Approximate Lipschitz stability for phaseless inverse scattering with background information

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Abstract

We prove approximate Lipschitz stability for monochromatic phaseless inverse scattering with background information in dimension $d \geq 2$. Moreover, these stability estimates are given in terms of non-overdetermined and incomplete data. Related results for reconstruction from phaseless Fourier transforms are also given. Prototypes of these estimates for the phased case were given in Novikov (2013 J. Inverse Ill-Posed Problems, 21, 813-823).

Keywords: Schrödinger equation, phaseless inverse scattering, phaseless Fourier transform, approximate Lipschitz stability, monochromatic and non-overdetermined data.

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1 Introduction

In this work we continue studies on phaseless inverse scattering for the stationary Schrödinger equation:

$$-\Delta\psi + V(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad E > 0, \tag{1.1}$$

where

$$V \in L^\infty(\mathbb{R}^d), \text{ and is compactly supported.} \tag{1.2}$$

For equation (1.1), under conditions (1.2), we consider the scattering solutions $\psi^+ = \psi^+(x,k)$, $k \in \mathbb{R}^d$, $k^2 = E$, such that

$$\psi^+(x,k) = e^{ikx} + \psi^{\rm sc}(x,k); \tag{1.3}$$

$$|x|^{(d-1)/2}\left(\frac{\partial}{\partial|x|} - i|k|\right)\psi^{\rm sc}(x,k) \to 0 \ \text{ as } |x| \to +\infty, \tag{1.4}$$

uniformly in $x/|x|$. In particular, we have that

$$\psi^{\rm sc}(x,k) = \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} c(d,|k|) f\left(k, |k|\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{(d+1)/2}}\right) \quad \text{as } |x| \to +\infty, \tag{1.5}$$

$$c(d,|k|) := -\pi i(-2\pi i)^{(d-1)/2}|k|^{(d-3)/2}, \text{ for } \sqrt{-2\pi i} = \sqrt{2\pi}e^{-i\pi/4}. \tag{1.6}$$

The coefficient $f = f[V]$ arising in (1.5) is known as the scattering amplitude for equation (1.1). In turn, $|f|^2$ is known as the differential scattering cross section for equation (1.1).

Note that f is defined on

$$\mathcal{M}_E = \{k, l \in \mathbb{R}^d : \ \ k^2 = l^2 = E\} = \mathbb{S}^{d-1}_{\sqrt{E}} \times \mathbb{S}^{d-1}_{\sqrt{E}}. \tag{1.7}$$

We recall that in quantum mechanics complex values of ψ^+ and f have no direct physical sense, whereas $|\psi^+|^2$ and $|f|^2$ admit probabilistic intepritations and can be measured (Born principle going back to [5]). In particular, $|f(k,l)|^2$ describes the probability density of scattering of the quantum mechanical particle with initial impulse k into direction $l/|l| \neq k/|k|$.

We consider the following monochromatic phaseless inverse scattering problem for equation (1.1) under assumptions (1.2):

Problem 1. *(A) Reconstruct a compactly supported potential v from the differential scattering cross section $|f[v+w]|^2$ given on some appropriate $\mathcal{M}' \subseteq \mathcal{M}_E$ for some known compactly supported background potential w sufficiently separated from v.*

(B) Reconstruct a compactly supported potential v from the differential scattering cross sections $|f[v]|^2$, $|f[v+w_1]|^2$, ..., $|f[v+w_n]|^2$ given on some appropriate $\mathcal{M}' \subseteq \mathcal{M}_E$ for some known compactly supported background potentials $w_1, ..., w_n$ sufficiently separated from v.

Actually, in Problem 1(A) we consinder the Schrödinger equation (1.1) with $V = v + w$, while in Problem 1(B) we consider $n+1$ Schrödinger equations (1.1) for $V = v + w_j$, $j = 1, ..., n$, and $V = v$.

Approximate reconstruction for Problem 1(A) in dimension $d \geq 2$ was developed in [20], [9].

Approximate reconstruction for Problem 1(B) in dimension $d \geq 2$ was developed, in particular, in [1], [2], [9], [18, 19], [20].

We also consider Problems 1(A), 1(B) in the Born approximation, when the phaseless scattering data are reduced to the phaseless Fourier transforms. In this respect, we continue, in particular, studies of [20, 9].

In the present work, we give the first approximate stability results for Problem 1(A) and Problem 1(B) for $n = 1$. Related results for reconstruction from phaseless Fourier transforms are also given. Prototypes of these estimates for the phased case were given in [16].

In addition to Problem 1, there are also other phaseless inverse problems for equation (1.1) and for related equations; see, for example, [3], [4], [7], [8], [10], [11], [12], [13], [14], [15], [21], [22], [23], [24], and references therein.

In particular, in connection with phaseless inverse scattering with background information for equation (1.1) at positive energies E for $d = 1$, we refer to [3] and references therein.

The main results of the present work are formulated in Section 2; see Theorems 2.1 and 2.2, Propositions 2.5 and 2.6. Preliminary results required for the proof of these theorems are given in Section 3. Theorems 2.1 and 2.2 are proved in Section 4. Propositions 2.5 and 2.6 are proved in Section 5.

2 Main results

We assume that:

$$v_1,\, v_2 \in L^\infty(\mathbb{R}^d), \quad d \geq 2, \tag{2.1a}$$

$$supp\, v_1,\, supp\, v_2 \subseteq D,\, supp\, w \subseteq \Omega, \tag{2.1b}$$

$$D,\, \Omega \text{ are open convex bounded domains}, D \cap \Omega = \varnothing. \tag{2.1c}$$

We also assume that:

$$v_1 - v_2 \in W^{m,1}(\mathbb{R}^d) \text{ for some } m > d, \tag{2.2}$$

$$\|v_j\|_\infty \leq N_1,\, j = 1, 2;\, \|v_1 - v_2\|_{m,1} \leq N_2, \tag{2.3}$$

where

$$W^{m,1}(\mathbb{R}^d) = \{u : \partial^J u \in L^1(\mathbb{R}^d),\, |J| \leq m\},$$
$$\|u\|_{m,1} = \max_{|J| \leq m} \|\partial^J u\|_{L^1(\mathbb{R}^d)}, \tag{2.4}$$

$$\|u\|_\infty = \|u\|_{L^\infty(\mathbb{R}^d)}. \tag{2.5}$$

Finally, we assume that:

$$\|w\|_\infty \leq N_1, \quad \widehat{w}(p) \geq c_1(1+|p|)^{-\beta}, \quad p \in \mathbb{R}^d, \tag{2.6}$$

for some $c_1 > 0$ and $\beta > d$. Here $\widehat{w}$ is Fourier transform of w, defined by

$$\widehat{u}(p) = \mathcal{F}u(p) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ip\cdot x} u(x) dx. \tag{2.7}$$

For examples of w satisfying (2.6), see [2], [25].

In addition to $\mathcal{M}_E$ defined by (1.7), we also consider its subsets $\Gamma_E^\tau \subset \mathcal{M}_E$, for $\tau \in (0, 1)$:

$$\Gamma_E^\tau = \left\{k = k_E(p),\, l = l_E(p)\, : p \in B_{2\tau\sqrt{E}}\right\},$$
$$k_E(p) = p/2 + (E - p^2/4)^{1/2}\gamma(p),\; l_E(p) = -p/2 + (E - p^2/4)^{1/2}\gamma(p), \tag{2.8}$$

where γ is a piecewise continuous vector–function on $\mathbb{R}^d$, $d \geq 2$, such that

$$|\gamma(p)| = 1,\, \gamma(p)p = 0,\; p \in \mathbb{R}^d. \tag{2.9}$$

Let $C(\mathcal{M}_E)$ denote continuous functions on $\mathcal{M}_E$, and $C(\Gamma_E^\tau)$ denotes their restrictions on Γ_E^τ.

Theorem 2.1. *Let functions v_1, v_2 satisfy assumptions* (2.1)–(2.6)*, and* $\operatorname{dist}(D,\Omega) > \operatorname{diam} D$*. Then, for any* $\varepsilon \in (0,1/2)$*,*

$$\|v_1 - v_2\|_{L^\infty(D)} \le C_1 E^{\frac{1}{2}-\varepsilon} \| |f[v_1+w]|^2 - |f[v_2+w]|^2 \|_{C(\Gamma_E^\tau)} + C_2 E^{-(\frac{1}{2}-\varepsilon)\frac{m-d}{\beta+d}}, \tag{2.10}$$

for $E \ge E_1 = E_1(D, N_1, \Omega, \beta, c_1, \tau, \varepsilon)$*, where* $\tau \in (0,1)$*,* E_1 *is defined in* (4.29)*,* $C_1 = 2K_2K_3$*,* $C_2 = 2K_1$*, and constants* $K_1 = \widetilde{K}_1(d,m)N_2$*,* $K_2 = c_1^{-1}\widetilde{K}_2(d,\beta,\varepsilon)$*,* $K_3 = K_3(d,\beta)$ *are defined in* (4.25)*,* (4.26)*.*

Theorem 2.2. *Let functions v_1, v_2 satisfy assumptions* (2.1)–(2.6)*. Then, for any* $\varepsilon \in (0,1/2)$*,*

$$\|v_1 - v_2\|_{L^\infty(D)} \le C_1 E^{\frac{1}{2}-\varepsilon} \| (|f[v_1+w]|^2 - |f[v_2+w]|^2) - (|f[v_1]|^2 - |f[v_2]|^2) \|_{C(\Gamma_E^\tau)} + C_2 E^{-(\frac{1}{2}-\varepsilon)\frac{m-d}{\beta+d}}, \tag{2.11}$$

for $E \ge E_2 = E_2(D, N_1, \Omega, \beta, c_1, \tau, \varepsilon)$*, where* $\tau \in (0,1)$*,* E_2 *is defined in* (4.39)*,* $C_1 = 2K_2K_3$*,* $C_2 = 2K_1$*, and constants* $K_1 = \widetilde{K}_1(d,m)N_2$*,* $K_2 = c_1^{-1}\widetilde{K}_2(d,\beta,\varepsilon)$*,* $K_3 = (d,\beta,\alpha)$ *are defined in* (4.25)*,* (4.26)*.*

Theorems 2.1 and 2.2 are proved in Section 4.

One can see that Theorem 2.1 is a stability result to Problem 1(A), while Theorem 2.2 is a stability result to Problem 1(B) for $n = 1$.

Remark 2.3. *For the phased case, the prototype of* (2.10)*,* (2.11) *is as follows:*

$$\|v_1 - v_2\|_{L^\infty(D)} \le A_1 E^{\frac{1}{2}} \|f_1 - f_2\|_{C(\Gamma_E^\tau)} + A_2 E^{-\frac{1}{2}\frac{m-d}{d}}, \tag{2.12}$$

where $\tau\sqrt{E} = \tau(E)\sqrt{E} = E^{1/(2d)}$*,* $A_1 = A_1(N_1, D)$*,* $A_2 = A_2(N_1, N_2, D, m)$*; see [16].*

Following [16], we say that (2.10), (2.11), (2.12) are approximate Lipschitz stability esimates.

One can see that the right hand sides of (2.10), (2.11), (2.12) are sums of two terms. The first one is Lipschitz term with respect to data difference, and the second one is approximate but decaying for high energies. In addition, its decay is very fast for large m, that is for smooth $v_1 - v_2$.

Remark 2.4. *The second (approximate) terms in estimates* (2.10)*,* (2.11) *are similar to the error estimates in formula (3.12) in [9] for iterative reconstructions* u_E^j*, when* $j \to \infty$*.*

Note that in formulas (2.10), (2.11) the norm of difference is taken on Γ_E^τ, where τ could be very small. In fact, τ can even decrease as $E \to +\infty$, but not very fast, that is $\tau(E)E^{1/2} \ge E^\gamma$, for $\gamma = \left(\frac{1}{2} - \varepsilon\right)\frac{1}{\beta+d}$. Therefore, Theorems 2.1, 2.2 can be considered as stability results for non-overdetermined and non-complete data.

We recall that, for the case of Born approximation for small V, scattering amplitude reduces to the Fourier transform:

$$f[V](k,l) \approx \widehat{V}(k-l), \quad (k,l) \in \mathcal{M}_E. \tag{2.13}$$

Therefore, Fourier analogs of Theorems 2.1, 2.2 can be summarized as the following result:

Proposition 2.5. *Let functions v_1, v_2, w satisfy assumptions* (2.1)–(2.6)*, and* $\operatorname{dist}(D,\,\Omega) > \operatorname{diam} D$*. Then, for any* $\varepsilon \in (0,1/2)$*,*

$$\|v_1 - v_2\|_{L^\infty(D)} \le C_1 E^{\frac{1}{2}-\varepsilon} \| |\mathcal{F}(v_1+w)|^2 - |\mathcal{F}(v_2+w)|^2 \|_{C(B_{2\tau\sqrt{E}})} + C_2 E^{-(\frac{1}{2}-\varepsilon)\frac{m-d}{\beta+d}}, \tag{2.14}$$

for $E \ge E_3 = E_3(D, N_1, \Omega, \beta, c_1, \tau, \varepsilon)$*. In addition, if the condition on supports is relaxed to* $\operatorname{dist}(D,\,\Omega) > 0$*, then*

$$\|v_1 - v_2\|_{L^\infty(D)} \le C_1 E^{\frac{1}{2}-\varepsilon} \| (|\mathcal{F}(v_1+w)|^2 - |\mathcal{F}(v_2+w)|^2) - (|\mathcal{F}v_1|^2 - |\mathcal{F}v_2|^2) \|_{C(B_{2\tau\sqrt{E}})} + C_2 E^{-(\frac{1}{2}-\varepsilon)\frac{m-d}{\beta+d}} \tag{2.15}$$

for $E \ge E_4 = E_4(D, N_1, \Omega, \beta, c_1, \tau, \varepsilon)$*. Here* C_1*,* C_2 *are the same as in* (2.10)*,* (2.11)*, and* E_3*,* E_4 *are defined in* (5.3)*.*

The estimates (2.14), (2.15) follow from (2.10), (2.11) up to values of E_3 and E_4.

Note that estimates (2.14), (2.15) have considerable similarity with some of results of [9] and, in particular, with estimate (1.11). These results of [9] can be specified also as the following approximate Lipschitz stability estimates:

Proposition 2.6. *Let functions v_1, v_2, w satisfy assumptions* (2.1)–(2.6)*, and* $\operatorname{dist}(D,\,\Omega) > \operatorname{diam} D$*. Then*

$$\|v_1 - v_2\|_{L^\infty(D)} \le C_1 \tau^{\beta+d} E^{\frac{\beta+d}{2}} \| |\mathcal{F}(v_1+w)|^2 - |\mathcal{F}(v_2+w)|^2 \|_{C(B_{2\tau\sqrt{E}})} + C_2 \tau^{-(m-d)} E^{-\frac{m-d}{2}}, \tag{2.16}$$

for $E \ge E_5$*. In addition, if the condition on supports is relaxed to* $\operatorname{dist}(D,\,\Omega) > 0$*, then*

$$\|v_1 - v_2\|_{L^\infty(D)} \le C_1 \tau^{\beta+d} E^{\frac{\beta+d}{2}} \| (|\mathcal{F}(v_1+w)|^2 - |\mathcal{F}(v_2+w)|^2) - (|\mathcal{F}v_1|^2 - |\mathcal{F}v_2|^2) \|_{C(B_{2\tau\sqrt{E}})} + C_2 \tau^{-(m-d)} E^{-\frac{m-d}{2}}, \tag{2.17}$$

for $E \ge E_5$*, where* $E_5 = E_5(\tau)$ *is defined in* (5.10)*;* $C_1 = C_1(d,\beta,c_1)$ *is defined in* (5.8)*,* $C_2 = C_2(d, m, N_1, N_2, D, \Omega, \beta, c_1)$ *is defined by* (5.9) *for* (2.16)*, and by* (5.11) *for* (2.17)*.*

Propositions 2.5 and 2.6 are proved in Section 5. These proofs are based on the explicit reconstruction formulas of [20]. The proof of Proposition 2.5 follows the scheme of proofs of Theorems 2.1, 2.2. The proof of Proposition 2.6 is more straightforward. This straightforward scheme can be also used for the case of Proposition 2.5. This approach leads to somewhat different constants C_1, C_2, E_4, E_5 in formulas (2.14), (2.15).

3 Preliminaries

3.1 Direct scattering

Starting from v, in order to find ψ^+ and f, one can use, in particular, the Lippmann-Schwinger integral equation

$$\psi^+(x,k) = e^{ikx} + \int_{\mathbb{R}^d} G^+(x-y,k)V(y)\psi^+(y,k)dy, \tag{3.1}$$

$$G^+(x,k) = -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{e^{i\xi x}d\xi}{\xi^2 - k^2 - i0}, \tag{3.2}$$

and the relation

$$f[V](k,l) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ily}V(y)\psi^+(y,k)dy, \tag{3.3}$$

where $x,k,l \in \mathbb{R}^d$, $k^2 = l^2 = E$; see, for example, [6].

To deal with equation (3.1) and formula (3.3), it is convenient to use the following Agmon estimate:

$$\|\Lambda^{-s}G^+(k)\Lambda^{-s}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le a_0(d,s)|k|^{-1}, \quad |k|\to\infty, \quad s > 1/2, \tag{3.4}$$

where Λ is the multiplication operator by the function $(1+|x|^2)^{1/2}$, $G^+(k)$ denotes integral operator with Schwartz kernel $G^+(x-y,k)$.

In particular, it follows from (3.4) that (3.1) is uniquely solvable in $L^\infty(\mathbb{R}^d)$ for fixed k, for $|k| = E^{1/2} \ge \rho_1(d,s,\|V\|_{\infty,s})$, where

$$\rho_1(d,s,N) = \max(2a_0(d,s/2)N, 1), \tag{3.5}$$

and the following estimate holds:

$$\|\Lambda^{-s/2}\psi^+(x,k) - \Lambda^{-s/2}e^{ikx}\|_{L^2(\mathbb{R}^d)} \le b_1(d,s)\|V\|_{\infty,s}|k|^{-1}, \tag{3.6}$$

for $|k| \ge \rho_1(d,s,\|V\|_{\infty,s})$, $k\in\mathbb{R}^d$. Here

$$\|u\|_{\infty,s} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^d}(1+|x|)^s|u(x)|,\ s>0. \tag{3.7}$$

We also have that

$$\begin{aligned} &f[V](k,l) = \widehat{V}(k-l) + \delta f[V](k,l),\\ &|\delta f[V](k,l)| \le b_2(d,s)(\|V\|_{\infty,s})^2E^{-1/2}, \end{aligned} \tag{3.8}$$

for $k,l\in\mathbb{R}^d$, $|k| = |l| \ge \rho_1(d,s,\|V\|_{\infty,s})$.

In connection with (3.4), (3.6), (3.8), see [17].

3.2 Estimates for direct scattering

We consider scattering potentials V_j of the form

$$V_j = v_j + w, \quad j = 1,2, \tag{3.9}$$

where v_1, v_2, w satisfy the assumptions of Section 2. Note that the following properties hold:

$$\begin{aligned} &\|V_j\|_\infty = \|v_j + w\|_\infty \le N_1, \quad \|V_2 - V_1\|_\infty = \|v_2 - v_1\|_\infty,\\ &supp\, V_j \subseteq (D\cup\Omega), \quad supp(V_1 - V_2) \subseteq D, \end{aligned} \tag{3.10}$$

for $j = 1,2$; see (2.1)-(2.6). Note also that:

$$\begin{aligned} &|V_2 - V_1| = |v_2 - v_1| \text{ are bounded on } D,\\ &V_2 = V_1 = w \qquad\qquad \text{on } \mathbb{R}^d \setminus D. \end{aligned} \tag{3.11}$$

We also consider

$$f_j := f[V_j] = f[v_j + w], \quad j = 1,2. \tag{3.12}$$

In view of (2.1), (2.3), (2.6), (3.8), (3.9), we have that

$$\begin{aligned} &f_j(k,l) = f[V_j](k,l) = \widehat{V}_j(k-l) + \delta f_j(k,l), \\ &|\delta f_j| \le a_1(D\cup\Omega) N_1^2 E^{-1/2}, \quad j=1,2, \end{aligned} \tag{3.13}$$

for $(k,l) \in \mathcal{M}_E$, $E \ge (\rho_1(d,s,\lambda_s(D\cup\Omega)N_1))^2$, where ρ_1 is defined in (3.5), $a_1(D\cup\Omega) = b_2(d,s)\lambda_s^2(D\cup\Omega)$, $s>d$, and

$$\lambda_s(\mathcal{U}) := (1+\max_{x\in\mathcal{U}}|x|)^s. \tag{3.14}$$

We also have the following Lemma (see [16]):

Lemma 3.1. *Let v_j, V_j, and $f_j = f[V_j]$, $j=1,2$, be as in (2.1), (3.9), (3.12). Then the following estimate holds:*

$$\begin{aligned} &f_2(k,l) - f_1(k,l) = \widehat{v}_2(k-l) - \widehat{v}_1(k-l) + \Delta(k,l), \\ &|\Delta(k,l)| \le a_2(D\cup\Omega) N_1 \|v_2 - v_1\|_\infty E^{-1/2}, \end{aligned} \tag{3.15}$$

for k, $l \in \mathbb{R}^d$, $\sqrt{E} = |k| = |l| \ge \rho_1(d,s,\lambda_s(D\cup\Omega)N_1)$ and some positive $a_2(D\cup\Omega)$. Here N_1, D, Ω are as in (3.10), (3.11).

Let $\mu(\mathcal{U})$ be Lebesgue measure of a domain $\mathcal{U} \subset \mathbb{R}^d$.

We also have the following Lemma, which will be used in Section 4.

Lemma 3.2. *Let $V_j = v_j + w$, $j=1,2$, be as in Lemma 3.1. Then*

$$\begin{aligned} &\left| |f[v_2+w](k,l)|^2 - |f[v_1+w](k,l)|^2 + |(\widehat{v}_1+\widehat{w})(k-l)|^2 - |(\widehat{v}_2+\widehat{w})(k-l)|^2 \right| \le \\ &\le 2\left((2\pi)^{-d}(a_1\mu(D) + a_2\mu(D\cup\Omega)) + a_1 a_2 N_1 E^{-1/2} \right) N_1^2 E^{-1/2} \|v_2 - v_1\|_\infty, \end{aligned} \tag{3.16}$$

$$(k,l) \in \mathcal{M}_E, \quad E^{1/2} \ge \rho_1(d,s,\lambda_s(D\cup\Omega)N_1), \tag{3.17}$$

and

$$\left| |\widehat{v}_2+\widehat{w}|^2 - |\widehat{v}_1+\widehat{w}|^2 \right|(p) \le 2(2\pi)^{-2d}\mu(D\cup\Omega)\mu(D) N_1 \|v_2 - v_1\|_\infty, \quad \forall p \in \mathbb{R}^d, \tag{3.18}$$

$$\left| |\widehat{v}_2+\widehat{w}|^2 - |\widehat{v}_1+\widehat{w}|^2 \right|(p) \le 2(2\pi)^{-d}\mu(D\cup\Omega) N_1 |\widehat{v}_2 - \widehat{v}_1|(p), \quad \forall p \in \mathbb{R}^d, \tag{3.19}$$

$$\left| |\widehat{v}_2+\widehat{w}|^2 - |\widehat{v}_2|^2 - |\widehat{v}_1+\widehat{w}|^2 + |\widehat{v}_1|^2 \right|(p) \le 2(2\pi)^{-2d}\mu(D)\mu(\Omega) N_1 \|v_2 - v_1\|_\infty, \quad \forall p \in \mathbb{R}^d, \tag{3.20}$$

$$\left| |\widehat{v}_2+\widehat{w}|^2 - |\widehat{v}_2|^2 - |\widehat{v}_1+\widehat{w}|^2 + |\widehat{v}_1|^2 \right|(p) \le 2(2\pi)^{-d}\mu(\Omega) N_1 |\widehat{v}_2 - \widehat{v}_1|(p), \quad \forall p \in \mathbb{R}^d, \tag{3.21}$$

where a_1, a_2 are as in (3.13), (3.15), D, Ω, N_1 are as in (3.10), ρ_1 is as in (3.5), λ_s is as in (3.14), $s>d$.

This Lemma is a variation of Lemma 3.6 in [9].

Proof of Lemma 3.2. Note that, for z_1, $z_2 \in \mathbb{C}$,

$$|z_2|^2 - |z_1|^2 = \overline{z}_2(z_2 - z_1) + z_1(\overline{z_2 - z_1}). \tag{3.22}$$

Using (3.22) for f_j in place of z_j, and (3.13), (3.15), we obtain

$$\begin{aligned} &|f_2|^2 - |f_1|^2 = \overline{f}_2(f_2 - f_1) + f_1(\overline{f_2 - f_1}) = (\overline{\widehat{V}_2} + \overline{\delta f_2})(\widehat{V_2} - \widehat{V}_1 + \Delta) + (\widehat{V}_1 + \delta f_1)(\overline{\widehat{V_2} - \widehat{V}_1 + \Delta}) = \\ &= |\widehat{V}_2|^2 - |\widehat{V}_1|^2 + \delta f_1(\overline{\widehat{V}_2 - \widehat{V}_1}) + \overline{\delta f_2}(\widehat{V}_2 - \widehat{V}_1) + \delta f_1\overline{\Delta} + \overline{\delta f_2}\Delta + \widehat{V}_1\overline{\Delta} + \overline{\widehat{V}_2}\Delta. \end{aligned} \tag{3.23}$$

From (3.23), using (3.10), (3.13), (3.15), (3.11), we conclude

$$\begin{aligned} &||f_2|^2 - |f_1|^2 - (|\widehat{V}_2|^2 - |\widehat{V}_1|^2)| \le 2a_1 N_1^2 E^{-1/2}|\widehat{V}_2 - \widehat{V}_1| + (|\widehat{V}_1| + |\widehat{V}_2|)a_2 N_1 \|V_2 - V_1\|_\infty E^{-1/2} + \\ &+ 2a_1 N_1^2 E^{-1/2} a_2 N_1 \|V_2 - V_2\|_\infty E^{-1/2} \le 2\left((2\pi)^{-d}(a_1\mu(D) + a_2\mu(D\cup\Omega)) + a_1 a_2 N_1 E^{-1/2} \right) N_1^2 E^{-1/2} \|V_2 - V_1\|_\infty. \end{aligned} \tag{3.24}$$

Formula (3.16) follows from (3.9), (3.24).

Using (2.1), (2.7), (3.11), (3.22), we obtain

$$\begin{aligned} &\left| |\widehat{v}_2+\widehat{w}|^2 - |\widehat{v}_1+\widehat{w}|^2 \right| = \left| (\overline{\widehat{v}_2+\widehat{w}})(\widehat{v}_2 - \widehat{v}_1) + (\widehat{v}_1+\widehat{w})(\overline{\widehat{v}_2 - \widehat{v}_1}) \right| \le \\ &\le 2(2\pi)^{-d}\mu(D\cup\Omega) N_1 |\widehat{v}_2 - \widehat{v}_1|(p) \le 2(2\pi)^{-2d}\mu(D\cup\Omega) N_1 \mu(D) \|v_2 - v_1\|_\infty. \end{aligned} \tag{3.25}$$

Analogously to (3.25), we obtain

$$\begin{aligned} &\left| |\widehat{v}_2+\widehat{w}|^2 - |\widehat{v}_2|^2 - |\widehat{v}_1+\widehat{w}|^2 + |\widehat{v}_1|^2 \right| \le \left| \overline{(\widehat{v}_2+\widehat{w})}\widehat{w} + \widehat{v}_2\overline{\widehat{w}} - \overline{(\widehat{v}_1+\widehat{w})}\widehat{w} - \widehat{v}_1\overline{\widehat{w}} \right| \le \\ &\le \left| \overline{(\widehat{v}_2 - \widehat{v}_1)}\widehat{w} + (\widehat{v}_2 - \widehat{v}_1)\overline{\widehat{w}} \right| \le 2(2\pi)^{-d}\mu(\Omega) N_1 |\widehat{v}_2 - \widehat{v}_1| \le 2(2\pi)^{-2d}\mu(D)\mu(\Omega) N_1 \|v_2 - v_1\|_\infty. \end{aligned} \tag{3.26}$$

Thus, (3.18)–(3.21) are also proved.

3.3 Phase retrieval formulas of [20]

Let $v = v_1$ and w be as in (2.1). Then

$$\widehat{v}(p) = (\widehat{w}(p))^{-1}\mathcal{F}\left(\chi_{D-\Omega}\cdot\left(\mathcal{F}^{-1}(|\mathcal{F}(v+w)|^2 - |\mathcal{F}(v)|^2) - W\right)\right), \quad p\in\mathbb{R}^d, \tag{3.27}$$

and if $\operatorname{dist}(D,\Omega) > \operatorname{diam} D$, then

$$\widehat{v}(p) = (\widehat{w}(p))^{-1}\mathcal{F}\left(\chi_{D-\Omega}\cdot\left(\mathcal{F}^{-1}(|\mathcal{F}(v+w)|^2) - W\right)\right), \quad p\in\mathbb{R}^d, \tag{3.28}$$

where

$$W(x) := (2\pi)^{-d}\int_{\mathbb{R}^d} w(x+y)\overline{w(y)}dy. \tag{3.29}$$

Formulas (3.27), (3.28) were given in Section 3 of [20].

In formulas (3.27), (3.28), $D-\Omega$ is defined by

$$D-\Omega = \{x-y,\, x\in D,\, y\in\Omega\}\subset\mathbb{R}^d, \tag{3.30}$$

and $\chi_{D-\Omega}$ is a function such that

$$\begin{cases} \chi_{D-\Omega}(x) = 1, & x\in D-\Omega,\\ \chi_{D-\Omega}(x) = 0, & \operatorname{dist}(x, D-\Omega) > \varepsilon,\\ \chi_{D-\Omega}(x)\in[0,1], & 0 < \operatorname{dist}(x, D-\Omega) < \varepsilon,\\ \chi_{D-\Omega}(x)\in C^\infty(\mathbb{R}^d), \end{cases} \tag{3.31}$$

for some

$$\varepsilon \in \begin{cases} (0, \operatorname{dist}(D-\Omega, B_{\operatorname{diam} D})), & \text{for Theorem 2.1,}\\ (0, \operatorname{dist}(D-\Omega, \Omega-D)), & \text{for Theorem 2.2.} \end{cases} \tag{3.32}$$

In particular, we have that

$$|\widehat{\chi}_{D-\Omega}(p)| \le \frac{C(\sigma)}{(1+|p|)^\sigma}, \quad \forall p\in\mathbb{R}^d, \tag{3.33}$$

for any $\sigma\ge 0$, and some $C(\sigma) = C(\chi_{D-\Omega},\sigma) > 0$; see formula (82) of [20].

4 Proof of Theorems 2.1 and 2.2

4.1 Proof of Theorem 2.1

We start with the following inequalities:

$$|v_1 - v_2|(x) \le \left|\int_{\mathbb{R}^d} e^{-ipx}(\widehat{v}_1(p) - \widehat{v}_2(p))dp\right| \le I_1(\kappa) + I_2(\kappa), \tag{4.1}$$

$$I_1(\kappa) := \int_{|p|\ge\kappa} |\widehat{v}_1(p) - \widehat{v}_2(p)|dp, \;\; I_2(\kappa) := \int_{|p|\le\kappa} |\widehat{v}_1(p) - \widehat{v}_2(p)|dp, \tag{4.2}$$

where $x\in D$, $\kappa\in(0,\tau\sqrt{E})$, $\tau\in(0,1)$. Here and below τ is the parameter of Theorem 2.1.

Estimating I_1 is as in [16]. Due to (2.2), (2.3), we have that

$$|\widehat{v}_1(p) - \widehat{v}_2(p)| \le a_3(m,d)N_2(1+|p|)^{-m}, \tag{4.3}$$

and, therefore,

$$I_1(\kappa) \le \frac{c(d)a_3(m,d)N_2}{m-d}\frac{1}{\kappa^{m-d}}, \tag{4.4}$$

where $c(d) = |\mathbb{S}^d|$.

Estimating I_2 is as follows. Due to (3.28), we have that

$$\begin{aligned}&|\widehat{v}_1(p)-\widehat{v}_2(p)|=|(\widehat{w}(p))^{-1}\mathcal{F}\left(\chi_{D-\Omega}(x)\cdot\mathcal{F}^{-1}(|\mathcal{F}(v_1+w)|^2-|\mathcal{F}(v_2+w)|^2)\right)|\leq\\&\leq|(\widehat{w}(p))^{-1}|\left(|\mathcal{F}\chi_{D-\Omega}|*||\mathcal{F}(v_1+w)|^2-|\mathcal{F}(v_2+w)|^2|\right),\end{aligned}\tag{4.5}$$

where $*$ denotes the convolution

$$\nu_1*\nu_2(x):=\int_{\mathbb{R}^d}\nu_1(x-y)\nu_2(y)dy,\tag{4.6}$$

for test-functions ν_1, ν_2. In (4.5) we also used the following property of the Fourier transform:

$$\mathcal{F}(\varphi_1\varphi_2)=(\mathcal{F}\varphi_1)*(\mathcal{F}\varphi_2),\tag{4.7}$$

for test-functions φ_1, φ_2.

Using formulas (2.6), (3.33), (4.5), we obtain that:

$$I_2(\kappa)\leq\int_{|p|\leq\kappa}c_1^{-1}(1+|p|)^\beta\left(|\mathcal{F}\chi_{D-\Omega}|*||\mathcal{F}(v_1+w)|^2-|\mathcal{F}(v_2+w)|^2|\right)dp\leq c_1^{-1}C(\sigma)(I_3(\kappa,\delta)+I_4(\kappa,\delta));\tag{4.8}$$

$$I_3(\kappa,\delta)=\int_{|p|\leq\kappa}(1+|p|)^\beta dp\int_{|p'|\leq\delta\sqrt{E}+\kappa}\frac{g(p')}{(1+|p-p'|)^\sigma}dp',\tag{4.9}$$

$$I_4(\kappa,\delta)=\int_{|p|\leq\kappa}(1+|p|)^\beta dp\int_{|p'|\geq\delta\sqrt{E}+\kappa}\frac{g(p')}{(1+|p-p'|)^\sigma}dp',\tag{4.10}$$

where

$$g(p'):=||\mathcal{F}(v_1+w)(p')|^2-|\mathcal{F}(v_2+w)(p')|^2|,\tag{4.11}$$

and $\delta\sqrt{E}\in(\kappa,2\tau\sqrt{E}-\kappa)$.

Applying formula (3.16) for $|p'|\leq B_{\delta\sqrt{E}+\kappa}$ and formula (3.18) for $|p'|\geq B_{\delta\sqrt{E}+\kappa}$, we obtain that

$$g(p')\leq\begin{cases}G_1, & \text{for } |p'|\leq\delta E^{1/2}+\kappa,\\ G_2, & \text{for } |p'|\geq\delta E^{1/2}+\kappa,\end{cases}\tag{4.12}$$

where

$$G_1:=\||f[v_1+w]|^2-|f[v_2+w]|^2\|_C+2\left((2\pi)^{-d}(a_1\mu(D)+a_2\mu(D\cup\Omega))+a_1a_2N_1E^{-1/2}\right)N_1^2E^{-1/2}\|v_1-v_2\|_\infty,\tag{4.13}$$

$$G_2:=2(2\pi)^{-2d}\mu(D)\mu(D\cup\Omega)N_1\|v_1-v_2\|_\infty.\tag{4.14}$$

Here and further $\|\cdot\|_C$ denotes the uniform norm for functions on $C(\Gamma_E^{(\delta+\kappa E^{-1/2})/2})$, and $\|\cdot\|_\infty$ is defined by (2.5).

Estimating I_3 *and* I_4. To estimate I_3 defined by (4.9), we use that

$$I_3\leq G_1(A_1+A_2),\tag{4.15}$$

where

$$A_1:=\int_{|p|\leq\kappa}(1+|p|)^\beta dp\int_{|p-p'|\leq\kappa}\frac{dp'}{(1+|p-p'|)^\sigma},\tag{4.16}$$

$$A_2:=\int_{|p|\leq\kappa}(1+|p|)^\beta dp\int_{|p-p'|\geq\kappa,\,|p'|\leq\delta E^{1/2}+\kappa}\frac{dp'}{(1+|p-p'|)^\sigma}.\tag{4.17}$$

Note that in (4.16) the condition $|p'|\leq\delta E^{1/2}+\kappa$ is fulfilled automatically, due to the choice of κ and δ.

We have that

$$\int_{|p-p'|\leq\kappa}\frac{dp'}{(1+|p-p'|)^\sigma}\leq\int_{0<r\leq\infty}\frac{c(d)r^{d-1}dr}{(1+r)^\sigma}=c(d)B(d,\sigma-d),\tag{4.18}$$

where $c(d)=|\mathbb{S}^d|$, B is the beta-function.

Therefore,

$$A_1 \le \int_{0<r_1\le\kappa} c(d) r_1^{d-1}(1+r_1)^\beta dr_1 \int_{0<r\le\infty} \frac{c(d) r^{d-1} dr}{(1+r)^\sigma} \le \frac{c^2(d) B(d,\sigma-d)}{\beta+d}(1+\kappa)^{\beta+d}. \tag{4.19}$$

In addition, for arbitrary $\alpha > 0$, such that $\sigma - d - \alpha \ge 1$, we have that,

$$\begin{aligned}
A_2 &\le \int_{|p|\le\kappa} (1+|p|)^\beta dp \int_{|p-p'|\ge\kappa} \frac{dp'}{(1+|p-p'|)^\sigma} \le \\
&\le \int_{|p|\le\kappa} (1+|p|)^\beta dp \int_{|p-p'|\ge\kappa} \frac{dp'}{(1+\kappa)^{\sigma-d-\alpha}(1+|p-p'|)^{d+\alpha}} \le \\
&\le \frac{1}{(1+\kappa)^{\sigma-d-\alpha}} \int_{|p|\le\kappa} (1+|p|)^\beta dp \int_{\mathbb{R}^d} \frac{dp'}{(1+|p-p'|)^{d+\alpha}} \le \\
&\le \frac{1}{(1+\kappa)^{\sigma-d-\alpha}} \int_{|p|\le\kappa} (1+|p|)^\beta dp \int_{r\in(0,\infty)} \frac{c(d) r^{d-1} dr}{(1+r)^{d+\alpha}} \le \\
&\le \frac{1}{(1+\kappa)^{\sigma-d-\alpha}} \int_{0<r\le\kappa} c(d) r^{d-1}(1+r)^\beta dr\, c(d) B(d,\alpha) \le \frac{c^2(d) B(d,\alpha)}{\beta+d} \frac{(1+\kappa)^{\beta+d}}{(1+\kappa)^{\sigma-d-\alpha}}.
\end{aligned} \tag{4.20}$$

Therefore, we estimate $A_1 + A_2$ of (4.15) as

$$A_1 + A_2 \le \frac{c^2(d)(1+\kappa)^{\beta+d}}{\beta+d} \left(B(d,\sigma-d) + \frac{B(d,\alpha)}{(1+\kappa)^{\sigma-d-\alpha}} \right). \tag{4.21}$$

In order to estimate I_4 defined by (4.10), we use (4.14) and obtain

$$\begin{aligned}
&I_4 \le G_2 \int_{|p|\le\kappa} (1+|p|)^\beta dp \int_{|p'|\ge\delta E^{1/2}+\kappa} \frac{dp'}{(1+|p-p'|)^\sigma} \le G_2 \int_{|p|\le\kappa} (1+|p|)^\beta dp \int_{r\in(0,\infty)} \frac{c(d) r^{d-1} dr}{(1+r)^\sigma} \le \\
&\le G_2 \int_{|p|\le\kappa} (1+|p|)^\beta dp \frac{c(d) B(d,\alpha)}{(1+\delta E^{1/2})^{\sigma-d-\alpha}} \le G_2 \frac{c^2(d) B(d,\alpha)}{\beta+d} \frac{(1+\kappa)^{\beta+d}}{(1+\delta E^{1/2})^{\sigma-d-\alpha}}.
\end{aligned} \tag{4.22}$$

Final part of the proof. Let

$$\kappa = E^\gamma, \quad \gamma = \left(\frac{1}{2} - \varepsilon\right) \frac{1}{\beta+d}, \quad \sigma = \left(\frac{1}{2} - \varepsilon\right)^{-1} (\beta+d)(d+1), \quad \alpha = d, \quad s = s(d) = d + 1/2, \quad \delta = \tau. \tag{4.23}$$

Then, for $E \ge \tau^{-\frac{1}{1/2-\gamma}}$, we have that $\kappa \in (0, \tau\sqrt{E})$ and $\delta\sqrt{E} \in (\kappa, 2\tau\sqrt{E} - \kappa)$.

Using formulas (4.1), (4.4), (4.8)–(4.10), (4.13), (4.14), (4.21), (4.22), and (4.23), we obtain that

$$\begin{aligned}
&\|v_1 - v_2\|_\infty \le K_1 E^{-\left(\frac{1}{2}-\varepsilon\right)\frac{m-d}{\beta+d}} + \\
&+ K_2 \left(K_3 E^{\frac{1}{2}-\varepsilon} \left(\| |f[v_1+w]|^2 - |f[v_2+w]|^2 \|_C + K_4 E^{-1/2} \|v_1 - v_2\|_\infty \right) + \frac{K_5 E^{\frac{1}{2}-\varepsilon} \|v_1 - v_2\|_\infty}{(1+\tau E^{1/2})^{\sigma-d-\alpha}} \right),
\end{aligned} \tag{4.24}$$

$$K_1 := K_1(d, m, N_2) = \frac{c(d) a_3(m,d) N_2}{m-d}, \quad K_2 := K_2(w,\sigma) = c_1^{-1} C(\sigma), \tag{4.25}$$

$$K_3 := K_3(d,\beta) \le 2\frac{c^2(d)}{\beta+d} \left(B(d,\sigma-d) + B(d,d) \right), \tag{4.26}$$

$$K_4 := K_4(D,\Omega,d,N_1) \le 2 \left((2\pi)^{-d} (a_1(d, D\cup\Omega)\mu(D) + a_2(D\cup\Omega)\mu(D\cup\Omega)) + \frac{a_1(d,D\cup\Omega) a_2(D\cup\Omega)}{2a_0(d, s(d)\lambda_{s(d)}(D\cup\Omega))} \right) N_1^2, \tag{4.27}$$

$$K_5 := K_5(D,\Omega,N_1,d,\beta) = 2(2\pi)^{-2d} \mu(D)\mu(D\cup\Omega) N_1 \frac{c^2(d) B(d,d)}{\beta+d}. \tag{4.28}$$

Let

$$E_1 = \max(E_{root},\, \tau^{-\frac{1}{1/2-\gamma}},\, \rho_1^2(d, s, \lambda_s(D\cup\Omega) N_1)), \tag{4.29}$$

where E_{root} is the maximal root of equation for E

$$K_2 K_3 K_4 E^{-\varepsilon} + K_2 K_5 E^{\frac{1}{2}-\varepsilon} (1+\tau E^{1/2})^{-\sigma+d+\alpha} = 1/2, \tag{4.30}$$

ρ_1 is defined in (3.5), λ_s is defined in (3.14). Note that E_{root} exists, since

$$\frac{1}{2} - \varepsilon - (\sigma - d - \alpha)/2 < 0. \tag{4.31}$$

Therefore, for $E \geq E_1$,

$$K_2K_3K_4E^{-\varepsilon} + K_2K_5E^{\frac{1}{2}-\varepsilon}(1+\tau E^{1/2})^{-\sigma+d+\alpha} \leq 1/2. \tag{4.32}$$

In view of (4.32), the coefficient with $\|v_2 - v_1\|_\infty$ in the right-hand side of (4.24) is less than 1/2. Therefore,

$$\|v_2 - v_1\|_\infty \leq 2K_1E^{-\left(\frac{1}{2}-\varepsilon\right)\frac{m-d}{\beta+d}} + 2K_2K_3E^{\frac{1}{2}-\varepsilon}\||f_2|^2 - |f_1|^2\|_C, \tag{4.33}$$

for $E \geq E_1$.

Note that $\tau\sqrt{E} \geq \kappa$, for $E \geq E_1$, see (4.23), (4.29). Therefore, $\Gamma_E^{(\tau+\kappa E^{-1/2})/2} \subseteq \Gamma_E^\tau$, and $\|\cdot\|_C = \|\cdot\|_{C(\Gamma_E^{(\tau+\kappa E^{-1/2})/2))}} \leq \|\cdot\|_{C(\Gamma_E^\tau)}$. This completes the proof.

4.2 Proof of Theorem 2.2

Proof of Theorem 2.2 is similar to the Proof of Theorem 2.1 up to the following changes:

- For estimate of I_2 we use formula (3.27) in place of (3.28).
- Formulas (4.5) and (4.11) are replaced by

$$\begin{aligned} &|\widehat{v}_1(p) - \widehat{v}_2(p)| \leq \\ &\leq |(\widehat{w}(p))^{-1}|\left(|\mathcal{F}\chi_{D-\Omega}| * \left(|(|\mathcal{F}(v_1+w)|^2 - |\mathcal{F}(v_2+w)|^2) - (|\mathcal{F}v_1|^2 - |\mathcal{F}v_2|^2)|\right)\right), \end{aligned} \tag{4.34}$$

$$g(p') := |(|\mathcal{F}(v_1+w)(p')|^2 - |\mathcal{F}(v_2+w)(p')|^2) - (|\mathcal{F}v_1(p')|^2 - |\mathcal{F}v_2(p')|^2)|. \tag{4.35}$$

- Taking into account (3.20), formulas (4.13), (4.14) are replaced by

$$\begin{aligned} G_1 &:= \|(|f[v_1+w]|^2 - |f[v_2+w]|^2) - (|f[v_1]|^2 - |f[v_2]|^2)\|_C + \\ &+ 4\left((2\pi)^{-d}(a_1\mu(D) + a_2\mu(D\cup\Omega)) + a_1a_2N_1E^{-1/2}\right)N_1^2E^{-1/2}\|v_1 - v_2\|_\infty, \end{aligned} \tag{4.36}$$

$$G_2 := 2(2\pi)^{-2d}\mu(D)\mu(\Omega)N_1\|v_1 - v_2\|_\infty. \tag{4.37}$$

- In formula (4.24), the term $\||f[v_1+w]|^2 - |f[v_2+w]|^2\|_C$ should be replaced by $\|(|f[v_1+w]|^2 - |f[v_2+w]|^2) - (|f[v_1]|^2 - |f[v_2]|^2)\|_C$.
- In formula (4.27) constant K_4 should be replaced by $2K_4$.
- Due to (4.37), in formula (4.28) constant K_5 should be replaced by

$$K_5 = 2(2\pi)^{-2d}\mu(D)\mu(\Omega)N_1\frac{c^2(d)B(d,d)}{\beta+d}. \tag{4.38}$$

- We define

$$E_2 := \max(E_{root},\, \tau^{-\frac{1}{1/2-\gamma}},\, \rho_1^2(d,s,\lambda_s(D\cup\Omega)N_1)), \tag{4.39}$$

  where E_{root} is the maximal root of equation for E

$$2K_2K_3K_4E^{-\varepsilon} + K_2K_5E^{\frac{1}{2}-\varepsilon}(1+\delta E^{1/2})^{-\sigma+d+\alpha} = 1/2, \tag{4.40}$$

  for K_2, K_3, K_4 defined by (4.25)-(4.27), and K_5 defined by (4.38).
- For $E \geq E_2$ we have the following formula in place of (4.32)

$$2K_2K_3K_4E^{-\varepsilon} + K_2K_5E^{\frac{1}{2}-\varepsilon}(1+\tau E^{1/2})^{-\sigma+d+\alpha} \leq 1/2, \tag{4.41}$$

  for K_2, K_3, K_4 defined by (4.25)-(4.27), and K_5 defined by (4.38).

5 Proof of Propositions 2.5 and 2.6

5.1 Proof of Proposition 2.5

We repeat the proofs of Theorems 2.1, 2.2 up to the following changes:

- In formulas (4.13), (4.36), G_1 should be replaced by

$$G_1 := \||\mathcal{F}(v_1+w)|^2 - |\mathcal{F}(v_2+w)|^2\|_{C(B_{\tau\sqrt{E}+E^\gamma})}, \tag{5.1}$$

$$G_1 := \|(|\mathcal{F}(v_1+w)|^2 - |\mathcal{F}(v_2+w)|^2) - (|\mathcal{F}v_1|^2 - |\mathcal{F}v_2|^2)\|_{C(B_{\tau\sqrt{E}+E^\gamma})}, \tag{5.2}$$

  respectively.

- Consequently, formula (4.27) should be replaced by $K_4 = 0$.

- We define E_3, E_4 as

$$\begin{aligned} E_3 &= \max(E_{root,\,3}, \tau^{-\frac{1}{1/2-\gamma}}, 1), \\ E_4 &= \max(E_{root,\,4}, \tau^{-\frac{1}{1/2-\gamma}}, 1), \end{aligned} \tag{5.3}$$

  where $E_{root,\,3}$, $E_{root,\,4}$, are the maximal roots of the equations for E :

$$\begin{aligned} K_2K_6E^{\frac{1}{2}-\varepsilon}(1+\tau E^{1/2})^{-\sigma+d+\alpha} &= 1/2, \\ K_2K_7E^{\frac{1}{2}-\varepsilon}(1+\tau E^{1/2})^{-\sigma+d+\alpha} &= 1/2, \end{aligned} \tag{5.4}$$

  respectively; if there are no roots, we take $E_{root,j} = 0$, for $j = 3$ or 4. Here, K_2 is as in (4.25), $K_6 = K_5$ defined in (4.28), and $K_7 = K_5$ defined in (4.38).

5.2 Proof of Proposition 2.6

We repeat the proof of Section 4.1, till to formula (4.21), where we use change (5.1). We estimate I_4 of (4.10) via formula (3.19) and formula (4.3) as follows:

$$\begin{aligned} I_4(\kappa,\delta) &\le \int_{|p|\le\kappa}(1+|p|)^\beta dp \int_{|p'|\ge\delta\sqrt{E}+\kappa} \frac{2(2\pi)^{-d}\mu(D\cup\Omega)N_1|\widehat{v}_2(p)-\widehat{v}_1(p)|}{(1+|p-p'|)^\sigma}dp' \le \\ &\le 2(2\pi)^{-d}\mu(D\cup\Omega)N_1 \int_{|p|\le\kappa}(1+|p|)^\beta dp \int_{|p'|\ge\delta\sqrt{E}+\kappa} \frac{a_3(m,d)N_2}{(1+|p-p'|)^\sigma(1+|p'|)^m}dp' \le \\ &\le \frac{2(2\pi)^{-d}\mu(D\cup\Omega)N_1}{(1+\tau\sqrt{E})^\sigma} \int_{|p|\le\kappa}(1+|p|)^\beta dp \int_{|p'|\ge\delta\sqrt{E}+\kappa} \frac{a_3(m,d)N_2}{(1+|p'|)^m}dp' \le \\ &\le \frac{2(2\pi)^{-d}\mu(D\cup\Omega)a_3(m,d)N_1N_2c^2(d)}{(1+\tau\sqrt{E})^{\sigma-\beta-d}(\beta+d)} \frac{1}{(1+\frac{3}{2}\tau\sqrt{E})^{m-d}(m-d)}. \end{aligned} \tag{5.5}$$

We fix our parameters as follows:

$$\kappa = (\tau/2)E^{1/2}, \quad \sigma = \beta+d+1, \quad \alpha = d, \quad \delta = \tau. \tag{5.6}$$

Using formulas (4.1), (4.4), (4.8)–(4.10), (4.21), (5.1), (5.5) and (5.6), we obtain that

$$\|v_1-v_2\|_\infty \le C_1\tau^{\beta+d}E^{\frac{\beta+d}{2}}\||\mathcal{F}(v_1+w)|^2 - \mathcal{F}(v_2+w)|^2\|_{C(B_{2\tau\sqrt{E}})} + C_2\tau^{-(m-d)}E^{-\frac{m-d}{2}}, \tag{5.7}$$

$$C_1 := C_1(d,\beta,c_1) \le c_1^{-1}C(\beta+d+1)\frac{c^2(d)}{\beta+d}\left(B(d,\beta+1) + \frac{B(d,d)}{2^{\beta-d+1}}\right), \tag{5.8}$$

$$C_2 := C_2(d,m,N_1,N_2,D,\Omega,\beta,c_1) \le \frac{2^{m-d}c(d)a_3(m,d)N_2}{m-d}\left(1 + \frac{2(2\pi)^{-d}c_1^{-1}c(d)C(\beta+d+1)\mu(D\cup\Omega)N_1}{3^{m-d+1}(\beta+d)}\right), \tag{5.9}$$

for

$$E \ge E_5 := E_5(\tau) = 4/\tau^2. \tag{5.10}$$

Formula (2.16) is proved.

Note that, increasing E_5, we can down constant C_2.

In order to prove formula (2.17), it is sufficient to replace (3.19) by (3.21) in (5.5), and (5.1) by (5.2) in (5.7). In this case C_2 should be defined as

$$C_2 := C_2(d,m,N_1,N_2,\Omega,\beta,c_1) \le \frac{2^{m-d}c(d)a_3(m,d)N_2}{m-d}\left(1 + \frac{2(2\pi)^{-d}c_1^{-1}c(d)C(\beta+d+1)\mu(\Omega)N_1}{3^{m-d+1}(\beta+d)}\right). \tag{5.11}$$

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