf{x}_{\{i\}}} = \sum_{j \in \{i\}} \varphi(x_j) \mathbf{e}_j$ . For  $i \in \{1, \dots, p\}$ , let define  $\mathbf{P}_i$  by

$$\mathbf{P}_i(f)(\mathbf{x}) = \varphi^{-1} \left[ \left\{ \left( \ell(\varphi_{\mathbf{x}_{\{i\}}}, \mathbf{0}_{-\{i\}}) \right)^\theta + \left( \varphi \circ f(\mathbf{1}_{\{i\}}, \mathbf{x}_{-\{i\}}) \right)^\theta \right\}^{1/\theta} \right] \quad (11)$$

completed by  $\mathbf{P}_{p+1} = \dots = \mathbf{P}_d = \mathbf{I}$ . They form a commuting set of idempotent maps. Then, testing

$$(\mathcal{H}_6) C(\mathbf{x}) = \varphi^{-1} \left[ \left\{ \left( \ell(\varphi_{\mathbf{x}_{\{1\}}}, \mathbf{0}_{-\{1\}}) \right)^\theta + \dots + \left( \ell(\mathbf{0}_{-\{p\}}, \varphi_{\mathbf{x}_{\{p\}}}) \right)^\theta \right\}^{1/\theta} \right]$$

can be rewritten  $(\mathcal{H}_6) C = \mathbf{M}_\theta(C)$  for  $\mathbf{M}_\mathcal{A}$  combining (2) and (11). As already explained in the first lines of Subsection 2.3.3, the combination of both inferring (here on  $\theta, \ell$  and  $\varphi$ , see [13]) and testing steps is theoretically more interesting. This goes however beyond the scope of this paper.

#### 2.4. When maps involved reduce to the identity under the null hypothesis

Another list of null hypotheses is drawn up below. The main difference with what precedes concerns the operators which are no more idempotent nor commuting in general. But they will simplify to the identity operator (which of course is idempotent) under the null hypothesis. Consequently,  $\mathbf{M}_\theta(C) = C$  and  $\mathbf{M}_\mathcal{A}(C) \equiv 0$  for any  $\mathcal{A} \in \mathcal{P}_d^*$ . Since any  $(\mathcal{H}^\mathcal{A})$  holds true,  $(\mathcal{H}) \subseteq \bigcap_{\mathcal{A}} (\mathcal{H}^\mathcal{A})$ .

##### 2.4.1. The max-stability assumption

For a given positive integer  $r$ , let us consider the null hypothesis

$$(\mathcal{H}_{7,r}) C(\mathbf{x}) = C^r(\mathbf{x}^{1/r}) \quad \forall \mathbf{x} \in [0, 1]^d.$$

The max-stability assumption is the intersection of any such null hypothesis, that is  $(\mathcal{H}_7) = \bigcap_{r \in \mathbb{N}^*} (\mathcal{H}_{7,r})$ .

**Remark 6.** *The presentation above is similar to the description done in Section 3 of [36]. Of course, the analysis in terms of specific maps injected in (2), and therefore in (1), is new.*

Let  $r_i$  be a positive integer. Consider the functional

$$\mathbf{P}_i(f)(\mathbf{x}) = f^{r_i}(\mathbf{x}^{1/r_i}), \quad (12)$$

which, in general, does not satisfy the idempotence assumption unless  $f$  satisfies  $(\mathcal{H}_7)$ , or more precisely  $(\mathcal{H}_{7,r_i})$ . In this case, the operator  $\mathbf{P}_i$  becomes the identity operator  $\mathbf{I}$ . As a consequence, under  $(\mathcal{H}_7)$  the copula  $C$  satisfies  $C = \mathbf{M}_\emptyset(C)$  where  $\mathbf{M}_A$  stands now for the combination of (2) with (12).

**Remark 7.** *In the bivariate setting,  $\mathbf{P}_1$  and  $\mathbf{P}_2$  should be chosen in accordance with Proposition 4.2 of [34]. Therein, the equivalence is proven between  $(\mathcal{H}_7)$  and the finite intersection  $(\mathcal{H}_{7,r_1}) \cap (\mathcal{H}_{7,r_2})$  for two positive reals  $r_1$  and  $r_2$  such that  $\log r_1 / \log r_2$  is irrational.*

#### 2.4.2. Exchangeability

Let  $\mathfrak{S}_d$  be the set of all permutations of  $\{1, \dots, d\}$  and set  $\mathbf{x}_\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(d)})$  for any  $\sigma \in \mathfrak{S}_d$ . Testing symmetry of the copula can be handled in a very similar way to the previous one. The null hypothesis of exchangeability of the random variables  $X_1, \dots, X_d$  can indeed be expressed as

$$(\mathcal{H}_8) \quad C(\mathbf{x}) = C(\mathbf{x}_\sigma) \quad \forall \mathbf{x} \in [0, 1]^d \text{ and } \forall \sigma \in \mathfrak{S}_d .$$

It is the (finite here) intersection over  $\mathfrak{S}_d$  of null hypotheses of the form

$$(\mathcal{H}_{8,\sigma}) \quad C(\mathbf{x}) = C(\mathbf{x}_\sigma) \quad \forall \mathbf{x} \in [0, 1]^d .$$

Let  $T_{1,d}$  denote the set consisting of the  $d - 1$  transpositions  $\tau_i = (1i)$  for  $i = 2, \dots, d$ . Noting that  $T_{1,d}$  generates  $\mathfrak{S}_d$ , it is also possible to write here that  $(\mathcal{H}_8) = \bigcap_{i=2}^d (\mathcal{H}_{8,\tau_i})$ . Consider the functional

$$\mathbf{P}_i(f)(\mathbf{x}) = f(\mathbf{x}_{\tau_i}) , \tag{13}$$

for  $i = 2, \dots, d$  and complete the collection with  $\mathbf{P}_1 = \mathbf{I}$  the identity map. The map in (13) is no more idempotent. Additionally,  $\mathbf{P}_k$  and  $\mathbf{P}_m$  do not commute in general. However, under the restriction that  $f$  is assumed to be symmetric,  $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$  forms a set of commuting projections since all reduce to the identity  $\mathbf{I}$ . Under  $(\mathcal{H}_8)$  the copula  $C$  satisfies  $C = \mathbf{M}_\emptyset(C)$  where  $\mathbf{M}_A$  combines (2) with (13).

**Remark 8.** *In the bivariate case, such a test has been first investigated by [22]. Testing exchangeability for copulas in arbitrary dimensions is studied in [29].*

#### 2.4.3. Summary

The interest of the preceding two lists (Subsections 2.3 and 2.4) is to show the extent to which the functional decomposition given through a set of operators applies successfully. It brings together, in a common scheme and writing, some very varied structural copula-based dependence hypotheses. However, in cases as those described in Subsection 2.4, the decomposition brings no additional

information. Note indeed that the composition  $\prod_{i=1}^d \mathbf{P}_i$  in the exchangeability context (13) is nothing other than the single  $\mathbf{P}(f)(\mathbf{x}) = f(\mathbf{x}_{\tau_1 \circ \dots \circ \tau_d})$ . Analogously, under the max-stability hypothesis, the composition  $\prod_{i=1}^d \mathbf{P}_i$  when  $\mathbf{P}_i$  is defined by (12) reduces to the single  $\mathbf{P}(f)(\mathbf{x}) = f^p(\mathbf{x}^{1/p})$  with  $p = r_1 \times \dots \times r_d$ . Consequently, the analysis of the terms  $\mathbf{M}_{\mathcal{A}}(C)$  will not add any information in comparison with the classical measure based on  $\mathbf{M}_{\emptyset}(C) - C$  or any combination of the latter (in the max-stability example, combining different values of  $r$ ).

### 3. Behavior and approximation of the testing process

The purpose of this section is to introduce the empirical testing processes. Consider a structural dependence hypothesis for copulas expressed as  $(\mathcal{H})C = \mathbf{M}_{\emptyset}(C)$ . Recall that  $\{\mathbf{M}_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$  is the set depending, through Formula (2), on a collection of operators  $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$  defined on  $\mathcal{F}$ . It is assumed that  $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$  are commuting and idempotent maps, at least when  $(\mathcal{H})$  holds true. This is the case of any previous example. Starting from a copula estimator  $C_n$ , it is natural to construct the testing process as  $(\sqrt{n}(C_n - \mathbf{M}_{\emptyset}(C_n)))(\mathbf{x})$ ,  $\mathbf{x} \in [0, 1]^d$  when considering  $(\mathcal{H})$ . This is precisely what is done in the literature. Nevertheless, since  $(\mathcal{H})$  implies any sub-hypothesis  $(\mathcal{H}^{\mathcal{A}})\mathbf{M}_{\mathcal{A}}(C) = 0$ , another choice is possible.

#### 3.1. Weak convergence of the empirical processes

Consider  $\mathbf{X}_1, \dots, \mathbf{X}_n$  a sample of  $d$ -variate observations of  $\mathbf{X}$  where  $\mathbf{X}_j$  stands for  $(X_{j1}, \dots, X_{jd})^T$ . Set  $\mathbf{U}_j = (F_1(X_{j1}), \dots, F_d(X_{jd}))$  for  $j \in \{1, \dots, n\}$ . The empirical cumulative distribution function based on  $\mathbf{U}_1, \dots, \mathbf{U}_n$  is denoted by  $G_n$  and we set  $\mathbb{G}_n = \sqrt{n}(G_n - C)$ . Under regular conditions, the empirical process  $\mathbb{G}_n$  converges weakly in  $\ell^\infty([0, 1]^d)$  to a tight centered Gaussian process  $\mathbb{G}_C$  concentrated on

$$\mathcal{C}_0 = \{h \in C([0, 1]^d) \text{ such that } h(\mathbf{1}) = 0 \text{ and } h(\mathbf{x}) = 0 \text{ if some components of } \mathbf{x} \text{ are equal to } 0\}. \quad (14)$$

Throughout the paper, we assume the existence and the paths continuity of

$$\mathbb{W}_C(\mathbf{x}) = \mathbb{G}_C(\mathbf{x}) - \sum_{i=1}^d \partial C_i(\mathbf{x}) \mathbb{G}_C(x_i, \mathbf{1}_{-i}), \quad \mathbf{x} \in [0, 1]^d. \quad (15)$$

We introduce and study in this paper the concatenated empirical testing process

$$\left( \sqrt{n}(C_n - \mathbf{M}_{\emptyset}(C_n)), \{\sqrt{n}\mathbf{M}_{\mathcal{A}}(C_n)\}_{\mathcal{A} \in \mathcal{P}_d^*} \right).$$

**Theorem 3.1.** *Consider  $(\mathcal{H})C = \mathbf{M}_{\emptyset}(C)$ , depending through (2) on a set of operators  $\mathbf{P}_1, \dots, \mathbf{P}_d$ . It is assumed, at least when  $(\mathcal{H})$  holds true, that*

- $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$  are commuting and idempotent maps.

- The associated maps  $\{\mathbf{M}_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$  are Hadamard-differentiable at  $C$  tangentially to  $\mathcal{C}_0$ .

Consider an empirical copula  $C_n$  such that, as  $n$  tends to infinity, the empirical copula process  $\sqrt{n}(C_n - C)$  converges weakly in  $\ell^\infty([0, 1]^d)$  to  $\mathbb{W}_C$  given in (15). Then, under  $(\mathcal{H})$  and as  $n$  tends to infinity, the joint empirical processes converge weakly in  $\{\ell^\infty([0, 1]^d)\}^{2^d}$  as following

$$\left( \sqrt{n}(C_n - \mathbf{M}_\emptyset(C_n)), \{\sqrt{n}\mathbf{M}_{\mathcal{A}}(C_n)\}_{\mathcal{A} \in \mathcal{P}_d^*} \right) \xrightarrow[n \rightarrow \infty]{w} \left( \mathbb{W}_C - \mathbf{M}'_\emptyset(C; \mathbb{W}_C), \{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C)\}_{\mathcal{A} \in \mathcal{P}_d^*} \right). \quad (16)$$

**Remark 9.** The last lines of Section 2 in [37] list carefully the conditions under which  $\sqrt{n}(C_n - C) \xrightarrow[n \rightarrow \infty]{w} \mathbb{W}_C$  in  $\ell^\infty([0, 1]^d)$  for the following list of well-known empirical copulas

- the non-parametric estimators

$$\tilde{C}_n(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \prod_{i=1}^d \mathbf{1}_{\{F_{n,j}(X_{ji}) \leq x_i\}},$$

and

$$\hat{C}_n(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \prod_{i=1}^d \mathbf{1}_{\{\hat{U}_{ji,n} \leq x_i\}},$$

where  $\hat{U}_{ji,n} = R_{ji,n}/(n+1)$  and  $R_{ji,n} = \text{rank of } X_{ji} \text{ among } X_{1i}, \dots, X_{ni}$ ,

- the empirical checkerboard copula

$$C_n^\#(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \prod_{i=1}^d \min\{\max\{nx_i - R_{ji,n}, 0\}, 1\},$$

- and the empirical beta copula,

$$C_n^\beta(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \prod_{i=1}^d F_{n,R_{ji,n}}(x_i).$$

where  $F_{n,r}$  stands for the pdf of the Beta distribution  $\mathcal{B}(r, n+1-r)$ .

**Remark 10.** Consider the null hypothesis  $(\mathcal{H}_1)$  where the independence among subvectors is investigated. Note that part of Theorem 8 in [35] is recovered by Theorem 3.1. Let us provide another example. When testing  $(\mathcal{H}_{7,r})$ , a sub-max-stability assumption, set  $\mathbf{P}_1(f)(\mathbf{x}) = f^r(\mathbf{x}^{1/r})$  and complete the collection with the identity map. Then  $\mathbf{M}_\emptyset = \mathbf{P}_1$  and the convergence of the left component in (16) is exactly the one studied in the limit theorem given by [36] in their Proposition 1.

The weak convergence of the empirical processes has just been proved but the covariance structures of the limiting processes  $\mathbb{D}_\emptyset := \mathbb{W}_C - \mathbf{M}'_\emptyset(C; \mathbb{W}_C)$  and  $\mathbb{D}_\mathcal{A} := \mathbf{M}'_\mathcal{A}(C; \mathbb{W}_C)$  depend on the unknown copula  $C$ . For this reason it is not always directly applicable for statistical testing. Multiplier bootstrap and subsampling have been introduced in the literature to reproduce independently the asymptotic behavior of such processes. Both are discussed in the rest of this section.

### 3.2. Subsampling empirical testing processes

To simplify notation, set  $D_{\emptyset,n} := \sqrt{n}(C_n - \mathbf{M}_\emptyset(C_n))$  and  $D_{\mathcal{A},n} := \sqrt{n}\mathbf{M}_\mathcal{A}(C_n)$  for  $\mathcal{A} \in \mathcal{P}_d^*$ , as well as  $\mathbb{C}_n := \sqrt{n}(C_n - C)$ . Theorem 3.1 has stated the weak convergence of  $\{D_{\mathcal{A},n}\}_{\mathcal{A} \in \mathcal{P}_d}$  to  $\{\mathbb{D}_\mathcal{A}\}_{\mathcal{A} \in \mathcal{P}_d}$  under some conditions, in particular the weak convergence of  $\mathbb{C}_n$ . The subsampling method is a first substitute to approximate the limiting distribution. See [42] and its adaptation in [37] that inspired this development.

Let  $b < n$  denote the size of the samples extracted from  $\mathbf{X}_1, \dots, \mathbf{X}_n$  and let  $B_{b,n}$  be the corresponding number of possible subsamples. Under the i.i.d. setting,  $B_{b,n} = \binom{n}{b}$  since the subsamples may be obtained without replacement. Since  $B_{b,n}$  might be too large, the complete enumeration could not be possible. The practical solution is to obtain its stochastic approximation through a  $N$ -sample of integers  $I_{1,n}, \dots, I_{N,n}$  independently extracted with replacement from  $\{1, \dots, B_{b,n}\}$ . The quantities of interest would be then computed for these  $N$  values of the index  $m$ . In the serial context, the sampling should preserve the dependence so that the subsamples must have consecutive index, and  $B_{b,n} = n - b + 1$ . Denote by  $C_b^{[m]}$  the replicates of the estimator  $C_n$  evaluated on such  $b$ -subsamples. Set now  $\mathbb{C}_b^{[m]} = \sqrt{b}(C_b^{[m]} - C_n)$  the associated replicates of the empirical copula processes. Set finally  $D_{\emptyset,b}^{[m]} = \sqrt{b}(C_b^{[m]} - \mathbf{M}_\emptyset(C_b^{[m]}))$  and  $D_{\mathcal{A},b}^{[m]} = \sqrt{b}\mathbf{M}_\mathcal{A}(C_b^{[m]})$ .

**Theorem 3.2.** *Consider  $(\mathcal{H}) C = \mathbf{M}_\emptyset(C)$ , depending through (2) on a set of operators  $\mathbf{P}_1, \dots, \mathbf{P}_d$ . It is assumed, at least when  $(\mathcal{H})$  holds true, that*

- $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$  are commuting and idempotent maps.
- The associated maps  $\{\mathbf{M}_\mathcal{A}\}_{\mathcal{A} \in \mathcal{P}_d}$  are Hadamard-differentiable at  $C$  tangentially to  $\mathcal{C}_0$ .
- The derivatives  $\{\mathbf{M}'_\mathcal{A}(C; \cdot)\}_{\mathcal{A} \in \mathcal{P}_d}$  are continuous on  $\ell^\infty([0, 1]^d)$ .

Let  $v_n \rightarrow 1$  and  $b_n \rightarrow \infty$  such that  $b_n = o(n)$ . Consider an empirical copula  $C_n$  such that  $(\mathbb{C}_n, v_n \mathbb{C}_{b_n}^{[I_{1,n}]}, v_n \mathbb{C}_{b_n}^{[I_{2,n}]})$  converges weakly in  $\{\ell^\infty([0, 1]^d)\}^3$ , as  $n$  tends to infinity, to  $(\mathbb{W}_C, \mathbb{W}_C^{[1]}, \mathbb{W}_C^{[2]})$  where  $\mathbb{W}_C^{[1]}$  and  $\mathbb{W}_C^{[2]}$  are independent copies of  $\mathbb{W}_C$ . Then, under  $(\mathcal{H})$  and as  $n$  tends to infinity,

$$\left( \{D_{\mathcal{A},n}\}_{\mathcal{A} \in \mathcal{P}_d}, \left\{ v_n D_{\mathcal{A},b_n}^{[I_{1,n}]} \right\}_{\mathcal{A} \in \mathcal{P}_d}, \left\{ v_n D_{\mathcal{A},b_n}^{[I_{2,n}]} \right\}_{\mathcal{A} \in \mathcal{P}_d} \right)$$

converges weakly in  $\{\ell^\infty([0, 1]^d)\}^{3 \times 2^d}$  to  $(\{\mathbb{D}_{\mathcal{A}}\}_{\mathcal{A}}, \{\mathbb{D}_{\mathcal{A}}^{[1]}\}_{\mathcal{A}}, \{\mathbb{D}_{\mathcal{A}}^{[2]}\}_{\mathcal{A}})$ , where  $\{\mathbb{D}_{\mathcal{A}}^{[2]}\}_{\mathcal{A} \in \mathcal{P}_d}$  and  $\{\mathbb{D}_{\mathcal{A}}^{[2]}\}_{\mathcal{A} \in \mathcal{P}_d}$  are independent copies of  $\{\mathbb{D}_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$ .

**Remark 11.** Previous result assumes the subsampling of the empirical copula process  $\mathbb{C}_n$ . Theorem 3.3 in [37] states such a convergence

- under i.i.d. observations,  $b/n \rightarrow \alpha \in [0, 1)$ ,  $v_n = (1 - b/n)^{-1/2}$  when  $C_n$  stands for  $\hat{C}_n, C_n^\#$  or  $C_n^\beta$ ,
- under strictly stationary alpha-mixing sequence with  $\alpha(k) = O(k^{-a})$  for some  $a > 1$ ,  $b/n \rightarrow 0$ ,  $v_n = 1$  when  $C_n$  stands for  $\tilde{C}_n$ ,
- under strictly stationary alpha-mixing sequence with  $\alpha(k) = O(a^k)$  for some  $a \in (0, 1)$ ,  $b/n \rightarrow 0$ ,  $v_n = 1$ , when  $C_n$  stands for  $\hat{C}_n, C_n^\#$  or  $C_n^\beta$ ,

assuming additionally classical regularity properties on the true copula  $C$ .

### 3.3. Weighted version of the subsampling methodology

A weight is used to emphasize the region where the copula differs from the tested copula. See [15] or more recently [2]. Following [1] and [2] for instance we provide, under stronger assumptions, the weak convergence with respect to stronger metrics. It also includes the validity of the subsampling methodology for the empirical testing process, as it has been done in [37].

**Theorem 3.3.** Consider  $(\mathcal{H})C = \mathbf{M}_\emptyset(C)$ , depending through (2) on a set of operators  $\mathbf{P}_1, \dots, \mathbf{P}_d$ . It is assumed, at least when  $(\mathcal{H})$  holds true, that

- $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$  are commuting and idempotent maps.
- The associated maps  $\{\mathbf{M}_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$  are Hadamard-differentiable at  $C$  tangentially to  $\mathcal{C}_0$ .
- The derivatives  $\{\mathbf{M}'_{\mathcal{A}}(C; \cdot)\}_{\mathcal{A} \in \mathcal{P}_d}$  are continuous and linear on  $\ell^\infty([0, 1]^d)$ .

Let  $v_n \rightarrow 1$  and  $b_n \rightarrow \infty$  such that  $b_n = o(n)$ . Let  $C_n$  be an empirical copula and  $q$  a weight function such that, as  $n$  tends to infinity,  $(\frac{\mathbb{C}_n}{q}, v_n \frac{\mathbb{C}_{b_n}^{[I_{1,n}]}}{q}, v_n \frac{\mathbb{C}_{b_n}^{[I_{2,n}]}}{q})$  converges weakly in  $\{\ell^\infty([0, 1]^d)\}^3$  to  $(\frac{\mathbb{W}_C}{q}, \frac{\mathbb{W}_C^{[1]}}{q}, \frac{\mathbb{W}_C^{[2]}}{q})$ , where  $\mathbb{W}_C^{[1]}$  and  $\mathbb{W}_C^{[2]}$  are independent copies of  $\mathbb{W}_C$ . Then, under  $(\mathcal{H})$  and as  $n$  tends to infinity,

$$\left( \{D_{\mathcal{A},n}/q\}_{\mathcal{A} \in \mathcal{P}_d}, \{v_n D_{\mathcal{A},b_n}^{[I_{1,n}]} / q\}_{\mathcal{A} \in \mathcal{P}_d}, \{v_n D_{\mathcal{A},b_n}^{[I_{2,n}]} / q\}_{\mathcal{A} \in \mathcal{P}_d} \right)$$

converges weakly in  $\{\ell^\infty([0, 1]^d)\}^{3 \times 2^d}$  to

$$\left( \{\mathbb{D}_{\mathcal{A}}/q\}_{\mathcal{A} \in \mathcal{P}_d}, \{\mathbb{D}_{\mathcal{A}}^{[1]}/q\}_{\mathcal{A} \in \mathcal{P}_d}, \{\mathbb{D}_{\mathcal{A}}^{[2]}/q\}_{\mathcal{A} \in \mathcal{P}_d} \right),$$

where  $\{\mathbb{D}_{\mathcal{A}}^{[2]}\}_{\mathcal{A} \in \mathcal{P}_d}$  and  $\{\mathbb{D}_{\mathcal{A}}^{[2]}\}_{\mathcal{A} \in \mathcal{P}_d}$  are independent copies of  $\{\mathbb{D}_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$ .

**Remark 12.** The subsampling of the process  $\mathbb{C}_n/q$  assumed in Theorem 3.3 has been proved in [37]. Let  $\vee$  and  $\wedge$  stand respectively for the maximum and the minimum. On  $[0, 1]^d$  let  $g$  be the weight function  $g(\mathbf{x}) = \wedge_{i=1}^d \left\{ x_i \wedge \vee_{k=1, k \neq i}^d (1 - x_k) \right\}$ . With  $w \in [0, 1/2)$  and  $q = g^w$ , see [37, Theorem 4.3] for the statement concerning both  $\hat{C}_n$  and  $\tilde{C}_n$  and [37, Theorem 4.5] for that involving  $C_n^\#$  and  $C_n^\beta$ .

### 3.4. The multiplier bootstrap procedure

In statistical inference on copulas, the multiplier bootstrap is a well-established resampling technique for approximating the distribution of the limiting process. It was initially proposed by [48] and further investigated in [44], [4], [49], [6] for instance. We adapt here [36].

For the sake of simplicity within this section, we assume that the original time series is serially independent. The strong mixing case can be handled in a manner analogous to that of [5] where further details may be found to construct the serial dependence of the multipliers. The multiplier bootstrap procedure relies on consistent estimators of the partial derivatives involved in (15). Consider then those based on finite-differencing of the empirical copula estimator  $\hat{C}_n$  defined in Remark 9. The associated empirical processes are then denoted by  $\hat{C}_n = \sqrt{n}(\hat{C}_n - C)$ ,  $\hat{D}_{\emptyset, n} = \sqrt{n}(\hat{C}_n - \mathbf{M}_\emptyset(\hat{C}_n))$  and  $\hat{D}_{\mathcal{A}, n} = \sqrt{n}\mathbf{M}_{\mathcal{A}}(\hat{C}_n)$ . Let  $h_n$  be a sequence tending to 0 such that  $\inf_n h_n \sqrt{n} > 0$ . Define

$$\hat{C}_n^{[i]}(\mathbf{x}) = \begin{cases} \frac{\hat{C}_n(\mathbf{x} + h_n \mathbf{e}_i) - \hat{C}_n(\mathbf{x} - h_n \mathbf{e}_i)}{2h_n} & x_i \in [h_n, 1 - h_n] \\ \frac{\hat{C}_n(x_1, \dots, x_{i-1}, h_n, x_{i+1}, \dots, x_d)}{h_n} & x_i \in [0, h_n) \\ \frac{\hat{C}_n(\mathbf{x}) - \hat{C}_n(x_1, \dots, x_{i-1}, 1 - h_n, x_{i+1}, \dots, x_d)}{h_n} & x_i \in (1 - h_n, 1] . \end{cases}$$

Now, let  $N$  be a large integer. Below, we define  $N$  processes that are asymptotically independent copies of  $\{\mathbb{D}_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$ . For any  $k$  in  $\{1, \dots, N\}$ , the multipliers  $\{Z_{j,n}^{(k)}\}_{j \in \mathbb{N}}$  are defined as independent sequences of independent and identically distributed random variables with mean 0 and variance 1, which are independent of  $\mathcal{X}$ . For any  $k$  in  $\{1, \dots, N\}$ , set

$$\hat{C}_n^{(k)}(\mathbf{x}) = G_n^{(k)}(\mathbf{x}) - \sum_{i=1}^d \hat{C}_n^{[i]}(\mathbf{x}) G_n^{(k)}(x_i, \mathbf{1}_{-i})$$

in which

$$G_n^{(k)}(\mathbf{x}) = \frac{1}{\sqrt{n}} \sum_{j=1}^n Z_{j,n}^{(k)} \left\{ \mathbf{1}_{\hat{\mathbf{U}}_j \leq \mathbf{x}} - \hat{C}_n(\mathbf{x}) \right\} .$$

Finally, set  $\hat{D}_{\emptyset, n}^{(k)} = \hat{C}_n^{(k)} - \mathbf{M}'_{\emptyset}(\hat{C}_n; \hat{C}_n^{(k)})$  and  $\hat{D}_{\mathcal{A}, n}^{(k)} = \mathbf{M}'_{\mathcal{A}}(\hat{C}_n; \hat{C}_n^{(k)})$ .

**Theorem 3.4.** Consider  $(\mathcal{H})C = \mathbf{M}_\emptyset(C)$ , depending through (2) on a set of operators  $\mathbf{P}_1, \dots, \mathbf{P}_d$ . It is assumed, at least when  $(\mathcal{H})$  holds true, that

- $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$  are commuting and idempotent maps.
- The associated maps  $\{\mathbf{M}_\mathcal{A}\}_{\mathcal{A} \in \mathcal{P}_d}$  are Hadamard-differentiable at  $C$  tangentially to  $\mathcal{C}_0$ .
- There exists some  $\varepsilon > 0$  such that for any  $\mathcal{A} \in \mathcal{P}_d$ ,  $(f, h) \mapsto \mathbf{M}'_\mathcal{A}(f; h)$  is continuous on  $\{f \in \ell^\infty([0, 1]^d), \|f - C\|_\infty < \varepsilon\} \times \ell^\infty([0, 1]^d)$ .

Assume that the copula  $C$  is such that for any  $i \in \{1, \dots, d\}$ ,  $C^{[i]}$  exists and is continuous on the set  $\{\mathbf{x} \in [0, 1]^d : 0 < x_i < 1\}$ . Then, under  $(\mathcal{H})$  and as  $n$  tends to infinity,

$$\left( \{\hat{D}_{\mathcal{A},n}\}_{\mathcal{A}}, \{\hat{D}_{\mathcal{A},n}^{(1)}\}_{\mathcal{A}}, \dots, \{\hat{D}_{\mathcal{A},n}^{(N)}\}_{\mathcal{A}} \right) \underset{n \rightarrow \infty}{\rightsquigarrow} \left( \{\mathbb{D}_\mathcal{A}\}_{\mathcal{A}}, \{\mathbb{D}_\mathcal{A}^{(1)}\}_{\mathcal{A}}, \dots, \{\mathbb{D}_\mathcal{A}^{(N)}\}_{\mathcal{A}} \right)$$

in  $\{\ell^\infty([0, 1]^d)\}^{2^{d(N+1)}}$  where  $\{\mathbb{D}_\mathcal{A}^{(k)}\}_{\mathcal{A} \in \mathcal{P}_d}$  for  $k = 1, \dots, N$  are independent copies of  $\{\mathbb{D}_\mathcal{A}\}_{\mathcal{A} \in \mathcal{P}_d}$ .

**Remark 13.** When  $(\mathcal{H}_{\tau,r})$  is under study, the previous convergence restricted to  $\mathcal{A} = \emptyset$  is in agreement with Proposition 2 of [36].

#### 4. The testing procedure

Natural measures of departure from the null hypothesis are Kolmogorov–Smirnov or Cramér–von Mises statistics. We give their expressions, and a new combination derived from the concatenated testing processes is introduced and studied. As these statistics are not distribution-free, the subsampling methodology, the multiplier bootstrap explained in the previous section or the parametric bootstrap will be needed to approximate p-values.

##### 4.1. Well-known families of test statistics

Consider  $(\mathcal{H})C = \mathbf{M}_\emptyset(C)$ , depending through (2) on the operators  $\mathbf{P}_1, \dots, \mathbf{P}_d$ . It is assumed, at least when  $(\mathcal{H})$  holds true, that  $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$  are commuting and idempotent maps. Recall that the definition of  $D_{\mathcal{A},n}$  is specific when  $\mathcal{A} = \emptyset$  since  $D_{\emptyset,n} = C_n - \mathbf{M}_\emptyset(C_n) = C_n - (\prod_{i=1}^d \mathbf{P}_i)(C_n)$  whereas, if  $\mathcal{A} \neq \emptyset$ , the definition is  $D_{\mathcal{A},n} = \mathbf{M}_\mathcal{A}(C_n) = (\prod_{i \in \mathcal{A}} (I - \mathbf{P}_i) \prod_{i \notin \mathcal{A}} \mathbf{P}_i)(C_n)$ .

We introduce the  $2^d$  Cramér–von Mises (CvM) statistics, derived from the functional decomposition, and associated limits as

$$I_{\mathcal{A},n,q} = \int_{[0,1]^d} \left\{ \frac{D_{\mathcal{A},n}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 d\mathbf{x} \quad \text{and} \quad \mathbb{I}_{\mathcal{A},q} = \int_{[0,1]^d} \left\{ \frac{\mathbb{D}_\mathcal{A}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 d\mathbf{x}$$

for any  $\mathcal{A} \in \mathcal{P}_d$ . Therein, the weight function  $q$  is the function already introduced in Subsection 3.3. Replacing in the CvM statistics  $I_{\mathcal{A},n,q}$  the Lebesgue product



measure  $d\mathbf{x}$  on  $[0, 1]^d$  by the empirical  $dC_n(\mathbf{x})$  is of greater interest. We thus introduce, for any  $\mathcal{A} \in \mathcal{P}_d$ , the alternative CvM statistics as

$$S_{\mathcal{A},n,q} = \int_{[0,1]^d} \left\{ \frac{D_{\mathcal{A},n}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 dC_n(\mathbf{x}) \quad \text{and} \quad \mathbb{S}_{\mathcal{A},q} = \int_{[0,1]^d} \left\{ \frac{\mathbb{D}_{\mathcal{A}}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 dC(\mathbf{x}).$$

Similarly, the Kolmogorov-Smirnov statistics and associated limits are defined, for any  $\mathcal{A} \in \mathcal{P}_d$ , by

$$U_{\mathcal{A},n,q} = \sup_{\mathbf{x} \in [0,1]^d} \left| \frac{D_{\mathcal{A},n}(\mathbf{x})}{q(\mathbf{x})} \right| \quad \text{and} \quad \mathbb{U}_{\mathcal{A},q} = \sup_{\mathbf{x} \in [0,1]^d} \left| \frac{\mathbb{D}_{\mathcal{A}}(\mathbf{x})}{q(\mathbf{x})} \right|.$$

When  $C_n$  is specifically defined as  $\hat{C}_n$  so that  $D_{\mathcal{A},n}$  becomes  $\hat{D}_{\mathcal{A},n}$ , as it has been done in Subsection 3.4, the previous statistics will be denoted by  $\hat{I}_{\mathcal{A},n,q}$ ,  $\hat{S}_{\mathcal{A},n,q}$  and  $\hat{U}_{\mathcal{A},n,q}$ . In view of bootstrap considerations,  $\hat{I}_{\mathcal{A},n,q}^{(k)}$ ,  $\hat{S}_{\mathcal{A},n,q}^{(k)}$  and  $\hat{U}_{\mathcal{A},n,q}^{(k)}$  are obtained while  $\hat{D}_{\mathcal{A},n}$  is replaced by  $\hat{D}_{\mathcal{A},n}^{(k)}$ . In view of subsampling considerations,  $I_{\mathcal{A},n,q}^{[I_m]}$ ,  $S_{\mathcal{A},n,q}^{[I_m]}$  and  $U_{\mathcal{A},n,q}^{[I_m]}$  (resp.  $\mathbb{I}_{\mathcal{A},q}^{[I_m]}$ ,  $\mathbb{S}_{\mathcal{A},q}^{[I_m]}$  and  $\mathbb{U}_{\mathcal{A},q}^{[I_m]}$ ) are obtained while  $D_{\mathcal{A},n}$  (resp.  $\mathbb{D}_{\mathcal{A}}$ ) is replaced by  $D_{\mathcal{A},n}^{[I_m]}$  (resp.  $\mathbb{D}_{\mathcal{A}}^{[I_m]}$ ).

**Corollary 4.1.** (i) Under the assumptions of Theorem 3.1, the random vectors  $\{I_{\mathcal{A},n,1}\}_{\mathcal{A}}$ ,  $\{S_{\mathcal{A},n,1}\}_{\mathcal{A}}$  and  $\{U_{\mathcal{A},n,1}\}_{\mathcal{A}}$  converge in distribution to  $\{\mathbb{I}_{\mathcal{A},1}\}_{\mathcal{A}}$ ,  $\{\mathbb{S}_{\mathcal{A},1}\}_{\mathcal{A}}$  and  $\{\mathbb{U}_{\mathcal{A},1}\}_{\mathcal{A}}$  respectively.

(ii) The assumptions of Theorem 3.2 imply the convergence in distribution of  $(\{I_{\mathcal{A},n,1}\}_{\mathcal{A}}, \{v_n^2 I_{\mathcal{A},b_n,1}^{[I_1,n]}\}_{\mathcal{A}}, \{v_n^2 I_{\mathcal{A},b_n,1}^{[I_2,n]}\}_{\mathcal{A}})$  to  $(\{\mathbb{I}_{\mathcal{A},1}\}_{\mathcal{A}}, \{\mathbb{I}_{\mathcal{A},1}^{[1]}\}_{\mathcal{A}}, \{\mathbb{I}_{\mathcal{A},1}^{[2]}\}_{\mathcal{A}})$ , where the latter is the concatenation of independent copies. Similar results hold true for the  $S$  and  $U$  families of test statistics (for  $U$  use  $v_n$  instead of  $v_n^2$ ).

(iii) The assumptions of Theorem 3.3 imply the convergence in distribution of  $(\{S_{\mathcal{A},n,q}\}_{\mathcal{A}}, \{v_n^2 S_{\mathcal{A},b_n,q}^{[I_1,n]}\}_{\mathcal{A}}, \{v_n^2 S_{\mathcal{A},b_n,q}^{[I_2,n]}\}_{\mathcal{A}})$  to  $(\{\mathbb{S}_{\mathcal{A},q}\}_{\mathcal{A}}, \{\mathbb{S}_{\mathcal{A},q}^{[1]}\}_{\mathcal{A}}, \{\mathbb{S}_{\mathcal{A},q}^{[2]}\}_{\mathcal{A}})$ , where the latter is the concatenation of independent copies. Similar results hold true for  $I$  and  $U$  families of test statistics (for  $U$  use  $v_n$  instead of  $v_n^2$ ).

(iv) Under the assumptions of Theorem 3.4,  $(\{\hat{U}_{\mathcal{A},n,1}\}_{\mathcal{A}}, \{\hat{U}_{\mathcal{A},n,1}^{(I_1,n)}\}_{\mathcal{A}}, \{\hat{U}_{\mathcal{A},n,1}^{(I_2,n)}\}_{\mathcal{A}})$  converges in distribution to  $(\{\mathbb{U}_{\mathcal{A},1}\}_{\mathcal{A}}, \{\mathbb{U}_{\mathcal{A},1}^{(1)}\}_{\mathcal{A}}, \{\mathbb{U}_{\mathcal{A},1}^{(2)}\}_{\mathcal{A}})$ , concatenating independent copies. Similar results hold true for  $\hat{I}$  and  $\hat{S}$  families of test statistics.

#### 4.2. New test statistics derived from the functional decomposition

Note that under independence or independence by block, the p-values associated with the above family of test statistics are also asymptotically mutually independent (with respect to  $\mathcal{A}$ ). As a consequence, individual critical values can be chosen to achieve an asymptotic global significance level. Furthermore, it is possible to combine individual p-values and get a global p-value thanks to the method *à la Fisher* (the resulting statistics are denoted by  $W_n$ ) as well as *à la Tippett* (the resulting statistics is denoted  $T_n$ ). For more details, we refer to the discussion in [26] or the paragraph ‘‘Combining p-values’’ in Section 3 of [35].

In general, for other types of null hypotheses, the asymptotic mutual independence is no more true. Even if our purpose in this paper is not to construct better statistics rather than a unified theory, new test statistics are defined. The use of the functional decomposition

$$\mathbf{M}_\emptyset(C) - C = \sum_{\mathcal{A} \in \mathcal{P}_p^*} \mathbf{M}_{\mathcal{A}}(C)$$

allows to improve the well-known statistics, that only considers the left hand member of the above equality, by defining instead a combination that takes into account both the left hand member and all the right hand member terms of the above equality.

Let  $w = \{w^{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$  be a vector of positive weights. The latter reflects the importance we put in the test  $(\mathcal{H}) C = \mathbf{M}_\emptyset(C)$  through  $w^\emptyset$ , or in the test  $(\mathcal{H}^{\mathcal{A}}) \mathbf{M}_{\mathcal{A}}(C) = 0$  through  $w^{\mathcal{A}}$ . We introduce the statistic and associated limit by

$$S_{w,n,q} = \sum_{\mathcal{A} \in \mathcal{P}_d} w^{\mathcal{A}} S_{\mathcal{A},n,q} \quad \text{and} \quad \mathbb{S}_{w,q} = \sum_{\mathcal{A} \in \mathcal{P}_d} w^{\mathcal{A}} \mathbb{S}_{\mathcal{A},q}. \quad (17)$$

Similarly, the test statistics  $I_{w,n,q}$  and  $U_{w,n,q}$  (and respective limits  $\mathbb{I}_{w,q}$  and  $\mathbb{U}_{w,q}$ ) are the weighted linear combination constructed from the collections  $I_{\mathcal{A},n,q}$  and  $U_{\mathcal{A},n,q}$  (and from  $\mathbb{I}_{\mathcal{A},q}$  and  $\mathbb{U}_{\mathcal{A},q}$  for the limits).

**Corollary 4.2.** *The results stated in Corollary 4.1 are inherited by the weighted combinations  $I_{w,n,q}$ ,  $S_{w,n,q}$  and  $U_{w,n,q}$ . To give an example, under the assumptions of Theorem 3.1, the test statistic  $S_{w,n,1}$  converges in distribution to  $\mathbb{S}_{w,1}$ .*

### 4.3. Practical implementation of the tests

In some particular cases, and when  $C_n$  is taken as  $\hat{C}_n$  it is possible to provide the expression of the statistics in terms of the pseudo-observations only. See for instance Section 4 in [25] when complete independence is under testing. See also Proposition 10 and Proposition 13 in [35] when testing concerns independence by blocks. Here, the calculations haven't been taken as far. Let  $R_{j,i,n}$  denote the rank of  $X_{ji}$  among  $X_{1i}, \dots, X_{ni}$  and set  $\hat{\mathbf{U}}_{j \cdot, n} = (R_{j1,n}/n, \dots, R_{jd,n}/n)$ . Then,

$$S_{\mathcal{A},n,q} = \frac{1}{n} \sum_{j=1}^n \left\{ \frac{D_{\mathcal{A},n}(\hat{\mathbf{U}}_{j \cdot, n})}{q(\hat{\mathbf{U}}_{j \cdot, n})} \right\}^2.$$

In general, we proceed by numerical approximation based on a grid. Let  $K$  be a large integer and let  $\mathbf{u}_1, \dots, \mathbf{u}_K$  be  $K$  uniformly spaced points on  $(0, 1)^d$ . Then  $I_{\mathcal{A},n,q} \simeq \frac{1}{K} \sum_{k=1}^K \{D_{\mathcal{A},n}(\mathbf{u}_k)/q(\mathbf{u}_k)\}^2$  and  $U_{\mathcal{A},n,q} \simeq \max_{k=1, \dots, K} |D_{\mathcal{A},n}(\mathbf{u}_k)/q(\mathbf{u}_k)|$ .

Taking into account Section 3, the p-values are approximately uniform on  $[0, 1]$  under the null hypothesis  $(\mathcal{H})$ . In the case of goodness-of-fit tests  $(\mathcal{H}) C = C_\theta$ , samples of reference should not be obtained from subsampling or multiplier bootstrapping but by parametric bootstrapping instead. Note that

---

**Algorithm 1:** Approximating the p-value (Illustrated with the  $S$  family)

---

**Compute**  $S_{w,n,q}^{(0)}$  the value of  $S_{w,n,q}$  on the original series  
**Generate** from subsampling  $D_{\mathcal{A},n,q}^{[k]}$  or multiplier bootstrapping  $\hat{D}_{\mathcal{A},n,q}^{(k)}$  for  $k = 1, \dots, N$   
**Compute**  $S_{w,n,q}^{(1)}, \dots, S_{w,n,q}^{(N)}$  the value of  $S_{w,n,q}$  on one of these processes  
**Define** an approximate p-value for the test statistic as following  

$$\frac{1}{N+1} \left( \frac{1}{2} + \sum_{k=1}^N \mathbf{1}\{S_{w,n,q}^{(k)} > S_{w,n,q}^{(0)}\} \right)$$

---

[27] establish its validity when the series are independent. In the current paper, parametric bootstrap can be applied under  $(\mathcal{H}_1)$  when  $p = d$ ,  $(\mathcal{H}_3)$  or  $(\mathcal{H}_4)$  for instance. See also [47] when studying some multivariate stochastic volatility models. The alternative version of the algorithm is given below.

---

**Algorithm 2:** Approximating the p-value (Goodness-of-fit tests)

---

**Compute** the pseudo-observations  $\mathbf{U}_1, \dots, \mathbf{U}_n$  on the original series,  $\theta_n = \hat{\theta}(\mathbf{U}_1, \dots, \mathbf{U}_n)$  and  $S_{w,n,q,[\theta_n]}$   
**for**  $k = 1$  to  $N$  **do**  
    **Generate**  $\mathbf{U}_1^{\{k\}}, \dots, \mathbf{U}_n^{\{k\}}$  from  $C_{\theta_n}$   
    **Compute**  $\theta_n^{\{k\}} = \hat{\theta}(\mathbf{U}_1^{\{k\}}, \dots, \mathbf{U}_n^{\{k\}})$  and  $S_{w,n,q,[\theta_n^{\{k\}]}$  on the parametric sample  
**end for**  
**Define** an approximate p-value for the test statistic as following

$$\frac{1}{N+1} \left( \frac{1}{2} + \sum_{k=1}^N \mathbf{1}\{S_{w,n,q,[\theta_n^{\{k\}]} > S_{w,n,q,[\theta_n]}\} \right)$$


---

#### 4.4. Numerical experiments

In this final section, we shall consider the use of the functional decomposition in two experiments and analyze the results. First, we explore the testing problem  $(\mathcal{H}_1)$  of block independence and specifically one of the practical settings imagined in [35]. We investigate then the goodness-of-fit test  $(\mathcal{H}_3)$  where two Archimedean copulas, namely Clayton and Gumbel, are opposed. These copulas have been chosen as they are part of the first example in the routine `gofCopula` of the R `copula` package [32].

##### 4.4.1. Independence between three continuous $r$ -dimensional random vectors

We adapt here Section 4 from [35] that implements testing procedures for  $(\mathcal{H}_1)$ . Let  $\mathbf{X} = (X_1, \dots, X_{12})$  and consider the 3 groups  $\{X_1, \dots, X_4\}$ ,  $\{X_5, \dots, X_8\}$  and  $\{X_9, \dots, X_{12}\}$  so that  $r = 4$ ,  $p = 3$  and  $d = 12$ . The dependence is described by the normal copula. The  $d \times d$  correlation matrices  $\Sigma$  are structured as follows:

	$X_1$	...	$X_4$	$X_5$	...	$X_8$	$X_9$	...	$X_{12}$
$X_1$	1		$\rho_{\text{intra}}$						
$\vdots$									
$X_4$	$\rho_{\text{intra}}$		1		$\rho_{\text{inter}}$			$\rho_{\text{inter}}$	
$X_5$				1		$\rho_{\text{intra}}$			
$\vdots$									
$X_8$		$\rho_{\text{inter}}$			$\rho_{\text{intra}}$	1		$\rho_{\text{inter}}$	
$X_9$							1		$\rho_{\text{intra}}$
$\vdots$									
$X_{12}$		$\rho_{\text{inter}}$			$\rho_{\text{inter}}$		$\rho_{\text{intra}}$		1

The quantity  $\rho_{\text{inter}}$  (resp.  $\rho_{\text{intra}}$ ) controls the amount of dependence among (resp. within) the three random vectors. Under the normal copula, the values  $\rho_{\text{inter}} \in \{0.000, 0.025, 0.050, \dots, 0.275, 0.300\}$  for  $\rho_{\text{intra}} = 0.5$  are considered. We generate 1000 samples composed of  $n = 200$  independent realizations of  $\mathbf{X}$ . Note that, in all the simulations, the number of randomized samples is set to 1000.

Table 1 shows the rejection rates of the null hypothesis, the proportion of times that the different tests reject ( $\mathcal{H}_1$ ), with respect to the value of  $\rho_{\text{inter}}$ . The significance level is arbitrarily set to 5% and measured on the first column of the table. The global Cramér-von-mises statistic  $I_n = I_{\emptyset, n, 1}$ , as well as  $W_n$  the test statistic *à la Fisher*, and  $T_n$  the test statistic *à la Tippett* are those studied in Figure 3 of [35], with the difference that  $n = 200$  here. Four additional measures of the form  $S_{w, n, 1}$  are included. Recall that taking into account the form of the null hypothesis ( $\mathcal{H}_1$ ) with  $p = 3$  blocks, the weights  $w$  have the following structure

$$w = (w^\emptyset, w^{\{1\}}, w^{\{2\}}, w^{\{3\}}, w^{\{12\}}, w^{\{13\}}, w^{\{23\}}, w^{\{123\}}).$$

More precisely, we consider

- $w_1 = (1, 0, 0, 0, 0, 0, 0, 0)$  that only measures the left hand term of the decomposition,
- $w_2 = (1, 0, 0, 0, 1, 1, 1, 1)$  that combines the left hand term of the decomposition with the right hand terms of order 2 and 3 (recall that the right hand terms associated with singletons all vanish),
- $w_3 = (0, 0, 0, 0, 1, 1, 1, 1)$  that only combines the non-null right hand terms of the decomposition,
- $w_4 = (8.253373, 0, 0, 0, 2.373714, 2.344580, 2.35670, 1.989524)$  that combines left and right hand terms proportionally to some variances. It is an empirical choice where each  $w^{\mathcal{A}}$  is proportional to the estimate of  $\text{var}(S_{\mathcal{A}, n, 1})$  obtained by block bootstrapping.

As can be seen in Table 1, the  $S$ -type statistics perform best among all the approaches, with one exception:  $S_{w_3, n, 1}$  appears twice below the values of  $W_n$ . Additionally, there is no uniformly better  $S$ , even if  $S_{w_2, n, 1}$  is roughly speaking the best choice globally. However, an empirical choice of the weights as defined in  $S_{w_4, n, 1}$  yields a better rate for the particular value  $\rho_{\text{inter}} = 0.05$ . Note that  $S_{w_1, n, 1}$  which is the usual CvM test statistic from the literature leads to results relatively close to those given by  $S_{w_2, n, 1}$ .

		0.00	0.025	0.05	$\rho_{\text{inter}}$			
					0.075	0.10	0.125	0.15
Test statistics	$W_n$	0.060	0.146	0.332	0.540	0.770	0.901	0.982
	$T_n$	0.047	0.110	0.234	0.404	0.584	0.766	0.932
	$I_n$	0.054	0.128	0.287	0.503	0.711	0.854	0.971
	$S_{w_1,n,1}$	0.049	0.176	0.376	0.673	0.875	0.964	0.994
	$S_{w_2,n,1}$	0.046	<b>0.181</b>	0.388	<b>0.691</b>	<b>0.883</b>	<b>0.971</b>	<b>0.995</b>
	$S_{w_3,n,1}$	0.041	0.124	0.311	0.572	0.819	0.953	0.992
	$S_{w_4,n,1}$	0.046	0.173	<b>0.400</b>	0.683	0.878	0.969	0.994

TABLE 1

Percentage of rejection of  $(\mathcal{H}_1)$  that stipulates the block independence of  $\{X_1, \dots, X_4\}$ ,  $\{X_5, \dots, X_8\}$  and  $\{X_9, \dots, X_{12}\}$  that comes from Normal copula with correlation matrices  $\Sigma$  whereas the null hypothesis  $(\mathcal{H}_1)$  is only true under the  $x$ -axis value  $\rho_{\text{inter}} = 0.00$ . The statistics  $W_n$ ,  $T_n$  and  $I_n$  are those presented in [35] and available in the routine `multIndepTest` of the R package `copula`. The statistics  $S_{w,n,1}$ , defined by (17), are evaluated for several weights  $w$  as indicated in the text.

#### 4.4.2. Goodness-of-fit tests

Turning to the Goodness-of-Fit tests  $(\mathcal{H}_3)$ , we consider the Clayton or the Gumbel family in a 3-dimensional setting. These classes will both be used as the generator of datasets or as the family being tested. To generate the original samples, three values of Kendall's  $\tau$  are chosen:  $\tau = .1$ ,  $\tau = .2$  and  $\tau = .3$ . Test statistics  $S_{w,n,1}$  given by (17) and where

$$w = (w^\emptyset, w^{\{1\}}, w^{\{2\}}, w^{\{3\}}, w^{\{12\}}, w^{\{13\}}, w^{\{23\}}, w^{\{123\}})$$

are computed for the weights:  $w_1 = (1, 0, 0, 0, 0, 0, 0, 0)$ ,  $w_2 = (1, 0, 0, 0, 1, 1, 1, 1)$  and  $w_3 = (0, 0, 0, 0, 1, 1, 1, 1)$ . The results are provided in Table 2 for  $n = 100$ . The first lines are dedicated to the test  $(\mathcal{H}_3)$  when  $\varphi$  is the Clayton copula. Similarly, Gumbel copula is tested in the last lines of the table. The parameter associated with the generator  $\varphi$  is estimated at each step as the mean of empirical Kendall's  $\tau$ . The parametric bootstrap described in Algorithm 2 with  $n_{\text{boot}} = 200$  is used to compute the p-value. The rejection rates are estimated through  $n_{\text{rep}} = 500$  repetitions of each experiment. Two characteristics are of interest: the empirical level might be close to the nominal level, arbitrarily fixed at 0.05, and the empirical power. We also add another procedure in Table 2. The line `gofCopula` corresponds to the results associated with the command `gofCopula(CopulTest, rCopula(n, CopulSimu), estim.method = "itau")`.

As already explained, the goal of the paper is not to find an overall best test statistics but rather about showing the interest of a functional decomposition chosen in accordance with the null hypothesis. The analysis of Table 2 is not straightforward but the results are interesting. First, on the right upper corner of the table, one can remark that  $S_{w_1,n,1}$  always dominates in discriminating true Gumbel from supposed Clayton. This becomes true for  $S_{w_2,n,1}$  in the left bottom corner, except when the dependence becomes stronger with  $\tau = 0.3$ , the more powerful is then  $S_{w_3,n,1}$ .

		Simulated copulas						
		Clayton			Gumbel			
		$\tau = .1$	$\tau = .2$	$\tau = .3$	$\tau = .1$	$\tau = .2$	$\tau = .3$	
Tested copulas	Clayton	$w_1$	0.056	0.044	0.048	<b>0.364</b>	<b>0.690</b>	<b>0.934</b>
		$w_2$	0.052	0.044	0.038	0.294	0.674	0.924
		$w_3$	0.044	0.060	0.046	0.160	0.494	0.852
		gofCopula	0.050	0.044	0.044	0.214	0.670	0.924
	Gumbel	$w_1$	0.282	0.720	0.922	0.070	0.048	0.056
		$w_2$	<b>0.306</b>	<b>0.766</b>	0.948	0.036	0.042	0.052
		$w_3$	0.254	0.710	<b>0.960</b>	0.024	0.042	0.048
		gofCopula	0.288	0.716	0.930	0.030	0.062	0.054

TABLE 2

Rejection rates of the null hypothesis. In the first lines, Clayton copula is being tested, whereas Gumbel copula is tested in the last lines of the table. The datasets are simulated for three different strengths of dependence, calibrated through the Kendall's  $\tau$ : .1, .2 and .3. The test statistics  $S_{w,n}$  are studied for three different weights:  $w_1, w_2$  and  $w_3$  (definition in the text). Additional parameters of the numerical study: sample size  $n = 100$ , parametric bootstrap size  $n_{\text{boot}} = 200$  and number of repetitions of the experiment  $n_{\text{rep}} = 500$ .

## 5. Concluding remarks

Identifying and modeling dependencies with copulas remain an important topic, which has become very popular over the last decades since it has been applied in almost every discipline. The aim in this paper is to provide a kind of unification of various papers, as [18], [26], [25], [35], [36] among others. All derive copula-based tests of the structure of dependence. The solution here is to dip them in the functional decomposition context of [38] (and its recent version of [41] which removes the linearity assumption) in order to reveal a common pattern. Then, the goal of the paper is not really to improve a methodology but to transform already known tools in particular cases of more general statements.

The numerical section provides two simple studies, one analyzing the independence among  $p = 3$  blocks in a ( $d =$ )12-dimensional setting and one examining particular Archimedean copula families in dimension  $d = 3$ . Of course, we conclude that there is no best procedure even if one can take advantage in analyzing the functional decomposition associated with the null hypothesis in order to derive powerful weighted test statistics.

The dimensions  $d$  or  $p$  are small in our experiments. Nevertheless, the current paper provides an interesting perspective on high dimensional problems. The practical implementation of the tests relies indeed on a trade-off between exhaustivity (all subsets of  $\mathcal{P}_d$ ) and dimensionality (exponential growth in  $d$ ). When  $d$  becomes larger, it could be interesting to use only part of the subsets. With the help of the weight  $w$  introduced in the definition of the test statistics, one can focus only in a given size of  $\mathcal{A}$  or in all sizes that do not exceed a given size. This way, we can control the underlying complexity of the method. The question will be then: how much this selection affects the corresponding power of the testing procedure?

## 6. Proofs

*Proof of Proposition 2.1.* Suppose  $\mathbf{M}_{\mathcal{A}}(C) \equiv 0$  for any non-empty subset  $\mathcal{A}$  of  $\{1, \dots, d\}$ . By application of (1) inherited from Proposition 1 of [41], one obtains  $C - \mathbf{M}_{\emptyset}(C) \equiv 0$  which is  $(\mathcal{H})$ . Reciprocally, if  $(\mathcal{H})$  holds true, then  $C = \mathbf{M}_{\emptyset}(C) = (\prod_{j=1}^d \mathbf{P}_j)(C)$ . Combined with (2), it yields by commutativity,

$$\begin{aligned} \mathbf{M}_{\mathcal{A}}(C) &= \left( \prod_{i \in \mathcal{A}} (\mathbf{I} - \mathbf{P}_i) \prod_{i \notin \mathcal{A}} \mathbf{P}_i \right) (C) = \left( \prod_{i \in \mathcal{A}} (\mathbf{I} - \mathbf{P}_i) \prod_{i \notin \mathcal{A}} \mathbf{P}_i \right) \left( \prod_{j=1}^d \mathbf{P}_j(C) \right) \\ &= \left( \prod_{i \in \mathcal{A}} (\mathbf{P}_i - \mathbf{P}_i^2) \prod_{i \notin \mathcal{A}} \mathbf{P}_i \right) \left( \prod_{j \notin \mathcal{A}} \mathbf{P}_j(C) \right) \end{aligned}$$

which vanishes for  $\mathcal{A} \neq \emptyset$ , since  $\mathbf{P}_i = \mathbf{P}_i^2$  by the idempotence assumption.  $\square$

*Proof of Theorem 3.1.* By assumption,  $\sqrt{n}(C_n - C) \xrightarrow[n \rightarrow \infty]{w} \mathbb{W}_C$  in  $\ell^\infty([0, 1]^d)$  and any  $\mathbf{M}_{\mathcal{A}}$  is Hadamard-differentiable at  $C$ . The functional version of the Delta method (see Section 3.9 of [55]) applied to  $f \mapsto (f, \mathbf{M}_{\emptyset}(f), \{\mathbf{M}_{\mathcal{A}}(f)\}_{\mathcal{A} \in \mathcal{P}_d^*})$  yields

$$\begin{aligned} (\sqrt{n}(C_n - C), \sqrt{n}(\mathbf{M}_{\emptyset}(C_n) - \mathbf{M}_{\emptyset}(C)), \{\sqrt{n}(\mathbf{M}_{\mathcal{A}}(C_n) - \mathbf{M}_{\mathcal{A}}(C))\}_{\mathcal{A} \in \mathcal{P}_d^*}) &\xrightarrow[n \rightarrow \infty]{w} \\ (\mathbb{W}_C, \mathbf{M}'_{\emptyset}(C; \mathbb{W}_C), \{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C)\}_{\mathcal{A} \in \mathcal{P}_d^*}) \end{aligned}$$

in  $\{\ell^\infty([0, 1]^d)\}^{2^d+1}$ . From the continuous mapping theorem applied to the functional  $T(f, g, \{h_{\mathcal{A}}\}_{\mathcal{A}}) = (f - g, \{h_{\mathcal{A}}\}_{\mathcal{A}})$ , we obtain the weak convergence of

$$(\sqrt{n}(C_n - \mathbf{M}_{\emptyset}(C_n)) - \sqrt{n}(C - \mathbf{M}_{\emptyset}(C)), \{\sqrt{n}(\mathbf{M}_{\mathcal{A}}(C_n) - \mathbf{M}_{\mathcal{A}}(C))\}_{\mathcal{A} \in \mathcal{P}_d^*})$$

in  $(\ell^\infty([0, 1]^d))^{2^d}$  to  $(\mathbb{W}_C - \mathbf{M}'_{\emptyset}(C; \mathbb{W}_C), \{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C)\}_{\mathcal{A} \in \mathcal{P}_d^*})$ . Now, when  $(\mathcal{H})$  holds true, the collection of maps  $\{\mathbf{P}_i\}_{i=1, \dots, d}$  is assumed to form an idempotent and commuting family. Consequently, Proposition 2.1 applies,  $\cap_{\mathcal{A} \in \mathcal{P}_d^*} (\mathcal{H}^{\mathcal{A}})$  holds true so that  $C - \mathbf{M}_{\emptyset}(C) = 0$  as well as  $\mathbf{M}_{\mathcal{A}}(C) = 0$  for any  $\mathcal{A} \in \mathcal{P}_d^*$ . Then, the left hand side of the last convergence reduces to the process under study.  $\square$

*Proof of Theorem 3.2.* Since by assumption both  $\mathbf{M}_{\emptyset}(C) = C$  and  $\mathbf{M}_{\mathcal{A}}(C) = 0$  for any subset  $\mathcal{A} \in \mathcal{P}_d^*$ , one can observe that,

$$D_{\emptyset, n} = \mathbb{C}_n - \sqrt{n} \left( \mathbf{M}_{\emptyset} \left( C + \frac{\mathbb{C}_n}{\sqrt{n}} \right) - \mathbf{M}_{\emptyset}(C) \right) \quad (18)$$

$$D_{\mathcal{A}, n} = \sqrt{n} \left( \mathbf{M}_{\mathcal{A}} \left( C + \frac{\mathbb{C}_n}{\sqrt{n}} \right) - \mathbf{M}_{\mathcal{A}}(C) \right) \quad (19)$$

$$D_{\emptyset, b}^{[m]} = \mathbb{C}_b^{[m]} + \sqrt{\frac{b}{n}} \mathbb{C}_n - \sqrt{b} \left( \mathbf{M}_{\emptyset} \left( C + \frac{\mathbb{C}_b^{[m]}}{\sqrt{b}} + \frac{\mathbb{C}_n}{\sqrt{n}} \right) - \mathbf{M}_{\emptyset}(C) \right) \quad (20)$$

$$D_{\mathcal{A}, b}^{[m]} = \sqrt{b} \left( \mathbf{M}_{\mathcal{A}} \left( C + \frac{\mathbb{C}_b^{[m]}}{\sqrt{b}} + \frac{\mathbb{C}_n}{\sqrt{n}} \right) - \mathbf{M}_{\mathcal{A}}(C) \right) \quad (21)$$

for  $m = I_{1,n}$  or  $I_{2,n}$  in the last two displays. By assumption, the weak convergence of  $(\mathbb{C}_n, v_n \mathbb{C}_b^{[I_{1,n}]}, v_n \mathbb{C}_b^{[I_{2,n}]})$  in  $\{\ell^\infty([0, 1]^d)\}^3$  to  $(\mathbb{W}_C, \mathbb{W}_C^{[1]}, \mathbb{W}_C^{[2]})$  holds true. Using the fact that  $b = o(n)$ , the following

$$\left( \mathbb{C}_n, v_n \mathbb{C}_b^{[I_{1,n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n, v_n \mathbb{C}_b^{[I_{2,n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right)$$

shares the same asymptotic behavior. Continuous mapping theorem with

$$(f_1, f_2, f_3) \mapsto (-\mathbf{M}'_\emptyset(C; f_1), -\mathbf{M}'_\emptyset(C; f_2), -\mathbf{M}'_\emptyset(C; f_3), \\ \{\mathbf{M}'_{\mathcal{A}}(C; f_1)\}_{\mathcal{A}}, \{\mathbf{M}'_{\mathcal{A}}(C; f_2)\}_{\mathcal{A}}, \{\mathbf{M}'_{\mathcal{A}}(C; f_3)\}_{\mathcal{A}})$$

then delivers the weak convergence of

$$\left( -\mathbf{M}'_\emptyset(C; \mathbb{C}_n), -\mathbf{M}'_\emptyset\left(C; v_n \mathbb{C}_b^{[I_{1,n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n\right), -\mathbf{M}'_\emptyset\left(C; v_n \mathbb{C}_b^{[I_{2,n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n\right), \right. \\ \left. \{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{C}_n)\}_{\mathcal{A}}, \left\{ \mathbf{M}'_{\mathcal{A}}\left(C; v_n \mathbb{C}_b^{[I_{1,n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n\right) \right\}_{\mathcal{A}}, \right. \\ \left. \left\{ \mathbf{M}'_{\mathcal{A}}\left(C; v_n \mathbb{C}_b^{[I_{2,n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n\right) \right\}_{\mathcal{A}} \right)$$

in  $\{\ell^\infty([0, 1]^d)\}^{3 \times 2^d}$  to

$$\left( -\mathbf{M}'_\emptyset(C; \mathbb{W}_C), -\mathbf{M}'_\emptyset(C; \mathbb{W}_C^{[1]}), -\mathbf{M}'_\emptyset(C; \mathbb{W}_C^{[2]}), \right. \\ \left. \{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C)\}_{\mathcal{A}}, \{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C^{[1]})\}_{\mathcal{A}}, \{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C^{[2]})\}_{\mathcal{A}} \right).$$

It remains to show that, for  $m = I_{1,n}$  or  $I_{2,n}$ ,

$$\sup_{[0,1]^d} |D_{\emptyset,n} - \mathbb{C}_n + \mathbf{M}'_\emptyset(C; \mathbb{C}_n)| \xrightarrow{\mathbb{P}} 0 \quad ; \quad \sup_{[0,1]^d} \left| \frac{D_{\mathcal{A},n}}{q} - \mathbf{M}'_{\mathcal{A}}(C; \mathbb{C}_n) \right| \xrightarrow{\mathbb{P}} 0, \\ \sup_{[0,1]^d} \left| v_n D_{\emptyset,b}^{[m]} - v_n \mathbb{C}_b^{[m]} - v_n \sqrt{\frac{b}{n}} \mathbb{C}_n + \mathbf{M}'_\emptyset\left(C; v_n \mathbb{C}_b^{[m]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n\right) \right| \xrightarrow{\mathbb{P}} 0, \\ \sup_{[0,1]^d} \left| v_n D_{\mathcal{A},b}^{[m]} - \mathbf{M}'_{\mathcal{A}}\left(C; v_n \mathbb{C}_b^{[m]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n\right) \right| \xrightarrow{\mathbb{P}} 0,$$

which are, once the equations (18), (19), (20) and (21) taken into account, only the consequences of what precedes combined with the functional Delta Method applied to the maps  $\mathbf{M}'_\emptyset$  or  $\mathbf{M}'_{\mathcal{A}}$ .  $\square$

*Proof of Theorem 3.3.* By assumption,  $\left(\frac{\mathbb{C}_n}{q}, v_n \frac{\mathbb{C}_b^{[I_{1,n}]}}{q}, v_n \frac{\mathbb{C}_b^{[I_{2,n}]}}{q}\right)$  converges weakly in  $\{\ell^\infty([0, 1]^d)\}^3$  to  $\left(\frac{\mathbb{W}_C}{q}, \frac{\mathbb{W}_C^{[1]}}{q}, \frac{\mathbb{W}_C^{[2]}}{q}\right)$ . Consequently, using the fact that  $b =$



$o(n)$ , the following  $\left( \frac{\mathbb{C}_n}{q}, v_n \frac{\mathbb{C}_b^{[I_{1,n}]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q}, v_n \frac{\mathbb{C}_b^{[I_{2,n}]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q} \right)$  shares the same asymptotic behaviour. Using the continuous mapping theorem with

$$(f_1, f_2, f_3) \mapsto (-\mathbf{M}'_\emptyset(C; f_1), -\mathbf{M}'_\emptyset(C; f_2), -\mathbf{M}'_\emptyset(C; f_3), \\ \{\mathbf{M}'_{\mathcal{A}}(C; f_1)\}_{\mathcal{A}}, \{\mathbf{M}'_{\mathcal{A}}(C; f_2)\}_{\mathcal{A}}, \{\mathbf{M}'_{\mathcal{A}}(C; f_3)\}_{\mathcal{A}})$$

we thus obtain that

$$\left( -\mathbf{M}'_\emptyset \left( C; \frac{\mathbb{C}_n}{q} \right), -\mathbf{M}'_\emptyset \left( C; v_n \frac{\mathbb{C}_b^{[I_{1,n}]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q} \right), -\mathbf{M}'_\emptyset \left( C; v_n \frac{\mathbb{C}_b^{[I_{2,n}]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q} \right), \right. \\ \left. \left\{ \mathbf{M}'_{\mathcal{A}} \left( C; \frac{\mathbb{C}_n}{q} \right) \right\}_{\mathcal{A}}, \left\{ \mathbf{M}'_{\mathcal{A}} \left( C; v_n \frac{\mathbb{C}_b^{[I_{1,n}]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q} \right) \right\}_{\mathcal{A}}, \right. \\ \left. \left\{ \mathbf{M}'_{\mathcal{A}} \left( C; v_n \frac{\mathbb{C}_b^{[I_{2,n}]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q} \right) \right\}_{\mathcal{A}} \right)$$

converges weakly in  $\{\ell^\infty([0, 1]^d)\}^{3 \times 2^d}$  to

$$\left( -\mathbf{M}'_\emptyset \left( C; \frac{\mathbb{W}_C}{q} \right), -\mathbf{M}'_\emptyset \left( C; \frac{\mathbb{W}_C^{[1]}}{q} \right), -\mathbf{M}'_\emptyset \left( C; \frac{\mathbb{W}_C^{[2]}}{q} \right), \right. \\ \left. \left\{ \mathbf{M}'_{\mathcal{A}} \left( C; \frac{\mathbb{W}_C}{q} \right) \right\}_{\mathcal{A}}, \left\{ \mathbf{M}'_{\mathcal{A}} \left( C; \frac{\mathbb{W}_C^{[1]}}{q} \right) \right\}_{\mathcal{A}}, \left\{ \mathbf{M}'_{\mathcal{A}} \left( C; \frac{\mathbb{W}_C^{[2]}}{q} \right) \right\}_{\mathcal{A}} \right)$$

which is equal to

$$\left( -\frac{\mathbf{M}'_\emptyset(C; \mathbb{W}_C)}{q}, -\frac{\mathbf{M}'_\emptyset(C; \mathbb{W}_C^{[1]})}{q}, -\frac{\mathbf{M}'_\emptyset(C; \mathbb{W}_C^{[2]})}{q}, \right. \\ \left. \left\{ \frac{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C)}{q} \right\}_{\mathcal{A}}, \left\{ \frac{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C^{[1]})}{q} \right\}_{\mathcal{A}}, \left\{ \frac{\mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C^{[2]})}{q} \right\}_{\mathcal{A}} \right)$$

by linearity of the Hadamard derivatives in their second argument. The result follows, when  $m = I_{1,n}$  or  $I_{2,n}$ , from

$$\sup_{[0,1]^d} \left| \frac{D_{\emptyset,n}}{q} - \frac{\mathbb{C}_n}{q} - \mathbf{M}'_\emptyset \left( C; \frac{\mathbb{C}_n}{q} \right) \right| \xrightarrow{\mathbb{P}} 0 \quad ; \quad \sup_{[0,1]^d} \left| \frac{D_{\mathcal{A},n}}{q} - \mathbf{M}'_{\mathcal{A}} \left( C; \frac{\mathbb{C}_n}{q} \right) \right| \xrightarrow{\mathbb{P}} 0, \\ \sup_{[0,1]^d} \left| v_n \frac{D_{\emptyset,b}^{[m]}}{q} - v_n \frac{\mathbb{C}_b^{[m]}}{q} - v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q} - \mathbf{M}'_\emptyset \left( C; v_n \frac{\mathbb{C}_b^{[m]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q} \right) \right| \xrightarrow{\mathbb{P}} 0, \\ \sup_{[0,1]^d} \left| v_n \frac{D_{\mathcal{A},b}^{[m]}}{q} - \mathbf{M}'_{\mathcal{A}} \left( C; v_n \frac{\mathbb{C}_b^{[m]}}{q} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{q} \right) \right| \xrightarrow{\mathbb{P}} 0. \quad \square$$

Before going through the rest of the proofs, one should remark that any statement of the current paper assumes that  $\mathbb{C}_n$  converges weakly to  $\mathbb{W}_C$  given in (15). It is assumed that the latter presents continuous paths. As a consequence,

$$\|\mathbb{C}_n / \sqrt{n}\|_\infty = \|C_n - C\|_\infty \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \quad (22)$$

*Proof of Theorem 3.4.* Set  $\hat{C}_n = \sqrt{n}(\hat{C}_n - C)$ . The regularity condition imposed to  $C$  unable us to use Proposition 3.2 of [49]. It follows

$$\left(\hat{C}_n, \hat{C}_n^{(1)}, \dots, \hat{C}_n^{(N)}\right) \xrightarrow[n \rightarrow \infty]{w} \left(\mathbb{W}_C, \mathbb{W}_C^{(1)}, \dots, \mathbb{W}_C^{(N)}\right) \quad (23)$$

in  $\{\ell^\infty([0, 1]^d)\}^{2^{dN+1}}$  where  $\mathbb{W}_C^{(1)}, \dots, \mathbb{W}_C^{(N)}$  are all independent copies of  $\mathbb{W}_C$ . The process under study involves  $\hat{D}_{\theta, n} = \hat{C}_n - \sqrt{n} \left( \mathbf{M}_\theta(C + \hat{C}_n/\sqrt{n}) - \mathbf{M}_\theta(C) \right)$ ,  $\hat{D}_{\mathcal{A}, n} = \sqrt{n} \left( \mathbf{M}_{\mathcal{A}}(C + \hat{C}_n/\sqrt{n}) - \mathbf{M}_{\mathcal{A}}(C) \right)$ ,  $\hat{D}_{\theta, n}^{(k)} = \hat{C}_n^{(k)} - \mathbf{M}'_\theta(C + \hat{C}_n/\sqrt{n}; \hat{C}_n^{(k)})$  and  $\hat{D}_{\mathcal{A}, n}^{(k)} = \mathbf{M}'_{\mathcal{A}}(C + \hat{C}_n/\sqrt{n}; \hat{C}_n^{(k)})$ . Taking into account (22) and (23), the desired joint convergence arises from both Hadamard-differentiability of the operators and continuity of their derivatives.  $\square$

*Proof of Corollary 4.1.* (i) Let us focus on the  $S$  family of test statistics. To integrate with respect to  $C_n$ , we adapt the proof of [36, Proposition 3]. Applying the continuous mapping theorem, a convergence is obtained while concatenating  $C$  with the vector under study in Theorem 3.1. Since  $\|C_n - C\|_\infty$  tends to zero in probability from (22), one can replace  $C$  by  $C_n$  on the left hand side of the concatenated convergence. Again, the continuous mapping theorem allows to conclude. (ii) - (iii) - (iv) as the proof of Corollary 4.2 are based on similar arguments as those used throughout the proof section.  $\square$

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