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# Clifford algebras for SE(3) and the Poincaré group

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**Abstract.** Non-homogeneous groups like the special Euclidean group SE(3) (of the translations and rotations in  $\mathbb{R}^3$ ) or the Poincaré group of Minkowski space-time  $\mathbb{R}^4$  also contain translations, which seems to preclude the possibility of building a representation based on Clifford algebra for them because representations are purely based on matrix multiplication. Summing of group elements is *a priori* not defined, only products are. However, with the use of homogeneous coordinates, SE(3) can be obtained as the group generated by an even number of reflections with respect to hyperplanes of  $\mathbb{R}^4$ . Similarly the Poincaré group is a group that can be generated by an even number of reflections with respect to hyperplanes of  $\mathbb{R}^5$ . However, this leads to some normalization problems and in order to avoid these, null vectors must be introduced. We work this out for the group SE(3) and for the Poincaré group. It leads to groups noted as SO(3,1,0) for SE(3) and SO(3,1,1) or SO(1,1,3) for the Poincaré group.

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## 1 Introduction

The present working document should be considered as a set of notes. There can be no illusion that its contents would be novel or original. It merely reflects my own attempt to get a feeling for the topic. However, the subject matter is probably little if at all known to physicists and by working it out in detail as we do here, it may become accessible to the physics community. For the development we will build further on some ideas that have been explained in [1, 2]. In this Introduction we will just recall some major ingredients of these ideas. It is explained in [1] in Fig.1 on p.6 and in [2], in Fig.1 on p.10 that a three-dimensional rotation can be obtained as a product of two reflections. The rotation angle  $\varphi$  of a rotation is twice the angle  $\varphi/2$  between two reflection planes, as explained in the Figures. SU(2) is a subgroup of the group generated by the reflections. It is the subgroup of group elements obtained from an even number of reflections. To obtain the  $2 \times 2$  representation matrices of SU(2) we first derive the expressions for those of the reflections. The representation matrices of other group elements are then obtained by matrix multiplication. The reflections are defined by the unit vectors  $\mathbf{a}$  that are orthogonal to their reflection planes. A reflection matrix is then given by:

$$\mathbf{A} = \mathbf{a} \cdot \boldsymbol{\sigma} = a_x \sigma_x + a_y \sigma_y + a_z \sigma_z = \begin{bmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -a_z \end{bmatrix}. \quad (1)$$

Here  $\sigma_x, \sigma_y, \sigma_z$  are the Pauli matrices. How Eq. 1 is derived by expressing  $\mathbf{A}^2 = \mathbb{1}$  is explained in [1] on pp. 5-7 and in [2] on pp. 9-11. Here  $\mathbb{1}$  is the  $2 \times 2$  unit matrix. Also the whole further construction of SU(2) is explained in [1, 2].

## 2 The special Euclidean group SE(3) of $\mathbb{R}^3$

### 2.1 The problem

The group SE(3) is also referred to as the group of rigid-body motions. It is the group which contains all translations and rotations of  $\mathbb{R}^3$ . In  $\mathbb{R}^3$  performing a translation after a rotation of a vector  $(x, y, z)$  is done in matrix form according to the school algebra:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \mathbf{R} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \mathbf{T}, \quad \text{where: } \mathbf{T} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \quad (2)$$

is the  $3 \times 1$  translation matrix and  $\mathbf{R}$  the  $3 \times 3$  rotation matrix. This way, it seems as though we cannot represent the group operations by pure matrix multiplications. We will now introduce an expedient that renders this possible.

## 2.2 The solution in three steps

### 2.2.1 Translations as a product of reflections

Let us consider the  $x$ -axis and lines  $x = x_1$  and  $x = x_2$  in the  $Oxy$  plane. A reflection of a point with coordinate  $x$  with respect to the line  $x = x_1$  will transform it into  $x_1 - (x - x_1) = 2x_1 - x$ . A further reflection with respect to  $x = x_2$  will transform it into  $2x_2 - (2x_1 - x) = 2(x_2 - x_1) + x$ . Hence, just like a rotation, a translation is also the product of two reflections, but now with respect to parallel planes. And the length  $2(x_2 - x_1)$  of the translation vector is twice the distance  $x_2 - x_1$  between the two planes, just like the rotation angle  $\varphi$  is twice the angle  $\varphi/2$  between the reflection planes. Therefore the situation is completely analogous to the one for rotations described in Section 1.

### 2.2.2 Homogeneous coordinates and projective geometry

We can write planes in  $\mathbb{R}^3$  with homogeneous coordinates  $(x, y, z, u) \in \mathbb{R}^4$  as:

$$a_x x + a_y y + a_z z + a_u u = 0. \quad (3)$$

This way  $(x, y, z, u)$  are then coordinates for points and  $(a_x, a_y, a_z, a_u)$  coordinates for planes and we have duality: When a point  $(x, y, z, u)$  belongs to a plane  $(a_x, a_y, a_z, a_u)$  considered as a set of points, the plane  $a_x, a_y, a_z, a_u$  belongs to the point  $(x, y, z, u)$  considered as the fan-like set of planes which contain it. Such a set is called a plane bundle in  $\mathbb{R}^3$  (and analogously a line bundle in  $\mathbb{R}^2$ ). In other words, Eq. 3 is an incidence relation. This is projective geometry. As the coordinates are determined up to a proportionality factor, we can choose the norm of both  $(x, y, z, u)$  and  $(a_x, a_y, a_z, a_u)$  to be 1.

Just as we can describe the homogeneous Lorentz group by  $4 \times 4$  matrices operating on  $4 \times 1$  column matrices representing vectors  $(x, y, z, ct) \in \mathbb{R}^4$  we can represent SE(3) by  $4 \times 4$  matrices operating on  $4 \times 1$  column matrices representing vectors  $(x, y, z, u) \in \mathbb{R}^4$  (see e.g. [3]). The playground for the group is therefore the vector space  $\mathbb{R}^4$ . However, what we develop in Clifford algebra is formulating the group theory on a different playground, *viz.* the group manifold. This way it becomes a formalism whereby group elements are represented by group automorphisms operating on group elements, as explained in [1, 2] for SU(2). Such a formalism does *a priori* not contain vectors. However, as explained in [1, 2] it is possible to incorporate the exterior algebra of vectors and multi-vectors into the formalism, illustrating that the group not only preserves length, but also oriented surfaces, volumes and hypervolumes.

By analogy with what we described in Section 1, we can now think of representing the (hyper)planes in a Clifford algebra by the unit vectors that are orthogonal to them. A unit vector will define the reflection with respect to the corresponding (hyper)plane. The vectors  $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z, \mathbf{e}_u$ , could be represented by the  $4 \times 4$  matrices  $\gamma_x, \gamma_y, \gamma_z, \gamma_u$ , adopted from:

$$\gamma_x = \begin{bmatrix} & \sigma_x \\ \sigma_x & \end{bmatrix}, \quad \gamma_y = \begin{bmatrix} & \sigma_y \\ \sigma_y & \end{bmatrix}, \quad \gamma_z = \begin{bmatrix} & \sigma_z \\ \sigma_z & \end{bmatrix}, \quad \gamma_u = \begin{bmatrix} & -i\mathbb{1} \\ +i\mathbb{1} & \end{bmatrix}, \quad \gamma_5 = \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix}. \quad (4)$$

They are generalizations to  $\mathbb{R}^4$  and  $\mathbb{R}^5$  of the Pauli matrices in  $\mathbb{R}^3$  for signatures  $+++$  or  $++++$  of the metric. They satisfy  $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \mathbb{1}$ . But this procedure is not going to work, because it leads to a normalization problem.<sup>1</sup> In fact, instead of  $\mathbf{a}^2 = 1$  as normalization condition we would have now  $\mathbf{a}^2 = 1$ . Two parallel planes  $a_x x + a_y y + a_z z + a_u u = 0$  and  $a_x x + a_y y + a_z z + a_u u = 0$  would this way lead to different values for  $\mathbf{a}^2$ . We would have to write them therefore as  $a_x x + a_y y + a_z z + a_u u = 0$  and  $b_x x + b_y y + b_z z + b_u u = 0$ , with  $\mathbf{a} \parallel \mathbf{b}$ . The vectors  $\mathbf{a}$  and  $\mathbf{b}$  would be proportional only. Furthermore, the product of the two reflections would be:

$$\begin{bmatrix} & -ib_u \mathbb{1} + [\mathbf{b} \cdot \boldsymbol{\sigma}] \\ +ib_u \mathbb{1} + [\mathbf{b} \cdot \boldsymbol{\sigma}] & \end{bmatrix} \begin{bmatrix} & -ia_u \mathbb{1} + [\mathbf{a} \cdot \boldsymbol{\sigma}] \\ +ia_u \mathbb{1} + [\mathbf{a} \cdot \boldsymbol{\sigma}] & \end{bmatrix} =$$

<sup>1</sup> We will use the notations  $\mathbf{a} \in \mathbb{R}^3$  while  $\mathbf{a} = (\mathbf{a}, a_u) \in \mathbb{R}^4$ . For the representation matrices we use  $\mathbf{A}$  for  $\mathbf{a}$  and  $\mathbf{A}$  for  $\mathbf{a}$ .

$$\left[ \begin{array}{c} (\mathbf{a} \star \mathbf{b})\mathbb{1} + \iota(a_u[\mathbf{b} \cdot \boldsymbol{\sigma}] - b_u[\mathbf{a} \cdot \boldsymbol{\sigma}]) \\ (\mathbf{a} \star \mathbf{b})\mathbb{1} - \iota(a_u[\mathbf{b} \cdot \boldsymbol{\sigma}] - b_u[\mathbf{a} \cdot \boldsymbol{\sigma}]) \end{array} \right], \quad (5)$$

where  $\mathbf{a} \star \mathbf{b} = a_x b_x + a_y b_y + a_z b_z + a_u b_u$ . It is hard to work with this. The problem is due to the fact that we normalize  $\mathbf{a}$  and  $\mathbf{b}$  to 1 in  $\mathbb{R}^4$ . This corresponds to introducing an artificial angle  $\psi_1$  such that  $|\mathbf{a}| = \cos \psi_1$  &  $|a_u| = \sin \psi_1$ , and *mutatis mutandis* a similar artificial angle  $\psi_2$  for  $\mathbf{b}$ . In reality the distance  $a_u$  of the plane from the origin  $O$  can grow arbitrarily large, while by normalization we represent the corresponding parameter by a number smaller than one.

### 2.2.3 Null vectors

The idea is therefore [4] to represent the fourth vector by a “null vector”, i.e. an isotropic vector that squares to zero:

$$\gamma_u = (\gamma_5 + \iota\gamma_4) = \left[ \begin{array}{cc} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & -\mathbb{1} \end{array} \right], \quad \text{with: } \gamma_u^2 = 0. \quad (6)$$

The definitions of  $\gamma_x$ ,  $\gamma_y$  and  $\gamma_z$  from Eq. 4 are maintained. This way, the  $u$ -components do not contribute to the norm, and two parallel planes yield the same normalization for  $\mathbf{a}$ . We can normalize then  $\mathbf{a}$  again to 1 and make  $a_u$  arbitrarily large. The null vector we have chosen is the analog in block form of  $\sigma_z + \iota\sigma_y$ , which corresponds to the isotropic vector  $(\mathbf{e}_z + \iota\mathbf{e}_y)$  in SU(2). Another isotropic vector is  $(\mathbf{e}_z - \iota\mathbf{e}_y)$ . We do not renormalize the isotropic vectors by a factor  $\frac{1}{\sqrt{2}}$  in order not to burden the notations and keep the equations more simple.<sup>2</sup> As the metric tensor is now:  $g_{xx} = 1$ ,  $g_{yy} = 1$ ,  $g_{zz} = 1$ ,  $g_{uu} = 0$ , the group is noted as SO(3,1,0), translating the fact that the signature of the metric tensor contains 3 times +1, 1 time 0, and 0 times -1.<sup>3</sup> There is no mention of this elegant method in the monograph by Lounesto [5] and as far as I can see (from the table of contents) also not in the work of Kim and Noz [6].

The most general vector or the most general hyperplane is then represented by:

$$\mathbf{A} = \left[ \begin{array}{cc} a_u \mathbb{1} & a_u \mathbb{1} + \mathbf{a} \cdot \boldsymbol{\sigma} \\ -a_u \mathbb{1} + \mathbf{a} \cdot \boldsymbol{\sigma} & -a_u \mathbb{1} \end{array} \right]. \quad (7)$$

The  $4 \times 4$  matrix  $\mathbf{A}$  represents both the normal vector  $\mathbf{a}$  and the reflection with respect to the plane defined by the normal vector  $\mathbf{a}$ , in perfect analogy with what is done in the construction in SU(2). The representation matrix of a vector  $\mathbf{v} = (x, y, z, u)$  is thus given by:  $\mathbf{V} = \mathbf{v} \cdot \boldsymbol{\gamma} + v_u \gamma_u$ . Reflections will transform a vector  $\mathbf{v}$  according to:

$$\mathbf{V} \rightarrow -\mathbf{A}\mathbf{V}\mathbf{A}^{-1} = -\mathbf{A}\mathbf{V}\mathbf{A}. \quad (8)$$

Of course for reflection operators  $\mathbf{A}^{-1} = \mathbf{A}$ , because reflections are involutions:  $\mathbf{A}^2 = \mathbb{1}$ . Mathematicians may reckon at this stage that the problem has been solved and move on. But applying a theory to a practical problem renders it less abstract and leads to a better understanding of it. We consider it therefore instrumental to show how this works out in practice, making sure that we develop a perfect dialogue between the geometry and the algebra.

### 2.3 Translations

We can now consider two reflection planes defined by  $\mathbf{B}$  and  $\mathbf{A}$ . They will be parallel if  $\mathbf{a} = (a_x, a_y, a_z) \parallel \mathbf{b} = (b_x, b_y, b_z)$ . We adopt the convention to consider the vectors  $\mathbf{a}$  as normalized to 1. The quantity  $a_u$  is then the distance of the plane from the origin  $O$ . With this convention we have then  $\mathbf{b} = \mathbf{a}$  and  $\mathbf{B}\mathbf{A}$  will be a translation. We obtain then:

$$\begin{aligned} \mathbf{T} = \mathbf{B}\mathbf{A} &= \left[ \begin{array}{cc} b_u \mathbb{1} & b_u \mathbb{1} + \mathbf{a} \cdot \boldsymbol{\sigma} \\ -b_u \mathbb{1} + \mathbf{a} \cdot \boldsymbol{\sigma} & -b_u \mathbb{1} \end{array} \right] \left[ \begin{array}{cc} a_u \mathbb{1} & a_u \mathbb{1} + \mathbf{a} \cdot \boldsymbol{\sigma} \\ -a_u \mathbb{1} + \mathbf{a} \cdot \boldsymbol{\sigma} & -a_u \mathbb{1} \end{array} \right] = \\ & \left[ \begin{array}{cc} (b_u - a_u)[\mathbf{a} \cdot \boldsymbol{\sigma}] + \mathbf{a}^2 \mathbb{1} & (b_u - a_u)[\mathbf{a} \cdot \boldsymbol{\sigma}] \\ (a_u - b_u)[\mathbf{a} \cdot \boldsymbol{\sigma}] & (a_u - b_u)[\mathbf{a} \cdot \boldsymbol{\sigma}] + \mathbf{a}^2 \mathbb{1} \end{array} \right]. \end{aligned} \quad (9)$$

<sup>2</sup> E.g. Eq. 11 would really become really more tedious and less elegant if we always carried along the normalization factor. Not normalizing the isotropic vector by the factor  $\frac{1}{\sqrt{2}}$  corresponds to not using an orthonormal basis. There is nothing wrong with that but in general it complicates things a bit. However, this will not play a rôle in this document because we will never consider the Hermitian norm of vectors. We will only treat vectors of  $\mathbb{R}^n$ . In any case, alle isotropic vectors have “Euclidean length zero”, such that normalization does not affect this “length” (but it does of course change their Hermitian length). The “length” is not really zero. It is just that the extrapolation of the Euclidian distance function from  $\mathbb{R}^n$  to  $\mathbb{C}^n$  is no longer a distance function. An isotropic vector is therefore also not truly orthogonal to itself, as we may have been told.

<sup>3</sup> In [4] the authors consider also conformal groups. However, their choice for the isotropic vector in Eq. 10.1 is wrong because  $PR + RP = \mathbb{1}$  which implies that the null vector is not orthogonal to all other basis vectors.

In summary:

$$\mathbf{T} = \mathbf{1} + (b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\sigma}] \cdot \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{1} & -\mathbf{1} \end{bmatrix} = \mathbf{1} + (b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\sigma}] \otimes \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}. \quad (10)$$

Here the non-boldface dot product just means that we multiply all  $2 \times 2$  blocks of  $\gamma_u$  with  $(b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\sigma}]$ . The symbol  $\mathbf{1}$  stands here for  $2 \times 2$  unit matrices. The expression for  $\mathbf{T}$  is thus of the form  $\mathbf{1} + \mathbf{Q}$ , where  $\mathbf{Q} = \gamma_u (b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\sigma}]$ . In this expression  $\mathbf{1}$  stands for the  $4 \times 4$  unit matrix. The matrix  $\mathbf{Q}$  is not invertible because its lines 1 and 3 are proportional. The same applies for its lines 2 and 4. But the matrix  $(\mathbf{1} + \mathbf{Q})$  is invertible and its inverse is  $(\mathbf{1} - \mathbf{Q})$ , because  $(\mathbf{1} + \mathbf{Q})(\mathbf{1} - \mathbf{Q}) = \mathbf{1} - \mathbf{Q}^2 = \mathbf{1}$ . The rotation matrices are invertible (see below) such that this shows that all group elements are represented by invertible matrices.<sup>4</sup> We can now calculate the effect of the translation:  $\mathbf{V} \rightarrow \mathbf{T}\mathbf{V}\mathbf{T}^{-1}$  on a vector  $\mathbf{V}$ :

$$\begin{aligned} \mathbf{T}\mathbf{V}\mathbf{T}^{-1} = & \begin{bmatrix} (b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\sigma}] + \mathbf{a}^2 \mathbf{1} & (b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\sigma}] \\ (a_u - b_u) [\mathbf{a} \cdot \boldsymbol{\sigma}] & (a_u - b_u) [\mathbf{a} \cdot \boldsymbol{\sigma}] + \mathbf{a}^2 \mathbf{1} \end{bmatrix} \begin{bmatrix} v_u \mathbf{1} & v_u \mathbf{1} + \mathbf{v} \cdot \boldsymbol{\sigma} \\ -v_u \mathbf{1} + \mathbf{v} \cdot \boldsymbol{\sigma} & -v_u \mathbf{1} \end{bmatrix} \times \\ & \begin{bmatrix} (a_u - b_u) [\mathbf{a} \cdot \boldsymbol{\sigma}] + \mathbf{a}^2 \mathbf{1} & (a_u - b_u) [\mathbf{a} \cdot \boldsymbol{\sigma}] \\ (b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\sigma}] & (b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\sigma}] + \mathbf{a}^2 \mathbf{1} \end{bmatrix}. \end{aligned} \quad (11)$$

This is rather cumbersome an expression. It is of the form:

$$(\mathbf{1} + \mathbf{Q})\mathbf{V}(\mathbf{1} - \mathbf{Q}) = \mathbf{V} + \mathbf{Q}\mathbf{V} - \mathbf{V}\mathbf{Q} - \mathbf{Q}\mathbf{V}\mathbf{Q}. \quad (12)$$

where  $\mathbf{Q} = \gamma_u (b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\gamma}]$  and  $\mathbf{V} = v_u \gamma_u + [\mathbf{v} \cdot \boldsymbol{\gamma}]$ . In principle  $\mathbf{Q}\mathbf{V} - \mathbf{V}\mathbf{Q}$  is a wedge product.<sup>5</sup> Now  $\mathbf{Q}$  contains a term  $\gamma_u$  that anti-commutes with  $[\mathbf{a} \cdot \boldsymbol{\gamma}]$ . Pulling this term towards the end by using anti-commutation relations we obtain  $-\mathbf{Q} = (b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\gamma}] \gamma_u$ . This will put the final term  $\gamma_u$  face to face with the term  $\gamma_u$  in  $\mathbf{V}$ , such that this term in  $\mathbf{V}$  disappears, because  $\gamma_u^2 = 0$ . We can then pull  $\gamma_u$  towards the end in the non-zero part of  $\gamma_u \mathbf{V}$  by anti-commutation. It will then be put face to face with the term  $\gamma_u$  in  $\mathbf{Q}$  which proves that  $\mathbf{Q}\mathbf{V}\mathbf{Q} = 0$ . Hence:

$$\mathbf{V} \rightarrow \mathbf{V} + (\mathbf{Q}\mathbf{V} - \mathbf{V}\mathbf{Q}). \quad (13)$$

Case 1. When  $\mathbf{a} \parallel \mathbf{v}$ , then  $\mathbf{v} = \mathbf{a}$ . Then  $\mathbf{Q}\mathbf{V} = \gamma_u (b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\gamma}]^2 = \gamma_u (b_u - a_u)$ . We obtain this way a translation, with amplitude  $2(b_u - a_u)$ , because  $-\mathbf{V}\mathbf{Q} = \mathbf{Q}\mathbf{V}$ :

$$\mathbf{V} \rightarrow \mathbf{V} + 2(b_u - a_u) \gamma_u. \quad (14)$$

Hence only the  $u$ -component of  $\mathbf{V}$  is changed. One adds twice the distance  $(b_u - a_u)$  between the two planes to the position parameter for the plane defined by  $\mathbf{V}$ . This is exactly what we described in Subsection 2.2.1. Let us now try to reason traditionally on the orthogonality of the vectors, which is not granted (see Footnote 5).  $\mathbf{Q}$  is then parallel to  $\mathbf{e}_u \wedge \mathbf{a}$ . Hence the expression  $\mathbf{Q}\mathbf{V} - \mathbf{V}\mathbf{Q}$  is parallel to  $(\mathbf{e}_u \wedge \mathbf{a}) \wedge \mathbf{a}$ . This is a vector parallel to  $\mathbf{e}_u$ .

Case 2. The other case occurs when  $\mathbf{a} \not\parallel \mathbf{v}$ . We obtain then:  $\mathbf{Q}\mathbf{V} = \gamma_u (b_u - a_u) \cos \alpha$ , where  $\mathbf{a} \cdot \mathbf{v} = \cos \alpha$ , where  $\alpha$  is the angle between the unit vectors  $\mathbf{a}$  and  $\mathbf{v}$ . In other words,  $\cos \alpha$  is the projection of  $\mathbf{a}$  on  $\mathbf{v}$ :

$$\mathbf{V} \rightarrow \mathbf{V} + 2(b_u - a_u) \cos(\alpha) \gamma_u. \quad (15)$$

These results show that the formalism works. Due to the funny conventions we introduced for the extra dimension and the vector  $\mathbf{e}_u$ , it was perhaps not obvious that this method would lead to the construction of a representation. We have now proved that this is indeed the case. By proceeding along the path we have outlined it becomes possible to describe the isometries of Euclidean space  $\mathbb{R}^3$  in a unified form.<sup>6</sup> The metric for a vector  $(x, y, z, u)$  will now be  $x^2 + y^2 + z^2$ . The transformation of a vector  $\mathbf{V}$  by an isometry  $\mathbf{I} \in \text{SE}(3)$  will be given by:

$$\mathbf{V} \rightarrow \mathbf{I}\mathbf{V}\mathbf{I}^{-1}. \quad (16)$$

The number of independent isometries is given by the number of independent bi-vectors:

<sup>4</sup> If  $\mathbf{b}$  and  $\mathbf{a}$  are not parallel, then  $\mathbf{B}\mathbf{A}$  will be a rotation. The rotation axis will be the intersection of the two planes and the rotation angle twice the angle between the two planes.

<sup>5</sup> We must be careful in drawing geometrical conclusions from the algebra. Reasoning on angles between  $\mathbf{e}_u$  and  $\mathbf{a}$  is somewhat ambiguous and could lead to wrong conclusions. In fact, intuitively  $\mathbf{e}_u$  is parallel to  $\mathbf{a}$  because one multiplies it by the amplitude of the translation to build the translation vector. But algebraically it is perpendicular to  $\mathbf{a}$ .

<sup>6</sup> In physics a screw is a rotation combined with a translation parallel to the rotation axis. When this parallelism is not present one should not call the group element a screw. It suffices then to call it an isometry.

$$\binom{4}{2} = 6. \quad (17)$$

This is in agreement with the dimension of the group  $SO(3,1,0)$  which is 6. This number can also be determined by counting the number of independent real parameters needed to characterize a transformed orthonormal basis.

## 2.4 Rotations around the origin

We will continue to use the notations  $\mathbf{A} = [\mathbf{a} \cdot \boldsymbol{\sigma}]$ , and  $\mathbf{a} \star \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + a_u b_u$ , introduced above. When  $a_u = b_u = 0$  and  $\mathbf{a} \not\parallel \mathbf{b}$  we will have two non-parallel planes through the origin and the product will be a rotation  $\mathbf{R} = \mathbf{B}\mathbf{A}$  around the origin:

$$\mathbf{R} = \mathbf{B}\mathbf{A} = \begin{bmatrix} & [\mathbf{b} \cdot \boldsymbol{\sigma}] \\ [\mathbf{b} \cdot \boldsymbol{\sigma}] & \end{bmatrix} \begin{bmatrix} & [\mathbf{a} \cdot \boldsymbol{\sigma}] \\ [\mathbf{a} \cdot \boldsymbol{\sigma}] & \end{bmatrix} = \begin{bmatrix} [\mathbf{b} \cdot \boldsymbol{\sigma}][\mathbf{a} \cdot \boldsymbol{\sigma}] & \\ & [\mathbf{b} \cdot \boldsymbol{\sigma}][\mathbf{a} \cdot \boldsymbol{\sigma}] \end{bmatrix} = \begin{bmatrix} \mathbf{B}\mathbf{A} & \\ & \mathbf{B}\mathbf{A} \end{bmatrix}. \quad (18)$$

In other words:

$$\mathbf{R} = \begin{bmatrix} \mathbf{R} & \\ & \mathbf{R} \end{bmatrix}, \quad (19)$$

where  $\mathbf{R} \in SU(2)$ . One should actually calculate also the effect of this rotation on a general vector, because this will be more involved than in  $\mathbb{R}^3$ , due to the fact that the vector can contain now an extra fourth component as illustrated by Eq. 7. This takes the form:

$$\begin{bmatrix} \mathbf{R} & \\ & \mathbf{R} \end{bmatrix} \left[ v_u \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & -\mathbb{1} \end{bmatrix} + \begin{bmatrix} & [\mathbf{v} \cdot \boldsymbol{\sigma}] \\ [\mathbf{v} \cdot \boldsymbol{\sigma}] & \end{bmatrix} \right] \begin{bmatrix} \mathbf{R}^{-1} & \\ & \mathbf{R}^{-1} \end{bmatrix} = \\ v_u \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & -\mathbb{1} \end{bmatrix} + \begin{bmatrix} & [\mathbf{R}(\mathbf{v}) \cdot \boldsymbol{\sigma}] \\ [\mathbf{R}(\mathbf{v}) \cdot \boldsymbol{\sigma}] & \end{bmatrix}. \quad (20)$$

The sum which occurs in this expression is meaningful because we are adding here vectors, for which a sum is defined. The whole group SE(3) can be generated by rotations around the origin  $O$  and translations. In reality, it is actually the group of translations and rotations around arbitrary points  $P \in \mathbb{R}^3$ . However, we retain the idea that the non-homogeneous group is an extension of the homogeneous group by translations.

## 2.5 General case: Two reflections generate a rotation around a point different from the origin

What remains to be done is treating the case  $a_u \neq 0$ ,  $b_u \neq 0$ ,  $\mathbf{a} \not\parallel \mathbf{b}$ . We have then the group element:

$$\mathbf{G} = \mathbf{B}\mathbf{A} = \begin{bmatrix} b_u \mathbb{1} & b_u \mathbb{1} + \mathbf{b} \cdot \boldsymbol{\sigma} \\ -b_u \mathbb{1} + \mathbf{b} \cdot \boldsymbol{\sigma} & -b_u \mathbb{1} \end{bmatrix} \begin{bmatrix} a_u \mathbb{1} & a_u \mathbb{1} + \mathbf{a} \cdot \boldsymbol{\sigma} \\ -a_u \mathbb{1} + \mathbf{a} \cdot \boldsymbol{\sigma} & -a_u \mathbb{1} \end{bmatrix} = \\ \begin{bmatrix} b_u [\mathbf{a} \cdot \boldsymbol{\sigma}] - a_u [\mathbf{b} \cdot \boldsymbol{\sigma}] + (\mathbf{a} \cdot \mathbf{b}) \mathbb{1} + \imath [(\mathbf{b} \wedge \mathbf{a}) \cdot \boldsymbol{\sigma}] & b_u [\mathbf{a} \cdot \boldsymbol{\sigma}] - a_u [\mathbf{b} \cdot \boldsymbol{\sigma}] \\ a_u [\mathbf{b} \cdot \boldsymbol{\sigma}] - b_u [\mathbf{a} \cdot \boldsymbol{\sigma}] & a_u [\mathbf{b} \cdot \boldsymbol{\sigma}] - b_u [\mathbf{a} \cdot \boldsymbol{\sigma}] + \imath [(\mathbf{b} \wedge \mathbf{a}) \cdot \boldsymbol{\sigma}] \end{bmatrix}. \quad (21)$$

This is just a rotation  $\mathbf{R}_P$  around another point  $P$  than the origin  $O$ . In principle  $\mathbf{R}_P = \mathbf{T}_{OP} \mathbf{R}_O \mathbf{T}_{PO}$ , which implies:  $\mathbf{R}_P \mathbf{T}_{OP} = \mathbf{T}_{OP} \mathbf{R}_O$ . It is tempting to write Eq. 21 algebraically as:

$$\begin{bmatrix} (\mathbf{a} \cdot \mathbf{b}) \mathbb{1} + \imath [(\mathbf{b} \wedge \mathbf{a}) \cdot \boldsymbol{\sigma}] & \\ & (\mathbf{a} \cdot \mathbf{b}) \mathbb{1} + \imath [(\mathbf{b} \wedge \mathbf{a}) \cdot \boldsymbol{\sigma}] \end{bmatrix} + [b_u [\mathbf{a} \cdot \boldsymbol{\sigma}] - a_u [\mathbf{b} \cdot \boldsymbol{\sigma}]] \cdot \gamma_u. \quad (22)$$

The first matrix is indeed the rotation matrix. But this equation is geometrically totally meaningless because the group elements do not build a vector space. The second matrix  $[b_u [\mathbf{a} \cdot \boldsymbol{\sigma}] - a_u [\mathbf{b} \cdot \boldsymbol{\sigma}]] \cdot \gamma_u$  is not a translation and not a rotation. Sums of representation matrices can be carried out algebraically but they do not have geometrical meaning [7]. The meaning of the vector represented by  $[b_u [\mathbf{a} \cdot \boldsymbol{\sigma}] - a_u [\mathbf{b} \cdot \boldsymbol{\sigma}]]$  can be understood as follows. Consider the two planes with equations:

$$\begin{cases} a_x x + a_y y + a_z z + a_u u = 0 & (L_1) \\ b_x x + b_y y + b_z z + b_u u = 0 & (L_2) \end{cases} \quad (23)$$

This set of equations defines the line  $\ell$  which is the intersection of the two planes. We can now replace one of the two equations by the equation  $b_u L_1 - a_u L_2$ . This equation corresponds to  $[b_u \mathbf{a} - a_u \mathbf{b}] \star [x, y, z, u] = 0$ , which defines a plane through the origin  $O$ , *viz.* the plane defined by  $\ell$  and  $O$ . The vector  $b_u \mathbf{a} - a_u \mathbf{b} = (b_u \mathbf{a} - a_u \mathbf{b}, 0)$  is orthogonal to this plane.

## 2.6 General case: Product of a translation and a rotation around the origin

### 2.6.1 Combination of a translation and a rotation around the origin: case $TR$

We noted that we can consider SE(3) as a group generated by rotations around the origin and translations. We check here the general form of such a product, independently from the way it was obtained (by two or four reflections).

$$TR = \left[ \mathbb{1} + (b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\sigma}] \cdot \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & -\mathbb{1} \end{bmatrix} \right] \begin{bmatrix} \mathbf{R} & \\ & \mathbf{R} \end{bmatrix}. \quad (24)$$

there will be two terms in a sum:  $\mathbf{R}$  and  $\gamma_u(b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\sigma}] \mathbf{R}$ . The sum is not meaningful because the two terms are group elements which cannot be added.<sup>7</sup> We can call it therefore a fake sum. We can consider (1) the plane  $\mathbf{A}$  defined by  $\mathbf{a}$  and the unit vector  $\mathbf{s}$  along the rotation axis and (2) the plane  $\mathbf{B}$  that contains  $\mathbf{s}$  and is at an angle  $\varphi/2$  with respect to  $(\mathbf{s}, \mathbf{a})$ . We obtain  $\mathbf{R}$  then as  $\mathbf{AB}$  such that then  $[\mathbf{a} \cdot \boldsymbol{\sigma}] \mathbf{R} = \mathbf{B}$ . Hence the product will always look as a fake sum of a rotation around the origin and a translation, as also emphasized in Eq. 22 and the true translation  $\mathbf{T}$  is never the term in this false sum.

### 2.6.2 Combination of translation and a rotation around the origin: case $RT$

The product:

$$RT = \begin{bmatrix} \mathbf{R} & \\ & \mathbf{R} \end{bmatrix} \left[ \mathbb{1} + (b_u - a_u) [\mathbf{a} \cdot \boldsymbol{\sigma}] \cdot \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & -\mathbb{1} \end{bmatrix} \right]. \quad (25)$$

will contain two terms in a sum:  $\mathbf{R}$  and  $(b_u - a_u) \mathbf{R} [\mathbf{a} \cdot \boldsymbol{\sigma}] \cdot \gamma_u$ . We can consider the plane  $\mathbf{A}$  defined by  $\mathbf{a}$  and  $\mathbf{s}$  and the plane  $\mathbf{B}$  that contains  $\mathbf{s}$  and is at an angle  $\varphi/2$  with respect to  $(\mathbf{s}, \mathbf{a})$ . We obtain  $\mathbf{R}$  then as  $\mathbf{BA}$  such that then  $\mathbf{R} [\mathbf{a} \cdot \boldsymbol{\sigma}] = \mathbf{B}$ . Hence the product will always look as a false sum of a rotation around the origin and a translation, as discussed in Eq. 22 and the translation is never the term in this false sum.

### 2.6.3 Finding the true translations within a product of two reflections

It follows from what precedes that it must be possible to write  $\mathbf{G}$  as  $\mathbf{G} = \mathbf{T}_1 \mathbf{R}$  and as  $\mathbf{G} = \mathbf{R} \mathbf{T}_2$ , where  $\mathbf{R}$  is a rotation around the origin. The rotation  $\mathbf{R}$  is defined by Eq. 22. We now want to find the true translations  $\mathbf{T}_1$  and  $\mathbf{T}_2$ . To determine  $\mathbf{T}_1$  and  $\mathbf{T}_2$  we must calculate  $\mathbf{T}_1 = \mathbf{G} \mathbf{R}^{-1}$  and  $\mathbf{T}_2 = \mathbf{R}^{-1} \mathbf{G}$ .

Case 1.  $\mathbf{T}_1 = \mathbf{G} \mathbf{R}^{-1}$  is given by:

$$\mathbb{1} + [\gamma_u \cdot [b_u [\mathbf{a} \cdot \boldsymbol{\sigma}] - a_u [\mathbf{b} \cdot \boldsymbol{\sigma}]]] \begin{bmatrix} \mathbf{AB} & \\ & \mathbf{AB} \end{bmatrix} = \mathbb{1} + [\gamma_u \cdot [b_u \mathbf{A} - a_u \mathbf{B}]] \begin{bmatrix} \mathbf{AB} & \\ & \mathbf{AB} \end{bmatrix} \quad (26)$$

In the algebra we encounter expressions  $\mathbf{AAB} = \mathbf{B}$  and  $\mathbf{BAB}$ :

$$\mathbf{T}_1 = \mathbb{1} + [[b_u \mathbf{B} - a_u \mathbf{BAB}] \cdot \gamma_u]. \quad (27)$$

This has indeed the form of a translation. The quantity  $-\mathbf{BAB}$  is a vector obtained from  $\mathbf{A}$  by a similarity transformation. It corresponds to  $\mathbf{A}$  reflected with respect to  $\mathbf{B}$ . As to the meaning of the vector  $[b_u \mathbf{B} - a_u \mathbf{BAB}]$  is concerned we can note that the intersection of two planes is not a point but a straight line. Whereas in  $\mathbb{R}^2$  we could have imagined that it is the position vector of the intersection point of two straight lines, it must be obvious that this can here not be the case.

Case 2.  $\mathbf{T}_2 = \mathbf{R}^{-1} \mathbf{G}$  is given by:

$$\mathbb{1} + \left[ \begin{bmatrix} \mathbf{AB} & \\ & \mathbf{AB} \end{bmatrix} [b_u [\mathbf{a} \cdot \boldsymbol{\sigma}] - a_u [\mathbf{b} \cdot \boldsymbol{\sigma}]] \cdot \gamma_u \right] = \mathbb{1} + \left[ \begin{bmatrix} \mathbf{AB} & \\ & \mathbf{AB} \end{bmatrix} [b_u \mathbf{A} - a_u \mathbf{B}] \cdot \gamma_u \right]. \quad (28)$$

In the algebra we encounter expressions  $\mathbf{ABB} = \mathbf{A}$  and  $\mathbf{ABA}$ :

$$\mathbf{T}_2 = \mathbb{1} + [[-a_u \mathbf{A} + b_u \mathbf{ABA}] \cdot \gamma_u]. \quad (29)$$

<sup>7</sup> Indeed for a group  $(G, \circ)$  only the composition law  $\circ$  is defined. Summing requires further structure in the form of vector space and this is in principle not the case. This will be especially important when the group manifold is curved (see [7]).

This has indeed the form of a translation. The quantity  $-\mathbf{A}\mathbf{B}\mathbf{A}$  is a vector obtained from  $\mathbf{B}$  by a similarity transformation. It corresponds to  $\mathbf{B}$  reflected with respect to  $\mathbf{A}$ .

We must now figure out the geometrical meaning of these results. The following are the relations between the vectors  $\mathbf{t}_1 \in \mathbb{R}^3$ ,  $\mathbf{t}_2 \in \mathbb{R}^3$  and their SU(2) representation matrices:

$$[\mathbf{t}_1 \cdot \boldsymbol{\sigma}] = b_u \mathbf{B} - a_u \mathbf{B}\mathbf{A}\mathbf{B} = \mathbf{B}(b_u \mathbf{B} - a_u \mathbf{A})\mathbf{B}. \quad (30)$$

$$[\mathbf{t}_2 \cdot \boldsymbol{\sigma}] = b_u \mathbf{A}\mathbf{B}\mathbf{A} - a_u \mathbf{A} = \mathbf{A}(b_u \mathbf{B} - a_u \mathbf{A})\mathbf{A}. \quad (31)$$

The vector  $b_u \mathbf{B} - a_u \mathbf{A}$  links the projections  $P_a$  and  $P_b$  of the origin  $O$  onto the planes  $\mathbf{a}$  and  $\mathbf{b}$ . The two vectors are thus the reflections of this connection vector  $P_a P_b$  with respect to the planes  $\mathbf{a}$  and  $\mathbf{b}$  through the origin. Now:

$$[\mathbf{t}_1 \cdot \boldsymbol{\sigma}] = \mathbf{B}\mathbf{A}[\mathbf{t}_2 \cdot \boldsymbol{\sigma}]\mathbf{A}\mathbf{B} = \mathbf{R}[\mathbf{t}_2 \cdot \boldsymbol{\sigma}]\mathbf{R}^{-1}. \quad (32)$$

We have  $\mathbf{T}_1 \mathbf{R} = \mathbf{R} \mathbf{T}_2$ , and this is reflected in the vectors  $\mathbf{t}_1$  and  $\mathbf{t}_2$ . The vectors  $\mathbf{t}_1$  and  $\mathbf{t}_2$  are exactly related the same way as the translations  $\mathbf{T}_1$  and  $\mathbf{T}_2$ . Furthermore there must be an exact symmetry operation  $(\mathbf{A}, \mathbf{B}) | (\mathbf{B}, \mathbf{A})$  between the expressions for  $\mathbf{t}_1$  and  $\mathbf{t}_2$  which is achieved by Eqs. 30-31. Intuitively we might have expected that  $\mathbf{T}_1 = \mathbf{T}_2$  and that the translation vector would be the vector  $OP$  where  $P$  is the projection of  $O$  on  $\ell$  within the plane defined by  $\ell$  and  $O$  but this is not the case because the formulation  $\mathbf{G} = \mathbf{T}_1 \mathbf{R}_O = \mathbf{R}_O \mathbf{T}_2$  is different from the formulation  $\mathbf{G} = \mathbf{R}_P = \mathbf{T}_{OP} \mathbf{R}_O \mathbf{T}_{PO}$ , which is equivalent to  $\mathbf{T}_{OP} \mathbf{R}_O = \mathbf{R}_P \mathbf{T}_{OP}$ .

We may note that  $b_u \mathbf{B} - a_u \mathbf{A}$  is not a symmetrical expression. In other words, the expressions are the symmetrizations of the connecting vector  $P_a P_b = b_u \mathbf{B} - a_u \mathbf{A}$ . The result may look counterintuitive but we should realize that our intuition is based on visual inspection within a fixed vector space  $\mathbb{R}^3$ , while we are dealing here with a language for group elements, i.e. functions, rather than vectors. These group operations entail each time changes of the basis for the vector space. E.g. in  $\mathbf{T}\mathbf{R}$  the translation  $\mathbf{T}$  is expressed with respect to a basis that has been turned by  $\mathbf{R}$ . This may illustrate that the dialogue between the geometry and the algebra is not always trivial. If we had stuck to the general abstract solution found in Subsection 2.2.3, we would not have discovered these issues.

We see that the algebra will only contain translations in the plane perpendicular to the rotation axis. However, this is due to the fact that we have only considered the product of two reflections  $\mathbf{A}$  and  $\mathbf{B}$ . Generating the composition of a rotation and a translation parallel to the rotation axis takes 4 reflections. The planes defined by just  $\mathbf{A}$  and  $\mathbf{B}$  are planes through the origin.

#### 2.6.4 Combined translations and rotations around the origin: Four-reflection case

The analysis in Subsection 2.6 is only valid for a transformation obtained from two reflections. When more than two reflections are needed to generate the transformation, things become more complicated. The number of reflections needed to obtain a group element is described by the Dieudonné-Cartan theorem [4]. The specific case is that of a rotation followed by a translation around the rotation axis. This requires four reflections. Let us note  $[\mathbf{s} \cdot \boldsymbol{\sigma}] = \mathbf{S}$ . The transformation would then be of the form:

$$\left[ \mathbf{1} + \begin{bmatrix} s_u \mathbf{S} & s_u \mathbf{S} \\ -s_u \mathbf{S} & -s_u \mathbf{S} \end{bmatrix} \right] \begin{bmatrix} \cos(\varphi/2) \mathbf{1} - \imath \sin(\varphi/2) \mathbf{S} & \\ & \cos(\varphi/2) \mathbf{1} - \imath \sin(\varphi/2) \mathbf{S} \end{bmatrix}. \quad (33)$$

as  $\mathbf{S}^2 = \mathbf{1}$  this will contain only entries with  $\mathbf{1}$  and  $\mathbf{S}$ :  $s_u \mathbf{S}(\cos(\varphi/2) \mathbf{1} - \imath \sin(\varphi/2) \mathbf{S}) = -\imath s_u \sin(\varphi/2) \mathbf{1} + s_u \cos(\varphi/2) \mathbf{S}$ . Hence we obtain:

$$\begin{bmatrix} \cos(\varphi/2) \mathbf{1} - \imath \sin(\varphi/2) \mathbf{S} & \\ & \cos(\varphi/2) \mathbf{1} - \imath \sin(\varphi/2) \mathbf{S} \end{bmatrix} + (s_u \cos(\varphi/2) \mathbf{S} - \imath s_u \sin(\varphi/2) \mathbf{1}) \cdot \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{1} & -\mathbf{1} \end{bmatrix}. \quad (34)$$

The second term is  $2\imath s_u \frac{\partial \mathbf{R}}{\partial \varphi} \cdot \gamma_u$ . We seem to recover this way the translation operator  $-\frac{1}{\imath} \frac{\partial}{\partial \varphi}$ . The transformation could be also of the form:

$$\begin{bmatrix} \cos(\varphi/2) \mathbf{1} - \imath \sin(\varphi/2) \mathbf{S} & \\ & \cos(\varphi/2) \mathbf{1} - \imath \sin(\varphi/2) \mathbf{S} \end{bmatrix} \left[ \mathbf{1} + \begin{bmatrix} s_u \mathbf{S} & s_u \mathbf{S} \\ -s_u \mathbf{S} & -s_u \mathbf{S} \end{bmatrix} \right]. \quad (35)$$

But because the expressions contain only  $\mathbf{1}$  and  $\mathbf{S}$  the product is commutative.



## 2.7 Conclusion

Apparently we can hardly draw any significant benefit from developing this way the representation theory for the isometries of  $\mathbb{R}^3$ . We might as well have studied the group by using the matrix formalism outlined in Eq. 2 or the four-dimensional matrix-formalism that operates on  $4 \times 1$  representations of vectors. It would have been more straightforward and simple. The essential point is however that we are not working with a language of vectors but a language of group elements. Quantum theory is written in the language of group elements [2], such that the application of the methods illustrated here to the Poincaré group remains useful.

## 3 The Poincaré group

### 3.1 Choice of Gamma matrices

One can generalise these ideas and methods to describe other non-homogeneous groups, like the non-homogeneous Galileo group or the Poincaré group by Clifford algebra. For the Poincaré group this will be a Clifford algebra which uses homogeneous coordinates in space-time and  $\mathbb{R}^5$ . We have worked this out in detail, such that it will be available for everybody. We do not know if this has ever been done before. The number of independent isometries, i.e. the number of independent bi-vectors  $\Gamma_\mu \Gamma_\nu$  ( $\mu < \nu$ ) will now be:

$$\binom{5}{2} = 10. \quad (36)$$

It is well known that the Poincaré group is ten-dimensional. The ten parameters define three boosts, three rotations and four space-time translations. In the Poincaré group we must consider all possible orders of compositions of boosts **B**, rotations **R** and translations **T**. There are six possible permutations of these operators: **TBR**, **TRB**, **BTR**, **RTB**, **BRT**, **RBT**. Each of these six decompositions of a general transformation could be different. A preliminary understanding of the homogeneous group is therefore always useful before one starts the study of the non-homogeneous group. This is because the non-homogeneous group is the extension of the homogeneous group by translations. In this respect, the combinations **TBR**, **TRB**, **BRT**, **RBT** should be easier to work out than the combinations **BTR**, **RTB**. Already the decompositions **BR** and **RB** and the differences between them in the homogeneous Lorentz group are rather difficult to establish.

The metric for the vector  $(x, y, z, ct, u)$  will be now  $x^2 + y^2 + z^2 - c^2t^2$  (or alternatively  $c^2t^2 - x^2 - y^2 - z^2$ ). This corresponds to SO(3,1,1) (or alternatively SO(1,1,3)). The gamma matrices:

$$\gamma_x = \begin{bmatrix} & \sigma_x \\ \sigma_x & \end{bmatrix}, \quad \gamma_y = \begin{bmatrix} & \sigma_y \\ \sigma_y & \end{bmatrix}, \quad \gamma_z = \begin{bmatrix} & \sigma_z \\ \sigma_z & \end{bmatrix}, \quad \gamma_t = \begin{bmatrix} & \mathbb{1} \\ -\mathbb{1} & \end{bmatrix}, \quad \gamma_5 = \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix}, \quad (37)$$

for the signature  $+++-$ , or:

$$\gamma_x = \begin{bmatrix} & \sigma_x \\ -\sigma_x & \end{bmatrix}, \quad \gamma_y = \begin{bmatrix} & \sigma_y \\ -\sigma_y & \end{bmatrix}, \quad \gamma_z = \begin{bmatrix} & \sigma_z \\ -\sigma_z & \end{bmatrix}, \quad \gamma_t = \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix}, \quad \gamma_5 = \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix}, \quad (38)$$

for the signature  $--+$ , are not sufficient to build an isotropic vector, because we need two real vectors that are orthogonal to the four other ones, that is we need a treatment in  $\mathbb{R}^6$ . We must therefore use the  $8 \times 8$  Gamma matrices which are designed for treating homogeneous groups of isometries in  $\mathbb{R}^6$  or  $\mathbb{R}^7$ :

$$\Gamma_x = \begin{bmatrix} & \gamma_x \\ \gamma_x & \end{bmatrix}, \quad \Gamma_y = \begin{bmatrix} & \gamma_y \\ \gamma_y & \end{bmatrix}, \quad \Gamma_z = \begin{bmatrix} & \gamma_z \\ \gamma_z & \end{bmatrix}, \quad \Gamma_t = \begin{bmatrix} & \gamma_t \\ \gamma_t & \end{bmatrix}, \\ \Gamma_5 = \begin{bmatrix} & \gamma_5 \\ \gamma_5 & \end{bmatrix}, \quad \Gamma_6 = \begin{bmatrix} & -\imath\mathbb{1} \\ \imath\mathbb{1} & \end{bmatrix}, \quad \Gamma_7 = \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix}. \quad (39)$$

We will use Eq. 38 combined with  $\Gamma_6$  and  $\Gamma_7$  to construct  $\Gamma_u$  in order to keep the expressions analogous to those in SO(3,1,0):

$$\Gamma_u = \Gamma_7 + \imath\Gamma_6 = \begin{bmatrix} & \mathbb{1} \\ -\mathbb{1} & -\mathbb{1} \end{bmatrix}. \quad (40)$$

This is then again analogous to  $\sigma_z + \imath\sigma_y$ . The vector  $\mathbf{v} = (x, y, z, ct, u)$  is then given by the  $8 \times 8$  matrix:

$$\mathbf{V} = \begin{bmatrix} u\mathbb{1} & u\mathbb{1} + ct\gamma_t + \mathbf{r}\cdot\boldsymbol{\gamma} \\ -u\mathbb{1} + ct\gamma_t + \mathbf{r}\cdot\boldsymbol{\gamma} & -u\mathbb{1} \end{bmatrix} \quad (41)$$

The blocks in it are here  $4 \times 4$  matrices and the same is true for the matrices  $\mathbb{1}$  inside them. In  $\text{SO}(3,1,0)$  and also here it will now no longer be possible to use a  $2 \times 2$  representation because the isometries will not have two zero blocks. This is also obvious from the fact that a  $2 \times 2$  representation can never accommodate for the 10 required independent real parameters. We have explained previously in [8] how we can obtain a boost as a product of space-time reflections. In other words, each of the three fundamental operations, boosts, rotations and translations can be obtained as a product of two space-time reflections.

### 3.2 The product of two general reflections

$$\begin{aligned} \mathbf{BA} &= \begin{bmatrix} u_b\mathbb{1} & u_b\mathbb{1} + ct_b\gamma_t + \mathbf{b}\cdot\boldsymbol{\gamma} \\ -u_b\mathbb{1} + ct_b\gamma_t + \mathbf{b}\cdot\boldsymbol{\gamma} & -u_b\mathbb{1} \end{bmatrix} \begin{bmatrix} u_a\mathbb{1} & u_a\mathbb{1} + ct_a\gamma_t + \mathbf{a}\cdot\boldsymbol{\gamma} \\ -u_a\mathbb{1} + ct_a\gamma_t + \mathbf{a}\cdot\boldsymbol{\gamma} & -u_a\mathbb{1} \end{bmatrix} \\ &= (ct_b\gamma_t + \mathbf{b}\cdot\boldsymbol{\gamma})(ct_a\gamma_t + \mathbf{a}\cdot\boldsymbol{\gamma}) \cdot \mathbb{1} + [u_b(ct_a\gamma_t + \mathbf{a}\cdot\boldsymbol{\gamma}) - u_a(ct_b\gamma_t + \mathbf{b}\cdot\boldsymbol{\gamma})] \cdot \Gamma_u. \end{aligned} \quad (42)$$

This is of the form  $\mathbf{BA} \cdot \mathbb{1} + (u_b\mathbf{A} - u_a\mathbf{B}) \cdot \Gamma_u$ . The first part describes calculations in the homogeneous Lorentz group, the second part the algebraic additions due to the extension to the Poincaré group. The unit matrix which appears in the first part is an  $8 \times 8$ -matrix. The translations are not given by the second part:

$$[u_b(ct_a\gamma_t + \mathbf{a}\cdot\boldsymbol{\gamma}) - u_a(ct_b\gamma_t + \mathbf{b}\cdot\boldsymbol{\gamma})], \quad (43)$$

but by:

Case 1:

$$\mathbb{1} + [u_b(ct_a\gamma_t + \mathbf{a}\cdot\boldsymbol{\gamma}) - u_a(ct_b\gamma_t + \mathbf{b}\cdot\boldsymbol{\gamma})] \mathbf{L}^{-1}, \quad (44)$$

Case 2:

$$\mathbb{1} + \mathbf{L}^{-1} [u_b(ct_a\gamma_t + \mathbf{a}\cdot\boldsymbol{\gamma}) - u_a(ct_b\gamma_t + \mathbf{b}\cdot\boldsymbol{\gamma})]. \quad (45)$$

Here  $\mathbf{L} = (ct_b\gamma_t + \mathbf{b}\cdot\boldsymbol{\gamma})(ct_a\gamma_t + \mathbf{a}\cdot\boldsymbol{\gamma}) \cdot \mathbb{1}$  is a general homogeneous Lorentz transformation. This follows exactly the same logic and formalisms as in Eqs. 27 and 29. We will now note  $\mathbf{A} = ct_a\gamma_t + \mathbf{a}\cdot\boldsymbol{\gamma}$  and  $\mathbf{B} = ct_b\gamma_t + \mathbf{b}\cdot\boldsymbol{\gamma}$ . We have then:

$$\mathbb{1} + [u_b\mathbf{A} - u_a\mathbf{B}] \mathbf{AB} = \mathbb{1} + [u_b\mathbf{B} - u_a\mathbf{BAB}] = \mathbb{1} + \mathbf{B} [u_b\mathbf{B} - u_a\mathbf{A}] \mathbf{B}. \quad (46)$$

$$\mathbb{1} + \mathbf{AB} [u_b\mathbf{A} - u_a\mathbf{B}] = \mathbb{1} + [u_b\mathbf{ABA} - u_a\mathbf{A}] = \mathbb{1} + \mathbf{A} [u_b\mathbf{B} - u_a\mathbf{A}] \mathbf{A}. \quad (47)$$

We recover again the vector  $P_a P_b = u_b \mathbf{b} - u_a \mathbf{a} = u_b \mathbf{b} - u_a \mathbf{a}$  and its symmetrization. However, this analysis is only valid for the case that the transformation is obtained from two reflections. When there are more than two reflections, things become more complicated.

### 3.3 Catalogue of possible cases

We have the following possible combinations for  $(a_u, a_t, \mathbf{a})$  and  $(b_u, b_t, \mathbf{b})$ :

$(a_{1u}, a_t, \mathbf{a})$	$(a_{2u}, a_t, \mathbf{a})$	translation
$(0, \cosh(\chi_1/2), \sinh(\chi_1/2) \mathbf{a})$	$(0, 1, \mathbf{0})$	boost
$(0, 0, \mathbf{a})$	$(0, 0, \mathbf{b})$	rotation around the origin $O$
$(a_u, 0, \mathbf{a})$	$(b_u, 0, \mathbf{b})$	rotation around a general point $P \in \mathbb{R}^3$
$(0, a_t, \mathbf{a})$	$(0, b_t, \mathbf{b})$	boost and rotation
$(a_{1u}, \cosh(\chi_1/2), \sinh(\chi_1/2) \mathbf{a})$	$(a_{2u}, 1, \mathbf{0})$	boost + translation
$(a_u, a_t, \mathbf{a})$	$(b_u, b_t, \mathbf{b})$	boost and rotation + translation

It takes four reflections to generate a boost and a rotation around the same axis. We consider the first four cases in the table as well understood. We will discuss the fifth case within the framework of the homogeneous Lorentz group using  $\text{SL}(2, \mathbb{C})$  matrices. Then we will discuss the combinations that contain translations.

### 3.4 Composition of two non-collinear boosts

#### 3.4.1 Variant *RB*

Calculating the composition of general Lorentz transformations leads to considerable algebra. The subject matter has been covered exhaustively by Ungar [9], but he introduces notations and terminology that render his presentation somewhat less accessible. The present document has no pretension of presenting something new. It just wants to derive the results in a more readable format. One theorem is that the composition of two boosts amounts to a boost and a rotation. But when one tries to prove this, the algebra becomes so involved that certain authors have expressed the opinion that working out the algebra would be of forbidding complexity. Especially when we use  $4 \times 4$  matrices operating on  $4 \times 1$  four-vectors of  $\mathbb{R}^4$  it becomes difficult to figure out what the rotation and the boost are. As we will see one can do it within the representation based on Clifford algebra, but it takes some thought about the strategy to follow in performing the calculations, else one can get really lost in an algebraic swamp. Even then, the calculations are in any case extremely tedious and unpleasant. We will carry out the calculations within the formalism of  $SL(2, \mathbb{C})$ . As we use the letters  $\mathbf{B}$  to notate boosts and  $\mathbf{R}$  to notate rotations, one may in a broader context have to make provisions in order to avoid confusion with the notation  $\mathbf{B}$  for the magnetic field. The expression for a boost  $\mathbf{B}(\mathbf{v})$  with velocity  $\mathbf{v} = v\mathbf{u}$  in  $SL(2, \mathbb{C})$  is given by:

$$\mathbf{B}(\mathbf{v}) = \sqrt{\frac{\gamma+1}{2}} \mathbb{1} - \sqrt{\frac{\gamma-1}{2}} [\mathbf{u} \cdot \boldsymbol{\sigma}]. \quad (48)$$

See e.g. reference [8] for a derivation based on Clifford algebra and reference [10], p.355, Eq. C.5 for a more *ad hoc derivation*. Here  $\mathbf{u}$  is the unit vector parallel with  $\mathbf{v}$  and pointing in the same direction. This expression is derived from the observation that in  $SL(2, \mathbb{C})$  a  $2 \times 2$  matrix  $\mathbf{V}$  representing a four-vector  $v_\mu$  transforms under a homogeneous Lorentz transformation with  $2 \times 2$ -matrix  $\mathbf{L}$  according to  $\mathbf{V} \rightarrow \mathbf{LVL}^\dagger$ . Knowing this and the expression for the transformed four-vector  $v'_\mu$  one can find the expression for the  $2 \times 2$  matrix  $\mathbf{B}$ . The fact that the transformation in  $SL(2, \mathbb{C})$  is  $\mathbf{V} \rightarrow \mathbf{LVL}^\dagger$  rather than  $\mathbf{V} \rightarrow \mathbf{LVL}^{-1}$ , follows from Eq. (4.3) in [10] for the expression of the four-vector in the  $4 \times 4$  Cartan-Weyl representation. The  $4 \times 4$  matrix of the four-vector functions also as a reflection operator and the set of all transformations one can obtain from an even number of reflections is the homogeneous Lorentz group. The representations of the Lorentz transformations are block-diagonal and from all this one can see that the transformation is  $\mathbf{V} \rightarrow \mathbf{LVL}^{-1}$  in the four-dimensional representation, but  $\mathbf{V} \rightarrow \mathbf{LVL}^\dagger$  in the two two-dimensional representation (which boils down to one of the two blocks on the diagonal).

We will now calculate the composition of boosts using the  $SL(2, \mathbb{C})$  formalism. We must thus calculate:

$$\begin{aligned} \mathbf{B}(\mathbf{v}_2)\mathbf{B}(\mathbf{v}_1) &= \left[ \sqrt{\frac{\gamma_2+1}{2}} \mathbb{1} - \sqrt{\frac{\gamma_2-1}{2}} [\mathbf{u}_2 \cdot \boldsymbol{\sigma}] \right] \left[ \sqrt{\frac{\gamma_1+1}{2}} \mathbb{1} - \sqrt{\frac{\gamma_1-1}{2}} [\mathbf{u}_1 \cdot \boldsymbol{\sigma}] \right] = \\ &= \underbrace{\left[ \sqrt{\frac{\gamma_2+1}{2}} \sqrt{\frac{\gamma_1+1}{2}} + \sqrt{\frac{\gamma_2-1}{2}} \sqrt{\frac{\gamma_1-1}{2}} (\mathbf{u}_1 \cdot \mathbf{u}_2) \right] \mathbb{1}}_{\textcircled{1}} - \underbrace{\sqrt{\frac{\gamma_1+1}{2}} \sqrt{\frac{\gamma_2-1}{2}} [\mathbf{u}_2 \cdot \boldsymbol{\sigma}]}_{\textcircled{2}} - \underbrace{\sqrt{\frac{\gamma_2+1}{2}} \sqrt{\frac{\gamma_1-1}{2}} [\mathbf{u}_1 \cdot \boldsymbol{\sigma}]}_{\textcircled{3}} - \underbrace{\sqrt{\frac{\gamma_2-1}{2}} \sqrt{\frac{\gamma_1-1}{2}} [(\mathbf{u}_1 \wedge \mathbf{u}_2) \cdot \boldsymbol{\sigma}]}_{\textcircled{4}}. \end{aligned} \quad (49)$$

Here  $\mathbf{u}_1 = \mathbf{v}_1/v_1$ ,  $\mathbf{u}_2 = \mathbf{v}_2/v_2$ . We note the unit vector perpendicular to the plane defined by  $\mathbf{u}_1$  and  $\mathbf{u}_2$  as  $\boldsymbol{\xi}$ . The angle between  $\mathbf{u}_1$  and  $\mathbf{u}_2$  will be called  $\theta$ , such that:  $\mathbf{u}_1 \cdot \mathbf{u}_2 = \cos \theta$  and  $\mathbf{u}_1 \wedge \mathbf{u}_2 = \sin \theta \boldsymbol{\xi}$ . The product of two boosts in Eq. 49 is in principle the product of a rotation and a boost. We must thus identify Eq. 49 with:

$$\begin{aligned} & \left[ \cos(\alpha/2) \mathbb{1} - \imath \sin(\alpha/2) [\boldsymbol{\xi} \cdot \boldsymbol{\sigma}] \right] \left[ \sqrt{\frac{\gamma+1}{2}} \mathbb{1} - \sqrt{\frac{\gamma-1}{2}} [\mathbf{u} \cdot \boldsymbol{\sigma}] \right] = \\ & \underbrace{\cos(\alpha/2) \sqrt{\frac{\gamma+1}{2}} \mathbb{1}}_{\textcircled{5}} - \underbrace{\cos(\alpha/2) \sqrt{\frac{\gamma-1}{2}} [\mathbf{u} \cdot \boldsymbol{\sigma}]}_{\textcircled{6}} - \underbrace{\imath \sin(\alpha/2) \sqrt{\frac{\gamma+1}{2}} [\boldsymbol{\xi} \cdot \boldsymbol{\sigma}]}_{\textcircled{7}} - \underbrace{\sin(\alpha/2) \sqrt{\frac{\gamma-1}{2}} [(\boldsymbol{\xi} \wedge \mathbf{u}) \cdot \boldsymbol{\sigma}]}_{\textcircled{8}} \end{aligned} \quad (50)$$

The rotation is indeed expected to happen in the plane defined by the two boost vectors. The rotation axis is thus the axis defined by  $\boldsymbol{\xi}$ . The rotation angle is called  $\alpha$ . By identifying the term  $\textcircled{7}$  in Eq 50 with the term  $\textcircled{4}$  in Eq. 49 we obtain:

$$\sin(\alpha/2) \sqrt{\frac{\gamma+1}{2}} = \sin \theta \sqrt{\frac{\gamma_2-1}{2}} \sqrt{\frac{\gamma_1-1}{2}}. \quad (51)$$

By identifying the term ⑤ with ① we obtain:

$$\cos(\alpha/2)\sqrt{\frac{\gamma+1}{2}} = \sqrt{\frac{\gamma_2+1}{2}}\sqrt{\frac{\gamma_1+1}{2}} + \sqrt{\frac{\gamma_2-1}{2}}\sqrt{\frac{\gamma_1-1}{2}} \cos \theta. \quad (52)$$

Squaring Eqs. 51 and 52 and subsequently summing them yields:

$$\frac{\gamma+1}{2} = \frac{\gamma_2-1}{2}\frac{\gamma_1-1}{2} + \frac{\gamma_2+1}{2}\frac{\gamma_1+1}{2} + \frac{1}{2}\sqrt{(\gamma_2^2-1)(\gamma_1^2-1)} \cos \theta, \quad (53)$$

or:

$$\gamma+1 = \gamma_1\gamma_2 + 1 + \sqrt{(\gamma_2^2-1)(\gamma_1^2-1)} \cos \theta. \quad (54)$$

We will regularly use the following identity:

$$\sqrt{\gamma^2-1} = \frac{\gamma v}{c}. \quad (55)$$

such that:

$$\boxed{\gamma = \gamma_1\gamma_2\left(1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right)} \quad (56)$$

We have then also:

$$\gamma-1 = \gamma_1\gamma_2\left(1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right) - 1, \quad \gamma+1 = \gamma_1\gamma_2\left(1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right) + 1. \quad (57)$$

Eq. 51 yields then:

$$\boxed{\sin(\alpha/2) = \sin \theta \sqrt{\frac{(\gamma_1-1)(\gamma_2-1)}{2\left[1 + \gamma_1\gamma_2\left(1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right)\right]}}} \quad (58)$$

On the other hand Eq. 52 yields:

$$\boxed{\cos(\alpha/2) = \sqrt{\frac{(\gamma_2+1)(\gamma_1+1)}{2\left[1 + \gamma_1\gamma_2\left(1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right)\right]}} + \sqrt{\frac{(\gamma_2-1)(\gamma_1-1)}{2\left[1 + \gamma_1\gamma_2\left(1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right)\right]}} \cos \theta} \quad (59)$$

We must still identify ② + ③ with ⑥ + ⑧. But this leads to very tedious calculations. Therefore we rather consider to obtain the result for  $\mathbf{u}$  from identifying:

$$\left[ \cos(\alpha/2)\mathbf{1} + \imath \sin(\alpha/2)[\boldsymbol{\xi} \cdot \boldsymbol{\sigma}] \right] \left[ \cos(\alpha/2)\mathbf{1} - \imath \sin(\alpha/2)[\boldsymbol{\xi} \cdot \boldsymbol{\sigma}] \right] \left[ \sqrt{\frac{\gamma+1}{2}}\mathbf{1} - \sqrt{\frac{\gamma-1}{2}}[\mathbf{u} \cdot \boldsymbol{\sigma}] \right] = \sqrt{\frac{\gamma+1}{2}}\mathbf{1} - \sqrt{\frac{\gamma-1}{2}}[\mathbf{u} \cdot \boldsymbol{\sigma}]. \quad (60)$$

It is convenient to keep for the moment the expression  $\sqrt{\frac{\gamma+1}{2}} \left[ \cos(\alpha/2)\mathbf{1} + \imath \sin(\alpha/2)[\boldsymbol{\xi} \cdot \boldsymbol{\sigma}] \right]$  in the calculation. In other words, we multiply both sides of Eq. 60 by  $\sqrt{\frac{\gamma+1}{2}}$ . This leads to:

$$\left[ \underbrace{\left[ \sqrt{\frac{\gamma_2+1}{2}}\sqrt{\frac{\gamma_1+1}{2}} + \sqrt{\frac{\gamma_2-1}{2}}\sqrt{\frac{\gamma_1-1}{2}} \cos \theta \right]}_{\textcircled{1}} \mathbf{1} + \underbrace{\left[ \sqrt{\frac{\gamma_2+1}{2}}\sqrt{\frac{\gamma_1+1}{2}} + \sqrt{\frac{\gamma_2-1}{2}}\sqrt{\frac{\gamma_1-1}{2}} \cos \theta \right]}_{\textcircled{5}} \mathbf{1} + \underbrace{\imath \sin \theta \sqrt{\frac{\gamma_2-1}{2}}\sqrt{\frac{\gamma_1-1}{2}}}_{\textcircled{6}} [\boldsymbol{\xi} \cdot \boldsymbol{\sigma}] \right] \times \left[ \underbrace{\left[ \sqrt{\frac{\gamma_2+1}{2}}\sqrt{\frac{\gamma_1+1}{2}} + \sqrt{\frac{\gamma_2-1}{2}}\sqrt{\frac{\gamma_1-1}{2}} \cos \theta \right]}_{\textcircled{1}} \mathbf{1} - \underbrace{\left[ \sqrt{\frac{\gamma_2+1}{2}}\sqrt{\frac{\gamma_1+1}{2}} + \sqrt{\frac{\gamma_2-1}{2}}\sqrt{\frac{\gamma_1-1}{2}} \cos \theta \right]}_{\textcircled{2}} [\mathbf{u}_2 \cdot \boldsymbol{\sigma}] - \underbrace{\left[ \sqrt{\frac{\gamma_2+1}{2}}\sqrt{\frac{\gamma_1+1}{2}} + \sqrt{\frac{\gamma_2-1}{2}}\sqrt{\frac{\gamma_1-1}{2}} \cos \theta \right]}_{\textcircled{3}} [\mathbf{u}_1 \cdot \boldsymbol{\sigma}] - \underbrace{\imath \sin \theta \sqrt{\frac{\gamma_2-1}{2}}\sqrt{\frac{\gamma_1-1}{2}}}_{\textcircled{4}} [\boldsymbol{\xi} \cdot \boldsymbol{\sigma}] \right] \quad (61)$$

This has now to be identified to:  $\frac{\gamma+1}{2}\mathbf{1} - \sqrt{\frac{\gamma^2-1}{4}}[\mathbf{u} \cdot \boldsymbol{\sigma}]$ . Working out the terms yields:

(5 + 6)(1 + 4)	$\left[ \sqrt{\frac{\gamma_2+1}{2}} \sqrt{\frac{\gamma_1+1}{2}} + \sqrt{\frac{\gamma_2-1}{2}} \sqrt{\frac{\gamma_1-1}{2}} \cos \theta \right]^2 + \left[ \sin \theta \sqrt{\frac{\gamma_2-1}{2}} \sqrt{\frac{\gamma_1-1}{2}} \right]^2 = \frac{\gamma_2+1}{2} \frac{\gamma_1+1}{2} + \frac{\gamma_2-1}{2} \frac{\gamma_1-1}{2} + 2\sqrt{\frac{\gamma_2-1}{4}} \sqrt{\frac{\gamma_1-1}{4}} \cos \theta$	(62)
6 2	$\frac{\gamma_2-1}{2} \sqrt{\frac{\gamma_1-1}{4}} \sin \theta [(\boldsymbol{\xi} \wedge \mathbf{u}_2) \cdot \boldsymbol{\sigma}]$	
6 3	$\frac{\gamma_1-1}{2} \sqrt{\frac{\gamma_2-1}{4}} \sin \theta [(\boldsymbol{\xi} \wedge \mathbf{u}_1) \cdot \boldsymbol{\sigma}]$	
5 2	$[\sqrt{\frac{\gamma_2+1}{2}} \sqrt{\frac{\gamma_1+1}{2}} + \sqrt{\frac{\gamma_2-1}{2}} \sqrt{\frac{\gamma_1-1}{2}} \cos \theta] [-\sqrt{\frac{\gamma_1+1}{2}} \sqrt{\frac{\gamma_2-1}{2}} [\mathbf{u}_2 \cdot \boldsymbol{\sigma}]] = -[\frac{\gamma_1+1}{2} \sqrt{\frac{\gamma_2-1}{4}} + \frac{\gamma_2-1}{2} \sqrt{\frac{\gamma_1-1}{4}} \cos \theta] [\mathbf{u}_2 \cdot \boldsymbol{\sigma}]$	
5 3	$[\sqrt{\frac{\gamma_2+1}{2}} \sqrt{\frac{\gamma_1+1}{2}} + \sqrt{\frac{\gamma_2-1}{2}} \sqrt{\frac{\gamma_1-1}{2}} \cos \theta] [-\sqrt{\frac{\gamma_2+1}{2}} \sqrt{\frac{\gamma_1-1}{2}} [\mathbf{u}_1 \cdot \boldsymbol{\sigma}]] = -[\frac{\gamma_2+1}{2} \sqrt{\frac{\gamma_1-1}{4}} + \frac{\gamma_1-1}{2} \sqrt{\frac{\gamma_2-1}{4}} \cos \theta] [\mathbf{u}_1 \cdot \boldsymbol{\sigma}]$	

We will express all quantities in the basis of  $\mathbf{u}_1$  and  $(\boldsymbol{\xi} \wedge \mathbf{u}_1)$ :

$$\mathbf{u}_2 = \cos \theta \mathbf{u}_1 + \sin \theta (\boldsymbol{\xi} \wedge \mathbf{u}_1), \quad \boldsymbol{\xi} \wedge \mathbf{u}_2 = \cos \theta (\boldsymbol{\xi} \wedge \mathbf{u}_1) - \sin \theta \mathbf{u}_1. \quad (63)$$

We can then regroup the terms as follows:

$\mathbb{1}$	$\frac{1}{2} [1 + \gamma_1 \gamma_2 (1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2})]$	(64)
$[\mathbf{u}_1 \cdot \boldsymbol{\sigma}]$	$-\left[ \frac{\gamma_2+1}{2} \sqrt{\frac{\gamma_1-1}{4}} + \frac{\gamma_1-1}{2} \sqrt{\frac{\gamma_2-1}{4}} \cos \theta \right] - \left[ \frac{\gamma_1+1}{2} \sqrt{\frac{\gamma_2-1}{4}} + \frac{\gamma_2-1}{2} \sqrt{\frac{\gamma_1-1}{4}} \cos \theta \right] \cos \theta - \frac{\gamma_2-1}{2} \sqrt{\frac{\gamma_1-1}{4}} \sin^2 \theta$	
$[(\boldsymbol{\xi} \wedge \mathbf{u}_1) \cdot \boldsymbol{\sigma}]$	$\frac{\gamma_1-1}{2} \sqrt{\frac{\gamma_2-1}{4}} \sin \theta - \left[ \frac{\gamma_1+1}{2} \sqrt{\frac{\gamma_2-1}{4}} + \frac{\gamma_2-1}{2} \sqrt{\frac{\gamma_1-1}{4}} \cos \theta \right] \sin \theta + \frac{\gamma_2-1}{2} \sqrt{\frac{\gamma_1-1}{4}} \sin \theta \cos \theta$	

After simplification this yields:

$\mathbb{1}$	$\frac{1}{2} [1 + \gamma_1 \gamma_2 (1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2})]$	(65)
$[\mathbf{u}_1 \cdot \boldsymbol{\sigma}]$	$-\frac{\gamma_2 \gamma_1}{2c} [(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{u}_1]$	
$[(\mathbf{u}_1 \wedge \boldsymbol{\xi}) \cdot \boldsymbol{\sigma}]$	$\frac{\gamma_2}{2c} v_2 \sin \theta$	

We remember that the identification we have to make is:

$$\frac{1}{2} [1 + \gamma_1 \gamma_2 (1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2})] \mathbb{1} - \frac{\gamma_2 \gamma_1}{2c} [(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{u}_1] [\mathbf{u}_1 \cdot \boldsymbol{\sigma}] + \frac{\gamma_2}{2c} [(\mathbf{u}_1 \wedge (\mathbf{u}_1 \wedge \mathbf{v}_2)) \cdot \boldsymbol{\sigma}] = \frac{\gamma+1}{2} \mathbb{1} - \frac{\gamma}{2c} [\mathbf{v} \cdot \boldsymbol{\sigma}] \quad (66)$$

where we recover the expression for  $\frac{\gamma+1}{2}$  and obtain the value for  $\mathbf{v}$  in implicit form:

$$\boxed{\gamma \mathbf{v} = \gamma_2 \gamma_1 [(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{u}_1] \mathbf{u}_1 - \gamma_2 (\mathbf{u}_1 \wedge (\mathbf{u}_1 \wedge \mathbf{v}_2))} \quad (67)$$

The algebra necessary to obtain the explicit form is really technical and uninspiring. To calculate the explicit expression for  $\mathbf{v}$  we can use the identity:

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}. \quad (68)$$

to obtain:

$$\mathbf{u}_1 \wedge (\mathbf{u}_1 \wedge \mathbf{v}_2) = (\mathbf{u}_1 \cdot \mathbf{v}_2) \mathbf{u}_1 - (\mathbf{u}_1 \cdot \mathbf{u}_1) \mathbf{v}_2 = v_{2\parallel} - \mathbf{v}_2 = -\mathbf{v}_{2\perp}. \quad (69)$$

Therefore:

$$\mathbf{v} = \frac{\gamma_1 \gamma_2 (\mathbf{v}_1 + \mathbf{v}_{2\parallel}) + \gamma_2 \mathbf{v}_{2\perp}}{\gamma_1 \gamma_2 (1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2})} = \frac{\mathbf{v}_1 + \mathbf{v}_{2\parallel}}{(1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2})} + \frac{\mathbf{v}_{2\perp}}{\gamma_1 (1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2})} = \frac{\mathbf{v}_1 + \mathbf{v}_{2\parallel}}{1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}} + \frac{\mathbf{v}_{2\perp}}{1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}} - \frac{\mathbf{v}_{2\perp}}{1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}} + \frac{\mathbf{v}_{2\perp}}{\gamma_1 (1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2})}, \quad (70)$$

such that:

$$\mathbf{v} = \frac{\mathbf{v}_1 + \mathbf{v}_2}{\left(1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right)} - \frac{\gamma_1 - 1}{\gamma_1} \frac{\mathbf{v}_{2\perp}}{1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}}. \quad (71)$$

Using:

$$\frac{\gamma_1 - 1}{\gamma_1} = \frac{\gamma_1^2 - 1}{(\gamma_1 + 1)\gamma_1} = \frac{1}{(\gamma_1 + 1)\gamma_1} \left( \frac{1}{1 - v_1^2/c^2} - 1 \right) = \frac{v_1^2}{c^2} \frac{\gamma_1}{\gamma_1 + 1}, \quad (72)$$

and re-injecting Eq. 69 we obtain then:

$$\mathbf{v} = \frac{\mathbf{v}_1 + \mathbf{v}_2}{1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}} - \frac{\gamma_1 - 1}{\gamma_1} \frac{\mathbf{v}_{2\perp}}{1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}} = \frac{\mathbf{v}_1 + \mathbf{v}_2}{\left(1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right)} + \frac{v_1^2}{c^2} \frac{\gamma_1}{\gamma_1 + 1} \frac{\mathbf{u}_1 \wedge (\mathbf{u}_1 \wedge \mathbf{v}_2)}{1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}}. \quad (73)$$

By recombining  $v_1 \mathbf{u}_1 = \mathbf{v}_1$  twice this becomes finally:

$$\boxed{\mathbf{v} = \frac{\mathbf{v}_1 + \mathbf{v}_2}{1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}} + \frac{1}{c^2} \frac{\gamma_1}{\gamma_1 + 1} \frac{\mathbf{v}_1 \wedge (\mathbf{v}_1 \wedge \mathbf{v}_2)}{1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}}.} \quad (74)$$

The forest of calculations in Eqs. 68 to 73 is particularly lacking any form of elegance. Of course we could have given the final result immediately with the kind of laconic comment we all have come across, and all have cursed, because it implies that the author is pulling your leg and that you have to spoil your precious time and energy in doing his tedious calculations all over. We have preferred to spare everybody the ordeal by reproducing the details. The final result is well-known. It is e.g. also mentioned on a website presumably written by Ungar in Wikipedia [11]. His calculations are based on the idea of decomposing the velocities in parallel and perpendicular components, which is thus quite a different angle of approach. Our derivation is more powerful because we have simultaneously calculated the rotation.

### 3.4.2 Variant $BR$

To calculate this variant we will not make the full calculations again. Enough is enough. We will rather use the previous results Eqs. 49 and 50, take the Hermitian conjugate of the expressions and substitute  $(\mathbf{v}_1, \mathbf{v}_2)|(\mathbf{v}_2, \mathbf{v}_1)$ . We have:

$$\mathbf{B}(\mathbf{v}_2) \mathbf{B}(\mathbf{v}_1) = \mathbf{R}(\alpha) \mathbf{B}(\mathbf{v}). \quad (75)$$

By Hermitian conjugation we obtain:

$$\mathbf{B}(\mathbf{v}_1) \mathbf{B}(\mathbf{v}_2) = \mathbf{B}(\mathbf{v}) \mathbf{R}(-\alpha) \quad (76)$$

because for boosts  $\mathbf{B}^\dagger = \mathbf{B}$  while for rotations  $\mathbf{R}^\dagger = \mathbf{R}^{-1}$ , and  $\mathbf{R}^{-1}(\alpha) = \mathbf{R}(-\alpha)$ . The substitution  $(\mathbf{v}_1, \mathbf{v}_2)|(\mathbf{v}_2, \mathbf{v}_1)$  transforms  $\theta$  into  $-\theta$  and therefore  $\alpha$  into  $-\alpha$ . It also changes  $\mathbf{v}$  in Eq. 74 into another boost velocity  $\mathbf{w}$ . We obtain then:

$$\mathbf{B}(\mathbf{v}_2) \mathbf{B}(\mathbf{v}_1) = \mathbf{B}(\mathbf{w}) \mathbf{R}(\alpha), \quad (77)$$

where:

$$\boxed{\mathbf{w} = \frac{\mathbf{v}_1 + \mathbf{v}_2}{1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}} - \frac{1}{c^2} \frac{\gamma_2}{\gamma_2 + 1} \frac{\mathbf{v}_2 \wedge (\mathbf{v}_1 \wedge \mathbf{v}_2)}{1 + \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}}.} \quad (78)$$

is obtained by performing the substitution  $(\mathbf{v}_1, \mathbf{v}_2)|(\mathbf{v}_2, \mathbf{v}_1)$  on Eq. 74.

### 3.5 Boost and translation

We will here adopt the notations from [8]. We have then for  $\mathbf{A}_2 \mathbf{A}_1$ :

$$\left[ \begin{array}{cc} u_2 \mathbf{1} & u_2 \mathbf{1} + \cosh(\chi/2) \gamma_t + \sinh(\chi/2) [\mathbf{u} \cdot \boldsymbol{\gamma}] \\ -u_2 \mathbf{1} + \cosh(\chi/2) \gamma_t + \sinh(\chi/2) [\mathbf{u} \cdot \boldsymbol{\gamma}] & -u_2 \mathbf{1} \end{array} \right] \left[ \begin{array}{cc} u_1 \mathbf{1} & u_1 \mathbf{1} + \gamma_t \\ -u_1 \mathbf{1} + \gamma_t & -u_1 \mathbf{1} \end{array} \right]$$

$$= (\cosh(\chi/2)) \gamma_t + \sinh(\chi/2) [\mathbf{u} \cdot \boldsymbol{\gamma}] \cdot \mathbf{1} + \left[ u_2 \gamma_t - u_1 [\cosh(\chi/2)) \gamma_t + \sinh(\chi/2) [\mathbf{u} \cdot \boldsymbol{\gamma}]] \right] \cdot \Gamma_u. \quad (79)$$

This is of the form  $(\mathbf{A}_2 \mathbf{A}_1) \cdot \mathbf{1} + (u_2 \mathbf{A}_1 - u_1 \mathbf{A}_2) \cdot \Gamma_u$ . Here  $\mathbf{A}_2$  and  $\mathbf{A}_1$  are reflections, which we have not written as  $\mathbf{A}$  and  $\mathbf{B}$  in order to avoid confusion with a boost  $\mathbf{B}$ . This corresponds to Eq. 42. The quantity  $(\mathbf{A}_2 \mathbf{A}_1) \cdot \mathbf{1}$  corresponds to a boost. The second part is as usual meaningless. It is this way only possible to obtain a translation in the direction of the boost, because  $P_1 P_2 = u_2 \mathbf{A}_2 - u_1 \mathbf{A}_1 \parallel \mathbf{u}$ . For more general translations, we need more than two reflections, or alternatively a product of a boost  $\mathbf{B}$  and a translation  $\mathbf{T}$ . We can then study how  $\mathbf{T}_1 \mathbf{B}$  relates to  $\mathbf{B} \mathbf{T}_2$ .

### 3.6 Boost, rotation and translation

This is the most general case. We can obtain it by using  $(u_a, ct_a, \mathbf{a})$  and  $(u_b, ct_b, \mathbf{b})$ , with  $\mathbf{a} \not\parallel \mathbf{b}$ ,  $ct_a \neq ct_b$  and  $u_a \neq u_b$ . One may then want to write the result under the form of a product that is one of the six permutations of the six possible products  $\mathbf{TBR}$ . This could lead to very involved calculations for which those in Subsection 3.4 might be only a small appetizer, a disheartening perspective. Remember that the complexity of the calculations in Subsection 3.4 was already deemed as almost unassailable by many authors as mentioned by Ungar [9]. We were lucky to able to simplify them by taking them on in a  $2 \times 2$  formalism. Now we would be obliged to use the  $8 \times 8$  formalism for the Poincaré group. (under the form of  $2 \times 2$  block structure). The strategy to obtain the decompositions would be the same as followed in Subsection 2.6.3. We also followed this strategy in Subsection 3.4. We have not worked out all cases, such that our presentation is not complete, but we still hope it will be of some utility for the reader.

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