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# About the Schwarz method in a geometrical multi-scale domain

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**Abstract.** A model parabolic linear partial differential equation in a geometrical multi-scale domain is studied. The domain consists of a two-dimensional central node, and several one-dimensional outgoing branches. The optimized Schwarz method in substructured form is adjusted to the problem. The convergence of the GMRES method to solve the interface system is stated and we look at the accuracy with which the solution of the problem can be approximated. Comparisons are made with related schemes also based on solving Robin problems at the node.

Mathematics Subject Classification : 35K05, 65F10, 65N08, 65N12, 65N55.

Mots clés : finite volume scheme, parabolic problem, multi-scale domain, domain decomposition, stability and convergence of numerical methods, optimized Schwarz methods, Robin interface condition.

## 1 Introduction

In this paper, we consider a model problem set in the geometrical multi-scale domain  $D_\varepsilon$  that consists of a two-dimensional (2D) central node  $\Omega(0)$  and  $p$  one-dimensional (1D) outgoing branches  $S_j, j = 1, \dots, p$  (see Figure 1). The node is delimited by segments  $\gamma_j, j = 1, \dots, p$ , at some distance  $\delta$  from the junction. Denote  $x^{e_j}$  the local coordinate in the direction of the  $j$ -th branch. A complete description can be found in [3]. The model problem is the heat equation in hybrid form :

$$\begin{aligned}
 \frac{\partial v_j}{\partial t}(x^{e_j}, t) - \frac{\partial^2 v_j}{\partial x^{e_j 2}}(x^{e_j}, t) &= f_j(x^{e_j}, t), \quad x^{e_j} \in (\delta, l_j), t \in (0, T), j = 1, \dots, p, \\
 v_j(x^{e_j}, 0) &= 0, \quad x^{e_j} \in (\delta, l_j), j = 1, \dots, p, \\
 v_j(l_j, t) &= 0, \quad t \in (0, T), j = 1, \dots, p, \\
 \frac{\partial u}{\partial t}(x, y, t) - \Delta u(x, y, t) &= f(x, y, t), \quad (x, y) \in \Omega(0), t \in (0, T), \\
 u(x, y, 0) &= 0, \quad (x, y) \in \Omega(0), \\
 \frac{\partial u}{\partial n}(x, y, t) &= 0, \quad (x, y) \in \partial\Omega(0) \setminus (\cup_{j=1}^p \gamma_j), t \in (0, T),
 \end{aligned} \tag{1}$$

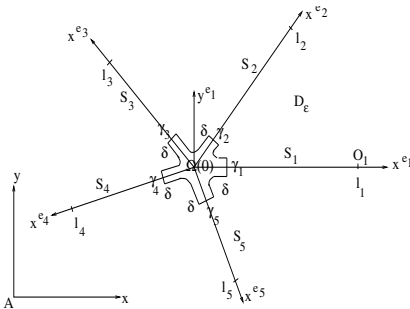


FIGURE 1 – The geometrical multi-scale domain  $D_\varepsilon$  (example with  $p = 5$ ).

supplemented by the transmission conditions

$$u(x, y, t) = v_j(\delta, t), (x, y) \in \gamma_j, j = 1, \dots, p, t \in (0, T), \quad (2)$$

$$\frac{1}{|\gamma_j|} \int_{\gamma_j} \frac{\partial u}{\partial n}(\cdot, t) d\gamma = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t), j = 1, \dots, p, t \in (0, T). \quad (3)$$

This is a multidomain formulation of the heat equation where the subdomains  $\Omega(0)$  and  $S_j, j = 1, \dots, p$ , do not overlap. The interfaces are the segments  $\gamma_j$ .

A monolithic finite volume scheme of hybrid dimension for solving directly the model problem was defined in [3]. Domain decomposition methods that result in interface systems at each time step, and that depend on the 1D and 2D separated problems, were also used. The interface systems were solved by using the matrix-free GMRES method. In the present paper, the boundary conditions for the 2D separated problem are only of Robin type. We extend and study in particular the optimized Schwarz method in substructured form (see [2], [1] and the references in it), since due to the heterogeneity of the domain, non-overlapping methods are particularly well suited. This work is to study the convergence and the accuracy of the GMRES method to solve the interface systems.

The paper is organized as follows. Section 2 presents the substructured form of the Schwarz method to solve the model problem. In Section 3, convergence of the GMRES method to solve the interface system is proved. In Section 4, we give numerical experiments.

## 2 The Schwarz scheme

To define the Schwarz scheme, we start from the monolithic scheme. The discretization is carried out in [3]. The description of the mesh of the 1D branches ( $N_j$  points  $x_i^{e_j}, i = 1, \dots, N_j$ , with  $x_0^{e_j} = \delta, x_{N_j+1}^{e_j} = l_j, j = 1, \dots, p$ ), and of the admissible mesh  $\mathcal{T}$  of the 2D node (with a family of points  $x_K$  chosen in each control volume  $K$ ) is not provided here for brevity reasons. The time step is  $k \in (0, T)$ , let  $t_n = nk$ . The value  $v_{j,i}^n$  is an approximation of  $v_j(x_i^{e_j}, t_n)$ . The value  $u_K^n$  is an approximation of  $u(x_K, t_n)$ .

The values on the right (subdomain  $S_j, j = 1, \dots, p$ ) and on the left (subdomain  $\Omega(0)$ ) of the interfaces are coincident due to the transmission conditions (2), and their approximations are both denoted by  $v_{j,0}^n$  in [3]. Let us now introduce and denote  $u_{\gamma,j}^n$  the approximation on the left. The numerical approximation of the fluxes is simply done by finite difference due to the orthogonality condition of the mesh  $\mathcal{T}$  :

$$F_{1/2}^{j,n+1} = -\frac{v_{j,1}^{n+1} - v_{j,0}^{n+1}}{h_{1/2}^{e_j}}, \quad (4)$$

$$F_{K,\sigma}^{n+1} = -\frac{m(\sigma)}{d_\sigma} (u_{\gamma,j}^{n+1} - u_K^{n+1}), \forall \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, \quad (5)$$

$j = 1, \dots, p$ . The discretization of the transmission conditions is

$$\begin{cases} u_{\gamma,j}^{n+1} = v_{j,0}^{n+1}, j = 1, \dots, p, \\ \frac{v_{j,1}^{n+1} - v_{j,0}^{n+1}}{h_{1/2}^{e_j}} = \frac{1}{|\gamma_j|} \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} (u_{\gamma,j}^{n+1} - u_K^{n+1}), j = 1, \dots, p. \end{cases} \quad (6)$$

Let  $q_j \neq 0, j = 1, \dots, p$ . Let us now define the quantities :

$$\begin{cases} r_j^{n+1} = \frac{v_{j,1}^{n+1} - v_{j,0}^{n+1}}{h_{1/2}^{e_j}} + q_j v_{j,0}^{n+1}, \text{ and } r_{-j}^{n+1} = -\frac{v_{j,1}^{n+1} - v_{j,0}^{n+1}}{h_{1/2}^{e_j}} + q_j v_{j,0}^{n+1}, \\ r_{0j}^{n+1} = \frac{1}{|\gamma_j|} \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} (u_{\gamma,j}^{n+1} - u_K^{n+1}) + q_j u_{\gamma,j}^{n+1}, \\ r_{-0j}^{n+1} = -\frac{1}{|\gamma_j|} \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} (u_{\gamma,j}^{n+1} - u_K^{n+1}) + q_j u_{\gamma,j}^{n+1}, \end{cases} \quad (7)$$

$j = 1, \dots, p$ , that discretize the following Robin quantities :

$$\pm \frac{\partial v_j}{\partial x^{e_j}}(\delta, t) + q_j v_j(\delta, t) \text{ and } \pm \frac{1}{|\gamma_j|} \int_{\gamma_j} \frac{\partial u}{\partial n}(\cdot, t) d\gamma + q_j u(x, y, t)|_{\gamma_j}, j = 1, \dots, p. \quad (8)$$

The notation for the Robin quantity is  $r$  when the normal derivative is directed from the 2D domain to the 1D domain, and  $r_-$  in the opposite direction.

The Schwarz method requires Robin quantities as unknowns on the left and right of the interfaces. Therefore our purpose is to make a change of unknowns. To begin with, if  $q_j \neq -1/h_{1/2}^{e_j}$ ,  $q_j \neq -s\tau_j, j = 1, \dots, p$ , we can easily check that

$$\begin{cases} F_{1/2}^{j,n+1} = -\frac{q_j v_{j,1}^{n+1} - r_{-j}^{n+1}}{q_j h_{1/2}^{e_j} + 1}, \\ r_j^{n+1} = \frac{q_j h_{1/2}^{e_j} - 1}{q_j h_{1/2}^{e_j} + 1} r_{-j}^{n+1} + \frac{2q_j}{q_j h_{1/2}^{e_j} + 1} v_{j,1}^{n+1}, \end{cases} \quad (9)$$

$$\begin{cases} F_{K,\sigma}^{n+1} = -\frac{m(\sigma)}{d_\sigma} \left( \frac{r_{0j}^{n+1} + \frac{1}{|\gamma_j|} \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} u_K^{n+1}}{q_j + s\tau_j} - u_K^{n+1} \right), \forall \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, \\ r_{-0j}^{n+1} = \frac{q_j - s\tau_j}{q_j + s\tau_j} r_{0j}^{n+1} + \frac{2q_j}{q_j + s\tau_j} \frac{1}{|\gamma_j|} \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} u_K^{n+1}, \end{cases} \quad (10)$$

$j = 1, \dots, p$ , where  $s\tau_j = \frac{1}{|\gamma_j|} \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma}$ , and that (6) is equivalent to

$$r_{0j}^{n+1} = r_j^{n+1}, \text{ and } r_{-j}^{n+1} = r_{-0j}^{n+1}, j = 1, \dots, p. \quad (11)$$

By introducing the constants  $\omega_j$  and  $\zeta_j$  which break down these equalities into

$$r_{0j}^{n+1} = \omega_j, r_j^{n+1} = \omega_j, \text{ and } r_{-j}^{n+1} = \zeta_j, r_{-0j}^{n+1} = \zeta_j, j = 1, \dots, p, \quad (12)$$

we arrive at the substructured version of the Schwarz scheme :

$$\begin{cases} \omega_j - S_{1D,j}^{R-R}(\zeta_j) = 0, j = 1, \dots, p, \\ (\zeta_1, \dots, \zeta_p) - S_{2D}^{mdRmdR-}(\omega_1, \dots, \omega_p) = 0, \end{cases} \quad (13)$$

that is considered in this paper, where the operator  $S_{1D,j}^{R-R}$  is defined by  $S_{1D,j}^{R-R}(\zeta_j) = r_j^{n+1}$ , where  $r_j^{n+1}$  is given by (9)<sub>2</sub> with  $(v_{j,i}^{n+1})$  solution of the 1D separated problem with the boundary condition  $r_{-j}^{n+1} = \zeta_j$ , and the operator  $S_{2D}^{mdRmdR-}$  is defined by  $S_{2D}^{mdRmdR-}(\omega_1, \dots, \omega_p) = (r_{-01}^{n+1}, \dots, r_{-0p}^{n+1})$ , where  $r_{-0j}^{n+1}$  is given by (10)<sub>2</sub> with  $(u_K^{n+1})$  solution of the 2D separated problem with the boundary condition  $r_{0j}^{n+1} = \omega_j, j = 1, \dots, p$ . The notation "mdR" is to remind us the **m**ean value of the normal **d**erivatives of the solution on  $\gamma_j$ . Note that in both cases, the normal derivative in the Robin type boundary condition is directed outward.

We choose to refer to each scheme discussed in this paper according to the input and output values of the operators that define it, starting with  $S_{2D}$ , and ending with  $S_{1D,j}$ . The Schwarz scheme (13) will be also referred to as  $mdRmdR-R-R(\omega, \zeta)$ .

### 3 Convergence

**Theorem 1** *Assuming  $q_j > 0, j = 1, \dots, p$ , the GMRES method to solve the interface system (13) is convergent. The method converges towards the Robin values given by the monolithic scheme to solve (1-2-3).*

The proof will be presented in a forthcoming paper. It consists of making the connection with the iterative Schwarz method. Then writing it as a Richardson iteration, thereafter limited to interfaces.

### 4 Numerical experiments

We solve the model problem (1-2-3) in a domain  $D_\varepsilon$  with one node and two branches ( $p = 2$ ), with  $|\gamma_1| = |\gamma_2| = \varepsilon = 0.2$ ,  $l_1 = l_2 = 1$ , and  $\delta = 0.5$ .

The problem that is solved is related to a polynomial function  $f$  that has already been considered in [3] and [5].

The tests have been done with  $k = 0.002$  (500 time iterations) and  $h = 0.02$ .

The solutions obtained by using successively the Schwarz method, and related interface systems based on Robin problems are compared with that of the monolithic scheme. The interface systems are solved by using the GMRES method, and it takes four ( $2 \times p$ , double trace formulation), even two ( $p$ , single trace formulation) iterations to achieve convergence. The GMRES method can therefore be considered here as a direct method. However computations are sometimes inaccurate. Our main concern is to better grasp this lack of accuracy. At the same time, schemes that work well are highlighted.

For the sake of simplicity, it is assumed that  $q_1 = q_2 = q$ . An overall error curve is examined. It is well known that existence and uniqueness of the separated problems with a Robin condition as (8) where the normal derivative is oriented towards the outside, is obtained as soon as  $q > 0$ . However it is interesting to look at what happens by reversing the orientation. For this, the study range includes both negative and positive values of  $q$ . We compare the results obtained for values of  $q$  between  $-200$  and  $200$ , from  $10$  to  $10$ ,  $q \neq 0$ . These values are chosen such that  $|q_j| < s\tau_j$ .

#### 4.1 Diagonal preconditioning. The variants.

We assess the stability and accuracy of the GMRES method to solve (13) by computing the  $L^2$  norm of the difference between the solution obtained from this reduced system and the solution of the monolithic scheme. This  $L^2$  error reaches  $10^{-14}$  when the reduced system is solved with a direct method instead of GMRES. In [5] where Dirichlet and Neumann separated problems are considered, it was noticed that this milestone of  $10^{-14}$  was not always reached with GMRES, but that a simple diagonal preconditioning of the interface systems improved the quality of the results.

Let us explain the setting up of this preconditioning for the Schwarz method. The linear system of the Schwarz scheme takes into account in all cases (9-10-12). Written in block form (as in [4]), let us consider the submatrix  $A_{\gamma_j}$  of the coefficients of  $\omega_j$  and  $\zeta_j$  in  $(9)_2$  and  $(10)_2$ . The diagonal preconditioning consists in making one coefficient per line of  $A_{\gamma_j}$  equal to 1 (the same choice for every  $j$ ). Since  $p = 2$ , four variants are tested, each one referred to by listing the name of the variables assigned a coefficient equal to one, first in  $(9)_2$  related to the 1D

TABLE 1 – The number of iterations in time where the GMRES method converges with non-zero relative residuals, and the  $L^2$  error.

-	Number of non-zero relative residuals				$L^2$ error			
	$\omega - \zeta$	$\omega - \omega$	$\zeta - \omega$	$\zeta - \zeta$	$\omega - \zeta$	$\omega - \omega$	$\zeta - \omega$	$\zeta - \zeta$
$q = -150$	20	0	0	10	$3.2 \cdot 10^{-10}$	$7.4 \cdot 10^{-13}$	$1.7 \cdot 10^{-12}$	$1.2 \cdot 10^{-10}$
$q = -50$	10	0	0	20	$5.1 \cdot 10^{-11}$	$7.4 \cdot 10^{-13}$	$2.6 \cdot 10^{-13}$	$5.5 \cdot 10^{-13}$
$q = 50$	500	2	0	500	$1.9 \cdot 10^{-6}$	$4.8 \cdot 10^{-11}$	$8.0 \cdot 10^{-14}$	$1.7 \cdot 10^{-5}$
$q = 150$	30	0	0	130	$2.8 \cdot 10^{-9}$	$1.4 \cdot 10^{-12}$	$6.2 \cdot 10^{-13}$	$1.3 \cdot 10^{-6}$

separated problem, and then in  $(10)_2$  related to the 2D separated problem :  $\omega - \zeta$ ,  $\omega - \omega$ ,  $\zeta - \omega$ ,  $\zeta - \zeta$ . The original scheme (13) is the variant  $\omega - \zeta$ .

The 1D separated problems of the Schwarz scheme are well defined if  $q_j \neq -1/h_{1/2}^{e_j} = -100$ . The 2D separated problem is well defined if  $q_j \neq -s\tau_j$ .

During the course of the calculations, it happens that the GMRES method converges with non-zero relative residuals to solve interface systems associated with some iterations in time. In this case the  $L^2$  error is well above  $10^{-14}$ . It is typical in the area of values where the separated problems are not defined and where the condition number are too large. Some values are highlighted in Table 1. Actually, an error as small as  $10^{-11}$  can already mean that the GMRES method converges badly. The variant  $\omega - \omega$  offers the widest range of values for  $q$  that can provide an accurate result. It produces results that can be seen from Figure 2a, for values of  $q = -200 + 10i$ ,  $i \in \{0, \dots, 40\}$ , except  $q = 0$  and  $q = -100$  replaced with  $q = -99, -1, 1$ . If  $90 < q < 200$ , the results are regular and the  $L^2$  error is less than  $10^{-12}$  and reaches  $10^{-13}$ . This is also the case (this was not expected by Theorem 1 which simply gives a sufficient condition) at the other end of the interval when  $-80 < q < -40$  and  $q < -120$ . So we are able to provide a range of values for  $q$ , for which GMRES for the Schwarz scheme gives accurate results.

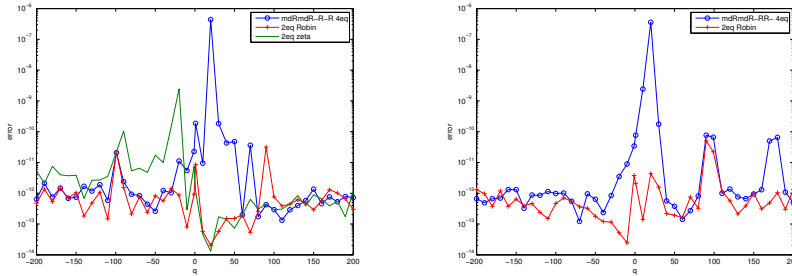


FIGURE 2 – Difference between the monolithic solution and the solution given a) by the scheme  $mdRmdR\_R\_R$  on the left and b) by the scheme  $mdRmdR\_RR\_$  on the right, with two or four unknowns.

## 4.2 Single trace formulation

A restriction of the system to one unknown per interface is another lever identified in [5] to reach the  $10^{-14}$  threshold for the  $L^2$  error. Indeed the matrices of the interface systems with two unknowns ( $p$  unknowns,  $p = 2$ , single trace formulation) have a better condition number. These methods are more classical (see for instance [4]). We construct the two unknowns schemes from

(13), by replacing  $\omega_j$  with a function of  $\zeta_j$ , or vice versa. The system of unknowns  $\omega_j$  is a Robin equation, the system of unknowns  $\zeta_j$  will be named a zeta equation.

After a diagonal preconditioning is implemented (interesting even for matrices of very small size  $2 \times 2$ ), it can be seen from Figure 2a that the two unknowns schemes are more stable than the four unknowns scheme for all the positive values of  $q$ , especially the zeta equation. The zeta equation gives accurate results for a larger choice of positive values of  $q$  than the Schwarz scheme, and allows to reach the expected threshold  $10^{-14}$  for the  $L^2$  error.

### 4.3 Schwarz-related schemes.

The Robin boundary conditions in the Schwarz scheme are expressed with the normal derivative directed outwards. Let us see what happens when we change the direction of the normal derivative for the 1D Robin problems. Consider the scheme  $mdRmdR_-RR_-(\omega, \zeta)$  defined by

$$\begin{cases} \zeta_j - S_{1D,j}^{RR_-}(\omega_j) = 0, j = 1, \dots, p, \\ (\zeta_1, \dots, \zeta_p) - S_{2D}^{mdRmdR_-}(\omega_1, \dots, \omega_p) = 0, \end{cases} \quad (14)$$

where the new operator  $S_{1D,j}^{R-R}$  is defined by  $S_{1D,j}^{RR_-}(\omega_j) = r_{-j}^{n+1}$ , where  $r_{-j}^{n+1}$  is given by (9)<sub>2</sub> with  $(v_{j,i}^{n+1})$  solution of the 1D separated problem with the boundary condition  $r_j^{n+1} = \omega_j$ . The problems 1D are well defined if  $q_j \neq 1/h_{1/2}^{e_j} = 100$ .

The results (with the variant  $\zeta - \omega$  that gives the best results) can be seen from Figure 2b for values of  $q = -200 + 10i, i \in \{0, \dots, 40\}$ , except  $q = 0$  and  $q = 100$  replaced with  $q = -1, 1, 99$ . The results are regular and accurate when  $q$  is negative, since henceforth the problem is defined when  $q = -100$  and then the permutation of the input/output variables of the 1D problems improves the stability. A sufficient condition for guaranteeing existence and uniqueness of such 1D problems is  $q < 0$ , whereas it is  $q > 0$  for the 2D problem. However, despite this contradiction, one sees here that valuable results are obtained with  $q < 0$ , indicating that the impact of the 1D part of the problem is far from negligible.

The Robin equation is the only one two unknowns scheme that follows. Once preconditioned, it allows to reach the expected threshold for the  $L^2$  error and is more stable and as accurate as the four unknowns scheme.

### 4.4 Dirichlet values as output.

To obtain the substructured Schwarz method, we wrote that the physical transmission conditions were equivalent to equalities between some Robin quantities from which we defined boundary conditions for the 1D and 2D separated problems. It is also possible to choose Dirichlet or Neumann quantities. These choices give rise to other interface systems of which many examples can be found in [5]. Without trying to be exhaustive, let us just look at the scheme  $mdRDDDR(\alpha, \omega)$ , defined by

$$\begin{cases} \omega_j - S_{1D,j}^{DR}(\alpha_j) = 0, j = 1, \dots, p \\ (\alpha_1, \dots, \alpha_p) - S_{2D}^{mdRD}(\omega_1, \dots, \omega_p) = 0, \end{cases} \quad (15)$$

where  $S_{1D,j}^{DR}$  solves a 1D Dirichlet separated problem and provides Robin values, and  $S_{2D}^{mdRD}$  solves the 2D Robin separated problem and provides Dirichlet values.

The variant  $\zeta - \zeta$  gives the best results which can be seen from Figure 3a. Except in the central area, an accuracy of  $10^{-12}$  is achieved for any negative value of  $q$ .

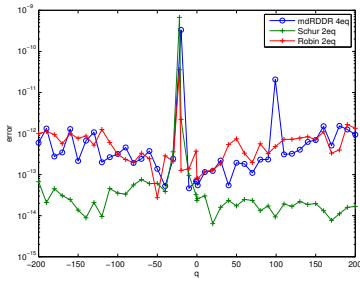


FIGURE 3 – Difference between the monolithic solution and the solution given by the scheme *mdRDDR* with two or four unknowns.

The Robin equation deriving from (15) is not significantly different from the four unknowns scheme except that there is no more oscillation for positive values of  $q$ . But the Schur equation, with the right preconditioning depending on  $q$ , allows us to achieve the desired objective : except in the central area, the error is in the order of  $10^{-14}$ . The range of values for  $q$  that give an accurate result is the largest obtained, for  $q < 0$  as well as for  $q > 0$ . The combination of Dirichlet boundary conditions (that solve the instability problems that arise for some negative values of  $q$  with the Schwarz scheme) and a single trace formulation leads to a clear improvement.

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