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A NEW CLASS OF CRITICAL SOLUTIONS FOR 1D CUBIC NLS

ANATOLE GUÉRIN

ABSTRACT. The aim of this article is to prove the existence of a new class of solutions of 1D cubic NLS with an initial data related to a sum of Dirac masses, of critical regularity $\mathcal{F}(L^\infty)$, and belonging to \dot{H}^s for any $s < -\frac{1}{2}$. This problem is motivated by the lack of result for critical regularity initial condition, and also by the study of the vortex filaments dynamics approximated by the binormal flow. Our result is based on a scattering approach, after performing a pseudo-conformal transformation, and on fine estimations of oscillatory integrals.

1. INTRODUCTION

1.1. The cubic NLS on \mathbb{R} .

The starting point of this article is the cubic nonlinear Schrödinger equation on \mathbb{R} :

$$(NLS) \quad i\partial_t u + \partial_x^2 u \pm |u|^2 u = 0.$$

This equation has been largely studied from theoretical and applied points of view, and appears in several areas of physics such as optics and plasma.

In the following, all the results will be valid for both focussing and defocussing cases. For simplicity we shall consider the focussing case.

Let us first recall the local well-posedness results in Sobolev spaces. It has been proven in [12] and [8] that the equation is well-posed in H^s for any $s \geq 0$. However, it is no longer the case when $s < 0$. For exemple, in [9] a norm inflation phenomena is pointed out.

Then, the critical Sobolev space associated with the scaling invariance $u\lambda(t, x) := \lambda u(\lambda^2 t, \lambda x)$ is $\dot{H}^{-\frac{1}{2}}$. For $s \leq -\frac{1}{2}$, it has been proven in [7] and [18] that norm inflation happens along with a loss of regularity, and in [21] that we have in fact a norm inflating phenomena around any data. For $-\frac{1}{2} < s < 0$, the control of Sobolev norms of Schwartz solutions on the torus and the line is proven in [17] and [19]. More recently, global well-posedness has been proven in [14] for any $s > -\frac{1}{2}$.

On the other hand, the critical Fourier-Lebesgue space associated to this equation is $\mathcal{F}(L^\infty)$, i.e. Fourier transform in L^∞ . The well-posedness has been proven for initial data with Fourier transform in L^p for $p < +\infty$ in [22],[13] and [10].

One would like to consider the initial value problem with data a sum of Dirac masses. Unfortunately, Dirac masses are in $\mathcal{F}(L^\infty)$ and borderline in $\dot{H}^{-\frac{1}{2}}$, and the problem is ill-posed for data $u_0 = \alpha\delta_0$. Indeed, in [16] the authors proved, using Galilean invariance,

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that there is either no weak solution or more than one to that Cauchy problem. More precisely, supposing uniqueness, the solution for $t > 0$ of (NLS) is:

$$(1) \quad u_\alpha(t, x) = \alpha e^{-i\alpha^2 \ln t} \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}},$$

but does not converges towards u_0 as t goes to zero.

However, this issue can be bypassed by a change of phase, which shows that

$$\psi_\alpha(t, x) = \alpha \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}}$$

is the solution of the renormalized equation (10) with initial condition $\alpha\delta_0$. As a matter of fact, similar problems have also been treated using renormalisation in context of Gibbs measures, such as in [20] or [6]. Once this obstruction has been identified, Banica and Vega constructed in [1] and [2] solutions of (NLS) that are smoother perturbations of u_α .

Now, we shall recall the results related to a sum of several Dirac masses. Let $q > \frac{1}{2}$, let $(\alpha_j)_{j \in \mathbb{Z}} \in l^{2,q}(\mathbb{C})$ ¹, and set:

$$(2) \quad M = \sum_{j \in \mathbb{Z}} |\alpha_j|^2.$$

It has been proven in Theorem 1.3 of [5] that there exists $T > 0$ and a unique solution of (NLS) on $(0, T]$ of the form

$$(3) \quad u_{\{\alpha_j\}}(t, x) = e^{-2iM \ln t} \sum_{j \in \mathbb{Z}} A_j(t) \frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}},$$

where

$$A_j(t) = e^{-i|\alpha_j|^2 \ln t} (\alpha_j + R_j(t)),$$

with $(R_j)_{j \in \mathbb{Z}}$ a family of functions such that for any $0 < \gamma < 1$:

$$(4) \quad \sup_{0 < \tau < T} \tau^{-\gamma} \|R_j(\tau)\|_{l^{2,q}} < C(T, \|\alpha_j\|_{l^{2,q}}).$$

In this article, we construct much more solutions of (NLS) as large perturbations of the particular one given by (3). The theorem involves the pseudo-conformal transformation given by:

$$\mathcal{T}(f)(t, x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} \bar{f} \left(\frac{1}{t}, \frac{x}{t} \right),$$

and the pseudo-conformal operator:

$$(5) \quad J(f)(t, x) = \left(\frac{x}{2} + it\nabla \right) f(t, x).$$

¹we define the space $l^{2,q}$ with $\|\alpha_j\|_{l^{2,q}}^2 = \sum_{j \in \mathbb{Z}} (1 + |j|)^{2q} |\alpha_j|^2$.

Theorem 1.1. *Let $s \in \mathbb{N}^*$, $(\alpha_j) \in l^{2,q}$ with $q - s > \frac{1}{2}$, $u_+ \in H^s \cap \dot{H}^{-2} \cap W^{1,s}$ such that*

$$(6) \quad \forall p \in \mathbb{Z} \quad \forall k \leq s \quad \frac{\widehat{u_+}(\cdot)}{\cdot + p/2} \in H^k(\mathbb{R}).$$

Let

$$u_1(t, x) = u_{\{\alpha_j\}} + e^{-2iM \ln t} \widehat{u_+} \left(-\frac{x}{2} \right).$$

Then, if $\|\alpha_j\|_{l^{2,q}}$ is small enough, there exists $T_1 < T$ and u a unique solution of (NLS) on $(0, T_1]$, such that:

$$u - u_{\{\alpha_j\}} - e^{-2iM \ln t} \mathcal{T} \left(e^{it\partial_x^2} u_+(x) \right) \in \mathcal{C}((0, T_1], L^2(\mathbb{R})),$$

with:

$$(7) \quad \forall k \in \llbracket 0, s \rrbracket, \left\| (-i)^k J^k \left[u(t, x) - u_{\{\alpha_j\}}(t, x) \right] - e^{-2iM \ln t} \mathcal{T} \left(e^{it\partial_x^2} \nabla^k u_+(x) \right) \right\|_{L^2} = \mathcal{O}(t^{\frac{1}{2}^-}).$$

Moreover, if $\nabla^k u_+ \in (x^4 L^2)$ for all $k \in \llbracket 0, s \rrbracket$, then:

$$u - u_1 \in \mathcal{C}((0, T_1], L^2(\mathbb{R})),$$

with:

$$(8) \quad \forall k \in \llbracket 0, s \rrbracket, \left\| J^k \left[u(t, x) - u_{\{\alpha_j\}}(t, x) \right] - e^{-2iM \ln t} \left(\frac{x}{2} \right)^k \widehat{u_+} \left(-\frac{x}{2} \right) \right\|_{L^2} = \mathcal{O}(t^{\frac{1}{2}^-}).$$

In particular, for $k = 0$ we have:

$$(9) \quad \|u - u_1\|_{L^2} = \mathcal{O}(t^{\frac{1}{2}^-}).$$

The condition (6) is not very restrictive and is only about $\widehat{u_+}$ being small enough at equidistant points. The fact that those points are in $\mathbb{Z}/2$ is not surprising since bounds in terms of $1/d(x, \mathbb{Z}/2)$ already appear in proofs of [5].

The proof of Theorem 1.1 starts by performing, as in [5], a change a phase

$$\psi(t, x) = e^{i2M \ln t} u(t, x),$$

leading to the Wick renormalisation of equation (NLS):

$$(10) \quad i\partial_t \psi + \partial_x^2 \psi + (|\psi|^2 - \frac{2M}{t})\psi = 0.$$

Then we perform a pseudo-conformal transformation:

$$(11) \quad \psi(t, x) = \mathcal{T}(v)(t, x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} \bar{v} \left(\frac{1}{t}, \frac{x}{t} \right).$$

The equation (NLS) is then equivalent to

$$(12) \quad i\partial_t v + \partial_x^2 v + \frac{1}{2t} (|v|^2 - 2M)v = 0,$$

and Theorem 1.1 rephrases for (12) as following. Denoting

$$(13) \quad v_1(t, x) = \mathcal{T}(e^{2iM \ln t} u_{\{\alpha_j\}}) + e^{it\partial_x^2} u_+(x) = A(t, x) + e^{it\partial_x^2} u_+(x),$$

where

$$(14) \quad A(t, x) = \sum_{j \in \mathbb{Z}} e^{-i|\alpha_j|^2 \ln t} (\overline{\alpha_j} + \overline{R_j}(\frac{1}{t})) e^{-i\frac{tj^2}{4} + i\frac{xj}{2}},$$

we want to find $t_0 > 0$ and v solution of (12) such that $v - v_1 \in \mathcal{C}([T_1, \infty), H^s(\mathbb{R}))$, with a decay given by

$$(15) \quad \forall k \in \llbracket 0, s \rrbracket, \quad \|\nabla^k(v - v_1)(t)\|_{L^2} = \mathcal{O}(t^{-\frac{1}{2}^-}).$$

This v will write

$$v(t) = \sum_{j \in \mathbb{Z}} e^{-i|\alpha_j|^2 \ln t} (\overline{\alpha_j} + \overline{R_j}(\frac{1}{t})) e^{-i\frac{tj^2}{4} + i\frac{xj}{2}} + e^{it\partial_x^2} u_+(x) + r(t, x).$$

for some remainder term r of decay given by (15). Note that this will prove the decay (7), but not (8) yet. This latter estimation is obtained by using Schrödinger linear evolution properties, as explained in section 2.5.

Although a bit computational, the proof can now be sum up as a fixed point theorem in an appropriate Sobolev space, using the Duhamel's formula as the functional, as in [1]. The expansion of the cubic non linearity will provide several terms that we will group in powers of $e^{it\partial_x^2} u_+$. The smallness hypothesis on $\|\alpha_j\|_{l^2, s}$ comes from the linear term. The term that will provide the weaker time decay is the quadratic one, with conjugated phases. Finally, we perform the reverse pseudo conformal transformation in section 2.5 to conclude with the proof.

Concerning the tools at our disposal, we will deal with oscillatory integrals, on which we often perform integration by parts after a Fourier transform, that gives sharper estimates than the use of Sobolev embeddings.

1.2. Link with vortex dynamics singularities.

In addition to the study of low regularity solutions of the 1-D cubic Schrödinger equation, this article is also motivated by the study of dynamics of vortex filaments. More precisely, we are referring to a model derived from Euler equations by Da Rios in 1906 in [11] called the binormal flow:

$$(BF) \quad \chi_t = \chi_x \wedge \chi_{xx},$$

where χ is an arc length parameterized curve in \mathbb{R}^3 and where the vortex is supposed to be located near $\chi(t)$. For further informations and references about this equation, see for instance the introduction of [5].

If T represents the tangent vector of a solution χ , then T solves the Schrödinger map with values in \mathbb{S}^2 :

$$(16) \quad T_t = T \wedge T_{xx}.$$

Moreover, Hasimoto constructed in [15] a correspondance between solutions of (NLS) and solutions of (BF) using the Frenet frame, considering the curvature of $\chi(t, x)$ as the modulus of the NLS-solution and the torsion as the derivative of its phase. This transformation stands in the spirit of the Madelung transform.

In [5], Banica and Vega showed that the solution (3) we are proving the stability depicts, via the Hasimoto transform, the evolution of a polygonal line through the binormal flow. In a few words, every Dirac mass correspond to the formation of a corner in $\chi(t)$ at arclength parameter $x = j \in \mathbb{Z}$, as the one developed by the self-similar solution of (BF) of curvature $\frac{\alpha_j}{\sqrt{t}}$. Hence, the odds are high that the class of solutions exhibited in Theorem 1.1 could correspond to the evolution of a curve with several corners through binormal flow. With the convergence rate of $\mathcal{O}(t^{\frac{1}{2}-})$ in Theorem 1.1, the method should be the same as for the case of one corner done in [3] and [4]. However, this method is quite intricate, and adding much more complicated terms when considering several Dirac masses makes this task dense enough to be the object of a future work.

2. SCATTERING IN SOBOLEV SPACES : PROOF OF THEOREM 1.1

The proof follows the same path as the proof of Theorem 1.4 in [1] and is based on a scattering argument. To do so, we introduce an appropriate functional ϕ on which we perform a fixed point argument. The analysis will be much more delicate here, due to the increased complexity of the function $A(t, x)$ defined by (14), (in [1], only one term is considered).

2.1. Fixed point functional.

To say that v solution of (12) is close to v_1 at infinity tends to consider the functional:

$$\phi : v(t, x) \mapsto v_1(t, x) + i \int_t^\infty e^{i(t-\tau)\partial_x^2} \left(-\frac{|v|^2 - 2M}{2\tau} v - (i\partial_t + \partial_{xx})v_1 \right) d\tau.$$

Since A solves (12), we have

$$(i\partial_t + \partial_{xx})v_1 = (i\partial_t + \partial_{xx})A + (i\partial_t + \partial_{xx})e^{it\partial_x^2}u_+ = -\frac{|A|^2 - 2M}{2t}A.$$

We will control the non linear term in v by comparing it to the nonlinearity of v_1 :

$$\begin{aligned} \phi(v) - v_1 = +i \int_t^\infty e^{i(t-\tau)\partial_x^2} & \left(-\frac{|v|^2 - 2M}{2\tau} v + \frac{|v_1|^2 - 2M}{2\tau} v_1 \right. \\ & \left. - \frac{|v_1|^2 - 2M}{2\tau} v_1 + \frac{|A|^2 - 2M}{2\tau} A \right) d\tau. \end{aligned}$$

We then develop the v_1 -cubic term:

$$\begin{aligned} (|v_1|^2 - M)v_1 &= ((A + e^{it\partial_x^2}u_+)(\bar{A} + e^{-it\partial_x^2}\bar{u}_+) - 2M)(A + e^{it\partial_x^2}u_+) \\ &= (|A|^2 - 2M)A + 2|A|^2 e^{it\partial_x^2}u_+ - 2M e^{it\partial_x^2}u_+ + A^2 e^{-it\partial_x^2}\bar{u}_+ + 2A|e^{it\partial_x^2}u_+|^2 \\ &\quad + \bar{A}(e^{it\partial_x^2}u_+)^2 + |e^{it\partial_x^2}u_+|^2 e^{it\partial_x^2}u_+, \end{aligned}$$

so we get

$$\begin{aligned} \phi(v) - v_1 = & + i \int_t^\infty e^{i(t-\tau)\partial_x^2} \left(-\frac{|v|^2 - 2M}{2\tau} v + \frac{|v_1|^2 - 2M}{2\tau} v_1 \right) d\tau \\ & + i \int_t^\infty e^{i(t-\tau)\partial_x^2} \left((-2|A|^2 e^{i\tau\partial_x^2} u_+ + 2M e^{i\tau\partial_x^2} u_+ - A^2 e^{-i\tau\partial_x^2} \overline{u_+} - 2A|e^{i\tau\partial_x^2} u_+|^2 \right. \\ & \quad \left. - \overline{A} e^{2i\tau\partial_x^2} u_+^2 - |e^{it\partial_x^2} u_+|^2 e^{i\tau\partial_x^2} u_+ \right) \frac{1}{\tau} d\tau. \end{aligned}$$

Finally, after expanding $|A|^2$, the functional ϕ writes

$$(17) \quad (\phi(v) - v_1)(t, x)$$

$$(18) \quad = + i \int_t^\infty e^{i(t-\tau)\partial_x^2} \left(\frac{(|v|^2 - 2M)v}{2\tau} - \frac{(|v_1|^2 - 2M)v_1}{2\tau} \right) d\tau$$

$$(19) \quad - 2i \int_t^\infty e^{i(t-\tau)\partial_x^2} \sum_{p \neq j} e^{-i(|\alpha_p|^2 - |\alpha_j|^2) \ln \tau} e^{i(p-j)\frac{\pi}{2}} \overline{\alpha_p} \alpha_j e^{i\frac{j^2 - p^2}{4}\tau} e^{i\tau\partial_x^2} u_+ \frac{d\tau}{\tau}$$

$$(20) \quad - 2i \int_t^\infty e^{i(t-\tau)\partial_x^2} \sum_{(p,j) \in \mathbb{Z}^2} e^{-i(|\alpha_p|^2 - |\alpha_j|^2) \ln \tau + i(p-j)\frac{\pi}{2}}$$

$$(21) \quad \times (\overline{\alpha_p} R_j + \overline{R_p} \alpha_j + \overline{R_p} R_j) e^{i\frac{j^2 - p^2}{4}\tau} e^{i\tau\partial_x^2} u_+ \frac{d\tau}{\tau}$$

$$(22) \quad - i \int_t^\infty e^{i(t-\tau)\partial_x^2} A^2 e^{-i\tau\partial_x^2} \overline{u_+} \frac{d\tau}{\tau}$$

$$(23) \quad - i \int_t^\infty e^{i(t-\tau)\partial_x^2} (2A|e^{i\tau\partial_x^2} u_+|^2 + \overline{A}(e^{i\tau\partial_x^2} u_+)^2) \frac{d\tau}{\tau}$$

$$(24) \quad - i \int_t^\infty e^{i(t-\tau)\partial_x^2} |e^{i\tau\partial_x^2} u_+|^2 e^{i\tau\partial_x^2} u_+ \frac{d\tau}{\tau}$$

$$= I(v) + J_a + J_b + J_c + J_d + J_e,$$

where $I(v)$ is the first term and the J_* are the following source terms. Since we will treat each integral's norm independently, we won't be regarding whether its sign is positive or negative. We won't either keep the $-$ in the phase as it doesn't affect the computation.

Thanks to their behavior, the R_j terms will be very easy to treat as well as J_e where we can largely exploit dispersion of the Schrödinger operator:

$$(25) \quad \|e^{it\partial_x^2} u_+\|_{L_x^\infty} \leq \frac{1}{\sqrt{t}} \|u_+\|_{L^1}.$$

Then, the fact that there is a phase depending on j or p in some of these integrals will produce a shift after an integration by parts and requires the hypothesis on $\widehat{u_+}/(\cdot + \mathbb{Z}/2)$. Finally, the limiting $1/t^{\frac{1}{2}}$ decay comes from using dispersion estimates for J_d .

We want to apply the fixed point theorem on $\phi : X_\delta \rightarrow X_\delta$, where

$$X_\delta = \{v \in \mathcal{C}([t_0, \infty), L^\infty(\mathbb{R})) / \|v - v_1\|_S \leq \delta\},$$

with

$$\|f\|_S = \sum_{0 \leq k \leq s} \sup_{t_0 \leq t} t^\mu \|\nabla^k(f)(t)\|_{L^2}$$

for μ and δ strictly positives, to be chosen later.

The first step is to verify that $\phi(X_\delta) \subset X_\delta$.

2.2. Estimates on the fixed point argument terms.

In this subsection, we give estimations for each of the terms listed above.

Lemma 2.1 (Estimation of $I(v)$). *Let I defined by (18):*

$$I(v) = i \int_t^\infty e^{i(t-\tau)\partial_x^2} \left(\frac{(|v|^2 - 2M)v}{2\tau} - \frac{(|v_1|^2 - 2M)v_1}{2\tau} \right) d\tau,$$

then,

$$\|I(v)\|_S \lesssim^2 \|v - v_1\|_S \left(\frac{\|\alpha_j\|_{l^{2,q}}^2}{\mu} + \frac{C(t_0, \|\alpha_j\|_{l^{2,s}})}{t_0^{2\gamma}(\mu + 2\gamma)} + \frac{\|u_+\|_{L^1}^2}{t_0(1 + \mu)} + \frac{\|v - v_1\|_S^2}{t_0^\mu} \right).$$

Note that we don't need any smallness hypothesis on u_+ if we choose t_0 large and $\|\alpha_j\|_{l^{2,q}}$ small.

Proof. The idea of the proof is to first apply Strichartz inhomogeneous inequality with the admissible couple $(\infty, 2)$ and then use dispersion inequality (25) and also (4) after usual upper-bounds:

$$\begin{aligned} & \sup_{t_0 \leq t} t^\mu \left\| \int_t^\infty e^{i(t-\tau)\partial_x^2} \left(\frac{(|v|^2 - 2M)v}{2\tau} - \frac{(|v_1|^2 - 2M)v_1}{2\tau} \right) d\tau \right\|_{L^2} \\ & \lesssim \sup_{t_0 \leq t} t^\mu \int_t^\infty \| |v|^2 v - |v_1|^2 v_1 - 2M(v - v_1) \|_{L^2} \frac{d\tau}{\tau} \\ & \lesssim \sup_{t_0 \leq t} t^\mu \int_t^\infty (2M + \|v_1\|_{L^\infty}^2 + \|v\|_{L^\infty}^2) \|v - v_1\|_{L^2} \frac{d\tau}{\tau} \\ & \lesssim \|v - v_1\|_S \sup_{t_0 \leq t} t^\mu \int_{t_0}^\infty (2M + \|v_1\|_{L^\infty}^2 + \|v\|_{L^\infty}^2) \frac{d\tau}{\tau^{1+\mu}} \\ & \lesssim \|v - v_1\|_S \left(\frac{2M}{\mu} + \sup_{t_0 \leq t} t^\mu \int_t^\infty 3\|v_1\|_{L^\infty}^2 + 2\|v - v_1\|_{L^\infty}^2 \frac{d\tau}{\tau^{1+\mu}} \right) \\ & \lesssim \|v - v_1\|_S \left(\frac{2M}{\mu} + \frac{6\|\alpha_j\|_{l^{2,q}}^2}{\mu} + \frac{C(t_0, \|\alpha_j\|_{l^{2,s}})}{t_0^{2\gamma}(\mu + 2\gamma)} + \frac{6}{1 + \mu} \frac{\|u_+\|_{L^1}^2}{t_0} + 2 \sup_{t_0 \leq t} t^\mu \int_t^\infty \|v - v_1\|_{L^\infty}^2 \frac{d\tau}{\tau^{1+\mu}} \right). \end{aligned}$$

Since $q > \frac{1}{2}$, we have also controlled in the last steps the l^1 norm with the $l^{2,s}$ norm of $(\alpha_j)_{j \in \mathbb{Z}}$, using Cauchy-Schwarz. Finally, we apply Gagliardo-Nirenberg's interpolation

$$(26) \quad |f|^2 \leq \|f\|_{L^2} \|f'\|_{L^2}$$

²Where \lesssim means " \leq up to a constant".

on the last integral:

$$\begin{aligned} \sup_{t_0 \leq t} t^\mu \int_t^\infty \|v - v_1\|_{L^\infty}^2 \frac{d\tau}{\tau^{1+\mu}} &\leq \sup_{t_0 \leq t} t^\mu \int_t^\infty \|v - v_1\|_{L^2} \|\nabla(v - v_1)\|_{L^2} \frac{d\tau}{\tau^{1+\mu}} \\ &= \frac{\|v - v_1\|_S^2}{t_0^\mu}, \end{aligned}$$

to conclude that

$$\|I(v)\|_S \lesssim \|v - v_1\|_S \left(\frac{\|\alpha_j\|_{L^{2,q}}^2}{\mu} + \frac{C(t_0, \|\alpha_j\|_{L^{2,s}})}{t_0^{2\gamma}(\mu + 2\gamma)} + \frac{\|u_+\|_{L^1}^2}{t_0(1+\mu)} + \frac{\|v - v_1\|_S^2}{t_0^\mu} \right).$$

□

We now have to control the gradient of this integral with the following lemma.

Lemma 2.2 (Estimation of $\nabla^k I(v)$). *Let I defined by (18):*

$$I(v) = i \int_t^\infty e^{i(t-\tau)\partial_x^2} \left(\frac{(|v|^2 - 2M)v}{2\tau} - \frac{(|v_1|^2 - 2M)v_1}{2\tau} \right) d\tau,$$

then, for $k \in \llbracket 1, s \rrbracket$:

$$\sup_{t_0 \leq t} t^\mu \|\nabla^k I(t)\|_{L^2} \lesssim \|v - v_1\|_S \left(\left(\frac{\|\alpha_j\|_{L^{2,q}}}{\mu} \right)^2 + \left(\frac{C(t_0, \|\alpha_j\|_{L^{2,s}})}{(\mu + \gamma)t_0^\gamma} \right)^2 + \left(\frac{\|u_+\|_{W^{1,s}}}{\sqrt{t_0}(\frac{1}{2} + \mu)} \right)^2 + \left(\frac{\delta}{2\mu t_0^\mu} \right)^2 \right).$$

Proof. To obtain this, we need the following computations. Using Gagliardo–Nirenberg interpolation (26), we have for $k \in \llbracket 0, s - 1 \rrbracket$:

$$(27) \quad |\nabla^k(v - v_1)(t)| \leq C \|\nabla^{k+1}(v - v_1)(t)\|_{L^2}^{\frac{1}{2}} \|\nabla^k(v - v_1)(t)\|_{L^2}^{\frac{1}{2}} \lesssim \frac{\delta}{t^\mu}.$$

Since ∇^k commute with the Schrödinger operator, we have

$$(28) \quad |\nabla^k e^{it\partial_x^2} u_+| \leq \frac{\|\nabla^k u_+\|_{L^1}}{\sqrt{t}}.$$

Thus, we can control the k^{th} gradient v , with $k \in \llbracket 0, s - 1 \rrbracket$, introducing v_1 :

$$(29) \quad |\nabla^k v(t)| \leq C |\nabla^k v_1(t)| + |\nabla^k(v - v_1)(t)| \lesssim \|\nabla^k A\|_{L^\infty} + \frac{\|\nabla^k u_+\|_{L^1}}{\sqrt{t}} + \frac{\delta}{t^\mu}$$

$$(30) \quad \lesssim \|\alpha_j\|_{L^{2,q}} + \frac{1}{t^\gamma} + \frac{\|\nabla^k u_+\|_{L^1}}{\sqrt{t}} + \frac{\delta}{t^\mu},$$

using that for each k , $q - k > \frac{1}{2}$ to control $\|\nabla^k A\|_{L^\infty}$ with the $l^{2,q}$ norm and (4). We then do the same with the k^{th} gradient of v^2 for $k \in \llbracket 1, s-1 \rrbracket$ using Leibniz formula:

$$(31) \quad |\nabla^k v^2| \leq \sum_{p=0}^k \binom{k}{p} |\nabla^p v| |\nabla^{k-p} v|$$

$$\lesssim \sum_{p=0}^k \binom{k}{p} \left(\|\nabla^p A\|_{L^\infty} + \frac{\|\nabla^p u_+\|_{L^1}}{\sqrt{t}} + \frac{\delta}{t^\mu} \right)$$

$$(32) \quad \times \left(\|\nabla^{k-p} A\|_{L^\infty} + \frac{\|\nabla^{k-p} u_+\|_{L^1}}{\sqrt{t}} + \frac{\delta}{t^\mu} \right).$$

Now, let $k \in \llbracket 1, s-1 \rrbracket$, the first step is to use $L^\infty L^2$ Strichartz estimates on I :

$$\sup_{t_0 \leq t} t^\mu \|\nabla^k I(t)\|_{L^2} \leq \sup_{t_0 \leq t} t^\mu \int_t^\infty \|\nabla^k (|v|^2 v - |v_1|^2 v_1)\|_{L^2} + \|2M \nabla^k (v - v_1)\|_{L^2} \frac{d\tau}{\tau}.$$

We compute the first term here:

$$\begin{aligned} & \nabla^k (|v|^2 v - |v_1|^2 v_1) \\ &= \sum_{p=0}^k \binom{k}{p} (\nabla^p \overline{v} \nabla^{k-p} v^2 - \nabla^p \overline{v_1} \nabla^{k-p} v_1^2 + \nabla^p \overline{v_1} \nabla^{k-p} v^2 - \nabla^p \overline{v} \nabla^{k-p} v_1^2) \\ &= \sum_{p=0}^k \binom{k}{p} (\nabla^p (\overline{v - v_1}) \nabla^{k-p} v^2 + \nabla^p (v^2 - v_1^2) \nabla^{k-p} \overline{v_1}) \\ &= \sum_{p=0}^k \binom{k}{p} (\nabla^p (\overline{v - v_1}) \nabla^{k-p} v^2 + \nabla^p [(v - v_1)(v + v_1)] \nabla^{k-p} \overline{v_1}) \\ &= \sum_{p=0}^k \binom{k}{p} \left(\nabla^p (\overline{v - v_1}) \nabla^{k-p} v^2 + \sum_{q=0}^p \binom{p}{q} (\nabla^q (v - v_1) \nabla^{p-q} (v + v_1)) \nabla^{k-p} \overline{v_1} \right). \end{aligned}$$

Therefore, using the preliminary computations:

$$\begin{aligned} & \sup_{t_0 \leq t} t^\mu \|\nabla^k I(t)\|_{L^2} \\ & \leq \frac{2M}{\mu} \|v - v_1\|_S + \sup_{t_0 \leq t} t^\mu \int_t^\infty \left(\sum_{p=0}^k \binom{k}{p} \|\nabla^p (v - v_1)\|_{L^2} \|\nabla^{k-p} v^2\|_{L^\infty} \right. \\ & \quad \left. + \sum_{p=0}^k \sum_{q=0}^p \binom{k}{p} \binom{p}{q} \|\nabla^q (v - v_1)\|_{L^2} \|\nabla^{k-p} v_1\|_{L^\infty} (\|\nabla^{p-q} v\|_{L^\infty} + \|\nabla^{p-q} v_1\|_{L^\infty}) \right) \frac{d\tau}{\tau} \end{aligned}$$

$$\begin{aligned}
&\lesssim \frac{2M}{\mu} \|v - v_1\|_S + \sup_{t_0 \leq t} t^\mu \int_t^\infty \sum_{p=0}^k \frac{\|v - v_1\|_S}{\tau^{1+\mu}} \sum_{j=0}^{k-p} \binom{k-p}{j} \left(\|\nabla^j A\|_{L^\infty} + \frac{\|\nabla^j u_+\|_{L^1}}{\sqrt{\tau}} + \frac{\delta}{\tau^\mu} \right) \\
&\quad \left(\|\nabla^{k-p-j} A\|_{L^\infty} + \frac{\|\nabla^{k-p-j} u_+\|_{L^1}}{\sqrt{\tau}} + \frac{\delta}{\tau^\mu} \right) \\
&\quad + \sup_{t_0 \leq t} t^\mu \int_t^\infty \sum_{p=0}^k \sum_{q=0}^p \frac{\|v - v_1\|_S}{\tau^{1+\mu}} \left(\|\nabla^{k-p} A\|_{L^\infty} + \frac{\|\nabla^{k-p} u_+\|_{L^1}}{\sqrt{\tau}} \right) \\
&\quad \left(\|\nabla^{p-q} A\|_{L^\infty} + \frac{\|\nabla^{p-q} u_+\|_{L^1}}{\sqrt{\tau}} + \frac{\delta}{\tau^\mu} + \|\nabla^q A\|_{L^\infty} + \frac{\|\nabla^q u_+\|_{L^1}}{\sqrt{\tau}} \right) d\tau.
\end{aligned}$$

We are now left with integrations:

$$\begin{aligned}
\sup_{t_0 \leq t} t^\mu \|\nabla^k I(t)\|_{L^2} &\leq \|v - v_1\|_S \left(\frac{2M}{\mu} + \left(\frac{\|\alpha_j\|_{l^{2,q}}^2}{\mu} + \frac{C(t_0, \|\alpha_j\|_{l^{2,s}})}{(\mu + 2\gamma)t_0^{2\gamma}} + \frac{\|u_+\|_{W^{1,s}}}{\sqrt{t_0}(\frac{1}{2} + \mu)} + \frac{C\delta}{2\mu t_0^{2\mu}} \right)^2 \right. \\
&\quad + \left(\frac{\|\alpha_j\|_{l^{2,q}}}{\mu} + \frac{1}{(\mu + \gamma)t_0^\gamma} + \frac{\|u_+\|_{W^{1,s}}}{\sqrt{t_0}(\frac{1}{2} + \mu)} + \frac{\delta}{2\mu t_0^{2\mu}} \right) \\
&\quad \left. \left(\frac{\|\alpha_j\|_{l^{2,q}}}{\mu} + \frac{C(t_0, \|\alpha_j\|_{l^{2,s}})}{(\mu + \gamma)t_0^\gamma} + \frac{\|u_+\|_{W^{1,s}}}{\sqrt{t_0}(\frac{1}{2} + \mu)} \right) \right),
\end{aligned}$$

wich implies the conclusion of the Lemma for $k \in \llbracket 1, s-1 \rrbracket$.

For $k = s$, we proceed similarly but instead of estimating

$$\sup_{t_0 \leq t} t^\mu \int_t^\infty \|v - v_1\|_{L^2} \|v\|_{L^\infty} \|\nabla^s v\|_{L^\infty} \frac{d\tau}{\tau}$$

by using Gagliardo-Nirenberg (26) on $\|\nabla^s v\|_{L^\infty}$, we rather write

$$\sup_{t_0 \leq t} t^\mu \int_t^\infty (\|v - v_1\|_{L^\infty} \|v\|_{L^\infty} \|\nabla^s(v - v_1)\|_{L^2} + \|v - v_1\|_{L^2} \|v\|_{L^\infty} \|\nabla^s v_1\|_{L^\infty}) \frac{d\tau}{\tau}.$$

Similarly, instead of estimating

$$\sup_{t_0 \leq t} t^\mu \int_t^\infty \|v - v_1\|_{L^\infty} \|v_1\|_{L^2} \|\nabla^s v\|_{L^\infty} \frac{d\tau}{\tau},$$

we rather write

$$\sup_{t_0 \leq t} t^\mu \int_t^\infty (\|v - v_1\|_{L^\infty} \|v_1\|_{L^\infty} \|\nabla^s(v - v_1)\|_{L^2} + \|v - v_1\|_{L^2} \|v_1\|_{L^\infty} \|\nabla^s v_1\|_{L^\infty}) \frac{d\tau}{\tau},$$

and obtain a similar bound. \square

Lemma 2.3 (Estimation of $\nabla^k J_a$). *Let J_a defined by (19):*

$$J_a(t, x) = -2i \int_t^\infty e^{i(t-\tau)\partial_x^2} \sum_{p \neq j} e^{-i(|\alpha_p|^2 - |\alpha_j|^2) \ln \tau} e^{i(p-j)\frac{x}{2}} \frac{x}{\alpha_p \alpha_j} e^{i\frac{j^2-p^2}{4}\tau} e^{i\tau\partial_x^2} u_+ \frac{d\tau}{\tau},$$

then

$$\sup_{t_0 < t} t^\mu \|\nabla^k J_a\|_{L^2} \leq C(\|\alpha_j\|_{l^{2,k}}) \sup_{t_0 \leq t} \frac{t^\mu}{t} \left\| \frac{\widehat{u}_+}{\cdot + \mathbb{Z}/2} \right\|_{H^k}.$$

Proof. The first step is to perform a Fourier transform on the space variable and set the domain exploit the linear phase in x :

$$\begin{aligned} \nabla^k J_a(t, x) &= e^{it\partial_x^2} \int_t^\infty \int_{\mathbb{R}} (i\xi)^k e^{i\tau\xi^2} e^{ix\xi} \sum_{p \neq j} \overline{\alpha_p} \alpha_j e^{-i(|\alpha_p|^2 - |\alpha_j|^2) \ln \tau} e^{-i\tau(\xi - \frac{j-p}{2})^2} \\ &\quad \times \widehat{u}_+(\xi - \frac{j-p}{2}) e^{i\frac{j^2-p^2}{4}\tau} d\xi \frac{d\tau}{\tau} \\ &= e^{it\partial_x^2} \sum_{p \neq j} \overline{\alpha_p} \alpha_j \int_{\mathbb{R}} (i\xi)^k e^{ix\xi} \widehat{u}_+(\xi - \frac{j-p}{2}) \int_t^\infty e^{i\tau(j-p)(\frac{j+p}{4} + \xi - \frac{j-p}{4})} e^{-i(|\alpha_p|^2 - |\alpha_j|^2) \ln \tau} \frac{d\tau}{\tau} d\xi \end{aligned}$$

As $p \neq j$, for all ξ except $\xi = \frac{p}{2}$ we now perform an integration by parts on the τ variable to gain integrability:

$$\begin{aligned} &\int_t^\infty e^{i\tau(j-p)(\xi - \frac{j-p}{2} + \frac{j}{2})} e^{-i(|\alpha_p|^2 - |\alpha_j|^2) \ln \tau} \frac{d\tau}{\tau} \\ &= -\frac{e^{it(j-p)(\xi - \frac{j-p}{2} + \frac{j}{2})} e^{-i(|\alpha_p|^2 - |\alpha_j|^2) \ln t}}{i(j-p)(\xi - \frac{j-p}{2} + \frac{j}{2}) t} + \int_t^\infty \frac{e^{i\tau(j-p)(\xi - \frac{j-p}{2} + \frac{j}{2})}}{i(j-p)(\xi - \frac{j-p}{2} + \frac{j}{2})} \left(\frac{e^{-i(|\alpha_p|^2 - |\alpha_j|^2) \ln \tau}}{\tau} \right)_\tau d\tau. \end{aligned}$$

Undoing the Fourier transform, we have:

$$\begin{aligned} \nabla^k J_a(t, x) &= \frac{e^{it\partial_x^2} i}{t} \sum_{p \neq j} \frac{e^{-i(|\alpha_p|^2 - |\alpha_j|^2) \ln t}}{j-p} \overline{\alpha_p} \alpha_j e^{i\frac{j^2-p^2}{4}t} \nabla^k [e^{i(p-j)\frac{x}{2}} e^{it\partial_x^2} u_+^j(x)] \\ &\quad + \frac{e^{it\partial_x^2}}{i} \int_t^\infty \sum_{p \neq j} \left(\frac{e^{-i(|\alpha_p|^2 - |\alpha_j|^2) \ln \tau}}{\tau} \right)_\tau \overline{\alpha_p} \alpha_j e^{i\frac{j^2-p^2}{4}\tau} \nabla^k [e^{i(p-j)\frac{x}{2}} e^{i\tau\partial_x^2} u_+^j(x)] d\tau, \end{aligned}$$

where

$$(33) \quad \widehat{u}_+^j(\xi) = \frac{\widehat{u}_+(\xi)}{\xi + \frac{j}{2}}.$$

To conclude, we now take the L^2 norm and use the fact that $(\alpha_j) \in l^{2,k}$ to obtain that:

$$\sup_{t_0 < t} t^\mu \|\nabla^k J_a\|_{L^2} \leq C(\|\alpha_j\|_{l^{2,k}}) \sup_{t_0 \leq t} \frac{t^\mu}{t} \left\| \frac{\widehat{u}_+(\cdot)}{\cdot + \mathbb{Z}/2} \right\|_{H^k}.$$

□

Lemma 2.4 (Estimation of $\nabla^k J_b$). *Let J_b defined by (21):*

$$J_b = -2i \int_t^\infty e^{i(t-\tau)\partial_x^2} \sum_{(p,j) \in \mathbb{Z}^2} e^{-i(|\alpha_p|^2 - |\alpha_j|^2) \ln \tau + i(p-j)\frac{\pi}{2}} (\overline{\alpha_p} R_j + \overline{R_p} \alpha_j + \overline{R_p} R_j) \\ \times e^{i\frac{j^2-p^2}{4}\tau} e^{i\tau\partial_x^2} u_+ \frac{d\tau}{\tau},$$

then

$$\sup_{t_0 \leq t} t^\mu \|\nabla^k J_b(t)\|_{L^2} \leq C(t_0, \|\alpha_j\|_{l^{2,k}}) \sup_{t_0 \leq t} \frac{t^\mu}{t^\gamma} \|u_+\|_{H^k}.$$

Proof. This proof is straightforward: we exploit the integrability provided by the decay (4) of the R_j 's:

$$\|\nabla^k J_b(t)\|_{L_x^2} \leq \int_t^\infty \sum_{(p,j) \in \mathbb{Z}^2} \|e^{-i(|\alpha_p|^2 - |\alpha_j|^2) \ln \tau} (\overline{\alpha_p} R_j + \overline{R_p} \alpha_j + \overline{R_p} R_j) \\ \times e^{i\frac{j^2-p^2}{4}\tau} \nabla^k [e^{i(p-j)\frac{\pi}{2}} e^{i\tau\partial_x^2} \widehat{u_+}(x)]\|_{L_x^2} \frac{d\tau}{\tau} \\ \leq \int_t^\infty \sum_{(p,j) \in \mathbb{Z}^2} C(t_0, \|\alpha_j\|_{l^{2,k}}) \|u_+\|_{H^k} \frac{d\tau}{\tau^{1+\gamma}},$$

so

$$\sup_{t_0 \leq t} t^\mu \|\nabla^k J_b(t)\|_{L^2} \leq C(t_0, \|\alpha_j\|_{l^{2,k}}) \sup_{t_0 \leq t} \frac{t^\mu}{t^\gamma} \|u_+\|_{H^k}.$$

□

Lemma 2.5 (Estimation of $\nabla^k J_c$). *Let J_c defined by (22):*

$$J_c = -i \int_t^\infty e^{i(t-\tau)\partial_x^2} A^2 e^{-i\tau\partial_x^2} u_+ \frac{d\tau}{\tau},$$

then

$$\sup_{t_0 \leq t} t^\mu \|\nabla^k J_c\|_{L^2} \leq C(t_0, \|\alpha_j\|_{l^{2,k}}) \left(\sup_{t_0 < t} \frac{t^\mu}{t} \left\| \frac{\widehat{u_+}(\cdot)}{\cdot + \mathbb{Z}/2} \right\|_{H^{k-1}} + \sup_{t_0 < t} \frac{t^\mu}{t^\gamma} \|u_+\|_{H^k} \right).$$

Proof. The proof is in two steps, according to the following decomposition:

$$J_c = - \int_t^\infty e^{i(t-\tau)\partial_x^2} \sum_{(j,p) \in \mathbb{Z}^2} \overline{\alpha_p} \alpha_j e^{-i\tau\frac{p^2+j^2}{4} + ix\frac{p+j}{2}} e^{-i(|\alpha_j|^2 - |\alpha_p|^2) \ln \tau} e^{-i\tau\partial_x^2} u_+ \frac{d\tau}{\tau} \\ - \int_t^\infty e^{i(t-\tau)\partial_x^2} \sum_{(j,p) \in \mathbb{Z}^2} (\overline{\alpha_p} R_j + \overline{R_p} \alpha_j + \overline{R_p} R_j) e^{-i\tau\frac{p^2+j^2}{4} + ix\frac{p+j}{2}} e^{-i(|\alpha_j|^2 + |\alpha_p|^2) \ln \tau} \\ \times e^{-i\tau\partial_x^2} u_+ \frac{d\tau}{\tau} \\ = -J_{c1} - J_{c2}$$

The diagonal term of J_{c_1} could be controlled with inhomogeneous Strichartz with the couple $(4, \infty)$ of dual $(\frac{4}{3}, \infty)$, but we rather use IBP in Fourier space on the whole term to obtain a better time decay:

$$\begin{aligned}
\nabla^k J_{c_1} &= \nabla^k \int_t^\infty e^{i(t-\tau)\partial_x^2} \sum_{(j,p) \in \mathbb{Z}^2} \overline{\alpha_p} \alpha_j e^{-i\tau \frac{p^2+j^2}{4} + ix \frac{p+j}{2}} e^{-i(|\alpha_j|^2 + |\alpha_p|^2) \ln \tau} e^{-i\tau \partial_x^2} \overline{u_+} \frac{d\tau}{\tau} \\
&= e^{-it\partial_x^2} \int_t^\infty \int_{\mathbb{R}} (i\xi)^k e^{i\tau\xi^2} e^{ix\xi} \sum_{(j,p) \in \mathbb{Z}^2} \overline{\alpha_p} \alpha_j e^{-i\tau \frac{p^2+j^2}{4}} e^{-i(|\alpha_j|^2 + |\alpha_p|^2) \ln \tau} e^{i\tau(\xi + \frac{p+j}{2})^2} \\
&\quad \times \widehat{u_+}(\xi + \frac{p+j}{2}) d\xi \frac{d\tau}{\tau} \\
&= e^{-it\partial_x^2} \sum_{(j,p) \in \mathbb{Z}^2} \overline{\alpha_p} \alpha_j \int_{\mathbb{R}} (i\xi)^k e^{ix\xi} \widehat{u_+}(\xi + \frac{p+j}{2}) \\
&\quad \times \int_t^\infty e^{-i\tau \frac{p^2+j^2}{4} - i(|\alpha_j|^2 + |\alpha_p|^2) \ln \tau + i\tau \frac{(p+j)^2}{4}} e^{2i\tau\xi^2} e^{i\tau\xi(p+j)} \frac{d\tau}{\tau} d\xi \\
&= e^{-it\partial_x^2} \sum_{(j,p) \in \mathbb{Z}^2} \overline{\alpha_p} \alpha_j \int_{\mathbb{R}} (i\xi)^k e^{ix\xi} \widehat{u_+}(\xi + \frac{p+j}{2}) \\
&\quad \times \int_t^\infty e^{-i(|\alpha_j|^2 + |\alpha_p|^2) \ln \tau} e^{i\tau(2\xi(\xi + \frac{p+j}{2}) + \frac{p+j}{2})} \frac{d\tau}{\tau} d\xi
\end{aligned}$$

We now perform for any ξ except the countably set of roots of the phase an IBP on the time integral:

$$\begin{aligned}
&\int_t^\infty e^{-i(|\alpha_j|^2 + |\alpha_p|^2) \ln \tau} e^{i\tau(2\xi(\xi + \frac{p+j}{2}) + \frac{p+j}{2})} \frac{d\tau}{\tau} \\
&= - \frac{e^{-i(|\alpha_j|^2 + |\alpha_p|^2) \ln t} e^{it(2\xi(\xi + \frac{p+j}{2}) + \frac{p+j}{2})}}{(2i\xi(\xi + \frac{p+j}{2}) + \frac{p+j}{2})t} + \int_t^\infty \frac{e^{i\tau(2\xi(\xi + \frac{p+j}{2}) + \frac{p+j}{2})}}{2\xi(\xi + \frac{p+j}{2}) + \frac{p+j}{2}} \left(\frac{e^{-i(|\alpha_j|^2 + |\alpha_p|^2) \ln \tau}}{\tau} \right)_{\tau} d\tau.
\end{aligned}$$

Undoing the Fourier transform, we have:

$$\begin{aligned}
\nabla^k J_{c_1} &= \frac{ie^{it\partial_x^2}}{2t} \sum_{(j,p) \in \mathbb{Z}^2} \overline{\alpha_j} \alpha_p e^{-it \frac{p^2+j^2}{4}} e^{-i(|\alpha_j|^2 - |\alpha_p|^2) \ln t} \nabla^{k-1} [e^{it \frac{(p+j)}{2} x} e^{-it\partial_x^2} \overline{u_+^{pj}}(x)] \\
&\quad + \frac{e^{it\partial_x^2}}{2i} \sum_{(j,p) \in \mathbb{Z}^2} \int_t^\infty \overline{\alpha_j} \alpha_p e^{-i\tau \frac{p^2+j^2}{4}} \left(\frac{e^{-i(|\alpha_j|^2 + |\alpha_p|^2) \ln \tau}}{\tau} \right)_{\tau} \nabla^{k-1} [e^{i\tau \frac{(p+j)}{2} x} e^{-i\tau\partial_x^2} \overline{u_+^{pj}}(x)].
\end{aligned}$$

where u_+^{pj} has been defined by (33). As for J_a , we conclude with this term by taking the L_2 norm on the space variable and use the fact that $(\alpha_j) \in l^{2,q}$:

$$\sup_{t_0 < t} t^\mu \|\nabla^k J_{c_1}\|_{L_x^2} \leq C(\|\alpha_j\|_{l^{2,k}}) \sup_{t_0 < t} \frac{t^\mu}{t} \left\| \frac{\widehat{u_+}(\cdot)}{\cdot + \mathbb{Z}/2} \right\|_{H^{k-1}}.$$

Finally, as for J_b we use (4):

$$\begin{aligned} \sup_{t_0 < t} t^\mu \|\nabla^k J_{c_2}\|_{L_x^2} &\leq C(t_0, \|\alpha_j\|_{l^{2,k}}) \sup_{t_0 < t} t^\mu \int_t^\infty \frac{1}{\tau^\gamma} \left\| \nabla^k \overline{u_+} \right\|_{L^2} \frac{d\tau}{\tau} \\ &\leq C(t_0, \|\alpha_j\|_{l^{2,k}}) \|u_+\|_{H^k} \sup_{t_0 < t} \frac{t^\mu}{t^\gamma}, \end{aligned}$$

hence:

$$\sup_{t_0 \leq t} t^\mu \|\nabla^k J_c\|_{L^2} \leq C(t_0, \|R_j\|_{l^{2,k}}) \left(\sup_{t_0 < t} \frac{t^\mu}{t} \left\| \frac{\widehat{u_+}(\cdot)}{\cdot + \mathbb{Z}/2} \right\|_{H^{k-1}} + \sup_{t_0 < t} \frac{t^\mu}{t^\gamma} \|u_+\|_{H^k} \right).$$

□

Lemma 2.6 (Estimation of $\nabla^k J_d$). *Let J_d defined by (23):*

$$J_d = -i \int_t^\infty e^{i(t-\tau)\partial_x^2} (2A|e^{i\tau\partial_x^2} u_+|^2 + \overline{A}(e^{i\tau\partial_x^2} u_+)^2) \frac{d\tau}{\tau},$$

then

$$\sup_{t_0 \leq t} t^\mu \|\nabla^k J_d\|_{L^2} \leq C(t_0, \|\alpha_j\|_{l^{2,k}}) \sup_{t_0 \leq t} \frac{t^\mu}{t^{\frac{1}{2}}} \|u_+\|_{L^1} \|u_+\|_{H^k}.$$

As announced, this is the limiting term for the time decay.

Proof. The proof is straightforward, we just use direct upper-bounds and dispersion estimate (25) on the Schrödinger group.

$$\begin{aligned} \sup_{t_0 \leq t} t^\mu \|\nabla^k J_d\|_{L^2} &\leq \sup_{t_0 \leq t} t^\mu \int_t^\infty \sum_{j \in \mathbb{Z}} (|\alpha_j| + |R_j|) \|\nabla^k [e^{i\frac{xj}{2}} |e^{i\tau\partial_x^2} u_+|^2 + e^{-i\frac{xj}{2}} (e^{i\tau\partial_x^2} u_+(x))^2]\|_{L^2} \frac{d\tau}{\tau} \\ &\leq C(t_0, \|\alpha_j\|_{l^{2,k}}) \sup_{t_0 \leq t} t^\mu \int_t^\infty \|u_+\|_{H^k} \|e^{i\tau\partial_x^2} u_+\|_{L^\infty} \frac{d\tau}{\tau} \\ &\leq C(t_0, \|\alpha_j\|_{l^{2,k}}) \sup_{t_0 \leq t} t^\mu \int_t^\infty \|u_+\|_{H^k} \frac{\|u_+\|_{L^1}}{\sqrt{\tau}} \frac{d\tau}{\tau} \\ &\leq C(t_0, \|\alpha_j\|_{l^{2,k}}) \sup_{t_0 \leq t} \frac{t^\mu}{t^{\frac{1}{2}}} \|u_+\|_{L^1} \|u_+\|_{H^k}. \end{aligned}$$

□

The last term J_e is easier to control, using again Schrödinger dispersion inequality.

Lemma 2.7 (Estimation of $\nabla^k J_e$). *Let J_e defined by (24):*

$$J_e = -i \int_t^\infty e^{i(t-\tau)\partial_x^2} |e^{i\tau\partial_x^2} u_+|^2 e^{i\tau\partial_x^2} u_+ \frac{d\tau}{\tau},$$

then

$$\sup_{t_0 \leq t} t^\mu \|\nabla^k J_e\|_{L^2} \lesssim \sup_{t_0 \leq t} \frac{t^\mu}{t} \|u_+\|_{L^1}^2 \|u_+\|_{H^s}$$

Proof. It is straightforward to write that:

$$\begin{aligned} \|J_e(t)\|_{H^s} &\leq \int_t^\infty \| |e^{i\tau\partial_x^2} u_+|^2 e^{i\tau\partial_x^2} u_+ \|_{H^s} \frac{d\tau}{\tau} \\ &\lesssim \int_t^\infty \|e^{i\tau\partial_x^2} u_+\|_{L^\infty}^2 \|e^{i\tau\partial_x^2} u_+\|_{H^s} \frac{d\tau}{\tau} \lesssim \frac{\|u_+\|_{L^1}^2 \|u_+\|_{H^s}}{t}. \end{aligned}$$

□

2.3. The stability.

Regarding the results of the previous part,

$$\|\phi(v) - v_1\|_S \leq C(t_0, u_+, \|\alpha_j\|_{l^{2,q}}) \left(\|v - v_1\|_S + \sup_{t_0 < t} \frac{t^\mu}{t} + \sup_{t_0 < t} \frac{t^\mu}{t^\gamma} + \sup_{t_0 < t} \frac{t^\mu}{t^{\frac{1}{2}}} \right),$$

so we have the stability inclusion $\phi(X_\delta) \subset X_\delta$ for $\|\alpha_j\|_{l^{2,q}}$ and δ small and t_0 large enough, and choosing $\mu < \max(\frac{1}{2}, \gamma)$.

2.4. The contraction.

In order to apply the fixed point, we still have to prove that ϕ contracts.

Let v and \tilde{v} in Y .

From

$$\phi(v) - \phi(\tilde{v}) = \int_t^\infty e^{i(t-\tau)\partial_x^2} \left(\frac{|v|^2 - 2M}{2\tau} v - \frac{|\tilde{v}|^2 - 2M}{2\tau} \tilde{v} \right) d\tau,$$

yields

$$\begin{aligned} &\sup_{t_0 \leq t} t^\mu \|\phi(v) - \phi(\tilde{v})\|_{L^2} \\ &\lesssim \sup_{t_0 \leq t} t^\mu \int_t^\infty \| |v|^2 v - |\tilde{v}|^2 \tilde{v} - 2M(v - \tilde{v}) \|_{L^2} \frac{d\tau}{\tau} \\ &\lesssim \sup_{t_0 \leq t} t^\mu \int_t^\infty (2M + \|v\|_{L^\infty}^2 + \|\tilde{v}\|_{L^\infty}^2) \|v - \tilde{v}\|_{L^2} \frac{d\tau}{\tau} \\ &\lesssim \|v - \tilde{v}\|_2 \left(\frac{2M}{\mu} + \sup_{t_0 \leq t} t^\mu \int_t^\infty (\|v(\tau)\|_{L^\infty}^2 + \|\tilde{v}(\tau)\|_{L^\infty}^2) \frac{d\tau}{\tau^{1+\mu}} \right) \\ &\lesssim \|v - \tilde{v}\|_2 \left(\frac{2M}{\mu} + \sup_{t_0 \leq t} t^\mu \int_t^\infty (2\|v - v_1\|_{L^\infty}^2 + 2\|\tilde{v} - v_1\|_{L^\infty}^2 + 4\|v_1\|_{L^\infty}^2) \frac{d\tau}{\tau^{1+\mu}} \right) \\ &\lesssim \|v - \tilde{v}\|_2 \left(\frac{2M}{\mu} + \sup_{t_0 \leq t} t^\mu \int_t^\infty (8\|A\|_{L^\infty}^2 + 8\frac{\|u_+\|_{L^1}^2}{\tau} + 4\frac{\delta^2}{\tau^{2\mu}}) \frac{d\tau}{\tau^{1+\mu}} \right) \\ &\lesssim \|v - \tilde{v}\|_2 \left(\frac{2M}{\mu} + \frac{8\|A\|_{L^\infty}}{\mu} + \frac{8\|u_+\|_{L^1}^2}{t_0(1+\mu)} + \frac{(2\delta)^2}{t_0^{2\mu}(1+2\mu)} \right). \end{aligned}$$

Moreover, for $k \in \llbracket 1, s-1 \rrbracket$:

$$\begin{aligned}
& \sup_{t_0 \leq t} t^\mu \|\nabla^k(\phi(v) - \phi(\tilde{v}))\|_{L^2} \\
& \leq \frac{2M}{\mu} \|v - \tilde{v}\|_S + \sup_{t_0 \leq t} t^\mu \int_t^\infty \left(\sum_{p=0}^k \binom{k}{p} \|\nabla^p(v - \tilde{v})\|_{L^2} \|\nabla^{k-p} v^2\|_{L^\infty} \right. \\
& \quad \left. + \sum_{p=0}^k \sum_{q=0}^p \binom{k}{p} \binom{p}{q} \|\nabla^q(v - \tilde{v})\|_{L^2} \|\nabla^{k-p} \tilde{v}\|_{L^\infty} (\|\nabla^{p-q} v\|_{L^\infty} + \|\nabla^{p-q} \tilde{v}\|_{L^\infty}) \right) \frac{d\tau}{\tau} \\
& \lesssim \|v - \tilde{v}\|_S \left(\frac{2M}{\mu} + \int_t^\infty \sum_{p=0}^k \binom{k}{p} \sum_{q=0}^{k-p} \binom{k-p}{q} \left(\|\nabla^q A\|_{L^\infty} + \frac{\|\nabla^q u_+\|}{\sqrt{\tau}} + \frac{\delta}{\tau^\mu} \right) \right. \\
& \quad \left. \left(\|\nabla^{k-p-q} A\|_{L^\infty} + \frac{\|\nabla^{k-p-q} u_+\|}{\sqrt{\tau}} + \frac{\delta}{\tau^\mu} \right) \frac{d\tau}{\tau^{1+\mu}} \right) \\
& \quad + \|v - \tilde{v}\|_S \int_t^\infty \sum_{p=0}^k \sum_{q=0}^p \binom{k}{p} \binom{p}{q} \left(\|\nabla^{k-p} A\|_{L^\infty} + \frac{\|\nabla^{k-p} u_+\|}{\sqrt{\tau}} + \frac{\delta}{\tau^\mu} \right) \\
& \quad 2 \left(\|\nabla^{p-q} A\|_{L^\infty} + \frac{\|\nabla^{p-q} u_+\|}{\sqrt{\tau}} + \frac{\delta}{\tau^\mu} \right) \frac{d\tau}{\tau}.
\end{aligned}$$

As for Lemma 2.2, we must avoid estimating

$$\sup_{t_0 \leq t} t^\mu \int_t^\infty \|v - \tilde{v}\|_{L^2} \|w_1\|_{L^\infty} \|\nabla^s w_2\|_{L^\infty} \frac{d\tau}{\tau},$$

by using Gagliardo-Nirenberg (26) on $\|\nabla^s v\|_{L^\infty}$. Therefore we rather write

$$\sup_{t_0 \leq t} t^\mu \int_t^\infty (\|v - \tilde{v}\|_{L^\infty} \|w_1\|_{L^\infty} \|\nabla^s(w_2 - v_1)\|_{L^2} + \|v - \tilde{v}\|_{L^2} \|w_1\|_{L^\infty} \|\nabla^s v_1\|_{L^\infty}) \frac{d\tau}{\tau},$$

for $(w_1, w_2) \in \{\tilde{v}, v\}^2$.

To conclude, we finally use the Gagliardo-Nirenberg interpolation inequality

$$\|v - \tilde{v}\|_{L^\infty} \leq \|v - \tilde{v}\|_{L^2}^{\frac{1}{2}} \|\nabla(v - \tilde{v})\|_{L^2}^{\frac{1}{2}} \leq \frac{\|v - \tilde{v}\|_S}{\tau^\mu}$$

so that

$$\begin{aligned}
& \sup_{t_0 \leq t} t^\mu \int_t^\infty (\|v - \tilde{v}\|_{L^\infty} \|w_1\|_{L^\infty} \|\nabla^s(w_2 - v_1)\|_{L^2} + \|v - \tilde{v}\|_{L^2} \|w_1\|_{L^\infty} \|\nabla^s v_1\|_{L^\infty}) \frac{d\tau}{\tau} \\
& \leq \sup_{t_0 \leq t} t^\mu \|v - \tilde{v}\|_S \int_t^\infty \|w_1\|_{L^\infty} \left(\frac{\|w_2\|_S}{\tau^\mu} + \|\nabla^s A\|_{L^\infty} + \frac{\|\nabla^k u_+\|_{L^1}}{\sqrt{\tau}} \right) \frac{d\tau}{\tau^{1+\mu}}.
\end{aligned}$$

Putting it together, we have that

$$\|\phi(v) - \phi(\tilde{v})\|_S \leq \|v - \tilde{v}\|_S C(t_0, \|\alpha_j\|_{l^2, q}, u_+),$$

therefore ϕ contracts for $\|\alpha_j\|_{l^{2,q}}$ and δ small and t_0 large enough, and solutions exists on $[t_0, \infty)$. This ends the first part of existence of solutions of Theorem 1.1.

2.5. End of the proof: estimates (7) to (9).

Let v the solution of (12) on $[t_0, \infty)$ constructed in Section 2. Note that we also dispose of the following estimate (15) on v :

$$(34) \quad \forall k \in \llbracket 0, s \rrbracket, \quad \|\nabla^k(v - v_1)(t)\|_{L^2} = \mathcal{O}(t^{-\frac{1}{2}^-}).$$

First we set $T_1 = \frac{1}{t_0}$ and use the pseudo conformal transformation (11) on v to obtain ψ , a solution of (10) on $(0, T_1]$ that satisfies:

$$v(t, x) = \mathcal{T}(\psi)(t, x) = \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}} \bar{\psi} \left(\frac{1}{t}, \frac{x}{t} \right).$$

Since v_1 is defined by (13), we have that:

$$v_1(t, x) = \mathcal{T}(e^{2iM \ln t} u_{\{\alpha_j\}}(t, x)) + e^{it\partial_x^2} u_+(x),$$

and estimate (34) becomes:

$$\forall k \in \llbracket 0, s \rrbracket, \quad \left\| \nabla^k \left[\mathcal{T} \left(\psi(t, x) - e^{2iM \ln t} u_{\{\alpha_j\}}(t, x) \right) - e^{it\partial_x^2} u_+(x) \right] \right\|_{L^2} = \mathcal{O}(t^{-\frac{1}{2}^-}).$$

Then we normalize back the solutions ψ of (10) to obtain an estimate on the solutions u of (NLS) on $(0, T_1]$. As \mathcal{T} preserves the L^2 norm, we use the fact that, for J defined in the introduction by (5) we have:

$$\nabla^k \mathcal{T} = \mathcal{T}(-i)^k J^k,$$

to obtain estimate (7):

$$\forall k \in \llbracket 0, s \rrbracket, \quad \left\| (-i)^k J^k [u(t, x) - u_{\{\alpha_j\}}(t, x)] - e^{-2iM \ln t} \mathcal{T} \left(e^{it\partial_x^2} \nabla^k u_+(x) \right) \right\|_{L^2} = \mathcal{O}(t^{\frac{1}{2}^-}).$$

Finally, when $\nabla^k u_+ \in (x^4 L^2)$ for all $k \in \llbracket 0, s \rrbracket$, the L^2 estimate (8) comes from Schrödinger evolution properties applied to the perturbation and has already been done in section 2.2 of [1]:

$$(35) \quad \left\| \mathcal{T} \left(e^{it\partial_x^2} \nabla^k u_+(x) \right) - \left(-i \frac{x}{2} \right)^k \widehat{u_+} \left(-\frac{x}{2} \right) \right\|_{L^2} = \mathcal{O}(t).$$

The last estimate (9) is exactly estimate (8) with $k = 0$ and Theorem 1.1 is proven.

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