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# A note on the duality of skew module codes.

D. Boucher \*

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## Abstract

We introduce a new notion of duality inspired from the paper *On the duality and the direction of polycyclic codes* by Adel Alahmadi, Steven Dougherty, André Leroy and Patrick Solé. We get that the dual of a central skew module code is a central skew module code.

## 1 Introduction.

Consider the finite field  $\mathbb{F}_q$  with  $q$  elements and a non-negative integer  $n$ . A linear code over  $\mathbb{F}_q$  of length  $n$  is a subspace of  $\mathbb{F}_q^n$ . The *dual* of a linear code  $C$  of length  $n$  over  $\mathbb{F}_q$  is defined as  $C^\perp = \{x \in \mathbb{F}_q^n \mid \forall y \in C, \langle x, y \rangle = 0\}$  where  $\langle \cdot, \cdot \rangle$  is an inner product over  $\mathbb{F}_q^n \times \mathbb{F}_q^n$ . The code  $C$  is *self-dual* if  $C$  is equal to  $C^\perp$ . Cyclic codes over  $\mathbb{F}_q$  form a class of linear codes who are invariant under a cyclic shift of coordinates. This cyclicity condition enables to describe a cyclic code as an ideal  $(g)/(X^n - 1)$  of  $\mathbb{F}_q[X]/(X^n - 1)$  where  $g$  is a monic divisor of  $X^n - 1$ . If we replace  $X^n - 1$  with a polynomial  $f \in \mathbb{F}_q[X]$  of degree  $n$  we get a polycyclic code. It is well known that the Euclidean dual of a cyclic code is a cyclic code and self-dual cyclic codes have been extensively studied ([10], [13], ...). However the dual of a polycyclic code is not polycyclic. In [1], an inner product is defined over  $\mathbb{F}_q^n$  in such a way that the dual of a polycyclic code is a polycyclic code. In this note, we will design a new notion of duality for polycyclic codes and skew module codes.

In Section 2, we give some generalities on skew module codes. In Section 3, we design a new notion of duality based on skew polynomials. In Section 4, we characterize self-dual skew module codes by an equation called self-dual skew module equation and in Section 5, we give some clues for the resolution of this equation when  $q = p^2$ .

## 2 Generalities on skew module codes.

For an automorphism  $\theta$  of  $\mathbb{F}_q$ , one considers the ring  $R = \mathbb{F}_q[X; \theta]$  where addition is defined to be the usual addition of polynomials and where multiplication is defined by the rule: for  $a$  in  $\mathbb{F}_q$

$$X \cdot a = \theta(a) X. \tag{1}$$

The ring  $R$  is called a skew polynomial ring or Ore ring (cf. [12]) and its elements are skew polynomials. When  $\theta$  is not the identity, the ring  $R$  is not commutative, it is a left and right

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Euclidean ring whose left and right ideals are principal. Left and right gcd and lcm exist in  $R$  and can be computed using the left and right Euclidean algorithms. The center of  $R$  is the commutative polynomial ring  $Z(R) = \mathbb{F}_q^\theta[X^m]$  where  $\mathbb{F}_q^\theta$  is the fixed field of  $\theta$  and  $m$  is the order of  $\theta$ .

**Definition 1 ([6])** Consider  $f$  in  $R$  of degree  $n$ . A  $\theta$ -module code or skew module code  $C$  is a  $R$ -sub-module on the left  $Rg/Rf \subset R/Rf$  where  $g$  is a right divisor of  $f$  in  $R$ . Its length is  $n = \deg(f)$  and its dimension is  $k = \deg(f) - \deg(g)$ . The skew polynomial  $g$  is a (skew) generator polynomial of  $C$ . If  $g$  is monic,  $g$  is the (monic) skew generator polynomial of  $C$ .

If  $f = X^n - a$  with  $a \in \mathbb{F}_q$ , one says that the code  $C$  is  $(\theta, a)$ -constacyclic. It is  $\theta$ -cyclic if  $a = 1$  and  $\theta$ -negacyclic if  $a = -1$ .

For  $x, y$  in  $\mathbb{F}_q^n$ ,  $\langle x, y \rangle_E := \sum_{i=1}^n x_i y_i$  is the (Euclidean) scalar product of  $x$  and  $y$ . The code  $C$  is (Euclidean) self-dual if  $C$  is equal to  $C^\perp$ . Assume that  $\sigma$  is an automorphism of  $\mathbb{F}_q$  of order 2. The (Hermitian) dual of a linear code  $C$  of length  $n$  over  $\mathbb{F}_q$  is defined as  $C^{\perp_H} = \{x \in \mathbb{F}_q^n \mid \forall y \in C, \langle x, y \rangle_H = 0\}$  where for  $x, y$  in  $\mathbb{F}_q^n$ ,  $\langle x, y \rangle_H := \sum_{i=1}^n x_i \sigma(y_i)$  is the (Hermitian) scalar product of  $x$  and  $y$ . The code  $C$  is (Hermitian) self-dual if  $C$  is equal to  $C^{\perp_H}$ .

If  $C$  is  $\theta$ -module code of length  $n$ , either it is  $\theta$ -constacyclic and then its (Euclidean) dual is a  $\theta$ -constacyclic code (Theorem 1 and Lemma 2 of [8]); either it is the shortened code of a  $\theta$ -cyclic code (of length  $N > n$ ) and its (Euclidean) dual is a punctured code of a  $\theta$ -cyclic code (Proposition 3 of [8]). Furthermore, in [11], the (Euclidean) dual of a  $\theta$ -module code (also called  $\theta$ -polycyclic code) is identified as a  $\theta$ -sequential code (see Theorem 2 of [11]).

In [1] an inner product is introduced in such a way that the dual of a polycyclic code (i.e. a  $\theta$ -module code for  $\theta = id$ ) is polycyclic. In this note, we want to design a new notion of duality such that the dual of a polycyclic code is a polycyclic code and the dual of a skew module code is a skew module code. Note that when  $\theta = id$ , the dual defined in this note is not the same as the one obtained in [1] (see Remark 1 and Remark 3).

We conclude this introductory part with some material on skew reciprocal polynomials which will be useful in this note.

**Definition 2 (Definition 3 of [8] or Definition 2 of [2])** Consider  $h = \sum_{i=0}^k h_i X^i \in R$  of degree  $k$ . The skew reciprocal polynomial of  $h$  is

$$h^* = \sum_{i=0}^k X^{k-i} \cdot h_i = \sum_{i=0}^{k-v} \theta^i(h_{k-i}) X^i$$

and the monic skew reciprocal polynomial of  $h$  is

$$h^\natural = \frac{1}{\theta^{k-v}(h_v)} h^* = X^{k-v} + \sum_{i=0}^{k-v-1} \frac{\theta^i(h_{k-i})}{\theta^{k-v}(h_v)} X^i$$

where  $v = \min\{i \mid h_i \neq 0\}$  is the valuation of  $h$ .

In what follows, we will denote

$$\theta : \begin{cases} R & \rightarrow R \\ \sum a_i X^i & \mapsto \sum \theta(a_i) X^i \end{cases} \quad \text{and} \quad \sigma : \begin{cases} R & \rightarrow R \\ \sum a_i X^i & \mapsto \sum \sigma(a_i) X^i. \end{cases}$$

**Lemma 1 (Lemma 1 of [8])** Consider  $f, g, h$  in  $R$  non zero.

1.  $(h \cdot g)^* = \theta^{\deg(h)}(g^*) \cdot h^*$ .
2. Let  $d$  be the degree of  $f$  and let  $v$  be the valuation of  $f$ , then  $(f^*)^* \cdot X^v = \theta^{d-v}(f)$ .

We will use a slight modification of the skew reciprocal polynomial : for  $h \in R$  of degree less than  $n$ , the  $n$ -skew reciprocal polynomial of  $h$  is

$$r_n(h) := X^{n-\deg(h)} \cdot h^* = \sum_{i=n-k}^{n-v} \theta^i(h_{n-i})X^i \quad (2)$$

where  $k$  is the degree of  $h$  and  $v$  is its valuation.

In particular, for  $g, h$  in  $R$  of degree less than  $n$  we have

$$r_n(g + h) = r_n(g) + r_n(h).$$

### 3 A notion of duality based on skew polynomials.

In [1], an inner product  $\langle \cdot, \cdot \rangle_f$  is defined over  $\mathbb{F}_q^n$  in the following way. Consider  $f$  in  $\mathbb{F}_q[X]$  of degree  $n$  and  $a = (a_0, \dots, a_{n-1}), b = (b_0, \dots, b_{n-1})$  in  $\mathbb{F}_q^n$ . Associate to  $a, b$  the polynomials  $a(X) = \sum_{i=0}^{n-1} a_i X^i$  and  $b(X) = \sum_{i=0}^{n-1} b_i X^i$  in  $\mathbb{F}_q[X]$ . The  $f$ -scalar product of  $a$  and  $b$  is defined as  $\langle a, b \rangle_f = Q(0)$  where  $Q(X)$  is the remainder in the division of  $a(X)b(X) \in \mathbb{F}_q[X]$  by  $f(X)$ . Inspired by the work of [1], we consider here a new map  $\langle \cdot, \cdot \rangle_{f,\theta,\sigma}$  where  $f(X)$  is a monic central polynomial of  $R = \mathbb{F}_q[X; \theta]$ ,  $\theta$  is an automorphism of  $\mathbb{F}_q$  and  $\sigma$  is an automorphism of  $\mathbb{F}_q$  such that  $\sigma^2 = id$ .

In what follows, we associate to  $a = (a_0, \dots, a_{n-1}) \in \mathbb{F}_q^n$  the skew polynomial  $a(X) = \sum_{i=0}^{n-1} a_i X^i$  in  $R$ . Furthermore we assume that  $f$  is a monic central polynomial.

**Definition 3** The map  $\langle \cdot, \cdot \rangle_{f,\theta,\sigma}$  from  $\mathbb{F}_q^n \times \mathbb{F}_q^n$  to  $\mathbb{F}_q$  is defined by : for  $a, b$  in  $\mathbb{F}_q^n$

$$\langle a, b \rangle_{f,\theta,\sigma} = P(0) \quad (3)$$

where  $P(X)$  is the remainder in the division on the right of the skew polynomial  $a(X) \cdot \sigma(r_n(b(X)))$  by  $f(X)$  and  $P(0)$  is the constant coefficient of  $P(X)$ .

**Remark 1** Consider  $\theta = \sigma = id$ ,  $f \in \mathbb{F}_q[X]$  of degree  $n$ ,  $a \in \mathbb{F}_q^n$  and  $b \in \mathbb{F}_q^n$ . We have  $\langle a, b \rangle_{f,id,\sigma} = P(0)$  where  $P(X)$  is the remainder in the division of  $a(X) \cdot r_n(b(X))$  by  $f(X)$  in  $\mathbb{F}_q[X]$ . Meanwhile the scalar product defined in [1]  $\langle \cdot, \cdot \rangle_f$  is defined by  $\langle a, b \rangle_f = Q(0)$  where  $Q(X)$  is the remainder in the division of  $a(X)b(X)$  by  $f(X)$  in  $\mathbb{F}_q[X]$ .

We recall that a  $\sigma$ -sesquilinear form (see [14]) on  $\mathbb{F}_q^n$  is defined as a map  $\langle \cdot, \cdot \rangle : \mathbb{F}_q^n \times \mathbb{F}_q^n \rightarrow \mathbb{F}_q$  such that if  $x, y, z \in \mathbb{F}_q^n$  and  $a \in \mathbb{F}_q$  then  $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$ ,  $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$ ,  $\langle ax, y \rangle = a \langle x, y \rangle$  and  $\langle x, ay \rangle = \langle x, y \rangle \sigma(a)$ .

**Proposition 1** The map  $\langle \cdot, \cdot \rangle_{f,\theta,\sigma}$  is a  $\sigma$ -sesquilinear form.

**Proof.** We denote  $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{f, \theta, \sigma}$ . Consider  $a, b, c$  in  $\mathbb{F}_q^n$  and  $\lambda$  in  $\mathbb{F}_q$ . We have  $(a(X) + b(X)) \cdot \sigma(r_n(c(X))) = a(X) \cdot \sigma(r_n(c(X))) + b(X) \cdot \sigma(r_n(c(X)))$  therefore  $\langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$ . We have  $r_n(b(X) + c(X)) = r_n(b(X)) + r_n(c(X))$  therefore  $a(X) \cdot \sigma(r_n(b(X) + c(X))) = a(X) \cdot \sigma(r_n(b(X))) + a(X) \cdot \sigma(r_n(c(X)))$  and  $\langle a, b + c \rangle = \langle a, b \rangle + \langle a, c \rangle$ .

Consider  $P(X)$  the remainder in the division on the right of  $a(X) \cdot \sigma(r_n(b(X)))$  by  $f(X)$  in  $R$ . Then  $\lambda P(X)$  is the remainder in the division on the right of  $\lambda a(X) \cdot \sigma(r_n(b(X)))$  by  $f(X)$  in  $R$  and we have  $\langle \lambda a, b \rangle = (\lambda P)(0) = \lambda \langle a, b \rangle$ . We have  $a(X) \sigma(r_n(\lambda b(X))) = a(X) \cdot \sigma(r_n(b(X))) \sigma(\lambda)$  and as  $f(X)$  is central, the remainder in the division of  $a(X) \cdot \sigma(r_n(b(X))) \lambda$  by  $f(X)$  on the right is  $P(X) \cdot \sigma(\lambda)$ , therefore  $\langle a, \lambda b \rangle = (P \cdot \sigma(\lambda))(0) = \sigma(\lambda) P(0) = \langle a, b \rangle \sigma(\lambda)$ .  
■

**Definition 4** Consider a linear code  $C$  over  $\mathbb{F}_q$  with length  $n$ .

The left dual of  $C$  for  $\langle \cdot, \cdot \rangle_{f, \theta, \sigma}$  is defined as

$$l(C) = l_{f, \theta, \sigma}(C) = \{x \in \mathbb{F}_q^n \mid \forall c \in C, \langle x, c \rangle_{f, \theta, \sigma} = 0\}. \quad (4)$$

The right dual of  $C$  is defined as

$$r(C) = r_{f, \theta, \sigma}(C) = \{x \in \mathbb{F}_q^n \mid \forall c \in C, \langle c, x \rangle_{f, \theta, \sigma} = 0\}. \quad (5)$$

In the proposition below, we make the link with the Euclidean dual and the Hermitian dual of linear codes.

**Proposition 2** Assume that  $f = X^n - \epsilon$  is a central polynomial with  $\epsilon \neq 0$ .

- If  $\sigma = id$ , then  $r(C) = l(C)$  is the dual  $C^\perp$  of  $C$  for the Euclidean scalar product.
- If  $\sigma$  has order 2, then  $r(C) = l(C)$  is the dual  $C^{\perp H}$  of  $C$  for the Hermitian scalar product.

**Proof.** Consider  $a, b \in \mathbb{F}_q^n$ . The constant coefficient of  $P(X) = a(X) \cdot \sigma(r_n(b(X))) = \sum_{i=0}^{n-1} a_i X^i \cdot \sum_{j=0}^{n-1} X^{n-j} \cdot \sigma(b_j) \in R/Rf$  is  $\epsilon \times \sum_{i=0}^{n-1} a_i \theta^{i+n-i}(\sigma(b_i)) = \epsilon \times \sum_{i=0}^{n-1} a_i \sigma(b_i)$ . Therefore for  $a, b \in \mathbb{F}_q^n$ ,  $\langle a, b \rangle_{f, \theta, \sigma} = 0 \Leftrightarrow \sum_{i=0}^{n-1} a_i \times \sigma(b_i) = 0$ .  
■

In what follows, we give an analogue of the MacWilliams formula for  $l(C)$  and  $r(C)$  inspired from [1].

**Lemma 2** If  $f(0) \neq 0$ , then the  $\sigma$ -sesquilinear form  $\langle \cdot, \cdot \rangle_{f, \theta, \sigma}$  is non-degenerate.

**Proof.** We denote  $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{f, \theta, \sigma}$  and  $F(X) = \frac{-1}{f(0)} \frac{f(X) - f(0)}{X}$ , which is well-defined as  $f(0) \neq 0$ . We have  $F(X) \cdot X = X \cdot F(X) = 1$  in  $R/Rf$ .

Consider  $a$  in  $\mathbb{F}_q^n$  and assume that for all  $b$  non-zero in  $\mathbb{F}_q^n$  we have  $\langle a, b \rangle = 0$ . Then for  $b = 1$ , we get  $\langle a, 1 \rangle = 0$  therefore  $a(0) = 0$  and  $a(X) = a'(X)X$ . Denote  $v$  the degree of the lowest term of  $F(X)$  and consider  $b(X) = r_n(F(X))$ . Then  $b(X) = X \cdot F^*(X)$ , therefore  $\deg(b) = 1 + n - 1 - v = n - v$  and  $b^*(X)X^v = (XF^*(X))^* X^v = (F^*(X))^* X^v = F(X)$ . We conclude that  $r_n(b(X)) = X^{n-(n-v)} b^*(X) = F(X)$ , therefore  $\langle a, b \rangle = a'(0) = 0$ . Repeating the operation, we obtain  $a = 0$ .

Consider  $b$  in  $\mathbb{F}_q^n$  and assume that for all  $a$  non-zero in  $\mathbb{F}_q^n$  we have  $\langle a, b \rangle = 0$ . For  $a = 1$  we get  $\langle 1, b \rangle = 0$ , therefore  $\sigma(r_n(b))(0) = 0$  and  $\sigma(r_n(b(X))) = Xb'(X)$ . For  $a(X) = F(X)$ ,

we get  $\langle F, b \rangle = 0$ , therefore  $F(X)\sigma(r_n(b(X))) = b'(X) \in R/Rf$  cancels at 0. Repeating the operation, we get  $\sigma(r_n(b)) = 0$  and  $b = 0$ .

■

The *weight enumerator* of a code  $C$  of length  $n$  over  $\mathbb{F}_q$  is

$$W_C(x, y) = \sum_{i=0}^n A_i x^{n-i} y^i \in \mathbb{Z}[x, y]$$

where  $A_i$  is the number of codewords of weight  $i$ .

Consider a function  $\phi$  defined over  $\mathbb{F}_q^n$ . Following [1], we consider two Fourier transforms  $\hat{\phi}_l$  and  $\hat{\phi}_r$  defined by

$$\hat{\phi}_l(c) = \sum_{d \in \mathbb{F}_q^n} \Psi(\langle c, d \rangle_{f, \theta, \sigma}) \phi(d)$$

and

$$\hat{\phi}_r(c) = \sum_{d \in \mathbb{F}_q^n} \Psi(\langle d, c \rangle_{f, \theta, \sigma}) \phi(d)$$

where  $\Psi$  is the character defined over  $\mathbb{F}_q = \mathbb{F}_{p^r}$  by  $\Psi(x) = w^{Tr(x)}$  with  $w$  a complex primitive root of unity of order the characteristic  $p$  of  $\mathbb{F}_q$  and  $Tr$  is the trace map from  $\mathbb{F}_q$  to  $\mathbb{F}_p$  defined by  $Tr(x) = x + x^p + \dots + x^{p^{r-1}}$ .

**Lemma 3** *Assume that  $f(0) \neq 0$ . Consider a linear code  $C$  of length  $n$  over  $\mathbb{F}_q$  and a function  $\phi$  defined over  $\mathbb{F}_q^n$ . We have the summation formulas*

$$\sum_{c \in l(C)} \phi(c) = \frac{1}{|C|} \sum_{c \in C} \hat{\phi}_l(c)$$

and

$$\sum_{c \in r(C)} \phi(c) = \frac{1}{|C|} \sum_{c \in C} \hat{\phi}_r(c).$$

**Proof.** We denote  $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{f, \theta, \sigma}$ . We have

$$\sum_{c \in C} \hat{\phi}_l(c) = \sum_{d \in l(C)} \phi(d) \sum_{c \in C} \Psi(\langle c, d \rangle) + \sum_{d \notin l(C)} \phi(d) \sum_{c \in C} \Psi(\langle c, d \rangle).$$

The first term is equal to  $\sum_{d \in l(C)} \phi(d) \sum_{c \in C} 1 = |C| \sum_{d \in l(C)} \phi(d)$ . Let us prove that the second term of this sum vanishes. Consider  $d \notin l(C)$  and  $\phi_d$  the map from  $C$  to  $\mathbb{F}_q$  which maps  $c$  to  $\langle c, d \rangle$ . This map is a morphism according to Proposition 1, therefore  $\sum_{c \in C} \Psi(\langle c, d \rangle) = |Ker(\phi_d)| \sum_{\alpha \in Im(\phi_d)} \Psi(\alpha)$ . Furthermore  $\langle \cdot, \cdot \rangle$  is non-degenerate and  $d \notin l(C)$  therefore  $Im(\phi_d) \neq \{0\}$ . We conclude using the orthogonality relation for group characters.

The same conclusion holds for  $\hat{\phi}_r$  because the map from  $C$  to  $\mathbb{F}_q$  which maps  $c$  to  $\langle d, c \rangle_{f, \theta, \sigma}$  is also a morphism.

■

**Proposition 3** Consider  $C$  a linear code over  $\mathbb{F}_q$  of length  $n$ . The weight enumerators of  $l(C) = l_{f,\theta,\sigma}(C)$  and  $r(C) = r_{f,\theta,\sigma}(C)$  are

$$W_{l(C)}(x, y) = \frac{1}{|C|} \sum_{c \in C} \sum_{d \in \mathbb{F}_q^n} \Psi(\langle c, d \rangle_{f,\theta,\sigma}) x^{n-w(d)} y^{w(d)}$$

and

$$W_{r(C)}(x, y) = \frac{1}{|C|} \sum_{c \in C} \sum_{d \in \mathbb{F}_q^n} \Psi(\langle d, c \rangle_{f,\theta,\sigma}) x^{n-w(d)} y^{w(d)}.$$

**Proof.** Apply Lemma 3 with  $\phi : c \mapsto x^{w(c)} y^{n-w(c)}$  ■

## 4 Central skew module codes and self-duality.

In this section we arrive to the main result of this note about the dual of a central skew module code, that means a code  $Rg/Rf$  where  $f$  is a monic central polynomial and  $g$  is a monic right divisor of  $f$  in  $R = \mathbb{F}_q[X; \theta]$ . We recall that  $\sigma$  is an automorphism of  $\mathbb{F}_q$  such that  $\sigma^2 = id$ .

**Proposition 4** Consider  $g$  in  $R$  and  $h \in R$  monic such that  $g \cdot h = h \cdot g = f$ . Assume that  $f = f^\natural$ . Consider the skew module code  $C = Rg/Rf$  with monic skew generator polynomial  $g$ . Then  $l(C) = r(C)$  is the skew module code  $RH/Rf$  where  $H = \sigma(h^\natural)$ .

**Proof.** Let us denote  $k$  the dimension of  $C$ . We have  $\deg(g) = n - k$  and  $\deg(h) = k$ . As  $f(0) \neq 0$ ,  $\deg(h^\natural) = \deg(h) = k$ . Consider  $i \in \{0, \dots, k-1\}$  and  $j \in \{0, \dots, n-k-1\}$ .

1. Consider  $H = \sigma^{-1}(h^\natural)$ . Let us prove that  $\langle X^i \cdot g, X^j \cdot H \rangle_{f,\theta,\sigma} = 0$ .

We have  $\langle X^i \cdot g, X^j \cdot H \rangle_{f,\theta,\sigma} = P(0)$  where  $P$  is the remainder in the division of  $(X^i \cdot g \cdot \sigma(r_n(X^j \cdot H)))$  by  $f$  on the right. Furthermore

$$\begin{aligned} X^i \cdot g \cdot \sigma(r_n(X^j \cdot H)) &= X^i \cdot g \cdot \sigma(X^{n-(k+j)}(X^j \cdot H)^*) \\ &= X^i \cdot g \cdot \theta^{n-k-j}(\theta^j(\sigma(H)^*))X^{n-(k+j)} \quad \text{because } (X^j \cdot H)^* = \theta^j(H^*) \\ &= X^i \cdot (g \cdot \theta^{n-k}(\sigma(H)^*))X^{n-(k+j)}. \end{aligned}$$

Furthermore  $\theta^{n-k}(\sigma(H)^*) = h \cdot \lambda$  where  $\lambda$  is a non-zero constant. As  $g \cdot h = f$  is central, we get  $P = 0$  therefore  $\langle X^i \cdot g, X^j \cdot H \rangle_{f,\theta,\sigma} = 0$ .

2. Consider  $H = \sigma(h^\natural)$ . Let us prove that  $\langle X^j \cdot H, X^i \cdot g \rangle_{f,\theta,\sigma} = 0$ . We have

$$\begin{aligned} X^j \cdot H \cdot \sigma(r_n(X^i \cdot g)) &= \sigma(X^j \cdot h^\natural \cdot X^{n-(n-k+i)}(X^i \cdot g)^*) \\ &= \sigma(X^j \cdot h^\natural \cdot \theta^{k-i}(\theta^i(g^*)))X^{k-i} \\ &= \sigma(X^j \cdot 1/\theta^k(h_0)h^* \cdot \theta^k(g^*))X^{k-i} \\ &= \sigma(X^j \cdot \theta^k(1/h_0)\theta^k(\theta^{n-k}(h^*) \cdot g^*))X^{k-i}. \end{aligned}$$

Furthermore  $\theta^{n-k}(h^*) \cdot g^* = (g \cdot h)^* = 0$  in  $R/Rf^*$ , therefore  $\langle X^j \cdot H, X^i \cdot g \rangle_{f^\natural,\theta,\sigma} = 0$ .

3. As  $\sigma = \sigma^{-1}$  we get  $\sigma(h^\natural) = \sigma^{-1}(h^\natural)$  therefore  $l(C) = r(C) = RH/Rf$  where  $H = \sigma(h^\natural)$ .

■

**Proposition 5 (Self-dual skew module equation)** Consider  $g$  in  $R$  and  $h \in R$  monic such that  $g \cdot h = h \cdot g = f$ . Consider the skew module code  $C = Rg/Rf$  with monic skew generator polynomial  $g$ . The code  $C$  is self-dual for  $\langle \cdot, \cdot \rangle_{f, \theta, \sigma}$  if, and only if, the skew polynomial  $h$  defined by  $g \cdot h = h \cdot g = f$  satisfies

$$\sigma(h^\natural) \cdot h = f. \quad (6)$$

In this case we have  $f = f^\natural$ .

**Remark 2** When  $f = X^n - \epsilon$  with  $\epsilon^2 = 1$ , the self-dual skew module equation (6) is called self-dual skew equation (Corollary 1 of [9]) and existence conditions were given in [4] for this equation. In next section, we will give some existence conditions to equation (6) when  $\theta$  has order 2 and  $f$  is any monic central element.

**Remark 3** When  $\theta = \sigma = id$ , one can check that there exists a self-dual polycyclic code for  $\langle \cdot, \cdot \rangle_{f, \theta, \sigma}$  if, and only if, the product of the self-reciprocal irreducible factors which divide  $f$  is a square. In particular, if  $f = X^n - 1$ , we recover that there exists a (Euclidean) self-dual cyclic code if and only if  $q$  is a power of 2 and  $n$  is even (see [10]). Note that in [1], self-dual polycyclic codes for  $\langle \cdot, \cdot \rangle_f$  are those for which  $f$  is a square (Theorem 3 of [1]) therefore when  $f = X^n - 1$ , self-dual polycyclic codes for  $\langle \cdot, \cdot \rangle_f$  are not (Euclidean) self-dual cyclic codes.

**Example 1** Consider  $R = \mathbb{F}_4[X; \theta]$  where  $\mathbb{F}_4 = \mathbb{F}_2(\alpha)$ ,  $\alpha^2 + \alpha + 1 = 0$  and  $\theta$  is the Frobenius automorphism. There are three central monic polynomials  $f$  of degree 8 satisfying  $f = f^\natural$ :  $X^8 + 1$ ,  $X^8 + X^4 + 1$  and  $X^8 + X^6 + X^4 + X^2 + 1$ . We consider the self dual skew module codes  $Rg/Rf$  for the scalar products  $\langle \cdot, \cdot \rangle_{f, \theta, id}$ . For  $f = X^8 + 1$  we obtain the three already known (Euclidean) self-dual  $\theta$ -cyclic codes. For  $f = X^8 + X^4 + 1$  there are 7 self-dual skew module codes  $Rg/Rf$  for the scalar products  $\langle \cdot, \cdot \rangle_{f, \theta, id}$ . For  $f = X^8 + X^6 + X^4 + X^2 + 1$ , we have 5 self-dual codes and give one of them here : consider  $h = X^4 + \alpha X^3 + \alpha X + \alpha$  and  $g = h^\natural = X^4 + \alpha X^3 + \alpha X + \alpha^2$ . The skew module code  $C = Rg/Rf$  is a  $[8, 4, 4]_4$  code. As  $h^\natural \cdot h = X^8 + X^6 + X^4 + X^2 + 1$ ,  $C$  is self-dual for the scalar product  $\langle \cdot, \cdot \rangle_{f, \theta, id}$ .

**Example 2** Consider  $R = \mathbb{F}_9[X; \theta]$  where  $\mathbb{F}_9 = \mathbb{F}_2(w)$ ,  $w^2 = w + 1$  and  $\theta$  is the Frobenius automorphism. There are six monic central polynomials  $f$  of degree 6 satisfying  $f = f^\natural$ :  $X^6 + 1$ ,  $X^6 - 1$ ,  $X^6 + X^4 + X^2 + 1$ ,  $X^6 + X^4 + 2X^2 + 2$ ,  $X^6 + 2X^4 + X^2 + 2$  and  $X^6 + 2X^4 + 2X^2 + 1$ . If  $f \in \{X^6 + 1, X^6 + X^4 + X^2 + 1, X^6 + 2X^4 + 2X^2 + 1\}$  there is no self-dual skew module code  $Rg/Rf$  for  $\langle \cdot, \cdot \rangle_{f, \theta, id}$ . Consider  $f = X^6 + 2X^4 + X^2 + 2$ . The skew polynomial  $g = X^3 + w^5 X^2 + w^3 X + w^2$  generates a  $[6, 3, 4]_9$  skew module code  $Rg/Rf$  which is self-dual for  $\langle \cdot, \cdot \rangle_{f, \theta, id}$ .

## 5 Self-dual central skew module codes over $\mathbb{F}_{p^2}$ .

Self-dual  $\theta$ -cyclic codes and  $\theta$ -negacyclic codes have been studied over  $\mathbb{F}_{p^2}$  in [2, 3]. Using and completing the material developed in these two previous works, we give here a necessary and sufficient condition of existence of self dual skew module codes  $Rg/Rf$  for  $\langle \cdot, \cdot \rangle_{f, \theta, \sigma}$  when  $f$  is a monic central polynomial and  $\sigma^2 = id$ .

Consider, for a monic central polynomial  $f(X^2) \in \mathbb{F}_p[X^2]$  the set :

$$\mathcal{H}_{f(X^2)}^{(\sigma)} := \{h \in R \mid h \text{ monic and } \sigma(h^\natural) \cdot h = f(X^2)\}.$$



Necessarily if  $\mathcal{H}_{f(X^2)}^{(\sigma)}$  is non empty then  $f = f^\natural$ .

**Proposition 6 (Proposition 2 of [3])** Consider  $\mathbb{F}_q$  a finite field with  $q = p^2$  elements where  $p$  is a prime number,  $\theta : x \mapsto x^p$  the Frobenius automorphism over  $\mathbb{F}_{p^2}$ ,  $R = \mathbb{F}_q[X; \theta]$ . Consider  $f(X^2) = f_1(X^2) \cdots f_r(X^2)$  where  $f_1(X^2), \dots, f_r(X^2)$  are pairwise coprime polynomials of  $\mathbb{F}_p[X^2]$  satisfying  $f_i^\natural = f_i$ . The application

$$\phi : \begin{cases} \mathcal{H}_{f_1(X^2)}^{(\sigma)} \times \cdots \times \mathcal{H}_{f_r(X^2)}^{(\sigma)} & \rightarrow \mathcal{H}_{f(X^2)}^{(\sigma)} \\ (h_1, \dots, h_r) & \mapsto \text{lcrm}(h_1, \dots, h_r) \end{cases}$$

is bijective.

**Example 3** Consider  $R = \mathbb{F}_4[X; \theta]$  where  $\mathbb{F}_4 = \mathbb{F}_2(\alpha)$ ,  $\alpha^2 + \alpha + 1 = 0$ ,  $\theta$  is the Frobenius automorphism and  $f(X^2) = X^{16} + X^{14} + X^{12} + X^{10} + X^8 + X^6 + X^4 + X^2 + 1 = (X^4 + X^2 + 1)(X^{12} + X^6 + 1)$ . Consider  $h_1 = X^2 + \alpha$  and  $h_2 = X^6 + \alpha^2 X^5 + \alpha X^4 + \alpha X^2 + \alpha^2 X + \alpha^2$ . We have  $h_1^\natural \cdot h_1 = X^4 + X^2 + 1$  and  $h_2^\natural \cdot h_2 = X^{12} + X^6 + 1$ . Consider  $h = \text{lcrm}(h_1, h_2) = X^8 + \alpha^2 X^7 + X^6 + X^5 + \alpha^2 X^4 + \alpha^2 X^3 + \alpha X^2 + X + \alpha$  then  $h^\natural \cdot h = f(X^2)$ . The skew module code  $Rg/Rf$  with skew generator polynomial  $g = h^\natural$  is a self-dual  $[16, 8, 6]_4$  for  $\langle \cdot, \cdot \rangle_{f, \theta, id}$  and we improve the best distance for all Euclidean self-dual  $\theta$ -cyclic codes of length 16 over  $\mathbb{F}_4$  (4 according to Section 4 of [7]).

Using the same construction, we get a self-dual  $[24, 12, 8]_4$  code for  $\langle \cdot, \cdot \rangle_{f, \theta, id}$  with  $f(X^2) = X^{24} + X^{22} + X^{12} + X^2 + 1$  and a self-dual  $[32, 16, 9]_4$  code for  $\langle \cdot, \cdot \rangle_{f, \theta, id}$  with  $f(X^2) = X^{32} + X^{22} + X^{20} + X^{18} + X^{16} + X^{14} + X^{12} + X^{10} + 1$ . The best Euclidean self-dual  $\theta$ -cyclic codes over  $\mathbb{F}_4$  of lengths 24 and 32 are  $[24, 12, 7]_4$  and  $[32, 16, 4]_4$  (Section 4 of [7]).

We now derive necessary and sufficient existence conditions for self-dual skew module codes defined over  $\mathbb{F}_{p^2}$ .

**Lemma 4** Consider  $\mathbb{F}_q$  a finite field with  $q = p^2$  elements where  $p$  is a prime number,  $\theta : x \mapsto x^p$  the Frobenius automorphism over  $\mathbb{F}_{p^2}$ ,  $\sigma \in \{id, \theta\}$ ,  $R = \mathbb{F}_q[X; \theta]$ . Consider  $f(X^2) = f^\natural(X^2)$  which is either irreducible in  $\mathbb{F}_p[X^2]$  or the product of two irreducible polynomials  $g(X^2) \neq g^\natural(X^2)$ . The set  $\mathcal{H}_{f(X^2)}^{(\sigma)}$  is non-empty if, and only if, one of the following conditions is fulfilled :

1.  $m$  is even;
2.  $m$  is odd and  $\deg_{X^2}(f(X^2)) > 1$ ;
3.  $m$  is odd,  $p = 2$  and  $f = X^2 + 1$ ;
4.  $m$  is odd,  $p$  is odd,  $\sigma = id$  and  $f = X^2 - (-1)^{\frac{p+1}{2}}$ ;
5.  $m$  is odd,  $p$  is odd,  $\sigma = \theta$  and  $f = X^2 + 1$ .

**Proof.**

1. If  $m$  is even then  $f^{m/2} \in \mathcal{H}_{f(X^2)^m}^{(\sigma)}$ .
2. If  $\deg_{X^2}(f(X^2)) > 1$ , according to Lemma 3.3 and Lemma 3.5 of [3], the set  $\mathcal{H}_{f(X^2)}^{(\sigma)}$  is non-empty. Consider  $H \in \mathcal{H}_{f(X^2)}^{(\sigma)}$ , then  $f^{(m-1)/2}H = Hf^{(m-1)/2} \in \mathcal{H}_{f(X^2)^m}^{(\sigma)}$ .

3. If  $m$  is odd,  $p = 2$  and  $f = X^2 + 1$ , then  $(X + 1)^m \in \mathcal{H}_{f(X^2)^m}^{(\sigma)}$ .
4. If  $m$  is odd,  $p$  is odd,  $\sigma = id$  and  $f = X^2 - \epsilon$ , with  $\epsilon^2 = 1$ , according to Proposition 2 of [2], the set  $\mathcal{H}_{f(X^2)^m}^{(\sigma)}$  is non-empty if and only if  $(-1)^{\frac{p+1}{2}} = \epsilon$ .
5. Assume that  $m$  is odd,  $p$  is odd,  $\sigma = \theta$  and  $f = X^2 - \epsilon$ , with  $\epsilon^2 = 1$ . The set  $\mathcal{H}_{f(X^2)^m}^{(\sigma)}$  is the disjoint union  $\sqcup_{j=0}^{(m-1)/2} f(X^2)^j \overline{\mathcal{H}^{(\sigma)}_{f(X^2)^{m-2j}}}$  where for  $i \geq 0$ ,  $\overline{\mathcal{H}^{(\sigma)}_{f(X^2)^i}} := \{h \in \mathcal{H}_{f(X^2)^i}^{(\sigma)} \mid f(X^2) \nmid h\}$  is the set of elements of  $\mathcal{H}_{f(X^2)^i}^{(\sigma)}$  which are not divisible by  $f(X^2) = X^2 - \epsilon$ . One can adapt the proof of Lemma 4.1 of [2] to get that for  $i \geq 1$ , the set  $\overline{\mathcal{H}^{(\sigma)}_{f(X^2)^i}}$  is non-empty if and only if  $\epsilon = -1$ . The conclusion follows.

■

**Proposition 7** Consider  $\mathbb{F}_q$  a finite field with  $q = p^2$  elements where  $p$  is an odd prime number,  $\theta : x \mapsto x^p$  the Frobenius automorphism over  $\mathbb{F}_{p^2}$ ,  $\sigma \in \{id, \theta\}$ ,  $R = \mathbb{F}_q[X; \theta]$ . Consider  $f(X^2) = f^{\natural}(X^2)$  in  $\mathbb{F}_p[X^2]$ . Consider  $m_1, m_2 \in \mathbb{N}$  such that  $f(X^2) = (X^2 - 1)^{m_1}(X^2 + 1)^{m_2}F(X^2)$  and  $F$  is not divisible by  $X^2 + 1$  or  $X^2 - 1$ . There exists a self-dual skew module code  $Rg/Rf$  for  $\langle \cdot, \cdot \rangle_{f, \theta, \sigma}$  if, and only if, one of these conditions is satisfied :

- $m_1$  and  $m_2$  are even;
- $m_1$  is odd,  $m_2$  is even,  $p \equiv 3 \pmod{4}$ ,  $\sigma = id$ ;
- $m_1$  is even,  $m_2$  is odd,  $p \equiv 1 \pmod{4}$ ,  $\sigma = id$ ;
- $m_1$  is even,  $m_2$  is odd,  $\sigma = \theta$ .

**Proof.** The proof is deduced from Proposition 6 and Lemma 4.

■

**Example 4** In Example 2, we have seen that there is no self-dual skew module code  $Rg/Rf$  over  $\mathbb{F}_9$  for the scalar product  $\langle \cdot, \cdot \rangle_{f, \theta, id}$  when  $f(X^2)$  is one of the following monic central polynomials :  $X^6 + 1 = (X^2 + 1)^3$ ,  $X^6 + X^4 + X^2 + 1 = (X^2 + 1)(X^4 + 1)$  and  $X^6 + 2X^4 + 2X^2 + 1 = (X^2 + 1)(X^2 - 1)^2$  while there exists a self-dual skew module code  $Rg/Rf$  over  $\mathbb{F}_9$  if  $f(X^2)$  is one of the following monic central polynomials :  $X^6 - 1 = (X^2 - 1)^3$ ,  $X^6 + X^4 + 2X^2 + 2 = (X^2 + 1)^2(X^2 - 1)$  and  $X^6 + 2X^4 + X^2 + 2 = (X^2 - 1)(X^4 + 1)$ .

## 6 Conclusion.

In this note, inspired by [1], we have constructed a new notion of duality for polycyclic codes and for central skew module codes. With this new notion of duality, we consider self-dual central  $\theta$ -module codes  $Rg/Rf$  for any monic central self-reciprocal skew polynomial  $f$ . When  $f = X^n - \epsilon$  with  $\epsilon^2 = 1$ , we get Euclidean and Hermitian self-dual  $\theta$ -constacyclic codes. When the order of  $\theta$  is 2, we give necessary and sufficient existence conditions of self-dual central skew module codes by using the results previously obtained in [2, 3]. It could be interesting to study these self-dual codes more deeply for any monic central skew polynomial  $f \neq X^n - \epsilon$ , especially when the automorphism  $\theta$  has an order  $\neq 1, 2, n$ .

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