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► **To cite this version:**

Viviana Grasselli. Dispersive equations on asymptotically conical manifolds: time decay in the low frequency regime. 2021. hal-03904202v1

**HAL Id: hal-03904202**

**<https://hal.science/hal-03904202v1>**

Preprint submitted on 16 Dec 2022 (v1), last revised 20 Feb 2024 (v3)

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# DISPERSIVE EQUATIONS ON ASYMPTOTICALLY CONICAL MANIFOLDS: TIME DECAY IN THE LOW FREQUENCY REGIME

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## Abstract

On an asymptotically conical manifold we prove time decay estimates for the flow of the Schrödinger, wave and Klein-Gordon equations via some continuity properties of the spectral measure. To keep the paper at a reasonable length we limit ourselves to the low energy part of the spectrum, which is the one that dictates the decay rates. With this paper we extend sharp estimates that are known in the asymptotically flat case (see Bouclet and Burq in [BB21]) to this more general geometric framework and therefore recover the same decay properties as for the euclidean case. The first step is to prove some resolvent estimates via a limiting absorption principle. It is at this stage that the proof of the previously mentioned authors fails, in particular when we try to recover a low frequency positive commutator estimate. Once the resolvent estimates are established we derive regularity for the spectral measure that in turn is applied to obtain the decay of the flows.

**Keywords:** Dispersive estimates, Spectral theory, Asymptotically conical manifold, Semiclassical analysis, Schrödinger operator

**2010 Mathematics Subject Classification:** 58J50, 35S99, 35Q40

**Acknowledgments:** This work received support from the University Research School EUR-MINT (State support by the National Research Agency' Future Investments program, reference number ANR-18-EURE-0023).

## 1 Introduction

In this paper we consider the Laplace-Beltrami operator on a quite general class of non compact manifolds with ends, which includes, among others, all compact perturbation of the euclidean metric. The geometric setting is the one of asymptotically conical manifolds that over the years has attracted the interest of a substantial community with the aim to recover some of the properties that hold in the flat case, such as resolvent or local energy decay estimates.

Broadly speaking we are interested in estimates on the operators

$$(P - \lambda \pm i0)^{-l} = \lim_{\varepsilon \rightarrow 0} (P - \lambda \pm i\varepsilon)^{-l}. \quad (1.1)$$

and the related evolutions  $e^{itP}$ ,  $e^{it\sqrt{P}}$ ,  $e^{it\sqrt{P}+1}$ ,  $\frac{\sin(t\sqrt{P})}{\sqrt{P}}$ . Here  $P$  is the Laplace Beltrami operator (possibly perturbed by a potential) on an asymptotically conical manifold.

We start by giving a brief review of known results in this kind of setting. We consider only the case of asymptotically conical manifolds, the reader can refer to the introduction of [BB21] for more details about the literature in the case of an asymptotically euclidean geometry.

In [GHS13a] results on the behaviour of the flow of the wave and Schrödinger equations are deduced by a study of the asymptotics of the spectral measure when the spectral parameter converges to 0. These properties of the spectral measure were later applied in [GHS13b] to prove bounds on spectral multipliers for the Laplace Beltrami operator with a decaying potential. In [Wan06] a similar strategy is used on a manifold which is exactly conical at infinity, where an asymptotic expansion of the resolvent is used to study the long time behaviour of the Schrödinger flow. For a particular class of initial data, dispersive estimates of the Schrödinger and wave flow are proved in [SSS08]. In [VW09], instead, the authors take a similar approach to the one of the present paper using positive commutator estimates to prove dispersive properties for the flow of the wave equation, although requiring a stronger decay on the potential. Some low frequency estimates of the resolvent in weighted  $L^2$  spaces can be found in [BR14b]. The estimates we will recover here on outgoing and ingoing resolvents hold for all powers of the resolvent and are sharp with respect to the behaviour in  $\lambda$ , unlike the ones presented in [BR14b] which only provide boundedness with respect to  $\lambda$ .

In particular, the aim of this work is to give some decay properties in weighted  $L^2$  spaces for the resolvent and the spectral measure of the Laplace-Beltrami operator on an asymptotically conical manifold. Our results are closely related to the ones presented in [BB21] where the geometry is the one of  $\mathbb{R}^n$ , up to an obstacle, with an asymptotically Euclidean metric. Indeed, we have proved here that even in the case of a manifold with an asymptotically conic end the same results as Theorem 1.2 and 1.3 in the paper by Bouclet and Burq hold.

We also consider the case of a non negative decaying potential, for which the proofs are easily generalised. In particular, we consider a potential with decay of the order  $\langle r \rangle^{-2-\varepsilon}$ ,  $r$  being the radial coordinate (precisely speaking  $W \in S^{-2-\varepsilon}$  in the sense of Definition 1.1).

Once these technical results are attained, it is straightforward to apply them to the Schrödinger, wave and Klein-Gordon flows in the low frequencies regime in order to obtain time decay estimates (see Theorem 1.3).

The strategy adopted here follows similar steps as the one presented in [BB21] up to the proof of low frequency exact Mourre estimates (Proposition 2.3). We refer to Section 2 for more details, only mentioning here that key point is to get rid of the compact remainder in a parameter dependent Mourre estimate.

The strategy used in [BB21] rests on the underlying Euclidean geometry and can not be applied to this more general context.

Indeed, in our case the fact that the operator has non constant coefficients will not allow us to commute derivatives with resolvents. Even though on the exact cone we will be able to recover some useful features of the flat case like the fact that  $\partial^{j,k}(-\Delta)^{-1}$  is a bounded operator, we will need a more careful spectral analysis to be able to apply similar properties. For example, in the exact conic case we will reduce the problem to dimension one thanks to a separation of variables argument and we will see how we can control our resolvent by studying the one dimensional resolvent corresponding to the spherical Laplacian.

This is also the reason for our assumption on the dimension, that we will assume here to be greater or equal than three (as opposed to [BB21] where all dimensions greater or equal than two are covered). This will allow us to use Hardy inequalities on  $L^2$  that will be necessary, for example in Appendix D.

Let  $n \geq 3$ , in the following we will consider and  $(S, \bar{g})$  an  $n - 1$ -dimensional closed Riemannian manifold with local coordinates  $(\theta_1, \dots, \theta_{n-1})$ , we will use these objects to define the angular part of conical manifolds.

We also need to define some notion of decay with respect to the radial variable.

**Definition 1.1.** Let  $f(r)$  an  $(h, k)$  tensor field, that is a section of the  $(h, k)$  tensor bundle  $(\otimes^h TS) \otimes$

( $\otimes^k T^*S$ ). Then  $f(r)$  is in the class  $S^{-\nu}$  if its coefficients verify the following inequality

$$|\partial_r^l \partial_\theta^\alpha f_{j_1 \dots j_k}^{i_1 \dots i_h}(r, \theta)| \lesssim \langle r \rangle^{-\nu-l}$$

locally uniformly in  $\theta$ , for all  $l \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^{n-1}$ .

In particular we will equip  $S$  either with the fixed metric  $\bar{g}$  or with a metric  $g(r)$  depending on the radial coordinate and which is a perturbation (in a  $S^{-\nu}$  sense) of  $\bar{g}$ , meaning that we assume

$$g(r) - \bar{g} \in S^{-\nu}. \quad (1.2)$$

The geometrical setting for all our analysis will be the following.

**Definition 1.2** (Asymptotically conical manifold). Let  $(M, G_M)$  a manifold of dimension  $n$  with  $K \subset M$  compact.  $M$  is said to be asymptotically conical if there exist  $R > 0$  and a diffeomorphism

$$\begin{aligned} \Omega : M \setminus K &\rightarrow (R, +\infty) \times S \\ m &\mapsto (r(m), \omega(m)) \end{aligned}$$

such that  $r : M \rightarrow [R, +\infty)$  is a proper function and the metric  $G_M$  is given by

$$G_M = \Omega^*(dr^2 + r^2g(r)).$$

*Remark 1.1.* Although with different notation, this is the same geometric framework as the one used in [IN10].

If  $\kappa : U_\kappa \subset S \rightarrow V_\kappa \subset \mathbb{R}^{n-1}$  are the coordinate charts on  $S$  we will denote by  $\Pi_\kappa, \Pi_\kappa^{-1}$  the pullback and pushforward on  $[R, +\infty) \times S$ ; moreover if  $(\varphi_\kappa)_\kappa$  is a partition of unity on  $S$  and  $\varphi$  a smooth cutoff on  $[R, +\infty)$  we will make use of the functions

$$\psi_\kappa(r, \omega) := \varphi(r)\varphi_\kappa(\omega) \in C_0^\infty([R, +\infty) \times U_\kappa)$$

which verify  $\sum_\kappa \psi_\kappa \equiv 1$  for large enough  $r$ .

*Remark 1.2.* A function on  $M \setminus K$  can be identified with a function on  $(R, +\infty) \times S$  thanks to  $\Omega$ . As we will basically always consider the corresponding quantities on  $(R, +\infty) \times S$  we will drop the composition by  $\Omega$ , which rigorously is the one that allows to pass from a point on the manifold to a point on  $(R, +\infty) \times S$ . This means that we will simply use the notation  $(r, \omega)$  for a point of  $M \setminus K$  and still denote by  $\psi_\kappa, \Pi_\kappa$  or  $\Pi_\kappa^{-1}$  the corresponding functions defined on  $M \setminus K$ .

In the subsequent table we group the different notations we introduced for the manifolds with their respective metrics, Hilbert spaces and the associated Laplace-Beltrami operators.

manifold	metric	Hilbert space	L-B operator
$M$	$G_M = \Omega^*(dr^2 + r^2g(r))$	$L^2(M)$	$P$
$[R, +\infty) \times S$	$\bar{G} = dr^2 + r^2\bar{g}$	$L_{\bar{G}}^2$	$-\Delta_0$

We also define

$$G = dr^2 + r^2g(r) \quad (1.3)$$

the perturbed metric on  $[R, +\infty) \times S$  and  $\|\cdot\|_{L_G^2}$  the  $L^2$  norm with respect to this metric.

We will denote by  $\|\cdot\|$  the norm of the operators on  $L^2(M)$ .

We remark that having quantified in (1.2) how much the metric  $g(r)$  deviates from  $\bar{g}$  we can compare the two operators  $-\Delta_0$  and  $P$  and also obtain that the norms of  $L_{\bar{G}}^2$  and  $L^2(M)$  are comparable, meaning that their quotient is bounded by constants from above and from below.

We will first obtain some estimates on weighted  $L^2$  spaces by Theorem 1.1 below that will then be used to prove the result on the regularity of the spectral measure stated in Theorem 1.2. Both these results are proved in the end of Section 3 in the free case. For how to adapt the proofs in the case of the addition of a potential the reader can refer to Section 5.

**Theorem 1.1.** *Let  $n \geq 3$ ,  $\lambda_0 > 0$ ,  $l \in \mathbb{N}$  and  $\alpha > l$ . Also let  $W$  a multiplicative potential that on  $M \setminus K$  is the multiplication by a real valued function in  $S^{-2-\varepsilon}$ . There exists a constant  $C$  such that for every  $\lambda \in (0, \lambda_0]$  it holds*

$$\|\langle r \rangle^{-\alpha} (P + W - \lambda \pm i0)^{-l} \langle r \rangle^{-\alpha}\| \leq C \lambda^{\min\{0, n/2-l\}}$$

if  $l \neq \frac{n}{2}$  and

$$\|\langle r \rangle^{-\alpha} (P + W - \lambda \pm i0)^{-\frac{n}{2}} \langle r \rangle^{-\alpha}\| \leq C |\log \lambda|$$

if  $n$  is even and  $l = \frac{n}{2}$ .

Let  $E_\Omega$  the indicator function of a set  $\Omega \subset \mathbb{R}$ , then for every  $u, v \in L^2(M)$  the map  $\Omega \mapsto (u, E_\Omega(P + W)v)$  is a well defined Borel measure. To say that we integrate  $\lambda$  with respect to this measure we write  $d(u, E_\lambda(P + W)v)$ . We call this measure the spectral measure of  $P + W$  associated to  $u$  and  $v$ . It satisfies the property

$$(u, f(P + W)v) = \int f(\lambda) d(u, E_\lambda(P + W)v)$$

for any Borel function  $f$ , or in short

$$f(P + W) = \int f(\lambda) dE_\lambda.$$

**Theorem 1.2.** *Let  $n \geq 3$ ,  $\lambda_0 > 0$ ,  $\alpha > k$  and  $E_\lambda$  the spectral measure of  $P + W$ . The function*

$$\lambda \mapsto \langle r \rangle^{-\alpha} \frac{dE_\lambda}{d\lambda} \langle r \rangle^{-\alpha}$$

is of class  $C^{k-1}((0, \lambda_0])$ . Moreover if  $\alpha > \frac{n}{2}$  then

$$\left\| \langle r \rangle^{-\alpha} \frac{d^j}{d\lambda^j} E_\lambda \langle r \rangle^{-\alpha} \right\| \lesssim \lambda^{\frac{n}{2}-j-1}.$$

These results can be applied to obtain decay estimates on the flow of the Schrödinger, wave and Klein-Gordon equations in the low frequency regime.

Indeed, by writing the flow of the equation through an integral against the spectral measure we can use the regularity results of Theorem 1.2 to obtain the desired estimates. The proofs carry out analogously to the ones in [BB21] (see Section 5 therein).

**Theorem 1.3.** *Let  $f \in C_0^\infty(\mathbb{R})$ . Then it holds*

i) *Schrödinger flow:*

$$\|\langle r \rangle^{-\alpha} f(P + W) e^{it(P+W)} \langle r \rangle^{-\alpha}\| \lesssim \langle t \rangle^{-\frac{\alpha}{2}}$$

for  $\alpha > \lceil \frac{n}{2} \rceil + 2$ .

ii) *Wave flow:*

$$\|\langle r \rangle^{-\alpha} f(P + W) \frac{\sin(t\sqrt{P+W})}{\sqrt{P+W}} \langle r \rangle^{-\alpha}\| \lesssim \langle t \rangle^{1-n}$$

and

$$\|\langle r \rangle^{-\alpha} f(P + W) e^{it\sqrt{P+W}} \langle r \rangle^{-\alpha}\| \lesssim \langle t \rangle^{-n}$$

for  $\alpha > n + 1$ .

iii) *Klein-Gordon flow:*

$$\|\langle r \rangle^{-\alpha} f(P + W) e^{it\sqrt{P+W+1}} \langle r \rangle^{-\alpha}\| \lesssim \langle t \rangle^{-\frac{\alpha}{2}}$$

for  $\alpha > \lceil \frac{n}{2} \rceil + 2$ .

The paper is organised as follows: in Section 2 we proceed to prove the existence of the limits in (1.1) where the crucial step (Section 4) will be to check a technical assumption ensuring that a strict Mourre inequality holds. This is the analogue in our context to what was done in [BB21]. Section 3 and 5 are then dedicated to the main results respectively in the free and potential case.

Throughout the whole paper we will have to consider convenient rescaled operators as follows.

Since we are interested in resolvents such as  $(P - \lambda \pm i0)^{-1}$  we consider the operator  $P/\lambda$  which it is convenient to study using rescaled pseudodifferential operators, that we now define.

**Definition 1.3.** A function  $a(r, \theta, \rho, \eta)$  is in  $\tilde{S}^{m, \mu}(\mathbb{R}^{2n})$  if and only if for every  $j, k \in \mathbb{N}, \alpha, \beta \in \mathbb{N}^{n-1}$  there exists a constant  $C$  such that

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\eta^\beta a(r, \theta, \rho, \eta)| \leq C \langle r \rangle^{m-j-|\beta|} \left( \langle \rho \rangle + \frac{\langle \eta \rangle}{\langle r \rangle} \right)^{\mu-k-|\beta|} \quad (1.4)$$

with

$$\langle r \rangle = \begin{cases} 1 & \text{if } r \text{ is in a compact set,} \\ r & \text{if } r \gg 1. \end{cases}$$

The seminorms of the space are given by the smallest constants verifying the inequality.

*Remark 1.3.* Although we are using the same notation as the radial coordinate on the manifold, here  $r$  is simply meant to denote the first variable of  $\mathbb{R}^n$ .

We consider the usual quantization of a symbol defined as

$$Op(a)f(r, \theta) = \frac{1}{(2\pi)^n} \int \int e^{i(r-r')\rho + i(\theta-\theta')\eta} a(r, \theta, \rho, \eta) f(r', \theta') dr' d\theta' d\rho d\eta$$

and we introduce the dilation operator with respect to  $r$  and its generator  $A$ , namely

$$e^{itA}u(r, \theta) = e^{\frac{tn}{2}} u(e^t r, \theta), \quad A := \frac{n}{2i} - ir\partial_r.$$

For a symbol  $a$  defined on  $\mathbb{R}^{2n}$  a rescaled pseudodifferential operator is defined as

$$Op_\lambda(a) := e^{i\tau A} Op(a) e^{-i\tau A}. \quad (1.5)$$

Taking  $\tau = \ln(\lambda^{\frac{1}{2}})$  implies a rescaling of the spatial variable by  $\lambda^{\frac{1}{2}}$ , in other words

$$Op_\lambda(a) = Op(a_\lambda)$$

with  $a_\lambda(\check{r}, \theta, \check{\rho}, \eta)$  and  $\check{r} := \lambda^{\frac{1}{2}} r, \check{\rho} := \lambda^{-\frac{1}{2}} \rho$ .

Analogously we define rescaled pseudodifferential operators on manifolds as

$$Op_{\lambda, \kappa}(a) \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega) := \Pi_\kappa Op_\lambda(a) \Pi_\kappa^{-1} \psi_\kappa \quad (1.6)$$

for a symbol  $a$  supported in  $[R, +\infty) \times V_\kappa \times \mathbb{R}^n$ . We remark that  $Op_{\lambda, \kappa}(a) \psi_\kappa(\lambda^{\frac{1}{2}} r, \omega)$  maps  $C_0^\infty(M)$  in the set of functions supported in  $[R, +\infty) \times U_\kappa$ .

For example, near infinity  $P$  agrees with the Laplace Beltrami operator on  $(R + \infty) \times S$  and therefore in local coordinates it is given by

$$P/\lambda = -\frac{\partial_r^2}{\lambda} - \frac{n-1}{r} \frac{\partial_r}{\lambda} - \frac{1}{\lambda r^2} \Delta_{g(r)} - \frac{\partial_r |g(r, \theta)|}{|g(r, \theta)|} \frac{\partial_r}{\lambda}. \quad (1.7)$$

In terms of pseudodifferential operators this can be written as

$$\frac{P}{\lambda} u = \sum_{\kappa} Op_{\lambda, \kappa}(a_{0, \lambda} + a_{1, \lambda}) \psi_\kappa u. \quad (1.8)$$

*Remark 1.4.* Considering rescaled pseudodifferential operators is convenient since it allows us to obtain a decay which is uniform with respect to  $\lambda$ , meaning that the symbols in  $Op_\lambda(\cdot)$  will belong to  $\lambda$ -independent subsets of  $\tilde{S}^{m,\mu}(\mathbb{R}^{2n})$  for some  $m$  and  $\mu$ . Indeed for  $\check{r} \gtrsim 1$  we obtain

$$a_{0,\lambda}(\check{r}, \theta, \check{\rho}, \eta) \in \tilde{S}^{0,2}, \quad a_{1,\lambda}(\check{r}, \theta, \check{\rho}, \eta) \in \tilde{S}^{-1,1},$$

where the bounds on the seminorms are uniform in  $\lambda$ . The expressions for the symbols are

$$\begin{aligned} a_{0,\lambda}(\check{r}, \theta, \check{\rho}, \eta) &:= -\check{\rho}^2 - \frac{1}{\check{r}^2} g^{j,k}(\lambda^{-\frac{1}{2}} r, \theta) \eta_j \eta_k, \\ a_{1,\lambda}(\check{r}, \theta, \rho, \eta) &:= -\frac{n-1}{\check{r}} \check{\rho} - \lambda^{-\frac{1}{2}} w(\lambda^{-\frac{1}{2}} \check{r}, \theta) \check{\rho} - \lambda^{-1} w_k(\lambda^{-\frac{1}{2}} \check{r}, \theta) \eta_k. \end{aligned}$$

## 2 Limiting absorption principle

As mentioned before, we focus first on the case of the operator without potential, since the addition of  $W$  does not add much difficulty to the proofs as we can see in Section 5.

The section will be devoted to the proof of the existence of the limits  $(P/\lambda - 1 \pm i0)^{-l}$  in weighted  $L^2$  spaces thanks to a limiting absorption principle (specifically Theorem 1 in [Gér08]).

We look for a conjugate operator for  $P/\lambda$ , that is a selfadjoint operator which verifies some positive commutator estimate and such that  $P/\lambda \in C^2(A^\lambda)$ , meaning that for all  $u \in L^2(M)$  the map

$$\mathbb{R} \ni t \mapsto e^{itA^\lambda} (P/\lambda + i)^{-1} e^{-itA^\lambda} u \in D(P) \quad (2.1)$$

is of class  $C^2$ .

To get selfadjointness we will construct a candidate  $A^\lambda$  as the generator of a unitary group. Let  $\chi \in C_0^\infty(\mathbb{R})$  that is equal to 1 in a large enough neighborhood of 0, starting from the generator of dilations  $\frac{rD_r + D_r r}{2}$  after localisation in the region  $\{|r| \geq \lambda^{-\frac{1}{2}} R\}$  we obtain

$$A_1^\lambda := \frac{(1-\chi)(\lambda^{\frac{1}{2}} r) r D_r + D_r r (1-\chi)(\lambda^{\frac{1}{2}} r)}{2} = (1-\chi)(\lambda^{\frac{1}{2}} r) \left( \frac{1}{2i} - ir\partial_r \right) + \frac{i}{2} r \partial_r \chi(\lambda^{\frac{1}{2}} r).$$

If we consider the group of transformations

$$U_t^\lambda \varphi(r, \theta) = |\det(\text{Jac } \phi_t^\lambda(r, \theta))|^{\frac{1}{2}} \varphi(\phi_t^\lambda(r, \theta)),$$

with  $\phi_t^\lambda$  the flow of the complete vector field  $((1-\chi)(\lambda^{\frac{1}{2}} r)r, 0, \dots, 0)$ , thanks to Theorem VIII.10 in [RS81] we can conclude that  $A_1^\lambda$  is essentially selfadjoint on  $C_0^\infty([R, +\infty) \times S)$  with respect to the measure induced by the metric  $\bar{G}$  and that its closure is the infinitesimal generator of  $U_t^\lambda$ .

Moreover conjugating  $U_t^\lambda$  by the function

$$y_S(r, \theta) := \frac{|g(\theta)|}{r^{n-1}|g(r, \theta)|}$$

we obtain a group  $y_S^{\frac{1}{2}} U_t^\lambda y_S^{-\frac{1}{2}}$  which is unitary with respect to the metric  $G$  defined in (1.3) and whose generator will be

$$\begin{aligned} A_1^\lambda &= (1-\chi)(\lambda^{\frac{1}{2}} r) \left( \frac{n}{2i} - ir\partial_r + \frac{1}{2i} r \frac{\partial_r |g(r, \theta)|}{|g(r, \theta)|} \right) + \frac{i}{2} r \partial_r \chi(\lambda^{\frac{1}{2}} r) \\ &= (1-\chi)(\lambda^{\frac{1}{2}} r) \tilde{A} + \frac{i}{2} r \partial_r \chi(\lambda^{\frac{1}{2}} r). \end{aligned} \quad (2.2)$$

To define a unitary group which acts on the Hilbert space where  $P$  is defined, namely  $L^2(M)$ , we set

$$e^{itA^\lambda} u := e^{itA_1^\lambda} \chi_{M \setminus K} u + \chi_K u$$

whose generator is an operator  $A^\lambda$  which is selfadjoint on  $L^2(M)$  and that is non zero only on the manifold end where it coincides with  $A_2^\lambda$ .

For the continuity of  $P/\lambda$  with respect to  $A^\lambda$ , which we recall is the property in (2.1), it is enough to prove that the operators

$$[P/\lambda, iA^\lambda](P/\lambda + i)^{-1}, \quad [[P/\lambda, iA^\lambda], iA^\lambda](P/\lambda + i)^{-1} \quad (2.3)$$

are bounded, where the commutators are appropriately defined in the sense of quadratic forms. Indeed, in general given an Hilbert space  $\mathcal{H}$  with  $T$ ,  $A$  selfadjoint and  $T$  bounded for the map

$$\mathbb{R} \ni t \mapsto e^{itA} T e^{-itA} u =: B(t)u \in \mathcal{H}$$

to be  $C^k(\mathbb{R})$  it is enough that  $k$ -th derivative  $\frac{d^k}{dt^k} B(t_0)$  is a bounded operator of  $\mathcal{L}(\mathcal{H})$  at some fixed  $t_0$ , which in turn is implied if

$$T, [T, iA], \quad ad_A^k(T) := [ad_A^{k-1}(T), iA]$$

are bounded operators, in the sense of quadratic forms.

In our specific case we need to control only the commutator and the first iterated commutator with  $T = (P/\lambda + i)^{-1}$  and  $A = A^\lambda$  and with some algebraic manipulations we see that it is equivalent to the boundedness of the operators in (2.3). To prove this we will exploit symbolic calculus and a parametrix for the operator  $(P/\lambda - z)^{-1}$  localised on the end of the manifold.

With standard techniques we can in fact prove the following.

**Theorem 2.1.** *Let  $\psi, \tilde{\psi}, \tilde{\tilde{\psi}} \in S^0$ , supported in  $(R, +\infty) \times U_k$  such that*

$$\tilde{\psi}\psi = \psi, \quad \tilde{\tilde{\psi}}\tilde{\psi} = \tilde{\psi}.$$

and  $z \in \mathbb{C} \setminus [0, \infty)$ . Then for any  $N \in \mathbb{N}$  there exist families of symbols  $b_{l,\lambda,z} \in \tilde{S}^{-l, -2-l}$  and  $r_{N,\lambda,z} \in \tilde{S}^{-N, -N}$  such that

$$\psi(\lambda^{\frac{1}{2}}r, \omega)(P/\lambda - z)^{-1} = \sum_{l=0}^{N-1} \psi(\lambda^{\frac{1}{2}}r, \omega) Op_{\lambda,k}(b_{l,\lambda,z}) \tilde{\psi}(\lambda^{\frac{1}{2}}r, \omega) + R_{\lambda,z}^N$$

with

$$R_{\lambda,z}^N = \psi(\lambda^{\frac{1}{2}}r, \omega) Op_{\lambda,k}(r_{N,\lambda,z}) \tilde{\tilde{\psi}}(\lambda^{\frac{1}{2}}r, \omega)(P/\lambda - z)^{-1}.$$

Moreover, all of the symbols also have seminorms uniformly bounded in  $\lambda$ .

Before going on with the computations we will state a useful property. Briefly, commutators between rescaled pseudodifferential operators essentially behave like commutators between differential operators, when we write them in symbolic form. We now give a formal description of the result, while the proof, being quite technical, is postponed to Appendix C.

**Definition 2.1** (Negligible operator of order  $N$ ). Let  $N \in \mathbb{N}$  and set

$$Q_N := \langle \lambda^{\frac{1}{2}}r \rangle^N \left( \frac{P}{\lambda} + 1 \right)^N = \sum_{\kappa} Op_{\lambda,\kappa}(\tilde{S}^{N, 2N}),$$

we say that an operator is negligible of order  $N$  if it is of the form  $Q_N^{-1} \mathcal{B} Q_N^{-1}$  for some bounded operator  $\mathcal{B}$  depending on  $\lambda$ .

*Remark 2.1.* The operator  $\mathcal{B}$  depends on  $\lambda$  since it will be the result of the composition of a rescaled pseudodifferential operator of negative order with  $Q_N$ . However, the symbols in  $Q_N$  have seminorms uniformly bounded with respect to  $\lambda$ .



The interest of this definition is that for such operators the arbitrary decay of  $Q_N^{-1}$  means that when composing either on the left or the right with a pseudodifferential operator of any order (such as differential operators or resolvents) will always result in a bounded operator. As for commutators, the following holds.

**Proposition 2.2.** *Let  $m, m', \mu, \mu'$  real numbers and the operators  $A, B$  on  $[R, +\infty) \times S$  defined as*

$$A := \sum_{\kappa} Op_{\lambda, \kappa}(a^{\kappa})\psi_{\kappa}, \quad B := \sum_{\kappa} Op_{\lambda, \kappa}(b^{\kappa})\psi_{\kappa}$$

with symbols in  $\tilde{S}^{m, \mu}$  and  $\tilde{S}^{m', \mu'}$  respectively and spatially supported in  $[R, +\infty) \times V_{\kappa}$ . Then for the commutator it holds

$$[A, B] = \sum_{\kappa} Op_{\lambda, \kappa}(\tilde{S}^{m+m'-1, \mu+\mu'-1})\psi_{\kappa} + \mathcal{R}_N$$

with  $\mathcal{R}_N$  an operator which is negligible of order  $N$  for any  $N \in \mathbb{N}$ .

*Remark 2.2.* When we will need to compose a negligible operator with an operator on the right, for example, we will make use of the decay on the right only so we will rather write  $Q_N^{-1}\mathcal{B}Q_N^{-1}$  in the form  $\sum_{\kappa} \mathcal{B}Op_{\lambda, \kappa}(\tilde{S}^{-M, -M})$  for a suitable bounded  $\mathcal{B}$  and  $M \in \mathbb{N}$ .

*Remark 2.3.* Although we interpreted commutators as derivatives of the map  $e^{itA}(P/\lambda + i)^{-1}e^{-itA}$  to perform computations we will rather use their symbolic form, as in (2.4) for example. Indeed, on  $C_0^{\infty}(M)$  (which is dense in  $D(P)$ ) we can prove that the derivative is indeed the commutator  $[(P/\lambda + i)^{-1}, iA^{\lambda}]$  which we can rewrite in terms of the commutator between  $P/\lambda$  and  $iA^{\lambda}$ . Now on smooth functions the action of  $P/\lambda$  and  $A^{\lambda}$  is the one of differential operators that we can write in the symbolic form used in (2.4).

Writing  $A^{\lambda}$  in its symbolic form as

$$A^{\lambda} = \sum_{\kappa} Op_{\lambda, \kappa}(\tilde{S}^{1,1} + \lambda^{\frac{\nu}{2}}\tilde{S}^{-\nu,0} + C_0^{\infty}(\mathbb{R} \setminus \{0\}))$$

and using Proposition 2.2 as well as Remark 2.2 we first find that

$$[P/\lambda, iA^{\lambda}] = [\sum_{\kappa} Op_{\lambda, \kappa}(\tilde{S}^{0,2}), \sum_l Op_{\lambda, l}(\tilde{S}^{1,1})] = \sum_{\kappa} (Op_{\lambda, \kappa}(\tilde{S}^{0,2}) + \mathcal{B}Op_{\lambda, \kappa}(\tilde{S}^{-M, -M})) \quad (2.4)$$

for some bounded operator  $\mathcal{B}$  and some positive  $M$ .

Combining with the informations provided by the parametrix we find out that  $[P/\lambda, iA^{\lambda}](P/\lambda + i)^{-1}$  is indeed a sum of bounded operators (symbols in  $\tilde{S}^{0,0}$  correspond to bounded operators of  $L^2(M)$ , see Proposition 3.4 in [BM16] and in particular inequality (3.13)).

An analogous result holds for the iterated commutator since we can still write

$$[[P/\lambda, iA^{\lambda}], iA^{\lambda}] = \sum_{\kappa} (Op_{\lambda, \kappa}(\tilde{S}^{0,2}) + \mathcal{B}Op_{\lambda, \kappa}(\tilde{S}^{-M, -M})) \quad (2.5)$$

and reason in the same way as before.

*Remark 2.4.* Actually, we remark here that we can iterate the argument since we can see that, going on from (2.5), any iterated commutator is of the form

$$ad_{A^{\lambda}}^k(P/\lambda) = \sum_{\kappa} (Op_{\lambda, \kappa}(\tilde{S}^{0,2}) + \mathcal{B}Op_{\lambda, \kappa}(\tilde{S}^{-M, -M})).$$

We can therefore apply Theorem 2.1 as before to conclude that  $ad_{A^{\lambda}}^k(P/\lambda)(P/\lambda + i)^{-1}$  is bounded, which implies that  $P/\lambda \in C^k(A^{\lambda})$  for any  $k$ .

Next, we will prove a positive commutator estimate for the  $P/\lambda$  and to this purpose we first need to make an assumption on a spectral localisation of the operator.

**Assumption 2.1.** For all  $\alpha > 0$  and  $\varepsilon > 0$  there exist  $\lambda_0 > 0$  and  $f \in C_0^\infty(\mathbb{R})$  equal to 1 in a neighborhood of 1 such that

$$\|\langle \lambda^{\frac{1}{2}} r \rangle^{-\alpha} f(P/\lambda)\| \leq \varepsilon$$

for all  $\lambda \in (0, \lambda_0]$ .

With this property, that we will check in Section 4, we can prove the desired inequality.

**Proposition 2.3.** *Let  $\lambda_0 > 0$  small enough, if Assumption 2.1 holds there exists  $I$  open bounded interval containing 1 such that*

$$\mathbb{1}_I(P/\lambda) i[P/\lambda, A^\lambda] \mathbb{1}_I(P/\lambda) \geq \mathbb{1}_I(P/\lambda).$$

for all  $\lambda \in (0, \lambda_0]$ .

Before going further on the proof of the result, we point out that thanks to this inequality coupled with the fact that  $P/\lambda \in C^2(A^\lambda)$  we can apply Theorem 1 in [Gér08] to state that

$$\sup_{\lambda \in (0, \lambda_0]} \sup_{\varepsilon > 0} \|\langle A^\lambda \rangle^{-s} (P/\lambda - 1 \pm i\varepsilon)^{-1} \langle A^\lambda \rangle^{-s}\| < \infty \quad (2.6)$$

for  $s > \frac{1}{2}$ , or equivalently

$$(A^\lambda + i)^{-s} (P/\lambda - 1 \pm i\varepsilon)^{-1} (A^\lambda + i)^{-s} \in \mathcal{L}(L^2(M))$$

with operator norms uniformly bounded in  $\lambda$ . Finally, thanks to the higher regularity of  $P/\lambda$  with respect to  $A^\lambda$  similar bounds can be proved for powers of the resolvent and therefore obtaining

$$(A^\lambda + i)^{-s} (P/\lambda - 1 \pm i0)^{-l} (A^\lambda + i)^{-s} \in \mathcal{L}(L^2(M))$$

for any  $s > l - \frac{1}{2}$  and still with norms uniformly bounded in  $\lambda$ . Indeed, our conjugate operator  $A^\lambda$  is, in particular, **uniformly conjugate** to  $P/\lambda$  according to Definition 5.1 in [BR14a] and, thanks to Remark 2.4, it is also **uniformly  $\infty$ -smooth** with respect to  $A^\lambda$  (see Definition 5.3 in [BR14a]). Following the ideas of [Jen85], it is proved in [BR14a] that with this kind of smoothness we have estimates for powers of the resolvent analogous to the ones in (2.6).

To prove Proposition 2.3 we will split the commutator in the part at infinity where  $P$  is of the form (1.8) and  $A^\lambda = \tilde{A}$ , according to (2.2), and treat the rest as a compactly supported perturbation (we recall that on the compact part of the manifold  $A^\lambda$  is simply zero). Namely we can write

$$i[P/\lambda, A^\lambda] = (1 - \tilde{\chi})(\lambda^{\frac{1}{2}} r) i[P/\lambda, \tilde{A}] + \tilde{\chi}(\lambda^{\frac{1}{2}} r) i[P/\lambda, A^\lambda] \quad (2.7)$$

for some smooth cutoff  $\tilde{\chi}$  such that  $\tilde{\chi} \equiv 1$  on the support of  $\chi$ . In local coordinates the Laplace-Beltrami operator  $-\Delta_0$  on the fixed cone  $([R, +\infty) \times S, \bar{G})$  is

$$-\partial_r^2 - \frac{n-1}{r} \partial_r - \frac{1}{r^2} \Delta_{\bar{g}}.$$

Recalling the local coordinates expression in (1.7) we notice that on the manifold end we can write  $P$  in function of  $\Delta_0$ . In doing so, thanks to the fact that  $g(r)$  is a perturbation of  $\bar{g}$  we can quantify the decay of the remaining part.

**Proposition 2.4.** *Let  $\lambda_0 > 0$  and  $\lambda \in (0, \lambda_0]$ , then*

$$(1 - \tilde{\chi})(\lambda^{\frac{1}{2}} r) P/\lambda = (1 - \tilde{\chi})(\lambda^{\frac{1}{2}} r) (-\Delta_0/\lambda) + \sum_{\kappa} \left( \lambda^{\frac{\nu}{2}} Op_{\lambda, \kappa}(\tilde{S}^{-\nu, 2}) + Op_{\lambda, \kappa}(\tilde{S}^{-1, 1}) \right) \psi_{\kappa}(\lambda^{\frac{1}{2}} r, \omega)$$

with symbols that belong to bounded subsets of  $\tilde{S}^{-\nu, 2}$  and  $\tilde{S}^{-1, 1}$  respectively.

Notably this will be useful since we are taking the commutator with  $\tilde{A}$  given by

$$\tilde{A} = A + \sum_{\kappa} \lambda^{\frac{\nu}{2}} Op_{\lambda, \kappa}(\tilde{S}^{-\nu, 0}) \psi_{\kappa}(\lambda^{\frac{1}{2}} r, \omega) = \sum_{\kappa} Op_{\lambda, \kappa}(\tilde{S}^{1, 1}) \psi_{\kappa}(\lambda^{\frac{1}{2}} r, \omega), \quad (2.8)$$

and this will allow us to take advantage of the fact that  $[-\Delta_0, iA] = 2(-\Delta_0)$  (recall that  $A = \frac{n}{2i} - ir\partial_r$ ).

*Proof of Proposition 2.3.* Using the expression of  $\tilde{A}$  given by (2.8) (which we recall is supported for  $r \gtrsim \lambda^{-\frac{1}{2}}$ ) and by Proposition 2.4 we can compute

$$\begin{aligned} [P/\lambda, \tilde{A}] &= \sum_{\kappa} [-\Delta_0/\lambda, A + \lambda^{\frac{\nu}{2}} Op_{\lambda, \kappa}(\tilde{S}^{-\nu, 0}) \psi_{\kappa}(\lambda^{\frac{1}{2}} r, \omega)] \\ &\quad + \sum_{\kappa} [(\lambda^{\frac{\nu}{2}} Op_{\lambda, \kappa}(\tilde{S}^{-\nu, 2}) + Op_{\lambda, \kappa}(\tilde{S}^{-1, 1})) \psi_{\kappa}(\lambda^{\frac{1}{2}} r, \omega), Op_{\lambda, \kappa}(\tilde{S}^{1, 1}) \psi_{\kappa}(\lambda^{\frac{1}{2}} r, \omega)] \\ &= 2(-\Delta_0/\lambda) + \sum_{\kappa} (\lambda^{\frac{\nu}{2}} Op_{\lambda, \kappa}(\tilde{S}^{-\nu, 2}) + Op_{\lambda, \kappa}(\tilde{S}^{-1, 1})) \psi_{\kappa}(\lambda^{\frac{1}{2}} r, \omega) \\ &\quad + \sum_{\kappa} \mathcal{B}Op_{\lambda, \kappa}(\tilde{S}^{-M, -M}) \psi_{\kappa}(\lambda^{\frac{1}{2}} r, \omega), \end{aligned}$$

since  $-\Delta_0/\lambda = \sum_{\kappa} Op_{\lambda, \kappa}(\tilde{S}^{0, 2}) \psi_{\kappa}(\lambda^{\frac{1}{2}} r, \omega)$  (we apply here the calculus rules given by Proposition 2.2 and the observation of Remark 2.2). Noticing that on the support of  $\psi_{\kappa}(\lambda^{\frac{1}{2}} r, \omega)$

$$\lambda^{\frac{\nu}{2}} Op_{\lambda, \kappa}(\tilde{S}^{-\nu, 2}) = \langle \lambda^{\frac{1}{2}} r \rangle^{-\nu} Op_{\lambda, \kappa}(\langle \lambda^{\frac{1}{2}} r \rangle^{\nu} \tilde{S}^{-\nu, 2}) = \langle \lambda^{\frac{1}{2}} r \rangle^{-\nu} Op_{\lambda, \kappa}(\tilde{S}^{0, 2}),$$

and similarly

$$Op_{\lambda, \kappa}(\tilde{S}^{-1, 1}) = \langle \lambda^{\frac{1}{2}} r \rangle^{-1} Op_{\lambda, \kappa}(\langle r \rangle \tilde{S}^{-1, 1}) = \langle \lambda^{\frac{1}{2}} r \rangle^{-1} Op_{\lambda, \kappa}(\tilde{S}^{0, 1}),$$

the quantity in (2.7) is then given by

$$\begin{aligned} i[P/\lambda, iA^{\lambda}] &= (1 - \tilde{\chi})(\lambda^{\frac{1}{2}} r) \left( 2P/\lambda + \sum_{\kappa} \langle \lambda^{\frac{1}{2}} r \rangle^{-\nu} Op_{\lambda, \kappa}(\tilde{S}^{0, 2}) + \langle \lambda^{\frac{1}{2}} r \rangle^{-1} Op_{\lambda, \kappa}(\tilde{S}^{0, 1}) \right) \\ &\quad + (1 - \tilde{\chi})(\lambda^{\frac{1}{2}} r) \sum_{\kappa} \mathcal{B}Op_{\lambda, \kappa}(\tilde{S}^{-M, -M}) \\ &\quad + \tilde{\chi}(\lambda^{\frac{1}{2}} r) [P/\lambda, iA^{\lambda}]. \end{aligned}$$

Moreover, up to a compactly supported perturbation we can commute  $(1 - \tilde{\chi})$  with any differential operator and in particular the pseudodifferential operators in the sum above are differential, given that they are the result of a commutator between differential operators. We have obtained

$$\begin{aligned} i[P/\lambda, A^{\lambda}] - 2P/\lambda &= -2\tilde{\chi}(\lambda^{\frac{1}{2}} r) P/\lambda \\ &\quad + \sum_{\kappa} \langle \lambda^{\frac{1}{2}} r \rangle^{-\nu} Op_{\lambda, \kappa}(\tilde{S}^{0, 2}) (1 - \tilde{\chi})(\lambda^{\frac{1}{2}} r) \\ &\quad + \sum_{\kappa} \langle \lambda^{\frac{1}{2}} r \rangle^{-1} Op_{\lambda, \kappa}(\tilde{S}^{0, 1}) (1 - \tilde{\chi})(\lambda^{\frac{1}{2}} r) \\ &\quad + (1 - \tilde{\chi})(\lambda^{\frac{1}{2}} r) \sum_{\kappa} \mathcal{B}Op_{\lambda, \kappa}(\tilde{S}^{-M, -M}) \\ &\quad + \psi(\lambda^{\frac{1}{2}} r) \end{aligned}$$

for some  $\psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$  with  $\psi(\lambda^{\frac{1}{2}}r)$  which includes the term  $\tilde{\chi}(\lambda^{\frac{1}{2}}r)[P/\lambda, iA^\lambda]$ . We take the composition with  $f(P/\lambda)$  on the right and on the left, where  $f$  satisfies Assumption 2.1. Noticing that by Theorem 2.1 and the spectral theorem

$$Op_{\lambda, \kappa}(\tilde{S}^{0,2})(1 - \tilde{\chi})(\lambda^{\frac{1}{2}}r)(P/\lambda + i)^{-1}, \quad Op_{\lambda, \kappa}(\tilde{S}^{0,1})(1 - \tilde{\chi})(\lambda^{\frac{1}{2}}r)(P/\lambda + i)^{-1},$$

and

$$(P/\lambda + i)f(P/\lambda), \quad P/\lambda(P/\lambda + i)f(P/\lambda)$$

are all bounded. we have the estimate

$$\|f(P/\lambda) \left( i[P/\lambda, A^\lambda] - 2P/\lambda \right) f(P/\lambda)\| \lesssim \|f(P/\lambda)\tilde{\chi}(\lambda^{\frac{1}{2}}r)(P/\lambda + i)^{-1}\| \quad (2.9)$$

$$+ \|f(P/\lambda)\langle \lambda^{\frac{1}{2}}r \rangle^{-\nu}\| + \|f(P/\lambda)\langle \lambda^{\frac{1}{2}}r \rangle^{-1}\| \quad (2.10)$$

$$+ \sum_{\kappa} \|f(P/\lambda)\mathcal{B}Op_{\lambda, \kappa}(\tilde{S}^{-M, -M})\| \quad (2.11)$$

$$+ \|f(P/\lambda)\psi(\lambda^{\frac{1}{2}}r)(P/\lambda + i)^{-1}\|. \quad (2.12)$$

Thanks Assumption 2.1, up to choosing  $f$  with small enough support we can make all the terms in the right hand side arbitrarily small. Indeed the assumption applies directly to the terms in (2.10), for (2.11) the decay in  $r$  is provided by  $Op_{\lambda, \kappa}(\tilde{S}^{-M, -M})$ , while for (2.9) and (2.12) we observe that we have compact support in  $r$  thanks to  $\tilde{\chi}$  and  $\psi$ . In particular it holds

$$\|f(P/\lambda) \left( i[P/\lambda, A^\lambda] - 2P/\lambda \right) f(P/\lambda)\| \leq \frac{1}{2}$$

for all  $\lambda \in (0, \lambda_0]$ . Now we simply have

$$f(P/\lambda)i[P/\lambda, A^\lambda]f(P/\lambda) \geq 2f(P/\lambda)P/\lambda f(P/\lambda) - \frac{1}{2} \geq \frac{3}{2}f^2(P/\lambda) - \frac{1}{2}$$

with the last inequality following from the fact that we can choose  $f$  with arbitrarily small support such that  $2f^2(x)x \geq \frac{3}{2}f^2(x)$ . At last, we choose  $I \subset \sigma(P)$  open bounded interval containing 1 and small enough such that  $f$  is constantly 1 on  $I$ , then  $f(x)\mathbb{1}_I(x) = \mathbb{1}_I(x)$  and applying  $\mathbb{1}_I(P/\lambda)$  on the right and left of the previous inequality we have

$$\mathbb{1}_I(P/\lambda)i[P/\lambda, A^\lambda]\mathbb{1}_I(P/\lambda) \geq \mathbb{1}_I^2(P/\lambda) = \mathbb{1}_I(P/\lambda),$$

concluding the proof.  $\square$

### 3 Main results

In this section we see how to prove the results stated in the introduction, having obtained from the previous computations that

$$(A^\lambda + i)^{-s}(P/\lambda - 1 \pm i0)^{-l}(A^\lambda + i)^{-s} \in \mathcal{L}(L^2(M)). \quad (3.1)$$

with operator norm uniformly bounded in  $\lambda$ .

We recall that this section is dedicated to the free case and details about the proofs for the operator  $P + W$  can be found in Section 5.

The first step into proving estimates for the resolvent in weighted spaces will be to look at the resolvent with some spectral localisations, such as

$$\|\langle r \rangle^{-\alpha} f(P/\lambda)(P - \lambda \pm i0)^{-l} \langle r \rangle^{-\alpha}\|$$

for some  $f \in C_0^\infty(\mathbb{R})$ . To be able to use (3.1) we will need to bound  $\langle r \rangle^{-\alpha} f(P/\lambda)(A^\lambda + i)^s$ , first we notice that we can rewrite such operators in a more convenient way.

**Lemma 3.1.** *Let  $g \in C_0^\infty(\mathbb{R})$ ,  $\lambda_0 > 0$  and  $s, N \in \mathbb{N}$ . There exist a family of symbols  $(\phi_{s,\lambda}^\kappa)_{\lambda \in (0,\lambda_0]} \in \tilde{S}^{s,-N}$ , a family of uniformly bounded operators  $(B_\lambda)_{\lambda \in (0,\lambda_0]}$  and  $(\tilde{\psi}_\kappa)_\kappa \in C^\infty([R, +\infty) \times S)$  supported in  $(R, +\infty) \times U_\kappa$  with  $\tilde{\psi}_\kappa \psi_\kappa \equiv 1$  such that*

$$(A^\lambda + i)^s g(P/\lambda) = \sum_{\kappa} Op_{\lambda,\kappa}(\phi_{s,\lambda}^\kappa) \tilde{\psi}_\kappa(\lambda^{\frac{1}{2}} r, \omega) + B_\lambda(P/\lambda + 1)^{-N}$$

for  $\lambda \in (0, \lambda_0]$ .

*Proof.* By Helffer-sjöstrand formula we write  $g(P/\lambda)$  in terms of its resolvent, then applying Theorem 2.1 thanks to the expression of the symbols in the parametrix and using the property of almost analytic extensions

$$\frac{1}{2\pi} \int \bar{\partial}_z \tilde{f}(z) (\lambda - z)^{-1-j} L(dz) = \frac{(-1)^j}{j!} f^{(j)}(\lambda)$$

we obtain symbols for  $g(P/\lambda)$  that have negative decay in space and compact support in the angular part. Namely we can write for any  $M \in \mathbb{N}$

$$\psi_\kappa(\lambda^{\frac{1}{2}} r, \omega) g(P/\lambda) = Op_{\lambda,\kappa}(\tilde{S}^{0,-M}) \tilde{\psi}_\kappa(\lambda^{\frac{1}{2}} r, \omega) + R_\lambda(P/\lambda + 1)^{-N} \quad (3.2)$$

where  $R_\lambda$  includes the integral of the remainder part given by the parametrix. Next we need to compose (3.2) on the left with powers of  $A^\lambda$  of order less or equal than  $s$ . By choosing the appropriate  $M$  ( $M = s + N$ ), we can conclude observing that  $(A^\lambda)^j$  will have symbols in  $\tilde{S}^{j,j}$  (see (2.8)) and that  $R_\lambda$  is of the form

$$R_\lambda = \frac{1}{\pi i} \int \bar{\partial}_z \tilde{f}(z) Op_{\lambda,\kappa}(\tilde{S}^{-L,-L})(P/\lambda - z)^{-1} (P/\lambda + 1)^N L(dz)$$

for any  $L \in \mathbb{N}$  and symbols that have seminorms growing polynomially in  $1/|Imz|^L$ . □

Now to be able to handle the powers of the resolvent appearing in the expression given by Lemma 3.1 we will prove a very useful result that will also be extensively used in the following section. First of all we have a preliminary result.

**Lemma 3.2.** *For any  $\lambda > 0$ ,  $p \in [1, 2]$  and integer  $N \geq 1 + \frac{n}{2} \left(\frac{1}{p} - \frac{1}{2}\right)$  it holds*

$$\|(P/\lambda + 1)^{-N}\|_{L^p(M) \rightarrow L^2(M)} \leq C \lambda^{\frac{n}{2} \left(\frac{1}{p} - \frac{1}{2}\right)}$$

for some constant  $C > 0$ .

The result is derived thanks to the behaviour of the flow of the heat equation  $e^{-tP}$

$$\|e^{-tP}\|_{L^p(M) \rightarrow L^2(M)} \leq C t^{-\frac{n}{2} \left(\frac{1}{p} - \frac{1}{2}\right)}, \quad p \in [1, 2] \quad (3.3)$$

which is due to the fact that a Nash type inequality holds. Namely, for all  $u \in C_0^\infty(M)$

$$\|u\|_{L^2(M)}^{1+\frac{2}{n}} \leq C_n \|u\|_{L^1(M)}^{\frac{2}{n}} \|P^{\frac{1}{2}} u\|_{L^2(M)} \quad (3.4)$$

for some  $C_n > 0$ .

The inequality is proved in detail in Appendix B, here we briefly record that it can be proved by first considering the full cone  $\mathbb{R}^+ \times S$  with fixed metric  $\bar{G}$  so that locally we can apply the Nash inequality on  $\mathbb{R}^n$ . The result still holds on  $M$  since on the compact part  $K$  we are locally on  $\mathbb{R}^n$ , while on the manifold end we use the inequality we have proved for the cone.

Then to derive (3.3) we can first prove  $\|e^{-tP}\|_{L^1(M) \rightarrow L^2(M)} \lesssim t^{-\frac{n}{4}}$  thanks to (3.4) and the fact that  $e^{-tP}$  preserves the sign and the  $L^1$  norm. To conclude, (3.3) follows from interpolating the previous bound with  $\|e^{-tP}\|_{L^2(M) \rightarrow L^2(M)} \lesssim 1$ , which is to Hille-Yosida theorem telling us that  $P$  generates a contraction semigroup  $e^{-tP}$ .

*Proof of Lemma 3.2.* We can follow the same proof described in Lemma 3.2 of [BB21]. Writing the resolvent via the heat kernel

$$(P/\lambda + 1)^{-N} = \frac{1}{N!} \int_0^\infty e^{-t(P/\lambda+1)} t^{N-1} dt$$

we apply (3.3) and the fact that, given the assumption on  $N$ ,  $e^{-t} t^{N-1-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}$  is integrable on  $\mathbb{R}^+$ .  $\square$

Finally, thanks to Lemma 3.2 we also easily obtain polynomial decay for powers of the resolvent.

**Lemma 3.3.** *Let  $n \geq 3$ , for all  $s \in [0, \frac{n}{4}] \cap [0, N)$  and  $\frac{\alpha}{2} > s$  it holds*

$$\|(P/\lambda + 1)^{-N} \langle r \rangle^{-\sigma}\| \lesssim \lambda^s$$

for any  $\lambda > 0$ .

*Proof.* The estimate follows from Lemma 3.2 and Hölder inequality to bound  $\langle r \rangle^{-\sigma}$  as an operator from  $L^2(M)$  to  $L^p(M)$ .  $\square$

Now, using the expression given in Lemma 3.1 we can easily derive the following proposition.

**Proposition 3.4.** *Let  $g \in C_0^\infty(\mathbb{R})$ ,  $\alpha \geq l$ ,  $\lambda_0 > 0$  and  $s \in (0, \frac{n}{4}] \cap (0, \frac{\alpha}{2})$ . Then*

$$\|(A^\lambda + i)^l g(P/\lambda) \langle r \rangle^{-\alpha}\| \lesssim \lambda^s$$

for  $\lambda \in (0, \lambda_0]$ .

*Proof.* From Lemma 3.1 and the fact that operators with symbols in  $\tilde{S}^{0,0}$  are bounded, paired with Lemma 3.3 to control  $\|B_\lambda(P/\lambda + 1)^{-N} \langle r \rangle^{-\alpha}\|$ . We underline that a power  $\lambda^{\frac{\alpha}{2}}$  is generated from the terms  $Op_{\lambda,\kappa}(\phi_{s,\lambda}^\kappa) \tilde{\psi}_\kappa(\lambda^{\frac{1}{2}} r, \omega) \langle r \rangle^{-\alpha}$  when moving the factor  $r^{-\alpha}$  into the rescaled pseudodifferential operator.  $\square$

Combining (3.1) with Proposition 3.4 we can straightforwardly obtain bounds on the spectrally localised resolvent.

**Theorem 3.5.** *Let  $f \in C_0^\infty(\mathbb{R})$ ,  $\lambda_0 > 0$  and  $l \in \mathbb{N}$ . If  $\alpha \geq l$  and  $s \in [0, \frac{n}{4}] \cap (0, \frac{\alpha}{2})$  then*

$$\|\langle r \rangle^{-\alpha} f(P/\lambda) (P - \lambda \pm i0)^{-l} \langle r \rangle^{-\alpha}\| \lesssim \lambda^{2s-l}$$

for  $\lambda \in (0, \lambda_0]$ .

At the aid of this theorem and with the same reasoning as in Proposition 4.4 of [BB21] we obtain a bound on the resolvent which is still localised but in a weaker way.

**Proposition 3.6.** *Let  $\lambda_0 > 0$ , there exists  $F \in C_0^\infty(\mathbb{R})$  equal to 1 near  $[0, \lambda_0]$  such that for  $l \in \mathbb{N} \setminus \{\frac{n}{2}\}$  and  $\alpha > l$  it holds*

$$\|\langle r \rangle^{-\alpha} F(P) (P - \lambda \pm i0)^{-l} \langle r \rangle^{-\alpha}\| \lesssim \lambda^{\min\{0, n/2-l\}},$$

if  $l = \frac{n}{2}$  instead

$$\|\langle r \rangle^{-\alpha} F(P) (P - \lambda \pm i0)^{-\frac{n}{2}} \langle r \rangle^{-\alpha}\| \lesssim |\log \lambda|$$

for  $\lambda \in (0, \lambda_0]$ .

Thanks to these preliminary steps we are now ready to prove the estimate without any localisation on the resolvent.

*Proof of Theorem 1.1.* Pick  $F$  as in the previous proposition, the result follows since Proposition 3.6 holds and  $(1 - F(P))(P - \lambda \pm i0)^{-l}$  is uniformly bounded in  $\lambda$  by the spectral theorem.  $\square$

As for the result on the spectral measure, we recall that thanks to Stone's formula

$$\frac{dE(\lambda)}{d\lambda} = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0} ((P - \lambda - i\varepsilon)^{-1} - (P - \lambda + i\varepsilon)^{-1}),$$

we can equivalently consider outgoing and incoming resolvents so to use the result we just established in Theorem 3.5.

*Proof of Theorem 1.2.* Let  $f \in C_0^\infty(\mathbb{R})$  bounded and supported around 1 then we have

$$(1 - f(P/\lambda)) \left( (P - \lambda - i\varepsilon)^{-1-j} - (P - \lambda + i\varepsilon)^{-1-j} \right) \rightarrow 0$$

in the strong topology as  $\varepsilon$  goes to 0. Hence it suffices to consider the terms

$$\frac{1}{2\pi i j!} \langle r \rangle^{-\alpha} f(P/\lambda) \left( (P - \lambda - i0)^{-1-j} - (P - \lambda + i0)^{-1-j} \right) \langle r \rangle^{-\alpha} \quad (3.5)$$

for  $j = 0, \dots, k-1$  to which Theorem 3.5 applies thanks to the assumption that  $\alpha > k$ .  $\square$

## 4 Proof of Assumption 2.1

A crucial step in the work presented up to now was to obtain the positive commutator estimate which allowed us to state that the outgoing and ingoing resolvents existed. Our main concern now is to prove that the Assumption 2.1 we made to obtain this result is valid for the operator  $P$  we are considering.

We will split the analysis into several steps by spatially localising the operator  $f(P/\lambda)$  as follows

$$\begin{aligned} f(P/\lambda) &= \chi f(P/\lambda) + (1 - \chi) f(P/\lambda) \chi + (1 - \chi) f(-\Delta_0/\lambda) (1 - \chi) \\ &\quad + (1 - \chi) f(P/\lambda) (1 - \chi) - (1 - \chi) f(-\Delta_0/\lambda) (1 - \chi) \end{aligned}$$

where  $\chi = \chi(r)$  is a smooth cutoff which is constantly 1 on  $K$  and zero for large  $r$ .

*Remark 4.1.* The difference in the second line of the expression is well defined. Indeed, thanks to the cutoff on the right we are restricting ourselves to functions supported on  $M \setminus K$  that can be identified with functions on  $(R, +\infty) \times S$  which is where both the actions of  $f(-\Delta_0/\lambda)$  and  $f(P/\lambda)$  make sense.

We recall that in Assumption 2.1 it is stated that for any  $\alpha$  the norm of  $\langle \lambda^{\frac{1}{2}} r \rangle^{-\alpha} f(P/\lambda)$  can be made arbitrarily small up to spectrally localising  $P/\lambda$  close to one. We will summarise here how each term is treated and where the relative statement can be found.

- $\chi f(P/\lambda)$ ,  $(1 - \chi) f(P/\lambda) \chi$ : their norm can be made arbitrarily small up to choosing  $\lambda$  sufficiently small. See Proposition 4.1, via Lemmas 3.2 and 3.3.
- $(1 - \chi) f(-\Delta_0/\lambda) (1 - \chi)$ : thanks to the multiplication by the decaying factor  $\langle \lambda^{\frac{1}{2}} r \rangle^{-\alpha}$  the norm can be made arbitrarily small up to choosing  $f$  with small enough support. See Proposition 4.4, via rescaling argument and Lemma 4.3.
- $(1 - \chi) f(P/\lambda) (1 - \chi) - (1 - \chi) f(-\Delta_0/\lambda) (1 - \chi)$ : the norm can be made arbitrarily small up to choosing  $\lambda$  sufficiently small. See (4.14), via Helffer-Sjöstrand formula, Lemma 4.5 and (4.13) (with intermediate steps in Lemmas 4.6, 4.8 and 4.11, where Lemma 3.3 is extensively used).

We will start by showing how to bound  $\chi f(P/\lambda)$  and  $(1 - \chi) f(P/\lambda) \chi$ .

**Proposition 4.1.** *Let  $f \in C_0^\infty(\mathbb{R})$  and  $\chi$  smooth cutoff on  $K$ , then*

$$\|\langle \lambda^{\frac{1}{2}} r \rangle^{-\alpha} (\chi f(P/\lambda) + (1 - \chi) f(P/\lambda) \chi)\| \lesssim \lambda^{\frac{\alpha}{4}}$$

for all  $\lambda > 0$ .

*Proof.* By the spectral theorem and Lemma 3.2 with  $p = 1$  we have

$$\|f(P/\lambda) \chi\| \lesssim \|(P/\lambda + 1)^{-N} \chi\| \lesssim \|(P/\lambda + 1)^{-N}\|_{L^1(M) \rightarrow L^2(M)} \lesssim \lambda^{\frac{\alpha}{4}}.$$

Same holds for  $\|\chi f(P/\lambda)\|$  and we conclude simply bounding  $\langle \lambda^{\frac{1}{2}} r \rangle^{-\alpha}$  by 1.  $\square$

Then, in the term  $(1 - \chi) f(-\Delta_0/\lambda) (1 - \chi)$  we can take advantage of the fact that on the exact cone  $([0, +\infty) \times S, \bar{G})$  we have scaling invariance.

**Lemma 4.2** (Rescaling on the fixed cone). *Let  $\|\cdot\|_{L_{\bar{G}}^2(\text{cone})}$  the norm with respect to the metric  $\bar{G}$  on the full cone  $[0, +\infty) \times S$ . Then for all  $\lambda > 0$*

$$\|\langle \lambda^{\frac{1}{2}} r \rangle^{-\alpha} f(-\Delta_0/\lambda)\|_{L_{\bar{G}}^2(\text{cone}) \rightarrow L_{\bar{G}}^2(\text{cone})} = \|\langle r \rangle^{-\alpha} f(-\Delta_0)\|_{L_{\bar{G}}^2(\text{cone}) \rightarrow L_{\bar{G}}^2(\text{cone})}.$$

*Proof.* By the results in Appendix A we can reduce ourselves to the half line  $(0, +\infty)$  and prove equivalently that

$$\|\langle \lambda^{\frac{1}{2}} r \rangle^{-\sigma} f(p_k/\lambda)\|_{\mathcal{L}(L^2(\mathbb{R}^+, r^{n-1} dr))} = \|\langle r \rangle^{-\sigma} f(p_k)\|_{\mathcal{L}(L^2(\mathbb{R}^+, r^{n-1} dr))},$$

where  $p_k$ , defined in (A.9), is given by

$$p_k = -\frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r} \partial_r + \frac{1}{r^2} \mu_k^2$$

with  $\lambda_k^2$  the  $k$ -th eigenvalue of  $\Delta_{\bar{G}}$ .

We can prove the equality by showing

$$\langle \lambda^{\frac{1}{2}} r \rangle^{-\sigma} f(p_k/\lambda) = T_\lambda \langle r \rangle^{-\sigma} f(p_k) T_\lambda^* \quad (4.1)$$

with  $T_\lambda$  unitary operator.  $\square$

Here and later we will need to compare the norm on the exact cone with the  $L^2$  norm of  $\mathbb{R}^n$  with respect to the Lebesgue measure, we give further details in the following remark.

*Remark 4.2* (The flat norm on  $\mathbb{R}^n$  and the one on the cone are comparable). The idea is to partition the angular part  $S$  into open sets that are diffeomorphic to open sets of  $\mathbb{S}^{n-1}$  and take advantage of the fact that norms on  $(0, +\infty) \times \mathbb{S}^{n-1}$ , with respect to the usual metric the metric  $dr^2 + r^2 d\sigma$ , are equivalent to norms on  $\mathbb{R}^n$ .

Let  $(\varphi_\kappa)_\kappa$  the partition of unity on  $S$  and  $\kappa_j : U_j \rightarrow V_j \subset \mathbb{R}^{n-1}$  the coordinate charts. In the same way let  $U$  an open set of  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$  and  $\varphi : U \rightarrow V \subset \mathbb{R}^{n-1}$  its coordinate chart.

Without loss of generality we can assume  $V_j \subset V$  so that it is well defined the diffeomorphism

$$\kappa = \varphi^{-1} \circ \kappa_j : S \supset U_j \rightarrow U \subset \mathbb{S}^{n-1}$$

through which  $u \in C_0^\infty(S)$  supported in  $U_j$  can be identified with  $u \circ \kappa^{-1} \in C_0^\infty(\mathbb{S}^{n-1})$ .

The metric tensors on  $S$  and  $\mathbb{S}^{n-1}$  are represented respectively by positive definite matrices such that

$$\bar{c}^{-1} I \leq (\bar{g}^{j,k}(\theta))_{j,k} \leq \bar{c} I, \quad c^{-1} I \leq (\sigma^{j,k}(x))_{j,k} \leq c I,$$



this means that up to multiplication by some bounded function we can pass from one metric to the other so we will say that integrals with respect to  $d\bar{g}$  or  $d\sigma$  are comparable and write

$$\int_{U_j \subset S} |\varphi_j u| d\bar{g} \simeq \int_{U \subset \mathbb{S}^{n-1}} |(\varphi_j u) \circ \kappa^{-1}| d\sigma$$

for  $u$  smooth on  $(0, +\infty) \times S$ .

We remark here that  $\kappa^{-1} = \kappa_j^{-1} \circ \varphi$  is only defined on  $\varphi^{-1}(V_j) \subset U$  since we need to require that  $\varphi$  maps elements into  $V_j \subset V$  which is where  $\kappa_j^{-1}$  is defined. However when we consider  $(\varphi_j \circ \kappa^{-1})(u \circ \kappa^{-1})$ , we can extend it to  $U$  by putting it at 0 outside of  $\varphi^{-1}(V_j)$ , since in this case  $(\varphi_j \circ \kappa^{-1})$  cuts off near the boundary of  $\varphi^{-1}(V_j)$ .

Considering the norm on the cone we have found that

$$\int_{(0, +\infty) \times U_j} \varphi_j u d\bar{G} \simeq \int_{(0, +\infty) \times U} (\varphi_j u \circ \kappa^{-1}) r^{n-1} dr d\sigma \simeq \int_{(0, +\infty) \times U \subset \mathbb{R}^n} (\varphi_j u) \circ \kappa^{-1} dx$$

where we still denote by  $\kappa$  the diffeomorphism  $(r, \omega) \mapsto (r, \kappa(\omega))$  through which we can identify a function on the cone (suitably supported) with a function on  $(0, +\infty) \times \mathbb{S}^{n-1}$ . In particular we see that the  $L^2$  norm with respect to the metric  $\bar{G}$  on the cone is equivalent to the one on  $\mathbb{R}^n$  where we pick the Lebesgue measure.

Having got rid of the dependence on  $\lambda$  thanks to Lemma 4.2, we can prove convergence in norm as the support of  $f$  shrinks to 1 and therefore write

**Lemma 4.3.** *For any  $\varepsilon > 0$  there exists  $f \in C_0^\infty(\mathbb{R})$  with  $f(1) = 1$  and small enough support around 1 such that*

$$\|\langle r \rangle^{-\alpha} f(-\Delta_0)\|_{L^2_{\bar{G}}(\text{cone}) \rightarrow L^2_{\bar{G}}(\text{cone})} \leq \varepsilon$$

for all  $\lambda > 0$ .

*Proof.* Let  $\tilde{f} \in C_0^\infty(\mathbb{R})$  a fixed function such that  $\tilde{f}$  is 1 near the support of  $f$  so that we can write

$$\langle r \rangle^{-\sigma} f(-\Delta_0) = \langle r \rangle^{-\sigma} \tilde{f}(-\Delta_0) f(-\Delta_0),$$

here if the support of  $f$  shrinks to  $\{1\}$  then  $f(-\Delta_0)$  converges strongly to 0, given that 1 is not an eigenvalue. Moreover  $\langle r \rangle^{-\sigma} \tilde{f}(-\Delta_0)$  is a fixed compact operator and therefore the composition converges to 0 in norm.

To prove compactness first let  $g \in C_0^1(\mathbb{R})$  supported around 0 and  $(\rho_\kappa)_\kappa$  a partition of unity of  $[0, +\infty) \times S$  with  $\text{supp} \rho_\kappa \subset [0, +\infty) \times U_\kappa$ . If  $(u_n)_n$  is a sequence of uniformly bounded functions in  $L^2_{\bar{G}}(\text{cone})$  we have

$$g(r) \tilde{f}(-\Delta_0) u_n = \sum_{\kappa} g(r) \rho_\kappa \tilde{f}(-\Delta_0) u_n \in H_0^1([0, +\infty) \times S)$$

where each term is supported in an open bounded set  $[0, \bar{R}] \times U_\kappa$  thanks to the fact that  $g$  has compact support and to the term of the partition of unity. Here, by  $H_0^1([0, +\infty) \times S)$  we mean the space defined by the performing the closure of  $C_0^\infty$  functions as in (A.3). Moreover the Sobolev regularity of  $g(r) \rho_\kappa \tilde{f}(-\Delta_0) u_n$  is given by the fact that  $\tilde{f}(-\Delta_0)$  has image in the domain of  $-\Delta_0$  which is contained in the Sobolev space (see (A.4)). By the spectral theorem and the uniform bound on  $(u_n)_n$  it follows

$$\|g(r) \rho_\kappa \tilde{f}(-\Delta_0) u_n\|_{L^2([0, \bar{R}] \times U_\kappa)} \lesssim \|g(r) \rho_\kappa \tilde{f}\|_\infty \|u_n\|_{L^2([0, \bar{R}] \times U_\kappa)} \lesssim 1.$$

Moreover by definition  $\nabla_{\bar{G}} = (\partial_r, 1/r \nabla_{\bar{g}})$  and from the fact that  $\rho_\kappa$  is a cutoff just with respect to the angular variable (and therefore of the form  $\rho_\kappa(r, \omega) = \rho_1(r) \rho_2(\omega)$  with  $\rho_1 \equiv 1$  on  $[0, +\infty)$ ) we obtain

$$|\nabla_{\bar{G}} g(r) \rho_\kappa \tilde{f}(-\Delta_0) u_n|^2 \lesssim |\nabla_{\bar{G}} (g \rho_\kappa) \tilde{f}(-\Delta_0) u_n|^2 + |g \rho_\kappa \nabla_{\bar{G}} \tilde{f}(-\Delta_0) u_n|^2$$

$$\begin{aligned} &\lesssim \|g' \rho_\kappa\|_\infty^2 |\tilde{f}(-\Delta_0)u_n|^2 + \|g \nabla_{\bar{g}} \rho_\kappa\| \|1/r \tilde{f}(-\Delta_0)u_n\|^2 \\ &\quad + \|g\|_\infty^2 |\nabla_{\bar{G}} \tilde{f}(-\Delta_0)u_n|^2. \end{aligned}$$

Now we can bound  $\|1/r \tilde{f}(-\Delta_0)u_n\|_{L^2([0, \bar{R}] \times U_\kappa)}$  by the  $L^2$  norm of the gradient via the Hardy inequality which paired with the equality  $\|\nabla_{\bar{G}} u\|_{L^2_{\bar{G}}} = \|(-\Delta_0)^{\frac{1}{2}} u\|_{L^2_{\bar{G}}}$  gives

$$\begin{aligned} \|\nabla_{\bar{G}} g(r) \tilde{f}(-\Delta_0)u_n \rho_\kappa\|_{L^2((0, \bar{R}) \times U_\kappa)}^2 &\lesssim \|\tilde{f}(-\Delta_0)u_n \rho_\kappa\|_{L^2((0, \bar{R}) \times U_\kappa)}^2 \\ &\quad + \|(-\Delta_0)^{\frac{1}{2}} \tilde{f}(-\Delta_0)u_n \rho_\kappa\|_{L^2((0, \bar{R}) \times U_\kappa)}^2 \\ &\lesssim \|u_n \rho_\kappa\|_{L^2((0, \bar{R}) \times U_\kappa)}^2 \\ &\lesssim 1 \end{aligned} \tag{4.2}$$

where to get to (4.2) we use again the spectral theorem.

So for each fixed  $\kappa$  the sequence  $(g(r) \tilde{f}(-\Delta_0)u_n \rho_\kappa)_n$  is uniformly bounded in  $H_0^1((0, \bar{R}) \times U_\kappa)$  and it will then be diffeomorphic to a uniformly bounded sequence of  $H_0^1(\Omega)$  with  $\Omega$  an open bounded subset of  $\mathbb{R}^n$  and where we take usual Lebesgue measure as discussed in the previous Remark 4.2. By compact Sobolev embedding we can then extract a subsequence converging on  $L^2(\Omega)$  and composing with the right diffeomorphism we can recover a subsequence of  $g(r) \tilde{f}(-\Delta_0)u_n \rho_\kappa$  that converges in  $L^2((0, \bar{R}) \times U_\kappa)$ . We have therefore proved compactness of  $g(r) \tilde{f}(-\Delta_0)$ .

Since the set of compact operators is closed with respect to norm convergence, considering  $\phi \in C_0^\infty(\mathbb{R})$  equal to 1 on  $B(0, 1)$  we write

$$\langle r \rangle^{-\sigma} \tilde{f}(-\Delta_0) = \langle r \rangle^{-\sigma} \phi\left(\frac{r}{R}\right) \tilde{f}(-\Delta_0) + \langle r \rangle^{-\sigma} (1 - \phi)\left(\frac{r}{R}\right) \tilde{f}(-\Delta_0)$$

for some large  $R$ , with the first term that is compact and the second one norm converging to 0 as  $R$  tends to  $\infty$ .  $\square$

Since on the support of  $(1 - \chi)$  the manifold is diffeomorphic to  $(R, +\infty) \times S$ , and therefore the norm of  $L^2(M)$  is comparable with the one of  $L^2_{\bar{G}}$ , the previous proposition implies the same result when we look at the norm of operators on  $L^2(M)$ .

**Proposition 4.4.** *For any  $\varepsilon > 0$  there exists  $f \in C_0^\infty(\mathbb{R})$  with  $f(1) = 1$  and small enough support such that*

$$\|(1 - \chi) \langle \lambda^{\frac{1}{2}} r \rangle^{-\alpha} f(-\Delta_0/\lambda) (1 - \chi)\| \leq \varepsilon$$

for any  $\lambda > 0$ .

The rest of the section will be dedicated to the analysis of the term localised on the end of the manifold, that is

$$D_f(\lambda) := (1 - \chi) f(P/\lambda) (1 - \chi) - (1 - \chi) f(-\Delta_0/\lambda) (1 - \chi), \tag{4.3}$$

in particular we will prove that  $D_f(\lambda)$  converges to 0 for any  $f \in C_0^\infty(\mathbb{R})$  as  $\lambda$  goes to 0. (We recall the term is well defined, see Remark 4.1.)

Using Helffer-Sjöstrand formula to compute functional calculus we will see we can reduce ourselves to comparing the resolvents, in particular setting

$$R_z(\lambda) := (1 - \chi) \left(\frac{P}{\lambda} - z\right)^{-1} (1 - \chi) - (1 - \chi) \left(\frac{-\Delta_0}{\lambda} - z\right)^{-1} (1 - \chi)$$

we need to consider

$$D_f(\lambda) = \frac{1}{\pi i} \int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z) R_z(\lambda) L(dz).$$

As we have seen in Proposition 2.4, on the support of  $(1 - \chi)$  we can compare  $P$  with  $-\Delta_0$  and more precisely, by expanding the expressions of  $\Delta_{g(r)}$  and  $\Delta_{\bar{g}}$  we can decompose the operator in a part on the fixed cone and a differential operator with decaying coefficients

$$(1 - \chi)P/\lambda = (1 - \chi)(-\Delta_0 - V)/\lambda,$$

where  $V$  is of the form

$$V := \sum_{\kappa} \varphi_{\kappa} \Pi_{\kappa} \left( a_{\kappa}(r, \theta) \partial_r + \sum_l b_{\kappa}^l(r, \theta) \frac{1}{r} \partial_l + \sum_{j,l} c_{\kappa}^{j,l}(r, \theta) \frac{1}{r^2} \partial_{j,l}^2 \right) \Pi_{\kappa}^{-1}$$

with

$$a_{\kappa} \in S^{-1-\nu}, \quad b_{\kappa}^l \in S^{-1-\nu}, \quad c_{\kappa}^{j,l} \in S^{-\nu}. \quad (4.4)$$

With this in mind, we will split  $D_f(\lambda)$  into several terms, since by algebraic manipulations we obtain

$$\begin{aligned} R_z(\lambda) &= -\frac{1}{\lambda} (P/\lambda - z)^{-1} [P, \chi] (P/\lambda - z)^{-1} (1 - \chi) \\ &\quad + \frac{1}{\lambda} (P/\lambda - z)^{-1} [P, \chi] (-\Delta_0/\lambda - z)^{-1} (1 - \chi) \\ &\quad + \frac{1}{\lambda} (P/\lambda - z)^{-1} (1 - \chi) V (-\Delta_0/\lambda - z)^{-1} (1 - \chi) \\ &=: R_z^1(\lambda) + R_z^2(\lambda) + R_z^V(\lambda). \end{aligned} \quad (4.5)$$

For  $R_z^1(\lambda)$  and  $R_z^2(\lambda)$  we can take advantage of the fact that the commutator is a differential operator of order one in the spatial variable with compactly supported coefficients and then use Lemma 3.3 to gain powers of  $\lambda$ .

**Lemma 4.5.** *Let  $f \in C_0^{\infty}(\mathbb{R})$ , there exists  $\delta > 0$  such that*

$$\left\| \int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z) R_z^i(\lambda) L(dz) \right\| \lesssim \lambda^{\delta}$$

for  $i = 1, 2$  and for all  $\lambda > 0$ .

*Remark 4.3* (Notation). To make the notation lighter, in the sequel we will omit the pullback and pushforward in the expression in local coordinates of  $V$  and  $[P, \chi]$ , meaning for example that we will still denote by  $a \partial_r$  the operator **on the manifold** that corresponds to the derivative with respect to the radial variable and to the multiplication by  $a$  on some open set of  $\mathbb{R}^n$ .

*Proof.* The commutator  $[P, \chi]$  is supported away from the compact part of the manifold, here we recall the expression of the operator in local coordinates is (1.7). The angular derivatives commute with  $\chi$ , so first of all we have

$$[P, \chi] = \sum_{\kappa} \varphi_{\kappa} (f_{1,\kappa}(r, \theta) + f_{2,\kappa}(r, \theta) \partial_r)$$

with  $f_{1,\kappa}, f_{2,\kappa} \in C_0^{\infty}(\mathbb{R}^n)$ . Then using this relation in the expression for  $R_z^1(\lambda)$  we apply Lemma 3.3 to each term. First, let  $\sigma \in \mathbb{R}$  and consider

$$\left\| \frac{1}{\lambda} (P/\lambda - z)^{-1} \langle r \rangle^{-\sigma} \varphi_{\kappa} f_{1,\kappa} \langle r \rangle^{2\sigma} \langle r \rangle^{-\sigma} (P/\lambda - z)^{-1} (1 - \chi) \right\|.$$

By Lemma 3.3 we can have estimates of the type

$$\|(P/\lambda - z)^{-1} \langle r \rangle^{-\sigma}\| \lesssim \frac{\langle z \rangle}{|\operatorname{Im} z|} \lambda^{\sigma} \quad (4.6)$$

for all  $s \in [0, \frac{\sigma}{4}] \cap [0, 1)$  such that  $s < \frac{\sigma}{2}$  and we also remark that

$$\|\varphi_\kappa f_{1,\kappa} \langle r \rangle^{2\sigma}\| \lesssim \|f_{1,\kappa} \langle r \rangle^{2\sigma}\|_{L^\infty} \lesssim 1$$

for any  $\sigma$ , given the compact support of  $f_{1,\kappa}$ . We can therefore freely choose the exponent  $\sigma$  and picking  $\sigma > 1$  allows us to find  $s \in (\frac{1}{2}, \frac{\sigma}{2})$  such that (4.6) holds. We observe that the  $\langle z \rangle / |Imz|$  factor in (4.6) is provided by

$$\|(P/\lambda - z)^{-1}(P/\lambda + 1)\| \lesssim \frac{\langle z \rangle}{|Imz|}.$$

Collecting all these informations together yields

$$\left\| \frac{1}{\lambda} (P/\lambda - z)^{-1} \varphi_\kappa f_{1,\kappa} (P/\lambda - z)^{-1} (1 - \chi) \right\| \lesssim \lambda^{2s-1} \frac{\langle z \rangle^2}{|Imz|^2}$$

with  $2s - 1 > 0$ . We decompose similarly the  $f_2$  term where this time we apply Lemma 3.3 only to the resolvent on the left. We choose again  $\sigma > 1$  so to obtain (4.6) for any  $s \in (\frac{1}{2}, \frac{\sigma}{2})$  and we still have

$$\|\varphi_\kappa f_{2,\kappa} \langle r \rangle^\sigma\| \lesssim \|f_{2,\kappa} \langle r \rangle^\sigma\|_{L^\infty} \lesssim 1.$$

Picking  $\tilde{f}_{2,\kappa}$  such that  $f_{2,\kappa} \tilde{f}_{2,\kappa} \equiv 1$  we can observe that

$$\begin{aligned} \left\| \tilde{f}_{2,\kappa} \partial_r (P/\lambda - z)^{-1} \right\| &\lesssim \left\| \tilde{f}_{2,\kappa} \partial_r P^{-\frac{1}{2}} \right\| \|P^{\frac{1}{2}} (P/\lambda - z)^{-1}\| \\ &\lesssim \lambda^{\frac{1}{2}} \|\nabla_G P^{-\frac{1}{2}}\|_{L_G^2 \rightarrow L_G^2} \\ &\lesssim \lambda^{\frac{1}{2}}. \end{aligned}$$

We conclude the proof, since

$$\left\| \frac{1}{\lambda} (P/\lambda - z)^{-1} \varphi_\kappa f_{2,\kappa} \partial_r (P/\lambda - z)^{-1} (1 - \chi) \right\| \lesssim \lambda^{s+\frac{1}{2}-1} \frac{\langle z \rangle}{|Imz|}$$

with  $s + \frac{1}{2} - 1 > 0$  and  $\bar{\partial}_z \tilde{f}(z) = O(|Imz|^M)$  for any  $M \geq 0$ .

The proof for  $R_z^2(\lambda)$  carries out in the same way.  $\square$

As opposed to  $R_z^1(\lambda)$  and  $R_z^2(\lambda)$ , in the case of  $R_z^V(\lambda)$  we have  $V$  which is a differential operator whose coefficients have only finite order decay, in particular the fact that  $\|\langle r \rangle^\alpha a_\kappa\|_{L^\infty}$  and  $\|\langle r \rangle^\alpha b_\kappa^l\|_{L^\infty}$  are finite only for  $\alpha \leq 1 + \nu$  will limit our choice of exponents when applying Lemma 3.3.

However this will not be a source of difficulty in the first order terms of  $V$  since we can still choose  $\sigma = \nu + 1 > 1$  and get additional powers of  $\lambda$  by bounding the operators  $\partial_r(-\Delta_0)^{-1}$  and  $\frac{1}{r} \partial_l(-\Delta_0)^{-1}$ .

On the contrary, for the second order term  $\|\langle r \rangle^\alpha c_\kappa^{j,l}\| \lesssim 1$  for  $\alpha \leq \nu$  will limit us to  $\sigma = \nu > 0$  and moreover we will only be able to obtain the boundedness of  $\partial_{j,l}^2(-\Delta_{\bar{g}})^{-1}$  (by projecting away from the 0 eigenspace and using an elliptic parametrix in Lemma 4.9). This represents an additional difficulty since we will then need to control operators like  $(-\Delta_{\bar{g}})/r^2(-\Delta_0/\lambda - z)^{-1}$ .

**Lemma 4.6** (Bound on first order terms I). *Let  $I_1$  defined by*

$$I_1 = \frac{1}{\lambda} (P/\lambda - z)^{-1} (1 - \chi) \varphi_\kappa (a_\kappa \partial_r) \left( \frac{-\Delta_0}{\lambda} - z \right)^{-1} (1 - \chi),$$

*then there exists  $\delta_1 > 0$  such that*

$$\|I_1\| \lesssim \lambda^{\delta_1} \frac{\langle z \rangle}{|Imz|}.$$

*Proof.* We start by mimicking the proof of Lemma 4.5, hence writing  $I_1$  as

$$\left\| \frac{1}{\lambda} (P/\lambda - z)^{-1} \langle r \rangle^{-\sigma} (1 - \chi) \varphi_\kappa \langle r \rangle^\sigma a_\kappa \partial_r (-\Delta_0/\lambda - z)^{-1} (1 - \chi) \right\|. \quad (4.7)$$

We recall that  $a_\kappa \in S^{-1-\nu}$  so the fact that  $\|\langle r \rangle^{\nu+1} a_\kappa\|_{L^\infty} \lesssim 1$  suggests that this time we choose  $\sigma = \nu + 1$  in Lemma 3.3, therefore giving us

$$\|(P/\lambda - z)^{-1} \langle r \rangle^{-\nu-1}\| \lesssim \frac{\langle z \rangle}{|Imz|} \lambda^s \quad (4.8)$$

for any  $s \in (\frac{1}{2}, \frac{\nu+1}{2})$ . We then proceed similarly to the previous proof, that is we estimate the quantity

$$\begin{aligned} \left\| \partial_r \left( \frac{-\Delta_0}{\lambda} - z \right)^{-1} (1 - \chi) \right\| &\lesssim \left\| \partial_r (-\Delta_0)^{-\frac{1}{2}} \right\|_{L^2_{\bar{G}} \rightarrow L^2_{\bar{G}}} \left\| (-\Delta_0)^{\frac{1}{2}} \left( \frac{-\Delta_0}{\lambda} - z \right)^{-1} \right\|_{L^2_{\bar{G}} \rightarrow L^2_{\bar{G}}} \\ &\lesssim \lambda^{\frac{1}{2}} \|\nabla_{\bar{G}} (-\Delta_0)^{-\frac{1}{2}}\|_{L^2_{\bar{G}} \rightarrow L^2_{\bar{G}}} \left\| \left( \frac{-\Delta_0}{\lambda} \right)^{\frac{1}{2}} \left( \frac{-\Delta_0}{\lambda} - z \right)^{-1} \right\|_{L^2_{\bar{G}} \rightarrow L^2_{\bar{G}}} \\ &\lesssim \lambda^{\frac{1}{2}}. \end{aligned}$$

The statement is proved since from the informations obtained up to now the bound on (4.7) is

$$\|I_1\| \lesssim \lambda^{s+\frac{1}{2}-1} \frac{\langle z \rangle}{|Imz|}$$

with  $s + \frac{1}{2} - 1 > 0$ . □

Once we have established

**Lemma 4.7.** *Let  $\varphi_\kappa$  a term of the partition of unity of  $S$ , then  $\varphi_\kappa \frac{1}{r} \partial_t (-\Delta_0)^{-\frac{1}{2}}$  is a bounded operator on  $L^2_{\bar{G}}$ .*

we can bound, with the exact same reasoning as in Lemma 4.6, the remaining first order part of  $R_z^V(\lambda)$ .

**Lemma 4.8** (Bound on first order terms II). *Let  $I_2$  defined by*

$$I_2 = \frac{1}{\lambda} (P/\lambda - z)^{-1} (1 - \chi) \varphi_\kappa b_\kappa^l \frac{1}{r} \partial_t (-\Delta_0/\lambda - z)^{-1} (1 - \chi),$$

*then there exists  $\delta_2 > 0$  such that*

$$\|I_2\| \lesssim \lambda^{\delta_2} \frac{\langle z \rangle}{|Imz|}.$$

*Proof of Lemma 4.7.* By ellipticity of the operator  $\Delta_{\bar{g}}$ , locally on coordinate patches we have the following lower bound

$$|\nabla_{\bar{g}} u|_{\bar{g}}^2 = \sum_{l,j} \bar{g}^{l,j}(\theta) \partial_l u \partial_j u \geq C_0 \sum_j |\partial_j u|^2 \geq C_0 |\partial_t u|^2$$

for some  $C_0 > 0$ , consequently for the operator on the manifold it holds

$$\int_S |\varphi_\kappa \partial_t u|^2 d\text{vol}_S = \int_{V_\kappa} |\partial_t u|^2 |\bar{g}(\theta)| d\theta \leq \frac{1}{C_0} \int_{V_\kappa} |\nabla_{\bar{g}} u|_{\bar{g}}^2 |\bar{g}(\theta)| d\theta \leq \frac{1}{C_0} \int_S |\varphi_\kappa \nabla_{\bar{g}} u|_{\bar{g}}^2 d\text{vol}_S$$

(we recall Remark 4.3 about the notation used). We can conclude, since we have found

$$\|\varphi_\kappa \frac{1}{r} \partial_t u\|_{L^2_{\bar{G}}} \lesssim \|\varphi_\kappa \frac{1}{r} \nabla_{\bar{g}} u\|_{L^2_{\bar{G}}} \lesssim \|\nabla_{\bar{G}} u\|_{L^2_{\bar{G}}} = \|(-\Delta_0)^{\frac{1}{2}} u\|_{L^2_{\bar{G}}}.$$

□

Now passing to consider the second order part in  $R_\lambda^V(z)$ , that is

$$I_3 := \frac{1}{\lambda} (P/\lambda - z)^{-1} (1 - \chi) \varphi_\kappa c_\kappa^{j,l} \frac{1}{r^2} \partial_{j,l}^2 (-\Delta_0/\lambda - z)^{-1} (1 - \chi)$$

we remark first the useful property

**Lemma 4.9.** *Let  $-\Delta_{\bar{g}}$  the Laplace Beltrami operator on  $(S, \bar{g})$ , if  $\varphi_\kappa$  is a term of the partition of unity of  $S$  then  $\varphi_\kappa \partial_{j,l}^2 (-\Delta_{\bar{g}})^{-1}$  is a bounded operator on  $L^2(S, d\bar{g})$ .*

*Proof.* Let  $u \in D(-\Delta_{\bar{g}})$  and  $\Pi_0$  the projection on  $\ker(-\Delta_{\bar{g}})$ , which is spanned by 1 and consequently  $\partial_{j,l}^2 \Pi_0 u = 0$ . Moreover  $\Pi_0$  can be written as  $f(-\Delta_{\bar{g}})$  for some  $f \in C_0^\infty(\mathbb{R})$  supported around 0 and with  $f(0) = 1$ , we consider

$$\begin{aligned} \varphi_\kappa \partial_{j,l}^2 u &= \varphi_\kappa \partial_{j,l}^2 (u - f(-\Delta_{\bar{g}})u) \\ &= \varphi_\kappa \partial_{j,l}^2 (-\Delta_{\bar{g}} + 1)^{-1} \frac{(-\Delta_{\bar{g}} + 1)}{(-\Delta_{\bar{g}})} (-\Delta_{\bar{g}})(1 - f)(-\Delta_{\bar{g}})u \\ &= \varphi_\kappa \partial_{j,l}^2 (-\Delta_{\bar{g}} + 1)^{-1} \frac{(-\Delta_{\bar{g}} + 1)(1 - f)(-\Delta_{\bar{g}})}{(-\Delta_{\bar{g}})} (-\Delta_{\bar{g}})u. \end{aligned} \quad (4.9)$$

With standard computations we can find a parameterix for the elliptic operator  $-\Delta_{\bar{g}}$ , namely there exist a family of symbols  $q^\kappa \in S^{-2}(\mathbb{R}_\theta^{n-1})$  and  $\mathcal{R}_N$  operator with symbol in  $S^{-N}(\mathbb{R}_\theta^{n-1})$  such that

$$(-\Delta_{\bar{g}} + 1) \left( \sum_\kappa \Pi_\kappa O p(q^\kappa) \Pi_\kappa^{-1} \right) = I + \mathcal{R}_N.$$

On the support of  $\varphi_\kappa$  the resolvent  $(-\Delta_{\bar{g}} + 1)^{-1}$  is a pseudodifferential operator of order minus two and the composition with the order two differential operator  $(\partial_{j,l}^2)$  results in a bounded operator of  $L^2(S)$ . By the spectral theorem  $\frac{(-\Delta_{\bar{g}} + 1)(1 - f)(-\Delta_{\bar{g}})}{(-\Delta_{\bar{g}})}$  is bounded and therefore the statement follows from (4.9) which yields

$$\|\varphi_\kappa \partial_{j,l}^2 u\|_{L^2(S)} \lesssim \|(-\Delta_{\bar{g}})u\|_{L^2(S)}.$$

□

As usual, we want to apply Lemma 3.3 to  $I_3$  and we will do so by taking advantage of the fact that  $\|\langle r \rangle^\nu c_\kappa^{j,l}\| \lesssim 1$  thanks to (4.4). However, applying Lemma 3.3 with  $\sigma = \nu > 0$  would provide a bound by  $\lambda^s$  with  $s \in (0, \frac{\nu}{2})$  which is worse than what we gained in the estimation of  $I_1$  and  $I_2$ , where taking  $\sigma = \nu + 1$  produced a higher power of  $\lambda$ , namely with exponent  $s > \frac{1}{2}$ .

We will then proceed differently by considering separately low and high angular frequencies. Let  $\phi$  a smooth cutoff function such that  $\phi \equiv 1$  on  $[0, n - 1]$ .

1. Consider

$$\frac{1}{\lambda} (P/\lambda - z)^{-1} c_\kappa^{j,l} \varphi_\kappa \partial_{j,l}^2 \phi(-\Delta_{\bar{g}})(1 - \chi) \frac{1}{r^2} \left( \frac{-\Delta_0}{\lambda} - z \right)^{-1}$$

where we have bounded contributions given by

$$\langle r \rangle^\nu c_\kappa^{j,l}, \quad (1 - \chi)(r) \frac{\langle r \rangle^2}{r^2}, \quad \varphi_\kappa \partial_{j,l}^2 (-\Delta_{\bar{g}})^{-1} \phi(-\Delta_{\bar{g}})(-\Delta_{\bar{g}})$$

(see (4.4), Lemma 4.9 and the support of  $\phi$ ) and we are left to consider

$$(P/\lambda - z)^{-1} \langle r \rangle^{-\nu-1} \quad \text{and} \quad \frac{\langle r \rangle^{-1}}{\lambda} (-\Delta_0/\lambda - z)^{-1}.$$

By Lemma 3.3 and Hardy inequality (see Propositions 2.2 in [BR14b]) we can handle these two remaining terms obtaining

$$\left\| (P/\lambda - z)^{-1} \langle r \rangle^{-\nu-1} \frac{1}{\lambda \langle r \rangle} (-\Delta_0/\lambda - z)^{-1} \right\| \lesssim \lambda^{s-\frac{1}{2}} \frac{\langle z \rangle^2}{|Imz|^2}$$

where we can choose  $s \in (\frac{1}{2}, \frac{\nu+1}{2})$ .

We notice here that we have used Lemma 3.3 in the same way as in the proof of Lemma 4.6, where we had obtained (4.8). However here  $c_\kappa^{j,l}$  has less decay than  $a_\kappa$  which leaves us with an extra growing term  $\langle r \rangle$  to handle (we cannot bound  $\langle r \rangle^{\nu+1} c_\kappa^{j,l}$ , but only  $\langle r \rangle^\nu c_\kappa^{j,l}$ ). For this reason we take advantage of the localisation  $(1 - \chi)$  that we use to write a bounded term where we collect all the factors depending on the radial variable.

2. The part localised at high angular frequencies is

$$\frac{1}{\lambda} (P/\lambda - z)^{-1} c_\kappa^{j,l} \varphi_\kappa \partial_{j,l}^2 (1 - \phi)(-\Delta_{\bar{g}})(1 - \chi) \frac{1}{r^2} (-\Delta_0/\lambda - z)^{-1}$$

where the operators

$$\langle r \rangle^\nu c_\kappa^{j,l}, \quad \varphi_\kappa \partial_{j,l}^2 (-\Delta_{\bar{g}})^{-1}$$

are bounded independently of  $\lambda$  and  $z$  thanks again to (4.4) and Lemma 4.9. Additionally, by Lemma 3.3

$$\|(P/\lambda - z)^{-1} \langle r \rangle^{-\nu}\| \lesssim \lambda^s \frac{\langle z \rangle}{|Imz|} \quad (4.10)$$

for some  $s \in (0, \frac{\nu}{2})$ . At this point we are left with

$$\left\| (1 - \phi)(-\Delta_{\bar{g}}) \frac{(-\Delta_{\bar{g}})}{\lambda r^2} \left( \frac{-\Delta_0}{\lambda} - z \right)^{-1} \right\| \lesssim \frac{\langle z \rangle}{|Imz|} \left\| (1 - \phi)(-\Delta_{\bar{g}}) \frac{(-\Delta_{\bar{g}})}{\lambda r^2} \left( \frac{-\Delta_0}{\lambda} + 1 \right)^{-1} \right\|$$

where, as opposed to item 1, the localisation by  $(1 - \phi)$  requires some extra care.

**Lemma 4.10.** *Let  $n \geq 3$  and  $\phi \in C_0^\infty(\mathbb{R})$  such that  $\phi \equiv 1$  on  $[0, n - 1]$ , then for  $\lambda \in (0, \lambda_0]$*

$$\left\| (1 - \phi)(-\Delta_{\bar{g}}) \frac{(-\Delta_{\bar{g}})}{\lambda r^2} \left( \frac{-\Delta_0}{\lambda} + 1 \right)^{-1} \right\|_{L_G^2 \rightarrow L_G^2} \lesssim_n 1.$$

*Proof.* By the results of Appendix A.3 (Proposition A.2 in particular) we can rather consider the one dimensional problem of bounding

$$\sup_{\mu_k^2 > n-1} \left\| \frac{\mu_k^2}{r^2} (p_k + \lambda)^{-1} \right\|_{L^2((R, +\infty), r^{n-1} dr) \rightarrow L^2((R, +\infty), r^{n-1} dr)} \quad (4.11)$$

where we recall that  $p_k$  is given by

$$p_k := -\frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r} \partial_r + \frac{1}{r^2} \mu_k^2$$

and  $(\mu_k^2, e_k)$  are eigenpairs of  $-\Delta_{\bar{g}}$ . To bound (4.11) we will use an analogous estimate where the  $\mu_k^2$  are replaced by the eigenvalues of the Laplace Beltrami operator on the unit sphere (see Appendix D). Indeed, we can separate the  $\mu_k^2$  by the eigenvalues of  $-\Delta_{S^{n-1}}$ , for whom we have the explicit expression  $\sigma_{j,n}^2 = j(j+n-2)$ , namely once we fix  $k$

$$\mu_k^2 \in (\sigma_{l,n}^2, \sigma_{l+1,n}^2] \quad \text{for a unique } l = l(k),$$

recalling that we restricted ourselves to the eigenvalues  $\mu_k^2 > n - 1$ . We can rewrite the operator  $p_k$  as

$$p_k = \underbrace{-\partial_r^2 - \frac{(n-1)}{r} \partial_r + \frac{\sigma_{l,n}^2}{r^2}}_{p_{l,n}} + \underbrace{\frac{\mu_k^2 - \sigma_{l,n}^2}{r^2}}_v$$

with  $v > 0$  and  $p_{l,n}$  which is the equivalent of  $p_k$  where the  $\lambda_k^2$  are replaced by the  $\sigma_{l,n}^2$ . If we express the resolvent in terms of the heat semigroup

$$(p_k + \lambda)^{-1} = \int_0^\infty e^{-t\lambda} e^{-tp_k} dt = \int_0^\infty e^{-t\lambda} e^{-t(p_{l,n}+v)} dt \quad (4.12)$$

since  $v$  and  $p_{l,n}$  do not commute of course  $e^{-t(p_{l,n}+v)} \neq e^{-tp_{l,n}} e^{-tv}$ , but from the Trotter product formula (see Theorem VIII.31 in [RS81]) we know that

$$e^{-t(p_{l,n}+v)} = \lim_{m \rightarrow +\infty} (e^{-\frac{t}{m} p_{l,n}} e^{-\frac{t}{m} v})^m$$

taking the limit in the strong sense. We can bound the kernel of  $(e^{-\frac{t}{m} p_{l,n}} e^{-\frac{t}{m} v})^m$  with the one of  $e^{-tp_{l,n}}$ , thanks to the non negativity of  $v$ , and like this we obtain a pointwise upper bound on  $(e^{-\frac{t}{m} p_{l,n}} e^{-\frac{t}{m} v})^m$  by  $e^{-tp_{l,n}}$ . Consequently, given that now  $e^{-t(p_{l,n}+v)}$  is bounded by  $e^{-tp_{l,n}}$ , by (4.12) we see that we control  $(p_k + \lambda)^{-1}$  with  $(p_{l,n} + \lambda)^{-1}$  for which the result in Corollary D.2 holds.

In particular from such corollary we obtain first that

$$\left\| \frac{\sigma_{l,n}^2}{r^2} (p_{l,n} + \lambda)^{-1} \right\|_{L^2((R,+\infty), r^{n-1} dr) \rightarrow L^2((R,+\infty), r^{n-1} dr)} \lesssim_n 1$$

uniformly in  $\lambda$ . Then since  $1 \leq \frac{\sigma_{l,n}^2}{\sigma_{l,n}^2} \leq m_n$  for some constant  $m_n$  it follows that  $\frac{\mu_k^2}{\sigma_{l,n}^2} \leq m_n$ , going back in (4.11) we have found

$$\sup_{\mu_k^2 > n-1} \left\| \frac{\mu_k^2}{r^2} (p_k + \lambda)^{-1} \right\| \lesssim m_n \left\| \frac{\sigma_{l,n}^2}{r^2} (p_{l,n} + \lambda)^{-1} \right\| \lesssim_n 1$$

where we are considering the norm of operators on  $L^2((0,+\infty), r^{n-1} dr)$ . □

With these two last steps we are able to bound the second order term.

**Lemma 4.11** (Bound on second order term). *Let  $I_3$  defined by*

$$I_3 = \frac{1}{\lambda} (P/\lambda - z)^{-1} (1 - \chi) \varphi_\kappa c_\kappa^{j,l} \frac{1}{r^2} \partial_{j,l}^2 (-\Delta_0/\lambda - z)^{-1} (1 - \chi),$$

then there exists  $\delta_3 > 0$  such that

$$\|I_3\| \lesssim \lambda^{\delta_2} \frac{\langle z \rangle^2}{|Imz|^2}.$$

*Proof.* Take  $\phi$  spectral localisation on the interval  $[0, n-1)$  and split  $I_3$  with the partition  $\phi(-\Delta_{\bar{g}})$  and  $(1 - \phi)(-\Delta_{\bar{g}})$ . Conclude by using item 1 (page 21) on the term localised on the angular frequencies on  $[0, n-1)$  and item 2 (page 22) together with Lemma 4.10 on the part localised at high angular frequencies. □

Eventually, thanks to Lemmas 4.6, 4.8 and 4.11 and the properties of almost analytic extensions we have

$$\left\| \int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z) R_z^V(\lambda) L(dz) \right\| \lesssim \lambda^\delta \quad (4.13)$$

for some  $\delta > 0$ , recalling the result of Lemma 4.5 and the definition of  $D_f(\lambda)$  in (4.3) we have found that

$$\|D_f(\lambda)\| \rightarrow 0 \text{ as } \lambda \rightarrow 0 \quad (4.14)$$

which concludes the proof of Assumption 2.1.



## 5 Adding a potential

We see here how we can generalize the arguments we presented in the case where we add a non negative decaying potential. Let

$$P_W := P + W$$

with  $W$  an operator that on  $M \setminus K$  is the multiplication by a real valued function  $W$  belonging to  $S^{-2-\varepsilon}$ . The robustness of the approach lies in the fact that the symbolic structure of the operator is not altered by the addition of such a potential. In particular, in local coordinates  $W$  can be represented by a pseudodifferential operator  $Op_\lambda(\tilde{S}^{-2-\varepsilon,0})$  and if  $\chi$  is a cutoff on the compact part of the manifold we can still write the operator under the form

$$(1 - \chi)P_W/\lambda = \sum_{\kappa} Op_{\lambda,\kappa}(a_{0,\lambda} + a_{1,\lambda}^W) \quad (5.1)$$

with  $a_{0,\lambda} \in \tilde{S}^{0,2}$ ,  $a_{1,\lambda}^W \in \tilde{S}^{-1,1}$  that have seminorms uniformly bounded with respect to  $\lambda$ .

We sketch here the main steps to obtain the corresponding results to Sections 2, 3 and 4.

- Since  $P_W$  has same symbolic structure as  $P$  the results of Section 2 hold with analogous proofs, provided we assume the equivalent of Assumption 2.1 replacing  $P$  by  $P_W$ . Therefore, by limiting absorption principle we obtain

$$(A^\lambda + i)^{-s}(P_W/\lambda - 1 \pm i0)^{-l}(A^\lambda + i)^{-s} \in \mathcal{L}(L^2(M))$$

for  $s > l - \frac{1}{2}$ .

- Once the existence of the outgoing and ingoing resolvents is established, all the proofs of Section 3 carry on in the exact same way for  $P_W$ .
- Lemmas 3.2 and 3.3 were crucial properties to be able to prove Assumption 2.1 and we have noticed in Section 4 that they derive from the behaviour of the heat flow.

Indeed, as we have done before we can use Trotter product formula (Theorem VIII.31 in [RS81]) to write

$$e^{-tP_W} = e^{-t(P+W)} = \lim_{m \rightarrow +\infty} (e^{-\frac{t}{m}P} e^{-\frac{t}{m}W})^m,$$

since  $W \geq 0$  we can bound the kernel of  $e^{-\frac{t}{m}P} e^{-\frac{t}{m}W}$  with the one of  $e^{-\frac{t}{m}P}$ . Therefore, if  $K_W(x, y, t)$  is the kernel of  $e^{-tP_W}$  pointwise it holds

$$K_W(x, y, t) \leq K_0(x, y, t)$$

which allows us to recover for  $e^{-tP_W}$  the same kind of estimates that we had for  $e^{-tP}$ .

*Remark 5.1.* Alternatively, we can recover the heat flow estimates by noticing that they follow from the Nash inequality which holds true for  $P_W$  also as we will see here. Thanks to the non negativity of  $W$  and the selfadjointness of  $P$  we can easily see that

$$\|P_W^{\frac{1}{2}}u\|^2 = \langle (P + W)u, u \rangle = \langle P^{\frac{1}{2}}u, P^{\frac{1}{2}}u \rangle + \langle Wu, u \rangle \geq \|P^{\frac{1}{2}}u\|^2$$

and therefore use the Nash inequality for  $P$  (namely (3.4)) to prove

$$\|u\|_{L^2(M)}^{1+\frac{2}{n}} \lesssim \|u\|_{L^1(M)}^{\frac{2}{n}} \|P_W^{\frac{1}{2}}u\|_{L^2(M)}.$$

We notice that the fact  $\|P_W^{\frac{1}{2}}u\| \geq \|P^{\frac{1}{2}}u\|$  implies that  $P_W$  has no 0 eigenvalue nor resonance, since 0 it is not an eigenvalue nor a resonance for  $P$  either.

Once we get to the proof of Assumption 2.1 the only relevant difference is when considering the term

$$D_f^W(\lambda) := (1 - \chi)f(P_W/\lambda)(1 - \chi) - (1 - \chi)f(-\Delta_0/\lambda)(1 - \chi),$$

applying Helffer Sjöstrand formula we pass to comparison between resolvents and we have an additional term involving  $W$  that is

$$R_z^W(\lambda) := \frac{1}{\lambda} (P_W/\lambda - z)^{-1} (1 - \chi)W(-\Delta_0/\lambda - z)^{-1} (1 - \chi) \quad (5.2)$$

and all the other terms can be bounded with the analogous of Lemma 3.3, hence providing equivalents to Lemma 4.5 and (4.13).

For (5.2) we exploit the boundedness of  $\|\langle r \rangle^{2+\varepsilon} w\|_{L^\infty}$  and applying Lemma 3.3 with  $\sigma = 1 + \varepsilon$  combined with Hardy inequality we obtain

$$\left\| \int_{\mathcal{C}} \bar{\partial}_z \tilde{f}(z) R_z^W(\lambda) L(dz) \right\| \lesssim \lambda^\delta$$

for some positive  $\delta$ . We can therefore conclude that  $\|D_f^W(\lambda)\|$  converges to 0 as  $\lambda$  goes to 0.

## A Operator on the exact cone and separation of variables

Let  $-\Delta_0$  the Friedrich extension of the Laplace Beltrami operator on the cone  $([R, +\infty) \times S, \bar{G})$  with Dirichlet boundary conditions that in local coordinates looks like

$$-\Delta_0 = -\frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \Delta_{\bar{g}}. \quad (A.1)$$

The quadratic form we need to define to derive the Friedrich extension is

$$q_0(u, v) := (\partial_r u, \partial_r v)_{L^2} + (1/r(-\Delta_{\bar{g}})^{\frac{1}{2}} u, 1/r(-\Delta_{\bar{g}})^{\frac{1}{2}} v)_{L^2_{\bar{G}}} \quad (A.2)$$

defined on elements of the space

$$\{ \text{closure of } C_0^\infty([R, +\infty) \times S) \text{ with respect to } \|u\|_+ := (q_0(u, u) + \|u\|_{L^2_{\bar{G}}}^2)^{\frac{1}{2}} \} \quad (A.3)$$

that we will denote by  $H_0^1([R, +\infty) \times S)$ . The domain of the Friedrich extension  $-\Delta_0$  then is

$$D(-\Delta_0) = \{ u \in H_0^1 \mid |q_0(u, v)| \leq C(u) \|v\|_{L^2_{\bar{G}}} \ \forall v \in H_0^1 \} \quad (A.4)$$

Since  $-\Delta_{\bar{g}}$  is selfadjoint on  $L^2(S)$ , we can consider an orthonormal basis  $(e_k)_k$  of the space such that

$$-\Delta_{\bar{g}} e_k = \mu_k^2 e_k, \quad 0 = \mu_0 \leq \mu_1 \leq \dots, \quad (A.5)$$

we can then decompose any function  $u \in L^2(S)$  on this basis

$$u(\omega) = \sum_{k \geq 0} \left( \int_S \bar{e}_k u \, d\text{vol}_S \right) e_k(\omega),$$

while a function of  $L^2_{\bar{G}}$  is of the form

$$u(r, \omega) = \sum_{k \geq 0} \left( \int_S \bar{e}_k(\cdot) u(r, \cdot) \, d\text{vol}_S \right) e_k(\omega) =: \sum_{k \geq 0} u_k(r) e_k(\omega). \quad (A.6)$$

In particular we can identify a function in  $L^2_{\bar{G}}$  with its one dimensional coefficients. It is straightforward to obtain the following.

**Proposition A.1.** *The map between Hilbert spaces*

$$L^2_{\overline{G}} \longrightarrow \bigoplus_{k \geq 0} L^2((R, +\infty), r^{n-1} dr)$$

$$u \mapsto (u_k)_k$$

is an isometry with

$$\|u\|_{L^2_{\overline{G}}}^2 = \sum_{k \geq 0} \|u_k\|_{L^2((R, +\infty), r^{n-1} dr)}^2. \quad (\text{A.7})$$

Moreover, prescribing the action of one dimensional operators on the coefficients  $u_k$  gives origin to well defined operators on the Hilbert space  $L^2_{\overline{G}}$ .

**Proposition A.2.** *Let  $(A_k)_k$  a bounded sequence of bounded operators on  $L^2((R, +\infty), r^{n-1} dr)$ . The operator  $A$  defined by*

$$Au(r, \omega) = \sum_k (A_k u_k)(r) e_k(\omega). \quad (\text{A.8})$$

is well defined on  $L^2_{\overline{G}}$  with norm

$$\|A\|_{L^2_{\overline{G}} \rightarrow L^2_{\overline{G}}} = \sup_k \|A_k\|_{L^2((R, +\infty), r^{n-1} dr) \rightarrow L^2((R, +\infty), r^{n-1} dr)}.$$

*Proof.* Follows directly from Proposition A.1.  $\square$

With Proposition A.2 in mind we want to reduce  $-\Delta_0$  to the action of suitable one dimensional operators. Indeed, once a function is represented with respect to the orthonormal basis  $(e_k)_k$  as in (A.6), the action of  $-\Delta_{\overline{g}}$  becomes multiplication by a scalar so going back to the expression in (A.1) this suggests we set

$$p_k := -\frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r} \partial_r + \frac{1}{r^2} \mu_k^2. \quad (\text{A.9})$$

We can prove that the sequence of one dimensional operators  $(p_k)_k$  corresponds exactly to  $-\Delta_0$ .

**Proposition A.3.** *Let  $u \in H_0^1([R, +\infty) \times S)$ , then  $u \in D(-\Delta_0)$  if and only if  $u_k \in D(p_k)$  for every  $k$  and*

$$\sum_k \|p_k u_k\|_{L^2((R, +\infty), r^{n-1} dr)}^2 < +\infty. \quad (\text{A.10})$$

Moreover

$$(-\Delta_0 u)_k = p_k u_k, \quad \text{from which } \|(-\Delta_0 u)\|_{L^2_{\overline{G}}}^2 = \sum_k \|p_k u_k\|_{L^2((R, +\infty), r^{n-1} dr)}^2.$$

*Proof.* Can be proved by direct computations, given that constructing  $p_k$  via Friedrich extension its associated quadratic form is

$$q_k(f, g) = \int_R^{+\infty} \left( \overline{f'} g' + \frac{1}{r^2} \mu_k^2 \overline{f} g \right) r^{n-1} dr,$$

with domain

$$h_{+1,k} := \left\{ \text{closure of } C_0^\infty([R, +\infty)) \text{ with respect to } \left( q_k(f, f) + \|f\|_{L^2((R, +\infty), r^{n-1} dr)}^2 \right)^{\frac{1}{2}} \right\},$$

while the domain of  $p_k$  is

$$D(p_k) = \{f \in h_{+1,k} \mid |q_k(f, g)| \leq C(f) \|g\|_{L^2((R, +\infty), r^{n-1} dr)} \quad \forall g \in h_{+1,k}\}.$$

$\square$

The result of Proposition A.3 extends to the case of functions of  $-\Delta_0$ , as we will see below, and therefore it will be of use to simplify the argument in estimating norms in Section 4.

**Proposition A.4.** *Let  $\varphi \in C_0^\infty(\mathbb{R})$  then*

$$\varphi(-\Delta_0)u(r, \omega) = \sum_k (\varphi(p_k)u_k)(r)e_k(\omega)$$

for any  $u \in L^2_{\overline{G}}$ .

*Proof.* The statement can be proved directly when  $\varphi(-\Delta_0)$  is a resolvent and can be generalised to any  $\varphi \in C_0^\infty(\mathbb{R})$  applying Hellfer-sjöstrand formula to compute the right hand side of the equality.  $\square$

## B Nash inequality

We show in this appendix how to obtain an inequality of the type

$$\|u\|_{L^2(M)}^{1+\frac{2}{n}} \lesssim \|u\|_{L^1(M)}^{\frac{2}{n}} \|P^{\frac{1}{2}}u\|_{L^2(M)}$$

using the analogous result which holds for the free operator  $-\Delta$  on  $\mathbb{R}^n$ , namely

$$\|u\|_{L^2(\mathbb{R}^n)}^{1+\frac{2}{n}} \lesssim \|u\|_{L^1(\mathbb{R}^n)}^{\frac{2}{n}} \|(-\Delta)^{\frac{1}{2}}u\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^1(\mathbb{R}^n)}^{\frac{2}{n}} \|\nabla u\|_{L^2(\mathbb{R}^n)} \quad (\text{B.1})$$

with a constant depending on  $n$ . (See [Nas58] p. 936).

We will proceed in two steps: first we will prove the inequality on a pure cone (thanks to (B.1)) and next we will pass onto  $M$  where on the manifold end,  $M \setminus K$ , we will use the result that holds for a pure a cone, while on the compact part we will exploit the fact that locally we are diffeomorphic to  $\mathbb{R}^n$  where (B.1) applies.

### B.1 Inequality on a fixed cone

Since we are considering the fixed cone  $((0, \infty) \times S, \overline{G})$  we can proceed as in Remark 4.2 to reduce ourselves to norms on  $\mathbb{R}^n$  for which holds.

**Lemma B.1.** *Let  $u \in C_0^\infty((0, +\infty) \times S)$ , then there exists  $C_n > 0$  such that*

$$\|u\|_{L^2((0, +\infty) \times S)}^{1+\frac{2}{n}} \leq C_n \|u\|_{L^1((0, +\infty) \times S)}^{\frac{2}{n}} \|\nabla_{\overline{G}}u\|_{L^2((0, +\infty) \times S)}.$$

*Proof.* Recall the equivalence between the  $L^2$  norms on  $((0, \infty) \times S, \overline{G})$  and the usual  $L^2$  norm with respect to the Lebesgue measure mentioned in Remark 4.2. Then for each term in  $u = \sum_j \varphi_j(\omega)u$  we can apply (B.1) and get

$$\|\varphi_j(\omega)u\|_{L^2((0, +\infty) \times S)}^{1+\frac{2}{n}} \simeq \|\varphi_j u\|_{L^2(\mathbb{R}^n)}^{1+\frac{2}{n}} \lesssim \|\varphi_j u\|_{L^1(\mathbb{R}^n)}^{\frac{2}{n}} \|\nabla_{\mathbb{R}^n}(\varphi_j u)\|_{L^2(\mathbb{R}^n)}$$

with

$$\|\varphi_j u\|_{L^1(\mathbb{R}^n)}^{\frac{2}{n}} \simeq \|\varphi_j u\|_{L^1((0, +\infty) \times S)}^{\frac{2}{n}} \lesssim \|u\|_{L^1((0, +\infty) \times S)}^{\frac{2}{n}}. \quad (\text{B.2})$$

Considering polar coordinates on  $(0, +\infty) \times \mathbb{S}^{n-1}$  the gradient is  $(\partial_r, 1/r \nabla_{\mathbb{S}^{n-1}})$ , so

$$\begin{aligned} \|\nabla_{\mathbb{R}^n}(\varphi_j u)\|_{L^2(\mathbb{R}^n)}^2 &\simeq \|\varphi_j \partial_r u\|_{L^2((0, +\infty) \times \mathbb{S}^{n-1})}^2 + \|1/r \nabla_{\mathbb{S}^{n-1}}(\varphi_j u)\|_{L^2((0, +\infty) \times \mathbb{S}^{n-1})}^2 \\ &\simeq \|\varphi_j(\omega) \partial_r u\|_{L^2((0, +\infty) \times S)}^2 + \|1/r \nabla_{\overline{G}}(\varphi_j(\omega)u)\|_{L^2((0, +\infty) \times S)}^2 \\ &\lesssim \|\partial_r u\|_{L^2((0, +\infty) \times S)}^2 + \|1/r \nabla_{\overline{G}}(\varphi_j(\omega)u)\|_{L^2((0, +\infty) \times S)}^2. \end{aligned} \quad (\text{B.3})$$

Now observing that  $\nabla_{\bar{g}} = (\bar{g}^{j,k}(\theta))_{j,k}(\partial_1, \dots, \partial_{n-1})^T$  when we rewrite  $\nabla_{\bar{g}}(\varphi_j(\omega)u)$  as  $[\nabla_{\bar{g}}, \varphi_j] + \varphi_j \nabla_{\bar{g}}$  the commutator

$$[\nabla_{\bar{g}}, \varphi_j(\omega)] = (\bar{g}^{j,k}(\theta))_{j,k} \nabla_{\theta}(\varphi_j \circ \kappa_j^{-1}) \in C_0^\infty(\mathbb{R}^{n-1}),$$

is the multiplication by a bounded function and hence a bounded operator. So we have found

$$\begin{aligned} \|1/r \nabla_{\bar{g}}(\varphi_j(\omega)u)\|_{L^2((0,+\infty) \times S)} &\leq \|1/r \varphi_j(\omega) \nabla_{\bar{g}} u\|_{L^2((0,+\infty) \times S)} + \|1/r [\nabla_{\bar{g}}, \varphi_j(\omega)]u\|_{L^2((0,+\infty) \times S)} \\ &\lesssim \|1/r \nabla_{\bar{g}} u\|_{L^2((0,+\infty) \times S)} + C \|u/r\|_{L^2((0,+\infty) \times S)} \\ &\lesssim \|1/r \nabla_{\bar{g}} u\|_{L^2((0,+\infty) \times S)} + C \|\partial_r u\|_{L^2((0,+\infty) \times S)} \end{aligned}$$

thanks to Hardy inequality (Propositions 2.2 and 3.5 in [BR14b]) in the last line. The statement follows combining (B.2), (B.3) together with this last estimate.  $\square$

## B.2 Inequality on the manifold

We split the analysis into the part near infinity and the compact one. From what we have found in the previous section, if  $\chi$  is a cutoff on the compact part at first we get

$$\begin{aligned} \|(1-\chi)u\|^{1+\frac{2}{n}} &\simeq \|(1-\chi)u\|_{L^2_{\bar{G}}}^{1+\frac{2}{n}} \lesssim \|(1-\chi)u\|_{L^1_{\bar{G}}}^{\frac{2}{n}} \|\nabla_{\bar{G}}((1-\chi)u)\|_{L^2((0,+\infty) \times S)} \\ &\simeq \|(1-\chi)u\|_{L^1(M \setminus K)}^{\frac{2}{n}} \|\nabla_{\bar{G}}((1-\chi)u)\|_{L^2((0,+\infty) \times S)} \\ &\leq \|u\|_{L^1(M)}^{\frac{2}{n}} \|\nabla_{\bar{G}}((1-\chi)u)\|_{L^2((0,+\infty) \times S)}. \end{aligned} \quad (\text{B.4})$$

The gradient we need to evaluate is

$$|\nabla_{\bar{G}}((1-\chi)u)|_{\bar{G}}^2 \lesssim |\chi' u|^2 + |(1-\chi)\partial_r u|^2 + |(1-\chi)/r \nabla_{\bar{g}} u|_{\bar{g}}^2 \quad (\text{B.5})$$

with  $\chi'$  compactly supported in  $[R, +\infty)$  so that  $\|\chi'\langle r \rangle\|_{L^\infty} \lesssim 1$  then passing to the  $L^2$  norm and using the Hardy inequality mentioned in the previous proof we can first bound the terms

$$\|\chi' u\|_{L^2_{\bar{G}}}^2 + \|(1-\chi)u\|_{L^2_{\bar{G}}}^2 \lesssim \|\langle r \rangle^{-1} u\|_{L^2_{\bar{G}}}^2 + \|\partial_r u\|_{L^2_{\bar{G}}}^2 \lesssim \|\partial_r u\|_{L^2_{\bar{G}}}^2. \quad (\text{B.6})$$

Then we will need to compare the gradient with respect to the fixed metric  $\bar{g}$  and to the metric  $g(r)$  via the following lemma.

**Lemma B.2.** *Let  $\bar{g}$  and  $g(r)$  two metrics on a closed manifold  $S$  satisfying the property in (1.2), then there exists a constant  $C > 0$  such that*

$$\|\nabla_{\bar{g}} u|_{\bar{g}}\|_{L^2(S, d\bar{g})} \leq C \|\nabla_{g(r)} u|_{g(r)}\|_{L^2(S, dg(r))}.$$

for  $r \gg 1$  and  $u \in H_0^1(S, d\bar{g}) \cap H_0^1(S, dg(r))$ .

*Proof.* Given the two gradients  $\nabla_{\bar{g}} = (\bar{g}^{j,k}(\theta))_{j,k} \nabla_{\theta}$  and  $\nabla_{g(r)} = (g(r, \theta)^{j,k})_{j,k} \nabla_{\theta}$  and property (1.2) for large  $r$  we get

$$\|\nabla_{\bar{g}} u|_{\bar{g}}\|_{L^2(S, d\bar{g})} \lesssim \left\| \sum_{\kappa} |\nabla_{\bar{g}}(\varphi_{\kappa} u)|_{\bar{g}} \right\|_{L^2(S, d\bar{g})} \lesssim \left\| \sum_{\kappa} |\nabla_{g(r)}(\varphi_{\kappa} u)|_{g(r)} \right\|_{L^2(S, dg(r))}.$$

Consider the kernel of the operator  $-\Delta_{g(r)}$ , which is spanned by 1, and let

$$\Pi_{g(r)}^0 := \text{projection on } \ker_{L^2}(-\Delta_{g(r)}),$$

in particular  $\nabla_{\bar{g}} u = \nabla_{\bar{g}}(u - \Pi_{g(r)}^0 u)$ . Now the gradient of  $u$  is

$$\|\nabla_{\bar{g}} u|_{\bar{g}}\|_{L^2(S, d\bar{g})} = \|\nabla_{\bar{g}}(u - \Pi_{g(r)}^0 u)|_{\bar{g}}\|_{L^2(S, d\bar{g})}$$

$$\begin{aligned}
&\lesssim \sum_{\kappa} \|\nabla_{g(r)}(\varphi_{\kappa}(u - \Pi_{g(r)}^0 u))|_{g(r)}\|_{L^2(S, dg(r))} \\
&\lesssim \sum_{\kappa} \|\varphi_{\kappa}|\nabla_{g(r)}(u - \Pi_{g(r)}^0 u)|_{g(r)}\|_{L^2(S, dg(r))} \\
&\quad + \sum_{\kappa} \|\nabla_{g(r)}\varphi_{\kappa}|_{g(r)}(u - \Pi_{g(r)}^0 u)\|_{L^2(S, dg(r))} \\
&\lesssim \|\nabla_{g(r)}u|_{g(r)}\|_{L^2(S, dg(r))} + \|u - \Pi_{g(r)}^0 u\|_{L^2(S, dg(r))}.
\end{aligned}$$

Let's write the projection on the 0 eigenspace  $\Pi_{g(r)}^0$  as  $f(-\Delta_{g(r)})$  with  $f \in C_0^\infty(\mathbb{R})$  such that  $f(0) = 1$  and supported around 0, then by the spectral theorem

$$\begin{aligned}
\|u - \Pi_{g(r)}^0 u\|_{L^2(S, dg(r))} &= \left\| \frac{(1-f)(-\Delta_{g(r)})}{(-\Delta_{g(r)})^{\frac{1}{2}}} (-\Delta_{g(r)})^{\frac{1}{2}} u \right\|_{L^2(S, dg(r))} \\
&\lesssim \|(-\Delta_{g(r)})^{\frac{1}{2}} u\|_{L^2(S, dg(r))} \\
&\simeq \|\nabla_{g(r)}u|_{g(r)}\|_{L^2(S, dg(r))}
\end{aligned}$$

from which the statement follows.  $\square$

Going back to (B.5) and thanks to (B.6) we obtain

$$\begin{aligned}
\|\nabla_{\bar{G}}((1-\chi)u)\|_{L^2((0,+\infty) \times S)}^2 &\lesssim \|\partial_r u\|_{L^2_{\bar{G}}}^2 + \|(1-\chi)/r|\nabla_{\bar{g}}u|_{\bar{g}}\|_{L^2_{\bar{G}}}^2 \\
&\lesssim \|\partial_r u\|_{L^2_{\bar{G}}}^2 + \|(1-\chi)/r|\nabla_{g(r)}u|_{g(r)}\|_{L^2_{\bar{G}}}^2 \\
&\lesssim \|\nabla_G u\|_{L^2_{\bar{G}}}^2 \simeq \|P^{\frac{1}{2}}u\|_{L^2(M \setminus K)}^2.
\end{aligned}$$

Applying this estimate to (B.4) we have proved the desired inequality on the manifold end

$$\|(1-\chi)u\|^{1+\frac{2}{n}} \lesssim \|u\|_{L^1(M)}^{\frac{2}{n}} \|P^{\frac{1}{2}}u\|_{L^2(M)}^2. \quad (\text{B.7})$$

Using a partition of unity for the remaining compact part we easily derive the following.

**Lemma B.3.** *Let  $u \in C_0^\infty(M)$  then it holds*

$$\|u\|_{L^2(M)}^{1+\frac{2}{n}} \leq C_n \|u\|_{L^1(M)}^{\frac{2}{n}} \|P^{\frac{1}{2}}u\|_{L^2(M)}$$

for some  $C_n > 0$ .

*Proof.* Write  $u$  as  $(1-\chi)u + \chi u$  and use (B.7) on  $(1-\chi)u$ . Then  $\chi u$  is supported on a compact set on which we can consider a finite covering  $(K_j)_j$  with associated partition of unity  $(\chi_j)_j$ . Each  $K_j \subset M$  is diffeomorphic to an open set of  $\mathbb{R}^n$ , via a diffeomorphism  $\psi_j$  and since the metric tensor on  $M$  is represented by a positive definite matrix we have

$$\int_M \chi_j(\chi u) dG \simeq \int_{\psi_j(K_j)} \chi_j(\chi u) \circ \psi_j^{-1} dx.$$

Applying again (B.1) it follows

$$\begin{aligned}
\|\chi_j(\chi u)\|_{L^2(M)}^{1+\frac{2}{n}} &\simeq \|\chi_j(\chi u)\|_{L^2(\mathbb{R}^n)}^{1+\frac{2}{n}} \lesssim \|\chi_j(\chi u)\|_{L^1(\mathbb{R}^n)}^{\frac{2}{n}} \|\nabla_{\mathbb{R}^n}(\chi_j \chi u)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|u\|_{L^1(M)}^{\frac{2}{n}} (\|\nabla_{\mathbb{R}^n}(\chi u)\|_{L^2(\mathbb{R}^n)} + \|\chi u\|_{L^2(\mathbb{R}^n)}) \\
&\simeq \|u\|_{L^1(M)}^{\frac{2}{n}} (\|\nabla_G(\chi u)\|_{L^2(M)} + \|\chi u\|_{L^2(M)})
\end{aligned}$$

$$\begin{aligned}
&\lesssim \|u\|_{L^1(M)}^{\frac{2}{n}} \|\nabla_G(\chi u)\|_{L^2(M)} \\
&\lesssim \|u\|_{L^1(M)}^{\frac{2}{n}} (\|\nabla_G u\|_{L^2(M)} + \|\chi' u\|_{L^2(M)}) \\
&\lesssim \|u\|_{L^1(M)}^{\frac{2}{n}} \|\nabla_G u\|_{L^2(M)}
\end{aligned}$$

where we have used Hardy inequality (Proposition 2.2 of [BR14b]) to estimate  $\|\chi u\|_{L^2(M)}$  since  $\chi\langle r \rangle$  is bounded and (2.4) in [BR14b] to bound  $\|\chi' u\|_{L^2(M)}$ .  $\square$

## C Commutators and symbolic calculus

We will see here how to use symbolic calculus to compute commutators between pseudodifferential operators on a manifold. First of all we point out that for  $Op_\lambda(\cdot)$  described in (1.5) the usual rules on the composition of pseudodifferential operators apply (for example see Proposition 3.1 in [BM16]).

As defined in (1.6), instead,  $Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa$  acts on functions supported on  $U_\kappa \subset M$  so summing up all the contributions  $\sum_\kappa Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa$  will be a pseudodifferential operator defined on the whole manifold. We also underline that we will consider operators with spatially localised symbols, namely  $supp a^\kappa \subset [R, +\infty) \times V_\kappa \times \mathbb{R}^n$ .

We proceed to the proof of the result mentioned in Proposition 2.2.

**Proposition C.1.** *Let  $m, m', \mu, \mu'$  real numbers and the operators  $A, B$  on  $[R, +\infty) \times S$  defined as*

$$A := \sum_\kappa Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa, \quad B := \sum_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa$$

with symbols in  $\tilde{S}^{m,\mu}$  and  $\tilde{S}^{m',\mu'}$  respectively and spatially supported in  $[R, +\infty) \times V_\kappa$ . Then for the commutator it holds

$$[A, B] = \sum_\kappa Op_{\lambda,\kappa}(\tilde{S}^{m+m'-1, \mu+\mu'-1})\psi_\kappa + \mathcal{R}$$

with  $\mathcal{R}_N$  an operator which is negligible of order  $N$  for any  $N \in \mathbb{N}$ .

*Proof.* In the following  $\mathcal{B}$  is a generic bounded operator that will be allowed to change from one line to the other.

The composition  $AB$  is given by a double sum, by the support properties of the symbols  $b^\kappa$   $Op_{\lambda,\kappa}(b^\kappa)$  will be supported in  $[R, +\infty) \times U_\kappa$ , while  $\psi_m$  localises in  $[R, +\infty) \times U_m$ . Hence for any chart such that  $U_m \cap U_\kappa = \emptyset$  the corresponding term in the sum is 0.

Next we start by looking at the easier case where we are composing two operators localised on the same chart. We denote by  $\psi_\kappa^{\mathbb{R}^n}$  the pushforward on  $\mathbb{R}^n$  of  $\psi_\kappa$  through any chart  $l$  whenever this quantity is well defined, that is  $\psi_\kappa^{\mathbb{R}^n} = \Pi_l^{-1}\psi_\kappa$ , and we will extensively use the relations

$$\psi_\kappa \Pi_l = \Pi_l \psi_\kappa^{\mathbb{R}^n}, \quad \Pi_l^{-1}\psi_\kappa = \psi_\kappa^{\mathbb{R}^n} \Pi_l^{-1}.$$

This said, we have

$$\begin{aligned}
Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa &= \Pi_\kappa Op_\lambda(a^\kappa) \Pi_\kappa^{-1}(\psi_\kappa \Pi_\kappa) Op_\lambda(b^\kappa) \Pi_\kappa^{-1}\psi_\kappa \\
&= \Pi_\kappa Op_\lambda(a^\kappa) \Pi_\kappa^{-1} \Pi_\kappa(\psi_\kappa^{\mathbb{R}^n} Op_\lambda(b^\kappa)) \Pi_\kappa^{-1}\psi_\kappa \\
&= \Pi_\kappa Op_\lambda(a^\kappa) Op_\lambda(\tilde{b}^\kappa) \Pi_\kappa^{-1}\psi_\kappa
\end{aligned}$$

where  $\tilde{b}^\kappa := \psi_\kappa^{\mathbb{R}^n} b^\kappa \in \tilde{S}^{m',\mu'}$  is still spatially supported on  $V_\kappa$ . Thanks to the multiplication on the right by  $\Pi_\kappa^{-1}\psi_\kappa$  the composition  $Op_\lambda(a^\kappa) Op_\lambda(\tilde{b}^\kappa)$  is applied to functions localised on  $V_\kappa$  so

we can harmlessly extend the symbols to 0 outside of their support which will give us rescaled pseudodifferential operators on  $\mathbb{R}^n$  to which usual composition formulas apply. Consequently

$$\begin{aligned} Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa &= \Pi_\kappa Op_\lambda(a^\kappa b^\kappa)\Pi_\kappa^{-1}\psi_\kappa + \Pi_\kappa Op_\lambda(\tilde{S}^{m+m'-1,\mu+\mu'-1})\Pi_\kappa^{-1}\psi_\kappa \\ &= Op_{\lambda,\kappa}(a^\kappa b^\kappa)\psi_\kappa + Op_{\lambda,\kappa}(\tilde{S}^{m+m'-1,\mu+\mu'-1})\psi_\kappa. \end{aligned} \quad (C.1)$$

Obviously the same holds for  $Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa Op_{\lambda,\kappa}(a^\kappa)\psi_\kappa$  so that when taking the commutator the term  $Op_{\lambda,\kappa}(a^\kappa b^\kappa)\psi_\kappa$  cancels with  $Op_{\lambda,\kappa}(b^\kappa a^\kappa)\psi_\kappa$ .

Now for the overlapping terms with  $m \neq \kappa$  let  $\tilde{\psi}_\kappa$  equal to 1 on the support of  $\psi_\kappa$ , by writing

$$\Pi_\kappa Op_\lambda(b^\kappa)\Pi_\kappa^{-1}\psi_\kappa = \tilde{\psi}_\kappa \Pi_\kappa Op_\lambda(b^\kappa)\Pi_\kappa^{-1}, \quad (C.2)$$

when we compose with  $Op_{\lambda,m}(a^m)\psi_m$  on the left the cutoff  $\psi_m \tilde{\psi}_\kappa \in C^\infty([R, +\infty) \times U_m \cap U_\kappa)$  localises in a region where both the charts  $\kappa$  and  $m$  are defined.

We pick smooth cutoffs  $\tilde{\psi}_\kappa$  and  $\tilde{\psi}_m$  such that

$$\tilde{\psi}_\kappa \tilde{\psi}_\kappa \equiv \tilde{\psi}_\kappa, \quad \tilde{\psi}_m \psi_m \equiv \psi_m$$

and since  $\tilde{\psi}_m^{\mathbb{R}^n} \equiv 1$  on the support of  $a^m$  we have

$$\begin{aligned} Op_{\lambda,m}(a^m)\psi_m \tilde{\psi}_\kappa &= \Pi_m \tilde{\psi}_m^{\mathbb{R}^n} Op_\lambda(a^m)\Pi_m^{-1}(\psi_m \tilde{\psi}_\kappa) \\ &= \tilde{\psi}_\kappa \tilde{\psi}_m \Pi_m Op_\lambda(a^m)\Pi_m^{-1}(\psi_m \tilde{\psi}_\kappa) \\ &\quad + (1 - \tilde{\psi}_\kappa)\tilde{\psi}_m \Pi_m Op_\lambda(a^m)\Pi_m^{-1}(\psi_m \tilde{\psi}_\kappa). \end{aligned} \quad (C.3)$$

On the intersection  $U_m \cap U_\kappa$  both the expression in local coordinates given by the chart  $m$  and the chart  $\kappa$  are well defined and we can pass from one to another by composing with smooth transition maps, like  $\kappa \circ m^{-1}$ . We notice that  $\Pi_m^{-1}(\psi_m \tilde{\psi}_\kappa)$  localises exactly in  $m(U_m \cap U_\kappa)$  which is where  $\kappa \circ m^{-1}$  is well defined and we have the relation

$$Op_\lambda(a^m)\Pi_m^{-1}(\psi_m \tilde{\psi}_\kappa) = \Pi_{\kappa \circ m^{-1}} Op_\lambda(\tilde{a}^\kappa)\Pi_{\kappa \circ m^{-1}}^{-1}\Pi_m^{-1}(\psi_m \tilde{\psi}_\kappa)$$

where  $\tilde{a}^\kappa$  is a symbol belonging to the same class of  $a^m$ . Indeed, we will prove in Proposition C.2 that conjugation by transition maps does not affect the decay of the symbol and that in particular it holds

$$Op_{\lambda,\kappa}(\tilde{a}^\kappa) = Op_{\lambda,\kappa}(\tilde{S}^{m,\mu}) + \mathcal{R}_N \quad (C.4)$$

with  $\mathcal{R}$  a negligible operator.

In the following sum we can first use (C.2) and then the expression found in (C.3)

$$\begin{aligned} \sum_\kappa \sum_m Op_{\lambda,m}(a^m)\psi_m Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa &= \sum_\kappa \sum_m Op_{\lambda,m}(a^m)\psi_m \tilde{\psi}_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa \\ &= \sum_\kappa \sum_m (\tilde{\psi}_\kappa \tilde{\psi}_m \Pi_m (\Pi_{\kappa \circ m^{-1}} Op_\lambda(\tilde{a}^\kappa)\Pi_{\kappa \circ m^{-1}}^{-1})\Pi_m^{-1}\psi_m \\ &\quad \tilde{\psi}_\kappa Op_{\lambda,\kappa}(b^\kappa))\psi_\kappa \\ &\quad + \sum_\kappa \sum_m (1 - \tilde{\psi}_\kappa)\tilde{\psi}_m Op_{\lambda,m}(a^m)(\psi_m \tilde{\psi}_\kappa) Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa. \end{aligned}$$

Noticing that  $\Pi_{\kappa \circ m^{-1}} = \Pi_m^{-1}\Pi_\kappa$  we can simplify some terms

$$\Pi_m \Pi_{\kappa \circ m^{-1}} = \Pi_\kappa, \quad \Pi_{\kappa \circ m^{-1}}^{-1} \Pi_m^{-1} = \Pi_\kappa^{-1},$$

yielding

$$\sum_\kappa \sum_m Op_{\lambda,m}(a^m)\psi_m Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa = \sum_\kappa \sum_m \tilde{\psi}_\kappa \tilde{\psi}_m \Pi_\kappa Op_\lambda(\tilde{a}^\kappa)\Pi_\kappa^{-1}\psi_m \tilde{\psi}_\kappa Op_{\lambda,\kappa}(b^\kappa)\psi_\kappa$$



$$+ \sum_{\kappa} \sum_m (1 - \tilde{\psi}_{\kappa}) \tilde{\psi}_m Op_{\lambda,m}(a^m) (\psi_m \tilde{\psi}_{\kappa}) Op_{\lambda,\kappa}(b^{\kappa}) \psi_{\kappa}. \quad (\text{C.5})$$

First, thanks to the support properties of  $\tilde{\psi}_m$  and  $\tilde{\psi}_{\kappa}$  and since the  $\psi_m$  sum up to one

$$\begin{aligned} & \sum_m \tilde{\psi}_{\kappa} \tilde{\psi}_m \Pi_{\kappa} Op_{\lambda}(\tilde{a}^{\kappa}) \Pi_{\kappa}^{-1} \psi_m \tilde{\psi}_{\kappa} \Pi_{\kappa} Op_{\lambda}(b^{\kappa}) \Pi_{\kappa}^{-1} \\ &= \Pi_{\kappa} Op_{\lambda}(\tilde{a}^{\kappa}) \Pi_{\kappa}^{-1} \sum_m \psi_m \tilde{\psi}_{\kappa} \Pi_{\kappa} Op_{\lambda}(b^{\kappa}) \Pi_{\kappa}^{-1} \\ &= \Pi_{\kappa} Op_{\lambda}(\tilde{a}^{\kappa}) \tilde{\psi}_{\kappa} Op_{\lambda}(b^{\kappa}) \Pi_{\kappa}^{-1} \end{aligned} \quad (\text{C.6})$$

$$= Op_{\lambda,\kappa}(\tilde{a}^{\kappa} b^{\kappa}) \tilde{\psi}_{\kappa} + Op_{\lambda,\kappa}(\tilde{S}^{m+m'-1, \mu+\mu'-1}) \tilde{\psi}_{\kappa} \quad (\text{C.7})$$

where in (C.6) we fall in the same case as in (C.1) so we use (C.4) and the properties of composition of pseudodifferential operators. For the term in (C.5) we have the part

$$\Pi_m (1 - \tilde{\psi}_{\kappa}^{\mathbb{R}^n}) \tilde{\psi}_m^{\mathbb{R}^n} Op_{\lambda}(a^m) \psi_m^{\mathbb{R}^n} \tilde{\psi}_{\kappa}^{\mathbb{R}^n} \Pi_m^{-1}$$

where we have a composition of pseudodifferential operators with disjoint supports, hence the usual formula for the composition produces the remainder only. This implies that we can write

$$R_m := (1 - \tilde{\psi}_{\kappa}) \tilde{\psi}_m Op_{\lambda,m}(a^m) \psi_m \tilde{\psi}_{\kappa} = \tilde{\psi}_m Op_{\lambda,m}(r)$$

with  $r \in \tilde{S}^{-4N, -4N}$  for any  $N \in \mathbb{N}$  and that

$$\sum_m R_m = Q_N^{-1} \mathcal{B} Q_N^{-1}.$$

Adding the contribution of  $Op_{\lambda}(b^{\kappa})$  gives us

$$\begin{aligned} \sum_m (1 - \tilde{\psi}_{\kappa}) \tilde{\psi}_m Op_{\lambda,m}(a^m) (\psi_m \tilde{\psi}_{\kappa}) Op_{\lambda,\kappa}(b^{\kappa}) &= \sum_m R_m Op_{\lambda,\kappa}(b^{\kappa}) \\ &= Q_N^{-1} \mathcal{B} Q_N^{-1} Op_{\lambda,\kappa}(b^{\kappa}) \Pi_{\kappa}^{-1} \\ &= Q_{N/2}^{-1} Q_{N/2}^{-1} \mathcal{B} Q_N^{-1} Op_{\lambda,\kappa}(b^{\kappa}) Q_{N/2} Q_{N/2}^{-1} \\ &= Q_{N/2}^{-1} \mathcal{B} Q_{N/2}^{-1} \end{aligned}$$

meaning that thanks to (C.7) we have found

$$\begin{aligned} \sum_{\kappa} \sum_m Op_{\lambda,m}(a^m) \psi_m Op_{\lambda,\kappa}(b^{\kappa}) \psi_{\kappa} &= Op_{\lambda,\kappa}(\tilde{a}^{\kappa} b^{\kappa}) \tilde{\psi}_{\kappa} + \\ &+ Op_{\lambda,\kappa}(\tilde{S}^{m+m'-1, \mu+\mu'-1}) \tilde{\psi}_{\kappa} + Q_N^{-1} \mathcal{B} Q_{N/2}^{-1}. \end{aligned}$$

By repeating the same procedure for the double sum that appears when taking the composition  $BA$  we obtain terms  $Op_{\lambda,\kappa}(b^{\kappa} \tilde{a}^{\kappa}) \tilde{\psi}_{\kappa}$ . Hence taking the difference  $AB - BA$  results in the statement.  $\square$

We prove now the invariance of the symbol classes by conjugation with a diffeomorphism. This will imply that passing from one chart to another, which means conjugating with a transition map the operators on  $\mathbb{R}^n$ , does not alter the decay of the symbols.

*Remark C.1* (Notation). To simplify the notations of the kernels we state the proposition for pseudodifferential operators, instead of the rescaled version  $Op_{\lambda}(a)$ . However, given that the kind of diffeomorphism we are considering leaves untouched the radial variable  $r$ , which is the only one affected by the rescaling, the result generalises easily for  $Op_{\lambda}(a)$ .

**Proposition C.2.** *Let  $\gamma : V \rightarrow W$  a diffeomorphism between open sets of  $\mathbb{R}^{n-1}$  with  $|\partial^\alpha \gamma| \leq C_\alpha$ ,  $|\partial^\alpha \gamma^{-1}| \leq c_\alpha$  and set  $\psi(r, \theta) = \chi(r)\varphi(\theta) \in C^\infty([R, +\infty) \times V)$  compactly supported in  $\theta$  and*

$$\begin{aligned} \Pi_\gamma : C^\infty([R, +\infty) \times W) &\rightarrow C^\infty([R, +\infty) \times V) \\ v &\mapsto v(r, \gamma(\theta)). \end{aligned}$$

If  $Op(a)$  has symbol  $a \in \tilde{S}^{m, \mu}$  with  $\text{supp } a \subset [R, +\infty) \times W$  then

$$\Pi_\gamma Op(a) \Pi_\gamma^{-1} \psi = Op(a_\gamma) \tilde{\psi} + \mathcal{R}_V$$

with  $a_\gamma \in \tilde{S}^{m, \mu}$ ,  $\tilde{\psi} \in C^\infty([R, +\infty) \times V)$  compactly supported in  $\theta$  and  $\mathcal{R}_V$  a pseudodifferential operator of negative order. In particular,  $\mathcal{R}_V$  is the pushforward on  $\mathbb{R}^n$  of a negligible operator of order  $N$  for any  $N \in \mathbb{N}$ .

Before proceeding with the proof we remark that the integral kernel of  $Op(a)$  is

$$K_a(r, \theta, r', \theta') = \frac{1}{(2\pi)^n} \int e^{i(r-r')\rho + i(\theta-\theta')\eta} a(r, \theta, \rho, \eta) d\rho d\eta$$

whereas the integral kernel of  $\Pi_\gamma Op(a) \Pi_\gamma^{-1} \psi$  is instead

$$\begin{aligned} K_\gamma(r, \theta, r', \theta') &= \frac{1}{(2\pi)^n} \int \left( e^{i(r-r')\rho + i(\gamma(\theta) - \gamma(\theta'))\eta} a(r, \gamma(\theta), \rho, \eta) \chi(r') \varphi(\theta') \right. \\ &\quad \left. |\text{Jac } \gamma|(\theta') \right) d\rho d\eta. \end{aligned}$$

To show that  $\Pi_\gamma Op(a) \Pi_\gamma^{-1} \psi$  is still a pseudodifferential operator, up to some remainder, we will need to write its kernel  $K_\gamma$  as an oscillating integral with phase  $i(r-r')\rho + i(\theta-\theta')\eta$  and a symbol  $a_\gamma$  depending only on the variables  $r, \theta, \rho$  and  $\eta$ .

*Proof.* As we said we want the oscillating term in the integral to be  $e^{i(r-r')\rho + i(\theta-\theta')\eta}$  so to linearise with respect to  $\theta$  we consider the Taylor expansion

$$\gamma(\theta) - \gamma(\theta') = (\theta - \theta') \int_0^1 d\gamma(\theta' + t(\theta - \theta')) dt =: (\theta - \theta') M(\theta, \theta')$$

with  $M(\theta, \theta')$  invertible matrix. Performing a change of variable in  $K_\gamma$  which sends  $\eta$  to  $M(\theta, \theta')^{-1}\eta$  yields

$$K_\gamma(r, \theta, r', \theta') = \frac{1}{(2\pi)^n} \cdot \int e^{i(r-r')\rho + i(\theta-\theta')\eta} A(r, \theta, r', \theta', \rho, \eta) d\rho d\eta \quad (\text{C.8})$$

where we have set

$$A(r, \theta, r', \theta', \rho, \eta) := a(r, \gamma(\theta), \rho, M(\theta, \theta')^{-1}\eta) \chi(r') \varphi(\theta') \tilde{M}(\theta, \theta').$$

In the integral (C.8) it now appears the oscillatory term in the desired form as commented before. To finally obtain the kernel of a pseudodifferential operator we must get rid of the dependence of  $A$  on  $(r', \theta')$ .

Let  $\Theta \in C_0^\infty(\mathbb{R}^n)$  a cutoff function such that  $\Theta \equiv 1$  near 0, we first consider the kernel  $K_\gamma$  localised around the diagonal  $\{r = r', \theta = \theta'\}$ , that is

$$\Theta(r - r', \theta - \theta') K_\gamma(r, \theta, r', \theta').$$

We use again a Taylor expansion: we expand  $A$  with respect to the variables  $r', \theta'$  around the point  $(r, \theta)$  up to order  $N$ , hence providing us with a polynomial of order  $N - 1$  plus a remainder term. In particular we can write

$$A(r, \theta, r', \theta', \rho, \eta) = A(r, \theta, r, \theta, \rho, \eta)$$

$$\begin{aligned}
& + \sum_{l=1}^{N-1} \sum_{j+|\alpha|=l} \mathcal{O}((r'-r)^j(\theta'-\theta)^\alpha)(\partial_{r'}^j \partial_{\theta'}^\alpha A)(r, \theta, r, \theta, \rho, \eta) \\
& + \sum_{j+|\alpha|=N} \mathcal{O}((r'-r)^j(\theta'-\theta)^\alpha) R^{j,\alpha}(r, \theta, r', \theta', \rho, \eta).
\end{aligned}$$

Here  $R^{j,\alpha}$  are the terms coming from the Taylor remainder: they are compactly supported in  $(r', \theta')$  and such that  $|\partial_\rho^k \partial_\eta^\beta R^{j,\alpha}| \lesssim \left(\langle \rho \rangle + \frac{\langle \eta \rangle}{\langle r \rangle}\right)^{\mu-k-|\beta|}$  (property which is inherited from  $a$ ).

We want to use this expansion in (C.8), observing that

$$(r'-r)^j(\theta'-\theta)^\alpha e^{i(r-r')\rho+i(\theta-\theta')\eta} = D_\rho^j D_\eta^\alpha e^{i(r-r')\rho+i(\theta-\theta')\eta}$$

we use this relation to perform integration by parts. Integral (C.8) then results in

$$\begin{aligned}
\Theta(r-r', \theta-\theta') K_\gamma(r, \theta, r', \theta') &= \Theta(r-r', \theta-\theta') \frac{1}{(2\pi)^n} \\
&\cdot \int e^{i(r-r')\rho+i(\theta-\theta')\eta} A(r, \theta, r, \theta, \rho, \eta) d\rho d\eta \\
&+ \Theta(r-r', \theta-\theta') \frac{1}{(2\pi)^n} \\
&\sum_l \sum_{(j,\alpha)} \int e^{i(r'-r)\rho+i(\theta'-\theta)\eta} (D_\rho^j D_\eta^\alpha \partial_{r'}^j \partial_{\theta'}^\alpha A)(r, \theta, r, \theta, \rho, \eta) d\rho d\eta \\
&+ \Theta(r-r', \theta-\theta') \frac{1}{(2\pi)^n} \\
&\sum_{j+|\alpha|=N} \int e^{i(r'-r)\rho+i(\theta'-\theta)\eta} D_\rho^j D_\eta^\alpha R^{j,\alpha}(r, \theta, r', \theta', \rho, \eta) d\rho d\eta.
\end{aligned}$$

Moreover we notice that by definition  $A$  preserves the decay of  $a$  with respect to  $r, \rho$  and  $\eta$ , meaning that

$$A(r, \theta, r, \theta, \rho, \eta) \in \tilde{S}^{m,\mu}, \quad (D_\rho^j D_\eta^\alpha \partial_{r'}^j \partial_{\theta'}^\alpha A)(r, \theta, r, \theta, \rho, \eta) \in \tilde{S}^{m-|\alpha|, \mu-|\alpha|-j}.$$

Up to the remainder term, we have obtained integrals which are the kernels of pseudodifferential operators of symbol  $\tilde{S}^{m,\mu}$  as we wanted.

For the remainders we can prove that they are kernels of negligible operators in the sense of Definition 2.1. The same can be proved for the contribution of  $(1-\Theta)K_\gamma$  and this will allow us to conclude the proof.

We now proceed to show that composing on the left and right with  $Q_S := \langle r \rangle^S (P+1)^S$  the Taylor remainders results in a bounded operator for any fixed large  $S$  (we still denote by  $P$  the corresponding expression in local coordinates).

Here the key point is that we are conjugating by  $Q_S$  an operator whose kernel is of the form

$$\int \Theta(r-r', \theta-\theta') e^{i(r'-r)\rho+i(\theta'-\theta)\eta} D_\rho^j D_\eta^\alpha R^{j,\alpha}(r, \theta, r', \theta', \rho, \eta) d\rho d\eta. \quad (\text{C.9})$$

This kernel is smooth, with derivatives in  $(r', \theta')$  which are compactly supported and such that in the integral we have arbitrary fast decay in  $\rho$  and  $\eta$ , since we recall  $j+|\alpha|=N$  implies

$$|D_\rho^j D_\eta^\alpha R^{j,\alpha}| \lesssim \left(\langle \rho \rangle + \frac{\langle \eta \rangle}{\langle r \rangle}\right)^{\mu-N}.$$

Applying  $Q_S$  on the right, where  $(P+1)^S$  is selfadjoint, would lead us to differentiate the kernel in  $r'$  and  $\theta'$ , while applying it on the left means taking derivatives with respect to  $r$  and  $\theta$ . The ensemble of these actions still results in a kernel which is bounded together with its derivatives,

thanks to the properties we just stated. Notably, the decay in  $\rho$  and  $\eta$  allows to compensate the growth which is generated when taking derivatives of the oscillating factor.

We can therefore apply Calderon-Vaillancourt theorem (Theorem 2.8.1 in [Mar02]) to conclude that the corresponding operator is bounded. (Actually here one needs to apply Calderon-Vaillancourt to a suitably conjugated operator in order to have boundedness with respect to the appropriate  $L^2$  norm, that is  $L_G^2$ .)

The same proof can be adapted to the contribution away from the diagonal after rewriting it as

$$(1 - \Theta)(r - r', \theta - \theta')K_\gamma(r, \theta, r', \theta') = \frac{(1 - \Theta)(r - r', \theta - \theta')}{(r - r')^M(\theta - \theta')^\beta} \int e^{i(r-r')\rho + i(\theta-\theta')\eta} (D_\rho^M D_\eta^\beta A)(r, \theta, r', \theta', \rho, \eta) d\rho d\eta$$

where we performed again integration by parts thanks to

$$(r - r')^M (\theta - \theta')^\beta e^{i(r-r')\rho + i(\theta-\theta')\eta} = D_\rho^M D_\eta^\beta e^{i(r-r')\rho + i(\theta-\theta')\eta}.$$

Indeed,  $D_\rho^M D_\eta^\beta A$  also decays as

$$|D_\rho^M D_\eta^\beta A| \lesssim \left( \langle \rho \rangle + \frac{\langle \eta \rangle}{\langle r \rangle} \right)^{\mu - M - |\beta|}$$

with arbitrary  $M$  and  $\beta$  and hence the same arguments used for (C.9) can be replicated to conclude that  $Q_S(1 - \Theta)K_\gamma Q_S$  is an integral kernel corresponding to a negligible operator.  $\square$

## D A uniform bound for the spherical Laplacian

Consider the Laplace Beltrami operator on the unit sphere

$$-\frac{\Delta_{\mathbb{S}^{n-1}}}{r^2} = -\Delta_{\mathbb{R}^n} + \partial_r^2 + \frac{(n-1)}{r} \partial_r, \quad (\text{D.1})$$

given the relation between the radial and euclidean coordinates  $r = |x|$  we can derive

$$\partial_r = \frac{1}{|x|} \sum_j x_j \partial_j, \quad \partial_r^2 = \frac{1}{|x|^2} \sum_k x_k \sum_j x_j \partial_{j,k}^2$$

and therefore rewrite the operator in terms of euclidean derivatives as

$$-\frac{\Delta_{\mathbb{S}^{n-1}}}{r^2} = -\Delta_{\mathbb{R}^n} + \sum_{j,k=1}^n \frac{x_j x_k}{|x|^2} \partial_{j,k}^2 + (n-1) \sum_{j=1}^n \frac{x_j}{|x|^2} \partial_j. \quad (\text{D.2})$$

**Proposition D.1.** *Let  $n \geq 3$ , there exists a constant  $C_n > 0$  such that*

$$\left\| \frac{-\Delta_{\mathbb{S}^{n-1}}}{r^2} (-\Delta_{\mathbb{R}^n} + \lambda)^{-1} \right\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C_n. \quad (\text{D.3})$$

for all  $\lambda \in (0, \lambda_0]$ .

*Proof.* We use the expression in (D.2). By the spectral theorem  $-\Delta_{\mathbb{R}^n} (-\Delta_{\mathbb{R}^n} + \lambda)^{-1}$  is bounded and by elliptic regularity results

$$\left\| \frac{x_j x_k}{|x|^2} \partial_{j,k}^2 (-\Delta_{\mathbb{R}^n} + \lambda)^{-1} \right\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \lesssim \left\| \partial_{j,k}^2 (-\Delta_{\mathbb{R}^n} + \lambda)^{-1} \right\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)}$$

$\lesssim 1$ .

Finally, since  $n \geq 3$  we can apply Hardy inequality to conclude

$$\begin{aligned} \left\| \frac{x_j}{|x|^2} \partial_j (-\Delta_{\mathbb{R}^n} + \lambda)^{-1} \right\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} &\leq \left\| \frac{1}{|x|} (-\Delta_{\mathbb{R}^n} + \lambda)^{-\frac{1}{2}} \right\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \\ &\quad \cdot \left\| (-\Delta_{\mathbb{R}^n} + \lambda)^{-\frac{1}{2}} \partial_j \right\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \\ &\leq \frac{2}{n-2} \left\| \nabla (-\Delta_{\mathbb{R}^n} + \lambda)^{-\frac{1}{2}} \right\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \\ &\leq \frac{2}{n-2}. \end{aligned}$$

□

Let  $(\sigma_{l,n}^2, s_{l,n}(\theta))_j$  the set of eigenpairs of  $-\Delta_{\mathbb{S}^{n-1}}$ , representing  $u \in L^2(\mathbb{R}^n)$  in spherical coordinates and decomposing the angular part along the basis of eigenfunctions we have

$$\frac{-\Delta_{\mathbb{S}^{n-1}}}{r^2} (-\Delta_{\mathbb{R}^n} + \lambda)^{-1} u(r, \theta) = \sum_j \frac{\sigma_{l,n}^2}{r^2} \left( -\partial_r^2 - \frac{(n-1)}{r} \partial_r + \frac{\sigma_{l,n}^2}{r^2} + \lambda \right)^{-1} u_{l,n}(r) s_{l,n}(\theta)$$

where we have used for  $-\Delta_{\mathbb{R}^n}$  the expression we can derive from (D.1).

The uniform bound of the previous proposition translates to the following bound on one dimensional operators.

**Corollary D.2.** *Let  $n \geq 3$  and  $\sigma_{l,n}^2$  the  $l$ -th eigenvalue of the spherical Laplacian on  $\mathbb{S}^{n-1}$ , then there exists  $C_n > 0$  such that*

$$\left\| \frac{\sigma_{l,n}^2}{r^2} \left( -\partial_r^2 - \frac{(n-1)}{r} \partial_r + \frac{\sigma_{l,n}^2}{r^2} + \lambda \right)^{-1} \right\|_{L^2(\mathbb{R}^+, r^{n-1} dr) \rightarrow L^2(\mathbb{R}^+, r^{n-1} dr)} \leq C_n$$

for all  $\lambda \in (0, \lambda_0]$ .

We remark that the operator that appears in the statement is exactly  $p_{l,n}$  that we defined in the proof of Lemma 4.10.

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