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Abstract

On an asymptotically conical manifold we prove time decay estimates for the flow of the Schrödinger, wave and Klein-Gordon equations via some differentiability properties of the spectral measure. To keep the paper at a reasonable length we limit ourselves to the low energy part of the spectrum, which is the one that dictates the decay rates. With this paper we extend sharp estimates that are known in the asymptotically flat case (see Bouclet and Burq in [BB21]) to this more general geometric framework and therefore recover the same decay properties as in the euclidean case. The first step is to prove some resolvent estimates via a limiting absorption principle. It is at this stage that the proof of the previously mentioned authors fails, in particular when we try to recover a low frequency positive commutator estimate. Once the resolvent estimates are established we derive regularity for the spectral measure that in turn is applied to obtain the decay of the flows.

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1 Introduction

In this paper we consider the Laplace-Beltrami operator on a quite general class of non compact manifolds with ends, which includes, among others, all compact perturbation of the euclidean metric. The geometric setting is the one of asymptotically conical manifolds that over the years has attracted the interest of a substantial community with the aim of recovering some of the properties that hold in the flat case, such as resolvent or local energy decay estimates.

Let P the Laplace-Beltrami operator on an asymptotically conical manifold and $W \geq 0$ a decaying multiplicative potential. Broadly speaking we are interested in estimates on the operators

$$(P + W - \lambda \pm i0)^{-l} = \lim_{\varepsilon \to 0} (P + W - \lambda \pm i\varepsilon)^{-l}. \tag{1.1}$$

and the related evolutions $e^{it(P+W)}$, $e^{it\sqrt{P+W}}$, $e^{it\sqrt{P+W}+1}$, $\frac{\sin(t\sqrt{P+W})}{\sqrt{P+W}}$.

In particular, in this work we give some decay properties in weighted L^2 spaces for the resolvent and the spectral measure of the Laplace-Beltrami operator, which are then applied to recover local energy decay. Our results are closely related to the ones presented in [BB21] where the geometry is the one of $\mathbb{R}^n$, up to an obstacle, with an asymptotically Euclidean metric. Indeed, we shall prove here that even in the case of a manifold with an asymptotically conic end the same results as Theorem 1.2 and 1.3 in [BB21] hold. As one can see in Section 5 in [BB21], these properties can then be used to prove decay on the evolution operators.

Let M an n dimensional manifold with $n \geq 3$. We assume M to be of the form $M = K \cup (M \setminus K)$ with K compact and $M \setminus K$ an infinite end which is asymptotically conical. For formal statements see Definitions 1.1 and 1.3. For the moment we just say that r is the radial coordinate on the manifold end, $\langle r \rangle$ is a positive smooth decaying function which is $O(r)$ for $r \gg 1$ and $\|\cdot\|$ the norm of operators on $L^2(M)$.

The main result of the paper is the following.

Theorem 1.1. *Let (M, G_M) an asymptotically conical manifold of dimension $n \geq 3$ and P the Laplace-Beltrami operator on M. Let $f \in C_0^\infty(\mathbb{R})$ and W a non negative multiplicative potential that on $M \setminus K$ agrees with a function in $S^{-2-\varepsilon}$ in the sense of Definition 1.1. Then*

i) Schrödinger flow:

$$\|\langle r \rangle^{-\alpha} f(P+W) e^{it(P+W)} \langle r \rangle^{-\alpha}\| \lesssim \langle t \rangle^{-\frac{n}{2}}$$

for $\alpha > \left[\frac{n}{2}\right] + 2$.

ii) Wave flow:

$$\|\langle r \rangle^{-\alpha} f(P+W) \frac{\sin(t\sqrt{P+W})}{\sqrt{P+W}} \langle r \rangle^{-\alpha}\| \lesssim \langle t \rangle^{1-n}$$

and

$$\|\langle r \rangle^{-\alpha} f(P+W) e^{it\sqrt{P+W}} \langle r \rangle^{-\alpha}\| \lesssim \langle t \rangle^{-n}$$

for $\alpha > n + 1$.

iii) Klein-Gordon flow:

$$\|\langle r \rangle^{-\alpha} f(P+W) e^{it\sqrt{P+W+1}} \langle r \rangle^{-\alpha}\| \lesssim \langle t \rangle^{-\frac{n}{2}}$$

for $\alpha > \left[\frac{n}{2}\right] + 2$.

Remark 1.1. In this work we only focus on the flow of the equations in the low frequency regime. Although this might seem a restriction, we point out that it is the low part of the spectrum which dictates the decay rate of the solutions. Indeed, for high frequency up to non trapping assumptions on the geodesic flow one can obtain arbitrary fast decay in time. See for example [Wan06] and [Vod04].

Remark 1.2. We will show in detail all the proofs in the case $W = 0$, since the addition of W requires only minor adaptations for which the reader can refer to Section 5.

Proving local energy decay for these equations, especially the wave and Klein-Gordon ones on manifolds, is a fundamental question in scattering theory which dates back to the work by Morawetz [Mor61], in which the author considers the flat wave equation outside of an obstacle. The topic is still the subject of recent works, such as the previously mentioned [BB21] or [Mor20], [MW21], in the latter the focus is on the influence of the decay rate of the metric on the decay rate of the solution. Indeed, the full picture of how the energy of the wave equation should decay for long range perturbations is not yet clear and it is therefore desirable to find approaches which are robust enough to allow for this type of perturbations.

We describe how our work compares to known results in this setting, while for an overview of results in the case of an asymptotically euclidean geometry the reader can refer to the introduction of [BB21].

In [GHS13] the authors consider a manifold with a scattering metric defined via a family of smooth metrics h. After the change of variable $r = 1/x$ and Taylor expansion of h the scattering

metric results in a particular case of Definition 1.3. More precisely, $g(r)$ would be given by the Taylor expansion of $h(1/r)$ around 0, ∂M corresponds to S and $h(0)$ to $\overline{g}$. There the authors take a geometric approach to obtain an expression for (1.1) as a sum of pseudodifferential operators and Legendre distributions. From this, they derive an explicit expansion for the Schwartz kernel of the spectral measure as $\lambda \to 0$, which is applied to obtain long time expansions for the Schrödinger and wave operators. The decay rate depends on the spectrum of the operator at infinity. In particular, using the notation of Section 1.1 of the present article, on the smallest eigenvalue of the Laplace-Beltrami operator on $(S, \overline{g})$ and the decay of the potential at infinity. In [GHS13] the potential is allowed to be negative, however with a control on the negative part (see condition (1.2) therein) and with a decay rate of -2. In this work the authors make the assumption of no zero resonance nor eigenvalue. While [GHS13] gives sharper estimates (and results in relation to Price's law) the geometric framework is more restrictive than ours.

A similar approach is used in [Wan06]. Here the result provides an asymptotic expansion for the resolvent, however allowing for the presence of zero eigenfunctions or resonant states, which unlike the Euclidean case are still present in higher dimension. The author first considers the operator on a manifold with an exactly conical end and then uses this result to treat the case of a perturbation by metric and by potential. The result also allows a decaying term of order -2, which can not be dealt with using perturbative arguments. We remark that to treat the model operator on the exact cone it is used a diagonalisation on the angular manifold similar to the one we present in Appendix A. In [Wan06] the author starts with a potential with decay as in the present work (condition (1.5) in [Wan06]), but a stronger decay and stronger weights are required to write the expansion of the Schrödinger evolution (cfr. Theorems 6.3 and 6.4). The improvement of our paper is the fact that we can allow a long range decay.

In [VW09], instead, in the case of a scattering manifold the authors take a similar approach to the one of the present paper using positive commutator estimates to prove dispersive properties for the flow of the wave equation. Although the positive commutator estimate is proved for potentials in the same class as the present work, decay of the flow is only recovered for potentials with a stronger decay.

The method used to prove Theorem 1.1 is the same as [BB21] and it relies on results on the regularity of the spectral measure and on quantification of estimates on the operators (1.1) (and its powers). We now state the two theorems from which Theorem 1.1 follows. We will prove these in Section 2, while the derivation of Theorem 1.1, being analogous to what is presented in Section 5 of [BB21], is omitted here.

First, recall the definition of spectral measure. Let E_Ω the indicator function of a set $\Omega \subset \mathbb{R}$, then for every $u, v \in L^2(M)$ the map $\Omega \mapsto (u, E_\Omega(P+W)v)$ is a well defined Borel measure. To say that we integrate λ with respect to this measure we write $d(u, E_\lambda(P+W)v)$. We call this measure the spectral measure of $P+W$ associated to u and v. It satisfies the property

$$(u, f(P+W)v) = \int f(\lambda) d(u, E_\lambda(P+W)v)$$

for any bounded Borel function f, or in short

$$f(P+W) = \int f(\lambda)\ dE_\lambda.$$

The following theorem gives us regularity results on E_λ. These can be used to prove Theorem 1.1, after writing the flow of the equation as an oscillatory integral against the spectral measure.

Theorem 1.2. *Let $n \geq 3, \lambda_0 > 0$, $\alpha > k$ and P, W as in Theorem 1.1. Let E_λ the spectral measure of $P+W$. The function*

$$\lambda \mapsto \langle r \rangle^{-\alpha} \frac{dE_\lambda}{d\lambda} \langle r \rangle^{-\alpha}$$

is of class $C^{k-1}((0,\lambda_0])$. *Moreover if* $\alpha > \frac{n}{2}$ *then*

$$\left\| \langle r \rangle^{-\alpha} \frac{d^j}{d\lambda^j} E_\lambda \langle r \rangle^{-\alpha} \right\| \lesssim \lambda^{\frac{n}{2}-j-1}$$

for all $j = 1, \dots, k-1$.

The strategy to prove Theorem 1.2 is to use Stone's formula to write the spectral measure in terms of the limiting values of the resolvent and then use the following uniform resolvent estimates.

Theorem 1.3. *Let* $n \geq 3$ $\lambda_0 > 0$, $l \in \mathbb{N}$, $\alpha > l$ *and* P, W *as in Theorem 1.1. There exists a constant* C *such that for every* $\lambda \in (0, \lambda_0]$ *it holds*

$$\| \langle r \rangle^{-\alpha} (P + W - \lambda \pm i0)^{-l} \langle r \rangle^{-\alpha} \| \leq C \lambda^{\min\{0, n/2 - l\}}$$

if $l \neq \frac{n}{2}$ *and*

$$\| \langle r \rangle^{-\alpha} (P + W - \lambda \pm i0)^{-\frac{n}{2}} \langle r \rangle^{-\alpha} \| \leq C |\log \lambda|$$

if n *is even and* $l = \frac{n}{2}$.

Remark 1.3. Some low frequency estimates on the resolvent in weighted L^2 spaces can be found in [BR14b]. The bounds we recover in Theorem 1.3 hold for all powers of the resolvent and are sharp with respect to the behaviour in λ, unlike the ones presented in [BR14b] which only provide boundedness with respect to λ.

The method in the proof of Theorem 1.3 follows similar steps as the one presented in [BB21] up to the proof of low frequency exact Mourre estimates (Proposition 3.2). We refer to Section 3 for more details, only mentioning here that key point is to get rid of the compact remainder in a parameter dependent Mourre estimate.

Indeed, the arguments used in [BB21] rest on the underlying Euclidean geometry and can not be applied to this more general context. In our case the fact that the operator has non constant coefficients will not allow us to commute derivatives with resolvents. Even though on the exact cone we will be able to recover some useful features of the flat case, like the fact that $\partial^{j,k}(-\Delta)^{-1}$ is a bounded operator, we will need a more careful spectral analysis to be able to apply similar properties. For example, in the exact conic case we will reduce the problem to dimension one thanks to separation of variables and we will see how we can control our resolvent by studying the one dimensional resolvent corresponding to the spherical Laplacian.

This is also the reason for our assumption on the dimension, that we take grater or equal than three (as opposed to [BB21] where all dimensions greater or equal than two are covered). This will allow us to use Hardy inequalities on L^2 that will be necessary, for example in Appendix D.

The rest of the paper is organised as follows: we conclude this introduction with precise formulations of the properties we require on the infinite end $M \setminus K$ and with definitions of the rescaled pseudodifferential operators we will use in the computations; in Section 2 we give the proofs of Theorems 1.2 and 1.3 under some conditions that we then prove in Sections 3 and 4; finally Section 5 describes how the arguments adapt to the case of addition of the potential W.

1.1 Definitions

Let $n \geq 3$, in the following we will consider $(S, \overline{g})$ an $n-1$ dimensional closed Riemannian manifold with local coordinates $(\theta_1, \dots, \theta_{n-1})$, we will use these objects to define the angular part of conical manifolds.

We also need to define some notion of decay with respect to the radial variable.

Definition 1.1. Let $f(r)$ a smooth function of r with values in the space of (h,k) tensor fields (i.e. sections of the (h,k) tensor bundle $(\otimes^h TS) \otimes (\otimes^k T^*S)$). Let θ local coordinates in a patch around

a point $\omega \in S$ and $f^{i_1,\dots i_h}_{j_1\dots j_k}(r,\theta)$ the coefficients of f with respect to a basis of $(\otimes^h T_\omega S)\otimes(\otimes^k T^*_\omega S)$. Then $f(r)$ is in the class $S^{-\nu}$ if

$$|\partial_r^l \partial_\theta^\alpha f^{i_1,\dots i_h}_{j_1\dots j_k}(r,\theta)| \lesssim \langle r\rangle^{-\nu-l}$$

locally uniformly in θ, for all $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{n-1}$.

We will equip S either with the fixed metric $\overline{g}$ or with a metric $g(r)$ depending on the radial coordinate and which is a perturbation (in a $S^{-\nu}$ sense) of $\overline{g}$, meaning that we assume

$$g(r) - \overline{g} \in S^{-\nu}. \tag{1.2}$$

The geometrical setting for all our analysis will be the following.

Definition 1.2 (Asymptotically conical manifold)**.** Let (M, G_M) a manifold of dimension n with $K \subset M$ compact. M is said to the asymptotically conical if there exist $R > 0$ and a diffeomorphism

$$\begin{aligned}\Omega : M \setminus K &\to (R,+\infty)\times S\\ m &\mapsto (r(m),\omega(m))\end{aligned}$$

such that $r : M \to [R,+\infty)$ is a proper function and the metric G_M is given by

$$G_M = \Omega^*(dr^2 + r^2 g(r)).$$

Remark 1.4. Although with different notation, this is the same geometric framework as the one used in [IN10].

If $\kappa : U_\kappa \subset S \to V_\kappa \subset \mathbb{R}^{n-1}$ are the coordinate charts on S we will denote by $\Pi_\kappa, \Pi_\kappa^{-1}$ the pullback and pushforward on $[R,+\infty)\times S$; moreover if $(\varphi_\kappa)_\kappa$ is a partition of unity on S and φ a smooth cutoff on $[R,+\infty)$ we will make use of the functions

$$\psi_\kappa(r,\omega) := \varphi(r)\varphi_\kappa(\omega) \in C_0^\infty([R,+\infty)\times U_\kappa)$$

which verify $\sum_\kappa \psi_\kappa \equiv 1$ for large enough r.

Remark 1.5. A function on $M \setminus K$ can be identified with a function on $(R,+\infty)\times S$ thanks to Ω. As we will basically always consider the corresponding quantities on $(R,+\infty)\times S$ we will drop the composition by Ω, which rigorously is the one that allows to pass from a point on the manifold to a point on $(R,+\infty)\times S$. This means that we will simply use the notation (r,ω) for a point of $M\setminus K$ and still denote by ψ_κ, Π_κ or Π_κ^{-1} the corresponding functions defined on $M\setminus K$.

In the subsequent table we group the different notations we introduce for the manifolds with their respective metrics, Hilbert spaces and the associated Laplace-Beltrami operators.

manifold	metric	Hilbert space	L-B operator
M	$G_M = \Omega^*(dr^2 + r^2 g(r))$	$L^2(M)$	P
$[R,+\infty)\times S$	$\overline{G} = dr^2 + r^2\overline{g}$	$L^2_{\overline{G}}$	$-\Delta_0$

We also define

$$G = dr^2 + r^2 g(r) \tag{1.3}$$

the perturbed metric on $[R,+\infty)\times S$ and $\|\cdot\|_{L^2_G}$ the L^2 norm with respect to this metric. We recall that $\|\cdot\|$ denotes the norm of the operators on $L^2(M)$.

We remark that having quantified in (1.2) how much the metric $g(r)$ deviates from $\overline{g}$ we can compare the two operators $-\Delta_0$ and P and also obtain that the norms of $L^2_{\overline{G}}$ and $L^2(M)$ are comparable, meaning that their quotient is bounded by constants from above and from below.

Throughout the whole paper we will have to consider convenient rescaled operators as follows. Since we are interested in resolvents such as $(P-\lambda\pm i0)^{-1}$ we consider the operator P/λ which it is convenient to study using rescaled pseudodifferential operators, that we now define.

Definition 1.3. A function $a(r,\theta,\rho,\eta)$ is in $\tilde{S}^{m,\mu}(\mathbb{R}^{2n})$ if and only if for every $j,k\in\mathbb{N},\alpha,\beta\in\mathbb{N}^{n-1}$ there exists a constant C such that

$$|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\eta^\beta a(r,\theta,\rho,\eta)|\leq C\langle r\rangle^{m-j-|\beta|}\left(\langle\rho\rangle+\frac{\langle\eta\rangle}{\langle r\rangle}\right)^{\mu-k-|\beta|} \tag{1.4}$$

with

$$\langle r\rangle:=\begin{cases}1 & \text{if } r \text{ is in a compact set,}\\ r & \text{if } r\gg 1.\end{cases}$$

The seminorms of the space are given by the smallest constants verifying the inequality.

Remark 1.6. Although we are using the same notation as the radial coordinate on the manifold, here r is simply meant to denote the first variable of $\mathbb{R}^n$.

We consider the usual quantization of a symbol defined as

$$Op(a)f(r,\theta)=\frac{1}{(2\pi)^n}\int\int e^{i(r-r')\rho+i(\theta-\theta')\eta}a(r,\theta,\rho,\eta)f(r',\theta')dr'd\theta'\ d\rho d\eta$$

and we introduce the dilation operator with respect to r and its generator A, namely

$$e^{itA}u(r,\theta)=e^{\frac{tn}{2}}u(e^t r,\theta),\qquad A:=\frac{n}{2i}-ir\partial_r.$$

For a symbol a defined on $\mathbb{R}^{2n}$ a rescaled pseudodifferential operator is defined as

$$Op_\lambda(a):=e^{i\tau A}Op(a)e^{-i\tau A}. \tag{1.5}$$

Taking $\tau=\ln(\lambda^{\frac{1}{2}})$ implies a rescaling of the spatial variable by $\lambda^{\frac{1}{2}}$, in other words

$$Op_\lambda(a)=Op(a_\lambda)$$

with $a_\lambda(\breve{r},\theta,\breve{\rho},\eta)$ and $\breve{r}:=\lambda^{\frac{1}{2}}r$, $\breve{\rho}:=\lambda^{-\frac{1}{2}}\rho$.

Analogously we define rescaled pseudodifferential operators on manifolds as

$$Op_{\lambda,\kappa}(a)\psi_\kappa(\lambda^{\frac{1}{2}}r,\omega):=\Pi_\kappa Op_\lambda(a)\Pi_\kappa^{-1}\psi_\kappa(\lambda^{\frac{1}{2}}r,\omega) \tag{1.6}$$

for a symbol a supported in $[R,+\infty)\times V_\kappa\times\mathbb{R}^n$. We remark that $Op_{\lambda,\kappa}(a)\psi_\kappa(\lambda^{\frac{1}{2}}r,\omega)$ maps $C_0^\infty(M)$ in the set of functions supported in $[R,+\infty)\times U_\kappa$.

For example, near infinity P agrees with the Laplace-Beltrami operator on $(R+\infty)\times S$ and therefore in local coordinates it is given by

$$P/\lambda=-\frac{\partial_r^2}{\lambda}-\frac{n-1}{r}\frac{\partial_r}{\lambda}-\frac{1}{\lambda r^2}\Delta_{g(r)}-\frac{\partial_r|g(r,\theta)|}{|g(r,\theta)|}\frac{\partial_r}{\lambda}. \tag{1.7}$$

In terms of pseudodifferential operators this can be written as

$$\frac{P}{\lambda}u=\sum_\kappa Op_{\lambda,\kappa}(a_{0,\lambda}+a_{1,\lambda})\psi_\kappa u \tag{1.8}$$

where the symbols are

$$\begin{aligned}a_{0,\lambda}(\breve{r},\theta,\breve{\rho},\eta)&:=-\breve{\rho}^2-\frac{1}{\breve{r}^2}g^{j,k}(\lambda^{-\frac{1}{2}}r,\theta)\eta_j\eta_k,\\ a_{1,\lambda}(\breve{r},\theta,\rho,\eta)&:=-\frac{n-1}{\breve{r}}\breve{\rho}-\lambda^{-\frac{1}{2}}w(\lambda^{-\frac{1}{2}}\breve{r},\theta)\breve{\rho}-\lambda^{-1}w_k(\lambda^{-\frac{1}{2}}\breve{r},\theta)\eta_k\end{aligned}$$

for some w,w_k depending on the metric $g(r)$ and such that $w\in S^{-1-\nu}$ and $w_k\in S^{-2}$.

Remark 1.7. Considering rescaled pseudodifferential operators is convenient since it allows us to obtain a decay which is uniform with respect to λ, meaning that the symbols in $Op_\lambda(\cdot)$ will belong to λ-independent subsets of $\tilde{S}^{m,\mu}(\mathbb{R}^{2n})$ for some m and μ. Indeed for $\breve{r} \gtrsim 1$ we obtain

$$a_{0,\lambda}(\breve{r},\theta,\breve{\rho},\eta) \in \tilde{S}^{0,2}, \quad a_{1,\lambda}(\breve{r},\theta,\breve{\rho},\eta) \in \tilde{S}^{-1,1},$$

where the bounds on the seminorms are uniform in λ.

Remark 1.8 (Notation). When using symbolic calculus we will often be interested in the decay properties of the symbols, rather than in their explicit expression. For this reason we will use the shorthand

$$Op_{\lambda,\kappa}(\tilde{S}^{m,\mu})$$

to denote a rescaled pseudodifferential operator with symbol in $\tilde{S}^{m,\mu}$.

2 Main results

As mentioned in the introduction, we focus on the case $W = 0$ and details about the proof for $P + W$ can be found in Section 5.

In this section we see how to prove the results in Theorems 1.2 and 1.3, under some conditions whose proof is postponed to Section 3. There we shall prove that there exists an operator A^λ selfadjoint on $(L^2(M), G_M)$ which satisfies Proposition 3.2, that is

$$\mathbb{1}_I(P/\lambda) i[P/\lambda, A^\lambda] \mathbb{1}_I(P/\lambda) \geq \mathbb{1}_I(P/\lambda)$$

for small enough positive λ and I an open neighborhood of 1.

For the construction and precise definition of A^λ we refer to Section 3. For the moment we only point out that A^λ is the generator of a unitary group and that in symbolic form it is given by

$$A^\lambda = \sum_\kappa Op_{\lambda,k}(\tilde{S}^{1,1}). \tag{2.1}$$

Applying Mourre theory in the following section we will be able to prove

$$(A^\lambda + i)^{-s}(P/\lambda - 1 \pm i0)^{-l}(A^\lambda + i)^{-s} \in \mathcal{L}(L^2(M)) \tag{2.2}$$

for any natural $s > l - \frac{1}{2}$ with operator norm uniformly bounded in λ for $\lambda \in (0, \lambda_0]$. Moreover we also obtain that the map

$$\tau \mapsto (A^\lambda + i)^{-s}(P/\lambda - \tau \pm i0)^{-l}(A^\lambda + i)^{-s} \tag{2.3}$$

is of class C^{l-1} in the interior of an interval where the positive commutator estimate holds.

The first step into proving Theorem 1.3 will be to look at the resolvent with a spectral localisation, such as

$$\|\langle r\rangle^{-\alpha} f(P/\lambda)(P - \lambda \pm i0)^{-l}\langle r\rangle^{-\alpha}\|$$

with $f \in C_0^\infty(\mathbb{R})$. To obtain this using (2.2) we will need to bound $\langle r\rangle^{-\alpha} f(P/\lambda)(A^\lambda + i)^s$ and we will use Helffer-Sjöstrand formula to evaluate $f(P/\lambda)$. The following property of the resolvent will then be useful.

Theorem 2.1. *Let $\psi, \tilde{\psi}, \tilde{\tilde{\psi}} \in S^0$, supported in $(R, +\infty) \times U_k$ such that*

$$\tilde{\psi}\psi = \psi, \qquad \tilde{\tilde{\psi}}\tilde{\psi} = \tilde{\psi}.$$

and $z \in \mathbb{C} \setminus [0, \infty)$. Then for any $N \in \mathbb{N}$ there exist families of symbols $b_{l,\lambda,z} \in \tilde{S}^{-l,-2-l}$ and $r_{N,\lambda,z} \in \tilde{S}^{-N,-N}$ such that

$$\psi(\lambda^{\frac{1}{2}} r, \omega)(P/\lambda - z)^{-1} = \sum_{l=0}^{N-1} \psi(\lambda^{\frac{1}{2}} r, \omega) Op_{\lambda,k}(b_{l,\lambda,z}) \tilde{\psi}(\lambda^{\frac{1}{2}} r, \omega) + R^N_{\lambda,z}$$

with

$$R^N_{\lambda,z} = \psi(\lambda^{\frac{1}{2}} r, \omega) Op_{\lambda,k}(r_{N,\lambda,z}) \tilde{\tilde{\psi}}(\lambda^{\frac{1}{2}} r, \omega)(P/\lambda - z)^{-1}.$$

Moreover, all of the symbols have seminorms uniformly bounded in λ.

The proof is simply by standard techniques for the construction of a parametrix. We use the previous result in the following lemma.

Lemma 2.2. *Let* $g \in C_0^\infty(\mathbb{R})$, $\lambda_0 > 0$ *and* $s, N \in \mathbb{N}$. *There exist a family of symbols* $(\phi^\kappa_{s,\lambda})_{\lambda\in(0,\lambda_0]} \in \tilde{S}^{s,-N}$, *a family of uniformly bounded operators* $(B_\lambda)_{\lambda\in(0,\lambda_0]}$ *and* $(\tilde{\psi}_\kappa)_\kappa \in C^\infty([R,+\infty)\times S)$ *supported in* $(R,+\infty)\times U_\kappa$ *with* $\tilde{\psi}_\kappa\psi_\kappa \equiv 1$ *such that*

$$(A^\lambda + i)^s g(P/\lambda) = \sum_\kappa Op_{\lambda,\kappa}(\phi^\kappa_{s,\lambda})\tilde{\psi}_\kappa(\lambda^{\frac{1}{2}} r, \omega) + B_\lambda(P/\lambda + 1)^{-N}$$

for $\lambda \in (0, \lambda_0]$.

Proof. By Helffer-sjöstrand formula we write $g(P/\lambda)$ in terms of its resolvent, then applying Theorem 2.1 we can replace the resolvent with the parametrix. We recall the property of almost analytic extensions

$$\frac{1}{2\pi}\int \overline{\partial}_z \tilde{g}(z)(\mu - z)^{-1-j} L(dz) = \frac{(-1)^j}{j!} g^{(j)}(\mu). \tag{2.4}$$

Thanks to the expression of the symbols in the parametrix and using (2.4) we obtain symbols in $g(P/\lambda)$ that have negative decay in space and compact support in the angular part. Namely, we can write for any $M \in \mathbb{N}$

$$\psi_\kappa(\lambda^{\frac{1}{2}} r, \omega) g(P/\lambda) = Op_{\lambda,\kappa}(\tilde{S}^{0,-M})\tilde{\psi}_\kappa(\lambda^{\frac{1}{2}} r, \omega) + R_{\lambda,z}(P/\lambda + 1)^{-N} \tag{2.5}$$

where $R_{\lambda,z}$ includes the integral of the remainder part given by the parametrix. Next we need to compose (2.5) on the left with powers $(A^\lambda)^j$ of order $j \le s$. By choosing the appropriate M ($M = s + N$), we can conclude observing that $(A^\lambda)^j$ will have symbols in $\tilde{S}^{j,j}$ (see (2.1)) and that $R_{\lambda,z}$ is

$$R_{\lambda,z} = \frac{1}{\pi i}\int \overline{\partial}_z \tilde{f}(z) Op_{\lambda,\kappa}(r^K_{\lambda,z})(P/\lambda - z)^{-1}(P/\lambda + 1)^N L(dz),$$

where $r^K_{\lambda,z} \in \tilde{S}^{-K,-K}$ has seminorms growing polynomially in $1/|Imz|^K$ for any $K \in \mathbb{N}$.

□

Remark 2.1. In the previous proof we used the fact that symbols in $\tilde{S}^{0,0}$ correspond to bounded operators of $L^2(M)$. For a proof in the case of the rescaled pesudodifferential operators we are using here see Proposition 3.4 in [BM16] and in particular inequality (3.13).

We now derive some useful properties in order to handle powers of the resolvent, as the one in the statement of Lemma 2.2.

Lemma 2.3. *Let* $p \in [1,2]$ *and an integer* $N \ge 1 + \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$. *There exists* $C > 0$ *such that*

$$\|(P/\lambda + 1)^{-N}\|_{L^p(M)\to L^2(M)} \le C\lambda^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}$$

for all $\lambda > 0$.

The result is derived thanks to the behaviour of the flow of the heat equation e^{-tP}

$$\|e^{-tP}\|_{L^p(M)\to L^2(M)} \le Ct^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}, \quad p \in [1,2] \tag{2.6}$$

which in turn is due to the fact that a Nash type inequality holds. Namely, for all $u \in C_0^\infty(M)$

$$\|u\|^{1+\frac{2}{n}}_{L^2(M)} \le C_n \|u\|^{\frac{2}{n}}_{L^1(M)} \|P^{\frac{1}{2}} u\|_{L^2(M)} \tag{2.7}$$

for some $C_n > 0$.

The inequality is proved in detail in Appendix B, here we briefly record that it is done by first considering the full cone $\mathbb{R}^+ \times S$ with fixed metric $\overline{G}$ so that locally we can apply the Nash inequality on $\mathbb{R}^n$. The result still holds on M since on the compact part K we can use a finite covering to reduce ourselves to $\mathbb{R}^n$, while on the manifold end we use the inequality obtained for the cone.

To derive (2.6) we can interpolate $\|e^{-tP}\|_{L^2(M)\to L^2(M)} \lesssim 1$, given by the Hille-Yosida theorem, and $\|e^{-tP}\|_{L^1(M)\to L^2(M)} \lesssim t^{-\frac{n}{4}}$ obtained from (2.7) and the fact that e^{-tP} preserves the sign and the L^1 norm.

Proof of Lemma 2.3. We can follow the same proof described in Lemma 3.2 of [BB21]. Writing the resolvent via the heat kernel

$$(P/\lambda + 1)^{-N} = \frac{1}{N!}\int_0^\infty e^{-t(P/\lambda+1)} t^{N-1} dt$$

we apply (2.6) and the fact that, given the assumption on N, $e^{-t}t^{N-1-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}$ is integrable on $\mathbb{R}^+$. □

Thanks to Lemma 2.3 we also easily obtain polynomial decay for powers of the resolvent. This lemma will also be used extensively in Section 4.

Lemma 2.4. *Let $n \geq 3$, for all $s \in [0, \frac{n}{4}] \cap [0, N)$ and $\frac{\sigma}{2} > s$ then there exists $C > 0$ such that*

$$\|(P/\lambda + 1)^{-N}\langle r\rangle^{-\sigma}\| \leq C\lambda^s$$

for all $\lambda > 0$.

Proof. The statement follows from Lemma 2.3 and Hölder inequality to bound $\langle r\rangle^{-\sigma}$ as an operator from $L^2(M)$ to $L^p(M)$. □

Now, using the expression given in Lemma 2.2 we can easily derive the following proposition.

Proposition 2.5. *Let $g \in C_0^\infty(\mathbb{R})$, $\alpha \geq l$, $\lambda_0 > 0$ and $s \in (0, \frac{n}{4}] \cap (0, \frac{\alpha}{2})$. Then*

$$\|(A^\lambda + i)^l g(P/\lambda)\langle r\rangle^{-\alpha}\| \lesssim \lambda^s$$

for $\lambda \in (0, \lambda_0]$.

Proof. From Lemma 2.2 and the fact that operators with symbols in $\tilde{S}^{0,0}$ are bounded, paired with Lemma 2.4 to control $\|B_\lambda(P/\lambda + 1)^{-N}\langle r\rangle^{-\alpha}\|$. We underline that a power $\lambda^{\frac{\alpha}{2}}$ is generated from the terms $Op_{\lambda,\kappa}(\phi^\kappa_{s,\lambda})\tilde{\psi}_\kappa(\lambda^{\frac{1}{2}}r, \omega)\langle r\rangle^{-\alpha}$ when moving the factor $\langle r\rangle^{-\alpha}$ into the rescaled pseudodifferential operator. □

Combining (2.2) with Proposition 2.5 we can straightforwardly obtain bounds on the spectrally localised resolvent.

Theorem 2.6. *Let $f \in C_0^\infty(\mathbb{R})$, $\lambda_0 > 0$ and $l \in \mathbb{N}$. If $\alpha \geq l$ and $s \in [0, \frac{n}{4}] \cap (0, \frac{\alpha}{2})$ then*

$$\|\langle r\rangle^{-\alpha} f(P/\lambda)(P - \lambda \pm i0)^{-l}\langle r\rangle^{-\alpha}\| \lesssim \lambda^{2s-l}$$

for $\lambda \in (0, \lambda_0]$.

With the aid of this theorem we obtain a bound on the resolvent which is still localised, but in a weaker way. The proof, being analogous to Proposition 4.4 in [BB21], is omitted here.

Proposition 2.7. *Let $\lambda_0 > 0$, there exists $F \in C_0^\infty(\mathbb{R})$ equal to 1 near $[0, \lambda_0]$ such that for $\alpha > l$ and $l \in \mathbb{N} \setminus \{\frac{n}{2}\}$*

$$\|\langle r\rangle^{-\alpha} F(P)(P - \lambda \pm i0)^{-l}\langle r\rangle^{-\alpha}\| \lesssim \lambda^{\min\{0, n/2-l\}}$$

for all $\lambda \in (0, \lambda_0]$. If $l = \frac{n}{2}$

$$\|\langle r\rangle^{-\alpha} F(P)(P - \lambda \pm i0)^{-\frac{n}{2}}\langle r\rangle^{-\alpha}\| \lesssim |\log \lambda|$$

for all $\lambda \in (0, \lambda_0]$.

Thanks to these preliminary steps we are now ready to prove the estimate without any localisation on the resolvent.

Proof of Theorem 1.3. Pick F as in the previous proposition, the result then follows from Proposition 2.7 since $(1 - F(P))(P - \lambda \pm i0)^{-l}$ is uniformly bounded in λ by the spectral theorem. □

As for the result on the spectral measure, we recall that thanks to Stone's formula

$$\frac{dE(\lambda)}{d\lambda} = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \left((P - \lambda - i\varepsilon)^{-1} - (P - \lambda + i\varepsilon)^{-1}\right),$$

we can equivalently consider outgoing and incoming resolvents so to use the result we just established in Theorem 2.6.

Proof of Theorem 1.2. Let $f \in C_0^\infty(\mathbb{R})$ bounded and supported around 1 then we have

$$(1 - f(P/\lambda)) \left((P - \lambda - i\varepsilon)^{-1-j} - (P - \lambda + i\varepsilon)^{-1-j}\right) \to 0$$

in the strong topology as ε goes to 0. It then suffices to consider the terms

$$\langle r\rangle^{-\alpha} f(P/\lambda) \left((P - \lambda - i0)^{-1} - (P - \lambda + i0)^{-1}\right) \langle r\rangle^{-\alpha},$$

thanks to the regularity of the map (2.3) and Lemma 2.2 we deduce that

$$\langle r\rangle^{-\alpha} f(P/\lambda) \frac{d}{d\lambda}(P - \lambda - i0)^{-1}\langle r\rangle^{-\alpha} = \langle r\rangle^{-\alpha}(P - \lambda - i0)^{-2}\langle r\rangle^{-\alpha}.$$

In general, for higher derivatives we have

$$\frac{1}{2\pi i j!}\langle r\rangle^{-\alpha} f(P/\lambda) \left((P - \lambda - i0)^{-1-j} - (P - \lambda + i0)^{-1-j}\right) \langle r\rangle^{-\alpha}. \tag{2.8}$$

For $j = 0, \ldots, k-1$ Theorem 2.6 applies thanks to the assumption that $\alpha > k$. □

3 Limiting absorption principle

The section will be devoted to the proof of the existence of the limits $(P/\lambda - 1 \pm i0)^{-l}$ in weighted L^2 spaces thanks to a limiting absorption principle (specifically Theorem 1 in [Gé08]). Consequently, the section mainly concerns the construction of a conjugate operator A^λ (Remark 3.1) and the proof of a positive commutator estimate (Proposition 3.2). This will be possible thanks to the condition stated in Assumption 3.1. Proving that this condition holds will be the aim of Section 4.

We look for A^λ, a conjugate operator for P/λ, that is a selfadjoint operator which verifies some positive commutator estimate and such that $P/\lambda \in C^2(A^\lambda)$, meaning that for all $u \in L^2(M)$ the map

$$\mathbb{R} \ni t \mapsto e^{itA^\lambda}(P/\lambda + i)^{-1} e^{-itA^\lambda} u \in D(P) \tag{3.1}$$

is of class C^2.

To get selfadjointness we will construct A^λ as the generator of a unitary group. Let $\chi \in C_0^\infty(\mathbb{R})$ equal to 1 in a large enough neighborhood of 0. Consider the generator of dilations $\frac{rD_r+D_rr}{2}$, after localisation in the region $\{|r| \geq \lambda^{-\frac{1}{2}}R\}$ we obtain

$$A_1^\lambda := \frac{(1-\chi)(\lambda^{\frac{1}{2}}r)rD_r + D_r r(1-\chi)(\lambda^{\frac{1}{2}}r)}{2} = (1-\chi)(\lambda^{\frac{1}{2}}r)\left(\frac{1}{2i} - ir\partial_r\right) + \frac{i}{2}r\partial_r\chi(\lambda^{\frac{1}{2}}r).$$

We define the group of transformations

$$U_t^\lambda \varphi(r,\theta) = \left|\det(\text{Jac }\phi_t^\lambda(r,\theta))\right|^{\frac{1}{2}}\varphi(\phi_t^\lambda(r,\theta)),$$

where ϕ_t^λ is the flow of the complete vector field $((1-\chi)(\lambda^{\frac{1}{2}}r)r, 0, \dots, 0)$. Thanks to Theorem VIII.10 in [RS81] we can conclude that A_1^λ is essentially selfadjoint on $C_0^\infty([R,+\infty)\times S)$ with respect to the measure induced by the metric $\overline{G}$ and that its closure is the infinitesimal generator of U_t^λ.

Moreover conjugating U_t^λ by the function

$$y_S(r,\theta) := \frac{|g(\theta)|}{r^{n-1}|g(r,\theta)|}$$

we obtain a group $y_S^{\frac{1}{2}}U_t^\lambda y_S^{-\frac{1}{2}}$ which is unitary with respect to the metric G defined in (1.3) and whose generator will be

$$\begin{aligned} A_2^\lambda =& (1-\chi)(\lambda^{\frac{1}{2}}r)\left(\frac{n}{2i} - ir\partial_r + \frac{1}{2i}r\frac{\partial_r|g(r,\theta)|}{|g(r,\theta)|}\right) + \frac{i}{2}r\partial_r\chi(\lambda^{\frac{1}{2}}r) \\ =:& (1-\chi)(\lambda^{\frac{1}{2}}r)\tilde{A} + \frac{i}{2}r\partial_r\chi(\lambda^{\frac{1}{2}}r). \end{aligned} \tag{3.2}$$

Remark 3.1 (Definition of A^λ). To define a unitary group which acts on the Hilbert space where P is defined, that is $L^2(M)$, we set

$$e^{itA^\lambda}u := e^{itA_2^\lambda}\chi_{M\setminus K}u + \chi_K u$$

whose generator is the operator A^λ, which is selfadjoint on $L^2(M)$ and non zero only on the manifold end where it coincides with A_2^λ.

For the continuity of P/λ with respect to A^λ, that is continuity of (3.1), it is enough to prove that the operators

$$[P/\lambda, iA^\lambda](P/\lambda+i)^{-1}, \qquad [[P/\lambda, iA^\lambda], iA^\lambda](P/\lambda+i)^{-1} \tag{3.3}$$

are bounded, where the commutators are appropriately defined in the sense of quadratic forms. Indeed, in general given an Hilbert space $\mathcal{H}$ with T, A selfadjoint and T bounded for the map

$$\mathbb{R} \ni t \mapsto e^{itA}Te^{-itA}u =: B(t)u \in \mathcal{H}$$

to be $C^k(\mathbb{R})$ it is enough that k-th derivative $\frac{d^k}{dt^k}B(t_0)$ is a bounded operator of $\mathcal{L}(\mathcal{H})$ for some fixed t_0. This in turn is implied if

$$T,\ ad_A^0(T) := [T, iA], \quad ad_A^j(T) := [ad_A^{j-1}(T), iA]$$

are bounded operators, in the sense of quadratic forms for $j = 1, \dots, k-1$.

In our specific case we need to control only the commutator and the first iterated commutator with $T = (P/\lambda+i)^{-1}$ and $A = A^\lambda$. With some algebraic manipulations we see that it is equivalent

to the boundedness of the operators in (3.3). To prove this we will exploit symbolic calculus and Theorem 2.1.

Before going on with the computations we will state a useful property. Briefly, commutators between rescaled pseudodifferential operators essentially behave like commutators between differential operators, when we write them in symbolic form. We now give a formal description of the result, while the proof, being quite technical, is postponed to Appendix C.

Definition 3.1 (Negligible operator of order N)**.** Let $N \in \mathbb{N}$ and set

$$Q_N := \langle \lambda^{\frac{1}{2}} r \rangle^N \left(\frac{P}{\lambda} + 1 \right)^N = \sum_\kappa Op_{\lambda,\kappa}(\tilde{S}^{N,2N}),$$

we say that an operator is negligible of order N if it is of the form $Q_N^{-1} \mathcal{B} Q_N^{-1}$ for some bounded operator $\mathcal{B}$ depending on λ.

Remark 3.2. The operator $\mathcal{B}$ depends on λ since it will be the result of the composition of a rescaled pseudodifferential operator of negative order with Q_N. However, the symbols in Q_N have seminorms uniformly bounded with respect to λ and therefore so will $\mathcal{B}$.

Proposition 3.1. *Let m, m', μ, μ' real numbers and the operators A, B on $[R, +\infty) \times S$ defined as*

$$A := \sum_\kappa Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa, \qquad B := \sum_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa$$

with symbols in $a^\kappa \in \tilde{S}^{m,\mu}$ and $b^\kappa \in \tilde{S}^{m',\mu'}$ spatially supported in $[R, +\infty) \times V_\kappa$. Then for the commutator it holds

$$[A, B] = \sum_\kappa Op_{\lambda,\kappa}(\tilde{S}^{m+m'-1,\mu+\mu'-1})\psi_\kappa + \mathcal{R}_N$$

with $\mathcal{R}_N$ an operator which is negligible of order N for any $N \in \mathbb{N}$.

Remark 3.3. Let $C = Op_{\lambda,\kappa}(c^\kappa)$ with $c^\kappa \in \tilde{S}^{m,\mu}$, from Proposition 3.1 we can always find N large enough such that $\mathcal{R}_N C$ and $C\mathcal{R}_N$ are still bounded. Indeed, the interest of Definition 3.1 for N arbitrary is that Q_N^{-1} provides infinite decay both in the spatial and phase variables. In practice we will consider compositions of commutators with resolvents (as in (3.3) or Section 4) and the remainder term will always stay bounded as we just remarked.

Remark 3.4. Let $\mathcal{B}$ bounded and $M > 0$. In the computations we will write negligible operators in the forms

$$\mathcal{B} \sum_\kappa Op_{\lambda,\kappa}(\tilde{S}^{-M,-M}), \qquad \sum_\kappa Op_{\lambda,\kappa}(\tilde{S}^{-M,-M})\mathcal{B}$$

when we will need decay only on the right or the left respectively.

Remark 3.5. Although we interpreted commutators as derivatives of the map $e^{itA}(P/\lambda+i)^{-1}e^{-itA}$, to perform computations we will rather use their symbolic form, (see (3.4) below). Indeed, on $C_0^\infty(M)$ (which is dense in $D(P)$) we can prove that the derivative of $e^{itA}(P/\lambda + i)^{-1}e^{-itA}$ is the commutator $[(P/\lambda+i)^{-1}, iA^\lambda]$, which we can rewrite in terms of the commutator between P/λ and iA^λ. Now on smooth functions the action of P/λ and A^λ is the one of differential operators, this allows us to write the symbolic form used in (3.4).

Writing A^λ in its symbolic form as

$$A^\lambda = \sum_\kappa Op_{\lambda,k}(\tilde{S}^{1,1} + \lambda^{\frac{\nu}{2}} \tilde{S}^{-\nu,0} + C_0^\infty(\mathbb{R} \setminus \{0\}))$$

and using Proposition 3.1 as well as Remark 3.4 we first find that

$$[P/\lambda, iA^\lambda] = [\sum_\kappa Op_{\lambda,\kappa}(\tilde{S}^{0,2}), \sum_l Op_{\lambda,l}(\tilde{S}^{1,1})] = \sum_\kappa (Op_{\lambda,\kappa}(\tilde{S}^{0,2}) + \mathcal{B}Op_{\lambda,\kappa}(\tilde{S}^{-M,-M})) \tag{3.4}$$

for some bounded operator $\mathcal{B}$ and some positive M.

Combining with the information provided by the parametrix we obtain that $[P/\lambda, iA^\lambda](P/\lambda + i)^{-1}$ is indeed a sum of bounded operators (recall Remark 2.1).

An analogous result holds for the iterated commutator since we can still write it in the form

$$[[P/\lambda, iA^\lambda], iA^\lambda] = \sum_\kappa (Op_{\lambda,\kappa}(\tilde{S}^{0,2}) + \mathcal{B}Op_{\lambda,\kappa}(\tilde{S}^{-M,-M})) \tag{3.5}$$

and reason in the same way as before.

Remark 3.6. Actually, we remark here that we can iterate the argument as many times as needed. Indeed, continuing from (3.5), any iterated commutator is of the form

$$ad^k_{A^\lambda}(P/\lambda) = \sum_\kappa (Op_{\lambda,\kappa}(\tilde{S}^{0,2}) + \mathcal{B}Op_{\lambda,\kappa}(\tilde{S}^{-M,-M})).$$

We can therefore apply Theorem 2.1 as before to conclude that $ad^k_{A^\lambda}(P/\lambda)(P/\lambda + i)^{-1}$ is bounded, which implies that $P/\lambda \in C^k(A^\lambda)$ for any k.

Next, we will prove a positive commutator estimate for P/λ for which we need the following property that will be checked in Section 4.

Assumption 3.1. For all $\alpha > 0$ and $\varepsilon > 0$ there exist $\lambda_0 > 0$ and $f \in C_0^\infty(\mathbb{R})$ equal to 1 in a neighborhood of 1 such that

$$\|\langle \lambda^{\frac{1}{2}} r\rangle^{-\alpha} f(P/\lambda)\| \leq \varepsilon$$

for all $\lambda \in (0, \lambda_0]$.

With this we can prove the desired inequality.

Proposition 3.2. *Let $\lambda_0 > 0$ small enough, if Assumption 3.1 holds there exists I open bounded interval containing 1 such that*

$$\mathbb{1}_I(P/\lambda) i[P/\lambda, A^\lambda]\mathbb{1}_I(P/\lambda) \geq \mathbb{1}_I(P/\lambda).$$

for all $\lambda \in (0, \lambda_0]$.

Before giving the proof of the positive commutator estimate, we point out that thanks to this inequality coupled with the fact that $P/\lambda \in C^2(A^\lambda)$ we can apply Theorem 1 in [Gé08]. As a result we have

$$\sup_{\lambda\in(0,\lambda_0]} \sup_{\varepsilon>0} \|\langle A^\lambda\rangle^{-s}(P/\lambda - 1 \pm i\varepsilon)^{-1}\langle A^\lambda\rangle^{-s}\| < \infty \tag{3.6}$$

for $s > \frac{1}{2}$, or equivalently

$$(A^\lambda + i)^{-s}(P/\lambda - 1 \pm i\varepsilon)^{-1}(A^\lambda + i)^{-s} \in \mathcal{L}(L^2(M))$$

with operator norms uniformly bounded in λ and we take $s \in \mathbb{N}$. Finally, thanks to the higher regularity of P/λ stated in Remark 3.6 similar bounds can be proved for powers of the resolvent, therefore obtaining

$$(A^\lambda + i)^{-s}(P/\lambda - 1 \pm i0)^{-l}(A^\lambda + i)^{-s} \in \mathcal{L}(L^2(M)) \tag{3.7}$$

for any natural $s > l - \frac{1}{2}$ with norms uniformly bounded in λ. Indeed, our conjugate operator A^λ is, in particular, **uniformly conjugate** to P/λ according to Definition 5.1 in [BR14a] and **uniformly ∞-smooth** with respect to A^λ (see Definition 5.3 in [BR14a]). Following the ideas of [Jen85], it is proved in [BR14a] that with this smoothness properties we have estimates for powers of the resolvent analogous to the ones in (3.6). This then implies (3.7). Finally, Theorem 5.8 in [BR14a] gives us the regularity of the map mentioned in (2.3).

To prove Proposition 3.2 we will split the commutator in the part at infinity, where P is of the form (1.8) and $A^\lambda = \tilde{A}$ (see (3.2)), and treat the rest as a compactly supported perturbation (we recall that on the compact part of the manifold A^λ is simply zero). Namely, we can write

$$i[P/\lambda, A^\lambda] = (1-\tilde{\chi})(\lambda^{\frac{1}{2}} r) i[P/\lambda, \tilde{A}] + \tilde{\chi}(\lambda^{\frac{1}{2}} r) i[P/\lambda, A^\lambda] \tag{3.8}$$

where $\tilde{\chi}$ is a smooth cutoff equal to one on the support of χ. In local coordinates the Laplace-Beltrami operator $-\Delta_0$ on the fixed half cone $([R, +\infty) \times S, \overline{G})$ is

$$-\partial_r^2 - \frac{n-1}{r}\partial_r - \frac{1}{r^2}\Delta_{\overline{g}}, \tag{3.9}$$

for more details on the definition of the operator see Appendix A. Recalling the local coordinates expression in (1.7) we notice that on the manifold end we can write P in function of $-\Delta_0$. In doing so, thanks to the fact that $g(r)$ is a perturbation of $\overline{g}$ we can quantify the decay of the remaining part.

Proposition 3.3. *Let $\lambda_0 > 0$ and $\lambda \in (0, \lambda_0]$, then*

$$\begin{aligned}(1-\tilde{\chi})(\lambda^{\frac{1}{2}} r) P/\lambda =& (1-\tilde{\chi})(\lambda^{\frac{1}{2}} r)(-\Delta_0/\lambda) \\ &+ \sum_\kappa \left(\lambda^{\frac{\nu}{2}} Op_{\lambda,\kappa}(\tilde{S}^{-\nu,2}) + Op_{\lambda,\kappa}(\tilde{S}^{-1,1}) \right) \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)\end{aligned}$$

with symbols belonging to bounded subsets of $\tilde{S}^{-\nu,2}$ and $\tilde{S}^{-1,1}$ respectively.

Notably this will be useful since we are taking the commutator with $\tilde{A}$ given by

$$\tilde{A} = A + \sum_\kappa \lambda^{\frac{\nu}{2}} Op_{\lambda,\kappa}(\tilde{S}^{-\nu,0}) \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega). \tag{3.10}$$

Indeed, writing P/λ in terms of $-\Delta_0/\lambda$ and $\tilde{A}$ in terms of $A = \frac{n}{2i} - ir\partial_r$ allows us to take advantage of the identity $[-\Delta_0, iA] = 2(-\Delta_0)$. This last property can be checked by direct computations given the expression in (3.9).

We will also write $\tilde{A}$ in the form

$$\tilde{A} = \sum_\kappa Op_{\lambda,\kappa}(\tilde{S}^{1,1}) \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega) \tag{3.11}$$

which will be useful to treat the commutators of $\tilde{A}$ with the perturbative terms.

Proof of Proposition 3.2. Given (3.9) we have $-\Delta_0/\lambda = \sum_\kappa Op_{\lambda,\kappa}(\tilde{S}^{0,2}) \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)$. By Proposition 3.3, (3.10) and (3.11) we can compute

$$\begin{aligned}[P/\lambda, \tilde{A}] =& \sum_\kappa [-\Delta_0/\lambda, A + \lambda^{\frac{\nu}{2}} Op_{\lambda,\kappa}(\tilde{S}^{-\nu,0}) \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)] \\ &+ \sum_\kappa [(\lambda^{\frac{\nu}{2}} Op_{\lambda,\kappa}(\tilde{S}^{-\nu,2}) + Op_{\lambda,\kappa}(\tilde{S}^{-1,1})) \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega), Op_{\lambda,\kappa}(\tilde{S}^{1,1}) \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)] \\ =& 2(-\Delta_0/\lambda) + \sum_\kappa (\lambda^{\frac{\nu}{2}} Op_{\lambda,\kappa}(\tilde{S}^{-\nu,2}) + Op_{\lambda,\kappa}(\tilde{S}^{-1,1})) \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega) \\ &+ \sum_\kappa \mathcal{B} Op_{\lambda,\kappa}(\tilde{S}^{-M,-M})) \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)\end{aligned}$$

(we apply here the calculus rules given by Proposition 3.1 and the observation in Remark 3.4). On the support of $\psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)$

$$\lambda^{\frac{\nu}{2}} Op_{\lambda,k}(\tilde{S}^{-\nu,2}) = \langle \lambda^{\frac{1}{2}} r \rangle^{-\nu} Op_{\lambda,k}(\langle \lambda^{\frac{1}{2}} r \rangle^{\nu} \tilde{S}^{-\nu,2}) = \langle \lambda^{\frac{1}{2}} r \rangle^{-\nu} Op_{\lambda,k}(\tilde{S}^{0,2}),$$

and similarly

$$Op_{\lambda,k}(\tilde{S}^{-1,1}) = \langle \lambda^{\frac{1}{2}} r\rangle^{-1} Op_{\lambda,k}(\langle r\rangle \tilde{S}^{-1,1}) = \langle \lambda^{\frac{1}{2}} r\rangle^{-1} Op_{\lambda,k}(\tilde{S}^{0,1}),$$

so the quantity in (3.8) is given by

$$\begin{aligned} i[P/\lambda, A^\lambda] =& (1-\tilde{\chi})(\lambda^{\frac{1}{2}} r)\left(2P/\lambda + \sum_\kappa \langle \lambda^{\frac{1}{2}} r\rangle^{-\nu} Op_{\lambda,k}(\tilde{S}^{0,2}) + \langle \lambda^{\frac{1}{2}} r\rangle^{-1} Op_{\lambda,k}(\tilde{S}^{0,1})\right) \\ &+ (1-\tilde{\chi})(\lambda^{\frac{1}{2}} r)\sum_\kappa \mathcal{B} Op_{\lambda,\kappa}(\tilde{S}^{-M,-M}) \\ &+ \tilde{\chi}(\lambda^{\frac{1}{2}} r)[P/\lambda, iA^\lambda]. \end{aligned}$$

Moreover, up to a compactly supported perturbation we can commute $(1-\tilde{\chi})$ with any differential operator and in particular the pseudodifferential operators in the sum above are differential (they are the result of a commutator between differential operators). We have obtained

$$\begin{aligned} i[P/\lambda, A^\lambda] - 2P/\lambda =& -2\tilde{\chi}(\lambda^{\frac{1}{2}} r)P/\lambda \\ &+ \sum_\kappa \langle \lambda^{\frac{1}{2}} r\rangle^{-\nu} Op_{\lambda,k}(\tilde{S}^{0,2})(1-\tilde{\chi})(\lambda^{\frac{1}{2}} r) \\ &+ \sum_\kappa \langle \lambda^{\frac{1}{2}} r\rangle^{-1} Op_{\lambda,k}(\tilde{S}^{0,1})(1-\tilde{\chi})(\lambda^{\frac{1}{2}} r) \\ &+ (1-\tilde{\chi})(\lambda^{\frac{1}{2}} r)\sum_\kappa \mathcal{B} Op_{\lambda,\kappa}(\tilde{S}^{-M,-M}) \\ &+ \psi(\lambda^{\frac{1}{2}} r) \end{aligned}$$

for $\psi \in C_0^\infty(\mathbb{R}\setminus\{0\})$ with $\psi(\lambda^{\frac{1}{2}} r)$ which includes the term $\tilde{\chi}(\lambda^{\frac{1}{2}} r)[P/\lambda, iA^\lambda]$. Take f satisfying Assumption 3.1 and compose $i[P/\lambda, A^\lambda] - 2P/\lambda$ on the right and on the left with $f(P/\lambda)$. Noticing that by Theorem 2.1 and the spectral theorem

$$Op_{\lambda,k}(\tilde{S}^{0,2})(1-\tilde{\chi})(\lambda^{\frac{1}{2}} r)(P/\lambda + i)^{-1}, \quad Op_{\lambda,k}(\tilde{S}^{0,1})(1-\tilde{\chi})(\lambda^{\frac{1}{2}} r)(P/\lambda + i)^{-1},$$

and

$$(P/\lambda + i)f(P/\lambda), \quad P/\lambda(P/\lambda + i)f(P/\lambda)$$

are all bounded, we have the estimate

$$\begin{aligned} \|f(P/\lambda)\left(i[P/\lambda, A^\lambda] - 2P/\lambda\right) f(P/\lambda)\| \lesssim& \|f(P/\lambda)\tilde{\chi}(\lambda^{\frac{1}{2}} r)(P/\lambda + i)^{-1}\| && (3.12) \\ &+ \|f(P/\lambda)\langle \lambda^{\frac{1}{2}} r\rangle^{-\nu}\| + \|f(P/\lambda)\langle \lambda^{\frac{1}{2}} r\rangle^{-1}\| && (3.13) \\ &+ \sum_\kappa \|f(P/\lambda)\mathcal{B} Op_{\lambda,\kappa}(\tilde{S}^{-M,-M})\| && (3.14) \\ &+ \|f(P/\lambda)\psi(\lambda^{\frac{1}{2}} r)(P/\lambda + i)^{-1}\|. && (3.15) \end{aligned}$$

Thanks Assumption 3.1 we can make all the terms in the right hand side arbitrarily small. Indeed, the assumption applies directly to the terms in (3.13), for (3.14) the decay in r is provided by $Op_{\lambda,\kappa}(\tilde{S}^{-M,-M})$, while for (3.12) and (3.15) we observe that we have compact support in r thanks to $\tilde{\chi}$ and ψ. In particular it holds

$$\|f(P/\lambda)\left(i[P/\lambda, A^\lambda] - 2P/\lambda\right) f(P/\lambda)\| \leq \frac{1}{2}$$

for all $\lambda \in (0, \lambda_0]$. Now we simply have

$$f(P/\lambda) i[P/\lambda, A^\lambda] f(P/\lambda) \geq 2f(P/\lambda)P/\lambda f(P/\lambda) - \frac{1}{2} \geq \frac{3}{2} f^2(P/\lambda) - \frac{1}{2}$$

with the last inequality obtained thanks to $2f^2(x)x \geq \frac{3}{2}f^2(x)$ (this is always true for f with small enough support as in Assumption 3.1). At last, we choose $I \subset \sigma(P)$ an open bounded interval containing 1 and small enough such that f is constantly 1 on I. Then $f(x)\mathbb{1}_I(x) = \mathbb{1}_I(x)$ and applying $\mathbb{1}_I(P/\lambda)$ on the right and left of the previous inequality we have

$$\mathbb{1}_I(P/\lambda)i[P/\lambda, A^\lambda]\mathbb{1}_I(P/\lambda) \geq \mathbb{1}_I^2(P/\lambda) = \mathbb{1}_I(P/\lambda),$$

concluding the proof.

□

4 Proof of Assumption 3.1

A crucial step in the work presented up to now was to obtain the positive commutator estimate which allowed us to state that the outgoing and ingoing resolvents exist. Our main concern now is to prove that the Assumption 3.1 we made to obtain this result is valid for the operator P we are considering.

We will split the analysis into several steps by spatially localising the operator $f(P/\lambda)$ as follows

$$\begin{aligned} f(P/\lambda) =& \chi f(P/\lambda) + (1-\chi)f(P/\lambda)\chi + (1-\chi)f(-\Delta_0/\lambda)(1-\chi) \\ &+ (1-\chi)f(P/\lambda)(1-\chi) - (1-\chi)f(-\Delta_0/\lambda)(1-\chi) \end{aligned}$$

where $\chi = \chi(r)$ is a smooth cutoff which is constantly 1 on K and zero for large r.

Remark 4.1. The difference in the second line of the expression is well defined. Indeed, thanks to the cutoff on the right we are restricting ourselves to functions supported on $M \setminus K$ that can be identified with functions on $(R, +\infty) \times S$ which is where both the actions of $f(-\Delta_0/\lambda)$ and $f(P/\lambda)$ make sense.

We recall that in Assumption 3.1 it is stated that for any α the norm of $\langle \lambda^{\frac{1}{2}} r\rangle^{-\alpha} f(P/\lambda)$ can be made arbitrarily small up to spectrally localising P/λ close to one. We will summarise here how each term is treated and where the relative statement can be found.

i) $\chi f(P/\lambda)$, $(1-\chi)f(P/\lambda)\chi$: their norm can be made arbitrarily small up to choosing λ sufficiently small. See Proposition 4.1, via Lemmas 2.3 and 2.4.

ii) $(1-\chi)f(-\Delta_0/\lambda)(1-\chi)$: thanks to the multiplication by the decaying factor $\langle \lambda^{\frac{1}{2}} r\rangle^{-\alpha}$ the norm can be made arbitrarily small up to choosing f with small enough support. See Proposition 4.4, via rescaling argument and Lemma 4.3.

iii) $(1-\chi)f(P/\lambda)(1-\chi) - (1-\chi)f(-\Delta_0/\lambda)(1-\chi)$: the norm can be made arbitrarily small up to choosing λ sufficiently small. In Section 4.2 see (4.14). Via Hellffer-Sjöstrand formula, Lemma 4.5 and (4.13) (with intermediate steps in Lemmas 4.6, 4.8 and 4.11, where Lemma 2.4 is extensively used).

4.1 Model operator and compact perturbations

In this first subsection we focus on the compactly supported terms of item *i)* and on the term given by the model operator $-\Delta_0/\lambda$ on the fixed cone, that is item *ii)*.

We will start by showing how to bound $\chi f(P/\lambda)$ and $(1-\chi)f(P/\lambda)\chi$.

Proposition 4.1. *Let $f \in C_0^\infty(\mathbb{R})$ and χ a smooth cutoff on K, then*

$$\|\langle \lambda^{\frac{1}{2}} r\rangle^{-\alpha}(\chi f(P/\lambda) + (1-\chi)f(P/\lambda)\chi)\| \lesssim \lambda^{\frac{n}{4}}$$

for all $\lambda > 0$.

Proof. By the spectral theorem and Lemma 2.3 with $p = 1$ we have

$$\|f(P/\lambda)\chi\| \lesssim \|(P/\lambda+1)^{-N}\chi\| \lesssim \|(P/\lambda+1)^{-N}\|_{L^1(M)\to L^2(M)} \lesssim \lambda^{\frac{n}{4}}.$$

Same holds for $\|\chi f(P/\lambda)\|$ and we conclude simply bounding $\langle\lambda^{\frac{1}{2}}r\rangle^{-\alpha}$ by 1. □

In the term $(1-\chi)f(-\Delta_0/\lambda)(1-\chi)$ we can take advantage of the fact that on the exact cone $([0,+\infty)\times S, \overline{G})$ we have scaling invariance.

Lemma 4.2 (Rescaling on the fixed cone)**.** *Let $\|\cdot\|_{L^2_{\overline{G}}(cone)}$ the norm with respect to the metric $\overline{G}$ on the full cone $[0,+\infty)\times S$. Then for all $\lambda > 0$*

$$\|\langle\lambda^{\frac{1}{2}}r\rangle^{-\alpha}f(-\Delta_0/\lambda)\|_{L^2_{\overline{G}}(cone)\to L^2_{\overline{G}}(cone)} = \|\langle r\rangle^{-\alpha}f(-\Delta_0)\|_{L^2_{\overline{G}}(cone)\to L^2_{\overline{G}}(cone)}.$$

Proof. Let λ_k^2 the k-th eigenvalue of $-\Delta_{\overline{g}}$ and p_k

$$p_k = -\frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r}\partial_r + \frac{1}{r^2}\mu_k^2$$

as defined in (A.8). By the results in Appendix A we can reduce ourselves to the half line $(0,+\infty)$ and prove equivalently that

$$\|\langle\lambda^{\frac{1}{2}}r\rangle^{-\sigma}f(p_k/\lambda)\|_{\mathcal{L}(L^2(\mathbb{R}^+,r^{n-1}dr))} = \|\langle r\rangle^{-\sigma}f(p_k)\|_{\mathcal{L}(L^2(\mathbb{R}^+,r^{n-1}dr))}.$$

The equality follows showing

$$\langle\lambda^{\frac{1}{2}}r\rangle^{-\sigma}f(p_k/\lambda) = T_\lambda\langle r\rangle^{-\sigma}f(p_k)T_\lambda^* \tag{4.1}$$

with T_λ unitary operator.

□

Here and later we will need to compare the norm on the exact cone with the L^2 norm on $\mathbb{R}^n$ with respect to the Lebesgue measure, we give further details in the following remark.

Remark 4.2 (The flat norm on $\mathbb{R}^n$ and the one on the cone are comparable). The idea is to partition the angular part S into open sets that are diffeomorphic to open sets of $\mathbb{S}^{n-1}$ and take advantage of the fact that the L^2 norm on $(0,+\infty)\times\mathbb{S}^{n-1}$, with respect to the usual metric $dr^2 + r^2 d\sigma$, is equivalent to the norm on $\mathbb{R}^n$.

Let $(\varphi_\kappa)_\kappa$ partition of unity on S and $\kappa_j : U_j \to V_j \subset \mathbb{R}^{n-1}$ the subordinate coordinate charts. In the same way let U an open set of $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $\varphi : U \to V \subset \mathbb{R}^{n-1}$ its coordinate chart.

Without loss of generality we can assume $V_j \subset V$ so that it is well defined the diffeomorphism

$$\kappa = \varphi^{-1}\circ\kappa_j : S \supset U_j \to U \subset \mathbb{S}^{n-1}$$

through which $u \in C_0^\infty(S)$ supported in U_j can be identified with $u\circ\kappa^{-1} \in C_0^\infty(\mathbb{S}^{n-1})$.

The metric tensors on S and $\mathbb{S}^{n-1}$ are represented respectively by positive definite matrices such that

$$\overline{c}^{-1}I \le (\overline{g}^{j,k}(\theta))_{j,k} \le \overline{c}I, \qquad c^{-1}I \le (\sigma^{j,k}(x))_{j,k} \le cI.$$

This means that up to multiplication by some bounded function we can pass from one metric to the other. We will say that integrals with respect to $d\overline{g}$ or $d\sigma$ are comparable and write

$$\int_{U_j\subset S} |\varphi_j u|\, d\overline{g} \simeq \int_{U\subset\mathbb{S}^{n-1}} |(\varphi_j u)\circ\kappa^{-1}|\, d\sigma$$

for u smooth on $(0,+\infty)\times S$.

We remark here that $\kappa^{-1} = \kappa_j^{-1} \circ \varphi$ is only defined on $\varphi^{-1}(V_j) \subset U$ since we need to require that φ maps elements into $V_j \subset V$ which is where κ_j^{-1} is defined. However when we consider $(\varphi_j \circ \kappa^{-1})(u \circ \kappa^{-1})$, we can extend it to U by setting it 0 outside of $\varphi^{-1}(V_j)$, since in this case $(\varphi_j \circ \kappa^{-1})$ cuts off near the boundary of $\varphi^{-1}(V_j)$.

Considering the norm on the cone we have found that

$$\int_{(0,+\infty)\times U_j} \varphi_j u \; d\overline{G} \simeq \int_{(0,+\infty)\times U} (\varphi_j u \circ \kappa^{-1}) \; r^{n-1} dr d\sigma \simeq \int_{(0,+\infty)\times U \subset \mathbb{R}^n} (\varphi_j u) \circ \kappa^{-1} \; dx$$

where we still denote by κ the diffeomorphism $(r, \omega) \mapsto (r, \kappa(\omega))$ through which we can identify a function on the cone (suitably supported) with a function on $(0, +\infty) \times \mathbb{S}^{n-1}$. In particular we see that the L^2 norm with respect to the metric $\overline{G}$ on the cone is equivalent to the one on $\mathbb{R}^n$ with the Lebesgue measure.

Having got rid of the dependence on λ thanks to Lemma 4.2, we can prove convergence in norm as the support of f shrinks to 1 and therefore write

Lemma 4.3. *For any $\varepsilon > 0$ there exists $f \in C_0^\infty(\mathbb{R})$ equal to one in a neighborhood of one and small enough support such that*

$$\| \langle r \rangle^{-\alpha} f(-\Delta_0) \|_{L^2_{\overline{G}}(cone) \to L^2_{\overline{G}}(cone)} \leq \varepsilon.$$

Proof. Let $\tilde{f} \in C_0^\infty(\mathbb{R})$ a fixed function such that $\tilde{f}$ is 1 near the support of f so that we can write

$$\langle r \rangle^{-\sigma} f(-\Delta_0) = \langle r \rangle^{-\sigma} \tilde{f}(-\Delta_0) f(-\Delta_0).$$

If the support of f shrinks to $\{1\}$ then $f(-\Delta_0)$ converges strongly to 0, given that 1 is not an eigenvalue. Moreover $\langle r \rangle^{-\sigma} \tilde{f}(-\Delta_0)$ is a fixed compact operator and therefore the composition converges to 0 in norm.

To prove compactness first let $g \in C_0^1(\mathbb{R})$ supported around 0 and $(\rho_\kappa)_\kappa$ a partition of unity of $[0, +\infty) \times S$ with $supp \rho_\kappa \subset [0, +\infty) \times U_\kappa$. If $(u_n)_n$ is a sequence of uniformly bounded functions in $L^2_{\overline{G}}(cone)$ we have

$$g(r) \tilde{f}(-\Delta_0) u_n = \sum_\kappa g(r) \rho_\kappa \tilde{f}(-\Delta_0) u_n \in H_0^1([0, +\infty) \times S)$$

where each term is supported in an open bounded set $(0, \overline{R}) \times U_\kappa$ thanks to the supports of g and ρ_κ. Here, by $H_0^1([0, +\infty) \times S)$ we mean the space defined by performing the closure of C_0^∞ functions as in (A.3). The Sobolev regularity of $g(r)\rho_\kappa \tilde{f}(-\Delta_0) u_n$ is given by the fact that $\tilde{f}(-\Delta_0)$ has image in the domain of $-\Delta_0$ which is contained in the Sobolev space (see (A.4)). By the spectral theorem and the uniform bound on $(u_n)_n$

$$\| g(r) \rho_\kappa \tilde{f}(-\Delta_0) u_n \|_{L^2((0,\overline{R}) \times U_\kappa)} \lesssim \| g(r) \rho_\kappa \tilde{f} \|_\infty \| u_n \|_{L^2(0,\overline{R}) \times U_\kappa)} \lesssim 1.$$

Moreover by definition $\nabla_{\overline{G}} = (\partial_r, 1/r \nabla_{\overline{g}})$ and from the fact that ρ_κ is a function of the angular variables only we obtain

$$\begin{aligned} |\nabla_{\overline{G}} \; (g(r) \rho_\kappa \tilde{f}(-\Delta_0) u_n)|^2 \lesssim & |\nabla_{\overline{G}} (g \rho_\kappa) \tilde{f}(-\Delta_0) u_n|^2 + |g \rho_\kappa \nabla_{\overline{G}} (\tilde{f}(-\Delta_0) u_n)|^2 \\ \lesssim & \| g' \|_\infty^2 |\rho_\kappa \tilde{f}(-\Delta_0) u_n|^2 + \| g \|_\infty |(\nabla_{\overline{g}} \rho_\kappa) 1/r \tilde{f}(-\Delta_0) u_n|^2 \\ & + \| g \|_\infty^2 |\rho_\kappa \nabla_{\overline{G}} (\tilde{f}(-\Delta_0) u_n)|^2. \end{aligned}$$

We can bound $\| 1/r \tilde{f}(-\Delta_0) u_n \|_{L^2((0,\overline{R}) \times U_\kappa)}$ with the L^2 norm of the gradient by Hardy inequality which paired with the equality $\| \nabla_{\overline{G}} u \|_{L^2_{\overline{G}}} = \| (-\Delta_0)^{\frac{1}{2}} u \|_{L^2_{\overline{G}}}$ gives

$$\| \nabla_{\overline{G}} \; (g(r) \rho_\kappa \tilde{f}(-\Delta_0) u_n) \|^2_{L^2((0,\overline{R}) \times U_\kappa)} \lesssim \| \tilde{f}(-\Delta_0) u_n \|^2_{L^2((0,\overline{R}) \times U_\kappa)}$$

$$
\begin{aligned}
&+ \|(-\Delta_0)^{\frac{1}{2}}\tilde{f}(-\Delta_0)u_n\|^2_{L^2((0,\overline{R})\times U_\kappa)} \\
&\lesssim \|u_n\|^2_{L^2((0,\overline{R})\times U_\kappa)} \qquad (4.2)\\
&\lesssim 1
\end{aligned}
$$

where to get to (4.2) we use again the spectral theorem.

So for each fixed κ the sequence $(g(r)\tilde{f}(-\Delta_0)u_n\rho_\kappa)_n$ is uniformly bounded in $H^1_0((0,\overline{R})\times U_\kappa)$. By Remark 4.2 it will then be diffeomorphic to a uniformly bounded sequence of $H^1_0(\Omega)$ with Ω an open bounded subset of $\mathbb{R}^n$. By compact Sobolev embedding we can then extract a subsequence converging on $L^2(\Omega)$ and composing with the right diffeomorphism we can recover a subsequence of $g(r)\tilde{f}(-\Delta_0)u_n\rho_\kappa$ that converges in $L^2((0,\overline{R})\times U_\kappa)$. We have therefore proved compactness of $g(r)\tilde{f}(-\Delta_0)$.

Since the set of compact operators is closed with respect to norm convergence, considering $\phi \in C^\infty_0(\mathbb{R})$ equal to 1 on $B(0,1)$ we write

$$
\langle r\rangle^{-\sigma}\tilde{f}(-\Delta_0) = \langle r\rangle^{-\sigma}\phi\left(\frac{r}{R}\right)\tilde{f}(-\Delta_0) + \langle r\rangle^{-\sigma}(1-\phi)\left(\frac{r}{R}\right)\tilde{f}(-\Delta_0)
$$

for some large R. The first term is compact and the second one converges to 0 in norm as R tends to ∞, hence $\langle r\rangle^{-\sigma}\tilde{f}(-\Delta_0)$ is a compact operator. □

Since on the support of $(1-\chi)$ the manifold is diffeomorphic to $(R,+\infty)\times S$, and therefore the norm of $L^2(M)$ is comparable with the one of $L^2_{\overline{G}}$, the previous propositions imply the result for the norm of operators on $L^2(M)$.

Proposition 4.4. *For any $\varepsilon > 0$ there exists $f \in C^\infty_0(\mathbb{R})$ equal to one in a neighborhood of one and with small enough support such that*

$$
\|(1-\chi)\langle\lambda^{\frac{1}{2}}r\rangle^{-\alpha}f(-\Delta_0/\lambda)(1-\chi)\| \leq \varepsilon
$$

for any $\lambda > 0$.

4.2 Perturbative terms on the infinite end

The rest of the section will be dedicated to the analysis of the term localised on the end of the manifold, that is

$$
D_f(\lambda) := (1-\chi)f(P/\lambda)(1-\chi) - (1-\chi)f(-\Delta_0/\lambda)(1-\chi), \qquad (4.3)
$$

in particular we will prove that $D_f(\lambda)$ converges to 0 for any $f \in C^\infty_0(\mathbb{R})$ as λ goes to 0. (We recall the term is well defined, see Remark 4.1.)

Using Helffer-Sjöstrand formula to compute functional calculus we can reduce ourselves to comparing two resolvents, in particular setting

$$
R_z(\lambda) := (1-\chi)\left(\frac{P}{\lambda}-z\right)^{-1}(1-\chi) - (1-\chi)\left(\frac{-\Delta_0}{\lambda}-z\right)^{-1}(1-\chi)
$$

we rewrite

$$
D_f(\lambda) = \frac{1}{\pi i}\int_{\mathbb{C}}\overline{\partial}_z\tilde{f}(z)R_z(\lambda)L(dz).
$$

As we have seen in Proposition 3.3, on the support of $(1-\chi)$ we can compare P with $-\Delta_0$. More precisely, by expanding the expressions of $\Delta_{g(r)}$ and $\Delta_{\overline{g}}$ we can decompose the operator in a part on the fixed half cone and a differential operator with decaying coefficients. Namely,

$$
(1-\chi)P/\lambda = (1-\chi)\left(-\Delta_0 - V\right)/\lambda,
$$

where V is of the form

$$V := \sum_{\kappa} \varphi_\kappa \Pi_\kappa \left(a_\kappa(r,\theta)\partial_r + \sum_l b^l_\kappa(r,\theta)\frac{1}{r}\partial_l + \sum_{j,l} c^{j,l}_\kappa(r,\theta)\frac{1}{r^2}\partial^2_{j,l} \right) \Pi_\kappa^{-1}$$

with

$$a_\kappa \in S^{-1-\nu}, \quad b^l_\kappa \in S^{-1-\nu}, \quad c^{j,l}_\kappa \in S^{-\nu}. \tag{4.4}$$

With this in mind, we will split $D_f(\lambda)$ into several terms, since by algebraic manipulations we obtain

$$\begin{aligned} R_z(\lambda) = &- \frac{1}{\lambda}(P/\lambda - z)^{-1}[P,\chi](P/\lambda - z)^{-1}(1-\chi) \\ &+ \frac{1}{\lambda}(P/\lambda - z)^{-1}[P,\chi](-\Delta_0/\lambda - z)^{-1}(1-\chi) \\ &+ \frac{1}{\lambda}(P/\lambda - z)^{-1}(1-\chi)V(-\Delta_0/\lambda - z)^{-1}(1-\chi) \\ =: &\, R^1_z(\lambda) + R^2_z(\lambda) + R^V_z(\lambda). \end{aligned} \tag{4.5}$$

Remark 4.3. For $R^1_z(\lambda)$ and $R^2_z(\lambda)$ we can take advantage of the fact that the commutator is a differential operator of order one in the spatial variable only with compactly supported coefficients. This allows us to use Lemma 2.4 to obtain a bound by a positive power of λ, see Lemma 4.5. Lemmas 4.6, 4.8 and 4.11 are the main results providing a bound for $R^V_z(\lambda)$ which allows to conclude obtaining (4.13) and consequently (4.14).

Lemma 4.5. *Let $f \in C^\infty_0(\mathbb{R})$, there exists $\delta > 0$ such that*

$$\left\| \int_{\mathbb{C}} \overline{\partial}_z \tilde{f}(z) R^i_z(\lambda) L(dz) \right\| \lesssim \lambda^\delta$$

for $i = 1, 2$ and for all $\lambda > 0$.

Remark 4.4 (Notation). To make the notation lighter, in the sequel we will omit the pullback and pushforward in the expression in local coordinates of V and $[P,\chi]$, meaning for example that we will still denote by $a\partial_r$ the operator **on the manifold** that, on an open set of $\mathbb{R}^n$, corresponds to the derivative with respect to the radial variable and to the multiplication by a.

Proof. The commutator $[P,\chi]$ is supported away from the compact part of the manifold, here we recall the expression of the operator in local coordinates is (1.7).The angular derivatives commute with χ, so first of all we have

$$[P,\chi] = \sum_\kappa \varphi_\kappa (f_{1,\kappa}(r,\theta) + f_{2,\kappa}(r,\theta)\partial_r)$$

with $f_{1,\kappa}, f_{2,\kappa} \in C^\infty_0(\mathbb{R}^n)$. Then using this relation in the definition of $R^1_z(\lambda)$ we apply Lemma 2.4 to each term. First, let $\sigma \in \mathbb{R}$ and consider

$$\left\| \frac{1}{\lambda}(P/\lambda - z)^{-1}\langle r\rangle^{-\sigma}\varphi_\kappa f_{1,\kappa}\langle r\rangle^{2\sigma}\langle r\rangle^{-\sigma}(P/\lambda - z)^{-1}(1-\chi) \right\|.$$

By Lemma 2.4 we can have estimates of the type

$$\|(P/\lambda - z)^{-1}\langle r\rangle^{-\sigma}\| \lesssim \frac{\langle z\rangle}{|Imz|}\lambda^s \tag{4.6}$$

for all $s \in [0, \frac{n}{4}] \cap [0,1)$ such that $s < \frac{\sigma}{2}$ and we also remark that

$$\|\varphi_\kappa f_{1,\kappa}\langle r\rangle^{2\sigma}\| \lesssim \|f_{1,\kappa}\langle r\rangle^{2\sigma}\|_{L^\infty} \lesssim 1$$

for any σ, given the compact support of $f_{1,\kappa}$. We can therefore freely choose the exponent σ and picking $\sigma > 1$ allows us to find $s \in \left(\frac{1}{2}, \frac{\sigma}{2}\right)$ such that (4.6) holds. We observe that the $\langle z\rangle/|Imz|$ factor in (4.6) is provided by

$$\|(P/\lambda - z)^{-1}(P/\lambda + 1)\| \lesssim \frac{\langle z\rangle}{|Imz|}.$$

Collecting all this information together yields

$$\left\| \frac{1}{\lambda} (P/\lambda - z)^{-1} \varphi_\kappa f_{1,\kappa} (P/\lambda - z)^{-1} (1-\chi) \right\| \lesssim \lambda^{2s-1} \frac{\langle z\rangle^2}{|Imz|^2}$$

with $2s - 1 > 0$. We treat similarly the f_2 term where this time we apply Lemma 2.4 only to the resolvent on the left. We we still have

$$\|\varphi_\kappa f_{2,\kappa}\langle r\rangle^\sigma\| \lesssim \|f_{2,\kappa}\langle r\rangle^\sigma\|_{L^\infty} \lesssim 1$$

and we choose again $\sigma > 1$ so to obtain (4.6) for any $s \in (\frac{1}{2}, \frac{\sigma}{2})$. Picking $\tilde{f}_{2,\kappa}$ such that $f_{2,\kappa}\tilde{f}_{2,\kappa} \equiv 1$

$$\begin{aligned}\left\| \tilde{f}_{2,\kappa} \partial_r (P/\lambda - z)^{-1} \right\| \lesssim & \|\tilde{f}_{2,\kappa}\partial_r P^{-\frac{1}{2}}\| \|P^{\frac{1}{2}} (P/\lambda - z)^{-1}\| \\ \lesssim & \lambda^{\frac{1}{2}} \|\nabla_G P^{-\frac{1}{2}}\|_{L^2_G \to L^2_G} \\ \lesssim & \lambda^{\frac{1}{2}}.\end{aligned}$$

We conclude the proof, since

$$\left\| \frac{1}{\lambda} (P/\lambda - z)^{-1} \varphi_\kappa f_{2,\kappa} \partial_r (P/\lambda - z)^{-1} (1-\chi) \right\| \lesssim \lambda^{s+\frac{1}{2}-1} \frac{\langle z\rangle}{|Imz|}$$

with $s + \frac{1}{2} - 1 > 0$ and $\overline{\partial}_z \tilde{f}(z) = O(|Imz|^M)$ for any $M \geq 0$.

The proof for $R^2_z(\lambda)$ carries out in the same way. □

As opposed to $R^1_z(\lambda)$ and $R^2_z(\lambda)$, in the case of $R^V_z(\lambda)$ we have V which is a differential operator whose coefficients have only finite order decay. In particular the fact that $\|\langle r\rangle^\alpha a_\kappa\|_{L^\infty}$ and $\|\langle r\rangle^\alpha b^l_\kappa\|_{L^\infty}$ are finite only for $\alpha \leq 1 + \nu$ will limit our choice of exponents when applying Lemma 2.4.

However this will not be a source of difficulty in the first order terms of V since we can still choose $\sigma = \nu + 1 > 1$ as in Lemma 4.5 and get additional powers of λ by bounding the operators $\partial_r(-\Delta_0)^{-1}$ and $\frac{1}{r}\partial_l(-\Delta_0)^{-1}$.

On the contrary, for the second order term we only have $\|\langle r\rangle^\alpha c^{j,l}_\kappa\| \lesssim 1$ for $\alpha \leq \nu$. This will limit us to $\sigma = \nu > 0$ and moreover we will only be able to obtain the boundedness of $\partial^2_{j,l}(-\Delta_{\overline{g}})^{-1}$ (projecting away from the 0 eigenspace and using an elliptic parametrix in Lemma 4.9). This represents an additional difficulty since we will then need to control operators like $(-\Delta_{\overline{g}})/r^2\,(-\Delta_0/\lambda - z)^{-1}$.

Lemma 4.6 (Bound on first order terms I)**.** *Let I_1 defined by*

$$I_1 = \frac{1}{\lambda} (P/\lambda - z)^{-1} (1-\chi)\varphi_\kappa(a_\kappa\partial_r) \left(\frac{-\Delta_0}{\lambda} - z\right)^{-1} (1-\chi),$$

there exists $\delta_1 > 0$ such that

$$\|I_1\| \lesssim \lambda^{\delta_1} \frac{\langle z\rangle}{|Imz|}$$

for all $\lambda > 0$.

Proof. We start by mimicking the proof of Lemma 4.5, hence writing I_1 as

$$\left\| \frac{1}{\lambda} \left(P/\lambda - z\right)^{-1} \langle r\rangle^{-\sigma} (1-\chi)\varphi_\kappa \langle r\rangle^{\sigma} a_\kappa \partial_r \left(-\Delta_0/\lambda - z\right)^{-1} (1-\chi) \right\|. \tag{4.7}$$

We recall that $a_\kappa \in S^{-1-\nu}$ so the fact that $\|\langle r\rangle^{\nu+1} a_\kappa\|_{L^\infty} \lesssim 1$ suggests that this time we choose $\sigma = \nu + 1$ in Lemma 2.4, therefore giving us

$$\| \left(P/\lambda - z\right)^{-1} \langle r\rangle^{-\nu-1} \| \lesssim \frac{\langle z\rangle}{|Imz|} \lambda^s \tag{4.8}$$

for any $s \in (\frac{1}{2}, \frac{\nu+1}{2})$. We then proceed similarly to the previous proof, that is we estimate the quantity

$$\begin{aligned}
\left\| \partial_r \left(\frac{-\Delta_0}{\lambda} - z\right)^{-1} (1-\chi) \right\| \lesssim & \|\partial_r (-\Delta_0)^{-\frac{1}{2}}\|_{L^2_{\overline{G}} \to L^2_{\overline{G}}} \left\| (-\Delta_0)^{\frac{1}{2}} \left(\frac{-\Delta_0}{\lambda} - z\right)^{-1} \right\|_{L^2_{\overline{G}} \to L^2_{\overline{G}}} \\
\lesssim & \lambda^{\frac{1}{2}} \|\nabla_{\overline{G}} (-\Delta_0)^{-\frac{1}{2}}\|_{L^2_{\overline{G}} \to L^2_{\overline{G}}} \left\| \left(\frac{-\Delta_0}{\lambda}\right)^{\frac{1}{2}} \left(\frac{-\Delta_0}{\lambda} - z\right)^{-1} \right\|_{L^2_{\overline{G}} \to L^2_{\overline{G}}} \\
\lesssim & \lambda^{\frac{1}{2}}.
\end{aligned}$$

The statement is proved since the bound on (4.7) is

$$\|I_1\| \lesssim \lambda^{s+\frac{1}{2}-1} \frac{\langle z\rangle}{|Imz|}$$

with $s + \frac{1}{2} - 1 > 0$.

□

Once we have established

Lemma 4.7. *Let φ_κ a term of the partition of unity of S, then $\varphi_\kappa \frac{1}{r} \partial_l (-\Delta_0)^{-\frac{1}{2}}$ is a bounded operator on $L^2_{\overline{G}}$.*

we can bound the remaining first order part of $R^V_z(\lambda)$ with the exact same reasoning of Lemma 4.6.

Lemma 4.8 (Bound on first order terms II)**.** *Let I_2 defined by*

$$I_2 = \frac{1}{\lambda} \left(P/\lambda - z\right)^{-1} (1-\chi)\varphi_\kappa b^l_\kappa \frac{1}{r} \partial_l \left(-\Delta_0/\lambda - z\right)^{-1} (1-\chi),$$

then there exists $\delta_2 > 0$ such that

$$\|I_2\| \lesssim \lambda^{\delta_2} \frac{\langle z\rangle}{|Imz|}$$

for all $\lambda > 0$.

Proof of Lemma 4.7. By ellipticity of the operator $-\Delta_{\overline{g}}$, locally on coordinate patches we have the following lower bound

$$|\nabla_{\overline{g}} u|^2_{\overline{g}} = \sum_{l,j} \overline{g}^{l,j}(\theta) \partial_l u \partial_j u \geq C_0 \sum_j |\partial_j u|^2 \geq C_0 |\partial_l u|^2$$

for some $C_0 > 0$. Consequently for the operator on the manifold it holds

$$\int_S |\varphi_\kappa \partial_l u|^2 \, d\mathrm{vol}_S = \int_{V_\kappa} |\partial_l u|^2 |\overline{g}(\theta)| d\theta \leq \frac{1}{C_0} \int_{V_\kappa} |\nabla_{\overline{g}} u|^2_{\overline{g}} |\overline{g}(\theta)| d\theta \leq \frac{1}{C_0} \int_S |\varphi_\kappa \nabla_{\overline{g}} u|^2_{\overline{g}} d\mathrm{vol}_S$$

(we recall Remark 4.4 about the notation used). We can conclude, since we have found

$$\|\varphi_\kappa 1/r\partial_l u\|_{L^2_{\overline{G}}} \lesssim \|\varphi_\kappa 1/r\nabla_{\overline{g}} u\|_{L^2_{\overline{G}}} \lesssim \|\nabla_{\overline{G}} u\|_{L^2_{\overline{G}}} = \|(-\Delta_0)^{\frac{1}{2}} u\|_{L^2_{\overline{G}}}.$$

□

Now passing to consider the second order part in $R^V_\lambda(z)$, that is

$$I_3 := \frac{1}{\lambda}\,(P/\lambda - z)^{-1}\,(1-\chi)\varphi_\kappa c^{j,l}_\kappa \frac{1}{r^2}\partial^2_{j,l}\,(-\Delta_0/\lambda - z)^{-1}\,(1-\chi)$$

we first remark a useful property.

Lemma 4.9. *Let $-\Delta_{\overline{g}}$ the Laplace-Beltrami operator on $(S,\overline{g})$, if φ_κ is a term of the partition of unity of S then $\varphi_\kappa\partial^2_{j,l}(-\Delta_{\overline{g}})^{-1}$ is a bounded operator on $L^2(S,d\overline{g})$.*

Proof. Let $u \in D(-\Delta_{\overline{g}})$ and Π_0 the projection on $\ker(-\Delta_{\overline{g}})$. The kernel is spanned by 1 and consequently $\partial^2_{j,l}\Pi_0 u = 0$. Moreover Π_0 can be written as $f(-\Delta_{\overline{g}})$ for some $f \in C^\infty_0(\mathbb{R})$ supported around 0 and with $f(0) = 1$, we consider

$$\begin{aligned}\varphi_\kappa\partial^2_{j,l}u =& \varphi_\kappa\partial^2_{j,l}(u - f(-\Delta_{\overline{g}})u)\\ =& \varphi_\kappa\partial^2_{j,l}(-\Delta_{\overline{g}}+1)^{-1}\frac{(-\Delta_{\overline{g}}+1)}{(-\Delta_{\overline{g}})}(-\Delta_{\overline{g}})(1-f)(-\Delta_{\overline{g}})u\\ =& \varphi_\kappa\partial^2_{j,l}(-\Delta_{\overline{g}}+1)^{-1}\frac{(-\Delta_{\overline{g}}+1)(1-f)(-\Delta_{\overline{g}})}{(-\Delta_{\overline{g}})}(-\Delta_{\overline{g}})u. \qquad (4.9)\end{aligned}$$

With standard computations we can find a parametrix for the elliptic operator $-\Delta_{\overline{g}}$, namely there exist a family of symbols $q^\kappa \in S^{-2}(\mathbb{R}^{n-1}_\theta)$ supported in open subsets of $\mathbb{R}^{n-1}$ and $\mathcal{R}_N$ pseudodifferential operator with symbol in $S^{-N}(\mathbb{R}^{n-1}_\theta)$ such that

$$(-\Delta_{\overline{g}}+1)\left(\sum_\kappa \Pi_\kappa Op(q^\kappa)\Pi^{-1}_\kappa\right) = I + \mathcal{R}_N.$$

On the support of φ_κ the resolvent $(-\Delta_{\overline{g}}+1)^{-1}$ is a pseudodifferential operator of order minus two and the composition with the order two differential operator $(\partial^2_{j,l})$ results in a bounded operator of $L^2(S)$. By the spectral theorem $\frac{(-\Delta_{\overline{g}}+1)(1-f)(-\Delta_{\overline{g}})}{(-\Delta_{\overline{g}})}$ is bounded and therefore the statement follows from (4.9) which yields

$$\|\varphi_\kappa\partial^2_{j,l}u\|_{L^2(S)} \lesssim \|(-\Delta_{\overline{g}})u\|_{L^2(S)}.$$

□

As usual, we want to apply Lemma 2.4 to I_3 and we will do so by taking advantage of the fact that $\|\langle r\rangle^\nu c^{j,l}_\kappa\| \lesssim 1$ thanks to (4.4). However, applying Lemma 2.4 with $\sigma = \nu > 0$ would provide a bound by λ^s with $s \in (0, \frac{\nu}{2})$ which is worse than what we gained in the estimations of I_1 and I_2, where taking $\sigma = \nu + 1$ produced a higher power of λ, namely with exponent $s > \frac{1}{2}$.

We will then proceed differently by considering separately low and high angular frequencies. Let ϕ a smooth cutoff function such that $\phi \equiv 1$ on $[0, n-1]$.

1. Consider
$$\frac{1}{\lambda}(P/\lambda - z)^{-1}c^{j,l}_\kappa\varphi_\kappa\partial^2_{j,l}\phi(-\Delta_{\overline{g}})(1-\chi)\frac{1}{r^2}\left(\frac{-\Delta_0}{\lambda} - z\right)^{-1}$$
where we have bounded contributions given by
$$\langle r\rangle^\nu c^{j,l}_\kappa, \quad (1-\chi)(r)\frac{\langle r\rangle^2}{r^2}, \quad \varphi_\kappa\partial^2_{j,l}(-\Delta_{\overline{g}})^{-1}\phi(-\Delta_{\overline{g}})(-\Delta_{\overline{g}})$$

(see (4.4), Lemma 4.9 and the support of ϕ) and we are left to consider

$$(P/\lambda - z)^{-1}\langle r\rangle^{-\nu-1} \quad \text{and} \quad \frac{\langle r\rangle^{-1}}{\lambda}\left(-\Delta_0/\lambda - z\right)^{-1}.$$

By Lemma 2.4 and Hardy inequality (see Propositions 2.2 in [BR14b]) we can handle these two remaining terms obtaining

$$\left\|(P/\lambda - z)^{-1}\langle r\rangle^{-\nu-1}\frac{1}{\lambda\langle r\rangle}\left(-\Delta_0/\lambda - z\right)^{-1}\right\| \lesssim \lambda^{s-\frac{1}{2}}\frac{\langle z\rangle^2}{|Imz|^2}$$

where we can choose $s \in (\frac{1}{2}, \frac{\nu+1}{2})$.
We notice here that we have used Lemma 2.4 in the same way as in the proof of Lemma 4.6, where we had obtained (4.8). However here $c_\kappa^{j,l}$ has less decay than a_κ which leaves us with an extra growing term $\langle r\rangle$ to handle (we cannot bound $\langle r\rangle^{\nu+1}c_\kappa^{j,l}$, but only $\langle r\rangle^{\nu}c_\kappa^{j,l}$). For this reason we take advantage of the localisation $(1-\chi)$ and we use it to write a bounded term where we collect all the factors depending on the radial variable.

2. The part localised at high angular frequencies is

$$\frac{1}{\lambda}(P/\lambda - z)^{-1}c_\kappa^{j,l}\varphi_\kappa\partial_{j,l}^2(1-\phi)(-\Delta_{\overline{g}})(1-\chi)\frac{1}{r^2}\left(-\Delta_0/\lambda - z\right)^{-1}$$

where the operators

$$\langle r\rangle^{\nu}c_\kappa^{j,l}, \qquad \varphi_\kappa\partial_{j,l}^2(-\Delta_{\overline{g}})^{-1}$$

are bounded independently of λ thanks to (4.4) and Lemma 4.9. Additionally, by Lemma 2.4

$$\left\|(P/\lambda - z)^{-1}\langle r\rangle^{-\nu}\right\| \lesssim \lambda^s\frac{\langle z\rangle}{|Imz|} \tag{4.10}$$

for some $s \in (0, \frac{\nu}{2})$. At this point we are left with

$$\left\|(1-\phi)(-\Delta_{\overline{g}})\frac{(-\Delta_{\overline{g}})}{\lambda r^2}\left(\frac{-\Delta_0}{\lambda} - z\right)^{-1}\right\| \lesssim \frac{\langle z\rangle}{|Imz|}\left\|(1-\phi)(-\Delta_{\overline{g}})\frac{(-\Delta_{\overline{g}})}{\lambda r^2}\left(\frac{-\Delta_0}{\lambda} + 1\right)^{-1}\right\|$$

where, as opposed to item 1, the localisation by $(1-\phi)$ requires some extra care.

Lemma 4.10. *Let $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi \equiv 1$ on $[0, n-1]$, then for $\lambda \in (0, \lambda_0]$*

$$\left\|(1-\phi)(-\Delta_{\overline{g}})\frac{(-\Delta_{\overline{g}})}{\lambda r^2}\left(\frac{-\Delta_0}{\lambda} + 1\right)^{-1}\right\|_{L^2_{\overline{G}}\to L^2_{\overline{G}}} \lesssim_n 1.$$

Proof. By the results of Appendix A (i.e. Proposition A.2) we can rather consider the one dimensional problem of bounding

$$\sup_{\mu_k^2 > n-1}\left\|\frac{\mu_k^2}{r^2}\left(p_k + \lambda\right)^{-1}\right\|_{L^2((R,+\infty),r^{n-1}dr)\to L^2((R,+\infty),r^{n-1}dr)} \tag{4.11}$$

where we recall

$$p_k := -\frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r}\partial_r + \frac{1}{r^2}\mu_k^2$$

and (μ_k^2, e_k) are eigenpairs of $-\Delta_{\overline{g}}$. To bound (4.11) we will use an estimate on an analogous quantity where μ_k^2 is replaced by the eigenvalue of the Laplace-Beltrami operator on the unit sphere (see Appendix D). Indeed, we can separate the values μ_k^2 with the eigenvalues of $-\Delta_{\mathbb{S}^{n-1}}$, for whom we have the explicit expression $\sigma_{j,n}^2 = j(j+n-2)$. That is, once we fix k

$$\mu_k^2 \in (\sigma_{l,n}^2, \sigma_{l+1,n}^2] \text{ for a unique } l = l(k),$$

recalling that we are only considering eigenvalues $\mu_k^2 > n-1$. We can then rewrite the operator p_k as

$$p_k = \underbrace{-\partial_r^2 - \frac{(n-1)}{r}\partial_r + \frac{\sigma_{l,n}^2}{r^2}}_{p_{l,n}} + \underbrace{\frac{\mu_k^2 - \sigma_{l,n}^2}{r^2}}_{v}$$

with $v > 0$ and $p_{l,n}$ which is the equivalent of p_k where the values μ_k^2 are replaced by $\sigma_{l,n}^2$. If we express the resolvent in terms of the heat semigroup we have

$$(p_k + \lambda)^{-1} = \int_0^\infty e^{-t\lambda} e^{-tp_k} dt = \int_0^\infty e^{-t\lambda} e^{-t(p_{l,n}+v)} dt, \tag{4.12}$$

since v and $p_{l,n}$ do not commute of course $e^{-t(p_{l,n}+v)} \neq e^{-tp_{l,n}} e^{-tv}$. However from Trotter product formula (see Theorem VIII.31 in [RS81]) we know that

$$e^{-t(p_{l,n}+v)} = \lim_{m\to+\infty} (e^{-\frac{t}{m}p_{l,n}} e^{-\frac{t}{m}v})^m$$

taking the limit in the strong sense. Here we can bound the kernel of $(e^{-\frac{t}{m}p_{l,n}} e^{-\frac{t}{m}v})^m$ with the one of $e^{-tp_{l,n}}$ thanks to the non negativity of v, hence obtaining a pointwise upper bound on $(e^{-\frac{t}{m}p_{l,n}} e^{-\frac{t}{m}v})^m$ by $e^{-tp_{l,n}}$. Consequently, given that now $e^{-t(p_{l,n}+v)}$ is bounded by $e^{-tp_{l,n}}$, by (4.12) we see that we control $(p_k + \lambda)^{-1}$ with $(p_{l,n} + \lambda)^{-1}$ for which the result in Corollary D.2 holds.

In particular from such corollary we obtain first that

$$\left\| \frac{\sigma_{l,n}^2}{r^2} (p_{l,n} + \lambda)^{-1} \right\|_{L^2((R,+\infty), r^{n-1}dr) \to L^2((R,+\infty), r^{n-1}dr)} \lesssim_n 1$$

uniformly in λ. Then since $1 \leq \frac{\sigma_{l+1,n}^2}{\sigma_{l,n}^2} \leq m_n$ for some constant m_n it follows that $\frac{\mu_k^2}{\sigma_{l,n}^2} \leq m_n$, going back in (4.11) we have found

$$\sup_{\mu_k^2 > n-1} \left\| \frac{\mu_k^2}{r^2} (p_k + \lambda)^{-1} \right\| \lesssim m_n \left\| \frac{\sigma_{l,n}^2}{r^2} (p_{l,n} + \lambda)^{-1} \right\| \lesssim_n 1$$

where we are considering again the norm of operators on $L^2((R,+\infty), r^{n-1}dr)$.

□

We are now able to bound the second order term.

Lemma 4.11 (Bound on second order term)**.** *Let I_3 defined by*

$$I_3 = \frac{1}{\lambda}(P/\lambda - z)^{-1}(1-\chi)\varphi_\kappa c_\kappa^{j,l} \frac{1}{r^2} \partial_{j,l}^2 (-\Delta_0/\lambda - z)^{-1} (1-\chi),$$

there exists $\delta_3 > 0$ such that

$$\|I_3\| \lesssim \lambda^{\delta_2} \frac{\langle z \rangle^2}{|Imz|^2}$$

for all $\lambda > 0$.

Proof. Take ϕ a spectral localisation on the interval $[0, n-1)$ and split I_3 with the partition $\phi(-\Delta_{\overline{g}})$ and $(1-\phi)(-\Delta_{\overline{g}})$. Conclude by using item 1 (page 23) on the term localised on the angular frequencies $[0, n-1)$ and item 2 (page 24) together with Lemma 4.10 on the part localised at high angular frequencies. □

Eventually, thanks to Lemmas 4.6, 4.8 and 4.11 and the properties of almost analytic extensions we have

$$\left\| \int_{\mathbb{C}} \overline{\partial}_z \tilde{f}(z) R_z^V(\lambda) L(dz) \right\| \lesssim \lambda^{\delta} \tag{4.13}$$

for some $\delta > 0$ and $\lambda > 0$. Recalling the result of Lemma 4.5 and the definition of $D_f(\lambda)$ in (4.3) we have found that

$$\|D_f(\lambda)\| \to 0 \quad \text{as} \quad \lambda \to 0 \tag{4.14}$$

which concludes the proof of Assumption 3.1.

5 Adding a potential

We explain here how to obtain Proposition 3.2, and consequently Theorems 1.3 and 1.2, when P is replaced by $P + W$.

Let

$$P_W := P + W$$

with $W \geq 0$ a multiplicative potential that on $M \setminus K$ is the multiplication by a function W belonging to $S^{-2-\varepsilon}$. The robustness of the approach lies in the fact that the symbolic structure of the operator is not altered by the addition of such a potential. In particular, in local coordinates W can be represented by a pseudodifferential operator $Op_\lambda(\tilde{S}^{-2-\varepsilon,0})$ and if χ is a cutoff on the compact part of the manifold we can still write the operator under the form

$$(1-\chi)P_W/\lambda = \sum_{\kappa} Op_{\lambda,\kappa}(a_{0,\lambda} + a_{1,\lambda}^W) \tag{5.1}$$

with $a_{0,\lambda} \in \tilde{S}^{0,2}, a_{1,\lambda}^W \in \tilde{S}^{-1,1}$ that have seminorms uniformly bounded with respect to λ.

We sketch the main steps to obtain the results corresponding to the ones in Sections 2, 3 and 4.

- Provided we have existence of the outgoing and ingoing resolvents, all the proofs of Section 2 carry on in the exact same way for P_W.
- Since P_W has same symbolic structure as P the results of Section 3 hold with analogous proofs, provided we assume the equivalent of Assumption 3.1 replacing P by P_W. Therefore, by limiting absorption principle we obtain

  $$(A^\lambda + i)^{-s}(P_W/\lambda - 1 \pm i0)^{-l}(A^\lambda + i)^{-s} \in \mathcal{L}(L^2(M))$$

  for $s > l - \frac{1}{2}$ and $s \in \mathbb{N}$.
- Lemmas 2.3 and 2.4 were crucial properties to be able to prove Assumption 3.1 and we have remarked that they derive from the behaviour of the heat flow.

  Equivalents of Lemmas 2.3 and 2.4 hold for the operator P_W. Indeed, as we have done before we can use Trotter product formula (Theorem VIII.31 in [RS81]) to write

  $$e^{-tP_W} = e^{-t(P+W)} = \lim_{m \to +\infty} (e^{-\frac{t}{m}P} e^{-\frac{t}{m}W})^m,$$

  since $W \geq 0$ we can bound the kernel of $e^{-\frac{t}{m}P} e^{-\frac{t}{m}W}$ with the one of $e^{-\frac{t}{m}P}$. Therefore, if $K_0(x,y,t)$ and $K_W(x,y,t)$ are the kernels of e^{-tP} and e^{-tP_W} respectively

  $$K_W(x,y,t) \leq K_0(x,y,t)$$

  which allows us to recover for e^{-tP_W} the same kind of estimates that we had for e^{-tP}.

Remark 5.1. Alternatively, we can recover the heat flow estimates by noticing that they follow from the Nash inequality which also holds true for P_W. Thanks to the non negativity of W and the selfadjointess of P we can easily see that

$$\|P_W^{\frac{1}{2}}u\|^2 = \langle (P+W)u, u\rangle = \langle P^{\frac{1}{2}}u, P^{\frac{1}{2}}u\rangle + \langle Wu, u\rangle \geq \|P^{\frac{1}{2}}u\|^2$$

and therefore use the Nash inequality for P (namely (2.7)) to prove

$$\|u\|_{L^2(M)}^{1+\frac{2}{n}} \lesssim \|u\|_{L^1(M)}^{\frac{2}{n}} \|P_W^{\frac{1}{2}}u\|_{L^2(M)}.$$

We notice that $\|P_W^{\frac{1}{2}}u\| \geq \|P^{\frac{1}{2}}u\|$ implies that P_W has no 0 eigenvalue nor resonance, since 0 it is not an eigenvalue nor a resonance for P either.

- Once we get to the proof of Assumption 3.1 the only relevant difference is when considering the term
$$D_f^W(\lambda) := (1-\chi)f(P_W/\lambda)(1-\chi) - (1-\chi)f(-\Delta_0/\lambda)(1-\chi).$$
Applying Helffer Sjöstrand formula we pass to comparison between resolvents and we have an additional term involving W that is
$$R_z^W(\lambda) := \frac{1}{\lambda}\left(P_W/\lambda - z\right)^{-1}(1-\chi)W(-\Delta_0/\lambda - z)^{-1}(1-\chi), \tag{5.2}$$
while all the other terms can be bounded with the analogous of Lemma 2.4. This leads to results equivalent to Lemma 4.5 and (4.13).
For (5.2) we exploit the boundedness of $\|\langle r\rangle^{2+\varepsilon}w\|_{L^\infty}$ and applying Lemma 2.4 with $\sigma = 1+\varepsilon$ we obtain
$$\left\| \int_{\mathbb{C}} \overline{\partial}_z \tilde{f}(z) R_z^W(\lambda) L(dz) \right\| \lesssim \lambda^\delta$$
for some positive δ and $\lambda > 0$. We can therefore conclude that $\|D_f^W(\lambda)\|$ converges to 0 as λ goes to 0, as in the conclusion of Section 4.

A Operator on the exact cone and separation of variables

Let $-\Delta_0$ the Friedrichs extension of the Laplace-Beltrami operator on the half cone $([R,+\infty)\times S, \overline{G})$ with Dirichlet boundary condition. In local coordinates this looks like

$$-\Delta_0 = -\frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_{\bar{g}}. \tag{A.1}$$

The quadratic from which we derive the Friedrichs extension is

$$q_0(u,v) := (\partial_r u, \partial_r v)_{L^2_{\overline{G}}} + (1/r(-\Delta_{\bar{g}})^{\frac{1}{2}}u, 1/r(-\Delta_{\bar{g}})^{\frac{1}{2}}v)_{L^2_{\overline{G}}}, \tag{A.2}$$

defined on elements of the space

$$\{\text{closure of } C_0^\infty([R,+\infty)\times S) \text{ with respect to } \|u\|_+ := (q_0(u,u) + \|u\|^2_{L^2_{\overline{G}}})^{\frac{1}{2}}\ \} \tag{A.3}$$

that we will denote by $H_0^1([R,+\infty)\times S) = H_0^1$. The domain of the Friedrichs extension $-\Delta_0$ then is

$$D(-\Delta_0) = \{u \in H_0^1 \mid |q_0(u,v)| \leq C(u)\|v\|_{L^2_{\overline{G}}}\ \forall v \in H_0^1\} \tag{A.4}$$

Since $-\Delta_{\bar{g}}$ is selfadjoint on $L^2(S)$, we can consider an orthonormal basis $(e_k)_k$ of the space such that

$$-\Delta_{\bar{g}} e_k = \mu_k^2 e_k, \qquad 0 = \mu_0 \leq \mu_1 \leq \dots,$$

we can then decompose any function $u \in L^2(S)$ on this basis

$$u(\omega) = \sum_{k\geq 0} \left(\int_S \overline{e_k} u \ d\mathrm{vol}_S \right) e_k(\omega).$$

A function of $L^2_{\overline{G}}$ is then of the form

$$u(r,\omega) = \sum_{k\geq 0} \left(\int_S \overline{e_k(\cdot)} u(r,\cdot) \ d\mathrm{vol}_S \right) e_k(\omega) =: \sum_{k\geq 0} u_k(r) e_k(\omega). \tag{A.5}$$

In particular we can identify a function in $L^2_{\overline{G}}$ with its one dimensional coefficients. It is straightforward to obtain the following.

Proposition A.1. *The map between Hilbert spaces*

$$\begin{aligned} L^2_{\overline{G}} &\longrightarrow \bigoplus_{k\geq 0} L^2((R,+\infty), r^{n-1}dr) \\ u &\mapsto (u_k)_k \end{aligned}$$

is an isometry with

$$\|u\|^2_{L^2_{\overline{G}}} = \sum_{k\geq 0} \|u_k\|^2_{L^2((R,+\infty),r^{n-1}dr)}. \tag{A.6}$$

Moreover, prescribing the action of one dimensional operators on the coefficients u_k gives origin to well defined operators on the Hilbert space $L^2_{\overline{G}}$.

Proposition A.2. *Let $(A_k)_k$ a bounded sequence of bounded operators on $L^2((R,+\infty), r^{n-1}dr)$. The operator A defined by*

$$Au(r,\omega) = \sum_k (A_k u_k)(r) e_k(\omega). \tag{A.7}$$

is well defined on $L^2_{\overline{G}}$ with norm

$$\|A\|_{L^2_{\overline{G}} \to L^2_{\overline{G}}} = \sup_k \|A_k\|_{L^2((R,+\infty),r^{n-1}dr) \to L^2((R,+\infty),r^{n-1}dr)}.$$

Proof. Follows directly from Proposition A.1. □

With Proposition A.2 in mind we want to reduce $-\Delta_0$ to the action of suitable one dimensional operators. Indeed, once a function is represented with respect to the orthonormal basis $(e_k)_k$ as in (A.5), the action of $-\Delta_{\bar{g}}$ becomes multiplication by a scalar so going back to (A.1) this suggests we set

$$p_k := -\frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r}\partial_r + \frac{1}{r^2}\mu_k^2. \tag{A.8}$$

We can prove that the sequence of one dimensional operators $(p_k)_k$ corresponds exactly to $-\Delta_0$.

Proposition A.3. *Let $u \in H^1_0([R,+\infty) \times S)$, then $u \in D(-\Delta_0)$ if and only if $u_k \in D(p_k)$ for every k and*

$$\sum_k \|p_k u_k\|^2_{L^2((R,+\infty),r^{n-1}dr)} < +\infty. \tag{A.9}$$

Moreover

$$(-\Delta_0 u)_k = p_k u_k, \quad \textit{from which} \quad \|(-\Delta_0)u\|^2_{L^2_{\overline{G}}} = \sum_k \|p_k u_k\|^2_{L^2((R,+\infty),r^{n-1}dr)}.$$

Proof. Can be proved by direct computations, given that defining p_k via Friedrichs extension its associated quadratic form is

$$q_k(f,g) = \int_R^{+\infty} \left(\overline{f'} g' + \frac{1}{r^2} \mu_k^2 \overline{f} g \right) \, r^{n-1} dr,$$

with domain

$$h_{+1,k} := \left\{ \text{closure of } C_0^\infty([R,+\infty)) \text{ with respect to } \left(q_k(f,f) + \|f\|^2_{L^2((R,+\infty), r^{n-1}dr)} \right)^{\frac{1}{2}} \right\},$$

and the domain of p_k is

$$D(p_k) = \{ f \in h_{+1,k} \mid |q_k(f,g)| \leq C(f) \|g\|_{L^2((R,+\infty), r^{n-1}dr)} \ \forall g \in h_{+1,k} \}.$$

□

The result of Proposition A.3 extends to the case of functions of $-\Delta_0$, as we will see below. This will be of use to simplify the argument in estimations of norms in Section 4.

Proposition A.4. *Let $\varphi \in C_0^\infty(\mathbb{R})$ then*

$$\varphi(-\Delta_0) u(r,\omega) = \sum_k (\varphi(p_k) u_k)(r) e_k(\omega)$$

for any $u \in L^2_{\overline{G}}$.

Proof. The statement can be proved directly when $\varphi(x) = (x-z)^{-1}$ with $z \in \mathbb{C} \setminus \mathbb{R}$ and can be generalised to any $\varphi \in C_0^\infty(\mathbb{R})$ applying Hellfer-sjöstrand formula to compute the right hand side of the equality. □

B Nash inequality

We show in this appendix how to obtain an inequality of the type

$$\|u\|^{1+\frac{2}{n}}_{L^2(M)} \lesssim_n \|u\|^{\frac{2}{n}}_{L^1(M)} \|P^{\frac{1}{2}} u\|_{L^2(M)}$$

using the analogous result which holds for the free operator $-\Delta$ on $\mathbb{R}^n$, namely

$$\|u\|^{1+\frac{2}{n}}_{L^2(\mathbb{R}^n)} \lesssim_n \|u\|^{\frac{2}{n}}_{L^1(\mathbb{R}^n)} \|(-\Delta)^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^n)} = \|u\|^{\frac{2}{n}}_{L^1(\mathbb{R}^n)} \|\nabla u\|_{L^2(\mathbb{R}^n)}. \tag{B.1}$$

See [Nas58] p. 936.

We will proceed in two steps: first we will prove the inequality on a pure cone (thanks to (B.1)) and next we will pass onto M. On the manifold end, $M \setminus K$, we will use the result that holds on a pure a cone, while on the compact part we will exploit the fact that locally the domain is diffeomorphic to $\mathbb{R}^n$ where (B.1) applies.

B.1 Inequality on a fixed cone

Since we are considering the fixed cone $((0,\infty) \times S, \overline{G})$ we can proceed as in Remark 4.2 to reduce ourselves to norms on $\mathbb{R}^n$ for which (B.1) holds. We recall the definition of $\| \cdot \|_{L^2_{\overline{G}}(cone)}$ given in Lemma 4.2, that is the norm on the full cone with respect to the fixed metric $\overline{g}$. In the same way we define $\| \cdot \|_{L^1_{\overline{G}}(cone)}$.

Lemma B.1. *Let $u \in C_0^\infty((0,+\infty)\times S)$, then there exists $C_n > 0$ such that*

$$\|u\|^{1+\frac{2}{n}}_{L^2_{\overline{G}}(cone)} \leq C_n \|u\|^{\frac{2}{n}}_{L^1((0,+\infty)\times S)} \|\nabla_{\overline{G}} u\|_{L^2((0,+\infty)\times S)}.$$

Proof. Recall the equivalence between the L^2 norm on $((0,\infty)\times S, \overline{G})$ and the usual L^2 norm on $\mathbb{R}^n$ with respect to the Lebesgue measure mentioned in Remark 4.2. Then for each term in $u = \sum_j \varphi_j(\omega) u$ we can apply (B.1) and get

$$\|\varphi_j(\omega)u\|^{1+\frac{2}{n}}_{L^2_{\overline{G}}(cone)} \simeq \|\varphi_j u\|^{1+\frac{2}{n}}_{L^2(\mathbb{R}^n)} \lesssim \|\varphi_j u\|^{\frac{2}{n}}_{L^1(\mathbb{R}^n)} \|\nabla_{\mathbb{R}^n}(\varphi_j u)\|_{L^2(\mathbb{R}^n)}$$

with

$$\|\varphi_j u\|^{\frac{2}{n}}_{L^1(\mathbb{R}^n)} \simeq \|\varphi_j u\|^{\frac{2}{n}}_{L^1_{\overline{G}}(cone)} \lesssim \|u\|^{\frac{2}{n}}_{L^1_{\overline{G}}(cone)}. \tag{B.2}$$

Considering polar coordinates on $(0,+\infty)\times\mathbb{S}^{n-1}$ the gradient is $(\partial_r, 1/r\nabla_{\mathbb{S}^{n-1}})$, so

$$\begin{aligned}
\|\nabla_{\mathbb{R}^n}(\varphi_j u)\|^2_{L^2(\mathbb{R}^n)} &\simeq \|\varphi_j \partial_r u\|^2_{L^2((0,+\infty)\times\mathbb{S}^{n-1})} + \|1/r\nabla_{\mathbb{S}^{n-1}}(\varphi_j u)\|^2_{L^2((0,+\infty)\times\mathbb{S}^{n-1})} \\
&\simeq \|\varphi_j(\omega)\partial_r u\|^2_{L^2_{\overline{G}}(cone)} + \|1/r\nabla_{\overline{g}}(\varphi_j(\omega)u)\|^2_{L^2_{\overline{G}}(cone)} \\
&\lesssim \|\partial_r u\|^2_{L^2_{\overline{G}}(cone)} + \|1/r\nabla_{\overline{g}}(\varphi_j(\omega)u)\|^2_{L^2_{\overline{G}}(cone)}.
\end{aligned} \tag{B.3}$$

Now observing that $\nabla_{\overline{g}} = (\overline{g}^{j,k}(\theta))_{j,k}\nabla_\theta$ when we rewrite $\nabla_{\overline{g}}(\varphi_j(\omega)u)$ as $[\nabla_{\overline{g}}, \varphi_j] + \varphi_j \nabla_{\overline{g}}$ the commutator

$$[\nabla_{\overline{g}}, \varphi_j(\omega)] = (\overline{g}^{j,k}(\theta))_{j,k}\nabla_\theta(\varphi_j \circ \kappa_j^{-1}) \in C_0^\infty(\mathbb{R}^{n-1}),$$

is the multiplication by a bounded function and hence a bounded operator. So we have found

$$\begin{aligned}
\|1/r\nabla_{\overline{g}}(\varphi_j(\omega)u)\|_{L^2((0,+\infty)\times S)} &\leq \|1/r\varphi_j(\omega)\nabla_{\overline{g}}u\|_{L^2_{\overline{G}}(cone)} + \| \ 1/r[\nabla_{\overline{g}}, \varphi_j(\omega)]u\|_{L^2_{\overline{G}}(cone)} \\
&\lesssim \|1/r\nabla_{\overline{g}}u\|_{L^2_{\overline{G}}(cone)} + C\|u/r\|_{L^2_{\overline{G}}(cone)} \\
&\lesssim \|1/r\nabla_{\overline{g}}u\|_{L^2_{\overline{G}}(cone)} + C\|\partial_r u\|_{L^2_{\overline{G}}(cone)}
\end{aligned}$$

thanks to Hardy inequality (Propositions 2.2 and 3.5 in [BR14b]) in the last line. The statement follows combining (B.2), (B.3) together with this last estimate. □

B.2 Inequality on the manifold

We split the analysis into the part near infinity and the compact one. From what we have found in the previous section, if $(1-\chi)$ is a cutoff on the manifold end at first we get

$$\begin{aligned}
\|(1-\chi)u\|^{1+\frac{2}{n}} &\simeq \|(1-\chi)u\|^{1+\frac{2}{n}}_{L^2_{\overline{G}}} \lesssim \|(1-\chi)u\|^{\frac{2}{n}}_{L^1_{\overline{G}}} \|\nabla_{\overline{G}}((1-\chi)u)\|_{L^2_{\overline{G}}} \\
&\simeq \|(1-\chi)u\|^{\frac{2}{n}}_{L^1(M\setminus K)} \|\nabla_{\overline{G}}((1-\chi)u)\|_{L^2_{\overline{G}}} \\
&\leq \|u\|^{\frac{2}{n}}_{L^1(M)} \|\nabla_{\overline{G}}((1-\chi)u)\|_{L^2_{\overline{G}}}.
\end{aligned} \tag{B.4}$$

The gradient we need to evaluate is

$$|\nabla_{\overline{G}}((1-\chi)u)|^2_{\overline{G}} \lesssim |\chi' u|^2 + |(1-\chi)\partial_r u|^2 + |(1-\chi)/r\nabla_{\overline{g}}u|^2_{\overline{g}} \tag{B.5}$$

with χ' compactly supported in $[R,+\infty)$ so that $\|\chi'\langle r\rangle\|_{L^\infty} \lesssim 1$. Evaluating the L^2 norm and using the Hardy inequality mentioned in the previous proof we can first bound the terms

$$\|\chi' u\|^2_{L^2_{\overline{G}}} + \|(1-\chi)\partial_r u\|^2_{L^2_{\overline{G}}} \lesssim \|\langle r\rangle^{-1}u\|^2_{L^2_{\overline{G}}} + \|\partial_r u\|^2_{L^2_{\overline{G}}} \lesssim \|\partial_r u\|^2_{L^2_{\overline{G}}}. \tag{B.6}$$

Then we will need to compare the gradient with respect to the fixed metric $\overline{g}$ with the one with respect to the metric $g(r)$ via the following lemma.

Lemma B.2. *Let $\overline{g}$ and $g(r)$ two metrics on a closed manifold S satisfying the property in* (1.2)*, then there exists a constant $C > 0$ such that*

$$\||\nabla_{\overline{g}} u|_{\overline{g}}\|_{L^2(S,d\overline{g})} \leq C\||\nabla_{g(r)} u|_{g(r)}\|_{L^2(S,dg(r))}.$$

for $r \gg 1$ and $u \in H^1_0(S, d\overline{g}) \cap H^1_0(S, dg(r))$.

Proof. Given the two gradients $\nabla_{\overline{g}} = (\overline{g}^{j,k}(\theta))_{j,k}\nabla_\theta$ and $\nabla_{g(r)} = (g(r,\theta)^{j,k})_{j,k}\nabla_\theta$ and property (1.2) for large r we get

$$\||\nabla_{\overline{g}} u|_{\overline{g}}\|_{L^2(S,d\overline{g})} \lesssim \|\sum_\kappa |\nabla_{\overline{g}}(\varphi_\kappa u)|_{\overline{g}}\|_{L^2(S,d\overline{g})} \lesssim \|\sum_\kappa |\nabla_{g(r)}(\varphi_\kappa u)|_{g(r)}\|_{L^2(S,dg(r))}.$$

Consider the kernel of the operator $-\Delta_{g(r)}$, which is spanned by 1, and let

$$\Pi^0_{g(r)} := \text{ projection on } \ker_{L^2}(-\Delta_{g(r)}),$$

in particular $\nabla_{\overline{g}} u = \nabla_{\overline{g}}(u - \Pi^0_{g(r)} u)$. Now the gradient of u is

$$\begin{aligned}
\||\nabla_{\overline{g}} u|_{\overline{g}}\|_{L^2(S,d\overline{g})} =& \||\nabla_{\overline{g}}(u - \Pi^0_{g(r)} u)|_{\overline{g}}\|_{L^2(S,d\overline{g})} \\
\lesssim& \sum_\kappa \||\nabla_{g(r)}(\varphi_\kappa(u - \Pi^0_{g(r)} u))|_{g(r)}\|_{L^2(S,dg(r))} \\
\lesssim& \sum_\kappa \|\varphi_\kappa|\nabla_{g(r)}(u - \Pi^0_{g(r)} u)|_{g(r)}\|_{L^2(S,dg(r))} \\
&+ \sum_\kappa \||\nabla_{g(r)}\varphi_\kappa|_{g(r)}(u - \Pi^0_{g(r)} u)\|_{L^2(S,dg(r))} \\
\lesssim& \||\nabla_{g(r)} u|_{g(r)}\|_{L^2(S,dg(r))} + \|u - \Pi^0_{g(r)} u\|_{L^2(S,dg(r))}.
\end{aligned}$$

Write the projection on the 0 eigenspace $\Pi^0_{g(r)}$ as $f(-\Delta_{g(r)})$ with $f \in C^\infty_0(\mathbb{R})$ such that $f(0) = 1$ and supported around 0, then by the spectral theorem

$$\begin{aligned}
\|u - \Pi^0_{g(r)} u\|_{L^2(S,dg(r))} =& \left\| \frac{(1-f)(-\Delta_{g(r)})}{(-\Delta_{g(r)})^{\frac{1}{2}}} (-\Delta_{g(r)})^{\frac{1}{2}} u \right\|_{L^2(S,dg(r))} \\
\lesssim& \|(-\Delta_{g(r)})^{\frac{1}{2}} u\|_{L^2(S,dg(r))} \\
\simeq& \||\nabla_{g(r)} u|_{g(r)}\|_{L^2(S,dg(r))}
\end{aligned}$$

and the statement follows. □

Going back to (B.5), thanks to (B.6) and Lemma B.2 we obtain

$$\begin{aligned}
\|\nabla_{\overline{G}}((1-\chi)u)\|^2_{L^2((0,+\infty)\times S)} \lesssim& \|\partial_r u\|^2_{L^2_{\overline{G}}} + \|(1-\chi)/r|\nabla_{\overline{g}} u|_{\overline{g}}\|^2_{L^2_{\overline{G}}} \\
\lesssim& \|\partial_r u\|^2_{L^2_G} + \|(1-\chi)/r|\nabla_{g(r)} u|_{g(r)}\|^2_{L^2_G} \\
\lesssim& \|\nabla_G u\|^2_{L^2_G} \simeq \|P^{\frac{1}{2}} u\|^2_{L^2(M\setminus K)}.
\end{aligned}$$

Applying this estimate to (B.4) we have proved the desired inequality on the manifold end

$$\|(1-\chi)u\|^{1+\frac{2}{n}} \lesssim \|u\|^{\frac{2}{n}}_{L^1(M)} \|P^{\frac{1}{2}} u\|^2_{L^2(M)}. \tag{B.7}$$

Using a partition of unity for the remaining compact part we derive Nash inequality for the full manifold.

Lemma B.3. *Let $u \in C_0^\infty(M)$ then there exists $C_n > 0$ such that*

$$\|u\|_{L^2(M)}^{1+\frac{2}{n}} \leq C_n \|u\|_{L^1(M)}^{\frac{2}{n}} \|P^{\frac{1}{2}} u\|_{L^2(M)}.$$

Proof. Write u as $(1-\chi)u + \chi u$ and use (B.7) on $(1-\chi)u$. Then χu is supported on a compact set on which we can consider a finite covering $(K_j)_j$ with associated partition of unity $(\chi_j)_j$. Each $K_j \subset M$ is diffeomorphic to an open set of $\mathbb{R}^n$, via a diffeomorphism ψ_j and since the metric tensor on M is represented by a positive definite matrix we have

$$\int_M \chi_j(\chi u)\, dG \simeq \int_{\psi_j(K_j)} \chi_j(\chi u) \circ \psi_j^{-1}\, dx.$$

Applying again (B.1) it follows

$$\begin{aligned}
\|\chi_j(\chi u)\|_{L^2(M)}^{1+\frac{2}{n}} &\simeq \|\chi_j(\chi u)\|_{L^2(\mathbb{R}^n)}^{1+\frac{2}{n}} \lesssim \|\chi_j(\chi u)\|_{L^1(\mathbb{R}^n)}^{\frac{2}{n}} \|\nabla_{\mathbb{R}^n}(\chi_j \chi u)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|u\|_{L^1(M)}^{\frac{2}{n}} \left(\|\nabla_{\mathbb{R}^n}(\chi u)\|_{L^2(\mathbb{R}^n)} + \|\chi u\|_{L^2(\mathbb{R}^n)} \right) \\
&\simeq \|u\|_{L^1(M)}^{\frac{2}{n}} \left(\|\nabla_G(\chi u)\|_{L^2(M)} + \|\chi u\|_{L^2(M)} \right) \\
&\lesssim \|u\|_{L^1(M)}^{\frac{2}{n}} \|\nabla_G(\chi u)\|_{L^2(M)} \\
&\lesssim \|u\|_{L^1(M)}^{\frac{2}{n}} \left(\|\nabla_G u\|_{L^2(M)} + \|\chi' u\|_{L^2(M)} \right) \\
&\lesssim \|u\|_{L^1(M)}^{\frac{2}{n}} \|\nabla_G u\|_{L^2(M)}
\end{aligned}$$

where we have used Hardy inequality (Proposition 2.2 of [BR14b]) to estimate $\|\chi u\|_{L^2(M)}$ and (2.4) in [BR14b] to bound $\|\chi' u\|_{L^2(M)}$.

□

C Commutators and symbolic calculus

We will see here how to use symbolic calculus to compute commutators between rescaled pseudodifferential operators on a manifold. First of all we point out that for $Op_\lambda(\cdot)$ described in (1.5) the usual rules on the composition of pseudodifferential operators apply (for example see Proposition 3.1 in [BM16]).

As defined in (1.6), $Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa$ acts on functions supported on $U_\kappa \subset M$ so summing up all the contributions $\sum_\kappa Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa$ will be a pseudodifferential operator defined on the whole manifold. We also underline that we will consider operators with spatially localised symbols, namely *supp* $a^\kappa \subset [R, +\infty) \times V_\kappa \times \mathbb{R}^n$.

We proceed to the proof of the result mentioned in Proposition 3.1.

Proposition C.1. *Let m, m', μ, μ' real numbers and the operators A, B on $[R, +\infty) \times S$ defined as*

$$A := \sum_\kappa Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa, \qquad B := \sum_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa$$

with $a^\kappa \in \tilde{S}^{m,\mu}$ and $b^\kappa \in \tilde{S}^{m',\mu'}$ and both spatially supported in $[R, +\infty) \times V_\kappa$. Then

$$[A, B] = \sum_\kappa Op_{\lambda,\kappa}(\tilde{S}^{m+m'-1,\mu+\mu'-1})\psi_\kappa + \mathcal{R}_N$$

with $\mathcal{R}_N$ an operator which is negligible of order N for any $N \in \mathbb{N}$.

Proof. In the following $\mathcal{B}$ is a generic bounded operator that will be allowed to change from one line to the other.

The composition AB is given by a double sum, by the support properties of the symbols $Op_{\lambda,\kappa}(b^\kappa)$ will be supported in $[R,+\infty)\times U_\kappa$, while ψ_m localises in $[R,+\infty)\times U_m$. Hence for any chart such that $U_m\cap U_\kappa=\emptyset$ the corresponding term in the sum, $Op_{\lambda,m}(a^m)\psi_m Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa$, is 0.

We start by looking at the easier case where we are composing two operators localised on the same chart. We denote by $\psi_\kappa^{\mathbb{R}^n}$ the pushforward on $\mathbb{R}^n$ of ψ_κ through any chart l whenever this quantity is well defined, that is $\psi_\kappa^{\mathbb{R}^n}=\Pi_l^{-1}\psi_\kappa$, and we will extensively use the relations

$$\psi_\kappa\Pi_l=\Pi_l\psi_\kappa^{\mathbb{R}^n},\qquad \Pi_l^{-1}\psi_\kappa=\psi_\kappa^{\mathbb{R}^n}\Pi_l^{-1}.$$

This said, we have

$$\begin{aligned}Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa =&\Pi_\kappa Op_\lambda(a^\kappa)\Pi_\kappa^{-1}(\psi_\kappa\Pi_\kappa)Op_\lambda(b^\kappa)\Pi_\kappa^{-1}\psi_\kappa\\ =&\Pi_\kappa Op_\lambda(a^\kappa)\Pi_\kappa^{-1}\Pi_\kappa(\psi_\kappa^{\mathbb{R}^n}Op_\lambda(b^\kappa))\Pi_\kappa^{-1}\psi_\kappa\\ =&\Pi_\kappa Op_\lambda(a^\kappa)Op_\lambda(\tilde{b}^\kappa)\Pi_\kappa^{-1}\psi_\kappa\end{aligned}$$

where $\tilde{b}^\kappa:=\psi_\kappa^{\mathbb{R}^n}b^\kappa\in\tilde{S}^{m',\mu'}$ is still spatially supported on V_κ. Thanks to the multiplication on the right by $\Pi_\kappa^{-1}\psi_\kappa$ the composition $Op_\lambda(a^\kappa)Op_\lambda(\tilde{b}^\kappa)$ is applied to functions localised on V_κ so we can harmlessly extend the symbols to 0 outside of their support. This, will give us a rescaled pseudodifferential operator on $\mathbb{R}^n$ to which usual composition formulas apply. Consequently

$$\begin{aligned}Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa =&\Pi_\kappa Op_\lambda(a^\kappa\tilde{b}^\kappa)\Pi_\kappa^{-1}\psi_\kappa+\Pi_\kappa Op_\lambda(\tilde{S}^{m+m'-1,\mu+\mu'-1})\Pi_\kappa^{-1}\psi_\kappa\\ =&Op_{\lambda,\kappa}(a^\kappa b^\kappa)\psi_\kappa+Op_{\lambda,\kappa}(\tilde{S}^{m+m'-1,\mu+\mu'-1})\psi_\kappa.\end{aligned}\tag{C.1}$$

Obviously the same holds for $Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa$ so that when taking the commutator the term $Op_{\lambda,\kappa}(a^\kappa\tilde{b}^\kappa)\psi_\kappa$ cancels with $Op_{\lambda,\kappa}(b^\kappa\tilde{a}^\kappa)\psi_\kappa$, where $b^\kappa\tilde{a}^\kappa=b^\kappa\psi_\kappa^{\mathbb{R}^n}a^\kappa=a^\kappa\tilde{b}^\kappa$.

Now for the overlapping terms with $m\neq\kappa$ let $\tilde{\psi}_\kappa$ equal to 1 on the support of ψ_κ, that we use to move the localisation on the left

$$\Pi_\kappa Op_\lambda(b^\kappa)\Pi_\kappa^{-1}\psi_\kappa=\tilde{\psi}_\kappa\Pi_\kappa Op_\lambda(b^\kappa)\Pi_\kappa^{-1}.\tag{C.2}$$

Then in

$$Op_{\lambda,m}(a^m)\psi_m Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa=Op_{\lambda,m}(a^m)\psi_m\tilde{\psi}_\kappa Op_{\lambda,\kappa}(b^\kappa)$$

the cutoff $\psi_m\tilde{\psi}_\kappa\in C^\infty([R,+\infty)\times U_m\cap U_\kappa)$ localises in a region where both the charts κ and m are defined.

We pick smooth cutoffs $\tilde{\tilde{\psi}}_\kappa$ and $\tilde{\psi}_m$ such that

$$\tilde{\tilde{\psi}}_\kappa\tilde{\psi}_\kappa\equiv\tilde{\psi}_\kappa,\qquad \tilde{\psi}_m\psi_m\equiv\psi_m$$

and since $\tilde{\psi}_m^{\mathbb{R}^n}\equiv 1$ on the support of a^m we have

$$\begin{aligned}Op_{\lambda,m}(a^m)\psi_m\tilde{\psi}_\kappa =&\Pi_m\tilde{\psi}_m^{\mathbb{R}^n}Op_\lambda(a^m)\Pi_m^{-1}(\psi_m\tilde{\psi}_\kappa)\\ =&\tilde{\tilde{\psi}}_\kappa\tilde{\psi}_m\Pi_m Op_\lambda(a^m)\Pi_m^{-1}(\psi_m\tilde{\psi}_\kappa)\\ &+(1-\tilde{\tilde{\psi}}_\kappa)\tilde{\psi}_m\Pi_m Op_\lambda(a^m)\Pi_m^{-1}(\psi_m\tilde{\psi}_\kappa).\end{aligned}\tag{C.3}$$

On the intersection $U_m\cap U_\kappa$ both the expression in local coordinates given by the chart m and the chart κ are well defined and we can pass from one to another by composing with smooth transition maps, like $\kappa\circ m^{-1}$. We notice that $\Pi_m^{-1}(\psi_m\tilde{\psi}_\kappa)$ localises exactly in $m(U_m\cap U_\kappa)$ which is where $\kappa\circ m^{-1}$ is well defined and we have the relation

$$Op_\lambda(a^m)\Pi_m^{-1}(\psi_m\tilde{\psi}_\kappa)=\Pi_{\kappa\circ m^{-1}}Op_\lambda(\tilde{a}^\kappa)\Pi_{\kappa\circ m^{-1}}^{-1}\Pi_m^{-1}(\psi_m\tilde{\psi}_\kappa)$$

where $\tilde{a}^\kappa$ is a symbol belonging to the same class of a^m. Indeed, we will prove in Proposition C.2 that conjugation by transition maps does not affect the decay of the symbol and that in particular it holds

$$Op_{\lambda,\kappa}(\tilde{a}^\kappa) = Op_{\lambda,\kappa}(\tilde{S}^{m,\mu}) + \mathcal{R}_N \tag{C.4}$$

with $\mathcal{R}_N$ a negligible operator.

In the following sum we can first use (C.2) and then the expression found in (C.3)

$$\begin{aligned}\sum_\kappa \sum_m Op_{\lambda,m}(a^m)\psi_m Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa &= \sum_\kappa \sum_m Op_{\lambda,m}(a^m)\psi_m\tilde{\psi}_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa \\ &= \sum_\kappa \sum_m (\tilde{\tilde{\psi}}_\kappa\tilde{\psi}_m\Pi_m(\Pi_{\kappa\circ m^{-1}}Op_\lambda(\tilde{a}^\kappa)\Pi^{-1}_{\kappa\circ m^{-1}})\Pi_m^{-1}\psi_m \\ &\qquad \tilde{\psi}_\kappa Op_{\lambda,\kappa}(b^\kappa))\psi_\kappa \\ &\quad + \sum_\kappa \sum_m (1-\tilde{\tilde{\psi}}_\kappa)\tilde{\psi}_m Op_{\lambda,m}(a^m)(\psi_m\tilde{\psi}_\kappa)Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa.\end{aligned}$$

Noticing that $\Pi_{\kappa\circ m^{-1}} = \Pi_m^{-1}\Pi_\kappa$ we can simplify some terms

$$\Pi_m\Pi_{\kappa\circ m^{-1}} = \Pi_\kappa, \qquad \Pi^{-1}_{\kappa\circ m^{-1}}\Pi_m^{-1} = \Pi_\kappa^{-1},$$

yielding

$$\begin{aligned}\sum_\kappa \sum_m Op_{\lambda,m}(a^m)\psi_m Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa &= \sum_\kappa \sum_m \tilde{\tilde{\psi}}_\kappa\tilde{\psi}_m\Pi_\kappa Op_\lambda(\tilde{a}^\kappa)\Pi_\kappa^{-1}\psi_m\tilde{\psi}_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa \\ &\quad + \sum_\kappa \sum_m (1-\tilde{\tilde{\psi}}_\kappa)\tilde{\psi}_m Op_{\lambda,m}(a^m)(\psi_m\tilde{\psi}_\kappa)Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa. \quad \text{(C.5)}\end{aligned}$$

First, thanks to the support properties of $\tilde{\psi}_m$ and $\tilde{\tilde{\psi}}_\kappa$ and since $(\psi_m)_m$ sum up to one

$$\begin{aligned}&\sum_m \tilde{\tilde{\psi}}_\kappa\tilde{\psi}_m\Pi_\kappa Op_\lambda(\tilde{a}^\kappa)\Pi_\kappa^{-1}\psi_m\tilde{\psi}_\kappa\Pi_\kappa Op_\lambda(b^\kappa)\Pi_\kappa^{-1} \\ &\quad = \Pi_\kappa Op_\lambda(\tilde{a}^\kappa)\Pi_\kappa^{-1}\sum_m \psi_m\tilde{\psi}_\kappa\Pi_\kappa Op_\lambda(b^\kappa)\Pi_\kappa^{-1} \\ &\quad = \Pi_\kappa Op_\lambda(\tilde{a}^\kappa)\tilde{\psi}_\kappa Op_\lambda(b^\kappa)\Pi_\kappa^{-1} \quad \text{(C.6)} \\ &\quad = Op_{\lambda,\kappa}(\tilde{a}^\kappa b^\kappa)\tilde{\psi}_\kappa + Op_{\lambda,\kappa}(\tilde{S}^{m+m'-1,\mu+\mu'-1})\tilde{\psi}_\kappa \quad \text{(C.7)}\end{aligned}$$

where (C.6) falls in the same case of (C.1) so we use (C.4) and the properties of composition of pseudodifferential operators. For the term in (C.5) the part

$$\Pi_m(1-\tilde{\tilde{\psi}}_\kappa^{\mathbb{R}^n})\tilde{\psi}_m^{\mathbb{R}^n}Op_\lambda(a^m)\psi_m^{\mathbb{R}^n}\tilde{\psi}_\kappa^{\mathbb{R}^n}\Pi_m^{-1}$$

is a composition of pseudodifferential operators with disjoint supports, hence the usual formula for the composition produces the remainder only. This implies that we can write

$$R_m := (1-\tilde{\tilde{\psi}}_\kappa)\tilde{\psi}_m Op_{\lambda,m}(a^m)\psi_m\tilde{\psi}_\kappa = \tilde{\tilde{\psi}}_m Op_{\lambda,m}(r)$$

with $r \in \tilde{S}^{-4N,-4N}$ for any $N \in \mathbb{N}$ and that

$$\sum_m R_m = Q_N^{-1}\mathcal{B}Q_N^{-1}.$$

Adding the contribution of $Op_\lambda(b^\kappa)$ gives us

$$\sum_m (1-\tilde{\tilde{\psi}}_\kappa)\tilde{\psi}_m Op_{\lambda,m}(a^m)(\psi_m\tilde{\psi}_\kappa)Op_{\lambda,\kappa}(b^\kappa) = \sum_m R_m Op_{\lambda,\kappa}(b^\kappa)$$

$$
\begin{aligned}
&= Q_N^{-1}\mathcal{B}Q_N^{-1}Op_{\lambda,\kappa}(b^\kappa)\\
&= Q_{N/2}^{-1}Q_{N/2}^{-1}\mathcal{B}Q_N^{-1}Op_{\lambda,\kappa}(b^\kappa)Q_{N/2}Q_{N/2}^{-1}\\
&= Q_{N/2}^{-1}\mathcal{B}Q_{N/2}^{-1}.
\end{aligned}
$$

Thanks to (C.7) we have found

$$
\begin{aligned}
\sum_\kappa\sum_m Op_{\lambda,m}(a^m)\psi_m Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa =& Op_{\lambda,\kappa}(\tilde{a}^\kappa b^\kappa)\tilde{\psi}_\kappa +\\
&+ Op_{\lambda,\kappa}(\tilde{S}^{m+m'-1,\mu+\mu'-1})\tilde{\psi}_\kappa + Q_N^{-1}\mathcal{B}Q_{N/2}^{-1}.
\end{aligned}
$$

We repeat the same procedure when evaluating the double sum produced by the composition BA, where in this case we obtain terms $Op_{\lambda,\kappa}(b^\kappa\tilde{a}^\kappa)\tilde{\psi}_\kappa$. Hence taking the difference $AB - BA$ results in the statement.

□

We prove now the invariance of the symbol classes by conjugation with a diffeomorphism. This will imply that passing from one chart to another, which means conjugating with a transition map the operators on $\mathbb{R}^n$, does not alter the decay of the symbols.

Remark C.1 (Notation). To simplify the notations of the kernels we state the proposition for pseudodifferential operators, instead of the rescaled version $Op_\lambda(a)$. However, given that the kind of diffeomorphism we are considering leaves untouched the radial variable r, which is the only one affected by the rescaling, the result generalises easily for $Op_\lambda(a)$.

Proposition C.2. *Let $\gamma : V \to W$ a diffeomorphism between open sets of $\mathbb{R}^{n-1}$ with $|\partial^\alpha\gamma| \leq C_\alpha$, $|\partial^\alpha\gamma^{-1}| \leq c_\alpha$ and set $\psi(r,\theta) = \chi(r)\varphi(\theta) \in C^\infty([R,+\infty)\times V)$ compactly supported in θ and*

$$
\begin{aligned}
\Pi_\gamma : C^\infty([R,+\infty)\times W) &\to C^\infty([R,+\infty)\times V)\\
v &\mapsto v(r,\gamma(\theta)).
\end{aligned}
$$

If $Op(a)$ has symbol $a \in \tilde{S}^{m,\mu}$ with supp $a \subset [R,+\infty)\times W$ then

$$
\Pi_\gamma Op(a)\Pi_\gamma^{-1}\psi = Op(a_\gamma)\tilde{\psi} + \mathcal{R}_V
$$

with $a_\gamma \in \tilde{S}^{m,\mu}$, $\tilde{\psi} \in C^\infty([R,+\infty)\times V)$ compactly supported in θ and $\mathcal{R}_V$ a pseudodifferential operator of negative order. In particular, $\mathcal{R}_V$ is the pushforward on $\mathbb{R}^n$ of a negligible operator of order N with arbitrary N.

Remark C.2. The integral kernel of $Op(a)$ is

$$
K_a(r,\theta,r',\theta') = \frac{1}{(2\pi)^n}\int e^{i(r-r')\rho+i(\theta-\theta')\eta}a(r,\theta,\rho,\eta)d\rho d\eta
$$

whereas the integral kernel of $\Pi_\gamma Op(a)\Pi_\gamma^{-1}\psi$ is instead

$$
\begin{aligned}
K_\gamma(r,\theta,r',\theta') = \frac{1}{(2\pi)^n}\int\Big(& e^{i(r-r')\rho+i(\gamma(\theta)-\gamma(\theta'))\eta}a(r,\gamma(\theta),\rho,\eta)\chi(r')\varphi(\theta')\\
&|\mathrm{Jac}\gamma|(\theta')\Big)d\rho d\eta.
\end{aligned}
$$

To show that $\Pi_\gamma Op(a)\Pi_\gamma^{-1}\psi$ is still a pseudodifferential operator, up to some remainder, we will need to write its kernel K_γ as an oscillating integral with phase $i(r-r')\rho + i(\theta-\theta')\eta$ and a symbol a_γ depending only on the variables r,θ,ρ and η.

Proof. As observed in the previous remark, we need the oscillating term in the integral K_γ to be $e^{i(r-r')\rho+i(\theta-\theta')\eta}$, so to linearize with respect to θ we consider the Taylor expansion

$$\gamma(\theta)-\gamma(\theta')=(\theta-\theta')\int_0^1 d\gamma(\theta'+t(\theta-\theta'))dt=:(\theta-\theta')M(\theta,\theta')$$

with $M(\theta,\theta')$ an invertible matrix. Performing a change of variable in K_γ which sends η to $M(\theta,\theta')^{-1}\eta$ yields

$$K_\gamma(r,\theta,r',\theta')=\frac{1}{(2\pi)^n}\cdot\int e^{i(r-r')\rho+i(\theta-\theta')\eta}A(r,\theta,r',\theta',\rho,\eta)d\rho d\eta \tag{C.8}$$

where we have set

$$A(r,\theta,r',\theta',\rho,\eta):=a(r,\gamma(\theta),\rho,M(\theta,\theta')^{-1}\eta)\chi(r')\varphi(\theta')\tilde{M}(\theta,\theta').$$

In (C.8) it now appears the oscillatory term in the desired form as commented before. To finally obtain the kernel of a pseudodifferential operator we must get rid of the dependence of A on (r',θ').

Let $\Theta\in C_0^\infty(\mathbb{R}^n)$ a cutoff function such that $\Theta\equiv 1$ near 0, we first consider the kernel K_γ localised around the diagonal $\{r=r',\theta=\theta'\}$, that is

$$\Theta(r-r',\theta-\theta')K_\gamma(r,\theta,r',\theta').$$

We use again a Taylor expansion: we expand A with respect to the variables r',θ' around the point (r,θ) up to order N, hence providing us with a polynomial of order $N-1$ plus a remainder term. In particular we can write

$$\begin{aligned}A(r,\theta,r',\theta',\rho,\eta)=&A(r,\theta,r,\theta,\rho,\eta)\\&+\sum_{l=1}^{N-1}\sum_{j+|\alpha|=l}\mathcal{O}((r'-r)^j(\theta'-\theta)^\alpha)(\partial_{r'}^j\partial_{\theta'}^\alpha A)(r,\theta,r,\theta,\rho,\eta)\\&+\sum_{j+|\alpha|=N}\mathcal{O}((r'-r)^j(\theta'-\theta)^\alpha)R^{j,\alpha}(r,\theta,r',\theta',\rho,\eta).\end{aligned}$$

Here $R^{j,\alpha}$ are the terms coming from the Taylor remainder: they are compactly supported in (r',θ') and such that $|\partial_\rho^k\partial_\eta^\beta R^{j,\alpha}|\lesssim\left(\langle\rho\rangle+\frac{\langle\eta\rangle}{\langle r\rangle}\right)^{\mu-k-|\beta|}$ (property which is inherited from a).

We want to use this expansion in (C.8) and do integration by parts after observing that

$$(r'-r)^j(\theta'-\theta)^\alpha e^{i(r-r')\rho+i(\theta-\theta')\eta}=D_\rho^jD_\eta^\alpha e^{i(r-r')\rho+i(\theta-\theta')\eta}.$$

Integral (C.8) then results in

$$\begin{aligned}\Theta(r-r',\theta-\theta')K_\gamma(r,\theta,r',\theta')=&\Theta(r-r',\theta-\theta')\frac{1}{(2\pi)^n}\\&\cdot\int e^{i(r-r')\rho+i(\theta-\theta')\eta}A(r,\theta,r,\theta,\rho,\eta)d\rho d\eta\\&+\Theta(r-r',\theta-\theta')\frac{1}{(2\pi)^n}\\&\sum_l\sum_{(j,\alpha)}\int e^{i(r'-r)\rho+i(\theta'-\theta)\eta}(D_\rho^jD_\eta^\alpha\partial_{r'}^j\partial_{\theta'}^\alpha A)(r,\theta,r,\theta,\rho,\eta)d\rho d\eta\\&+\Theta(r-r',\theta-\theta')\frac{1}{(2\pi)^n}\\&\sum_{j+|\alpha|=N}\int e^{i(r'-r)\rho+i(\theta'-\theta)\eta}D_\rho^jD_\eta^\alpha R^{j,\alpha}(r,\theta,r',\theta',\rho,\eta)d\rho d\eta.\end{aligned}$$

Moreover we notice that by definition A preserves the decay of a with respect to r, ρ and η, meaning that

$$A(r,\theta,r,\theta,\rho,\eta) \in \tilde{S}^{m,\mu}, \qquad (D_\rho^j D_\eta^\alpha \partial_{r'}^j \partial_{\theta'}^\alpha A)(r,\theta,r,\theta,\rho,\eta) \in \tilde{S}^{m-|\alpha|,\mu-|\alpha|-j}.$$

Up to the remainder term, we have obtained integrals which are the kernels of pseudodifferential operators of symbol $\tilde{S}^{m,\mu}$ as we wanted.

For the remainders we can prove that they are kernels of negligible operators in the sense of Definition 3.1. The same can be proved for the contribution of $(1-\Theta)K_\gamma$ and this will allow us to conclude the proof.

To do this we need to show that composing on the left and right with $Q_S := \langle r\rangle^S (P+1)^S$ the Taylor remainder results in a bounded operator for any fixed large S.

Here the key point is that we are conjugating by Q_S an operator whose kernel is of the form

$$\int \Theta(r-r',\theta-\theta') e^{i(r'-r)\rho + i(\theta'-\theta)\eta} D_\rho^j D_\eta^\alpha R^{j,\alpha}(r,\theta,r',\theta',\rho,\eta) d\rho d\eta. \tag{C.9}$$

This kernel is smooth, with derivatives in (r',θ') which are compactly supported and such that in the integral we have arbitrary fast decay in ρ and η, since we recall $j+|\alpha| = N$ implies

$$|D_\rho^j D_\eta^\alpha R^{j,\alpha}| \lesssim \left(\langle\rho\rangle + \frac{\langle\eta\rangle}{\langle r\rangle}\right)^{\mu-N}.$$

Applying Q_S on the right, where $(P+1)^S$ is selfadjoint, would lead us to differentiate the kernel in r' and θ', while applying it on the left means taking derivatives with respect to r and θ. The ensemble of these actions still results in a kernel which is bounded together with its derivatives, thanks to the properties we just stated. Notably, the decay in ρ and η allows to compensate the growth which is generated when taking derivatives of the oscillating factor.

We can therefore apply Calderon-Vaillancourt theorem (Theorem 2.8.1 in [Mar02]) to conclude that the corresponding operator is bounded. (Actually here one needs to apply Calderon-Vaillancourt to a suitably conjugated operator in order to have boundedness with respect to the appropriate L^2 norm, that is L^2_G.)

The same reasoning can be adapted to the contribution away from the diagonal, after rewriting it as

$$\begin{aligned}(1-\Theta)(r-r',\theta-\theta')K_\gamma(r,\theta,r',\theta') =& \frac{(1-\Theta)(r-r',\theta-\theta')}{(r-r')^M(\theta-\theta')^\beta} \\ & \int e^{i(r-r')\rho + i(\theta-\theta')\eta}(D_\rho^M D_\eta^\beta A)(r,\theta,r',\theta',\rho,\eta) d\rho d\eta\end{aligned}$$

where we performed again integration by parts thanks to

$$(r-r')^M(\theta-\theta')^\beta e^{i(r-r')\rho+i(\theta-\theta')\eta} = D_\rho^M D_\eta^\beta e^{i(r-r')\rho+i(\theta-\theta')\eta}.$$

Indeed, $D_\rho^M D_\eta^\beta A$ also decays as

$$|D_\rho^M D_\eta^\beta A| \lesssim \left(\langle\rho\rangle + \frac{\langle\eta\rangle}{\langle r\rangle}\right)^{\mu-M-|\beta|}$$

with arbitrary M and β and hence the same arguments used for (C.9) can be replicated to conclude that $Q_S(1-\Theta)K_\gamma Q_S$ is an integral kernel of a bounded operator.

□

D A uniform bound for the spherical Laplacian

Consider the Laplace-Beltrami operator on the unit sphere

$$-\frac{\Delta_{\mathbb{S}^{n-1}}}{r^2} = -\Delta_{\mathbb{R}^n} + \partial_r^2 + \frac{(n-1)}{r}\partial_r, \tag{D.1}$$

given the relation between the radial and euclidean coordinates $r = |x|$ we can derive

$$\partial_r = \frac{1}{|x|}\sum_j x_j \partial_j, \qquad \partial_r^2 = \frac{1}{|x|^2}\sum_k x_k \sum_j x_j \partial_{j,k}^2$$

and therefore rewrite the operator in terms of euclidean derivatives as

$$-\frac{\Delta_{\mathbb{S}^{n-1}}}{r^2} = -\Delta_{\mathbb{R}^n} + \sum_{j,k=1}^{n} \frac{x_j x_k}{|x|^2}\partial_{j,k}^2 + (n-1)\sum_{j=1}^{n}\frac{x_j}{|x|^2}\partial_j. \tag{D.2}$$

Proposition D.1. *Let $n \geq 3$ and $\lambda_0 > 0$, there exists a constant $C_n > 0$ such that*

$$\left\| \frac{-\Delta_{\mathbb{S}^{n-1}}}{r^2}\left(-\Delta_{\mathbb{R}^n} + \lambda\right)^{-1} \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C_n. \tag{D.3}$$

for all $\lambda \in (0, \lambda_0]$.

Proof. We use the expression in (D.2). By the spectral theorem $-\Delta_{\mathbb{R}^n}\left(-\Delta_{\mathbb{R}^n} + \lambda\right)^{-1}$ is bounded and by elliptic regularity results

$$\begin{aligned}\left\| \frac{x_j x_k}{|x|^2}\partial_{j,k}^2\left(-\Delta_{\mathbb{R}^n} + \lambda\right)^{-1} \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} &\lesssim \left\|\partial_{j,k}^2\left(-\Delta_{\mathbb{R}^n} + \lambda\right)^{-1}\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \\ &\lesssim 1.\end{aligned}$$

Finally, since $n \geq 3$ we can apply Hardy inequality to conclude

$$\begin{aligned}\left\| \frac{x_j}{|x|^2}\partial_j\left(-\Delta_{\mathbb{R}^n} + \lambda\right)^{-1} \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} &\leq \left\| \frac{1}{|x|}\left(-\Delta_{\mathbb{R}^n} + \lambda\right)^{-\frac{1}{2}} \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \\ &\quad \cdot \left\| \left(-\Delta_{\mathbb{R}^n} + \lambda\right)^{-\frac{1}{2}}\partial_j \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \\ &\leq \frac{2}{n-2}\left\| \nabla\left(-\Delta_{\mathbb{R}^n} + \lambda\right)^{-\frac{1}{2}} \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \\ &\leq \frac{2}{n-2}.\end{aligned}$$

□

Let $(\sigma_{l,n}^2, s_{l,n}(\theta))_j$ the eigenpairs of $-\Delta_{\mathbb{S}^{n-1}}$, representing $u \in L^2(\mathbb{R}^n)$ in spherical coordinates and decomposing the angular part along the basis of eigenfunctions we have

$$\frac{-\Delta_{\mathbb{S}^{n-1}}}{r^2}\left(-\Delta_{\mathbb{R}^n} + \lambda\right)^{-1} u(r,\theta) = \sum_j \frac{\sigma_{l,n}^2}{r^2}\left(-\partial_r^2 - \frac{(n-1)}{r}\partial_r + \frac{\sigma_{l,n}^2}{r^2} + \lambda\right)^{-1} u_{l,n}(r) s_{l,n}(\theta)$$

where we have used for $-\Delta_{\mathbb{R}^n}$ the expression deriving from (D.1).

The uniform bound of the previous proposition translates to the following bound on one dimensional operators.

Corollary D.2. *Let $n \geq 3$ and $\sigma^2_{l,n}$ the $l-$th eigenvalue of the spherical Laplacian on $\mathbb{S}^{n-1}$, then there exists $C_n > 0$ such that*

$$\left\| \frac{\sigma^2_{l,n}}{r^2} \left(-\partial_r^2 - \frac{(n-1)}{r}\partial_r + \frac{\sigma^2_{l,n}}{r^2} + \lambda \right)^{-1} \right\|_{L^2(\mathbb{R}^+, r^{n-1}dr) \to L^2(\mathbb{R}^+, r^{n-1}dr)} \leq C_n$$

for all $\lambda \in (0, \lambda_0]$.

We remark that the operator that appears in the statement is exactly $p_{l,n}$ that we defined in the proof of Lemma 4.10.

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