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A combinatorial description of stability for toric vector bundles

Lucie DEVEY

Abstract

This paper aims to give an algorithm for the slope stability of toric vector bundles (or equivariant reflexive sheaves). We also provide a visualisation of stability in terms of their parliaments of polytopes, a generalization of Newton polytopes for toric vector bundles by Di Rocco, Jabbusch, and Smith. We finally define subparliaments of polytopes and identify them with parliaments of equivariant subbundles.

Introduction

Stability has been introduced in the context of classifying vector bundles, by constructing moduli spaces of semistable bundles or sheaves. Using the Harder–Narasimhan filtration, any vector bundle \mathcal{E} can be built up from semistable coherent sheaves. In 1963, Mumford extended the works of Grothendieck (see [Gro57] Theorem 2.1) and Atiyah (see [Ati57]) by classifying vector bundles on curves (see [Mum63]). In this paper, we consider the generalisation of Mumford-stability to vector bundles on varieties of any dimension called slope-stability and introduced in 1972 by Takemoto ([Tak72]). This paper aims to study the toric case.

Consider a toric variety X with torus T . A toric vector bundle over X is a vector bundle \mathcal{E} over X , equipped with a T -action such that the projection map $\pi : \text{Spec}(\text{Sym } \mathcal{E}) \rightarrow X$ is T -equivariant and T acts linearly on the fibers.

Our main result is an algorithm (as wished in [DDK19]) to test the slope-stability of toric vector bundles, or more generally equivariant reflexive sheaves \mathcal{E} . See Subsection 1.4 for the description of the algorithm and Appendix A for the algorithm implemented in Macaulay2. So far an algorithm existed only for the special case of tangent bundles (see [DDK19] and [HNS19]).

In a second section, we work on visualizing the stability of a toric vector bundle on its parliament of polytopes, a construction of Sandra Di Rocco, Kelly Jabbusch, and Gregory G. Smith (see [DJS14]) generalizing the construction of the Newton polytope of a toric line bundle. The parliament of polytopes $PP_{\mathcal{E}}$ is a collection of convex polytopes $(P_e)_{e \in G(\mathcal{E})}$ indexed by the elements in the ground set of a matroid $M(\mathcal{E})$ associated to \mathcal{E} representable in $E \simeq \mathbb{C}^r$: the fiber of \mathcal{E} over the identity. The wealth of information about a toric vector bundle \mathcal{E} contained in its parliament is astounding. For instance, lattice points in the parliament of polytopes for \mathcal{E} correspond to a T -equivariant generating set for the space of global sections of \mathcal{E} (see [DJS14] Proposition 1.1). In addition, we can also recover some positivity properties of \mathcal{E} , such as global generation (see [DJS14] Theorem 1.2), ampleness (see [DJS14] Corollary 6.7), or bigness (see [Nø20] Theorem 7.5). In our stability study using parliaments, we define in Definition 2.16 for any equivariant saturated subsheaf of \mathcal{E} , a polytope called the average polytope. It is the Newton polytope of $c_1(\mathcal{F})/\text{rk}(\mathcal{F})$. For any polarisation α of X , we define a total order $<_{\alpha}$ between

polytopes, such that comparing average polytopes of subsheaves corresponds to comparing their α -slopes. To check stability, we need to compare the slope of \mathcal{E} with the slope of its subsheaves. Checking the equivariant saturated subsheaves is sufficient for general toric bundles (see [Koo08], and [DDK19] or [HNS19]). These nice results allowed the same authors to have a finite check for the stability of tangent bundles. We prove that for toric bundles of rank less or equal to 3, it suffices to check the flats of the matroid of any parliament of polytopes of \mathcal{E} define subsheaves that check stability (see Section 2.4).

In Section 3, we develop the theory of parliament of polytopes by defining subparliaments and identifying them with parliaments of equivariant subbundles.

Theorem 1. *The subparliaments of the parliaments of \mathcal{E} are the parliaments of the equivariant subbundles of \mathcal{E} .*

In Section 4, we explain how to detect the stability of restrictions to invariant curves on the parliament. We give examples of (semi)stable toric bundles with non stable restrictions to toric curves. The last part of this article explores the geometric information that can be reconstructed from the parliament of polytopes of a toric vector bundle. In particular, we translate some results of Payne (see [Pay07] Proposition 3.4, Corollary 3.6) and Klyachko (see [Kly89] Corollary 1.2.4) to the language of parliaments of polytopes.

Proposition 2. *The data of the parliament of polytopes of a globally generated equivariant vector bundle \mathcal{E} , up to the translation of each direct component, and quotiented by $GL_r(\mathbb{C})$, enables us to reconstruct the isomorphism class of the vector bundle \mathcal{E} .*

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1 Stability of equivariant vector bundles

Let X be a smooth complete toric variety of dimension d with fan Σ and torus T . The lattice of characters and its dual lattice are denoted by M and N . We denote by $\Sigma(k)$ the cones of dimension k . Let \mathcal{E} be a given rank r equivariant vector bundle over X and E be the fiber over the identity of the torus. Let $\{v_0, \dots, v_{n-1}\}$ be the set of primitive generators of the rays ρ of Σ . (See Chapter [CLS11] for more fundamentals on toric varieties.)

In this section, we give an algorithm to check whether an equivariant reflexive sheaf \mathcal{E} is stable with respect to a given polarisation.

1.1 Equivariant vector bundles and Klyachko classification

A combinatorial description of equivariant vector bundles has been given by Klyachko in Theorem 0.1.1 of [Kly89].

Theorem 1.1. *There exists an equivalence of categories between the category of rank r -equivariant vector bundles \mathcal{E} on X and the category of compatible decreasing \mathbb{Z} -filtrations $(E^\rho(j))_{\rho \in \Sigma(1)}$*

$$E^\rho(j) = \begin{cases} E & \text{if } j \leq A_1^\rho \\ H_\rho & \text{if } A_1^\rho < j \leq A_2^\rho \\ \dots & \\ \text{Span}(u_\rho) & \text{if } A_{r-1}^\rho < j \leq A_r^\rho \\ \{0\} & \text{if } A_r^\rho < j \end{cases} .$$

of a dimension r \mathbb{C} -vector space E together with a compatibility condition as follows. There exist decompositions of E into 1-dimensional vector spaces L_u^σ satisfying

$$\forall \sigma \in \Sigma(d), \exists (L_u^\sigma)_{u \in \mathbf{u}(\sigma)} \text{ s.t. } E = \bigoplus_{u \in \mathbf{u}(\sigma)} L_u^\sigma \text{ and } \forall \rho \preceq \sigma, E^\rho(j) = \sum_{\langle u, v_\rho \rangle \geq j} L_u^\sigma . \quad (\text{CC})$$

The sets $u(\sigma) \subset M$ are uniquely determined by \mathcal{E} as they satisfy

$$\forall \sigma \in \Sigma(d), \quad \mathcal{E}|_{U_\sigma} \cong \bigoplus_{u \in \mathbf{u}(\sigma)} \mathcal{O}_X(\text{div}(u))|_{U_\sigma},$$

they are called associated characters of \mathcal{E} .

Remark 1.2. We might have $A_k^\rho = A_{k+1}^\rho$. In that case, the filtration $(E^\rho(j))_j$ does not contain any $r - k$ -dimensional vector space.

Perling extends this classification in Theorem 5.19 of [Per01].

Theorem 1.3. *There is an equivalence of categories between the category of equivariant reflexive sheaves \mathcal{F} on X of rank r , and the category of n decreasing \mathbb{Z} -filtrations of a dimension r \mathbb{C} -vector space F where we do not impose the compatible condition.*

Definition 1.4. *We will call the sequences*

$$A^\rho(\mathcal{E}) := (A_1^\rho, \dots, A_r^\rho) \quad \text{for all } \rho \in \Sigma(1)$$

the jumping loci of \mathcal{E} .

1.2 Definitions

We start with the definition of slope for coherent sheaves which leads us to the definition of stability. The slope of a coherent sheaf \mathcal{E} depends on the choice of a polarization α , where α is either H^{d-1} for an ample divisor H or more generally a movable curve.

Definition 1.5. *Let X be a smooth complete toric variety of dimension d and let α be a polarization of X . The slope of any coherent sheaf \mathcal{E} with respect to α is*

$$\mu_\alpha(\mathcal{E}) := \frac{c_1(\mathcal{E}) \cdot \alpha}{\text{rk}(\mathcal{E})} \in \mathbb{Q},$$

where c_1 is the first Chern class of \mathcal{E} .

The dependence on α is often omitted.

Definition 1.6. *A coherent sheaf \mathcal{E} is α -semistable if for any nonzero subsheaf $\mathcal{F} \subseteq \mathcal{E}$, the respective slopes satisfy the inequality*

$$\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E}).$$

It is α -stable if, in addition, for any nonzero subsheaf $\mathcal{F} \subsetneq \mathcal{E}$, the strict inequality $\mu_\alpha(\mathcal{F}) < \mu_\alpha(\mathcal{E})$ holds.

A coherent sheaf \mathcal{E} is α -polystable if it is a direct sum of α -stable sheaves.

Lemma 1.7. *As explained in Remark 2.4 of [HNS19], to prove that some equivariant bundle \mathcal{E} is α -semistable, it is sufficient to verify that $\mu_\alpha(\mathcal{F}) < \mu_\alpha(\mathcal{E})$ holds for every equivariant saturated subsheaf \mathcal{F} .*

We need the following combinatorial description of equivariant saturated subsheaves of \mathcal{E} .

Theorem 1.8 (Proposition 2.3 of [HNS19] or Corollary 0.0.2. [DDK20]). *Let \mathcal{E} be a rank r -equivariant vector bundle on a smooth complete toric variety X . Via the Klyachko classification, \mathcal{E} corresponds to a \mathbb{Z} -filtration $(E^\rho(j))_{j \in \mathbb{Z}}$ of $E \cong \mathbb{C}^r$ for each ray $\rho \in \Sigma(1)$. The equivariant saturated subsheaves \mathcal{F} of \mathcal{E} are then in one-to-one correspondence with the subfiltrations*

$$(F^\rho(j) = E^\rho(j) \cap F)_{j \in \mathbb{Z}} \quad \text{of} \quad (E^\rho(j))_{j \in \mathbb{Z}},$$

for some vector subspace $F \subseteq E$.

1.3 Polarization, Weights and Slope

Usually, the slope of a coherent sheaf on X is defined with respect to some ample divisor H called a polarization of X . More precisely, it involves the self-intersection product H^{d-1} . In [GKP14]), to do birational geometry, Greb, Kebekus, and Peternell generalize slopes to movable curves on any complex, projective manifold X .

Let us explain it in the toric case. The set of polarizations of X can be extended from ample divisors H to movable divisors L on X . This way the movable divisor L polarizes X as well as the blow-up of X making L ample. In [LX16], Lehmann and Xiao studied movable curves on toric varieties.

Proposition 1.9. *A movable curve α on X , positive along a spanning set of rays¹, can be written as the positive self-intersection of a unique big and movable divisor L_α*

$$\alpha = \langle L_\alpha^{d-1} \rangle,$$

where $\langle \rangle$ is the Boucksom positive product².

Proof. See Theorem 3.12, Lemma 4.1 and Theorem 4.2 of [LX16]. □

To compute the positive intersection product of big toric divisor classes L_1, \dots, L_p , we consider a higher birational model on which L_1, \dots, L_p are nef after removing some positive linear combination of exceptional divisors D_i . Then $\langle L_1, \dots, L_p \rangle$ is defined as

$$(L_1 - D_1) \cdot \dots \cdot (L_p - D_p).$$

As such, the positive intersection product restricts to the intersection product on ample classes. Moreover, it satisfies that $\langle L^d \rangle = \text{Vol}(L)$ for any big divisor.

By Minkowski's theorem for polytopes (see [Sch93] Section 7), the data of a polarization α is equivalent to the data of its weights, defined as follows.

¹that is to say $\alpha \cdot D_\rho > 0$ for rays' generators v_ρ generating $N_{\mathbb{R}}$. It is equivalent to α having a positive volume for a certain volume function that they define

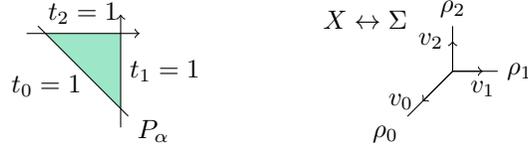
²developed in [BDPP04] and constructed algebraically in [BFJ06]

Definition 1.10. Let us denote by P_{L_α} the moment polytope of L_α (see Proposition 1.3) and write f_ρ for the volume of the face of P_{L_α} which has external normal vector v_ρ . We call the weights of α the numbers

$$\text{for } \rho \in \Sigma(1), \quad t_\rho = f_\rho \frac{(d-1)!}{\|v_\rho\|}.$$

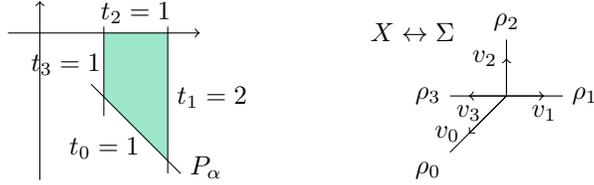
Note that we have $t_\rho \geq 0$.

Example 1.11. On \mathbb{P}^2 , the polarization $\alpha = D_0 = \langle D_0^{d-1} \rangle$ has weights $(1, 1, 1)$.



◇

Example 1.12. The polarization $\alpha = 2D_1 - D_3$ on $\text{Bl}_{[0:1:0]}\mathbb{P}^2$ has weights $(1, 2, 1, 1)$.



◇

We consider now a rank r -equivariant reflexive sheaf \mathcal{E} on X defined by the filtrations

$$E^\rho(j) = \begin{cases} E & \text{if } j \leq A_1^\rho \\ \cdots & \\ \text{Span}(u_\rho) & \text{if } A_{r-1}^\rho < j \leq A_r^\rho \\ \{0\} & \text{if } A_r^\rho < j. \end{cases}$$

We reformulate the slope of \mathcal{E} . In his original paper [Kly89], in Remark 3.2.4, Klyachko expressed the Chern classes of any equivariant vector bundle³. Kool extended this result, computing the first Chern class of any equivariant coherent sheaf (see Corollary 3.18 of [Koo08]). One can show his formula is equivalent to the following.

Proposition 1.13. *The first Chern class of \mathcal{E} is given by*

$$c_1(\mathcal{E}) = \sum_{\rho \in \Sigma(1)} \left(\sum_{k=1}^r A_k^\rho \right) D_\rho.$$

Proposition 1.14. *If the polarisation α has weights $(t_\rho)_{\rho \in \Sigma(1)}$ (see Definition 1.10), then the slope of \mathcal{E} is*

$$\mu_\alpha(\mathcal{E}) = \sum_{\rho} \left(\frac{1}{r} \sum_{k=1}^r A_k^\rho \right) t_\rho.$$

Proof. In Section 4 of [LX16], Lehmann and Xiao show that the t_ρ are the intersection

³See also Proposition 3.1 of [Pay07]

numbers $D_\rho \cdot \alpha$. We then have that

$$\mathrm{rk}(\mathcal{E})\mu_\alpha(\mathcal{E}) = c_1(\mathcal{E}) \cdot \alpha = \sum_\rho \left(\sum_{k=1}^r A_k^\rho \right) D_\rho \cdot \alpha = \sum_\rho \left(\sum_{k=1}^r A_k^\rho \right) t_\rho .$$

□

1.4 A finite check of stability and algorithm

The following theorem has been pointed out to me by Diane Maclagan. Fix a polarization α .

Theorem 1.15. *Let \mathcal{E} be any equivariant reflexive sheaf on a toric variety X . There exists a finite set I of equivariant saturated subsheaves \mathcal{F} of \mathcal{E} such that \mathcal{E} is α -stable (resp. α -semistable) if and only if*

$$\mu_\alpha(\mathcal{F}) < \mu_\alpha(\mathcal{E}) \quad (\text{resp. } \mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E})) ,$$

for any $\mathcal{F} \in I$.

Proof. By Proposition 1.14, the slope of an equivariant reflexive sheaf \mathcal{E} only depends on its jumping loci $(A^\rho(\mathcal{E}))_{\rho \in \Sigma(1)}$.

Second, if \mathcal{F} is an equivariant saturated subsheaf of \mathcal{E} (it is enough to check those by Lemma 1.7), then for each ray ρ , its ρ -jumping loci form subsequences of those of \mathcal{E} . This fact follows easily from Theorem 1.8 and by observing that

$$E^\rho(j_1) = E^\rho(j_2) \Rightarrow E^\rho(j_1) \cap F = E^\rho(j_2) \cap F$$

holds for any subspace $F \subset E$.

Let us conclude by the facts that $\Sigma(1)$ is finite and for each ray $\rho \in \Sigma(1)$, there is finitely many subsequences of the jumping loci $A^\rho(\mathcal{E}) = (A_1^\rho, \dots, A_r^\rho)$ of $(E^\rho(j))_{j \in \mathbb{Z}}$. As a consequence, the finite set I may be composed of one representative \mathcal{F} (if it exists) for each choice of subsequences of the jumping loci $(A^\rho(\mathcal{E}))_\rho$. □

In the following, we present the algorithm in Appendix A that checks the stability of a given toric reflexive sheaf \mathcal{E} .

The main mission of the algorithm is to check for a given vector $(d_{\rho,k})_{\rho \in \Sigma(1), k \in \{1, \dots, r\}}$ whether there exists an \mathcal{F} such that

$$\dim(E_{r-k+1}^\rho \cap F) \geq d_{\rho,k} \quad \forall \rho \in \Sigma(1), k \in \{1, \dots, r\} \quad (1)$$

holds.

This can be achieved by constructing, for every $\rho \in \Sigma(1)$ and $k \in \{1, \dots, r\}$, a matrix $M_{\rho,k}$ being the concatenation of two matrices

$$M_{\rho,k} = M_1 \mid M_2 .$$

The first matrix $M_1 = (x_{a,b})_{a \in \{1, \dots, r\}, b \in \{1, \dots, l\}}$ contains the variables $x_{1,1}, \dots, x_{r,l}$ that are the coordinates of vectors defining our future F (if it exists). The second one M_2 contains the coordinates of vectors defining E_{r-k+1}^ρ .

Let I be the ideal defined by the $k+l-d_{\rho,k}+1$ minors of $M_{\rho,k}$ and J be the ideal defined by the l -minors of M_1 .

Proposition 1.16. *There exists an equivariant saturated subsheaf \mathcal{F} of \mathcal{E} such that (1) holds, if and only if, $\sqrt{I+J} \neq \sqrt{I}$.*

Proof. Let us call F the vector space spanned by the columns of M_1 even though it may have a dimension less than l . The inequality $\dim(E_{r-k+1}^\rho \cap F) \geq d_{\rho,k}$ is equivalent to the matrix $M_{\rho,k}$ having rank

$$\text{rk}(M_{\rho,k}) = \dim(E_{r-k+1}^\rho + F) = \dim(E_{r-k+1}^\rho) + \dim(F) - \dim(E_{r-k+1}^\rho \cap F) \leq k + l - d_{\rho,k} .$$

As a consequence, F satisfies (1) if and only if $F \in V(I)$. Moreover, remark that F has dimension l if and only if its matrix M_1 has rank l which means that $F \notin V(J)$.

Let us conclude by saying that $\sqrt{I+J} \neq \sqrt{I}$ implies, by the Nullstellensatz, that $V(I) \cap V(J) = V(I+J) \neq V(I)$ and so $V(I) \setminus (V(I) \cap V(J))$ is non empty. \square

Let us continue with a lemma.

Lemma 1.17. *Consider two vector spaces F and F_0 of the same dimension l . Assume that for every ray $\rho \in \Sigma(1)$, for any $j \in \mathbb{Z}$, we have that*

$$\dim(F \cap E^\rho(j)) \geq \dim(F_0 \cap E^\rho(j)) .$$

Then the slope of \mathcal{F} is greater than or equal to the slope of \mathcal{F}_0

$$\mu(\mathcal{F}) \geq \mu(\mathcal{F}_0) .$$

Proof. If $(A_k^\rho)_{\rho,k}$ and $(B_k^\rho)_{\rho,k}$ are the jumping loci of \mathcal{F} and \mathcal{F}_0 , then for all $k \in \{1, \dots, l\}$, $\rho \in \Sigma(1)$,

$$\begin{cases} A_k^\rho = \min \{j \mid \dim(F_0 \cap E^\rho(j)) \leq l - k + 1\} \\ B_k^\rho = \min \{j \mid \dim(F \cap E^\rho(j)) \leq l - k + 1\} \end{cases}$$

satisfy $A_k^\rho \geq B_k^\rho$. We conclude by Proposition 1.14. \square

Remark now that from the data of $\dim(E_{r-k+1}^\rho \cap F)$, we can recover the jumping loci and the slope of \mathcal{F} . Similarly, we may define the slope $\mu_\alpha((d_{\rho,k})_{\rho,k})$ associated to the vector $(d_{\rho,k})_{\rho \in \Sigma(1), k \in \{1, \dots, r\}}$. Checking that \mathcal{E} is stable amounts to checking that the slopes defined by the vectors $(d_{\rho,k})_{\rho \in \Sigma(1), k \in \{1, \dots, r\}}$ such that there exists such an \mathcal{F} satisfying (1) are less than $\mu_\alpha(\mathcal{E})$. Indeed, by Lemma 1.17, the slope $\mu_\alpha(\mathcal{F})$ would be greater or equal to $\mu_\alpha((d_{\rho,k})_{\rho,k})$ and the condition

$$\mu_\alpha((d_{\rho,k})_{\rho,k}) < \mu_\alpha(\mathcal{E})$$

should be satisfied.

Remark 1.18. Note that as the last step requires Gröbner bases, our algorithm has high complexity. In the next section, we reduce the complexity (from exponential to linear) in some special cases. See subsections 2.4, 2.5.

2 Parliaments of polytopes and stability

Parliaments of polytopes were introduced in [DJS14] in order to give explicit polyhedral interpretations of properties to equivariant vector bundles. The parliaments of polytopes $PP_{\mathcal{E}}$ of \mathcal{E} are composed of at least r polytopes linked with some combinatorial data: the polytopes are labeled by elements of a representable matroid of rank r .

We first recall some matroid terminology. See [Kat14] for more fundamentals.

2.1 Matroid terminology

Matroids are a generalization of the notion of linear independence in vector spaces.

Definition 2.1. A matroid M is the data of a finite set G and a collection \mathcal{B} , of subsets of G , called bases, satisfying the following properties:

B1: \mathcal{B} is nonempty ;

B2: (basis exchange property) If $A, B \in \mathcal{B}$ are distinct and $a \in A \setminus B$, then there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

We call G the ground set and \mathcal{B} the set of bases of the matroid $M = (G, \mathcal{B})$.

Definition 2.2. An isomorphism of matroids $\varphi : M_1 = (G_1, \mathcal{B}_1) \rightarrow M_2 = (G_2, \mathcal{B}_2)$ is a bijection from G_1 to G_2 such that

$$A \in \mathcal{B}_1 \Leftrightarrow \varphi(A) \in \mathcal{B}_2 .$$

Definition 2.3. A representable matroid of rank r is a matroid isomorphic to

$$M = (G, \mathcal{B}) ,$$

where G is a finite subset of some dimension r -vector space E and \mathcal{B} is the set of bases of E formed by vectors in G . We say that M is represented in E .

Definition 2.4. A flat of a matroid (G, \mathcal{B}) represented in E , is a subset $f \subset G \subset E$ such that

$$\text{Span}(f) \cap G = f .$$

2.2 Parliaments of polytopes

The construction of parliaments of polytopes of \mathcal{E} requires the Klyachko classification.

Definition 2.5. The r elements in $\mathbf{u}(\sigma)$ are called the associated characters of σ .

Remark 2.6. The compatibility condition (CC) implies the existence of a basis B_σ for every maximal cone $\sigma \in \Sigma(d)$, given by a generator of L_u^σ for each $u \in \mathbf{u}(\sigma)$. We call B_σ a compatible basis. Nevertheless, a compatible basis may not be unique (see Example 4.4 of [DJS14]) although $\mathbf{u}(\sigma)$ always is.

Remark 2.7. An important feature (equivalent to (CC)) is that for any maximal cone $\sigma \in \Sigma(d)$, \mathcal{E} splits equivariantly on U_σ as

$$\mathcal{E}|_{U_\sigma} \simeq \bigoplus_{u \in \mathbf{u}(\sigma)} \mathcal{O}_X(\text{div}(u))|_{U_\sigma} .$$

A parliament of polytopes of a vector bundle \mathcal{E} is a certain collection of polytopes

$$\{(P_e, e) \mid e \in G(\mathcal{E}) \subseteq E\} ,$$

where $G(\mathcal{E})$ is the ground set of a matroid $M(\mathcal{E})$ defined in the following manner. The matroids, called $M(\mathcal{E})$, associated to \mathcal{E} can be seen as minimal matroids generating

$$L(\mathcal{E}) = \left\{ \bigcap_{\rho \in \Sigma(1)} E^\rho(j_\rho) \mid (j_\rho)_{\rho \in \Sigma(1)} \in \mathbb{Z}^{n+1} \right\}$$

as a meet-subsemilattice (as a partially ordered subset with a meet i.e. a greatest lower bound). Despite Proposition 3.1 [DJS14], this matroid is not unique⁴. However, the ground sets $G(\mathcal{E})$ generating the matroids $M(\mathcal{E})$ are the possible outputs of Algorithm 3.2 of [DJS14].

Algorithm 3.2 Computing $G(\mathcal{E})$

```

 $r \leftarrow$  the dimension of the largest linear subspace of  $L(\mathcal{E})$ 
 $G \leftarrow \emptyset$ 
for  $k = 1$  to  $r$  do
  for  $k$ -dimensional linear subspace  $V \in L(\mathcal{E})$  do
     $G' \leftarrow G \cap V$ 
    if  $\text{Span}(G') \subsetneq V$  then
       $G \leftarrow G \sqcup CB_V$ ,  $CB_V$  is a basis of a complement to  $\text{Span}(G')$  in  $V$ 
    end if
  end for
end for
return  $G$ 

```

Additionally, if G_E^1 and G_E^2 are ground sets resulting from Algorithm 3.2 applied to $L(\mathcal{E})$ then the same number of elements is added at Step V . Indeed, CB_V^1, CB_V^2 are bases of a complement to $\text{Span}(G'^1), \text{Span}(G'^2)$, which have same dimension

$$\dim(\text{Span}(G'^\varepsilon)) = \dim\left(\text{Span}(V' \cap V)_{V' \cap V \subsetneq V}\right) \quad \forall \varepsilon \in \{1, 2\}$$

in V . The equality holds because, as $L(\mathcal{E})$ is stable by intersection, the vector spaces $V' \cap V$ that are different from V , are steps of the algorithm and are already generated by G'^ε at Step V . Therefore, there exists a map φ sending G_E^1 to G_E^2 , we call it of type (\star) .

Definition 2.8. A map $\varphi : M(G_E^1) \rightarrow M(G_E^2)$ of type (\star) is a bijection

$$\varphi : G_E^1 \rightarrow G_E^2$$

which respects Algorithm 3.2 for \mathcal{E} , that is to say which sends CB_V^1 to CB_V^2 at Step V of Algorithm 3.2.

We use the following notation for a ground set $G(\mathcal{E})$ of $M(\mathcal{E})$

$$G(\mathcal{E}) = \{e_0, \dots, e_l\} \quad \text{for some } l \geq r.$$

Definition 2.9. Any output $G(\mathcal{E})$ from Algorithm 3.2 gives rise to a parliament of polytopes of \mathcal{E} . These are the sets of indexed polytopes $(P_e)_{e \in G(\mathcal{E})}$ defined as

$$P_e := \{m \in M_{\mathbb{R}} \mid \forall \rho \in \Sigma(1), \langle m, v_\rho \rangle \leq \max\{j \mid e \in E^\rho(j)\}\},$$

where $G(\mathcal{E})$ is seen modulo isomorphism of matroids of type (\star) .

Example 2.10. [Example 3.6 in [KM16]] To a fixed toric vector bundle, we associated a finite number of parliaments of polytopes, corresponding to each isomorphism class of matroids obtained by Algorithm 3.2. Here we give some counterexamples of the uniqueness of $M(\mathcal{E})$. Consider the canonical basis $\{e_1, e_2, e_3\}$ of \mathbb{C}^3 .

⁴see Example 2.10 for counterexamples

For the first counterexample, let X be the projective plane \mathbb{P}^2 and let \mathcal{E} be the toric bundle of rank 3 defined by the following filtrations

$$E^{\rho_i}(j) = \begin{cases} \mathbb{C}^3 & \text{if } j \leq 0 \\ V_i & \text{if } 0 < j \leq 1 \\ \{0\} & \text{if } 1 < j \end{cases} ,$$

where $V_0 = \text{Span}(e_1)$, $V_1 = \text{Span}(e_1 + e_2)$ and $V_2 = \text{Span}(e_2, e_3)$. The meet-subsemilattice is

$$L_1(\mathcal{E}) = \{\{0\}, \text{Span}(e_1), \text{Span}(e_1 + e_2), \text{Span}(e_2, e_3), \mathbb{C}^3\} .$$

Two non-isomorphic matroids are given by $G_1 = \{e_1, e_1 + e_2, e_2, e_3\}$ and $G_2 = \{e_1, e_1 + e_2, e_2 + e_3, e_3\}$. Here the non-uniqueness comes from $L_1(\mathcal{E})$ nonbeing stable by sums.

A second example can be found on \mathbb{P}^2 , let \mathcal{E} be defined by the following filtrations

$$E^{\rho_i}(j) = \begin{cases} \mathbb{C}^3 & \text{if } j \leq 0 \\ V_i & \text{if } 0 < j \leq 1 \\ \text{Span}(e_1) & \text{if } 1 < j \leq 2 \\ \{0\} & \text{if } 2 < j \end{cases} ,$$

where $V_0 = \text{Span}(e_1, e_2)$, $V_1 = \text{Span}(e_1, e_3)$ and $V_2 = \text{Span}(e_1, e_2 + e_3)$. The meet-subsemilattice is

$$L_2(\mathcal{E}) = \{\{0\}, \text{Span}(e_1), \text{Span}(e_1, e_2), \text{Span}(e_1, e_3), \text{Span}(e_1, e_2 + e_3), \mathbb{C}^3\} .$$

Two non-isomorphic matroids are given by $G_1 = \{e_1, e_2, e_2 + e_3, e_3\}$ and $G_2 = \{e_1, e_1 + e_2, e_2 + e_3, e_3\}$. Here the non-uniqueness comes from $L_2(\mathcal{E}) \cup \{G_2\}$ nonbeing stable by iterated sums and intersections. \diamond

Remark 2.11. By Remark 1.3, an equivariant reflexive sheaf is defined by filtrations as toric vector bundles. We extend the definition of parliaments of toric bundles (Definition 2.9) to preparliaments of equivariant saturated subsheaves of toric bundles. A preparliament that satisfies the compatibility condition (CC) will be a parliament, and the corresponding equivariant saturated subsheaf a toric bundle. We keep the notation $PP_{\mathcal{F}}$ for preparliaments.

Proposition 2.12. *A preparliament is a set of indexed polytopes constructed from a family of decreasing separated-exhaustive \mathbb{Z} -filtrations $((E^{\rho}(j))_{j \in \mathbb{Z}})_{\rho \in \Sigma(1)}$. Let us denote the $|\Sigma(1)| \times r$ hyperplanes used in the construction of the polytopes by*

$$(H_{\rho, k} = (\langle \cdot, v_{\rho} \rangle = A_k^{\rho}))_{\rho \in \Sigma(1), k \in \{1, \dots, r\}} .$$

A parliament is a preparliament such that for any fixed maximal cone σ , there exists r associated characters $u \in \mathbf{u}(\sigma)$ at the intersection of some hyperplanes

$$(H_{\rho, k_{\rho, u}})_{\rho \in \sigma(1)} \quad \text{with } k_{\rho, u} \in \{1, \dots, r\} ,$$

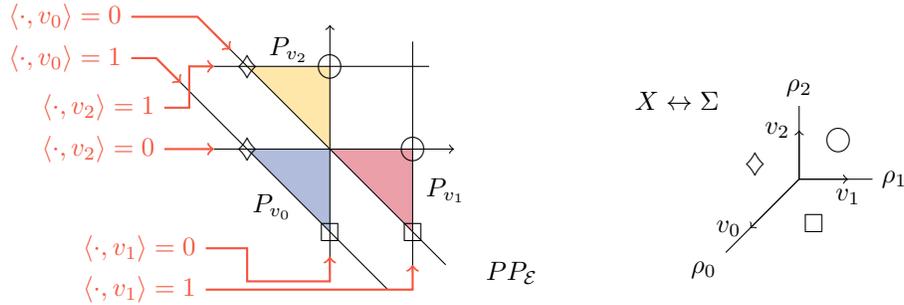
such that these hyperplanes are part of those defining some polytope $P_{L_u^{\sigma}}$ in $PP_{\mathcal{E}}$ and in the construction of the r associated characters of $\mathbf{u}(\sigma)$, each hyperplane should be used exactly once :

$$\forall \rho \in \sigma(1), \quad \{k_{\rho, u} \mid u \in \mathbf{u}(\sigma)\} = \{1, \dots, r\} .$$

Example 2.13. Consider $X = \mathbb{P}^2$ and its tangent bundle $\mathcal{E} = \mathcal{T}_X$. The \mathbb{Z} -filtrations are

$$\text{for } i = 0, 1, 2, E^{\rho_i}(j) = \begin{cases} \mathbb{C}^2 & \text{if } j \leq 0 \\ \text{Span}(v_i) & \text{if } 0 < j \leq 1 \\ \{0\} & \text{otherwise} \end{cases} .$$

The ground set is $G(\mathcal{E}) = \{v_0, v_1, v_2\} = \Sigma(1)$ and the parliament $PP_{\mathcal{E}}$ is the following.

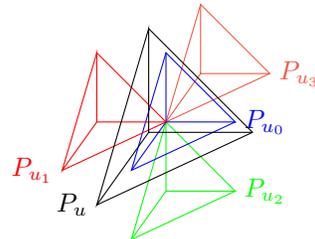


We associate to each maximal cone σ a symbol (say \square) and represent the associated characters $\mathbf{u}(\sigma)$ by r symbols \square on the parliament. The compatibility condition (CC) is verified. \diamond

Example 2.14. A contrario, the equivariant reflexive sheaf on P^3 defined by the filtrations

$$\text{for } i = 0, 1, 2, E^{\rho_i}(j) = \begin{cases} \mathbb{C}^3 & \text{if } j \leq 0 \\ \text{Span}(u_i, u) & \text{if } 0 < j \leq 1 \\ \text{Span}(u_i) & \text{if } 1 < j \leq 2 \\ \{0\} & \text{otherwise} \end{cases} ,$$

where u_0, u_1, u_2, u_3 belong to a same plane P and u does not, is not a toric bundle, but we can still construct its preparliament of polytopes. It contains five polytopes indexed by u_0, u_1, u_2, u_3 and u , but there is no compatible basis for any maximal cone $\sigma \in \Sigma(3)$.



\diamond

Remark 2.15. We may define a virtual (pre)parliament of polytopes of an equivariant reflexive sheaf \mathcal{E} . It means that each polytope in the parliament is considered virtual. This way, we are keeping a trace of the defining hyperplanes even if \mathcal{E} does not have any positivity property.

2.3 Stability of equivariant vector bundles

Most positivity properties of a toric vector bundle can be detected on its parliaments of polytopes. We may expect other properties such as stability to be visualisable on the parliaments. This notion is important, especially in the spirit of constructing moduli

spaces of toric vector bundles. In this section, we give an algorithm for the (semi)stability of a toric vector bundle \mathcal{E} in terms of its parliament of polytopes $PP_{\mathcal{E}}$. In this section, we assume, for simplicity, that all the (pre)parliaments are virtual (see Remark 2.15).

We consider a nonzero equivariant saturated subsheaf \mathcal{F} of an equivariant reflexive sheaf \mathcal{E} on a toric variety X . By Theorem 1.8, it is defined by filtrations

$$F^\rho(j) = \begin{cases} F & \text{if } j \leq A_1^\rho \\ \dots & \\ \text{Span}(u_\rho) & \text{if } A_{l-1}^\rho < j \leq A_l^\rho \\ \{0\} & \text{if } A_l^\rho < j. \end{cases}$$

where $F^\rho(j) = E^\rho(j) \cap F$, the $(E^\rho(j))_{\rho \in \Sigma(1), j \in \mathbb{Z}}$ being the defining filtrations of \mathcal{E} . The slope of \mathcal{F} with respect to α may be visualized using the notion of average polytope.

Definition 2.16. We define the average polytope $P_{\mathcal{F}}$ of \mathcal{F} to be the moment polytope associated to the divisor

$$\frac{c_1(\mathcal{F})}{l} = \sum \left(\frac{1}{l} \sum_{k=1}^l A_k^\rho \right) D_\rho.$$

Notation 2.17. We may also use the notation $P_{\mathcal{F}}$ if it appears to be more convenient.

The average polytope $P_{\mathcal{F}}$ of \mathcal{F} is a visualization of the slopes of \mathcal{F} . In the following proposition, we recover, for any polarization α of X , the α -slope of \mathcal{F} , only using the average polytope $P_{\mathcal{F}}$.

Proposition 2.18. The α -slope of an equivariant saturated subsheaf $\mathcal{F} \subset \mathcal{E}$ with average polytope $P_{\mathcal{F}} = P_D$ where $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ is

$$\mu_\alpha(\mathcal{F}) = \sum_{\rho \in \Sigma(1)} a_\rho t_\rho,$$

where α has weights $(t_\rho)_{\rho \in \Sigma(1)}$. Geometrically, the α -slope is the mixed volume of the average polytope $P_{\mathcal{F}}$ with the polytope P_{L_α} encoding the polarization

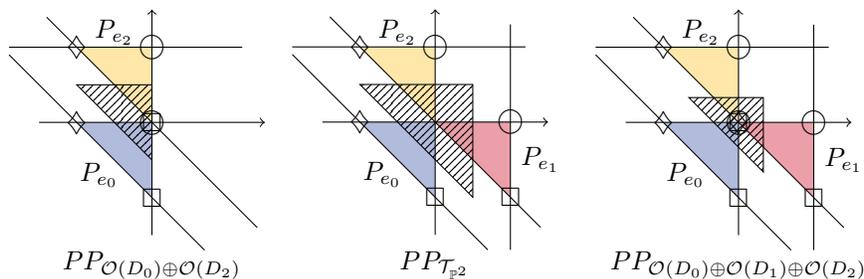
$$\mu_\alpha(\mathcal{F}) = V(P_{\mathcal{F}}, \underbrace{P_{L_\alpha}, \dots, P_{L_\alpha}}_{d-1 \text{ times}})^5.$$

Proof. See Proposition 1.14 and Definition 2.16. □

Example 2.19. The average polytopes of

$$\mathcal{F} = \mathcal{O}(D_0) \oplus \mathcal{O}(D_2), \quad \mathcal{F} = \mathcal{T}_{\mathbb{P}^2} \quad \text{and} \quad \mathcal{F} = \mathcal{O}(D_0) \oplus \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$$

are the hashed polytopes in the following pictures



⁵This beautiful remark has been pointed out to me by Christian Haase.

The difference between the two last pictures is the following. The hyperplanes forming a '+' in the middle have different multiplicities (1 or 2). Indeed, the tangent bundle $\mathcal{T}_{\mathbb{P}^2}$ has rank $l = 2$ whereas the splitting bundle $\mathcal{O}(D_0) \oplus \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$ has rank $l = 3$. The slopes with respect to the polarization α with weights $(1, 1, 1)$ (see Example 1.11) are

$$\begin{cases} \mu_\alpha(\mathcal{O}(D_0) \oplus \mathcal{O}(D_2)) = \frac{1}{2} + 0 + \frac{1}{2} = 1; \\ \mu_\alpha(\mathcal{T}_{\mathbb{P}^2}) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}; \\ \mu_\alpha(\mathcal{O}(D_0) \oplus \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1. \end{cases}$$

◇

To compare slopes between sheaves with respect to a polarisation α amounts to compare average polytopes using the following order.

Definition 2.20. Let X be a smooth complete toric variety and α a polarization with weights

$$t_\rho := D_\rho \cdot \alpha = f_\rho \frac{(d-1)!}{\|v_\rho\|} \quad \text{for every ray } \rho \in \Sigma.$$

We define a total order on moment polytopes of X by saying

$$P_1 <_\alpha P_2 \iff \sum_\rho a_1^\rho t_\rho < \sum_\rho a_2^\rho t_\rho \quad (\text{or } \iff V(P_1, P_{L_\alpha}^{d-1}) < V(P_2, P_{L_\alpha}^{d-1})),$$

where, for $j \in \{1, 2\}$, P_j is the moment polytope of the divisor $\sum_\rho a_j^\rho D_\rho$, and $V(\cdot, \dots, \cdot)$ is the mixed volume of polytopes.

Using Definition 2.20, Theorem 1.8, and Lemma 1.7, we may reformulate the definition of stability (Definition 1.6) of toric vector bundles into the following proposition.

Proposition 2.21. An equivariant reflexive sheaf \mathcal{E} is α -stable (resp. α -semistable) if and only if for any nonzero subspace $F \subsetneq E$, we have

$$P_F <_\alpha P_E \quad (\text{resp. } P_F \leq_\alpha P_E),$$

where P_F is the moment polytope of the preparliament $PP_{\mathcal{F}}$ obtained by considering the parliament $PP_{\mathcal{E}}$ and keeping only the hyperplanes corresponding to the jumping loci of \mathcal{F} .

Let us reformulate Theorem 1.15 in terms of parliaments of polytopes.

Theorem 2.22. Let \mathcal{E} be any equivariant reflexive sheaf on a toric variety X . There exists a finite matroid $M(\mathcal{E})^S$ representable in E such that \mathcal{E} is α -stable (resp. α -semistable) if and only if

$$P_F <_\alpha P_E \quad (\text{resp. } P_F \leq_\alpha P_E),$$

for any space $\{0\} \subsetneq F = \text{Span}(f) \subsetneq E$ given by a flat f of $M(\mathcal{E})^S$.

Remark 2.23. The matroid $M(\mathcal{E})^S$ does not depend on α .

2.4 Case of rank less or equal to 3

The purpose of the following theorem is to prove that if \mathcal{E} has rank less or equal to 3, then the matroid $M(\mathcal{E})$ of any parliament of \mathcal{E} checks stability.

Theorem 2.24. Let X be a smooth complete toric variety, let \mathcal{E} be a toric vector bundle on X , of rank less or equal to 3, and let $(PP_{\mathcal{E}}, G(\mathcal{E}))$ be any of its parliament. Then \mathcal{E} is

α -(semi)stable if and only if for any nonzero flat $f \subsetneq G(\mathcal{E})$, we have

$$P_{\mathcal{F}} <_{\alpha} P_{G(\mathcal{E})} \quad (\text{resp. } P_{\mathcal{F}} \leq_{\alpha} P_{G(\mathcal{E})}),$$

where \mathcal{F} is the equivariant saturated sheaf corresponding by Theorem 1.8 to the linear subspace $\text{Span}(f) \subset E$.

Proof. By Lemma 1.17, it is enough to show that for any vector space $\{0\} \subsetneq F_0 \subsetneq E$, there exists a flat $f \subsetneq G(\mathcal{E})$ with $F = \text{Span}(f)$ of the same dimension than F_0 and satisfying

$$\dim(F \cap V) \geq \dim(F_0 \cap V) \quad \text{for all } V \in L(\mathcal{E}).$$

In the case where $\dim(F_0) = 1$, take F generated by a vector of $G(\mathcal{E})$ in $\bigcap_{V \text{ st } F_0 \cap V \neq \{0\}} V$. It is possible because $\bigcap_{F_0 \cap V \neq \{0\}} V$ is nonzero (F_0 being included into it) and $G(\mathcal{E})$ has been constructed such that $\text{Span}(G(\mathcal{E})) \cap \bigcap_{V \text{ st } F_0 \cap V \neq \{0\}} V = \bigcap_{V \text{ st } F_0 \cap V \neq \{0\}} V$. It concludes the proof for bundles of rank 2.

It remains the case where $\dim(F_0) = 2$ and $\text{rk}(\mathcal{E}) = 3$. Let us denote by

$$V_1^1, \dots, V_a^1, V_1^2, \dots, V_b^2$$

the vector spaces of $L(\mathcal{E})$ such that $\dim(F_0 \cap V_k^{\varepsilon}) = \varepsilon$. Remark that b is either equal to 1 (and $V_1^2 = E$) or 2 (and $\{V_1^2, V_2^2\} = \{F, E\}$). If $b = 2$ then there exists a flat $f \subseteq G(\mathcal{E})$ such that $\text{Span}(f) = F_0$ and we are done. Assume thus that $b = 1$ and let us distinguish cases.

- If there exists $k_1, k_2 \in \{1, \dots, a\}$ such that $V_{k_1}^1, V_{k_2}^1$ of dimension 1, then F_0 is also generated by a flat

$$f = \{e_1, e_2\}$$

of $G(\mathcal{E})$, where $V_{k_1}^1 = \text{Span}(e_1)$, $V_{k_2}^1 = \text{Span}(e_2)$. We can take $F = F_0$.

- If there exists exactly one $V_{k_1}^1$ of dimension 1, then take a generator $e_1 \in G(\mathcal{E})$ of $V_{k_1}^1$ and any other vector $e_2 \in G(\mathcal{E})$. We define F as $F = \text{Span}(e_1, e_2)$. As for all $k \neq k_1$, we have that $\dim(V_k^1) \geq 2$, $\dim(F) = 2$ and both V_k^1 and F are included in E of dimension 3, it holds that

$$\dim(F \cap V_k^1) \geq 1 = \dim(F_0 \cap V_k^1).$$

- If for all $k \in \{1, \dots, a\}$, we have $\dim(V_k^1) \geq 2$ then, by our last argument, any dimension 2-space $F = \text{Span}(f)$ generated by a flat of $G(\mathcal{E})$ satisfies

$$\dim(F \cap V_k^1) \geq 1 = \dim(F_0 \cap V_k^1) \quad \text{for all } k \in \{1, \dots, a\}.$$

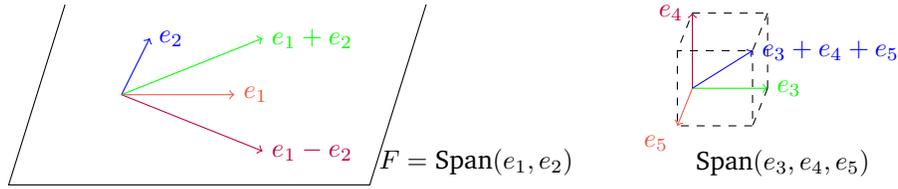
□

Example 2.25. In general, the flats of a matroid $M(\mathcal{E})$ are not enough to know that \mathcal{E} is not (semi)stable. Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ and the equivariant reflexive sheaf defined by the filtrations

$$\text{for } \rho \in \Sigma(1), \quad E^{\rho_i}(j) = \begin{cases} \mathbb{C}^5 & \text{if } j < 0 \\ V_i & \text{if } 0 \leq j < 1 \\ \{0\} & \text{otherwise} \end{cases},$$

where the vector spaces V_i are defined as $V_0 = \text{Span}(e_1 + e_2, e_3)$, $V_1 = \text{Span}(e_1 - e_2, e_4)$,

$V_2 = \text{Span}(e_1, e_5)$ and $V_3 = \text{Span}(e_2, e_3 + e_4 + e_5)$.



A matroid that can be obtained by Algorithm 3.2 is given by the ground set

$$G = \{e_1 + e_2 + e_3, e_1 + e_2 - e_3, e_1 - e_2 + 2e_4, e_1 - e_2 - 2e_4, \\ e_1 + 3e_5, e_1 - 3e_5, e_2 + 4e_3 + 4e_4 + 4e_5, e_2 - 4e_3 - 4e_4 - 4e_5\}.$$

Consider the vector space $F = \text{Span}(e_1, e_2)$. The slope of \mathcal{E} and \mathcal{F} are

$$\mu(\mathcal{E}) = \frac{2+2+2+2}{5} = \frac{8}{5} \quad \text{and} \quad \mu(\mathcal{F}) = \frac{1+1+1+1}{2} = 2.$$

Consequently, \mathcal{E} is unstable, but it cannot be seen by looking at the flats of G . Indeed, we have that

- A flat f of dimension $l = 1, 2$ or 3 satisfies

$$\text{Span}(V_i \cap f) = V_i \cap \text{Span}(f)$$

and its associated saturated subsheaf \mathcal{F} has thus slope $\mu(\mathcal{F}) = l/l = 1$.

- A flat of dimension 4 either

$$\text{Span}(V_i \cap f) = V_i \cap \text{Span}(f)$$

and its associated saturated subsheaf has slope $4/4 = 1$, or is of the form $V_i + V_j = \text{Span}(e_1, e_2, e'_1, e'_2)$ with e'_1, e'_2 in G and its associated saturated subsheaf \mathcal{F} has slope $\mu(\mathcal{F}) = \frac{6}{4} = \frac{3}{2}$. \diamond

Remark 2.26. We may enhance Example 2.25 by proving that all the flats of all the matroids $M(\mathcal{E})$ of a parliament are not enough to check the (semi)stability of \mathcal{E} . Indeed, we may force the ground set to be exactly the set G of Example 2.25. It suffices to incorporate 1-dimensional spaces $V = \text{Span}(g)$ for any $g \in G$ into the filtrations (take for instance $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$).

2.5 A generalisation of the case of tangent bundles

Let X be a smooth complete toric variety of dimension d and let \mathcal{E} be any equivariant reflexive sheaf on X with Klyachko filtrations given by vector spaces of dimension $0, 1$ and d and general jumping loci A_j^ρ . For instance, the tangent bundle $\mathcal{E} = \mathcal{T}_X$ on X satisfies these conditions as its Klyachko description is

$$\text{for } \rho \in \Sigma(1), E^\rho(j) = \begin{cases} \mathbb{C}^d & \text{if } j < 0 \\ \text{Span}(v_\rho) & \text{if } 0 \leq j < 1 \\ \{0\} & \text{otherwise} \end{cases}.$$

Note however that the tangent bundle case has already been treated in [HNS19] or [DDK19].

We prove that $M(\mathcal{E})$ is a matroid that checks stability. The only matroid $M(\mathcal{E})$ associated to \mathcal{E} has ground set given by $G(\mathcal{E}) = \{e_\rho \mid \rho \in \Sigma(1)\}$, where the spaces $\text{Span}(e_\rho)$ are the 1-dimensional vector spaces appearing in the Klyachko filtrations of \mathcal{E} .

Proposition 2.27. *The equivariant reflexive sheaf \mathcal{E} is α -(semi)stable if and only if for any nonzero flat $f \subseteq G(\mathcal{E})$, we have*

$$P_{\mathcal{F}} <_{\alpha} P_{\mathcal{E}} \quad (\text{resp. } P_{\mathcal{F}} \leq_{\alpha} P_{\mathcal{E}}),$$

where \mathcal{F} is the equivariant saturated sheaf corresponding by Theorem 1.8 to the linear subspace $\text{Span}(f) \subset E$.

Proof. The vector spaces in

$$L(\mathcal{E}) = \left\{ \bigcap_{\rho \in \Sigma(1)} E^{\rho}(j_{\rho}) \mid (j_{\rho})_{\rho \in \Sigma(1)} \in \mathbb{Z}^{n+1} \right\}$$

are of dimension either 0, 1 or d . Hence, any saturated subsheaf $\mathcal{F}_0 \subset \mathcal{E}$ corresponds to a subspace $F_0 \subset \mathbb{C}^d$ having intersection with elements of $L(\mathcal{E})$ either

$$F_0 \cap V = \{0\}, \quad F_0 \cap V = V \text{ and } \dim(V) = 1, \quad \text{or } F_0 \cap V = F_0.$$

Each $V \in L(\mathcal{E})$ of dimension 1 has a generator g_V in $G(\mathcal{E})$. Let us consider a flat f of $G(\mathcal{E})$ obtained starting with the set of generators $g_V \in G(\mathcal{E})$ of the vector spaces $V \in L(\mathcal{E})$ of dimension 1 included in F_0 , and adjoining elements of G such that the vector space $F := \text{Span}(f)$ satisfies $\dim(F) = \dim(F_0)$. By Lemma 1.17, we have the inequality

$$\mu(\mathcal{F}) \geq \mu(\mathcal{F}_0),$$

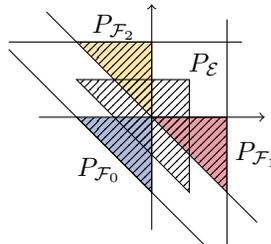
and the flats of $G(\mathcal{E})$ are enough to check the (semi)stability of \mathcal{E} . \square

Remark 2.28. In the tangent bundle case, we recover the criterion for stability given in Proposition 1.2. of [HNS19].

Example 2.29. Consider the tangent bundle $\mathcal{E} = \mathcal{T}_{\mathbb{P}^2}$ on \mathbb{P}^2 . The nontrivial flats of $G(\mathcal{E}) = \{v_0, v_1, v_2\}$ are

$$f_0 = \{v_0\}, \quad f_1 = \{v_1\} \text{ and } f_2 = \{v_2\}.$$

They correspond to the three nontrivial equivariant saturated subsheaves $\mathcal{F}_0, \mathcal{F}_1$ and \mathcal{F}_2 of \mathcal{E} . Let us look at the average polytopes:



By Theorem 2.24, the fact that $P_{\mathcal{F}_0}, P_{\mathcal{F}_1}, P_{\mathcal{F}_2} <_{\alpha} P_{\mathcal{E}}$ proves that \mathcal{E} is α -stable. \diamond

3 Parliaments of equivariant subbundles

In this section, we first define subparliaments of polytopes and identify them with parliaments of equivariant subbundles. Second, we give an example of a toric subbundle which is not a direct factor.

Let us state our definition of subparliament of a parliament of polytopes.

Definition 3.1. *Let $f = G(\mathcal{E}) \cap F$ be a flat of the matroid $M(\mathcal{E})$ such that there exists a compatible basis $(B_\sigma)_{\sigma \in \Sigma(d)}$ for \mathcal{E} satisfying*

$$\text{for all } \sigma \in \Sigma(d), \quad \#B_\sigma \cap f = \dim F .$$

Then we call f a compatible flat, and we call the subset of polytopes of $PP_{\mathcal{E}}$ indexed by elements of $f = F \cap G(\mathcal{E})$

$$\{(P_e, e) \mid e \in f\}$$

a subparliament of the parliament of polytopes \mathcal{E} .

Equivariant subbundles have been combinatorially described in [KD18]. We translate their results in terms of parliaments of polytopes.

Proposition 3.2. *Via the Klyachko classification, a rank r -equivariant vector bundle \mathcal{E} on X corresponds to a \mathbb{Z} -filtration $(E^\rho(j))_{j \in \mathbb{Z}}$ of $E \cong \mathbb{C}^r$ for each $\rho \in \Sigma(1)$.*

The equivariant subbundles \mathcal{F} of \mathcal{E} are then in one-to-one correspondence with the subfiltrations

$$(F^\rho(j) = E^\rho(j) \cap F)_{j \in \mathbb{Z}} \quad \text{of} \quad (E^\rho(j))_{j \in \mathbb{Z}},$$

for some vector subspace $F \subseteq E$ such that there exist ground sets $G(\mathcal{F})$ and $G(\mathcal{E})$ resulting from Algorithm 3.2 for $L(\mathcal{F})$ and $L(\mathcal{E})$ with

$$G(\mathcal{E}) \cap F = G(\mathcal{F})$$

and such that there exist compatible bases $(B_\sigma)_{\sigma \in \Sigma(d)}$ for \mathcal{E} which, restricted to F , satisfy the compatibility condition (CC) for \mathcal{F} .

Proof. By Proposition 4.1.1 of [KD18], the equivariant subbundles \mathcal{F} of \mathcal{E} are then in one-to-one correspondence with the subfiltrations

$$(F^\rho(j) = E^\rho(j) \cap F)_{j \in \mathbb{Z}} \quad \text{of} \quad (E^\rho(j))_{j \in \mathbb{Z}},$$

for some vector subspace $F \subseteq E$ such that $\{F, \{E^\rho(j)\}_{\rho \in \sigma(1)}\}$ of E forms a distributive lattice.

Let us show that there exist ground sets $G(\mathcal{F})$, $G(\mathcal{E})$ obtained by Algorithm 3.2 for $L(\mathcal{F})$, $L(\mathcal{G})$ with

$$G(\mathcal{E}) \cap F = G(\mathcal{F}) .$$

Consider the ground set $G(\mathcal{E})$ obtained by taking a maximal number of vectors in F in Algorithm 3.2. If $V_1, V_2 \in L(\mathcal{E})$, let us say that $V_1 < V_2$ if V_1 appears before V_2 in Algorithm 3.2. We denote by G_V the intersection of V with the temporary ground set at the beginning of Step $V \in L(\mathcal{E})$. During Step V , we need to add a basis CB_V of any complement of $\text{Span}(G_V)$ in V to the ground set. We choose this basis to be composed of

a maximal number of vectors in F so that we have

$$\begin{aligned}
V \cap F &= (\text{Span}(G_V) + \text{Span}(CB_V)) \cap F \\
&= \left(\sum_{\substack{V'=V_1 \cap V \\ V_1 < V}} V' + \text{Span}(CB_V) \right) \cap F \\
&= \left(\sum_{\substack{V'=V_1 \cap V \\ V_1 < V}} V' \right) \cap F + \text{Span}(CB_V \cap F) \\
&= \sum_{\substack{V'=V_1 \cap V \\ V_1 < V}} (V' \cap F) + \text{Span}(CB_V \cap F) .
\end{aligned}$$

The last step comes from $\{F, \{E^\rho(j)\}_{\rho \in \sigma(1)}\}$ forming a distributive lattice of E . By induction on the $V \in L(\mathcal{E})$ appearing in Algorithm 3.2, $G_V \cap F$ generates every $(V_1 \cap V) \cap F$ for $V_1 < V$. We finally obtain that $G(\mathcal{E}) \cap F$ generates every $V \cap F$ for $V \in L(\mathcal{E})$.

Now if we denote by $G_{V \cap F}^{\mathcal{F}}$ the intersection of $V \cap F$ with the temporary ground set at the beginning of Step $V \cap F \in L(\mathcal{F})$ of Algorithm 3.2 (applied to \mathcal{F}), then

$$V \cap F = \text{Span}(G_{V \cap F}^{\mathcal{F}}) + \text{Span}(CB_{V \cap F}^{\mathcal{F}}) = \sum_{\substack{V' \cap F = (V_1 \cap F) \cap (V \cap F) \\ (V_1 \cap F) < (V \cap F)}} (V' \cap F) + \text{Span}(CB_{V \cap F}^{\mathcal{F}}) .$$

Moreover, since $V_1 < V \Rightarrow (V_1 \cap F < V \cap F)$ and $(V_1 \cap F < V \cap F) \Rightarrow (V_1 \cap V < V)$, we have

$$\begin{aligned}
&\{V' \cap F \mid V' = V_1 \cap V \text{ and } V_1 < V\} \\
&= \{V' \cap F \mid V' \cap F = (V_1 \cap F) \cap (V \cap F) \text{ and } (V_1 \cap F) < (V \cap F)\} .
\end{aligned}$$

We may take $CB_{V \cap F}^{\mathcal{F}}$ to be $CB_V \cap F$, and we finally obtain $G(\mathcal{F}) = G(\mathcal{E}) \cap F$.

Moreover, for each cone $\sigma \in \Sigma(d)$, any compatible basis B_σ^F of F extends to a compatible basis B_σ of E . Indeed, to construct a compatible basis B_σ (resp. B_σ^F), we apply Algorithm 3.2 to

$$L_\sigma(\mathcal{E}) = \left\{ \bigcap_{\rho \subset \sigma} E^\rho(j_\rho) \mid (j_\rho)_{\rho \subset \sigma} \right\} \quad (\text{resp. } L_\sigma(\mathcal{F}) = \{V \cap F \mid V \in L_\sigma(\mathcal{E})\}) ,$$

taking CB_V included in $G(\mathcal{E})$ (resp. taking $CB_{V \cap F}^{\mathcal{F}}$ included in $G(\mathcal{E}) \cap F$). Extending the bases B_σ^F is possible because $G(\mathcal{E})$ generates each $V \in L_\sigma(\mathcal{E}) \subset L(\mathcal{E})$ as well as $G(\mathcal{F}) = G(\mathcal{E}) \cap F$ generates each $V \cap F \in L_\sigma(\mathcal{F}) \subset L(\mathcal{F})$. \square

Definition 3.3. Consider a toric vector bundle \mathcal{E} and a parliament of polytopes $PP_\mathcal{E}$ of \mathcal{E} . A subparliament of polytopes of $PP_\mathcal{E}$ is a set of the form

$$PP_\mathcal{F} = \{(P_e, e) \mid e \in G(\mathcal{E}) \cap F\} \subset PP_\mathcal{E} ,$$

for some vector space F , which is a parliament meaning that it satisfies the compatibility condition (see Proposition 2.12).

Theorem 3.4. Let \mathcal{E} be a toric vector bundle and let \mathcal{F} be the equivariant subbundle of \mathcal{E} corresponding by Klyachko's theorem to the filtrations $(E^\rho(j) \cap F)_{j \in \mathbb{Z}}$ for each $\rho \in \Sigma(1)$.

Then, there exists a parliament $PP_{\mathcal{E}}$ of \mathcal{E} such that a parliament of \mathcal{F} is a subparliament of $PP_{\mathcal{E}}$

$$PP_{\mathcal{F}} = \{(P_e, e) \mid e \in G(\mathcal{E}) \cap F\} \subset PP_{\mathcal{E}}.$$

Proof. Consider a ground set $G(\mathcal{E})$ for the parliament of \mathcal{E} obtained via Algorithm 3.2 by taking at each step a maximal number of elements in F . Then by Proposition 3.2, $G(\mathcal{F}) = G(\mathcal{E}) \cap F$ is a possible output for Algorithm 3.2 applied to $L(\mathcal{F})$. Now if $e \in G(\mathcal{F}) = G(\mathcal{E}) \cap F$ then the polytope $P_e \in PP_{\mathcal{F}}$ and

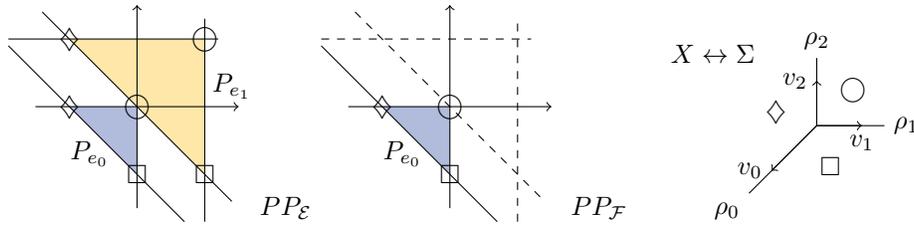
$$\begin{aligned} P_e &= \{m \in M_{\mathbb{R}} \mid \forall \rho \in \Sigma(1), \langle m, v_{\rho} \rangle \leq \max \{j \mid e \in E^{\rho}(j) \cap F\}\} \\ &= \{m \in M_{\mathbb{R}} \mid \forall \rho \in \Sigma(1), \langle m, v_{\rho} \rangle \leq \max \{j \mid e \in E^{\rho}(j)\}\} \end{aligned}$$

is the same as the one in the parliament of \mathcal{E} . \square

Example 3.5. Consider the toric variety $X = \mathbb{P}^2$ with its T -invariant divisors D_0, D_1 and D_2 and the splitting, equivariant, rank 2-vector bundle $\mathcal{E} = \mathcal{O}(D_0) \oplus \mathcal{O}(D_1 + D_2)$ on X . The corresponding \mathbb{Z} -filtrations of $E = \mathbb{C}^2$ are of the form

$$E^{\rho_i}(j) = \begin{cases} \mathbb{C}^2 & \text{if } j \leq 0 \\ \text{Span}(e_i) & \text{if } 0 < j \leq 1 \quad (\text{for all } i = 0, 1, 2), \\ \{0\} & \text{otherwise} \end{cases},$$

with $e_1 = e_2$, and where $\text{Span}(e_1)$ and $\text{Span}(e_0)$ are different lines in \mathbb{C}^2 . The ground set is $G(\mathcal{E}) = \{e_1, e_0\}$. The toric subbundle $\mathcal{F} = \mathcal{O}(D_0) \subset \mathcal{E}$ corresponds to the flat $f = \{e_0\} \subset G(\mathcal{E})$.

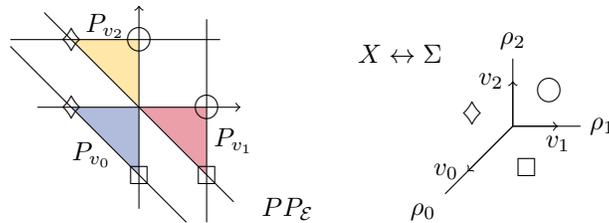


The compatibility condition (CC) for \mathcal{E} is satisfied on taking

$$B_{\sigma} = \{L_u^{\sigma} \mid u \in \mathbf{u}(\sigma)\} = \{e_1, e_0\} \quad \text{for all } \sigma \in \Sigma(d).$$

We see that for all $\sigma \in \Sigma(d)$, $B_{\sigma} \cap f = \{e_0\}$ generates $F = \text{Span}(f)$, the flat $f = \{e_0\}$ is thus compatible. The parliament $PP_{\mathcal{F}}$ is a subparliament of $PP_{\mathcal{E}}$. \diamond

Example 3.6. Let us come back to Example 2.13 with $X = \mathbb{P}^2$ and its tangent bundle $\mathcal{E} = \mathcal{T}_{\mathbb{P}^2}$.



There exists no nontrivial equivariant subbundle \mathcal{F} of \mathcal{E} because the nontrivial flats are $f_0 = \{v_0\}$, $f_1 = \{v_1\}$ and $f_2 = \{v_2\}$ and are not compatible. \diamond

Corollary 3.7 (Theorem 1). *The subparliaments of the parliaments of \mathcal{E} are the parliaments*

of the equivariant subbundles of \mathcal{E} .

Classifying toric vector bundle requires to understand Harder–Narasimhan filtrations. Let us define an equivariant vector bundle \mathcal{E} with a nontrivial equivariant subbundle \mathcal{F} that we cannot factorize in a direct sum of \mathcal{E} .

Example 3.8. Consider the equivariant vector bundle \mathcal{E} on \mathbb{P}^2 defined by the filtrations

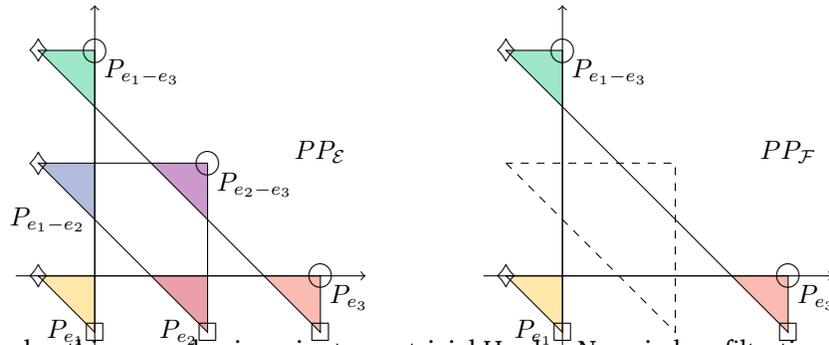
$$E^{\rho_0}(j) = \begin{cases} \mathbb{C}^3 & \text{if } j < -3 \\ \text{Span}(e_1, e_2) & \text{if } -3 \leq j < -1 \\ \text{Span}(e_1) & \text{if } -1 \leq j < 1 \\ \{0\} & \text{otherwise} \end{cases}, \quad E^{\rho_1}(j) = \begin{cases} \mathbb{C}^3 & \text{if } j < 0 \\ \text{Span}(e_2, e_3) & \text{if } 0 \leq j < 2 \\ \text{Span}(e_3) & \text{if } 2 \leq j < 4 \\ \{0\} & \text{otherwise} \end{cases}$$

and $E^{\rho_2}(j) = \begin{cases} \mathbb{C}^3 & \text{if } j < 0 \\ \text{Span}(e_1 - e_3, e_1 - e_2) & \text{if } 0 \leq j < 2 \\ \text{Span}(e_1 - e_3) & \text{if } 2 \leq j < 4 \\ \{0\} & \text{otherwise} \end{cases}$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. The unique possible ground set of $M(\mathcal{E})$ is

$$G(\mathcal{E}) = \{e_1, e_2, e_3, e_1 - e_2, e_1 - e_3, e_2 - e_3\}.$$

Taking $F = \text{Span}(e_2)^\perp$, we obtain an equivariant subbundle \mathcal{F} .



In particular, this example gives rise to nontrivial Harder–Narasimhan filtrations

$$0 \subseteq \mathcal{F} \subseteq \mathcal{E},$$

that we may visualise on the parliament of polytopes $PP_{\mathcal{E}}$. ◇

4 Stability of restrictions to invariant curves

The α -semistability of the restrictions $\mathcal{E}|_C$ of a toric vector bundle \mathcal{E} to a torus invariant curve is much easier to view on the parliament of \mathcal{E} . Moreover, it is often useful to compare the α -semistability of a toric vector bundle \mathcal{E} to the semistability of its restrictions $\mathcal{E}|_C$ to any torus invariant curve.

Remark 4.1. Theorem 2.5 from [BLG13] furnishes a sufficient condition to deduce from the semistability of a toric vector bundle \mathcal{E} , the semistability of its restriction $\mathcal{E}|_C$ to any invariant curves C : the characteristic class of \mathcal{E} is 0

$$\Delta(\mathcal{E}) = c_2(\mathcal{E}) - \frac{r-1}{2r}c_1(\mathcal{E})^2 = 0.$$

In this section, we explain how to see the (semi)stability of the restrictions $\mathcal{E}|_C$ of a vector

bundle \mathcal{E} to a torus invariant curve C , given a parliament $PP_{\mathcal{E}}$. In [DJS14] Subsection 3.1, Di Rocco, Jabbusch and Smith recover the parliaments of restrictions $\mathcal{E}|_C$ to torus invariant curves from the parliaments of polytopes of \mathcal{E} .

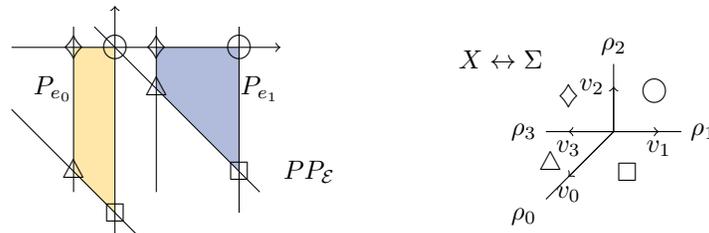
Proposition 4.2. *By the cone-orbit correspondence, torus invariant curves correspond to a cone $\tau \in \Sigma(d-1)$. Since X is complete, there are two maximal cones σ and σ' in $\Sigma(d)$ containing τ .*

If $PP_{\mathcal{E}}$ is a parliament of \mathcal{E} , then a parliament of polytopes of $\mathcal{E}|_C$ is composed of the projection on τ^\perp of the line segments parallel to τ^\perp joining the associated characters in $\mathbf{u}(\sigma)$ and $\mathbf{u}(\sigma')$ (renormalized by $1/||u_\tau||$).

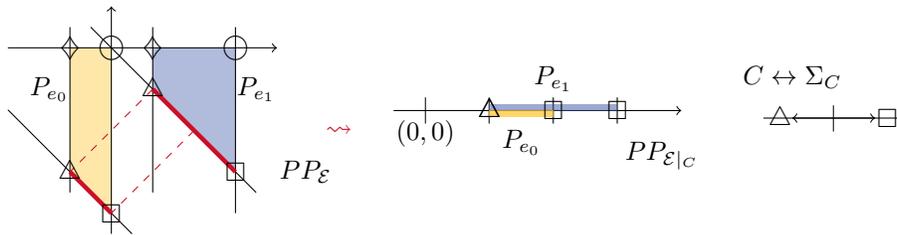
Example 4.3. Consider $X = \text{Bl}_{[0:1:0]}(\mathbb{P}^2)$ and let \mathcal{E} be the equivariant vector bundle

$$\mathcal{E} = \mathcal{O}_X(4D_0 + D_3) \oplus \mathcal{O}_X(3D_1 - D_3)$$

on X . The unique parliament of polytopes of \mathcal{E} is



The parliament of \mathcal{E} restricted to the torus invariant curve D_0 is obtained the following way.



The semistability of such restriction $\mathcal{E}|_C$ should be easier to understand as torus invariant curves C are isomorphic to \mathbb{P}^1 . Moreover, in 1957, Grothendieck classified vector bundles on \mathbb{P}^1 .

Theorem 4.4 ([Gro57] Theorem 2.1). *Every holomorphic vector bundle \mathcal{E} on \mathbb{P}^1 is holomorphically isomorphic to a direct sum of line bundles:*

$$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(d_r).$$

The problem of semistability is then reduced to verify if $d_1 = \dots = d_r$. This can be achieved by looking at the parliaments of polytopes of \mathcal{E} . The operation on parliaments corresponding to restricting the toric bundle to an invariant curve is explained in Proposition 4.2. We obtain the following result.

Theorem 4.5. *Let C be a torus invariant curve on X . C corresponds to a wall $\tau \in \Sigma(d-1)$ between two maximal cones $\sigma, \sigma' \in \Sigma(d)$. A toric vector bundle \mathcal{E} is semistable if and only*

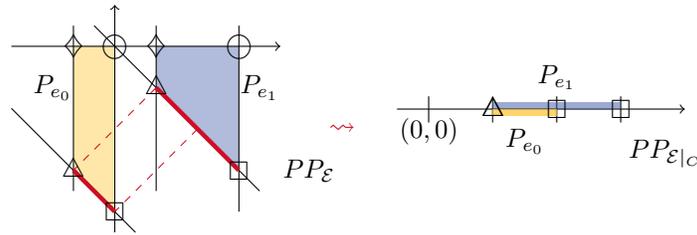
if the normalized lattice distance between associated characters $u \in \mathbf{u}(\sigma)$ and $u' \in \mathbf{u}(\sigma')$ in the one-dimensional lattice $(\tau^\perp + u) \cap M$ is the same for any pair u, u' .

The following example concerns a semistable vector bundle such that its restriction to an invariant curve is not semistable.

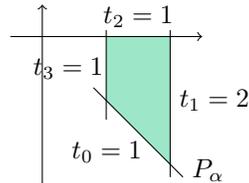
Example 4.6. Consider $X = \text{Bl}_{[0:1:0]}(\mathbb{P}^2)$ and let \mathcal{E} be the equivariant vector bundle

$$\mathcal{E} = \mathcal{O}_X(4D_0 + D_3) \oplus \mathcal{O}_X(3D_1 - D_3)$$

on X . The parliament \mathcal{E}_{D_0} of \mathcal{E} restricted to the torus invariant curve D_0 is obtained the following way.



The bundle $\mathcal{E}|_{D_0}$ would be stable if the segments P_{e_0} and P_{e_1} were of the same length. In this example, there is no torus invariant curve C such that $\mathcal{E}|_C$ is semistable. A contrario, taking the movable curve $\alpha = 2D_1 - D_3$, as in Example 1.12, with its corresponding $(t_\rho)_{\rho \in \Sigma(1)}$



gives \mathcal{E} is α -polystable: $\mu(\mathcal{L}_0) = 4t_0 + t_3 = 5$ is equal to $\mu(\mathcal{L}_1) = 3t_1 - t_3 = 5$. \diamond

Example 4.7. For any choice of movable curve α , the tangent bundle $\mathcal{T}_{\mathbb{P}^2}$ of \mathbb{P}^2 is an example of α -stable equivariant vector bundle \mathcal{E} with a nonstable restriction to a torus invariant curve. \diamond

5 Alternative definition of parliaments

We propose definitions for the parliaments of polytopes of a toric vector bundle \mathcal{E} . For each definition, the data of a parliament of polytopes of some globally generated equivariant vector bundle \mathcal{E} corresponds to some isomorphism class of \mathcal{E} (corresponding to the title of the subsection).

Remark 5.1. If we consider virtual polytopes as proposed in Remark 2.15, the condition of global generateness becomes superfluous.

The naive definition for a parliament of polytopes of a toric vector bundle \mathcal{E} would be

$$PP_{\mathcal{E}} := \{P_e \mid e \in G(\mathcal{E})\}$$

which does not keep a trace of the label of each polytope. This definition does not even allow us to distinguish a rank 2- from a rank 3-vector bundle on \mathbb{P}^2 : the tangent bundle

$\mathcal{T}_{\mathbb{P}^2}$ and the splitting bundle $\mathcal{O}_{\mathbb{P}^2}(D_0) \oplus \mathcal{O}_{\mathbb{P}^2}(D_1) \oplus \mathcal{O}_{\mathbb{P}^2}(D_2)$ have the same parliament of polytopes.

5.1 Equivariant isomorphism class of framed toric vector bundle

The definition we gave in Definition 2.9 for a parliament of polytopes of a toric vector bundle \mathcal{E} was $PP_{\mathcal{E}} := \{(P_e, e) \mid e \in G(\mathcal{E})\}$ where $G(\mathcal{E})$ is defined modulo isomorphism of type (\star) .

Definition 5.2. *A framed equivariant vector bundle is a toric vector bundle with a choice of isomorphism $E \cong \mathbb{C}^r$.*

A morphism of framed equivariant vector bundles is a morphism of equivariant vector bundles compatible with the framing.

Proposition 5.3. *The data of a parliament of polytopes*

$$PP_{\mathcal{E}} := \{(P_e, e) \mid e \in G(\mathcal{E})\} / (\star)$$

of a globally generated equivariant vector bundle \mathcal{E} is equivalent to knowing the equivariant isomorphism class of the framed equivariant vector bundle \mathcal{E} .

Proof. In [Pay07] Proposition 3.4, the equivariant classes of a framed equivariant vector bundle is uniquely determined by the following

$$(\{\mathbf{u}(\sigma)\}_{\sigma \in \Sigma(d)}, \{Fl(\rho)\}_{\rho \in \Sigma(1)}),$$

where $Fl(\rho)$ is the flag appearing in the filtration $(E^\rho(j))_j$. From the parliament we recover the flag

$$Fl(\rho) : \{0\} \subset E^\rho(A_r^\rho) \subset \dots \subset E^\rho(A_1^\rho) = E$$

by considering all different vector spaces

$$E_{\rho,j} = \sum_{\substack{\langle u, v_\rho \rangle \geq j, \\ u \in P_e \text{ for some } e \in G(\mathcal{E})}} e.$$

Indeed, if $e \in E_{\rho,j}$ then there exists $u \in P_e$ such that $\langle u, v_\rho \rangle \geq j$ and by definition of P_e , we obtain that $e \in E^\rho(j)$. And conversely, by Theorem 1.2 of [DJS14], as \mathcal{E} is taken globally generated, any $u \in \mathbf{u}(\sigma)$ belongs to some polytope and by (CC), we have $E^\rho(j) \subseteq E_{\rho,j}$. \square

5.2 Equivariant isomorphism class of toric vector bundle

We may not want to deal with framings anymore. Let us thus quotient by the action of $GL_r(\mathbb{C})$ on \mathbb{C}^r .

Proposition 5.4 (Proposition 2). *The data of a parliament of polytopes*

$$PP_{\mathcal{E}} := \{(P_e, e) \mid e \in G(\mathcal{E})\} / (\star) / GL_r(\mathbb{C})$$

of a globally generated equivariant vector bundle \mathcal{E} is equivalent to knowing the equivariant isomorphism class of the equivariant vector bundle \mathcal{E} .

Remark 5.5. Here morphisms do not have to preserve the framings anymore.

Proof of the proposition. It follows from [Pay07] Corollary 3.6 that the equivariant classes of a toric vector bundle are uniquely determined by

$$\left(\{\mathbf{u}(\sigma)\}_{\sigma \in \Sigma(d)}, \mathcal{O}_{\{Fl(\rho)\}_{\rho \in \Sigma(1)}} \right),$$

where $\mathcal{O}_{\{Fl(\rho)\}_{\rho \in \Sigma(1)}}$ is the $GL_r(\mathbb{C})$ -orbit of the flag given by the filtration $(E^\rho(j))_j$. We conclude by Proposition 5.3.

5.3 Isomorphism class of toric vector bundle

Finally, we may want an object that represents the isomorphism class of \mathcal{E} and not its equivariant isomorphism class. For that, we need to quotient by the group T of compositions of translations for each direct component.

Proposition 5.6. *The data of a parliament of polytopes*

$$PP_{\mathcal{E}} := \{(P_e, e) \mid e \in G(\mathcal{E})\} / (\star) / GL_r(\mathbb{C}) / T,$$

of a globally generated equivariant vector bundle \mathcal{E} is equivalent to knowing the isomorphism class of \mathcal{E} .

Proof. It is known that any line bundle $\mathcal{O}_X(D)$ on toric variety is isomorphic to an equivariant line bundle $\mathcal{O}_X(D_T) = \mathcal{O}_X(D) \otimes \chi^u$ for a unique $u \in M$.

This result has been generalized by Klyachko in ([Kly89] Corollary 1.2.4). If two equivariant vector bundles \mathcal{E} and \mathcal{F} are isomorphic then there exists characters χ_1, \dots, χ_m such that

$$\mathcal{E}_i \otimes \chi_i \text{ and } \mathcal{F}_i \text{ are equivariantly isomorphic,}$$

where $\mathcal{E} = \mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_m$ and $\mathcal{F} = \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_m$ are some direct decompositions of \mathcal{E} and \mathcal{F} .

As a level of parliaments, tensoring by a character χ^u corresponds to translating the parliament of $u \in M$. \square

A Algorithm

The following algorithm, that you may run in Macaulay2, requires two inputs: a toric bundle in Klyachko's description, for instance, you may consider

```
E = tangentBundle (projectiveSpaceFan 2) ;
```

and a polarisation given by its weights $(t_\rho)_{\rho \in \Sigma(1)}$ as for example

```
H = rho -> 1 ;
```

The program goes as follows.

```
loadPackage "ToricVectorBundles";
loadPackage "PositivityToricBundles";
loadPackage "NormalToricVarieties";
```

```
E = tangentBundle (projectiveSpaceFan 2);
H = rho -> 1;
Sigma1 = rays E;
N = length(Sigma1);
r = rank E;
```

```
Slope = (poss, l) -> -sum(flatten flatten apply(toList(0 .. N-1),
rho -> H(rho) * entries(submatrix((filtration E)#((rays E)#rho),
poss(rho)))))/l;
```

```
SlopeE = -sum(flatten flatten apply( toList(0 .. N-1),
rho -> H(rho)*entries((filtration E)#((rays E)#rho))))/r;
```

```
l = 1
while l < r do (
  D = subsets(r, l);
  R = QQ[x_{1,1} .. x_{r,1}];
  possibility = 0;
  while possibility < ((#D)^N) do (
    poss = rho -> D#( (possibility//(#D)^(rho) )%(#D) );
    M = (rho, a) -> matrix(pack(x_{1,1}..x_{r,1}, l) |
    submatrix( (base E)#((rays E)#rho),
    toList(0..(poss(rho))#a) );
    Irhoa = (rho, a) -> minors(l+(poss(rho))#a-a+1, M(rho, a));
    I = ideal ();
    for rho from 0 to N-1 do (
      for a from 0 to l-1 do
        I=I+Irho(rho, a) );
    J = minors (l, matrix(pack(x_{1,1}..x_{r,1}, l) );
    if radical(I+J)!=radical(I) and Slope(poss, l) >= SlopeE
    then break print ( apply(toList(0..N-1),
    rho -> poss(rho)), l) );
    possibility = possibility +1 );
  l = l+1; );
if (l == r and possibility == ((#D)^N)) then print "E is stable"
```

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