Ergodic quasi-exchangeable stationary processes are isomorphic to Bernoulli processes

Doureid Hamdan

Ergodic quasi-exchangeable stationary processes are isomorphic to Bernoulli processes

Doureid Hamdan

Abstract

A discrete time process, with law μ, is quasi-exchangeable if for any finite permutation σ of time indices, the law μ^σ of the resulting process is equivalent to μ. For a quasi-exchangeable stationary process, our main results are (1) if the process is ergodic then it is isomorphic to a Bernoulli process and (2) if the family of all Radon-Nikodym derivatives $\{\frac{d\mu^\sigma}{d\mu}\}$ is uniformly integrable then the process is a mixture of Bernoulli processes, which generalizes De Finetti's Theorem. We give application of (1) to some determinantal processes.

1 Introduction, Notation

According to the generalization, by Hewitt and Savage [9], of De Finetti's classical Theorem, an exchangeable sequence $(X_n)_{n\geq 1}$ of random variables, with values in a presentable space, is a mixture of sequences of independent identically distributed random variables, meaning that the law of the process (X_n) is an integral of the laws of some family of independent processes (The family of presentable spaces include the polish or locally compact spaces).
In the present paper we consider, more generally, the class of stationary quasi-exchangeable sequences of random variables (Definition 2 below). For a stationary quasi-exchangeable sequence $(X_n)_{n\geq 1}$, we prove mainly two facts:
(1) Theorem 1: if the dynamical system generated by (X_n) is ergodic, then it is isomorphic to a Bernoulli system.
Our result (1) implies, Theorem 3, that the discrete time stationary quasi-invariant determinantal processes are isomorphic to Bernoulli processes. These determinantal processes

Keywords: Ergodic processes, quasi exchangeable sequence, translation-invariant determinantal process, Bernoulli system, De Finetti's Theorem.
2010 Mathematics Subject Classification: Primary 28D05, 37A05, 37A50, 60G09, 60G10. Secondary 60G55, 60J10.

contain the discrete time translation-invariant determinantal processes in the family considered by Bufetov in [4] (for example, the discrete sine process of Borodin Okounkov and Olshanski).
Also, we give(Corollary 2, Remark 7) a simple proof of De Finetti's Theorem, when the state space K is compact.
(2) Under the additional hypothesis that the family

$$\{\frac{d\mu^\sigma}{d\mu} : \sigma \ \textit{is a permutation of only finitely many coordinates}\}$$

of Radon-Nikodym derivatives, be uniformly integrable, we prove(Theorem 2) that the process is a mixture of i.i.d. sequences. This generalizes the De Finetti's Theorem.
We establish, first, some definitions and notations.

Definition 1
A sequence $(X_n)_{n\geq 1}$ *of random variables is exchangeable if the law* P^σ *of the process* $(X_{\sigma(n)})_{n\geq 1}$ *is equal to the law* P *of the process* $(X_n)_{n\geq 1}$*, for every permutation* σ *of the set* $\mathbb{N}$ *of natural numbers, which leaves fixed all but a finite number of integers.*

Suppose that for any $n \geq 1$, X_n takes values in the measurable space $(K, \mathcal{F})$. Then the sequence $(X_n)_{n\geq 1}$ is exchangeable, if and only if

$$P(X_1 \in A_1, ..., X_n \in A_n) = P(X_{\tau(1)} \in A_1, ..., X_{\tau(n)} \in A_n), \tag{1}$$

holds for all $n \geq 1$, $A_1, ..., A_n \in \mathcal{F}$ and any permutation τ of $\{1, ..., n\}$, or equivalently, if and only if for any permutation σ of $\{1, ..., n\}$ ($\tau = \sigma^{-1}$),

$$P(X_1 \in A_1, ..., X_n \in A_n) = P(X_1 \in A_{\sigma(1)}, ..., X_n \in A_{\sigma(n)}). \tag{2}$$

In the particular case where τ is defined by

$$\tau(k) = k + 1, \ \textit{for} \ 1 \leq k \leq n - 1, \ \ \textit{and} \ \ \tau(n) = 1,$$

and when $A_n = K$, equality (1) reads

$$P(X_1 \in A_1, ..., X_{n-1} \in A_{n-1}) = P(X_2 \in A_1, ..., X_n \in A_{n-1}) \qquad (\textit{shift-inv}). \tag{3}$$

which proves that the law P of an exchangeable sequence is invariant by the unilateral shift on $(K^{\mathbb{N}}, \mathcal{F}^{\otimes\mathbb{N}})$.

Let

$$\Omega := K^{\mathbb{Z}}, \ \ \textit{endowed with the product sigma algebra} \ \ \mathcal{B} := \mathcal{F}^{\otimes\mathbb{Z}}, \ \ \textit{and}$$
$$X'_n(\omega) = \omega_n, \ \ \textit{for all integer} \ n \in \mathbb{Z} \ \ \textit{and all} \ \ \omega \in \Omega.$$

Suppose that $(X_n)_{n\geq 1}$, with law P, is exchangeable. Define the measure μ on $(\Omega, \mathcal{B})$, which extends P, by setting for all $l, k \geq 0$, and $A_{-l}, ..., A_k \in \mathcal{F}$,

$$\mu(X'_{-l} \in A_{-l}, ..., X'_k \in A_k) := P(X_1 \in A_{-l}, ..., X_{k+l+1} \in A_k).$$

Then, by (3), for all $p \geq 1$,

$$\mu(X'_{-l} \in A_{-l}, ..., X'_k \in A_k) = P(X_p \in A_{-l}, ..., X_{k+l+p} \in A_k). \qquad (\textit{shift-inv})'. \tag{4}$$

so that μ extends to a probability measure on Ω, which is also invariant by the shift transformation $S:\ (S\omega)_n = \omega_{n+1},\ n \in \mathbb{Z},\ \omega \in \Omega$.
Let G be the group of all permutations of $\mathbb{Z}$ and $H \subset G$, be the subgroup of all permutations with finite support:

$$\sigma \in H \iff \sigma \in G,\ \textit{and}\ \exists N,\ \sigma(n) = n, \forall n, \mid n \mid \geq N. \qquad (a_1)$$

Let H_N be the subgroup of of all permutations which leaves fixed any n in $\{n \in \mathbb{Z} :\mid n \mid > N\}$:

$$H_N := \{\sigma \in G : \sigma(n) = n, \forall n,\ \textit{with}\ \mid n \mid > N\}, \qquad (a_2)$$

so that $H = \bigcup_N H_N$.

For any $\tau \in G$, let the transformation $T_\tau : \Omega \to \Omega$, be defined for all $\omega \in \Omega$, by

$$(T_\tau(\omega))_n = \omega_{\tau(n)},\ \forall n \in \mathbb{Z}.$$

Then T_τ is $\mathcal{B}$-measurable, and, when K is a topological space, T_τ is continuous for the product topology on Ω. Also for all σ and τ in H,

$$T_{\tau\circ\sigma} = T_\sigma \circ T_\tau, \tag{5}$$

from which follows that

$$T_\sigma^{-1} = T_{\sigma^{-1}}. \tag{6}$$

Now for every $\sigma \in H$, any N and $A_1, ..., A_{2N+1} \in \mathcal{F}$, one can see that the following equality holds

$$\mu(X'_{\sigma(-N)} \in A_1, ..., X'_{\sigma(N)} \in A_{2N+1}) = \mu(X'_{-N} \in A_1, ..., X'_N \in A_{2N+1}) \qquad (\textit{exchang}). \tag{7}$$

and that it is also equivalent to

$$\mu \circ T_\sigma^{-1}(X'_{-N} \in A_1, ..., X'_N \in A_{2N+1}) = \mu(X'_{-N} \in A_1, ..., X'_N \in A_{2N+1}). \tag{8}$$

In conclusion, the preceding shows that the exchangeability of the process $(X_n)_{n\geq 1}$ is the same as the exchangeability of the process $(X'_n)_{n\in\mathbb{Z}}$ and it is also equivalent to the invariance of μ, the law of $(X'_n)_{n\in\mathbb{Z}}$, by the transformation T_τ, for all $\tau \in H$, and in particular, implies, as noted before, the invariance of μ by the shift S. Henceforth, the process (X_n)

we consider will be indexed by $\mathbb{Z}$, and furthermore X_n will be the n^{th} coordinate function on $\Omega := K^{\mathbb{Z}}$.

A slight generalization of exchangeability is given by the following

Definition 2
We say that a sequence $(X_n)_{n\in\mathbb{Z}}$ of random variables, with law μ, is quasi exchangeable if $\mu \circ T_\sigma^{-1}$ is equivalent to μ, for all permutation $\sigma \in H$.
We note that also invariant or symmetric, are used in place of exchangeable, so that quasi invariance, quasi exchangeability, and quasi symmetry are all equivalent terms.

In this case we denote the Radon-Nikodym derivative of $\mu \circ T_\sigma^{-1}$ with respect to μ, by ϕ_σ

$$\phi_\sigma := \frac{d\mu \circ T_\sigma^{-1}}{d\mu}. \tag{9}$$

Since we are only interested exclusively in stationary sequences of random variables, the following remark may justify Definition 2.

Remark 1
Naturally we shall say that a unilateral sequence $(X_n)_{n\geq 1}$ of random variables is quasi-exchangeable if its law P is equivalent to the law P^σ of the sequence $(X_{\sigma(n)})_{n\geq 1}$, for every permutation σ of the set $\mathbb{N}$ of natural numbers, which leaves fixed all but a finite number of integers.
Then, if $(X_n)_{n\geq 1}$, with law P, is quasi-exchangeable and stationary, the sequence $(X'_n)_{n\in\mathbb{Z}}$, with law μ defined by (4), will be shift invariant and quasi-exchangeable.

Definition 3
Let $(X_n)_{n\in\mathbb{Z}}$ be a quasi exchangeable process, with law μ. If the family $\{\frac{d\mu\circ T_\sigma^{-1}}{d\mu} : \sigma \in H\}$ of all Radon-Nikodym derivatives is uniformly integrable, we say that the process X is quasi-exchangeable with uniformly integrable densities.

We shall also use the following notations.
If L and s are integers with $L \geq 0$ and $s \geq 1$, and $A_{-L}, ..., A_s$ are measurable subsets of K, we set

$$\Pi(A_{-L}, ..., A_0) := \{\omega \in \Omega : \omega_{-L} \in A_{-L}, ..., \omega_0 \in A_0\} \tag{10}$$
$$F(A_1, ..., A_s) := \{\omega \in \Omega : \omega_1 \in A_1, ..., \omega_s \in A_s\}. \tag{11}$$

and for all $I \subset \mathbb{Z}$,

$\mathcal{A}_I :=$ *the smallest algebra containing the sets* $\{\omega \in \Omega : \omega_j \in A\}, j \in I, A \in \mathcal{F}$,

and $\mathcal{B}_I :=$ *the sigma algebra generated by* $\mathcal{A}_I$.

In the following particular cases:

$$I = \{n \in \mathbb{Z} : n \leq p\} \quad \mathcal{A}_I \quad \textit{is denoted } \mathcal{A}_{\leq p}$$
$$I = \{n \in \mathbb{Z} : n \geq p\} \quad \mathcal{A}_I \quad \textit{is denoted } \mathcal{A}_{\geq p}, \ \forall p \in \mathbb{Z},$$
$$I = \mathbb{Z} \quad \mathcal{A}_I \quad \textit{is denoted } \mathcal{A}.$$

The same notation will be used for $\mathcal{B}_I$, in particular $\mathcal{B}_{\mathbb{Z}} = \mathcal{B}$.
Similarly, if μ is a measure on Ω, then μ_I denotes the restriction of μ to the sigma-algebra $\mathcal{B}_I$. Also, $M_1(Y)$ will denote the set of all Radon probability measures on the topological space Y, and if $(Z, \mathcal{G}, m)$ is a probability space and $\mathcal{G}_1$ is a sub sigma-algebra of $\mathcal{G}$, the conditional expectation of a function $f \in L^1(Z, \mathcal{G}, m)$ given $\mathcal{G}_1$ will be denoted by $E_m[f \mid \mathcal{G}_1]$, or, if there is no confusion on the measure m, simply by $E[f \mid \mathcal{G}_1]$.
Also, the smallest sigma-algebra which contains a finite number of sigma-algebras $\mathcal{F}_1, ..., \mathcal{F}_n$ [respectively a sequence $\mathcal{F}_n$ of sigma-algebras] is denoted by $\mathcal{F}_1 \vee \mathcal{F}_2 \vee ... \vee \mathcal{F}_n$ or by $\vee_{j=1}^n \mathcal{F}_j$ [respectively $\bigvee_n \mathcal{F}_n$]. The complement of a subset A is denoted A^c.

2 The main results

We shall first recall some definitions and some results which will be useful in this section.

Defintion 4
The probability spaces $(\Omega_1, \mathcal{F}_1, m_1)$, $(\Omega_2, \mathcal{F}_2, m_2)$ *are said to be isomorphic if there exist* $M_1 \in \mathcal{F}_1$, $M_2 \in \mathcal{F}_2$ *with* $m_1(M_1) = m_2(M_2) = 1$ *and an invertible measure preserving trasformation* $\phi : M_1 \to M_2$ *(where for* $i = 1, 2$, M_i *is equiped with the induced sigma-algebra* $\{M_i \cap B : B \in \mathcal{F}_i\}$, *and the restriction of the measure* m_i *to this sigma-algebra.)*

Defintion 5
A probability space $(\Omega, \mathcal{F}, m)$ *is a Lebesgue (or Lebesgue-Rohlin) space if it is isomorphic to a probability space which is the disjoint union of at most countable number of points* $\{y_1, y_2, ...\}$ *each of positive measure and the space* $([0, s], \mathcal{L}([0, s], \lambda)$ *where* $\mathcal{L}([0, s])$ *is the sigma-algebra of Lebesgue measurable subsets of* $[0, s]$, λ *is the Lebesgue measure and* $s = 1 - \sum_{n=1}^{\infty} m(\{y_n\})$.

Often in ergodic theory all probability spaces are assumed to be Lebesgue spaces. Note that a countable product of Lebesgue spaces is a Lebesgue space. Note also that if E is a complete separable metric space and $\mathcal{B}_m(E)$ denotes the completion of the Borel sigma-algebra $\mathcal{B}(E)$ of E under a probability measure m, then $(E, \mathcal{B}_m(E), m)$ is a Lebesgue space.

Definition 6
Let $(\Omega, \mathcal{F}, m)$ *be a probability space. The relation* $=_m$ *defined on* $\mathcal{F}$ *by* $A =_m B \iff m(A\Delta B) = 0$ *is an equivalence relation. For any* $A \in \mathcal{F}$, *let* $\tilde{A}$ *denotes the equivalence class to which* A *belongs, and* $\tilde{\mathcal{F}} := \{\tilde{A} : A \in \mathcal{F}\}$. *Then* $\tilde{\mathcal{F}}$ *is a Boolean* σ*-algebra under the operations of complementation, union and intersection inherited from* $\mathcal{F}$ *. The measure* m *induces a measure* $\tilde{m}$ *on* $\tilde{\mathcal{F}}$ *by* $\tilde{m}(\tilde{A}) = m(A)$. *The pair* $(\tilde{\mathcal{F}}, \tilde{m})$ *is the measure algebra corresponding to the probability space* $(\Omega, \mathcal{F}, m)$ *.*

Note that the function $d(\tilde{A}, \tilde{B}) := m(A\Delta B)$, defines a distance(the Nikodym distance) on $\tilde{\mathcal{F}}$. We say that $\tilde{\mathcal{F}}$ is separable if the metric space $(\tilde{\mathcal{F}}, d)$ is separable.

Defintion 7
The measure algebras $\tilde{\mathcal{F}}_1$ *and* $\tilde{\mathcal{F}}_2$ *of the probability spaces* $(\Omega_1, \mathcal{F}_1, m_1)$, $(\Omega_2, \mathcal{F}_2, m_2)$ *are said to be isomorphic if there is a bijection* $\Phi : \tilde{\mathcal{F}}_2 \to \tilde{\mathcal{F}}_1$ *which preserves complements, countable unions and intersections and satisfies* $\tilde{m}_1(\Phi\tilde{A}) = \tilde{m}_2(\tilde{A}), \forall \tilde{A} \in \tilde{\mathcal{F}}_2$.
The probability spaces are said to be conjugate if their measure algebras are isomorphic.

Definition 8
Let, for $i = 1, 2$ $(\Omega_i, \mathcal{F}_i, m_i)$ *a probability space and* $T_i : \Omega_i \to \Omega_i$ *a measure preserving transformation.*
a) We say that T_1 *is isomorphic to* T_2 *if*
(a_1) *there exist* $M_1 \in \mathcal{F}_1$, $M_2 \in \mathcal{F}_2$ *with* $m_1(M_1) = m_2(M_2) = 1$ *such that* $T_1 M_1 \subset M_1$, $T_2 M_2 \subset M_2$ *and*
(a_2) *there is an invertible measure preserving transformation* $\phi : M_1 \to M_2$ *with* $\phi(T_1 x) = T_2\phi(x), \forall x \in M_1$.
b) If (a_1) *holds and* (a_2) *holds with* ϕ *not necessarily invertible we say that* T_2 *is a factor of* T_1.

Clearly isomorphism between measure preserving transformations is stronger than conjugacy. For Lebesgue spaces the two notions coincide.

Remark 2
suppose that T_2 is a factor of T_1 and let ϕ the factor map. Let

$$\Phi : \tilde{\mathcal{F}}_2 \to \tilde{\mathcal{F}}_1, \ \ \Phi(\tilde{A}) = \widetilde{\phi^{-1}(A)}, \forall A \in \mathcal{F}_2. \qquad (a_3)$$

Then Φ is an injetive homomorphism of measure algebras. In fact it preserves complements and countable unions and hence countable intersections: this follows from the fact that ϕ^{-1} preserves these operations.

The following implications which hold true for all $A, B \in \mathcal{F}_2$,

$$\Phi(\tilde{A}) = \Phi(\tilde{B}) \iff \phi^{-1}(A) =_{m_1} \phi^{-1}(B) \iff \int \mid 1_{\phi^{-1}A} - 1_{\phi^{-1}B} \mid dm_1 = 0$$

$$\iff \int \mid 1_A - 1_B \mid dm_2 = 0 \iff \tilde{A} = \tilde{B},$$

show that Φ is well defined and injective.
It follows that Φ is a bijection from $\tilde{\mathcal{F}}_2$ onto its image $\Phi(\tilde{\mathcal{F}}_2)$. Also we have

$$\tilde{m_1}(\Phi(\tilde{A})) = m_1(\phi^{-1}(A) = m_2(A) = \tilde{m_2}(\tilde{A}),\ \forall A \in \mathcal{F}_2.$$

We conclude that the two measure algebras $(\tilde{\mathcal{F}}_2, \tilde{m_2})$ and $(\Phi(\tilde{\mathcal{F}}_2), \tilde{m_1})$ are isomorphic, the isomorphism being given by Φ as in (a_3) above.

Defnition 9: ([24], Définition 4.11, p. 105)
Let $(K, \mathcal{F}, m)$ be a probability space. Let $(\Omega, \mathcal{G}, \mu) = \prod_{n=-\infty}^{\infty}(K, \mathcal{F}, m)$ and T be the shift: $(T\omega)_n = \omega_{n+1}$, $\forall \omega \in \Omega, n \in \mathbb{Z}$. The dynamical system $(\Omega, \mathcal{G}, \mu, T)$ is called the Bernoulli shift with state space $(K, \mathcal{F}, m)$.

Remark 3
A Bernoulli shift with Lebesgue state space is called a generalized Bernoulli shift in [18], and also the following equivalent definition is given there:

Dfinition 9'
Let T be a one-to-one, invertible measure-preserving transformation on a separable measure algebra $\tilde{\mathcal{G}}$ corresponding to the Lebesgue probability space $(\Omega, \mathcal{G}, \mu)$. We say that T is a generalized Bernoulli shift if there is a sub-σ-algebra $\mathcal{E}$ such that the $T^n\mathcal{E}$, $n \in \mathbb{Z}$,
(i) are independent in the sense that for every $n \geq 1$ and $A_{-n} \in T^{-n}\mathcal{E}, ..., A_n \in T^n\mathcal{E}$ we have $\mu(\bigcap_{j=-n}^{n} A_j) = \prod_{j=-n}^{n} \mu(A_j)$, and
(ii) generate the full measure algebra in that for each A in $\mathcal{G}$, there is a sequence of sets $A_n \in \vee_{j=-n}^{n} T^j\mathcal{E}$, such that $\lim_n \mu(A \Delta A_n) = 0$.

We shall use the following Theorem:

Theorem A([18], Theorem 2)
Let T be a one-to-one invertible measure preserving transformation on a measure algebra $\mathcal{L}$. If $\mathcal{L}$ is the increasing union of invariant sub-σ-algebras $\mathcal{L}_i$ such that T restricted to each $\mathcal{L}_i$ is a Bernoulli shift, then T is a generalized Bernoulli shift.

Remark 4
One should note that, with the notation in Theorem A, $\mathcal{L} = \tilde{\mathcal{F}}$ where $\mathcal{F} = \vee_n \mathcal{L}_n$, for in

fact this measure algebra $\tilde{\mathcal{F}}$ is the smallest measure algebra containing the union $\cup_n \mathcal{L}_n$.

Theorem A is reformulated in [24] as

Theorem A' ([24], Theorem 4.29, p.106)
Let $(X, \mathcal{B}, m)$ be a Lebesgue space and $T : X \to X$ an invertible measure preserving transformation. Then, if $\mathcal{F}_n$ is an increasing sequence of sub–σ-algebras of $\mathcal{B}$ with $T\mathcal{F}_n = \mathcal{F}_n$, which generates $\mathcal{B}$, and if the factor transformation associated with each $\mathcal{F}_n$ is conjugate to a Bernoulli automorphism then $(X, \mathcal{B}, T, m)$ is conjugate to a Bernoulli automorphism.

Definition 10 ([12], Définition 3)
Let K be a finite set, $\Omega := K^{\mathbb{Z}}$, S the shift on Ω, and μ a shift invariant probability measure on Ω.
The system $(\Omega, \mathcal{B}(\Omega), S, \mu)$ is said to be $\leq\leq$faiblement de Bernoulli$\geq\geq$ if the two measures μ and $\mu_{\leq -1} \times \mu_{\geq 0}$ coincide on the two sided tail sigma algebra $\mathcal{T} := \bigcap_{n\geq 0}(\bigvee_{k\leq -n} \mathcal{B}_k \vee \bigvee_{k\geq n} \mathcal{B}_k)$.

Let $\nu := \mu_{\leq -1} \times \mu_{\geq 0}$. Then

$$\nu \circ S^{-m} = \mu_{\leq -1-m} \times \mu_{\geq -m}. \qquad (a_4)$$

In fact, for any $l, s \in \mathbb{Z}$, with $l \leq s$, and any subsets $A_l, ...A_s$ of K, let $C := \{\omega \in \Omega : \omega_l \in A_l, ..., \omega_s \in A_s\}$, which we write as $C = (\omega_l \in A_l, ..., \omega_s \in A_s)$, so that $S^{-m}C = (\omega_{l+m} \in A_l, ..., \omega_{m+s} \in A_s)$.
If $0 \leq l + m$, we have $S^{-m}C \in \mathcal{B}_{\geq 0}$, and then

$$\begin{aligned}\nu(S^{-m}C) &= \mu_{\geq 0}(S^{-m}C) = \mu(S^{-m}C) = \mu(C) = \mu(\omega_l \in A_l, ..., \omega_s \in A_s) \\ &= \mu_{\geq -m}(\omega_l \in A_l, ..., \omega_s \in A_s) = \mu_{\geq -m}(C) = \mu_{\leq -m-1} \times \mu_{\geq -m}(C).\end{aligned}$$

Similarly, if $m + s \leq -1$ we have $S^{-m}C \in \mathcal{B}_{\leq -1}$, and then also

$$\nu(S^{-m}C) = \mu_{\leq -m-1} \times \mu_{\geq -m}(C).$$

If $l + m < 0$ and $m + s \geq 0$, then

$$\begin{aligned}\nu(S^{-m}C) &= \nu(\omega_{l+m} \in A_l, ..., \omega_{-1} \in A_{-m-1}, \omega_0 \in A_{-m}, ..., \omega_{s+m} \in A_s) \\ &= \mu(\omega_{l+m} \in A_l, ..., \omega_{-1} \in A_{-m-1})\mu(\omega_0 \in A_{-m}, ..., \omega_{s+m} \in A_s) \\ &= \mu(\omega_l \in A_l, ..., \omega_{-m-1} \in A_{-m-1})\mu(\omega_{-m} \in A_{-m}, ..., \omega_s \in A_s) \\ &= \mu_{\leq -m-1} \times \mu_{\geq -m}(C),\end{aligned}$$

which ends the proof of (a_4).

Also, from

$$S^{-1}\mathcal{B}_k = \mathcal{B}_{k+1},$$

one can see that

$$S^{-1}\mathcal{T} = \mathcal{T},$$

and therefore, if μ and $\mu_{\leq -1} \times \mu_{\geq 0}$ coincide on $\mathcal{T}$, we get, by (a_4), for any $E \in \mathcal{T}$,

$$\mu(E) = \mu(S^{m+1}E) = \nu(S^{m+1}E) = \mu_{\leq m} \times \mu_{\geq m+1}(E).$$

Hence we have the following

Remark 5
The following are equivalent

(i) The system (Ω, S, μ) is faiblement de Bernoulli.
(ii) for any m, μ and $\mu_{\leq m} \times \mu_{\geq m+1}$ coincide on $\mathcal{T}$.
(iii) For some m, μ and $\mu_{\leq m} \times \mu_{\geq m+1}$ coincide on $\mathcal{T}$.

In the following theorem, which is the main result of this section, we suppose that K is a (Polish) complete separable metric space and $\mathcal{F} = \mathcal{B}(K)$, the Borel sigma algebra of K.

Theorem 1
Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary quasi exchangeable process, with law μ, such that the dynamical system (Ω, S, μ) is ergodic. Then the process is isomorphic to a Bernoulli process.
In the particular case where the state space K is finite the system (Ω, S, μ) is "faiblement de Bernoulli".

Proof We consider first the case where the state space K is finite, and then we prove that (Ω, S, μ) is "faiblement de Bernoulli". For this, we shall use the ergodicity and the quasi-exchangeability under a particular infinite family of permutations, in order to find a sequence of measures converging to $\mu_{\leq 0} \times \mu_{\geq 1}$ on all cylinders and such that any measure in the sequence, coincides with μ on the two-sided tail sigma-algebra $\mathcal{T} := \bigcap_n (\mathcal{B}_{\leq -n} \vee \mathcal{B}_{\geq n})$. The details follow.
For all natural numbers k and P, such that $1 \leq k < P$, let us consider the permutation (involution) $\sigma := \sigma_{P,k} \in H$ which translates the "interval" $I := \mathbb{N} \cap [1, ..., k]$ by P, translates the "interval" $P + I = \mathbb{N} \cap [P + 1, ..., P + k]$ by $-P$, and leaves fixed all $n \in \mathbb{Z}$, which are not in the disjoint union $I \cup (P + I)$, that is, which is defined by

$$\sigma(j) = P + j, \ \textit{and} \ \sigma(P + j) = j, \ \textit{if} \ 1 \leq j \leq k$$
$$\sigma(n) = n \ \textit{if} \ n \notin \{1, ..., k\} \cup \{P + j : j = 1, ..., k\},$$

so that

$$T_\sigma(\omega)_n = \omega_n, \ \textit{if} \ n \notin \{1, ..., k\} \cup \{P + j : j = 1, ..., k\}$$
$$T_\sigma(\omega)_j = \omega_{P+j} \quad \textit{and} \quad T_\sigma(\omega)_{P+j} = \omega_j \quad \textit{for} \quad 1 \leq j \leq k.$$

Note first that for every $M \in \mathcal{T}$, and any $\tau \in H$, we have $T_\tau^{-1} M = M$. Next let $P_0 \in \mathbb{N}$ such that $1 \leq P_0 - k$, be fixed, and recall that $\phi_\sigma := \frac{d\mu \circ T_\sigma^{-1}}{d\mu}$. For all integers $L \geq 0$, and m with $1 \leq m \leq P_0 - k$, consider any measurable subsets $A_{-L}, ..., A_k$, $B_1, ..., B_k$, $C_1, ..., C_m$ of K, and also any $M \in \mathcal{T}$. Let $P \geq P_0$. Using the notation as in (10) and (11), and setting

$$E = \Pi(A_{-L}, ..., A_0) := \{\omega \in \Omega : \omega_{-L} \in A_{-L}, ..., \omega_0 \in A_0\}, \tag{12}$$

the equality

$$\mu(T_\sigma^{-1}(M \cap E \cap F(A_1, ..., A_k) \cap S^{-k}F(C_1, ..., C_m) \cap S^{-P}F(B_1, ..., B_k)) = \\ \mu(M \cap E \cap F(B_1, ..., B_k) \cap S^{-k}F(C_1, ..., C_m) \cap S^{-P}F(A_1, ..., A_k))$$

which holds by the definitions of σ and T_σ, reads also, in view of quasi invariance, as

$$\int_{M \cap E \cap F(A_1,...,A_k) \cap S^{-k}F(C_1,...,C_m)} 1_{F(B_1,...,B_k)} \circ S^P \phi_\sigma d\mu = \\ \int_{M \cap E \cap F(B_1,...,B_k) \cap S^{-k}F(C_1,...,C_m)} 1_{F(A_1,...,A_k)} \circ S^P d\mu. \tag{13}$$

For every integer $Q \geq P_0$, consider the following two functions $\xi^Q_{B_1,...,B_k} \in L^1(\mu)$ and $\psi^Q_{A_1,...,A_k} \in L^\infty(\mu)$, defined by

$$\xi^Q_{B_1,...,B_k} := \frac{1}{Q} \sum_{P=P_0}^{Q} 1_{F(B_1,...,B_k)} \circ S^P \phi_{\sigma_{P,k}},$$

$$\textit{and} \quad \psi^Q_{A_1,...,A_k} := \frac{1}{Q} \sum_{P=P_0}^{Q} 1_{F(A_1,...,A_k)} \circ S^P.$$

Then, by (13), we obtain

$$\int_{M \cap E \cap F(A_1,...,A_k) \cap S^{-k}F(C_1,...,C_m)} \xi^Q_{B_1,...,B_k} d\mu = \\ \int_{M \cap E \cap F(B_1,...,B_k) \cap S^{-k}F(C_1,...,C_m)} \psi^Q_{A_1,...,A_k} d\mu,$$

which we write as

$$\int_{M \cap E \cap S^{-k}F(C_1,...,C_m)} 1_{F(A_1,...,A_k)} \xi^Q_{B_1,...,B_k} d\mu = \\ \int_{M \cap E \cap S^{-k}F(C_1,...,C_m)} 1_{F(B_1,...,B_k)} \psi^Q_{A_1,...,A_k} d\mu.$$

Since this last equality holds true for all $L \geq 1$, $1 \leq m \leq P_0 - k$, all E as in (12) and all $C_1, ..., C_m$, and any $M \in \mathcal{T}$, we have the equality

$$E[1_{F(A_1,...,A_k)} \xi^Q_{B_1,...,B_k} \mid \mathcal{T} \vee \mathcal{B}_{\leq 0} \vee \mathcal{B}_{\{k+1,...,P_0\}}] = E[1_{F(B_1,...,B_k)} \psi^Q_{A_1,...,A_k} \mid \mathcal{T} \vee \mathcal{B}_{\leq 0} \vee \mathcal{B}_{\{k+1,...,P_0\}}]]. \tag{14}$$

Now, k, P_0 being fixed, by ergodicity, the sequence $(\psi^Q_{A_1,...,A_k})_{Q\geq P_0}$ converges in $L^2(\mu)$ norm to the constant $\mu(F(A_1,...,A_k))$. Then, by Cauchy Schwartz for example, the sequence $(1_{F(B_1,...,B_k)}\psi^Q_{A_1,...,A_k})_{Q\geq P_0}$ converges in $L^2(\mu)$ norm also to $\mu(F(A_1,...,A_k))1_{F(B_1,...,B_k)}$. It follows that the sequence in the left side of (14) converges in norm $L^2(\mu)$,
to $\mu(F(A_1,...,A_k))E[1_{F(B_1,...,B_k)} \mid \mathcal{T}\vee\mathcal{B}_{\leq 0}\vee\mathcal{B}_{\{k+1,...,P_0\}}]$.

In particular,

$$\lim_Q \sup_{M\in\mathcal{T}\vee\mathcal{B}_{\leq 0}\vee\mathcal{B}_{\{k+1,...,P_0\}}} \mid \int_{M\cap F(A_1,...,A_k)} \xi^Q_{B_1...,B_k}d\mu - \mu(M\cap F(B_1,...,B_k))\mu(F(A_1,...,A_k)) \mid = 0. \quad (15)$$

The sequence $(\xi^Q_{(B_1,...,B_k)})_{Q\geq P_0}$, is bounded in $L^1(\mu)$:

$$|| \xi^Q_{(B_1,...B_k)} ||_1 \leq || \frac{1}{Q}\sum_{P=P_0}^{Q} \phi_{\sigma_{P,k}} ||_1 = \frac{Q-k}{Q} \leq 1.$$

Then it is bounded in the bidual $L^{\infty *}(\mu)$ of $L^1(\mu)$. Hence, by Alaoglu-Bourbaki Theorem([7], Theorem 2, p.424), this sequence has at least one weak-star cluster point. Let $\eta^k_{(B_1,...,B_k)} \in L^{\infty *}(\mu)$ be such a cluster point. Then $\eta^k_{B_1,...,B_k}$ is positive, because for each Q, $\xi^Q_{(B_1,...,B_k)}$ is positive. Also, for any $x^* \in L^\infty(\mu)$, there exists a subsequence of natural numbers $(Q_j)_{j\geq 1} = (Q_j(x^*))_{j\geq 1}$, which may depend on x^*, converging to infinity such that

$$\eta^k_{(B_1,...,B_k)}(x^*) = \lim_j \xi^{Q_j(x^*)}_{B_1,...,B_k}(x^*).$$

Taking, in particular, $x^* = x^*_0$, where $x^*_0 = 1_{M\cap F(A_1,...,A_k)}$, we obtain

$$\eta^k_{(B_1,...,B_k)}(M\cap F(A_1,...,A_k)) = \lim_j \int_{M\cap F(A_1,...,A_k)} \xi^{Q_j(x^*_0)}_{B_1,...,B_k}d\mu. \quad (16)$$

Then, in view of (15), the limit in (16) is independent of the sequence $(Q_j(x^*_0))$, and the following equality holds

$$\eta^k_{B_1,...,B_k}(M\cap F(A_1,...,A_k)) = \mu(M\cap F(B_1,...,B_k))\mu(F(A_1,...,A_k)) \quad (17)$$

for all $M \in \mathcal{T}\vee\mathcal{B}_{\leq 0}\vee\mathcal{B}_{\{k+1,...,P_0\}}$.
In the particular case, where $B_1 = ... = B_k = K$, let us denote $\eta^k_{B_1,...,B_k}$ simply by η^k. Then, by (17),

$$\eta^k(M\cap F(A_1,...,A_k)) = \mu(M)\mu(F(A_1,...,A_k)), \forall M \in \mathcal{T}\vee\mathcal{B}_{\leq 0}\vee\mathcal{B}_{\{k+1,...,P_0\}}, \quad (18)$$

which means that, under η^k, the algebra $\mathcal{A}_{\{1,...,k\}}$ and the sigma algebra $\mathcal{T}\vee\mathcal{B}_{\leq 0}\vee\mathcal{B}_{\{k+1,...,P_0\}}$ are independent(Note that as element of $L^{\infty *}(\mu)$, η^k may have a non null purely finitely additive part). Set

$$\mathcal{M} = \mathcal{T}\vee\mathcal{B}_{\leq 0}.$$

Then by (18), we have in particular

$$\eta^k(A \cap M) = \mu(A)\mu(M),\ \forall A \in \mathcal{A}_{\{1,\dots,k\}},\ \forall M \in \mathcal{M}. \tag{19}$$

Also, if we denote by $\mathcal{D}_k$, the smallest algebra containing $\mathcal{M} \cup \mathcal{A}_{\{1,\dots,k\}}$, then $\mathcal{D}_k \subset \mathcal{D}_{k+1}$, and (19) implies that η^{k+1} extends η^k from $\mathcal{D}_k$ to $\mathcal{D}_{k+1}$. By induction we then have, for every $n \geq 0$,

$$\eta^{k+n}(A \cap M) = \mu(A)\mu(M),\ \forall A \in \mathcal{A}_{\{1,\dots,k\}},\ \forall M \in \mathcal{M}. \tag{20}$$

Now, the sequence $(\eta^k)_{k\geq 1}$ is norm bounded in $L^{\infty *}(\mu)$, since by positivity and by (18) we have

$$||\ \eta^k\ ||_{L^{\infty *}(\mu)} = \eta^k(\Omega) = 1.$$

Then $(\eta^k)_{k\geq 1}$ has at least one weak-star cluster point. By (20), every such cluster point, say η, satisfies

$$\eta(A \cap M) = \mu(A)\mu(M),\ \forall A \in \mathcal{A}_{\{1,\dots,k\}},\ \forall k \geq 1,\ \forall M \in \mathcal{M}. \tag{21}$$

which means

$$\eta(A \cap M) = \mu(A)\mu(M),\ \forall A \in \mathcal{A}_{\geq 1},\ \ \forall M \in \mathcal{M}.$$

In particular

$$\eta(A \cap B) = \mu(A)\mu(B),\ \ \forall A \in \mathcal{B}_{\leq 0},\ \ \forall B \in \mathcal{A}_{\geq 1}. \tag{22}$$

Notice that (22) implies that η and $\mu_{\leq 0} \times \mu_{\geq 1}$ coincide on all cylinders. We recall that, for any subset $I \subset \mathbb{Z}$, μ_I denotes the restriction of μ to the sigma algebra $\mathcal{B}_I$. Also (21) implies that η and μ coincide on $\mathcal{T}$.
It follows that if $\mathcal{L}$ is the algebra generated by $\mathcal{M} \cup \mathcal{A}_{\geq 1}$, then clearly $\mathcal{L}$ contains $\mathcal{A}$ and, by (21), that η and $\mu_{\leq 0} \times \mu_{\geq 1}$ coincide also on $\mathcal{L}$, so that η is countably additive on $\mathcal{L}$. We also note that $\mathcal{B}$ is the smallest sigma-algebra containing $\mathcal{L}$. Then the unique countably additive measure $\tilde{\eta}$ extending η to the sigma algebra $\mathcal{B}$, is the measure $\mu_{\leq 0} \times \mu_{\geq 1}$.
Then, in particular, $\tilde{\eta}$ satisfies

$$\tilde{\eta}(A \cap B) := \mu(A)\mu(B),\ \ A \in \mathcal{B}_{\leq 0},\ \ B \in \mathcal{B}_{\geq 1}. \tag{23}$$

Since $\mathcal{T} \subset \mathcal{M}$, we also have, by (21), as noted before,

$$\tilde{\eta}(M) = \eta(M) = \mu(M),\ \ \forall M \in \mathcal{T}. \tag{24}$$

In conclusion, the countably additive measure $\tilde{\eta}$ on $\mathcal{B}$ satisfies the equalities (23) and (24), which means, in view of Remark 5, when the state space K is finite, that the system (Ω, S, μ) is "faiblement de Bernoulli". Since in this case, "faiblement de Bernoulli" is equivalent to weak Bernoulli ([12], Proposition 2) and also, a system which is weak Bernoulli is isomorphic

to a Bernoulli system [17], the proof is complete in this finite case.

To end the proof in the general case, we establish first the following Lemma.

Lemma 1
Let $\Omega = K^{\mathbb{Z}}$, S the shift on Ω and μ be an S-invariant probability measure on Ω. Let X_n be the n^{th} coordinate function on Ω and $\Pi = \{A_0, ..., A_{k-1}\}$ be a finite measurable partition of K.
Let $\Omega_1 := \{0, ..., k-1\}^{\mathbb{Z}}$, S_1 the shift on Ω_1, Y_n the n^{th} coordinate function on Ω_1 and $\theta : \Omega \to \Omega_1$ the factor map defined by:

$$\theta(x) = y \iff (S^n x)_0 \in A_{y(n)}, \forall n \in \mathbb{Z} \iff x_n \in A_{y(n)}, \forall n \in \mathbb{Z},$$

for any $x = (x_n)_{n\in\mathbb{Z}} \in \Omega$, that is

$$\theta(x)(n) = j \iff (S^n x)_0 \in A_j \iff x_n \in A_j.$$

Let $\nu = \mu \circ \theta^{-1}$. Then
(1) the quasi-exchangeability[respectively exchangeability] of the process $X = (X_n)$ implies the same property for the process $Y = (Y_n)$.
(2) the ergodicity of the process (X_n) implies the ergodicity of (Y_n).

Proof: (1) Note first that ν is the law of the process Y. For any permutation σ, let the transformation $R_\sigma : \Omega_1 \to \Omega_1$, defined by

$$R_\sigma(y)(n) = y(\sigma(n)), \forall n \in \mathbb{Z}, \forall y \in \Omega_1.$$

Then the quasi-exchangeability[respectively exchangeability] of (Y_n) follows from the commutation relationship

$$R_\sigma \circ \theta = \theta \circ T_\sigma$$

because if this relationship holds we will get

$$\nu \circ R_\sigma^{-1} = \mu \circ \theta^{-1} \circ R_\sigma^{-1} = \mu \circ T_\sigma^{-1} \circ \theta^{-1}$$

and then

$$\nu \circ R_\sigma^{-1}(A) = 0 \iff (\mu \circ T_\sigma^{-1})(\theta^{-1}A) = 0 \iff \mu(\theta^{-1}A) = 0 \iff \nu(A) = 0.$$

[Respectively

$$\nu \circ R_\sigma^{-1}(A) = (\mu \circ T_\sigma^{-1})(\theta^{-1}A) = \mu(\theta^{-1}A) = \nu(A).]$$

The commutation relation is a consequence of the following

$$\begin{aligned} R_\sigma(\theta(x))(n) = j &\iff \theta(x)(\sigma(n)) = j \iff x_{\sigma(n)} \in A_j \\ &\iff T_\sigma(x)(n) \in A_j \iff \theta(T_\sigma(x))(n) = j \end{aligned}$$

where $x \in \Omega, n \in \mathbb{Z}$, and $j = 0, 1, ..., k-1$.
(2) The dynamical system (Ω_1, S_1, ν) is a factor of (Ω, S, μ), for in fact for all $n \in \mathbb{Z}, x \in \Omega$ and $0 \leq j < k$, the following implications

$$(\theta(Sx))_n = j \iff (Sx)_n \in A_j \iff x_{n+1} \in A_j \qquad \textit{and}$$
$$(S_1(\theta x))_n = j \iff (\theta x)_{n+1} = j \iff x_{n+1} \in A_j$$

show the equality $\theta \circ S = S_1 \circ \theta$. Then (2) follows because a factor of an ergodic system is ergodic.

Corollary 1
Under the same conditions of the above Lemma, if the process X with law μ is quasi-exchangeable and the system (Ω, S, μ) is ergodic, then the system $(\Omega_1, S_1, \mu \circ \theta^{-1})$ is isomorphic to a Bernoulli system.

Proof The result follows immediately by Lemma 1 from the finite case.

We continue the proof of the theorem in the general case where K is a polish space and $\mathcal{F} = \mathcal{B}(K)$. To do this, for every finite partition $\Pi = \{A_0, ..., A_{k-1}\}$ of K, where each A_j is in $\mathcal{F}$, consider the corresponding time zero partition $\mathcal{P} = \mathcal{P}(\Pi)$ of $\Omega := K^{\mathbb{Z}}$, defined by

$$\mathcal{P} = \mathcal{P}(\Pi) := \{\{\omega \in K^{\mathbb{Z}} : \omega_0 \in A_0\}, ..., \{\omega \in K^{\mathbb{Z}} : \omega_0 \in A_{k-1}\}\},$$

and let

$$\mathcal{P}_S := \bigvee_{-\infty}^{\infty} S^n \mathcal{P}$$

be the corresponding invariant sigma-algebra.
Denote the cells of the time zero partition $\mathcal{P}(\Pi)$ by $P_0, ..., P_{k-1}$, specifically

$$P_j := \{\omega \in \Omega : \omega_0 \in A_j\}, \ \forall j = 0, ..., k-1,$$

and note then that for every $n \in \mathbb{Z}$,

$$S^n P_j = \{S^n \omega : \omega_0 \in A_j\} = \{\omega \in K^{\mathbb{Z}} : \omega_{-n} \in A_j\}.$$

On the other hand with the notation as in corollary 1, the dynamical system $(\Omega_1, \mathcal{B}(\Omega_1), S_1, \nu)$ is $^{\leq\leq}$ faiblement de Bernoulli$^{\geq\geq}$ and hence isomorphic to a Bernoulli system. Then, Ω_1 being compact metric so that $(\Omega_1, \mathcal{B}(\Omega_1), \nu)$ is a Lebesgue space, it follows according to Definition 5', that there exists a sub sigma algebra $\mathcal{E}$ of $\mathcal{B}(\Omega_1)$ which satisfies the conditions (i) and (ii) in that definition. Also, by Lemma 1, $(\Omega_1, \mathcal{B}(\Omega_1), S_1, \nu)$ is a factor of (Ω, S, μ), the factor map being θ, so that $\theta^{-1}\mathcal{E}$ satisfies the same two conditions (i) and (ii) above in (Ω, S, μ).
Clearly, because that the following implications

$$\theta(x)(n) = j \iff x_n \in A_j \iff (S^n x)_0 \in A_j \iff S^n x \in P_j \iff x \in S^{-n} P_j$$

hold for every $n \in \mathbb{Z}$, and any $j = 0, ..., k-1$, we have, for any $a \in \Omega_1 = \{0, 1, ..., k-1\}^{\mathbb{Z}}$, and every cylinder $C := \{y \in \Omega_1 : y_{-p} = a_{-p}, ..., y_q = a_q\}$,

$$\theta(x) \in C \iff \theta(x)(n) = a_n, \forall n = -p, ..., q \iff x \in \cap_{n=-p,...,q} S^{-n} P_{a_n}.$$

Then $\theta^{-1}\mathcal{B}(\Omega_1) = \vee_{n\in\mathbb{Z}} S^n\mathcal{P}$, and therefore $\theta^{-1}(\mathcal{E}) \subset \vee_{n\in\mathbb{Z}} S^n\mathcal{P}$.
It follows then, according to Remark 2, that the two measure algebras $(\tilde{\mathcal{P}}_S, \tilde{\mu})$ and $(\widetilde{\mathcal{B}(\Omega_1)}, \widetilde{\mu \circ \theta^{-1}})$ are isomorphic. Hence $(K^{\mathbb{Z}}, S, \mathcal{P}_S, \mu)$ is conjugate to a Bernoulli shift.

We prove now the following claim.
Claim
There exists an increasing sequence $\Pi_n := \{A_0^n, ..., A_{p_n}^n\}$ *of finite partitions of* K *whose elements are in* $\mathcal{F}$, *such that* $\mathcal{F} = \vee_n \sigma(\Pi_n)$.

In the following proof of this claim we shall use the notation:

$\sigma(D) :=$ *the smallest sigma algebra containing the family* D *of subsets of a set, and*
$\sigma(V) :=$ *the smallest sigma algebra for which the random variable* V *is measurable.*

We observe first that it suffices to prove this claim when the state space is a complete separable metric space, for in fact if $f : K_1 \to K_2$ is a homeomorphism, and $\Pi_1 = \{A_0, ..., A_{k-1}\}$ is a finite partition of K_1, by Borel sets, then $\{f(A_0), ..., f(A_{k-1})\}$ is a partiition of K_2 by Borel sets also, and evidently $\mathcal{B}(K_2) = f(\mathcal{B}(K_1)) := \{f(A) : A \in \mathcal{B}((K_1)\}$.
Suppose then that K is complete separable metric, and note that the Borel sigma algebra of a separable metric space is separable (meaning that it is generated by an at most countable family of sets) and countably separated (in that there exists an at most countable family of sets separating the points); a countable base of open sets satisfies these two conditions. Moreover, there exists a subset M in $[0, 1]$ and a one-to-one mapping $f : K \to M$ which is $(\mathcal{B}(K), \mathcal{B}(M))$-measurable such that $\mathcal{B}(K) = \{f^{-1}(A) : A \in \mathcal{B}(M)\}$ [[1], Theorem 6.5.8.] (One can also see that M is a Borel set in $[0, 1]$ but this is not needed for the proof).
Let $D_n = \{[\frac{2^n-1}{2^n}, 1]\} \cup \{[\frac{k}{2^n}, \frac{k+1}{2^n}[: k = 0, 1, ..., 2^n - 2\}$ be the dyadic partition of $[0, 1]$, and $D'_n = \{M \cap A : A \in D_n\}$ be the induced partition on M. Then the sequence $\mathcal{Q}_n := \{f^{-1}(B) : B \in D'_n\}$ of finite partitions of K, by Borel sets, satisfies $\mathcal{Q}_n \subset \mathcal{Q}_{n+1}$. Because $\bigvee_n \sigma(D_n) = \mathcal{B}([0, 1])$, we have $\bigvee_n \sigma(D'_n) = \mathcal{B}(M)$ and thus $\bigvee_n \mathcal{Q}_n = \mathcal{B}(K)$. The claim is proved.
Let then Π_n be as in the preceding claim, and set $\mathcal{P}_n := \mathcal{P}(\Pi_n) = \{\{\omega \in K^{\mathbb{Z}} : \omega_0 \in A_0^n\}, ..., \{\omega \in K^{\mathbb{Z}} : \omega_0 \in A_{p_n}^n\}\}$. Then

$$S^j\mathcal{P}_n = \{\{\omega \in K^{\mathbb{Z}} : \omega_{-j} \in A_0^n\}, ..., \{\omega \in K^{\mathbb{Z}} : \omega_{-j} \in A_{p_n}^n\}\}.$$

Now, recalling that X_j is the j^{th} coordinate, then for every $s = 0, ..., p_n$, we have $X_{-j}^{-1}(A_s^n) =$

$\{\omega \in \Omega : \omega_{-j} \in A_s^n\}$ so that $S^j\mathcal{P}_n = X_{-j}^{-1}(\Pi_n)$ and clearly hence

$$\bigvee_n S^j\mathcal{P}_n = \bigvee_n X_{-j}^{-1}(\Pi_n) = X_{-j}^{-1}(\bigvee_n \Pi_n) = X_{-j}^{-1}(\mathcal{B}(K)) = \sigma(X_{-j}) = \mathcal{B}_{\{-j\}}.$$

Now, for every j, $(\mathcal{P}_n)_S \supset S^j\mathcal{P}_n$ and then $\bigvee_n(\mathcal{P}_n)_S \supset \mathcal{B}_{\{-j\}}$, so that

$$\mathcal{F}^{\otimes\mathbb{Z}} \supset \bigvee_n(\mathcal{P}_n)_S \supset \bigvee_j \sigma(X_{-j}) = \bigvee_j \mathcal{B}_{\{-j\}} = \mathcal{F}^{\otimes\mathbb{Z}},$$

that is the increasing sequence of invariant sub-σ-algeras $(\mathcal{P}_n)_S$ generates the full σ-algebra $\mathcal{F}^{\otimes\mathbb{Z}} = \mathcal{B}(K^{\mathbb{Z}})$.
In conclusion we have proved that this sequence of the invariant sigma algebras $(\mathcal{P}_n)_S$ satisfies the two conditions (i) and (ii) in Theorem A, so that, according to that Theorem, the system $(\Omega, S, \mathcal{B}(\Omega), \mu)$ is isomorphic to a Bernoulli system. □

Recall that the exchangeability means the equalities $\mu \circ T_\sigma^{-1} = \mu$, for all $\sigma \in H$, so that the following result (Theorem 2) generalizes De Finetti's Theorem.

The following Lemma will be used in the proof of Theorem 2. We leave its proof to the reader.

Lemma 2([10], ex. (12,63), p. 186)
Let $(X, \mathcal{B}, \nu)$ be a measure space and f be a ν-measurable function. Then there exists a $\mathcal{B}$-measurable function g, which is equal to f, ν almost everywhere.
In particular
If μ is a Borel (respectively Baire) measure on a topological space and if $f \in L^1(\mu)$ then there exists a Borel(respectively Baire) function g such that $f = g$ μ almost everywhere.

Theorem 2
Let $X = (X_n)_{n\in\mathbb{Z}}$ be a stationary process. Then the following properties are equivalent
(1) X is exchangeable.
(2) X is quasi-exchangeable with uniformly integrable densities.
(3) X is a mixture of Bernoulli processes.

Proof The equivalence (1) $\iff$ (3) is the generalization by Hewitt-Savage of De Finetti's Theorem to presentable spaces. The implication (1) $\Rightarrow$ (2) is trivial. We show that (2) implies (3). For this, let μ denotes the law of the process, and for every $N \geq 1$, consider the measure

$$\nu_N := \frac{1}{card(H_N)} \sum_{\sigma\in H_N} \mu \circ T_\sigma^{-1} = (\frac{1}{card(H_N)} \sum_{\sigma\in H_N} \phi_\sigma)\mu,$$

where, as in (a_2), $H_N := \{\sigma \in H : \sigma(n) = n, \forall n, \mid n \mid > N\}$, and $card(H_N) = (2N+1)!$ denotes its cardinality.

Scheme of the proof: We prove first that any cluster point ν of the sequence $(\nu_N)_{N\geq 1}$ is invariant by T_σ, for any permutation $\sigma \in H$ (and thus the Hewitt-Savage generalization of De Finetti's Theorem applies to that cluster point ν). The hypothesis implies that ν is absolutely continuous with respect to μ, and using the definition of ν, we prove that also μ is absolutely continuous with respect to ν and we conclude using the Hewitt-Savage mentioned Theorem(or also by Corollary 2 and Remark 7 below).
The details are as follows.
Let ν be a cluster point of the sequence $(\nu_N)_{N\geq 1}$, and $N_k \to \infty$ with $\nu = \lim_k \nu_{N_k}$. Notice that the uniform integrability of the family $\{\phi_\sigma : \sigma \in H\}$ implies the same property for the family $\{\frac{1}{card(H_N)} \sum_{\sigma\in H_N} \phi_\sigma : N \geq 1\}$, so that, if $f_N := \frac{1}{card(H_N)} \sum_{\sigma\in H_N} \phi_\sigma$, then (f_{N_k}) is also uniformly integrable, so that according to Dunford-Pettis Theorem, (f_{N_k}) admits a subsequence which we still denote (f_{N_k}), which converges weakly in $L^1(\mu)$, to some $f \in L^1(\mu)$. It follows that the sequence of probability measures ν_{N_k} converges to $\nu := f\mu$, in the sense that

$$\forall A \in \mathcal{B},\ \ \nu(A) = \lim_k \nu_{N_k}(A).$$

Now, because, for every $\tau \in H$, there is $k_0(\tau)$ such that $\tau \in H_{N_k}$, for any $k \geq k_0(\tau)$, the measure ν is invariant by T_τ, for all $\tau \in H$. In fact let A be a measurable set, and $\tau \in H$. Then, with $M := N_k$, and using (5), the following equalities hold

$$\begin{aligned}\nu_{N_k}(T_\tau^{-1}A) &= \frac{1}{card(H_M)} \sum_{\sigma\in H_M} \mu \circ T_\sigma^{-1}(T_\tau^{-1}A) = \frac{1}{card(H_M)} \sum_{\sigma\in H_M} \mu \circ (T_\tau T_\sigma)^{-1}(A) \\ &= \frac{1}{card(H_M)} \sum_{\sigma\in H_M} \mu \circ T_{\sigma\circ\tau}^{-1}(A) = \frac{1}{card(H_M)} \sum_{\sigma\in (H_M)\tau} \mu \circ T_\sigma^{-1}(A).\end{aligned}$$

But, as noted before, for k big enough, $\tau \in H_{N_k} = H_M$, so that $H_M = (H_M)\tau$, since H_M is a group, and then

$$\nu_{N_k}(T_\tau^{-1}A) = \frac{1}{card(H_M)} \sum_{\sigma\in (H_M)\tau} \mu \circ T_\sigma^{-1}(A) = \frac{1}{card(H_M)} \sum_{\sigma\in H_M} \mu \circ T_\sigma^{-1}(A) = \nu_{N_k}(A)$$

and consequently $\nu(T_\tau^{-1}(A)) = \nu(A)$.

It follows that ν is also invariant by the shift S, (hence it will be equal to μ, if μ was ergodic, and thus μ will be invariant by H, hence μ will be Bernoulli, by the Hewitt-Savage generalization of De Finetti's Theorem, or by Proposition 1 below).

Suppose now that the system is not necessarily ergodic. By a slight adaptation of the Hewitt-Savage generalization of De Finetti's Theorem [9] or also, in the case where K is compact, by Corollary 2 and Remark 7 below, being invariant by all T_τ, the measure ν is given by an average of independent measures:

$$\begin{aligned}&\textit{there exists a probability measure} \quad \beta \in M_1(\tilde{K}), \quad \textit{such that} \\ &\nu(A) = \int_{\tilde{K}} (\tilde{\pi}(A)) d\beta(\tilde{\pi}) = \int_{\tilde{K}} (\int_{K^{\mathbb{Z}}} 1_A d\tilde{\pi}) d\beta(\tilde{\pi}) \qquad (25)\end{aligned}$$

for all $A \in \mathcal{B}_a(K^{\mathbb{Z}})$, the Baire sigma algebra of $K^{\mathbb{Z}}$, where $\tilde{K} = M_1(K^{\mathbb{Z}})$ and where, for any probability measure π on K, the probability measure $\tilde{\pi}$ denotes the corresponding product measure on $K^{\mathbb{Z}}$: $\tilde{\pi} := \pi^{\otimes\mathbb{Z}}$.

By repeated use of Lebesgue dominated convergence theorem and monotone convergence theorem, we deduce from (25) that, for any $\mathcal{B}_a(K^{\mathbb{Z}})$ measurable and ν-integrable h,

$$\nu(h) = \int_{\tilde{K}} d\beta(\tilde{\pi}) \int_{K^{\mathbb{Z}}} h d\tilde{\pi}. \tag{26}$$

Also, $\nu = f\mu$ is absolutely continuous with repect to μ, and thus, f is S-invariant μ almost everywhere.
Now the following implications, which hold for all $\sigma \in H$,

$$\begin{aligned} \nu \circ T_\sigma^{-1} = \nu &\iff f \circ T_\sigma^{-1}\mu \circ T_\sigma^{-1} = f\mu \iff f \circ T_\sigma^{-1}\phi_\sigma\mu = f\mu \\ &\iff f \circ T_\sigma^{-1}\phi_\sigma = f \quad \mu\ a.e. \iff f\phi_\sigma \circ T_\sigma = f \circ T_\sigma \quad \mu\ a.e. \end{aligned}$$

(the last implications use the equivalence of μ, $\mu \circ T_\sigma^{-1}$ and $\mu \circ T_\sigma$) imply that the set $A_0 := \{f = 0\}$ is mod(μ) invariant by T_σ, for all σ. Then, in particular

$$\mu(T_\sigma^{-1}(A_0)) = \mu(A_0),$$

so that

$$0 = \nu(A_0) = \lim_k \nu_{N_k}(A_0) = \lim_k \frac{1}{card(H_{N_k})} \sum_{\sigma \in H_{N_k}} \mu(T_\sigma^{-1}(A_0)) = \mu(A_0).$$

Then ν is equivalent to μ so that, for some S-invariant $g \in L^1(\nu)$, $\mu = g\nu$.
Now, according to Lemma 2 above, for some E with $\nu(E) = 1$, $g = g_1$ on E and g_1 is $\mathcal{B}_a(K^{\mathbb{Z}})$ measurable. Then, setting $F := \cap_{n\in\mathbb{Z}} S^n E$, we obtain $g = g \circ S^n = g_1 \circ S^n = g_1$ on F. We note that $g_1 \circ S^n$ is $\mathcal{B}_a(K^{\mathbb{Z}})$ measurable for every n. Then, for any $A \in \mathcal{B}_a(K^{\mathbb{Z}})$, and any n,

$$\begin{aligned} \nu(g1_A) = \nu(g_1 1_A 1_F) = \nu(g_1 \circ S^n 1_A 1_F) = \nu(g_1 \circ S^n 1_A) \ \textit{since}\ g_1 1_F = g_1 \circ S^n 1_F \\ \textit{but by (26)}, \quad \nu(g_1 \circ S^n 1_A) = \int_{\tilde{K}} d\beta(\tilde{\pi}) \int_{K^{\mathbb{Z}}} g_1 \circ S^n 1_A d\tilde{\pi}, \end{aligned}$$

and then, since this last equality holds for every n, we have

$$\nu(g1_A) = \lim_n \int_{\tilde{K}} d\beta(\tilde{\pi}) \int_{K^{\mathbb{Z}}} g_1 \circ S^n 1_A d\tilde{\pi}.$$

Then, using Lebesgue bounded convergence Theorem and the mixing property of the Bernoulli system $(K^{\mathbb{Z}}, S, \tilde{\pi})$, we obtain

$$\nu(g1_A) = \int_{\tilde{K}} d\beta(\tilde{\pi}) \int_{K^{\mathbb{Z}}} g_1 d\tilde{\pi} \int_{K^{\mathbb{Z}}} 1_A d\tilde{\pi}, \tag{27}$$

because

$$v_n(\tilde{\pi}) := \int_{K^{\mathbb{Z}}} g_1 \circ S^n 1_A d\tilde{\pi} \to \int_{K^{\mathbb{Z}}} g_1 d\tilde{\pi} \int_{K^{\mathbb{Z}}} 1_A d\tilde{\pi}$$

and

$$\mid v_n(\tilde{\pi}) \mid \leq \int_{K^{\mathbb{Z}}} g_1 d\tilde{\pi}, \qquad \int_{\tilde{K}} d\beta(\tilde{\pi}) \int_{K^{\mathbb{Z}}} g_1 d\tilde{\pi} = \nu(g_1) = 1.$$

Now setting $v(\tilde{\pi}) := \int_{K^{\mathbb{Z}}} g_1 d\tilde{\pi}$, and $\beta_1 = v\beta$, the equality (27) reads

$$\nu(g1_A) = \int_{\tilde{K}} d\beta(\tilde{\pi}) v(\tilde{\pi}) \int_{K^{\mathbb{Z}}} 1_A d\tilde{\pi},$$

that is

$$\mu(A) = \nu(g1_A) = \int_{\tilde{K}} d\beta_1(\tilde{\pi}) \int_{K^{\mathbb{Z}}} 1_A d\tilde{\pi} = \int_{\tilde{K}} \tilde{\pi}(A) d\beta_1(\tilde{\pi})$$

and this ends the proof, because

$$\int_{\tilde{K}} d\beta_1(\tilde{\pi}) = \int_{\tilde{K}} d\beta_1(\tilde{\pi}) \tilde{\pi}(1) = \nu(g) = \mu(1) = 1. \square$$

The simplicity of the proof together with the absence of topological hypothesis on the state space K, justify the following particular case of De Finetti's theorem.

Proposition 1
If μ is invariant by T_σ for all $\sigma \in H$, and ergodic for the shift S, then μ is the product measure.

Proof Let $N \geq 1$, and $A_1, ..., A_{N+1}$ be measurable subsets of K. For any $P > N+1$, let $\tau := \tau_{N,P} \in H$, be the transposition defined by

$$\tau_{N,P}(n) = n, \forall n \notin \{N+1, P\}, \qquad \tau_{N,P}(N+1) = P \quad and \quad \tau_{N,P}(P) = N+1.$$

Then, because $\mu \circ T^{-1}_{\tau_{N,P}} = \mu$, the following equality

$$\mu(\omega_0 \in A_0, ..., \omega_{N+1} \in A_{N+1}) = \mu(\omega_0 \in A_0, ..., \omega_N \in A_N, \omega_P \in A_{N+1})$$

holds for all $P > N+1$. Hence

$$\mu(\omega_0 \in A_0, ..., \omega_{N+1} \in A_{N+1}) = \frac{1}{Q-N-1} \sum_{P=N+2}^{Q} \mu(\omega_0 \in A_0, ..., \omega_N \in A_N, \omega_P \in A_{N+1})$$

holds for all $Q > N+1$, and shows then, by ergodicity, that

$$\mu(\omega_0 \in A_0, ..., \omega_{N+1} \in A_{N+1}) = \mu(\omega_0 \in A_0, ..., \omega_N \in A_N)\mu(\omega_0 \in A_{N+1}).$$

The proof is achieved.□

Recall that, if μ is a probability measure on Ω, then a measurable set A is μ-almost everywhere shift-invariant if $1_A = 1_A \circ S$ in $L^1(\mu)$, and also that it is μ-almost everywhere exchangeable if $1_A = 1_A \circ T_\sigma$ in $L^1(\mu)$, for any $\sigma \in H$.
Let us use the following notation

$$\mathcal{S} = \textit{ the convex set of all exchangeable probability measures on } \Omega = K^{\mathbb{Z}}.$$

We have seen that $\mathcal{S}$ is a subset of $M_1(\Omega, S)$, the space of all shift invariant probability measures on Ω. It is clear, when K is a compact space, that $\mathcal{S}$ is closed in $M_1(\Omega, S)$ for the weak star topology $\sigma(C(K)^*, C(K))$. Then $\mathcal{S}$ is convex compact for this topology. We need the following Lemma [16], which is easy to prove.

Lemma 3
For all $\mu \in \mathcal{S}$, the sigma-algebra $\mathcal{I}_\mu$ of μ-invariant sets is equal to the sigma-algebra $\mathcal{E}_{xch}$ of μ-exchangeable sets:

$$\mathcal{I}_\mu = \mathcal{E}_{xch}.$$

Lemma 4
The set of measures which are extreme points of $\mathcal{S}$ is the set of all Bernoulli measures.

Proof Let $\mu \in \mathcal{S}$ be an extreme point. By Proposition 1, it suffices to show that μ is ergodic for the shift S. Suppose that μ is not ergodic. Then there exists an invariant set A, with $\mu(A) \in]0,1[$. Then $A \in \mathcal{E}_{xch}$ also, because $\mathcal{I}_\mu = \mathcal{E}_{xch}$, by Lemma 3. Set then $\mu_1 = \frac{1}{\mu(A)} 1_A \mu$ and $\mu_2 = \frac{1}{\mu(A^c)} 1_{A^c} \mu$. Then $\mu_1, \mu_2 \in \mathcal{S}$, $\mu_1 \neq \mu_2$ and $\mu = \mu(A)\mu_1 + \mu(A^c)\mu_2$, so that μ is not extremal in $\mathcal{S}$, contradicting the hypothesis, and we conclude that μ is ergodic.□

Remark 6
It follows from Lemma 4 that the set of exreme points of $\mathcal{S}$ is closed.

Corollary 2
Let (X_n) be an exchangeable sequence with values in a compact metrizable space K, with law μ. Then there exists a probability measure η supported on the Bernoulli measures such that for every Borel subset A of $K^{\mathbb{Z}}$,

$$\mu(A) = \int_{Ber} \tilde{\pi}(A) d\eta(\tilde{\pi})$$

where Ber stands for the set of all Bernoulli probability measures on $\Omega = K^{\mathbb{Z}}$. In particular for any $k \geq 1$, all Borel sets $A_1, ..., A_k$ in K,

$$\mu(X_1 \in A_1, ..., X_k \in A_k) = \int_{Ber} \pi(X_1 \in A_1)...\pi(X_1 \in A_k) d\eta(\tilde{\pi}).$$

Proof
Since Ω is compact metrizable $M_1(\Omega, S)$ is compact metrizable. Then $\mathcal{S}$ is compact metrizable also, and so, by Choquet Theorem ([19], p.14), there is a probability measure η on $\mathcal{S}$, which represents μ and is supported by the extreme points of $\mathcal{S}$ and this ends the proof by Lemma 4. □

Remark 7
Under the conditions as in the Corollary above, but assuming only the state space K to be compact Hausdorff, the same conclusion holds, with Borel replaced by Baire.

Corollary 3
If the state space K is compact metrizable, the set $\mathcal{S}$ is a simplex.

Proof For any $\mu \in \mathcal{S}$, the decomposition $\mu = \int_{\mathcal{E}} \xi d\eta(\xi)$, on the ergodic measures, which is unique since $M_1(\Omega, S)$ is a simplex, is, by Corollary 2, given(represented) by a measure η concentrated on the subset Ber. Then, because, by Lemma 4, Ber is the set of extreme points of $\mathcal{S}$, we conclude that every $\mu \in \mathcal{S}$ is the barycenter of a unique measure which is supported by the extreme points of $\mathcal{S}$.
Since $\mathcal{S}$, being a closed subset of the compact metrizable space $M_1(\Omega, S)$, is metrizable, it follows then, by Choquet Theorem([19], p. 60), that $\mathcal{S}$ is a simplex.□

3 Application: Bernoullicity of some Determinantal Processes

We recall some properties of simple point processes and determinantal processes. For more details we refer to [6, 20, 21, 23, 5, 11].
A simple random point process with phase a locally compact Polish space E is a probability measure P on the measurable space $(Conf(E), \mathcal{B}(Conf(E))$, where $Conf(E)$ is the space of locally finite subsets of E, endowed with the vague topology of measures, by identifying every $\xi \in Conf(E)$ with the atomic integer valued measure $\sum_{x \in \xi} \delta_x$.
Let λ be a measure on E. A locally integrable function $\rho^{(n)}$ on the cartesian product E^n, is

called a n-point correlation function of P if

$$\int_{Conf_{(E)}} \prod_{j=1}^{m} \frac{\xi(B_j)!}{(\xi(B_j)-k_j)!} dP(\xi) = \int_{B_1^{k_1}\times ...\times B_m^{k_m}} \rho^{(n)}(x_1,...,x_n) d\lambda(x_1)...d\lambda(x_n),$$

for all disjoint bounded Borel subsets $B_1, ..., B_m$ of E, and all $k_1, ..., k_m \in \mathbb{N}$, with $k_1 + ... + k_m = n$([23] Def. 2 , [5] Def 2, [11])).
The point process P, all of whose correlation functions $\rho^{(n)}$ exist, is called determinantal if there exists a function $k : E \times E \to \mathbb{R}$, such that for all n and $x_1, ..., x_n \in E$,

$$\rho^{(n)}(x_1, ..., x_n) = det[(k(x_i, x_j))_{i,j=1,...,n}].$$

The function k above is called the correlation kernel of the process P.

In [20, 21] (see also [23] Theorem 3 p. 934) it is proved that if K is a bounded symmetric integral operator on $L^2(E, \lambda)$ with kernel k, which is also of locally trace class and with spectrum contained in $[0, 1]$, then there exists a unique probability measure P on $Conf(E)$, such that for every nonnegative continuous function ψ with compact support,

$$\int_{Conf(E)} \exp(\langle -\psi, \xi\rangle) dP(\xi) = Det(I - \sqrt{(1-e^{-\psi})}K\sqrt{(1-e^{-\psi})}) \qquad (d_1)$$

where the determinant is the Fredholm determinant (see [22]), and where $\sqrt{(1-e^{-\psi})}K\sqrt{(1-e^{-\psi})}$ denotes the integral operator with kernel

$$L(x, y) = \sqrt{1-e^{-\psi(x)}}k(x, y)\sqrt{1-e^{-\psi(y)}},$$

and moreover the correlation functions of P are given by

$$\rho^{(n)}(x_1, ..., x_n) = det(k(x_i, x_j))_{i,j=1,...,n}.$$

In the particular case, where E is countable(in fact $E = \mathbb{Z}$) and λ is the counting measure, which is relevant to our purpose, by identifying a subset A of E with its indicator function 1_A, we can take the configuration space to be equal $\{0, 1\}^E$ and then (d_1) is equivalent to (d_2) below ([21] Theorem 1.1, see also [13] and [15] p. 319)

$$P(\{\omega \in \{0,1\}^E : \omega(e) = 1, \forall e \in A\}) = P(\{\xi \subset E : A \subset \xi\}) = det(k(x, y))_{x,y\in A} \qquad (d_2)$$

for any finite subset A of E.

We have the following result

Theorem 3
Any stationary discrete time quasi-invariant determinantal process $X = (X_n)_{n\in\mathbb{Z}}$, with phase

space $\mathbb{Z}$ is isomorphic to a Bernoulli system.

Proof Let μ be the law of the process X. Since $\Omega := \{0,1\}^{\mathbb{Z}}$ is the configuration space of the process X, μ is a shift invariant probability measure on Ω. It follows from Theorem 7 in [23], that a translation invariant determinantal random point field, with one-particle space $E = \mathbb{Z}$, is mixing of any order and then in particular it is ergodic, meaning in our setting that the system (Ω, S, μ) is ergodic. Hence the process X satisfies the hypothesis in Theorem 1, and therefore it is isomorphic to a Bernoulli process. □

In [4] Bufetov considered a class of determinantal processes with phase space F, where $F = \mathbb{R}$ (the continuous case), or $F =$ a countable subset of $\mathbb{R}$, without accumulation points(the discrete case), corresponding to projection operators with integrable kernels. In the discrete case, he proved([4], Theorem 1.6) that they are quasi-invariant, which means quasi-exchangeable. It follows then from the preceding Theorem 3, that the processes in this class, corresponding to the phase space $F = \mathbb{Z}$, and which are translation invariant, are isomorphic to Bernoulli systems.
This applies, in particular, to the discrete sine process. Recall that the discrete sine kernel which is translation invariant kernel on the lattice $\mathbb{Z}$ is defined by

$$S(x,y;a) = S_1(x-y,a), \quad x,y \in \mathbb{Z}, \quad \textit{where}$$
$$S_1(x,a) := \frac{sin(x(arccos(a/2))}{\pi x}, \quad x \in \mathbb{Z}, x \neq 0$$
$$S_1(0,a) = \frac{arccos(a/2)}{\pi}.$$

where a is a real number $(-2 \leq a \leq 2)$([2] p. 486).

4 The family of quasi symmetric processes is strictly contained in the Bernoulli processes

There exists processes generating dynamical systems isomorphic to Bernoulli and which are not quasiexchangeable.

Before giving some examples, let us, when K is a finite set $K := \{0, ..., k-1\}$, denote the cylinder

$$C := \{\omega \in K^{\mathbb{Z}} : \omega_0 = x_0, ..., \omega_n = x_n\}, \qquad \textit{simply by} \qquad [[x_0, x_1, ..., x_n]]. \tag{28}$$

Then, for any permutation $\sigma \in H$, with $\tau := \sigma^{-1}$, and such that the support of σ is included

in $\{0, 1, ..., n\}$,

$$T_\sigma^{-1} C = [[x_{\tau(0)}, x_{\tau(1)}, ..., x_{\tau(n)}]]$$

so that, for a stationary Markov chain defined by a stochastic matrix $\Pi = (\Pi_{i,j})_{i;j=0,...,k-1}$ and an invariant probability vector $p = (p(0), p(1), ..., p(k-1))$, we have

$$\mu(C) = p(x_0)\Pi_{x_0,x_1}\Pi_{x_1,x_2}...\Pi_{x_{n-1},x_n} \qquad \textit{and}$$
$$\mu(T_\sigma^{-1} C) = p(x_{\tau(0)})\Pi_{x_{\tau(0)},x_{\tau(1)}}\Pi_{x_{\tau(1)},x_{\tau(2)}}....\Pi_{x_{\tau(n-1)},x_{\tau(n)}}.$$

(In general,

$$C = \{\omega : \omega_j \in A_j, j \in J\} \Rightarrow T_\sigma^{-1} C = \{\omega : \omega_k \in A_{\tau(k)}, k \in \sigma(J)\}.)$$

As a simple example, consider the stationary Markov chain with state space $K := \{0, 1\}$, defined by the matrix Π given by

$$\begin{pmatrix} t & 1-t \\ 1 & 0 \end{pmatrix}$$

and the invariant row probability vector p (meaning $p\Pi = p$)

$$p = \frac{1}{2-t}(1, 1-t).$$

Now the matrix Π^2 is equal to

$$\begin{pmatrix} t^2 + 1 - t & t(1-t) \\ t & 1-t \end{pmatrix}$$

so that for all t with $0 < t < 1$, Π is irreducible and aperiodic, and then this Markov chain is isomorphic to a Bernoulli shift.

For $n = 2$, for example, let $\tau = \sigma^{-1}$, be the transposition defined by

$$\tau(0) = 1, \qquad \tau(1) = 0, \qquad \tau(n) = n, \forall n \neq 0, 1$$

and take $C := [[0, 1, 1]]$ so that if $D := T_\sigma^{-1} C$, then $T_\sigma^{-1} D = C$, and hence the following holds

$$\mu(C) \leq \mu([[1, 1]]) = 0, \quad T_\sigma^{-1} C = [[1, 0, 1]] \quad \textit{and then} \quad \mu(T_\sigma^{-1} C) \neq 0,$$
$$\textit{also} \quad \mu(D) \neq 0 \quad \textit{and} \quad \mu(T_\sigma^{-1} D) = 0,$$

and proves that μ [resp. $\mu \circ T_\sigma^{-1}$] is not absolutely continuous with respect to $\mu \circ T_\sigma^{-1}$ [resp. μ]

More generally, we have the following

Proposition 2
Any stationary Markov chain with finite state space K, with irreducible and aperiodic transition matrix Π having at least one zero entry, is isomorphic to Bernoulli but it is not quasi-exchangeable.

Proof Observe first, due to the irreducibility and aperiodicity of Π, that if p is the row invariant probability vector, then all the coordinates of p are > 0. Let $i_0, j_0 \in K$, such that $\Pi_{i_0,j_0} = 0$. Then the following holds

$$\exists a, \Pi_{j_0,a} > 0, \ \exists j, \Pi_{i_0,j} > 0, \ \exists n, (\Pi^n)_{a,i_0} > 0$$
$$\textit{that is} \quad \exists x_1, ..., x_{n-1}, \quad \Pi_{a,x_1}\Pi_{x_1,x_2}...\Pi_{x_{n-1},i_0} > 0.$$

Let, with notation as in (28),

$$C := [[j, a, x_1, ..., x_{n-1}, i_0, j_0]],$$
$$D := [[j_0, a, x_1, ..., x_{n-1}, i_0, j]],$$

and $\sigma \in H$, be the transposition defined by

$$\sigma(p) = p, \quad \forall p \notin \{0, n+2\} \quad \textit{and}$$
$$\sigma(0) = n+2, \quad \sigma(n+2) = 0.$$

Then

$$T_\sigma^{-1}C = D \quad \textit{and} \quad T_\sigma^{-1}D = C.$$

But

$$\mu(C) = \mu(j)\Pi_{j,a}\Pi_{a,x_1}...\Pi_{x_{n-1},i_0}\Pi_{i_0,j_0} = 0, \ \textit{and}$$
$$\mu(D) = \mu(j_0)\Pi_{j_0,a}\Pi_{a,x_1}...\Pi_{x_{n-1},i_0}\Pi_{i_0,j} \neq 0$$

so that, $\mu \circ T_\sigma^{-1}[$ respectively $\mu]$ is not absolutely continuous with respect to $\mu[$ respectively $\mu \circ T_\sigma^{-1}]$, because

$$\mu(C) = 0, \quad \mu(T_\sigma^{-1}C) \neq 0,$$
$$\mu(T_\sigma^{-1}D) = 0, \quad \mu(D) \neq 0.$$

The proof is achieved because any mixing Markov chain is isomorphic to a Bernoulli System. □

Acknowledgments. Jean-Paul Thouvenot asked me the question whether or not quasi-invariance implies complete positive entropy. He suggested many improvements and additions. I am greatly indebted to him.
I am deeply grateful to the referee for his many comments and valuable suggestions that improved the article and its presentation.

References

1. Bogachev, V. I.: Measure Theory, Vollume II, Springer-Verlag Berlin Heidelberg (2007)
2. Borodin, A., Okounkov, A., Olshanski, G.: Asymptotics of Plancherel measures for symmetric groups. J. Amer. Math. Soc. 13, 481-515 (2000)
3. Borodin, A., Olshanski, G.: Random Partitions and the gamma kernel. Adv. Math. 194,141-202 (2005)
4. Bufetov, A. I.: Quasi-Symmetries of Determinantal Point Processes. Ann. Prob., Vol. 46. N0. 2, 956-1003 (2018)
5. Camilier, I., Decreusefond, L.: Quasi-invariance and integration by parts for determinantal and permanental processes. Journal of Functional Analysis, 259, 268-300 (2010)
6. Daley, D. J., Vere-Jones, D.: An Introduction to the Theory of Point Processes, Volume II: General Theory and Structure. Second Edition. Springer (2008)
7. Dunford, N., Schwartz, J. T.: Linear operators, Part I. General Theory, Interscience publishers, New York (1958)
8. Friedman, N. A., Ornstein, D. S.: On isomorphism of weak Bernoulli transformations. Adv. in Math. 5, 365-394 (1970)
9. Hewitt, E., Savage, L. J.: Symmetric measures on Cartesian products. Trans. Amer. Math. Soc. 80, 470-501 (1955)
10. Hewitt, E., Stromberg, K.: Real and Abstract Analysis. Springer-Verlag, Berlin. Heidelberg. New York(1969).
11. Johansson, K.: Random Matrices and Determinantal Processes. arXiv:math-ph/0510038v1 10 Oct 2005 (2005)
12. Ledrappier, F.: Sur la Condition de Bernoulli Faible et ses Applications. Lect. Notes in Math., vol 532, 152-159, Springer-Verlag (1976)
13. Lyons, R.: Determinantal probability measures. Publ. Math. Inst. Hautes Etudes Sci. 98, 167-212 (2003)
14. Lyons, R., Steif, J. E.: Stationary Determinantal Processes: Phase Multiplicity, Bernoullicity, Entropy, and Domination. Duke Matth. J. Vol. 120, N0. 3, 515-575 (2003)
15. Olshanski, G.: Difference Operators and Determinantal Point Processes. Functional Analysis and Its Applications, Vol. 42, No.4, pp.317-329 (2008)
16. Olshen, R. A.: The Coincidence of Measure Algebras under an Exchangeable Probability. Z. Wahrscheinlichkeitstheorie verw. Geb. 18, 153-158 (1971)
17. Ornstein, D.: Bernoulli shifts with the same entropy are isomorphic. Adv. Math. 4,337-352 (1970)
18. Ornstein, D.: Two Bernoulli shifts with infinite entropy are isomorphic. Adv. Math. 5, 339-348 (1971)
19. Phelps, R. R.: Lectures on Choquet's Theorem. Lecture notes in mathematics 1757. Second Edition. Springer, Berlin, New York (2001)
20. Shirai, T., Takahashi, Y.: Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes. Journal of Functional Analysis 205, 414-463 (2003)
21. Shirai, T., Takahashi, Y.: Random point fields associated with certain Fredholm determinants II: Fermion Shifts and their Ergodic and Gibbs Properties.The Annals of Probability Vol. 31, N0. 3, 1533-1564 (2003)

22. Simon, B.: Notes on Infinite Determinants of Hilbert Space operators. Adv. Math. 24, 244-273 (1977)
23. Soshnikov, A.: Determinantal random point fields. Russ. Math. Surv. 55 923, 107-160 (2000)
24. Walters, P.: An Introduction to Ergodic Theory, Springer-Verlag New York Heidelberg Berlin, (1982).

Sorbonne Université, UMR 8001, Laboratoire de Probabilités, Statistique et Modélisation, Boîte courrier 158, 4 Place Jussieu, F-75752 Paris Cedex 05, France.
E-mail: doureid.hamdan@sorbonne-universite.fr
doureid.hamdan@upmc.fr