



**HAL**  
open science

# Testing copula-based dependence hypotheses: a proofreading based on functional decompositions

Cécile Mercadier

► **To cite this version:**

Cécile Mercadier. Testing copula-based dependence hypotheses: a proofreading based on functional decompositions. 2022. hal-03905791v1

**HAL Id: hal-03905791**

**<https://hal.science/hal-03905791v1>**

Preprint submitted on 19 Dec 2022 (v1), last revised 23 Aug 2023 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Testing copula-based dependence hypotheses: a proofreading based on functional decompositions

Cécile Mercadier

December 19, 2022

## Abstract

Tests of multivariate independence may rely on asymptotically independent Cramér-von Mises statistics derived from a Möbius decomposition of the empirical copula process. We generalize this approach to some other copula-based assumptions, with the help of a functional decomposition based on commuting idempotent maps. As soon as the null hypothesis reflects the stability of the copula under the action of the composition of such operators, the methodology applies. The asymptotic joint distribution of the terms in the decomposition of the empirical copula process is established under the null hypothesis. Since the latter depends on the unknown copula being tested, we adapt the subsampling procedure to our setting and recall that the multiplier bootstrap as well as the parametric bootstrap also apply to approximate p-values. The benefit in deriving test statistics from a functional decomposition, defined in accordance with the dependence assumption under study, are illustrated and discussed through simulations.

## 1 Introduction

The nature and strength of cross-sectional dependence is of crucial importance to understand economic or environmental systems. One possible measure relies on copulas, which have become popular over the last decades. In this paper, we review and provide a new light on the extant literature for some testing problems.

Consider  $\mathcal{X}_n = \{\mathbf{X}_1, \dots, \mathbf{X}_n\}$  a sample of  $d$ -variate observations where  $\mathbf{X}_j$  stands for  $(X_{j1}, \dots, X_{jd})^T$ . At first, one may think that  $\mathcal{X}_n$  consists of independent copies of a  $d$ -dimensional random vector  $\mathbf{X} = (X_1, \dots, X_d)^T$ . However, most of the results hold true for some strict stationary time series. We assume that the cumulative distribution function (c.d.f.)  $F$  of the representative vector  $\mathbf{X}$  has continuous univariate margins denoted by  $F_1, \dots, F_d$ . There exists then a unique copula  $C : [0, 1]^d \rightarrow [0, 1]$ , that is a  $d$ -dimensional c.d.f. with standard uniform margins such that

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)), \quad \forall \mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

This representation, due to Sklar (1959), illustrates that the copula  $C$  characterizes the dependence between the components of  $\mathbf{X}$ .

The present work is concerned with testing structural hypotheses for the copula  $C$ . There exists indeed a large number of copula families and testing procedures help guide the choice of the most appropriate. Tests based on empirical copula processes have been successfully proposed in the literature. Let us cite for instance Deheuvels (1981), Kojadinovic and Holmes (2009), Kojadinovic, Segers, and Yan (2011), Genest, Nešlehová, Rémillard, and Murphy (2019) or Bücher and Pakzad (2022) that handled the independence, serial independence, independence by blocks, or broader classes such as extreme value copulas. Whereas earlier papers focus on one hypothesis at once, the aim of this paper is to demonstrate that several structural hypotheses for dependence share a common pattern. Our contribution is thus to unify part of the theory.

Our procedure could be roughly illustrated by the pioneer idea of Deheuvels (1981) which reveals the independence through the Möbius decomposition of the empirical process. The null hypothesis is thus equivalent to the intersection of a finite set of assumptions since all secondary terms of the decomposition vanish. We generalize this method by applying another functional decomposition, chosen in accordance with the structural assumption being tested. Indeed, for a given structural form of dependence, the null hypothesis is *often* characterized by the stability of the copula under the action of a transformation  $M_\emptyset$ . It leads to the test  $(\mathcal{H}) C = M_\emptyset(C)$  against its negation. Under various interesting examples, the transformation  $M_\emptyset$  can be obtained as the composition of several operators. This common functional pattern offers the possibility of generalizing the writing as unifying the method.

In the literature, some papers already propose a common procedure to a list of copula-based dependence

hypotheses. Based on an arbitrary finite set of points in  $[0, 1]^d$ , Li and Genton (2013) reduce the weak convergence of the renormalized empirical copula to a multivariate normal convergence. This is strongly different to our tools. The method in Quesy (2016) consists in rewriting the null hypotheses with quadratic functionals. But it still differs from our methodology. So, as far as we are aware, our generalization is completely new.

The remainder of this paper is organized as follows. Section 2 recalls the functional decomposition based on operators and makes it explicit in the context of dependence structures. A review of copula estimators and associated regularity conditions is dressed in Section 3, which also includes a first asymptotic statement. Section 4 explains how it is possible to construct independent copies of some limiting processes: both subsampling and multiplier bootstrap procedures are presented. Section 5 is devoted to a straightforward practical implementation of the theoretical results: test statistics, practical definition of a p-value (including parametric bootstrap) and numerical experiments are discussed. Concluding remarks are given in Section 6 and proofs of Section 4 are postponed up to Section 7. A list of references ends the paper.

## 2 Dependence structures and associated operators

Our aim in this section is to outline a functional decomposition based on commuting idempotent maps which will be the basis of our further developments. The link with the null hypothesis is the stability of the copula under a composition of some of these operators. Two lists of examples are collected in the second and third part: one listing the cases where the operators are always commuting and idempotent and one where all the operators reduce to the identity under the null hypothesis. Each dependence structure being tested is associated with its corresponding set of operators.

### 2.1 The functional decomposition of Kuo et al. (2010)

The functional decomposition mentioned here has a long story that is nicely described in van der Vaart (1998). To provide a short presentation, let us start by quoting Hoeffding (1948). His pioneer work uses  $L^2$  projections to decompose and study  $U$ -statistics. But it is in Hoeffding (1961) that the author proposes a recursive construction, based on conditional expectations, of what can be called the Hoeffding decomposition. Its first terms, depending on combinations of measurable functions of only one variable, corresponds to the Hajek projection. Efron and Stein (1981) seems to be the first reference with a clear statement and proof of the Hoeffding decomposition. It appears also in Sobol' (1993), with its own proof, with a major impact in the field of Global Sensitivity Analysis. This explains why his name is now attached to the first one. The generalization of the Hoeffding-Sobol decomposition, allowing the combination of commuting, idempotent and linear operators instead of conditional expectations, is due to Kuo, Sloan, Wasilkowski, and Woźniakowski (2010) and extended in Mercadier, Roustant, and Genest (2022). In the latter, the authors have relaxed the linear condition of the maps that are no more projections. We use below their formalism.

Let  $\mathcal{F}$  be the linear space of real-valued functions acting on  $[0, 1]^d$ . For each  $i \in \{1, \dots, d\}$ , let  $P_i : \mathcal{F} \rightarrow \mathcal{F}$  be an idempotent operator. We assume that the collection of functionals  $P_1, \dots, P_d$  commutes. Let  $I : \mathcal{F} \rightarrow \mathcal{F}$  denote the identity map. Set  $\mathcal{P}_d$  as the collection of all subsets of  $\{1, \dots, d\}$  and  $\mathcal{P}_d^* = \mathcal{P}_d \setminus \emptyset$ . Fix  $\mathcal{A} \in \mathcal{P}_d$ . Note that  $-\mathcal{A}$  stands for  $\{1, \dots, d\} \setminus \mathcal{A}$ . The composition of the maps  $P_i$  for  $i \in \mathcal{A}$  will be denoted as  $\prod_{i \in \mathcal{A}} P_i$  or  $P_{\mathcal{A}}$ , and equals  $I$  in the case where  $\mathcal{A} = \emptyset$ . From Proposition 1 of Mercadier et al. (2022), starting with such a collection  $(P_i)_{i \in \{1, \dots, d\}}$ , every  $f \in \mathcal{F}$  can be decomposed as

$$f = \sum_{\mathcal{A} \in \mathcal{P}_d} M_{\mathcal{A}}(f), \quad (1)$$

for the operator

$$M_{\mathcal{A}} = \left( \prod_{i \in \mathcal{A}} (I - P_i) \prod_{i \notin \mathcal{A}} P_i \right) \quad (2)$$

which can be obtained recursively

$$M_{\mathcal{A}} = P_{-\mathcal{A}} - \sum_{\mathcal{B} \subsetneq \mathcal{A}} M_{\mathcal{B}} \quad (3)$$

or with the following third equivalent formula

$$M_{\mathcal{A}} = \sum_{\mathcal{B} \subsetneq \mathcal{A}} (-1)^{|\mathcal{A} \setminus \mathcal{B}|} P_{-\mathcal{B}}. \quad (4)$$

The composition operator  $M_{\mathcal{A}}$  is such that  $P_i \circ M_{\mathcal{A}} = M_{\mathcal{A}} \circ P_i = M_{\mathcal{A}}$  whenever  $i \notin \mathcal{A}$  and vanishes otherwise. Another way of writing the equation (1) is

$$f - M_{\emptyset}(f) = \sum_{\mathcal{A} \in \mathcal{P}_d^*} M_{\mathcal{A}}(f), \quad (5)$$

where from (2) one knows that  $M_{\emptyset} = \prod_{i=1}^d P_i$ , the composition of all functionals. The main objective of this section is to identify, for some structural dependence null hypotheses ( $\mathcal{H}$ ), their associated set of operators  $\{P_1, \dots, P_d\}$  that allows to write

$$(\mathcal{H}) C = M_{\emptyset}(C) \quad \text{against} \quad (\mathcal{K}) C \neq M_{\emptyset}(C).$$

## 2.2 When maps involved are always commuting and idempotent

We first discuss a list of null hypotheses, describing a type of dependence or partially characterizing the copula. The complete independence, the independence by blocks, a weak form of associativity as well as particular Archimedean or Archimax copula are treated. Each associated alternative, denoted by ( $\mathcal{K}$ ) above, represents the negation of the null hypothesis.

### 2.2.1 Complete independence

The copula approach, for testing mutual independence of the components of  $\mathbf{X}$ , starts with the very well known contribution of Deheuvels (1981). Subsequent analyses are recalled below in Remark 1. Set  $\mathcal{F}_1 = \{f \in \mathcal{F}, f(\mathbf{1}) \neq 0\}$ . Note that  $\mathcal{F}_1$  contains the set of copulas. Define, for  $i \in \{1, \dots, d\}$ , the map  $P_i : \mathcal{F}_1 \rightarrow \mathcal{F}_1$  as following

$$P_i(f)(\mathbf{x}) = x_i \cdot f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d) / f(1, \dots, 1). \quad (6)$$

One could check that  $\{P_1, \dots, P_d\}$  defines a collection of commuting and idempotent operators. Moreover, the case of the complete independence

$$(\mathcal{H}_1) C(x_1, \dots, x_d) = x_1 \cdots x_d$$

can be written as  $(\mathcal{H}_1) C = M_{\emptyset}(C)$  where  $M_{\mathcal{A}}$  stands here for the combination of (2) with (6).

*Remark 1.* As already mentioned in Mercadier et al. (2022), the functional decomposition (1) associated with the map (2) and the projection (6) is known as the Möbius decomposition. As explained and extended in Genest and Rémillard (2004) or in the nice introduction of Genest, Quessy, and Remillard (2007), the pioneer work of Deheuvels (1981) makes already use, without naming it, of the Möbius decomposition. Some results for the case where the number of variables is of the same size, or even larger, than the sample size, have recently been obtained by Bücher and Pakzad (2022). However, none of these works do a presentation in terms of commuting idempotent maps. This is new.

### 2.2.2 Independence among subvectors

Let us structure the random vector  $\mathbf{X}$  as the concatenation of  $p$  subvectors  $\mathbf{X} = (\mathbf{X}_{\{1\}}, \dots, \mathbf{X}_{\{p\}})$  of dimension  $d_1, \dots, d_p$ . Therefore,  $d = d_1 + \dots + d_p$ . It is possible to generalize what precedes to the independence by blocks. For  $i \in \{1, \dots, p\}$ , set

$$P_i(f)(\mathbf{x}) = \frac{f(\mathbf{1}_{\{1\}}, \dots, \mathbf{1}_{\{i-1\}}, \mathbf{x}_{\{i\}}, \mathbf{1}_{\{i+1\}}, \dots, \mathbf{1}_{\{p\}}) \cdot f(\mathbf{x}_{\{1\}}, \dots, \mathbf{x}_{\{i-1\}}, \mathbf{1}_{\{i\}}, \mathbf{x}_{\{i+1\}}, \dots, \mathbf{x}_{\{p\}})}{f(\mathbf{1}_{\{1\}}, \dots, \mathbf{1}_{\{p\}})}, \quad (7)$$

and, to avoid a specific treatment, complete this collection with  $P_i = I$  the identity map for  $i \in \{p+1, \dots, d\}$ . It is possible to prove that (7) defines a commuting set of idempotent maps. The assertion  $(\mathcal{H}_2) \mathbf{X}_{\{1\}}, \dots, \mathbf{X}_{\{p\}}$  are independent, or equivalently

$$(\mathcal{H}_2) C(\mathbf{x}) = C(\mathbf{x}_{\{1\}}, \mathbf{1}_{\{2\}}, \dots, \mathbf{1}_{\{p\}}) \cdots C(\mathbf{1}_{\{1\}}, \dots, \mathbf{1}_{\{i-1\}}, \mathbf{x}_{\{i\}}, \mathbf{1}_{\{i+1\}}, \dots, \mathbf{1}_{\{p\}}) \cdots C(\mathbf{1}_{\{1\}}, \dots, \mathbf{1}_{\{p-1\}}, \mathbf{x}_{\{p\}})$$

can be rewritten as  $(\mathcal{H}_2) C = M_{\emptyset}(C)$  where  $M_{\mathcal{A}}$  stands now for the combination of (2) and (7).

*Remark 2.* Even if the presentation is not in terms of commuting idempotent maps, the reader should be aware that Kojadinovic and Holmes (2009) have successfully handled the question of testing  $(\mathcal{H}_2)$  with the help of the extended Möbius decomposition. The latter is the one which combines (1), (2) and (7). As already mentioned in Mercadier et al. (2022), Lemma 6 in Kojadinovic and Holmes (2009) is then a particular case of (1).

### 2.2.3 Associativity

According to Stupňanová and Kolesárová (2011), a function  $f : [0, 1]^d \rightarrow [0, 1]$  is said associative whenever for  $(x_1, \dots, x_d, \dots, x_{2d-1}) \in [0, 1]^{2d-1}$  it holds

$$f(f(x_1, \dots, x_d), x_{d+1}, \dots, x_{2d-1}) = f(x_1, f(x_2, \dots, x_{d+1}), x_{d+2}, \dots, x_{2d-1}) \dots = f(x_1, \dots, x_{d-1}, f(x_d, \dots, x_{2d-1})).$$

Any archimedean copula  $C$  is associative. In particular, under archimedeanity

$$C(x_1, \dots, x_i, C(1, \dots, 1, x_{i+1}, \dots, x_d), 1, \dots, 1) = C(x_1, \dots, x_d), \quad \forall \mathbf{x} = (x_1, \dots, x_d) \in [0, 1]^d, \forall i \in \{1, \dots, d-1\}.$$

We thus propose to consider the following map

$$P_i(f)(\mathbf{x}) = f(x_1, \dots, x_i, f(1, \dots, 1, x_{i+1}, \dots, x_d), 1, \dots, 1), \quad (8)$$

completed by  $P_d = I$ . Note that  $P_{d-1} = I$  too. This is clearly an arbitrary choice of operators in comparison with natural previous examples. To be more explicit,

$$P_i(f)(\mathbf{x}) = f(1, \dots, 1, f(x_1, \dots, x_i, 1, \dots, 1), x_{i+1}, \dots, x_d),$$

would also be convenient, and it is not the only one. Set  $(\mathbf{1}_{-\mathcal{A}}, \mathbf{x}_{\mathcal{A}}) = (\mathbf{x}_{\mathcal{A}}, \mathbf{1}_{-\mathcal{A}}) = \sum_{i=1}^d x_i \mathbf{e}_i \mathbf{1}_{i \in \mathcal{A}} + \mathbf{e}_i \mathbf{1}_{i \notin \mathcal{A}}$ . Restricted to functions  $f$  satisfying  $f(\mathbf{1}_{-i}, x_i) = x_i$  for any  $i$  in  $\{1, \dots, d\}$  which of course includes copulas, each operator above is idempotent and the family commutes. We do not characterize archimedeanity this way, only a weak version. However, it remains interesting to investigate

$$(\mathcal{H}_3) C = M_{\theta}(C) \quad (9)$$

when  $M_{\mathcal{A}}$  is given by the combination of (2) and (8). It is not so obvious what  $C = M_{\theta}(C)$  means here. It can be written  $C(\mathbf{x}) = C(x_1, z_2, 1, \dots, 1)$  with  $z_i = C(1, \dots, 1, x_i, z_{i+1})$  for  $i = 2, \dots, d-1$  and  $z_d = x_d$ .

*Remark 3.* Testing archimedeanity consists in testing associativity as well as an appropriate condition on the behavior of  $C$  on the diagonal. For complete details we refer to Bücher, Dette, and Volgushev (2012).

### 2.2.4 Describing a given Archimedean copula

The next null hypotheses need some preliminary remarks. Goodness-of-fit tests arise when  $C$  is unknown but assumed to belong to a particular class  $(\tilde{\mathcal{H}}) C \in \{C_{\theta}, \theta \in \Theta\}$  where  $\Theta$  is an open subset of  $\mathbb{R}^D$  for some integer  $D \geq 1$ . Natural tests consist in measuring a “distance” between the empirical copula and an estimation of  $C$  obtained under  $(\tilde{\mathcal{H}})$ . We skip here the details for the inference on the parameter and we focus below only on the testing part of the procedure. The null hypothesis becomes  $(\mathcal{H}) C = C_{\theta_0}$  where the reader should think  $C_{\theta_0}$  as  $C_{\hat{\theta}_n}$ , where  $\hat{\theta}_n$  estimates  $\theta$  in  $\Theta$ . Of course, convergence of the practical procedure with respect to  $(\tilde{\mathcal{H}})$  needs appropriate regularity conditions on both the parametric family and the sequence of estimators  $\hat{\theta}_n$ . We refer to Genest et al. (2009) for an pioneer review and discussion on combining both testing steps.

We focus here on testing a given Archimedean copula which differs from testing Archimedeanity. Fix  $\varphi$  the generator of interest that is a non-negative, continuous, strictly decreasing and convex function defined on  $[0, 1]$  satisfying  $\varphi\{1\} = 0$ . Its pseudo-inverse, denoted  $\varphi^{[-1]}$ , is defined as the usual inverse on  $[0, \varphi\{0\}]$  and equals 0 elsewhere. See McNeil and Neslehova (2009) for a complete characterization of the generator  $\varphi$ .

For  $i \in \{1, \dots, d\}$ , consider the functional

$$P_i(f)(\mathbf{x}) = \varphi^{[-1]} [\varphi\{f(1, \dots, 1, x_i, 1, \dots, 1)\} + \varphi\{f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d)\} - \varphi\{f(1, \dots, 1)\}]. \quad (10)$$

It forms a collection of commuting idempotent maps. Testing the specific Archimedean copula generated by  $\varphi$

$$(\mathcal{H}_4) C(\mathbf{x}) = \varphi^{[-1]} [\varphi\{x_1\} + \dots + \varphi\{x_d\}] \quad (11)$$

can be rewritten as  $(\mathcal{H}_4) C = M_{\theta}(C)$  where  $M_{\mathcal{A}}$  combines (2) with maps given in (10).

*Remark 4.* The symmetric logistic extreme value copula is a particular case. Recall first that a copula  $C$  is an extreme value copula if it can be written

$$C(\mathbf{x}) = \exp[-\ell\{-\ln(x_1), \dots, -\ln(x_d)\}] \quad (12)$$

for  $\ell : [0, \infty]^d \rightarrow [0, \infty]$  a so-called stable tail dependence function. See for instance Chapter 7 of de Haan and Ferreira (2006) for more details on the  $\ell$  function. It is called the symmetric logistic extreme value copula model when there exists a real  $\theta \in [1, \infty[$  such that  $\ell(x_1, \dots, x_d) = (x_1^{\theta} + \dots + x_d^{\theta})^{1/\theta}$ . Testing the symmetric logistic extreme value model

$$(\mathcal{H}_5) C(\mathbf{x}) = \exp\left[-\left\{(-\ln(x_1))^{\theta} + \dots + (-\ln(x_d))^{\theta}\right\}^{1/\theta}\right]$$

corresponds to  $(\mathcal{H}_5) C = M_{\theta}(C)$  as soon as  $M_{\mathcal{A}}$  stands for (2) with maps (11) and substituting  $\varphi(t) = (-\ln(t))^{\theta}$ .

### 2.2.5 Extension to Archimedean by blocks

A possible extension consists in mixing previous sections. Recall that the random vector  $\mathbf{X}$  might be seen as the concatenation of  $p$  subvectors  $\mathbf{X}_{\{1\}}, \dots, \mathbf{X}_{\{p\}}$  of dimension  $d_1, \dots, d_p$  where  $d = d_1 + \dots + d_p$ . The independence by blocks of Section 2.2.2 could be replaced by an Archimedean structure by blocks. Let  $\varphi$  a generator as described in Section 2.2.4. Then the null hypothesis

$$(\mathcal{H}_6) C(\mathbf{x}) = \varphi^{-1} [\varphi(C(\mathbf{x}_{\{1\}}, \mathbf{1}_{-\{1\}})) + \dots + \varphi(C(\mathbf{1}_{-\{p\}}, \mathbf{x}_{\{p\}}))] ]$$

could be written as  $(\mathcal{H}_6) C = M_\emptyset(C)$  when  $M_{\mathcal{A}}$  calls (2) while using the following functionals

$$P_i(f)(\mathbf{x}) = \varphi^{-1} [\varphi(f(\mathbf{x}_{\{i\}}, \mathbf{1}_{-\{i\}})) - \varphi(f(\mathbf{1})) + \varphi(f(\mathbf{x}_{-\{i\}}, \mathbf{1}_{\{i\}}))] . \quad (13)$$

Nested Archimedean copulas such as  $(\mathcal{H}_6)$  have an easy interpretation. Only  $p$ -uplets of variables, each belonging to one of the  $p$  blocks, are completely specified: Their dependence structure follows the Archimedean copula generated by  $\varphi$ . The dependence within any groups of variables belonging partially to the same block is not fixed.

### 2.2.6 Extension to some specific Archimax copulas

As before, consider  $\varphi$  a generator associated with an archimedean structure. And consider  $\ell : [0, \infty]^d \rightarrow [0, \infty]$  a stable tail dependence function. Recall from Charpentier, Fougères, Genest, and Nešlehová (2014) or more recently from Chatelain, Fougères, and Nešlehová (2020) that

$$C(\mathbf{x}) = \varphi^{-1} [\ell(\varphi(x_1), \dots, \varphi(x_d))] ]$$

is called an archimax copula. We restrict here the form of  $\ell$  as following

$$\ell(x_1, \dots, x_d) = g^{-1} [g\{\ell(\mathbf{x}_{\{1\}}, \mathbf{0}_{-\{1\}})\} + \dots + g\{\ell(\mathbf{0}_{-\{p\}}, \mathbf{x}_{\{p\}})\} ] , \quad (14)$$

where  $g$  is a continuous bijection from  $\mathbb{R}_+$  to  $\mathbb{R}_+$  satisfying  $g(1) = 1$ . From Theorem 6 in Ressel (2022), one knows that  $g(x) = x^\theta$  for some  $\theta \geq 1$ . For the sake of simplicity, set  $\varphi_{\mathbf{x}_{\{i\}}} = \sum_{j \in \{i\}} \varphi(x_j) \mathbf{e}_j$ . For  $i \in \{1, \dots, p\}$ , let define the operator  $P_i$  by

$$P_i(f)(\mathbf{x}) = \varphi^{-1} \left[ \left\{ \left( \ell(\varphi_{\mathbf{x}_{\{i\}}}, \mathbf{0}_{-\{i\}}) \right)^\theta + \left( \varphi \circ f(\mathbf{1}_{\{i\}}, \mathbf{x}_{-\{i\}}) \right)^\theta \right\}^{1/\theta} \right] \quad (15)$$

completed with  $P_{p+1} = \dots = P_d = I$ . They form a commuting set of idempotent maps. Then, testing

$$(\mathcal{H}_7) C(\mathbf{x}) = \varphi^{-1} \left[ \left\{ \left( \ell(\varphi_{\mathbf{x}_{\{1\}}}, \mathbf{0}_{-\{1\}}) \right)^\theta + \dots + \left( \ell(\mathbf{0}_{-\{p\}}, \varphi_{\mathbf{x}_{\{p\}}}) \right)^\theta \right\}^{1/\theta} \right]$$

can be rewritten  $(\mathcal{H}_7) C = M_\emptyset(C)$  for  $M_{\mathcal{A}}$  combining (2) and (15).

### 2.2.7 First summary

Let  $\{P_1, \dots, P_d\}$  be a collection of commuting idempotent functionals. The previous examples all illustrate the situation where the null hypothesis has the form  $(\mathcal{H}) C = (\prod_{i=1}^d P_i)(C)$ . Let  $M_{\mathcal{A}}$  denote the combination of (2) with the set of operators involved in the definition of  $(\mathcal{H})$ . From (5), recall that  $M_\emptyset = \prod_{i=1}^d P_i$  and that

$$C - M_\emptyset(C) = \sum_{\mathcal{A} \in \mathcal{P}_d^*} M_{\mathcal{A}}(C) .$$

As a consequence, the summation  $\sum_{\mathcal{A} \in \mathcal{P}_d^*} M_{\mathcal{A}}(C)$  vanishes when the null hypothesis  $(\mathcal{H})$  holds true. It is thus interesting to consider for any  $\mathcal{A} \in \mathcal{P}_d^*$  the null sub-hypothesis

$$(\mathcal{H}^{\mathcal{A}}) M_{\mathcal{A}}(C) = 0 . \quad (16)$$

A relevant question is to analyze whether any  $(\mathcal{H}^{\mathcal{A}})$  holds true under the null hypothesis  $(\mathcal{H})$ . What is its link exactly with the intersection? In the next proposition, we answer part of the question.

**Proposition 1.** *Let  $P_1, \dots, P_d$  be a commuting collection of idempotent operators on  $\mathcal{F}$ . Then, the null hypothesis satisfies the equality*

$$(\mathcal{H}) = \bigcap_{\mathcal{A} \in \mathcal{P}_d^*} (\mathcal{H}^{\mathcal{A}}) .$$

*Remark 5.* When the null hypothesis ( $\mathcal{H}_2$ ) is under study (namely the independence among subvectors), note that Proposition 7 in Kojadinovic and Holmes (2009) (restricted to  $H$  being a copula) can be viewed as a consequence of Proposition 1.

*Proof of Proposition 1.* Suppose  $M_{\mathcal{A}}(C) \equiv 0$  for any non empty subset  $\mathcal{A}$  of  $\{1, \dots, d\}$ . By application of (1) inherited from Proposition 1 of Mercadier et al. (2022), one obtains  $C - M_{\emptyset}(C) \equiv 0$  which is ( $\mathcal{H}$ ). Reciprocally, if ( $\mathcal{H}$ ) holds true, then  $C = M_{\emptyset}(C) = (\prod_{j=1}^d P_j)(C)$ . Combined with (2), it yields by commutativity,

$$M_{\mathcal{A}}(C) = \left( \prod_{i \in \mathcal{A}} (I - P_i) \prod_{i \notin \mathcal{A}} P_i \right) (C) = \left( \prod_{i \in \mathcal{A}} (I - P_i) \prod_{i \notin \mathcal{A}} P_i \right) \left( \prod_{j=1}^d P_j(C) \right) = \left( \prod_{i \in \mathcal{A}} (P_i - P_i^2) \prod_{i \notin \mathcal{A}} P_i \right) \left( \prod_{j \notin \mathcal{A}} P_j(C) \right)$$

which vanishes, whenever  $\mathcal{A}$  is non empty, since  $P_i = P_i^2$  by idempotence assumption.  $\square$

## 2.3 When maps involved reduce to the identity under the null hypothesis

Another list of null hypotheses is drawn up below. The main difference with what precedes concerns the operators which are no more idempotent nor commuting in general. But they simplify to the identity operator (which of course is idempotent) under the null hypothesis.

### 2.3.1 The max-stability assumption

For a given positive integer  $r$ , let us consider the null hypothesis

$$(\mathcal{H}_{8,r}) C(\mathbf{x}) = C^r(\mathbf{x}^{1/r}) \quad \forall \mathbf{x} \in [0, 1]^d .$$

The max-stability assumption is the intersection of any such null hypothesis, that is  $(\mathcal{H}_8) = \cap_{r \in \mathbb{N}^*} (\mathcal{H}_{8,r})$ .

*Remark 6.* The presentation above is similar to the description done in Section 3 of Kojadinovic, Segers, and Yan (2011). Of course, the analysis in terms of specific maps injected in (2), and therefore in (1), is new.

Let  $r_i$  be a positive integer. Consider the functional

$$P_i(f)(\mathbf{x}) = f^{r_i}(\mathbf{x}^{1/r_i}) , \tag{17}$$

which, in general, does not satisfy idempotence assumption unless  $f$  satisfies  $(\mathcal{H}_8)$ , or more precisely  $(\mathcal{H}_{8,r_i})$ . In this case, the operator  $P_i$  becomes the identity operator  $I$ . As a consequence, under  $(\mathcal{H}_8)$  the copula  $C$  satisfies  $C = M_{\emptyset}(C)$  where  $M_{\mathcal{A}}$  stands now for the combination of (2) with (17).

*Remark 7.* In the bivariate setting,  $P_1$  and  $P_2$  should be chosen in accordance with Proposition 4.2 of Klement, Mesiar, and Pap (2005). It indeed is proven therein the equivalence between  $(\mathcal{H}_8)$  and the *finite* intersection  $(\mathcal{H}_{8,r_1}) \cap (\mathcal{H}_{8,r_2})$  for two positive reals  $r_1$  and  $r_2$  such that  $\log r_1 / \log r_2$  is irrational.

### 2.3.2 Exchangeability

Let  $\mathfrak{S}_d$  be the set of all permutations of  $\{1, \dots, d\}$  and denote  $\mathbf{x}_{\sigma} = (x_{\sigma(1)}, \dots, x_{\sigma(d)})$  for any  $\sigma \in \mathfrak{S}_d$ . Testing symmetry of the copula can be handled in a very similar way to the previous one. The null hypothesis of exchangeability of the random variables  $X_1, \dots, X_d$  can indeed be expressed as

$$(\mathcal{H}_9) \quad C(\mathbf{x}) = C(\mathbf{x}_{\sigma}) \quad \forall \mathbf{x} \in [0, 1]^d \text{ and } \forall \sigma \in \mathfrak{S}_d .$$

It is the (finite here) intersection over  $\mathfrak{S}_d$  of null hypotheses of the form

$$(\mathcal{H}_{9,\sigma}) \quad C(\mathbf{x}) = C(\mathbf{x}_{\sigma}) \quad \forall \mathbf{x} \in [0, 1]^d .$$

Let  $T_{1,d}$  denote the set consisting of the  $d - 1$  transpositions  $\tau_i = (1i)$  for  $i = 2, \dots, d$ . Noting that  $T_{1,d}$  generates  $\mathfrak{S}_d$ , it is also possible to write here that

$$(\mathcal{H}_9) = \cap_{i=2}^d (\mathcal{H}_{9,\tau_i}) .$$

Consider the functional

$$P_i(f)(\mathbf{x}) = f(\mathbf{x}_{\tau_i}) , \tag{18}$$

for  $i = 2, \dots, d$  and complete the collection with  $P_1 = I$  the identity map. The map in (18) is no more idempotent. Additionally,  $P_k$  and  $P_m$  do not commute in general. However, under the restriction that  $f$  is assumed to be symmetric,  $\{P_1, \dots, P_d\}$  forms a set of commuting projections since all reduce to the identity  $I$ . Under  $(\mathcal{H}_9)$  the copula  $C$  satisfies  $C = M_{\emptyset}(C)$  where  $M_{\mathcal{A}}$  combines (2) with (18).

*Remark 8.* In the bivariate case, such test has been first investigated by Genest, Nešlehová, and Quessy (2012). Testing exchangeability for copulas in arbitrary dimensions is deeply studied in Harder and Stadtmüller (2017).

### 2.3.3 Second summary

The setting associated with the second list of examples differs from the first one. The maps  $P_1, \dots, P_d$  are not commuting nor idempotent in general. And a comparison between  $(\mathcal{H})$  and  $(\mathcal{H}^A)$  loses its sense. However, applied to a copula  $C$  satisfying the null hypothesis  $(\mathcal{H})$ , all operators of the previous list reduce to the identity. Consequently,  $M_\emptyset(C) = C$  and  $M_{\mathcal{A}}(C) \equiv 0$  for any  $\mathcal{A} \in \mathcal{P}_d^*$ . Hence any  $(\mathcal{H}^A)$  holds true. Thus we can only write here  $(\mathcal{H}) \subseteq \cap_{\mathcal{A}}(\mathcal{H}^A)$ .

The interest of the preceding two lists is to show the extent to which the functional decomposition given through a set of operators applies successfully. It brings together, in a common scheme and writing, some very varied structural copula-based dependence hypotheses. However, in cases as those described in the second list, the decomposition brings no additional information. Note indeed that the composition  $\prod_{i=1}^d P_i$  in the exchangeability context (18) is nothing other than the single  $P(f)(\mathbf{x}) = f(\mathbf{x}_{\tau_1 \circ \dots \circ \tau_d})$ . Analogously, under the max-stability hypothesis, the composition  $\prod_{i=1}^d P_i$  when  $P_i$  is defined by (17) reduces to the single  $P(f)(\mathbf{x}) = f^p(\mathbf{x}^{1/p})$  with  $p = r_1 \times \dots \times r_d$ . As a consequence, the analysis of the terms  $M_{\mathcal{A}}(C)$  will not add any information in comparison with the classical measure based on  $M_\emptyset(C) - C$  or any combination of the latter (for instance obtained with different values of  $r$  in the max-stability example).

## 2.4 Focus on Hadamard-differentiability

This section may be omitted in a first reading. Let us recall the notion of differentiability for operators. Then, it will be briefly applied to the maps  $P_i$  under study.

A functional  $T : \mathcal{D} \subseteq \ell^\infty([0, 1]^d) \rightarrow \mathcal{E}$  is Hadamard differentiable at  $f \in \mathcal{D}$  tangentially to a set  $\mathcal{C}_0$  if there exists a map  $T'(f; \cdot) : \mathcal{C}_0 \rightarrow \mathcal{E}$  such that  $T'(f; h)$  is the limit with respect to the Kolmogorov distance of  $(T(f + t_n h_n) - T(f)) / t_n$  as  $n$  tends to infinity, for any  $h_n$  converging to  $h \in \mathcal{C}_0$  and for any sequence  $t_n$  of reals converging to 0 such that  $f + t_n h_n \in \mathcal{D}$ .

Now, set

$$\mathcal{C}_0 = \{h \in C([0, 1]^d) \text{ such that } h(\mathbf{1}) = 0 \text{ and } h(\mathbf{x}) = 0 \text{ if some components of } \mathbf{x} \text{ are equal to } 0\}. \quad (19)$$

Let us begin by very simple calculations. By linearity of the map defined in (18),  $P'_i(f; h) = P_i(h)$ . For the operator  $P_i$  given in (17), we get  $P'_i(f; h)(\mathbf{x}) = r_i h(\mathbf{x}^{1/r_i}) f^{r_i-1}(\mathbf{x}^{1/r_i})$ , which is consistent with the limiting expression in Proposition 1 of Kojadinovic et al. (2011).

The map defined by (6), in the context of complete independence testing, is Hadamard-differentiable at every  $f \in \mathcal{F}_1$  tangentially to  $\mathcal{C}_0$  with derivative  $P'_i(f; h)(\mathbf{x}) = x_i (h(\mathbf{x}_{-i}, 1_i) / f(\mathbf{1}) - h(\mathbf{1}) f(x_i, \mathbf{1}_{-i}) / f(\mathbf{1})^2)$ . Applied to any copula  $C$ , it simplifies to  $P'_i(C; h)(\mathbf{x}) = x_i (h(\mathbf{x}_{-i}, 1_i) - h(\mathbf{1}) x_i)$ , and reduces for  $h \in \mathcal{C}_0$  to

$$P'_i(C; h)(\mathbf{x}) = x_i h(\mathbf{x}_{-i}, 1_i).$$

The mapping (7) is Hadamard-differentiable at every  $f \in \mathcal{F}_1$  tangentially to  $\mathcal{C}_0$  with derivative  $P'_i(f; h)(\mathbf{x}) = h(\mathbf{1}_{-\{i\}}, \mathbf{x}_{\{i\}}) f(\mathbf{x}_{-\{i\}}, \mathbf{1}_{\{i\}}) / f(\mathbf{1}) + h(\mathbf{x}_{-\{i\}}, \mathbf{1}_{\{i\}}) f(\mathbf{1}_{-\{i\}}, \mathbf{x}_{\{i\}}) / f(\mathbf{1}) - h(\mathbf{1}) f(\mathbf{x}_{-\{i\}}, \mathbf{1}_{\{i\}}) f(\mathbf{1}_{-\{i\}}, \mathbf{x}_{\{i\}}) / f(\mathbf{1})^2$ . It simplifies, for any copula  $C$  and  $h$  in  $\mathcal{C}_0$ , to

$$P'_i(C; h)(\mathbf{x}) = h(\mathbf{1}_{-\{i\}}, \mathbf{x}_{\{i\}}) C(\mathbf{x}_{-\{i\}}, \mathbf{1}_{\{i\}}) + h(\mathbf{x}_{-\{i\}}, \mathbf{1}_{\{i\}}) C(\mathbf{1}_{-\{i\}}, \mathbf{x}_{\{i\}}).$$

Still in this context of testing independence by blocks, Kojadinovic and Holmes (2009) give explicitly the Hadamard-derivative of the map  $M_{\mathcal{A}}$  in their Lemma 4.

The maps defined in (8) are Hadamard-differentiable on  $\mathcal{F}^* = \{f \in \mathcal{F}, f \text{ is differentiable}\}$  with

$$P'_i(f; h)(\mathbf{x}) = h(1, \dots, 1, x_{i+1}, \dots, x_d) \partial_{i+1} f(x_1, \dots, x_i, f(1, \dots, 1, x_{i+1}, \dots, x_d), 1, \dots, 1).$$

From Theorem 2.2 in McNeil and Nešlehová (2009), one knows that any Archimedean copula is associated to a generator at least  $d-2$  differentiable. Assume differentiability of  $\varphi$  and  $\varphi^{[-1]}$  on  $[0, 1]$  and  $(0, \varphi(0))$  respectively. Then (10) is Hadamard-differentiable at every  $f \in \mathcal{F}^{[0,1]} := \{f \in \mathcal{F}, f(\mathbf{x}) \in [0, 1] \text{ for any } \mathbf{x} \in [0, 1]^d\}$  tangentially to  $\mathcal{C}_0$ . Observe that any copula lies in  $\mathcal{F}^{[0,1]}$ . Note that

$$\varphi \circ P_i(f)(\mathbf{x}) = \varphi\{f(1, \dots, 1, x_i, 1, \dots, 1)\} + \varphi\{f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d)\} - \varphi\{f(\mathbf{1})\}$$

so that

$$\begin{aligned} (\varphi \circ P_i)'(f; h)(\mathbf{x}) &= (h \times \varphi' \circ f)(1, \dots, 1, x_i, 1, \dots, 1) \\ &\quad + (h \times \varphi' \circ f)(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d) - (h \times \varphi' \circ f)(\mathbf{1}). \end{aligned}$$



Now, since  $P_i = \varphi^{[-1]} \circ \varphi \circ P_i$ , one obtains

$$\begin{aligned} P_i'(f; h)(\mathbf{x}) &= (\varphi^{[-1]})' (\varphi \circ P_i(f)(\mathbf{x})) \times (\varphi \circ P_i)' (f; h)(\mathbf{x}) \\ &= \frac{(h \times \varphi' \circ f)(1, \dots, 1, x_i, 1, \dots, 1) + (h \times \varphi' \circ f)(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d) - (h \times \varphi' \circ f)(\mathbf{1})}{\varphi'(\varphi^{[-1]}[\varphi\{f(1, \dots, 1, x_i, 1, \dots, 1)\} + \varphi\{f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d)\} - \varphi\{f(\mathbf{1})\}])}. \end{aligned}$$

When computed for any copula  $C$  and any  $h \in \mathcal{C}_0$ , it becomes

$$P_i'(C; h)(\mathbf{x}) = \frac{h \times \varphi'(x_i) + (h \times \varphi' \circ C)(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d)}{\varphi'(P_i(C)(\mathbf{x}))}.$$

### 3 Inference and limit theorem

The purpose of this section is to introduce the testing processes. Consider a structural dependence hypothesis for copulas expressed as  $(\mathcal{H}) C = M_\emptyset(C)$ . Recall that  $\{M_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$  is the set depending, through Formula (2), on a collection of operators  $\{P_1, \dots, P_d\}$  defined on  $\mathcal{F}$ . It is assumed that  $\{P_1, \dots, P_d\}$  are commuting and idempotent maps, at least when  $(\mathcal{H})$  holds true. Starting from a copula estimator  $C_n$ , it is natural to construct the test process as  $(\sqrt{n}(C_n - M_\emptyset(C_n)))(\mathbf{x})$ ,  $\mathbf{x} \in [0, 1]^d$  when considering  $(\mathcal{H})$ . This is precisely what is done in the literature. However, since  $(\mathcal{H})$  implies any sub-hypothesis  $(\mathcal{H}^{\mathcal{A}}) M_{\mathcal{A}}(C) = 0$  (and sometimes equals their intersection) as explained in Section 2, another choice is possible. For the sake of clarity, set  $\mathbb{D}_{\emptyset, n} = \sqrt{n}(C_n - M_\emptyset(C_n))$  and  $\mathbb{D}_{\mathcal{A}, n} = \sqrt{n}M_{\mathcal{A}}(C_n)$ . The empirical process

$$(\mathbb{D}_{\emptyset, n}, \{\mathbb{D}_{\mathcal{A}, n}\}_{\mathcal{A} \in \mathcal{P}_d^*})$$

forms an appropriate ingredient to test  $(\mathcal{H})$ . We address in the current section its asymptotic behavior.

#### 3.1 Copula estimators

Since the true copula is not assumed to be known, it is a crucial step to furnish copula estimators. Many proposals have been made over the last decades. The list that we present below is highly inspired by the one provided in Section 2 of Kojadinovic and Stemičkovskaya (2019). Fix  $\mathbf{x} = (x_1, \dots, x_d) \in [0, 1]^d$  and consider the estimation of  $C$  at point  $\mathbf{x}$ . The ordinary non-parametric estimator has been constructed in Deheuvels (1979) by plug-in principle as  $F_n(F_{n1}^-(x_1), \dots, F_{nd}^-(x_d))$  where  $F_n$  is the empirical distribution function of  $\mathbf{X}_1, \dots, \mathbf{X}_n$  and the  $F_{ni}^-$ 's are the marginal quantile functions. There are several asymptotically equivalent definitions of the empirical copula. From Genest, Ghoudi, and Rivest (1995), we consider another definition and set

$$\tilde{C}_n(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \prod_{i=1}^d \mathbf{1}_{\{F_{nj}(X_{ji}) \leq x_i\}}.$$

Let  $R_{ji, n}$  denotes the rank of  $X_{ji}$  among  $X_{1i}, \dots, X_{ni}$  for  $i = 1, \dots, d$  and  $j = 1, \dots, n$ . Set  $\hat{U}_{ji, n} = R_{ji, n}/(n+1)$  the pseudo-observations. Depending on the possible presence of ties in the sample, the ordinary version  $\tilde{C}_n$  may coincide with the empirical copula given by

$$\hat{C}_n(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \prod_{i=1}^d \mathbf{1}_{\{\hat{U}_{ji, n} \leq x_i\}}.$$

The latter has been introduced by Ruschendorf (1976), and a bound for its approximation of  $C$  was provided in Deheuvels (1980). The previous estimates do not induce continuous maps on  $[0, 1]^d$  but only piecewise constant functions. Since the estimation target  $C$  is assumed continuous, two alternative smoothed versions follow. The empirical checkerboard copula is a multilinear extension of the empirical copula defined by

$$C_n^\#(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \prod_{i=1}^d \min\{\max\{nx_i - R_{ji, n}, 0\}, 1\}.$$

We refer to Carley and Taylor (2002); Genest, Nešlehová, and Rémillard (2017); Kojadinovic and Stemičkovskaya (2019) and references therein. And, in order to smooth the indicator function in the definition of  $\hat{C}_n$ , set  $F_{n, r}$  the probability distribution function of the Beta distribution  $\mathcal{B}(r, n+1-r)$ . The empirical beta copula, first

introduced in (4.1) of Segers, Sibuya, and Tsukahara (2017), and also studied in Berghaus and Segers (2018) and Kiriliouk, Segers, and Tsukahara (2021) for instance, is given by

$$C_n^\beta(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \prod_{i=1}^d F_{n,R_{j,i,n}}(x_i).$$

These estimators represent obviously a partial summary of the literature, but are widely used in practice. The notation  $C_n$  will denote one of them. And the corresponding empirical copula process will be denoted  $\mathbb{C}_n = \sqrt{n}(C_n - C)$ . If needed, particular choices of the empirical copula process can be identified as follows  $\tilde{\mathbb{C}}_n = \sqrt{n}(\tilde{C}_n - C)$ ,  $\hat{\mathbb{C}}_n = \sqrt{n}(\hat{C}_n - C)$ ,  $\mathbb{C}_n^\# = \sqrt{n}(C_n^\# - C)$  or  $\mathbb{C}_n^\beta = \sqrt{n}(C_n^\beta - C)$ .

### 3.2 Regularity conditions

Before going through our asymptotic statement, we need to introduce a list of regularity conditions. Let consider the space  $\ell^\infty([0, 1]^d)$  of all uniformly bounded real-valued functions  $f$  defined on  $[0, 1]^d$  equipped with the topology induced by the Kolmogorov norm  $\|f\|_\infty = \sup_{\mathbf{x} \in [0, 1]^d} |f(\mathbf{x})|$ . We describe here a setting where the weak convergence of the empirical copula processes is valid. The framework presented below reveals the progress made over the past forty years in order to refine the assumptions. See Ruschendorf (1976), Deheuvels (1979), Deheuvels (1980), Deheuvels (1981), van der Vaart and Wellner (1996), Fermanian et al. (2004), Genest and Rémillard (2004), Segers (2012), Bücher and Volgushev (2013), Bormann et al. (2016), Berghaus et al. (2017), Berghaus and Segers (2018) and Kojadinovic and Stemiakovskaya (2019).

$\mathcal{R}_{\mathcal{X}_n}$  For any  $i = 1, \dots, d$  the  $i$ th component sample  $X_{1i}, \dots, X_{ni}$  does not contain ties. *When the sample  $\mathbf{X}_1, \dots, \mathbf{X}_n$  is a collection of independent copies of  $\mathbf{X}$ , the assumption  $\mathcal{R}_{\mathcal{X}_n}$  holds true.*

$\mathcal{R}_{\mathbb{G}_n}$  For any  $j \in \{1, \dots, n\}$ , set  $\mathbf{U}_j = (F_1(X_{j1}), \dots, F_d(X_{jd}))$ . The empirical cumulative distribution function based on  $\mathbf{U}_1, \dots, \mathbf{U}_n$  is denoted by  $G_n$  and we set  $\mathbb{G}_n = \sqrt{n}(G_n - C)$ . The empirical process  $\mathbb{G}_n$  is assumed to converge weakly in  $\ell^\infty([0, 1]^d)$  to a tight, centered Gaussian process  $\mathbb{G}_C$  concentrated on  $\mathcal{C}_0$ , already defined in (19). *In the independent context, the condition  $\mathcal{R}_{\mathbb{G}_n}$  is valid from the Donsker's Theorem. In the serial context however, one knows since Bücher (2015) that  $\mathcal{R}_{\alpha, \text{pow}}$  allows to derive the convergence required on  $\mathbb{G}_n$ .*

$\mathcal{R}_{\dot{C}}$  For  $i \in \{1, \dots, d\}$ , the first-order partial derivative of  $C$  with respect to its  $i$ th argument, denoted  $\dot{C}_i$ , exists and is continuous on the set  $V_i = \{\mathbf{x} \in [0, 1]^d : 0 < x_i < 1\}$ .

$\mathcal{R}_{\ddot{C}}$  For every  $i_1, i_2 \in \{1, \dots, d\}$ , the second-order partial derivative of  $C$  with respect to its  $i_1$ th and  $i_2$ th arguments, denoted  $\ddot{C}_{i_1 i_2}$ , exists and is continuous on  $V_{i_1} \cap V_{i_2}$ . Moreover, there exists a constant  $K > 0$  such that,

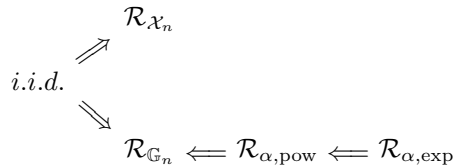
$$\left| \ddot{C}_{i_1 i_2}(\mathbf{x}) \right| \leq K \min \left\{ \frac{1}{x_{i_1}(1-x_{i_1})}, \frac{1}{x_{i_2}(1-x_{i_2})} \right\}.$$

$\mathcal{R}_{\alpha, \text{pow}}$   $(\mathbf{X}_n)_{n \in \mathbb{Z}}$  is a strictly stationary time series assumed to be an alpha-mixing sequence with  $\alpha(k) = O(k^{-a})$ , as  $k$  tends to infinity, for some  $a > 1$ . Recall that for  $k$  positive integer

$$\alpha(k) = \sup \{ |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|, A \in \sigma(\mathbf{X}_j, j \leq n), B \in \sigma(\mathbf{X}_{j+k}, j \geq n), n \in \mathbb{Z} \}.$$

$\mathcal{R}_{\alpha, \text{exp}}$   $(\mathbf{X}_n)_{n \in \mathbb{Z}}$  is a strictly stationary time series assumed to be an alpha-mixing sequence with  $\alpha(k) = O(a^k)$ , as  $k$  tends to infinity, for some  $a \in (0, 1)$ .

The following diagram allows a quick overview of the links between the assumptions under consideration:



### 3.3 Weak convergence of the empirical processes

The next result unifies the limit theorem associated with several testing problems. Before, we set

$$\mathbb{W}_C(\mathbf{x}) = \mathbb{G}_C(\mathbf{x}) - \sum_{i=1}^d \partial C_i(\mathbf{x}) \mathbb{G}_C(x_i, \mathbf{1}_{-i}), \quad \mathbf{x} \in [0, 1]^d, \quad (20)$$

where  $(x_i, \mathbf{1}_{-i})$  denotes the vector with  $x_i$  in the  $i$ -th component and the other components equal to 1 and where  $\mathbb{G}_C$  refers to the limiting process from the regularity condition  $(\mathcal{R}_{\mathbb{G}_n})$ .

**Theorem 2.** Assume that  $(\mathcal{H})C = M_\emptyset(C)$ , depending through (2) on a commuting and idempotent set of operators  $P_1, \dots, P_d$ , holds true. The associated maps  $\{M_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$  are assumed to be Hadamard-differentiable at  $C$  tangentially to  $\mathcal{C}_0$ . Assume  $\mathcal{R}_{\tilde{C}}$  for i.i.d. observations, whereas the conditions  $\mathcal{R}_{\mathcal{X}_n} - \mathcal{R}_{\mathbb{G}_n} - \mathcal{R}_{\tilde{C}}$  are required in the serial context. Let  $C_n$  stand for  $\tilde{C}_n, \hat{C}_n, C_n^\beta$  or  $C_n^\#$ . As  $n$  tends to infinity, the joint empirical processes converge weakly in  $\{\ell^\infty([0, 1]^d)\}^{2^d}$

$$\left( \mathbb{D}_{\emptyset, n}, \{\mathbb{D}_{\mathcal{A}, n}\}_{\mathcal{A} \in \mathcal{P}_d^*} \right) \xrightarrow[n \rightarrow \infty]{w} \left( \mathbb{D}_\emptyset := \mathbb{W}_C - M'_\emptyset(C; \mathbb{W}_C), \{\mathbb{D}_{\mathcal{A}} := M'_{\mathcal{A}}(C; \mathbb{W}_C)\}_{\mathcal{A} \in \mathcal{P}_d^*} \right). \quad (21)$$

Let us provide first some comments. The convergence of the first component holds true under the Hadamard differentiability of  $M_\emptyset$  only. Similarly, the convergence associated to a non empty subset  $\mathcal{A}$  only requires  $(\mathcal{H}^{\mathcal{A}})$  and Hadamard differentiability of  $M_{\mathcal{A}}$ . Finally, recall from Proposition 1, that if the set  $\{P_1, \dots, P_d\}$  always forms a commuting family of idempotent operators, then  $(\mathcal{H}) = \cap_{\mathcal{A} \in \mathcal{P}_d^*} (\mathcal{H}^{\mathcal{A}})$ , otherwise  $(\mathcal{H}) \subseteq \cap_{\mathcal{A} \in \mathcal{P}_d^*} (\mathcal{H}^{\mathcal{A}})$  only.

*Remark 9.* Consider the null hypothesis  $(\mathcal{H}_2)$  where the independence among subvectors is investigated. Note that part of Theorem 8 in Kojadinovic and Holmes (2009) is recovered by Theorem 2. Let us provide another example. When testing  $(\mathcal{H}_{6,r})$ , a sub-max-stability assumption, set  $P_1(f)(\mathbf{x}) = f^r(\mathbf{x}^{1/r})$  and complete the collection with the identity map. Then  $M_\emptyset = P_1$  and the convergence of the left component in (21) is exactly the one studied in the limit theorem given by Kojadinovic et al. (2011) in their Proposition 1.

*Proof of Theorem 2.* Section 2 of Kojadinovic and Stemikovskaya (2019) both recalls and states that under the regularity conditions assumed here, and for any choice  $C_n$  among the list of copula estimators given in the introduction, as  $n$  tends to infinity, the empirical copula process  $\mathbb{C}_n = \sqrt{n}(C_n - C)$  converges weakly in  $\ell^\infty([0, 1]^d)$  to the tight, centered Gaussian process  $\mathbb{W}_C$  given in (20). Note that by composition of Hadamard-differentiable maps,  $M_\emptyset$  and any  $M_{\mathcal{A}}$  are also Hadamard-differentiable. The functional version of the Delta method (see Section 3.9 of van der Vaart and Wellner 1996) applied to  $T(f) = (f, M_\emptyset(f), \{M_{\mathcal{A}}(f)\}_{\mathcal{A} \in \mathcal{P}_d^*})$  yields

$$\left( \sqrt{n}(C_n - C), \sqrt{n}(M_\emptyset(C_n) - M_\emptyset(C)), \{\sqrt{n}(M_{\mathcal{A}}(C_n) - M_{\mathcal{A}}(C))\}_{\mathcal{A} \in \mathcal{P}_d^*} \right) \xrightarrow[n \rightarrow \infty]{w} \left( \mathbb{W}_C, M'_\emptyset(C; \mathbb{W}_C), \{M'_{\mathcal{A}}(C; \mathbb{W}_C)\}_{\mathcal{A} \in \mathcal{P}_d^*} \right).$$

The result follows from the continuous mapping theorem applied to  $T(f, g, \{h_{\mathcal{A}}\}_{\mathcal{A}}) = (f - g, \{h_{\mathcal{A}}\}_{\mathcal{A}})$  and using the fact that, under  $(\mathcal{H})$ , both  $M_\emptyset(C) = C$  and  $M_{\mathcal{A}}(C) = 0$  for any  $\mathcal{A} \in \mathcal{P}_d^*$ .  $\square$

## 4 Approximating the limiting processes

The weak convergence of the empirical processes has just been proved but the covariance structures of the limiting processes  $\mathbb{D}_\emptyset = \mathbb{W}_C, M'_\emptyset(C; \mathbb{W}_C)$  and  $\mathbb{D}_{\mathcal{A}} = M'_{\mathcal{A}}(C; \mathbb{W}_C)$  depend on the unknown copula  $C$ . For this reason it is not always directly applicable for statistical testing. Multiplier bootstrap and subsampling have been introduced in the literature to reproduce independently the asymptotic behavior of such processes. Both are discussed in this section. Note that Genest and Rémillard (2008) establishes the validity of the parametric bootstrap method for goodness-of-fit tests of copulas when the series are independent. In the current paper, parametric bootstrap can be applied under  $(\mathcal{H}_1)$ ,  $(\mathcal{H}_4)$  or  $(\mathcal{H}_5)$  for instance. See also Rémillard (2017) when studying some multivariate stochastic volatility models.

### 4.1 Subsampling empirical test processes

The subsampling method is a first substitute to approximate the distribution of  $\mathbb{D}_\emptyset$  and  $\mathbb{D}_{\mathcal{A}}$ . See Politis and Romano (1994) and its recent adaptation in Kojadinovic and Stemikovskaya (2019) that we servilely follow and adapt below.

Let  $b < n$  denote the size of the samples extracted from  $\mathcal{X}_n = \{\mathbf{X}_1, \dots, \mathbf{X}_n\}$  and let  $B_{b,n}$  be the corresponding number of possible subsamples. Under the i.i.d. setting,  $B_{b,n} = \binom{n}{b}$  since the subsamples may be obtained without replacement. Since  $B_{b,n}$  might be too large, the complete enumeration could not be possible. The practical solution is to obtain its stochastic approximation through a  $N$ -sample of integers  $I_{1,n}, \dots, I_{N,n}$  independently extracted with replacement from  $\{1, \dots, B_{b,n}\}$ . The quantities of interest would be then computed for these  $N$  values of the index  $m$ . In the serial context, the sampling should preserve the dependence so that the subsamples must have consecutive index, and  $B_{b,n} = n - b + 1$ . Denote by  $\mathcal{X}_b^{[m]}$  for  $m \in \{1, \dots, B_{b,n}\}$  such  $b$ -subsamples on which are computed  $\tilde{C}_b^{[m]}, \hat{C}_b^{[m]}, C_b^{\#, [m]}$  and  $C_b^{\beta, [m]}$  the replicates of the estimators  $\tilde{C}_n, \hat{C}_n, C_n^\#$  and  $C_n^\beta$ . As before, the notation  $C_b^{[m]}$  will be used when  $C_n$  represents one of the copula estimators. Set now  $\tilde{\mathbb{C}}_b^{[m]} = \sqrt{b}(\tilde{C}_b^{[m]} - \tilde{C}_n)$ ,  $\hat{\mathbb{C}}_b^{[m]} = \sqrt{b}(\hat{C}_b^{[m]} - \hat{C}_n)$ ,  $\mathbb{C}_b^{\#, [m]} = \sqrt{b}(C_b^{\#, [m]} - C_n^\#)$  and  $\mathbb{C}_b^{\beta, [m]} = \sqrt{b}(C_b^{\beta, [m]} - C_n^\beta)$  the

associated replicates of the empirical copula processes, and  $C_b^{[m]} = \sqrt{b}(C_b^{[m]} - C_n)$  its generic form. All of them are already introduced in Kojadinovic and Stemikovskaya (2019). Coming back to our specific setting, set also

$$\begin{aligned}\tilde{\mathbb{D}}_{\emptyset,b}^{[m]} &= \sqrt{b} \left( \tilde{C}_b^{[m]} - M_{\emptyset}(\tilde{C}_b^{[m]}) \right) \quad \text{and} \quad \tilde{\mathbb{D}}_{\mathcal{A},b}^{[m]} = \sqrt{b} M_{\mathcal{A}}(\tilde{C}_b^{[m]}), \\ \hat{\mathbb{D}}_{\emptyset,b}^{[m]} &= \sqrt{b} \left( \hat{C}_b^{[m]} - M_{\emptyset}(\hat{C}_b^{[m]}) \right) \quad \text{and} \quad \hat{\mathbb{D}}_{\mathcal{A},b}^{[m]} = \sqrt{b} M_{\mathcal{A}}(\hat{C}_b^{[m]}), \\ \mathbb{D}_{\emptyset,b}^{\#, [m]} &= \sqrt{b} \left( C_b^{\#, [m]} - M_{\emptyset}(C_b^{\#, [m]}) \right) \quad \text{and} \quad \mathbb{D}_{\mathcal{A},b}^{\#, [m]} = \sqrt{b} M_{\mathcal{A}}(C_b^{\#, [m]}), \\ \mathbb{D}_{\emptyset,b}^{\beta, [m]} &= \sqrt{b} \left( C_b^{\beta, [m]} - M_{\emptyset}(C_b^{\beta, [m]}) \right) \quad \text{and} \quad \mathbb{D}_{\mathcal{A},b}^{\beta, [m]} = \sqrt{b} M_{\mathcal{A}}(C_b^{\beta, [m]}).\end{aligned}$$

Again,  $\mathbb{D}_{\mathcal{A},b}^{[m]}$  represents one of them.

**Theorem 3.** Assume that  $(\mathcal{H})C = M_{\emptyset}(C)$ , depending through (2) on a commuting and idempotent set of operators  $P_1, \dots, P_d$ , holds true. The associated maps  $\{M_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$  are assumed to be Hadamard-differentiable at  $C$  tangentially to  $C_0$  and that  $\{M'_{\mathcal{A}}(C; \cdot)\}_{\mathcal{A} \in \mathcal{P}_d}$  are continuous on  $\ell^\infty([0, 1]^d)$ . Assume Condition  $\mathcal{R}_{\tilde{C}}$  holds true. Under one of the following three situations

- i.i.d. setting,  $b/n \rightarrow \alpha \in [0, 1)$  and  $v_n = (1 - b/n)^{-1/2}$  when  $\mathbb{D}$  stands for  $\tilde{\mathbb{D}} = \hat{\mathbb{D}}, \mathbb{D}^{\#}$  or  $\mathbb{D}^{\beta}$ ,
  - $\mathcal{R}_{\alpha, \text{pow}}$  holds true,  $b/n \rightarrow 0$  and  $v_n = 1$  when  $\mathbb{D}$  stands for  $\tilde{\mathbb{D}}$ ,
  - $\mathcal{R}_{\mathcal{X}_n}$  and  $\mathcal{R}_{\alpha, \text{pow}}$  hold true,  $b/n \rightarrow 0$  and  $v_n = 1$ , when  $\mathbb{D}$  stands for  $\hat{\mathbb{D}}, \mathbb{D}^{\#}$  or  $\mathbb{D}^{\beta}$ ,
- then

$$\left( \mathbb{D}_{\emptyset, n}, v_n \mathbb{D}_{\emptyset, b}^{[I_{1, n}]}, v_n \mathbb{D}_{\emptyset, b}^{[I_{2, n}]}, \left\{ \mathbb{D}_{\mathcal{A}, n} \right\}_{\mathcal{A} \in \mathcal{P}_d^*}, \left\{ v_n \mathbb{D}_{\mathcal{A}, b}^{[I_{1, n}]} \right\}_{\mathcal{A} \in \mathcal{P}_d^*}, \left\{ v_n \mathbb{D}_{\mathcal{A}, b}^{[I_{2, n}]} \right\}_{\mathcal{A} \in \mathcal{P}_d^*} \right)$$

converges weakly in  $\{\ell^\infty([0, 1]^d)\}^{3 \times 2^d}$  to

$$\left( \mathbb{D}_{\emptyset}, \mathbb{D}_{\emptyset}^{[1]}, \mathbb{D}_{\emptyset}^{[2]}, \left\{ \mathbb{D}_{\mathcal{A}} \right\}_{\mathcal{A} \in \mathcal{P}_d^*}, \left\{ \mathbb{D}_{\mathcal{A}}^{[1]} \right\}_{\mathcal{A} \in \mathcal{P}_d^*}, \left\{ \mathbb{D}_{\mathcal{A}}^{[2]} \right\}_{\mathcal{A} \in \mathcal{P}_d^*} \right),$$

where  $\mathbb{D}_{\emptyset}^{[1]}, \mathbb{D}_{\emptyset}^{[2]}$  are independent copies of  $\mathbb{D}_{\emptyset}$  and  $\mathbb{D}_{\mathcal{A}}^{[1]}, \mathbb{D}_{\mathcal{A}}^{[2]}$  are independent copies of  $\mathbb{D}_{\mathcal{A}}$ .

Following Berghaus, Bücher, and Volgushev (2017) and Berghaus and Segers (2018) for instance we provide, under stronger assumptions, the weak convergence with respect to stronger metrics. It also includes the validity of the subsampling methodology for the empirical copula processes, as it has been done in Kojadinovic and Stemikovskaya (2019). Let  $\vee$  and  $\wedge$  stand respectively for the maximum and the minimum. The weight function considered in the literature is

$$g(\mathbf{x}) = \bigwedge_{i=1}^d \left\{ x_i \wedge \bigvee_{\substack{k=1 \\ k \neq i}}^d (1 - x_k) \right\}, \quad \mathbf{x} \in [0, 1]^d. \quad (22)$$

**Theorem 4.** Assume that  $(\mathcal{H})C = M_{\emptyset}(C)$ , depending through (2) on a commuting and idempotent set of operators  $P_1, \dots, P_d$ , holds true. The associated maps  $\{M_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$  are assumed to be Hadamard-differentiable at  $C$  tangentially to  $C_0$  and that  $\{M'_{\mathcal{A}}(C; \cdot)\}_{\mathcal{A} \in \mathcal{P}_d}$  are continuous and linear on  $\ell^\infty([0, 1]^d)$ . Assume both Conditions  $\mathcal{R}_{\tilde{C}}$  and  $\mathcal{R}_{\tilde{C}}$  hold true. Under one of the two following situations

- i.i.d. setting,  $b/n \rightarrow \alpha \in [0, 1)$ ,  $w \in [0, 1/2)$  and  $v_n = (1 - b/n)^{-1/2}$
  - $\mathcal{R}_{\mathcal{X}_n}$  and  $\mathcal{R}_{\alpha, \text{exp}}$  hold true,  $b/n \rightarrow 0$ ,  $w \in [0, 1/2)$  and  $v_n = 1$ ,
- then, when  $\mathbb{D}$  stands for  $\mathbb{D}^{\#}$  or  $\mathbb{D}^{\beta}$ ,

$$\left( \frac{\mathbb{D}_{\emptyset, n}}{g^w}, v_n \frac{\mathbb{D}_{\emptyset, b}^{[I_{1, n}]}}{g^w}, v_n \frac{\mathbb{D}_{\emptyset, b}^{[I_{2, n}]}}{g^w}, \left\{ \frac{\mathbb{D}_{\mathcal{A}, n}}{g^w} \right\}_{\mathcal{A} \in \mathcal{P}_d^*}, \left\{ v_n \frac{\mathbb{D}_{\mathcal{A}, b}^{[I_{1, n}]}}{g^w} \right\}_{\mathcal{A} \in \mathcal{P}_d^*}, \left\{ v_n \frac{\mathbb{D}_{\mathcal{A}, b}^{[I_{2, n}]}}{g^w} \right\}_{\mathcal{A} \in \mathcal{P}_d^*} \right)$$

converges weakly in  $\{\ell^\infty([0, 1]^d)\}^{3 \times 2^d}$  to

$$\left( \frac{\mathbb{D}_{\emptyset}}{g^w}, \frac{\mathbb{D}_{\emptyset}^{[1]}}{g^w}, \frac{\mathbb{D}_{\emptyset}^{[2]}}{g^w}, \left\{ \frac{\mathbb{D}_{\mathcal{A}}}{g^w} \right\}_{\mathcal{A} \in \mathcal{P}_d^*}, \left\{ \frac{\mathbb{D}_{\mathcal{A}}^{[1]}}{g^w} \right\}_{\mathcal{A} \in \mathcal{P}_d^*}, \left\{ \frac{\mathbb{D}_{\mathcal{A}}^{[2]}}{g^w} \right\}_{\mathcal{A} \in \mathcal{P}_d^*} \right),$$

where  $\mathbb{D}_{\emptyset}^{[1]}, \mathbb{D}_{\emptyset}^{[2]}$  are independent copies of  $\mathbb{D}_{\emptyset}$  and  $\mathbb{D}_{\mathcal{A}}^{[1]}, \mathbb{D}_{\mathcal{A}}^{[2]}$  are independent copies of  $\mathbb{D}_{\mathcal{A}}$ .

## 4.2 The multiplier bootstrap procedure

In statistical inference on copulas, the multiplier bootstrap is a well-established resampling technique for approximating the distribution of the limiting process. It was initially proposed by Scaillet (2005) and further investigated in Rémillard and Scaillet (2009), Bücher and Dette (2010), Segers (2012), Bücher and Ruppert (2013) for instance.

For the sake of simplicity within this section, we assume that the original time series is serially independent. The strong mixing case can be handled in a manner analogous to that of Bücher and Kojadinovic (2016) where further details may be found to construct the serial dependence of the multipliers. The multiplier bootstrap procedure relies on consistent estimators of the partial derivatives involved in (20). Consider then those based on finite-differencing of the empirical copula estimator. Let  $h_n$  be a sequence tending to 0 such that  $\inf_n h_n \sqrt{n} > 0$ . Define

$$\hat{C}_n^{[i]}(\mathbf{x}) = \begin{cases} \frac{\hat{C}_n(\mathbf{x} + h_n \mathbf{e}_i) - \hat{C}_n(\mathbf{x} - h_n \mathbf{e}_i)}{2h_n} & x_i \in [h_n, 1 - h_n] \\ \frac{\hat{C}_n(x_1, \dots, x_{i-1}, h_n, x_{i+1}, \dots, x_d)}{h_n} & x_i \in [0, h_n) \\ \frac{\hat{C}_n(\mathbf{x}) - \hat{C}_n(x_1, \dots, x_{i-1}, 1 - h_n, x_{i+1}, \dots, x_d)}{h_n} & x_i \in (1 - h_n, 1]. \end{cases}$$

Now, let  $N$  be a large integer. Below, for any  $\mathcal{A} \in \mathcal{P}_d$  we define  $N$  processes that are asymptotic independent copies of  $\mathbb{D}_{\mathcal{A}}$ . The multipliers  $\{Z_{j,\mathcal{A},n}^{(k)}\}_{j \in \mathbb{N}}$  for  $k = 1, \dots, N$  are defined as independent sequences of independent and identically distributed random variables with mean 0 and variance 1, which are independent of  $\mathcal{X}$ . For  $k \in \{1, \dots, N\}$ , and any  $\mathcal{A} \in \mathcal{P}_d$ , set

$$\hat{\mathbb{C}}_{\mathcal{A},n}^{(k)}(\mathbf{x}) = \mathbb{G}_{\mathcal{A},n}^{(k)}(\mathbf{x}) - \sum_{i=1}^d \hat{C}_n^{[i]}(\mathbf{x}) \mathbb{G}_{\mathcal{A},n}^{(k)}(\mathbf{x}^{(i)})$$

in which

$$\mathbb{G}_{\mathcal{A},n}^{(k)}(\mathbf{x}) = \frac{1}{\sqrt{n}} \sum_{j=1}^n Z_{j,\mathcal{A},n}^{(k)} \left\{ \mathbf{1}_{\hat{\mathbf{U}}_j \leq \mathbf{x}} - \hat{C}_n(\mathbf{x}) \right\}.$$

Then, one can prove that

$$\left( \hat{C}_n, \left\{ \hat{C}_{\mathcal{A},n}^{(1)}, \dots, \hat{C}_{\mathcal{A},n}^{(N)} \right\}_{\mathcal{A} \in \mathcal{P}_d} \right) \xrightarrow[n \rightarrow \infty]{w} \left( \mathbb{W}_C, \left\{ \mathbb{W}_{\mathcal{A},C}^{(1)}, \dots, \mathbb{W}_{\mathcal{A},C}^{(N)} \right\}_{\mathcal{A} \in \mathcal{P}_d} \right) \quad (23)$$

in  $\ell^\infty([0, 1]^d)^{2^{d(N+1)}}$  where  $\{\mathbb{W}_{\mathcal{A},C}^{(1)}, \dots, \mathbb{W}_{\mathcal{A},C}^{(N)}\}_{\mathcal{A} \in \mathcal{P}_d}$  are all independent copies of  $\mathbb{W}_C$ . In consequence, set for any  $k \in \{1, \dots, N\}$ , both  $\hat{\mathbb{D}}_{\emptyset,n}^{(k)} = \hat{C}_{\emptyset,n}^{(k)} - M'_\emptyset(\hat{C}_n; \hat{C}_{\emptyset,n}^{(k)})$  and  $\hat{\mathbb{D}}_{\mathcal{A},n}^{(k)} = M'_{\mathcal{A}}(\hat{C}_n; \hat{C}_{\mathcal{A},n}^{(k)})$  for any  $\mathcal{A} \in \mathcal{P}_d$ . From the continuous mapping theorem and the fact that  $\hat{C}_n$  converges uniformly in probability to  $C$ , one deduces that

$$\left( \{\hat{\mathbb{D}}_{\mathcal{A},n}^{(1)}, \hat{\mathbb{D}}_{\mathcal{A},n}^{(1)}, \dots, \hat{\mathbb{D}}_{\mathcal{A},n}^{(N)}\}_{\mathcal{A} \in \mathcal{P}_d} \right) \xrightarrow[n \rightarrow \infty]{w} \left( \{\mathbb{D}_{\mathcal{A}}, \mathbb{D}_{\mathcal{A}}^{(1)}, \dots, \mathbb{D}_{\mathcal{A}}^{(N)}\}_{\mathcal{A} \in \mathcal{P}_d} \right)$$

in  $\ell^\infty([0, 1]^d)^{2^{d(N+1)}}$  where  $\{\mathbb{D}_{\mathcal{A}}^{(1)}, \dots, \mathbb{D}_{\mathcal{A}}^{(N)}\}_{\mathcal{A} \in \mathcal{P}_d}$  are independent copies of  $\{\mathbb{D}_{\mathcal{A}}\}_{\mathcal{A} \in \mathcal{P}_d}$ .

*Remark 10.* When  $(\mathcal{H}_{8,r})$  is under study, the previous convergence restricted to  $\mathcal{A} = \emptyset$  is in agreement with Proposition 1 of Kojadinovic et al. (2011).

## 5 The test statistics

Natural measures of departure from the null hypothesis are Kolmogorov–Smirnov or Cramér–von Mises statistics, easily derived from the previous test processes. We recall their expressions and derive associated convergence results. As these statistics are not distribution-free, the subsampling methodology, the multiplier bootstrap explained in the previous section or the parametric bootstrap will be needed to approximate p-values.

### 5.1 Statistics derived from the test process and its functional decomposition

Assume that Conditions of Theorem 2 hold true and let  $C_n$  stand for one of the copula estimators considered therein. The Cramér–von Mises statistic derived from the testing process is given by

$$I_{\emptyset,n} = \int_{[0,1]^d} \{\mathbb{D}_{\emptyset,n}(\mathbf{x})\}^2 d\mathbf{x} = n \int_{[0,1]^d} \{C_n(\mathbf{x}) - M_\emptyset(C_n)(\mathbf{x})\}^2 d\mathbf{x} = n \int_{[0,1]^d} \{C_n(\mathbf{x}) - (\prod_{i=1}^d P_i)(C_n)(\mathbf{x})\}^2 d\mathbf{x},$$

and the  $2^d - 1$  Cramér–von Mises statistics obtained from the functional decomposition are given by

$$I_{\mathcal{A},n} = \int_{[0,1]^d} \{\mathbb{D}_{\mathcal{A},n}(\mathbf{x})\}^2 d\mathbf{x} = n \int_{[0,1]^d} \{M_{\mathcal{A}}(C_n)(\mathbf{x})\}^2 d\mathbf{x} = n \int_{[0,1]^d} \left\{ \left( \prod_{i \in \mathcal{A}} (I - P_i) \prod_{i \notin \mathcal{A}} P_i \right) (C_n)(\mathbf{x}) \right\}^2 d\mathbf{x},$$

for any  $\mathcal{A} \in \mathcal{P}_d^*$ . For the sake of brevity, we define from now the case  $\mathcal{A} = \emptyset$  and  $\mathcal{A} \neq \emptyset$  through a unique formula, valid for any  $\mathcal{A} \in \mathcal{P}_d$  even if the definition of  $\mathbb{D}_{\emptyset,n}$  and  $\mathbb{D}_{\mathcal{A},n}$  differ. So, similarly to the CVM statistics above, the Kolmogorov-Smirnov statistics are defined by

$$U_{\mathcal{A},n} = \sup_{\mathbf{x} \in [0,1]^d} |\mathbb{D}_{\mathcal{A},n}(\mathbf{x})|^2, \quad \forall \mathcal{A} \in \mathcal{P}_d.$$

By application of the continuous mapping theorem combined with Theorem 2 we obtain that the random vector  $(\{I_{\mathcal{A},n}\}_{\mathcal{A} \in \mathcal{P}_d}, \{U_{\mathcal{A},n}\}_{\mathcal{A} \in \mathcal{P}_d})$  converges in distribution to

$$\left( \left\{ \int_{[0,1]^d} \mathbb{D}_{\mathcal{A}}^2(\mathbf{x}) d\mathbf{x} \right\}_{\mathcal{A} \in \mathcal{P}_d}, \left\{ \sup_{\mathbf{x} \in [0,1]^d} \mathbb{D}_{\mathcal{A}}^2(\mathbf{x}) \right\}_{\mathcal{A} \in \mathcal{P}_d} \right).$$

In some particular cases, and when  $C_n$  is taken as  $\hat{C}_n$  it is possible to provide the expression of the statistics in terms of the pseudo-observations only. See for instance Section 4 in Genest et al. (2007) when complete independence is under testing. See also Proposition 10 and Proposition 13 in Kojadinovic and Holmes (2009) when testing concerns independence by blocks. In general, we proceed by numerical approximation based on a grid to calculate the test statistics. Let  $K$  be a large integer and let  $\mathbf{w}_1, \dots, \mathbf{w}_K$  be  $K$  uniformly spaced points on  $(0,1)^d$ . Then for any  $\mathcal{A} \in \mathcal{P}_d$ , the integral and the supremum can be approximated by

$$I_{\mathcal{A},n} \simeq \frac{1}{K} \sum_{k=1}^K \{\mathbb{D}_{\mathcal{A},n}(\mathbf{w}_k)\}^2 \quad \text{and} \quad U_{\mathcal{A},n} \simeq \max_{k=1, \dots, K} |\mathbb{D}_{\mathcal{A},n}(\mathbf{w}_k)|^2.$$

Of greater interest is the possibility of replacing in the Cramér–von Mises statistics  $I_{\mathcal{A},n}$  the Lebesgue product measure  $d\mathbf{x}$  on  $[0,1]^d$  by the empirical  $d\tilde{C}_n(\mathbf{x})$ . We thus introduce, for any  $\mathcal{A} \in \mathcal{P}_d$ , the alternative Cramér–von Mises statistics as

$$S_{\mathcal{A},n} = \int_{[0,1]^d} \{\mathbb{D}_{\mathcal{A},n}(\mathbf{x})\}^2 d\tilde{C}_n(\mathbf{x}).$$

Following the proof of Theorem 3 in Kojadinovic et al. (2011), the same ideas allow to state that under assumptions of Theorem 2

$$(\{S_{\mathcal{A},n}\}_{\mathcal{A} \in \mathcal{P}_d}) \xrightarrow[n \rightarrow \infty]{w} \left( \left\{ \int_{[0,1]^d} \mathbb{D}_{\mathcal{A}}^2(\mathbf{x}) dC(\mathbf{x}) \right\}_{\mathcal{A} \in \mathcal{P}_d} \right).$$

Set  $\hat{\mathbf{U}}_{j,\cdot,n} = (R_{j1,n}/n, \dots, R_{jd,n}/n)$  where we recall that  $R_{ji,n}$  denotes the rank of  $X_{ji}$  among  $X_{1i}, \dots, X_{ni}$ . The approximations of the previous statistics are then for any  $\mathcal{A} \in \mathcal{P}_d$

$$S_{\mathcal{A},n} = \frac{1}{n} \sum_{j=1}^n \{\mathbb{D}_{\mathcal{A},n}(\hat{\mathbf{U}}_{j,\cdot,n})\}^2.$$

Finally, to increase the power of the Cramér von Mises statistics, it might be useful to introduce a weight function. See De Wet (1980), or more recently Berghaus and Segers (2018), where a weight function emphasizing the tails is introduced when copula differ from the independence copula in the tails mainly. Assume now that the hypotheses of Theorem 4 hold true. Let us consider a weight function  $q$  such that  $g/q$  is bounded over  $[0,1]^d$ . The weighted versions of the statistics are then as follows

$$\begin{aligned} I_{\mathcal{A},n,q} &= \int_{[0,1]^d} \left\{ \frac{\mathbb{D}_{\mathcal{A},n}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 d\mathbf{x} \simeq \frac{1}{k} \sum_{k=1}^K \left\{ \frac{\mathbb{D}_{\mathcal{A},n}(\mathbf{w}_k)}{q(\mathbf{w}_k)} \right\}^2 \\ S_{\mathcal{A},n,q} &= \int_{[0,1]^d} \left\{ \frac{\mathbb{D}_{\mathcal{A},n}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 d\tilde{C}_n(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^n \left\{ \frac{\mathbb{D}_{\mathcal{A},n}(\hat{\mathbf{U}}_{j,\cdot,n})}{q(\hat{\mathbf{U}}_{j,\cdot,n})} \right\}^2 \\ U_{\mathcal{A},n,q} &= \sup_{\mathbf{x} \in [0,1]^d} \left\{ \frac{\mathbb{D}_{\mathcal{A},n}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 \simeq \max_{k=1, \dots, K} \left\{ \frac{\mathbb{D}_{\mathcal{A},n}(\mathbf{w}_k)}{q(\mathbf{w}_k)} \right\}^2. \end{aligned}$$

One can easily obtain that  $(\{I_{\mathcal{A},n,q}\}_{\mathcal{A} \in \mathcal{P}_d}, \{S_{\mathcal{A},n,q}\}_{\mathcal{A} \in \mathcal{P}_d}, \{U_{\mathcal{A},n,q}\}_{\mathcal{A} \in \mathcal{P}_d})$  converges in distribution to

$$\left( \left\{ \int_{[0,1]^d} \left\{ \frac{\mathbb{D}_{\mathcal{A}}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 d\mathbf{x} \right\}_{\mathcal{A} \in \mathcal{P}_d}, \left\{ \int_{[0,1]^d} \left\{ \frac{\mathbb{D}_{\mathcal{A}}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 dC(\mathbf{x}) \right\}_{\mathcal{A} \in \mathcal{P}_d}, \left\{ \sup_{\mathbf{x} \in [0,1]^d} \frac{\mathbb{D}_{\mathcal{A}}(\mathbf{x})}{q(\mathbf{x})} \right\}_{\mathcal{A} \in \mathcal{P}_d} \right).$$

## 5.2 Practical implementation of the tests

Let  $w = (\{w^A\}_{A \in \mathcal{P}_d})$  be a (normalized) vector of positive weights. The latter represents the importance we put in the decision to the equality  $C = M_\theta(C)$  through  $w^\emptyset$ , or to  $M_A(C) = 0$  through  $w^A$  since our setting provides  $(\mathcal{H}) \subset \cap_{A \in \mathcal{P}_d} (\mathcal{H}^A)$ . The test statistic

$$S_{w,n} = \sum_{A \in \mathcal{P}_d} w^A S_{A,n} = \frac{1}{n} \sum_{j=1}^n \sum_{A \in \mathcal{P}_d} w^A \{\mathbb{D}_{A,n}(\hat{\mathbf{U}}_{j,\cdot,n})\}^2 \quad (24)$$

should not be too large. The computation of the associated p-value is explained below. Similarly, the statistics  $I_{w,n}$  or  $U_{w,n}$  could be constructed from the collections  $I_{A,n}$  or  $U_{A,n}$ . It could also be adapted for the  $q$ -weighted versions of these statistics.

---

### Algorithm 1: Approximating the p-value

---

**Compute**  $S_{w,n}^{(0)}$  the value of  $S_{w,n}$  on the original series

**Generate** from subsampling  $\mathbb{D}_{A,n}^{[k]}$  or from multiplier bootstrapping  $\hat{\mathbb{D}}_{A,n}^{(k)}$  for  $k = 1, \dots, N$

**Compute**  $S_{w,n}^{(1)}, \dots, S_{w,n}^{(N)}$  the value of  $S_{w,n}$  on one of these processes

**Define** an approximate p-value for the test statistic as following  $\frac{1}{N+1} \left( \frac{1}{2} + \sum_{k=1}^N \mathbf{1}\{S_{w,n}^{(k)} > S_{w,n}^{(0)}\} \right)$

---

Taking into account both Section 3 and Section 4, the p-values obtained from the statistics  $S_{w,n}$  (as well as p-values computed from  $I$  or  $U$  statistics) are approximately uniform on  $[0, 1]$  under the null hypothesis. In the case of goodness-of-fit tests  $(\mathcal{H}) C = C_\theta$ , samples of reference should not be obtained from subsampling or multiplier bootstrapping but by parametric bootstrapping instead. The alternative version of the algorithm is given below.

---

### Algorithm 2: Approximating the p-value (Goodness-of-fit tests)

---

**Compute** the pseudo-observations  $\mathbf{U}_1, \dots, \mathbf{U}_n$  on the original series,  $\theta_n = \hat{\theta}(\mathbf{U}_1, \dots, \mathbf{U}_n)$  and  $S_{w,n,\theta_n}$

**for**  $k = 1$  to  $N$  **do**

**Generate**  $\mathbf{U}_1^{\{k\}}, \dots, \mathbf{U}_n^{\{k\}}$  from  $C_{\theta_n}$

**Compute**  $\theta_n^{\{k\}} = \hat{\theta}(\mathbf{U}_1^{\{k\}}, \dots, \mathbf{U}_n^{\{k\}})$  and  $S_{w,n,\theta_n^{\{k\}}}$  on the parametric sample

**end for**

**Define** an approximate p-value for the test statistic as following  $\frac{1}{N+1} \left( \frac{1}{2} + \sum_{k=1}^N \mathbf{1}\{S_{w,n,\theta_n^{\{k\}}} > S_{w,n,\theta_n}\} \right)$

---

## 5.3 Numerical experiments

In this final section, we shall consider the use of the functional decomposition in two experiments and analyze the results. First, we explore the testing problem  $(\mathcal{H}_2)$  of block independence and specifically one of the practical settings imagined in Kojadinovic and Holmes (2009). We investigate then the goodness-of-fit test  $(\mathcal{H}_4)$  where two archimedean copulas, namely Clayton and Gumbel, are opposed. These copulas have been chosen as they are part of the first example in the routine `gofCopula` of the R `copula` package (Hofert et al., 2022).

### 5.3.1 Independence between three continuous $r$ -dimensional random vectors

We adapt below Section 4 from Kojadinovic and Holmes (2009) that implements testing procedures under  $(\mathcal{H}_2)$ . Note that under independence or independence by block, the p-values associated with the statistics  $I_{w,n}$  for weights of the form  $w = \mathbf{1}_A$  are also asymptotically mutually independent. As a consequence, individual critical values can be chosen to achieve an asymptotic global significance level. Furthermore, it is possible to combine individual p-values and get a global p-value thanks to the method *à la Fisher* (the resulting statistics is denoted  $W_n$ ) as well as *à la Tippett* (the resulting statistics is denoted  $T_n$ ). For more details, we refer to the discussion in Genest and Rémillard (2004) or the paragraph ‘‘Combining p-values’’ in Section 3 of Kojadinovic and Holmes (2009).

Let  $\mathbf{X} = (X_1, \dots, X_{12})$  and consider the  $p = 3$  groups  $\{X_1, \dots, X_4\}$ ,  $\{X_5, \dots, X_8\}$  and  $\{X_9, \dots, X_{12}\}$ . The dependence is described by the normal copula. The  $d \times d$  correlation matrices  $\Sigma$  are structured as follows:

	$X_1$	...	$X_4$	$X_5$	...	$X_8$	$X_9$	...	$X_{12}$
$X_1$	1		$\rho_{\text{intra}}$						
$\vdots$									
$X_4$	$\rho_{\text{intra}}$		1		$\rho_{\text{inter}}$			$\rho_{\text{inter}}$	
$X_5$				1		$\rho_{\text{intra}}$			
$\vdots$									
$X_8$		$\rho_{\text{inter}}$		$\rho_{\text{intra}}$		1		$\rho_{\text{inter}}$	
$X_9$							1		$\rho_{\text{intra}}$
$\vdots$									
$X_{12}$		$\rho_{\text{inter}}$			$\rho_{\text{inter}}$		$\rho_{\text{intra}}$		1

The quantity  $\rho_{\text{inter}}$  (resp.  $\rho_{\text{intra}}$ ) controls the amount of dependence among (resp. within) the three random vectors. The values  $\rho_{\text{inter}} \in \{0.000, 0.025, 0.050, \dots, 0.275, 0.300\}$  for  $\rho_{\text{intra}} = 0.5$  are considered under the normal copula. We generate 1000 samples composed of  $n = 200$  independent realizations of  $\mathbf{X}$ . Note that, in all the simulations, the number of randomized samples is set to 1000.

Figure 1 shows the rejection rates of the null hypothesis, the proportion of times that the different tests reject ( $\mathcal{H}_2$ ), with respect to the value of  $\rho_{\text{inter}}$ . The significance level is arbitrarily set to 5% and plotted in the figure. The global Cramér-von-mises statistic  $I_n = I_{0,n}$ , as well as  $W_n$  the test statistic *à la Fisher*, and  $T_n$  the test statistic *à la Tippett* are those studied in Figure 3 of Kojadinovic and Holmes (2009), with the difference that  $n = 200$  here. Four additional measures of the form  $S_{w,n}$  are included. Recall that taking into account the form of the null hypothesis ( $\mathcal{H}_2$ ) with  $p = 3$  blocks, the weights  $w$  have the following structure

$$w = (w^\emptyset, w^{\{1\}}, w^{\{2\}}, w^{\{3\}}, w^{\{12\}}, w^{\{13\}}, w^{\{23\}}, w^{\{123\}}).$$

More precisely, we consider

- $w_1 = (1, 0, 0, 0, 0, 0, 0, 0)$  that only measures the left hand term of the decomposition,
- $w_2 = (1, 0, 0, 0, 1, 1, 1, 1)$  that combines the left hand term of the decomposition with the right hand terms of order 2 and 3 (recall that the right hand terms associated with singletons all vanish),
- $w_3 = (0, 0, 0, 0, 1, 1, 1, 1)$  that only combines the non-null right hand terms of the decomposition,
- $w_4 = (8.253373, 0, 0, 0, 2.373714, 2.344580, 2.35670, 1.989524)$  that combines left and right hand terms proportionally to some variances. It is an empirical choice where each  $w^A$  is proportional to the estimate of  $\text{var}(S_{A,n})$  obtained by block bootstrapping.

As can be seen in Figure 1, the  $S$ -type statistics perform best among all the approaches, with one exception:  $S_{w_3,n}$  appears twice below the values of  $W_n$ . Additionally, there is no uniformly better  $S$ , even if  $S_{w_2,n}$  is roughly speaking the best choice globally. However, an empirical choice of the weights as defined in  $S_{w_4,n}$  yields a better rate for the particular value  $\rho_{\text{inter}} = 0.05$ . Note that  $S_{w_1,n}$  which is the usual  $S$ -version from the literature leads to results relatively close to those given by  $S_{w_2,n}$ . As a conclusion, and even if our purpose in this paper is not to construct better statistics rather than a unified theory, one should retain the following. The use of the functional decomposition

$$M_\emptyset(C) - C = \sum_{\mathcal{A} \in \mathcal{P}_p^*} M_{\mathcal{A}}(C)$$

allows to improve the well-known Cramér-von-Mises statistics  $S_{w_1,n} = S_{\emptyset,n}$  (that only considers the left hand member of the above equality) by defining instead  $S_{w_2,n} = \sum_{\mathcal{A} \in \mathcal{P}_p} S_{\mathcal{A},n}$  (that takes into account both the left hand member and all the right hand member terms of the above equality).



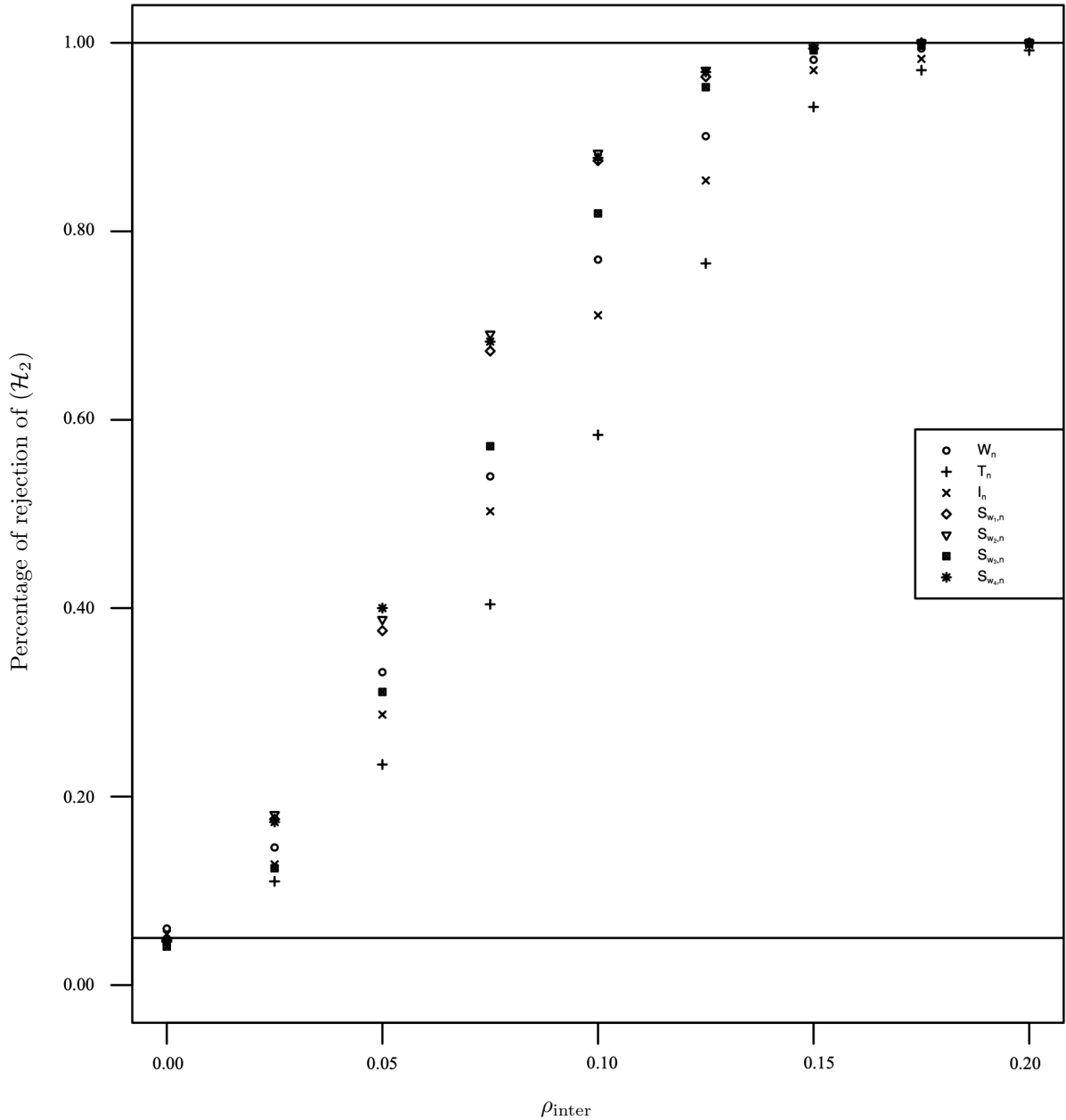


Figure 1: Percentage of rejection of  $(\mathcal{H}_2)$  that stipulates the block independence of  $\{X_1, \dots, X_4\}$ ,  $\{X_5, \dots, X_8\}$  and  $\{X_9, \dots, X_{12}\}$  that comes from Normal copula with correlation matrices  $\Sigma$  whereas the null hypothesis  $(\mathcal{H}_2)$  is only true under the  $x$ -axis value  $\rho_{\text{inter}} = 0.00$ . The statistics  $W_n$ ,  $T_n$  and  $I_n$  are those presented in Kojadinovic and Holmes (2009) and available in the routine `multIndepTest` of the R package `copula`. The statistics  $S_{w,n}$  as defined in (24) are evaluated for several weights  $w$  as indicated in the text.

### 5.3.2 Goodness-of-fit tests

Turning to the Goodness-of-Fit tests  $(\mathcal{H}_4)$ , we consider the Clayton or the Gumbel family in a 3-dimensional setting. These classes will both be used as the generator of datasets or as the family being tested. To generate the original samples, three values of Kendall's  $\tau$  are chosen:  $\tau = .1$ ,  $\tau = .2$  and  $\tau = .3$ . Test statistics  $S_{w,n}$  given by (24) and where

$$w = (w^\emptyset, w^{\{1\}}, w^{\{2\}}, w^{\{3\}}, w^{\{12\}}, w^{\{13\}}, w^{\{23\}}, w^{\{123\}})$$

are computed for different weights:  $w_1 = (1, 0, 0, 0, 0, 0, 0, 0)$ ,  $w_2 = (1, 0, 0, 0, 1, 1, 1, 1)$ ,  $w_3 = (0, 0, 0, 0, 1, 1, 1, 1)$ . The results are provided in Table 1 for  $n = 100$ . The first lines are dedicated to the test  $(\mathcal{H}_4)$  when  $\varphi$  is the Clayton copula. Similarly, Gumbel copula is tested in the last lines of the table. The parameter associated with the generator  $\varphi$  is estimated at each step as the mean of empirical Kendall's  $\tau$ . The parametric bootstrap described in Algorithm 2 with  $n_{\text{boot}} = 200$  is used to compute the p-value. The rejection rates are

estimated through  $n_{\text{rep}} = 500$  repetitions of each experiment. Two characteristics are of interest: the empirical level might be close to the nominal level, arbitrarily fixed at 0.05, and the empirical power. We also add another procedure in Table 1. The line `gofCopula` corresponds to the results associated with the command `gofCopula(copul_test, rCopula(n, copul_simu), N=nboot, estim.method = "itau")$p.value`.

		Copulas being simulated						
		Clayton			Gumbel			
		$\tau = .1$	$\tau = .2$	$\tau = .3$	$\tau = .1$	$\tau = .2$	$\tau = .3$	
Copulas being tested	Clayton	$w_1$	0.056	0.044	0.048	0.364	0.690	0.934
		$w_2$	0.052	0.044	0.038	0.294	0.674	0.924
		$w_3$	0.044	0.060	0.046	0.160	0.494	0.852
		gofCopula	0.050	0.044	0.044	0.214	0.670	0.924
	Gumbel	$w_1$	0.282	0.720	0.922	0.070	0.048	0.056
		$w_2$	0.306	0.766	0.948	0.036	0.042	0.052
		$w_3$	0.254	0.710	0.960	0.024	0.042	0.048
		gofCopula	0.288	0.716	0.930	0.030	0.062	0.054

Table 1: Rejection rates of the null hypothesis. In the first lines, Clayton copula is being tested, whereas Gumbel copula is tested in the last lines of the table. The datasets are simulated for three different strengths of dependence, calibrated through the Kendall's  $\tau$ : .1, .2 and .3. The test statistics  $S_{w,n}$  are studied for three different weights:  $w_1, w_2$  and  $w_3$  (definition in the text). Additional parameters of the numerical study: sample size  $n = 100$ , parametric bootstrap size  $n_{\text{boot}} = 200$  and number of repetitions of the experiment  $n_{\text{rep}} = 500$ .

As already explained, the goal of the paper is not to find an overall best test statistics but to show the interest of a functional decomposition chosen in accordance with the null hypothesis. The analysis of Table 1 is not straightforward but the results are interesting. First, on the right upper corner of the table, one can remark that  $S_{w_1,n}$  always dominates in discriminating true Gumbel from supposed Clayton. This becomes true for  $S_{w_2,n}$  in the left bottom corner, except when the dependence becomes stronger with  $\tau = 0.3$ , the more powerful is then  $S_{w_3,n}$ .

## 6 Concluding remarks

Identifying and modeling dependencies with copulas remain an important topic, which has become very popular over the last decades since it is applied in almost every discipline. The aim in this paper is to provide a kind of unification of various papers, as Deheuvels (1981), Genest and Rémillard (2004), Genest et al. (2007), Kojadinovic and Holmes (2009), Kojadinovic et al. (2011) among others. All derive copula-based tests of the structure of dependence. The solution here is to dip them in the functional decomposition context of Kuo et al. (2010) (and its recent version of Mercadier et al. (2022) which removes the linearity assumption) in order to reveal a common pattern. Then, the goal of the paper is not really to improve a methodology but to transform already known tools in particular cases of more general statements.

The numerical section provides two simple studies, one analyzing in a ( $d =$ )12-dimensional setting the independence among  $p = 3$  blocks and one looking at particular archimedean copula families in dimension  $d = 3$ . Of course, we conclude that there exists no best procedure even if one can take advantage in analyzing the functional decomposition associated with the null hypothesis in order to derive powerful weighted test statistics.

The dimensions  $d$  or  $p$  are small in our experiments. So the current paper provides an interesting perspective for high dimensional problems. The practical implementation of the tests relies indeed on a trade-off between exhaustivity (all subsets of  $\mathcal{P}_d$ ) and dimensionality (exponential growth in  $d$ ). When  $d$  becomes larger, it could be interesting to use only part of the subsets. With the help of the weight  $w$  introduced in the definition of the test statistics, one can focus only in a given size of  $\mathcal{A}$  or in all sizes that do not exceed a given size. We can control this way the underlying complexity of the method. The question will be then: how much this selection affects the corresponding power of the test procedure?

## 7 Proofs of Section 4

*Proof of Theorem 3.* Let  $\mathbb{C}_b^{[m]}$  stands for  $\tilde{\mathbb{C}}_b^{[m]}$ ,  $\hat{\mathbb{C}}_b^{[m]}$ ,  $\mathbb{C}_b^{\#, [m]}$  or  $\mathbb{C}_b^{\beta, [m]}$  under the first item, only  $\tilde{\mathbb{C}}_b^{[m]}$  under the second item and  $\hat{\mathbb{C}}_b^{[m]}$ ,  $\mathbb{C}_b^{\#, [m]}$  or  $\mathbb{C}_b^{\beta, [m]}$  under the third one. Since by assumption both  $M_\emptyset(C) = C$  and  $M_{\mathcal{A}}(C) = 0$  for any subset  $\mathcal{A} \in \mathcal{P}_d^*$ , one can observe that,

$$\mathbb{D}_{\emptyset, n} = \mathbb{C}_n - \sqrt{n} \left( M_\emptyset \left( C + \frac{\mathbb{C}_n}{\sqrt{n}} \right) - M_\emptyset(C) \right) \quad (25)$$

$$\mathbb{D}_{\mathcal{A}, n} = \sqrt{n} \left( M_{\mathcal{A}} \left( C + \frac{\mathbb{C}_n}{\sqrt{n}} \right) - M_{\mathcal{A}}(C) \right) \quad (26)$$

$$\mathbb{D}_{\emptyset, b}^{[m]} = \mathbb{C}_b^{[m]} + \sqrt{\frac{b}{n}} \mathbb{C}_n - \sqrt{b} \left( M_\emptyset \left( C + \frac{\mathbb{C}_b^{[m]}}{\sqrt{b}} + \frac{\mathbb{C}_n}{\sqrt{n}} \right) - M_\emptyset(C) \right) \quad (27)$$

$$\mathbb{D}_{\mathcal{A}, b}^{[m]} = \sqrt{b} \left( M_{\mathcal{A}} \left( C + \frac{\mathbb{C}_b^{[m]}}{\sqrt{b}} + \frac{\mathbb{C}_n}{\sqrt{n}} \right) - M_{\mathcal{A}}(C) \right) \quad (28)$$

for  $m = I_{1, n}$  or  $I_{2, n}$  in the last two displays. From Theorem 3.3 in Kojadinovic and Stemikovskaya (2019), one knows that in each of these situations  $\left( \mathbb{C}_n, v_n \mathbb{C}_b^{[I_{1, n}]} , v_n \mathbb{C}_b^{[I_{2, n}]} \right)$  converges weakly in  $\ell^\infty([0, 1]^d)^3$  to  $(\mathbb{W}_C, \mathbb{W}_C^{[1]}, \mathbb{W}_C^{[2]})$  where  $\mathbb{W}_C^{[1]}$  and  $\mathbb{W}_C^{[2]}$  are independent copies of  $\mathbb{W}_C$ . As a consequence, using the fact that  $b = o(n)$ , the following

$$\left( \mathbb{C}_n, v_n \mathbb{C}_b^{[I_{1, n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n, v_n \mathbb{C}_b^{[I_{2, n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right)$$

shares the same asymptotic behavior. Continuous mapping theorem with the map

$$(f_1, f_2, f_3) \mapsto (-M'_\emptyset(C; f_1), -M'_\emptyset(C; f_2), -M'_\emptyset(C; f_3), \{M'_{\mathcal{A}}(C; f_1)\}_{\mathcal{A}}, \{M'_{\mathcal{A}}(C; f_2)\}_{\mathcal{A}}, \{M'_{\mathcal{A}}(C; f_3)\}_{\mathcal{A}})$$

then delivers the weak convergence of

$$\left( -M'_\emptyset(C; \mathbb{C}_n), -M'_\emptyset \left( C; v_n \mathbb{C}_b^{[I_{1, n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right), -M'_\emptyset \left( C; v_n \mathbb{C}_b^{[I_{2, n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right), \right. \\ \left. \{M'_{\mathcal{A}}(C; \mathbb{C}_n)\}_{\mathcal{A}}, \left\{ M'_{\mathcal{A}} \left( C; v_n \mathbb{C}_b^{[I_{1, n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right) \right\}_{\mathcal{A}}, \left\{ M'_{\mathcal{A}} \left( C; v_n \mathbb{C}_b^{[I_{2, n}]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right) \right\}_{\mathcal{A}} \right)$$

in  $\ell^\infty([0, 1]^d)^{3 \times 2^d}$  to

$$\left( -M'_\emptyset(C; \mathbb{W}_C), -M'_\emptyset(C; \mathbb{W}_C^{[1]}), -M'_\emptyset(C; \mathbb{W}_C^{[2]}), \{M'_{\mathcal{A}}(C; \mathbb{W}_C)\}_{\mathcal{A}}, \{M'_{\mathcal{A}}(C; \mathbb{W}_C^{[1]})\}_{\mathcal{A}}, \{M'_{\mathcal{A}}(C; \mathbb{W}_C^{[2]})\}_{\mathcal{A}} \right).$$

It remains to show that, for  $m = I_{1, n}$  or  $I_{2, n}$ ,

$$\sup_{[0, 1]^d} |\mathbb{D}_{\emptyset, n} - \mathbb{C}_n + M'_\emptyset(C; \mathbb{C}_n)| \xrightarrow{\mathbb{P}} 0, \\ \sup_{[0, 1]^d} \left| v_n \mathbb{D}_{\emptyset, b}^{[m]} - v_n \mathbb{C}_b^{[m]} - v_n \sqrt{\frac{b}{n}} \mathbb{C}_n + M'_\emptyset \left( C; v_n \mathbb{C}_b^{[m]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right) \right| \xrightarrow{\mathbb{P}} 0, \\ \sup_{[0, 1]^d} \left| \frac{\mathbb{D}_{\mathcal{A}, n}}{g^w} - M'_{\mathcal{A}}(C; \mathbb{C}_n) \right| \xrightarrow{\mathbb{P}} 0, \\ \sup_{[0, 1]^d} \left| v_n \mathbb{D}_{\mathcal{A}, b}^{[m]} - M'_{\mathcal{A}} \left( C; v_n \mathbb{C}_b^{[m]} + v_n \sqrt{\frac{b}{n}} \mathbb{C}_n \right) \right| \xrightarrow{\mathbb{P}} 0,$$

which are, once the equations (25), (26), (27) and (28) taken into account, only the consequences of what precedes combined with the functional Delta Method applied to the maps  $M_\emptyset$  or  $M_{\mathcal{A}}$ .  $\square$

*Proof of Theorem 4.* Starting from Theorem 4.5 in Kojadinovic and Stemikovskaya (2019), one knows that

$$\left( \frac{\mathbb{C}_n}{g^w}, v_n \frac{\mathbb{C}_b^{[I_{1, n}]} }{g^w}, v_n \frac{\mathbb{C}_b^{[I_{2, n}]} }{g^w} \right)$$

converges weakly in  $\ell^\infty([0, 1]^d)^3$  to

$$\left( \frac{\mathbb{W}_C}{g^w}, \frac{\mathbb{W}_C^{[1]}}{g^w}, \frac{\mathbb{W}_C^{[2]}}{g^w} \right).$$

Consequently, using the fact that  $b = o(n)$ , the following

$$\left( \frac{\mathbb{C}_n}{g^w}, v_n \frac{\mathbb{C}_b^{[I_{1,n}]} }{g^w} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{g^w}, v_n \frac{\mathbb{C}_b^{[I_{2,n}]} }{g^w} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{g^w} \right)$$

shares the same asymptotic behaviour. Using the continuous mapping theorem with the map

$$(f_1, f_2, f_3) \mapsto (-M'_\emptyset(C; f_1), -M'_\emptyset(C; f_2), -M'_\emptyset(C; f_3), \{M'_A(C; f_1)\}_A, \{M'_A(C; f_2)\}_A, \{M'_A(C; f_3)\}_A)$$

we thus obtain that

$$\begin{aligned} & \left( -M'_\emptyset \left( C; \frac{\mathbb{C}_n}{g^w} \right), -M'_\emptyset \left( C; v_n \frac{\mathbb{C}_b^{[I_{1,n}]} }{g^w} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{g^w} \right), -M'_\emptyset \left( C; v_n \frac{\mathbb{C}_b^{[I_{2,n}]} }{g^w} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{g^w} \right), \right. \\ & \left. \left\{ M'_A \left( C; \frac{\mathbb{C}_n}{g^w} \right) \right\}_A, \left\{ M'_A \left( C; v_n \frac{\mathbb{C}_b^{[I_{1,n}]} }{g^w} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{g^w} \right) \right\}_A, \left\{ M'_A \left( C; v_n \frac{\mathbb{C}_b^{[I_{2,n}]} }{g^w} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{g^w} \right) \right\}_A \right) \end{aligned}$$

converges weakly in  $\ell^\infty([0, 1]^d)^{3 \times 2^d}$  to

$$\left( -M'_\emptyset \left( C; \frac{\mathbb{W}_C}{g^w} \right), -M'_\emptyset \left( C; \frac{\mathbb{W}_C^{[1]}}{g^w} \right), -M'_\emptyset \left( C; \frac{\mathbb{W}_C^{[2]}}{g^w} \right), \left\{ M'_A \left( C; \frac{\mathbb{W}_C}{g^w} \right) \right\}_A, \left\{ M'_A \left( C; \frac{\mathbb{W}_C^{[1]}}{g^w} \right) \right\}_A, \left\{ M'_A \left( C; \frac{\mathbb{W}_C^{[2]}}{g^w} \right) \right\}_A \right)$$

which is equal to

$$\left( -\frac{M'_\emptyset(C; \mathbb{W}_C)}{g^w}, -\frac{M'_\emptyset(C; \mathbb{W}_C^{[1]})}{g^w}, -\frac{M'_\emptyset(C; \mathbb{W}_C^{[2]})}{g^w}, \left\{ \frac{M'_A(C; \mathbb{W}_C)}{g^w} \right\}_A, \left\{ \frac{M'_A(C; \mathbb{W}_C^{[1]})}{g^w} \right\}_A, \left\{ \frac{M'_A(C; \mathbb{W}_C^{[2]})}{g^w} \right\}_A \right)$$

by linearity of the Hadamard derivatives. It may be shown, for  $m = I_{1,n}$  or  $I_{2,n}$ , that

$$\begin{aligned} & \sup_{[0,1]^d} \left| \frac{\mathbb{D}_{\emptyset,n}}{g^w} - \frac{\mathbb{C}_n}{g^w} - M'_\emptyset \left( C; \frac{\mathbb{C}_n}{g^w} \right) \right| \xrightarrow{\mathbb{P}} 0, \\ & \sup_{[0,1]^d} \left| v_n \frac{\mathbb{D}_{\emptyset,b}^{[m]}}{g^w} - v_n \frac{\mathbb{C}_b^{[m]}}{g^w} - v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{g^w} - M'_\emptyset \left( C; v_n \frac{\mathbb{C}_b^{[m]}}{g^w} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{g^w} \right) \right| \xrightarrow{\mathbb{P}} 0, \\ & \sup_{[0,1]^d} \left| \frac{\mathbb{D}_{A,n}}{g^w} - M'_A \left( C; \frac{\mathbb{C}_n}{g^w} \right) \right| \xrightarrow{\mathbb{P}} 0, \\ & \sup_{[0,1]^d} \left| v_n \frac{\mathbb{D}_{A,b}^{[m]}}{g^w} - M'_A \left( C; v_n \frac{\mathbb{C}_b^{[m]}}{g^w} + v_n \sqrt{\frac{b}{n}} \frac{\mathbb{C}_n}{g^w} \right) \right| \xrightarrow{\mathbb{P}} 0, \end{aligned}$$

which are, once the equations (25), (26), (27) and (28) taken into account, only the consequences of the application of the Delta Method with the maps  $M'_\emptyset$  or  $M'_A$  and the preceding convergences.  $\square$

## References

- B. Berghaus and J. Segers. Weak convergence of the weighted empirical beta copula process. *Journal of Multivariate Analysis*, 166:266–281, 2018.
- B. Berghaus, A. Bücher, and S. Volgushev. Weak convergence of the empirical copula process with respect to weighted metrics. *Bernoulli*, 23(1):743 – 772, 2017.
- C. Bormann, J. Schaumburg, and M. Schienle. Beyond Dimension two: A Test for Higher-Order Tail Risk. *Journal of Financial Econometrics*, 14(3):552–580, 2016.
- A. Bücher. A Note on Weak Convergence of the Sequential Multivariate Empirical Process Under Strong Mixing. Lidam reprints isba, Université catholique de Louvain, Institute of Statistics, Biostatistics and Actuarial Sciences (ISBA), January 2015.
- A. Bücher and H. Dette. A note on bootstrap approximations for the empirical copula process. *Statistics & Probability Letters*, 80(23-24):1925–1932, 2010.

- A. Bücher and I. Kojadinovic. A dependent multiplier bootstrap for the sequential empirical copula process under strong mixing. *Bernoulli*, 22(2):927–968, 2016.
- A. Bücher and M. Ruppert. Consistent testing for a constant copula under strong mixing based on the tapered block multiplier technique. *Journal of Multivariate Analysis*, 116:208–229, 2013.
- A. Bücher and C. Pakzad. Testing for independence in high dimensions based on empirical copulas, 2022. URL <https://arxiv.org/abs/2204.01803>.
- A. Bücher and S. Volgushev. Empirical and sequential empirical copula processes under serial dependence. *Journal of Multivariate Analysis*, 119:61–70, 2013.
- A. Bücher, H. Dette, and S. Volgushev. A test for archimedeanity in bivariate copula models. *Journal of Multivariate Analysis*, 110:121–132, 2012. Special Issue on Copula Modeling and Dependence.
- H. Carley and M. D. Taylor. *A New Proof of Sklar’s Theorem*, pages 29–34. Springer Netherlands, Dordrecht, 2002.
- A. Charpentier, A.-L. Fougères, C. Genest, and J.G. Nešlehová. Multivariate Archimax copulas. *Journal of Multivariate Analysis*, 126(C):118–136, 2014.
- S. Chatelain, A.-L. Fougères, and J. G. Nešlehová. Inference for Archimax copulas. *The Annals of Statistics*, 48(2):1025 – 1051, 2020.
- L. de Haan and A. Ferreira. *Extreme value theory. An introduction*. Springer Series in Operations Research and Financial Engineering. New York, NY: Springer., 2006.
- T. De Wet. Cramér-von mises tests for independence. *Journal of Multivariate Analysis*, 10(1):38–50, 1980.
- P. Deheuvels. La fonction de dépendance empirique et ses propriétés. un test non paramétrique d’indépendance. *Bulletins de l’Académie Royale de Belgique*, 65(1):274–292, 1979.
- P. Deheuvels. Non parametric tests of independence. In Jean-Pierre Raoult, editor, *Statistique non Paramétrique Asymptotique*, pages 95–107, Berlin, Heidelberg, 1980. Springer Berlin Heidelberg.
- P. Deheuvels. An asymptotic decomposition for multivariate distribution-free tests of independence. *Journal of Multivariate Analysis*, 11(1):102–113, 1981.
- B. Efron and C. Stein. The jackknife estimate of variance. *The Annals of Statistics*, 9(3):586–596, 1981.
- J.-D. Fermanian, D. Radulovic, and M. Wegkamp. Weak convergence of empirical copula processes. *Bernoulli*, 10(5): 847–860, 2004.
- C. Genest and B. Remillard. Tests of independence and randomness based on the empirical copula process. *Test*, 13(2): 335–370, 2004.
- C. Genest and B. Remillard. Validity of the parametric bootstrap for goodness-of-fit testing in semiparametric models. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 44(6):1096 – 1127, 2008.
- C. Genest, K. Ghoudi, and L.-P. Rivest. A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, 82(3):543–552, 1995.
- C. Genest, J.-F. Quessy, and B. Remillard. Asymptotic local efficiency of Cramér-von Mises tests for multivariate independence. *The Annals of Statistics*, 35(1):166–191, 2007.
- C. Genest, Remillard B., and D. Beaudoin. Goodness-of-fit tests for copulas: A review and a power study. *Insurance: Mathematics and Economics*, 44(2):199–213, 2009.
- C. Genest, J. Nešlehová, and J.-F. Quessy. Tests of symmetry for bivariate copulas. *Annals of the Institute of Statistical Mathematics*, 64(4):811–834, 2012.
- C. Genest, J. G. Nešlehová, and B. Remillard. Asymptotic behavior of the empirical multilinear copula process under broad conditions. *Journal of Multivariate Analysis*, 159:82–110, 2017.
- C. Genest, J. G. Nešlehová, B. Remillard, and O. A. Murphy. Testing for independence in arbitrary distributions. *Biometrika*, 106(1):47–68, 2019.
- J. Hajek. Asymptotic Normality of Simple Linear Rank Statistics Under Alternatives. *The Annals of Mathematical Statistics*, 39(2):325 – 346, 1968.
- M. Harder and U. Stadtmüller. Testing exchangeability of copulas in arbitrary dimension. *Journal of Nonparametric Statistics*, 29(1):40–60, 2017.
- W. Hoeffding. A class of statistics with asymptotically normal distribution. *Annals of Mathematical Statistics*, 19: 293–325, 1948.

- W. Hoeffding. The strong law of large numbers for u-statistics. *Institute of Statistics Mimeo Series, University of North Carolina, Chapel Hill, N. C.*, 302, 1961.
- M. Hofert, I. Kojadinovic, M. Maechler, and J. Yan. *copula: Multivariate Dependence with Copulas*, 2022. URL <https://CRAN.R-project.org/package=copula>. R package version 1.1-0.
- A. Kiriliouk, J. Segers, and H. Tsukahara. *Resampling Procedures with Empirical Beta Copulas*, pages 27–53. Springer Singapore, Singapore, 2021.
- E. P. Klement, R. Mesiar, and E. Pap. Transformations of copulas. *Kybernetika (Prague)*, 41(4):425–434, 2005.
- I. Kojadinovic and M. Holmes. Tests of independence among continuous random vectors based on Cramér-von Mises functionals of the empirical copula process. *Journal of Multivariate Analysis*, 100(6):1137–1154, 2009.
- I. Kojadinovic and K. Stemikovskaya. Subsampling (weighted smooth) empirical copula processes. *Journal of Multivariate Analysis*, 173:704–723, 2019. ISSN 0047-259X.
- I. Kojadinovic, J. Segers, and J. Yan. Large-sample tests of extreme-value dependence for multivariate copulas. *The Canadian Journal of Statistics / La Revue Canadienne de Statistique*, 39(4):703–720, 2011.
- F. Y. Kuo, I. H. Sloan, G. W. Wasilkowski, and H. Woźniakowski. On decompositions of multivariate functions. *Mathematics of Computation*, 79(270):953–966, 2010.
- B. Li and M. G. Genton. Nonparametric identification of copula structures. *Journal of the American Statistical Association*, 108(502):666–675, 2013.
- A. J. McNeil and J. G. Nešlehová. Multivariate Archimedean copulas, d-monotone functions and  $\ell_1$ -norm symmetric distributions. *The Annals of Statistics*, 37(5B):3059 – 3097, 2009.
- C. Mercadier, O. Rostant, and C. Genest. Linking the Hoeffding–Sobol and Möbius formulas through a decomposition of Kuo, Sloan, Wasilkowski, and Woźniakowski. *Statistics & Probability Letters*, 185, 2022.
- D. N. Politis and J. P. Romano. Large Sample Confidence Regions Based on Subsamples under Minimal Assumptions. *The Annals of Statistics*, 22(4):2031 – 2050, 1994.
- J.-F. Quessy. A general framework for testing homogeneity hypotheses about copulas. *Electronic Journal of Statistics*, 10(1):1064 – 1097, 2016.
- B. Rémillard and O. Scaillet. Testing for equality between two copulas. *Journal of Multivariate Analysis*, 100(3):377–386, 2009.
- P. Ressel. Stable tail dependence functions – some basic properties. *Dependence Modeling*, 10(1):225–235, 2022.
- L. Ruschendorf. Asymptotic distributions of multivariate rank order statistics. *The Annals of Statistics*, 4(5):912–923, 1976.
- B. Rémillard. Goodness-of-fit tests for copulas of multivariate time series. *Econometrics*, 5(1), 2017.
- O. Scaillet. A kolmogorov-smirnov type test for positive quadrant dependence. *Canadian Journal of Statistics*, 33(3): 415–427, 2005.
- J. Segers. Asymptotics of empirical copula processes under non-restrictive smoothness assumptions. *Bernoulli*, 18(3): 764–782, 2012.
- J. Segers, M. Sibuya, and H. Tsukahara. The empirical beta copula. *Journal of Multivariate Analysis*, 155(C):35–51, 2017.
- M. Sklar. Fonctions de répartition à  $n$  dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, 8:229–231, 1959.
- I. M. Sobol’. Sensitivity estimates for nonlinear mathematical models. *Mathematical Modeling and Computational Experiment. Model, Algorithm, Code*, 1(4):407–414 (1995), 1993.
- A. Stupňanová and A. Kolesárová. Associative n-dimensional copulas. *Kybernetika*, 47(1):93–99, 2011.
- A. W. van der Vaart. *Asymptotic Statistics*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 1998.
- A. W. van der Vaart and J. A. Wellner. *Weak Convergence and Empirical Processes: With Applications to Statistics*. Springer New York, New York, NY, 1996.