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# A combinatorial description of stability for toric vector bundles

Lucie DEVEY

## Abstract

The aim of this paper is to discuss a combinatorial characterisation of stability for toric vector bundle in the terms of their parliaments of polytopes, a generalization of Newton polytopes for toric vector bundles by Di Rocco, Jabbusch and Smith. We also define subparliaments of polytopes and identify them with parliaments of equivariant subbundles.

## Introduction

Stability has been introduced in the perspective of classifying vector bundles, by constructing moduli spaces of semistable bundles or sheaves. Using the Harder–Narasimhan filtration, any vector bundle  $\mathcal{E}$  can be built up from semistable coherent sheaves. In 1963, Mumford extended the works of Grothendieck (see [Gro57] Theorem 2.1) and Atiyah (see [Ati57]) by classifying vector bundles on curves (see [Mum63]). In this paper, we consider a generalisation of Mumford-stability to vector bundles on varieties of any dimension, called slope-stability and introduced in 1972 by Takemoto ([Tak72]). The problem of classification of vector bundles is still completely open. The aim of this paper is to study the toric case.

Consider a toric variety  $X$  with torus  $T$ . A toric vector bundle  $\mathcal{E}$  over  $X$  is a locally free  $\mathcal{O}_X$ -module of finite rank  $r$ , equipped with a  $T$ -action such that the projection map  $\pi : \text{Spec}(\text{Sym } \mathcal{E}) \rightarrow X$  is  $T$ -equivariant and  $T$  acts linearly on the fibres. We denote the fiber over the identity by  $E$ . In [DJS14], Sandra Di Rocco, Kelly Jabbusch and Gregory G. Smith generalized the construction of the Newton polytope of a line bundle, associating to any equivariant vector bundle  $\mathcal{E}$  its parliament of polytopes  $PP_{\mathcal{E}}$  : a collection of convex polytopes  $(P_e)_{e \in G(\mathcal{E})}$  indexed by the elements in the ground set of a matroid  $M(\mathcal{E})$  associated to  $\mathcal{E}$  representable in the vector space  $E \simeq \mathbb{C}^r$ . The wealth of information about a toric vector bundle  $\mathcal{E}$  contained in its parliament is astounding. For instance, lattice points in the parliament of polytopes for  $\mathcal{E}$  correspond to a torus-equivariant generating set for the space of global sections of  $\mathcal{E}$  (see [DJS14] Proposition 1.1). In addition, we can also recover some positivity properties of  $\mathcal{E}$ , such as global generatedness (see [DJS14] Theorem 1.2), ampleness (see [DJS14] Corollary 6.7), or bigness (see [Nø20] Theorem 7.5).

Our main result is a first step to having an algorithm (as wished in [DDK19]) for the slope-stability of toric vector bundles. We start by defining in Definition 2.12 for any equivariant saturated subsheaf  $\mathcal{F}$  of  $\mathcal{E}$ , a polytope called the average polytope  $P_{\mathcal{F}}$  which can be visualized on the parliament of  $\mathcal{E}$ . It is the Newton polytope of  $c_1(\mathcal{F})/\text{rk}(\mathcal{F})$ . In Definition 2.17, for any polarisation  $\alpha$  of  $X$ , we define a total order  $<_{\alpha}$  between polytopes, such that comparing average polytopes  $P_{\mathcal{F}_1}$  and  $P_{\mathcal{F}_2}$  corresponds to comparing the  $\alpha$ -slopes of their respective sheaves  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . To check stability, we need to compare

the slope of  $\mathcal{E}$  with the slope of its subsheaves. It has been proved, first that equivariant saturated subsheaves of  $\mathcal{E}$  are in one-to-one correspondence with vector subspaces  $F \subset E$  and second that it is enough to check the slopes of those subsheaves are less than the slope of  $\mathcal{E}$  (see [DDK19] or [HNS19]). These nice results allowed the same authors to have a finite check for the stability of tangent bundles. In this paper, we give the existence of a finite matroid  $M(\mathcal{E})^S$  on which the flats correspond to a finite family of subsheaves  $\mathcal{F}$  sufficiently varied to check stability of any toric vector bundle.

**Theorem (2.19).** *Let  $X$  be a smooth complete toric variety. For any toric vector bundle  $\mathcal{E}$  on  $X$ , there exists a finite matroid  $M(\mathcal{E})^S$  such that  $\mathcal{E}$  is  $\alpha$ -(semi)stable if and only if, for any nonzero flat  $f \subsetneq G(\mathcal{E})^S$  of  $M(\mathcal{E})^S$ , we have*

$$P_{\mathcal{F}} <_{\alpha} P_{\mathcal{E}} \quad (\text{resp. } P_{\mathcal{F}} \leq_{\alpha} P_{\mathcal{E}}),$$

where  $\mathcal{F}$  is the equivariant saturated sheaf corresponding by Theorem 2.4 to the linear subspace  $\langle f \rangle \subset E$ .

Moreover, we give a finite check of stability for any toric bundle of rank less or equal to 3: in that case, the matroid of any parliament of polytopes of  $\mathcal{E}$  checks stability.

In another part of the article, we prove that any equivariant subbundle  $\mathcal{F}$  of a toric vector bundle  $\mathcal{E}$  corresponds to a flat of a matroid  $M(\mathcal{E})$  compatible with the Klyachko filtrations of  $\mathcal{E}$ . We define subparliaments of polytopes and identify them with parliaments of equivariant subbundles.

**Corollary (3.7).** *The subparliaments of the parliaments of  $\mathcal{E}$  are the parliaments of the equivariant subbundles of  $\mathcal{E}$ .*

The last part of this article consists of exploring the geometric information that can be reconstructed from the parliament of polytopes of a toric vector bundle. In particular, we translate results of Payne (see [Pay07] Proposition 3.4, Corollary 3.6) and Klyachko (see [Kly89] Corollary 1.2.4) in terms of parliaments of polytopes.

**Proposition (5.6).** *The data of the parliament of polytopes of a globally generated equivariant vector bundle  $\mathcal{E}$ , up to translation of each direct component and quotiented by  $GL_r(\mathbb{C})$ , enables us to reconstruct the isomorphism class of the vector bundle  $\mathcal{E}$ .*

We start in Section 1 by recalling facts about matroids, giving the construction of parliaments of polytopes from [DJS14] and fixing notation.

Section 2 furnishes the definition of average polytopes that allows us to visualize the slope of an equivariant saturated sheaf. It also contains the most important result of the article (Theorem 2.19): the existence, for any equivariant vector bundle  $\mathcal{E}$ , of a finite matroid checking combinatorially the slope-stability of  $\mathcal{E}$  using one of its parliaments of polytopes. The cases of tangent bundles and of vector bundles of rank 3 are also discussed in this section.

In Section 3, given the parliament of polytopes of a toric vector bundle, we describe the parliaments of its equivariant subbundles.

In Section 4, we treat the stability of the restriction of an equivariant bundle  $\mathcal{E}$  to a torus invariant curve in terms of the parliament  $PP_{\mathcal{E}}$ . We also give examples of  $\alpha$ -(semi)stable equivariant vector bundle with non (semi)stable restrictions to torus invariant curves.

In Section 5, we discuss the definition of parliament of polytopes and state what data is encoded in the parliament of an equivariant bundle.

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## 1 Parliaments of polytopes and matroid terminology

Consider a smooth complete toric variety  $X$  of dimension  $d$  with fan  $\Sigma$  and torus  $T$ . The lattice of characters and its dual lattice are denoted by  $M$  and  $N$ . Some matroids thereafter will have the same notation  $M$ , we hope that it will not cause confusions. We denote by  $\Sigma(k)$  the cones of dimension  $k$ . Let  $\mathcal{E}$  be a given rank  $r$  equivariant vector bundle over  $X$  with fiber over the identity of the torus  $E$ . Let us denote by  $\{v_0, \dots, v_{n-1}\}$  the set of vectors generating the rays  $\rho_i$  of  $\Sigma$ . (See Chapter [CLS11] for more fundamentals on toric varieties.) Parliaments of polytopes were introduced in [DJS14] in order to give explicit polyhedral interpretations of properties to equivariant vector bundles. The parliaments of polytopes  $PP_{\mathcal{E}}$  of  $\mathcal{E}$  are composed of at least  $r$  polytopes linked with some combinatorial data: the polytopes are labelled by elements of a representable matroid of rank  $r$ . We first recall some matroid terminology. See [Kat14] for more fundamentals on matroids.

### 1.1 Matroid terminology

Matroids are a generalization of the notion of linear independence in vector spaces.

**Definition 1.1.** *A matroid  $M$  is the data of a finite set  $G$  and a collection  $\mathcal{B}$ , of subsets of  $G$ , called bases, satisfying the following properties:*

*B1:  $\mathcal{B}$  is nonempty ;*

*B2: (basis exchange property) If  $A, B \in \mathcal{B}$  are distinct and  $a \in A \setminus B$ , then there exists  $b \in B \setminus A$  such that  $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$ .*

*We call  $G$  the ground set and  $\mathcal{B}$  the set of bases of the matroid  $M = (G, \mathcal{B})$ .*

**Definition 1.2.** *An isomorphism of matroids  $\varphi : M_1 = (G_1, \mathcal{B}_1) \rightarrow M_2 = (G_2, \mathcal{B}_2)$  is a bijection from  $G_1$  to  $G_2$  such that*

$$A \in \mathcal{B}_1 \Leftrightarrow \varphi(A) \in \mathcal{B}_2 .$$

**Definition 1.3.** *A representable matroid of rank  $r$  is a matroid isomorphic to*

$$M = (G, \mathcal{B}) ,$$

*where  $G$  is a finite subset of some dimension  $r$ -vector space  $E$  and  $\mathcal{B}$  is the set of bases of  $E$  formed by vectors in  $G$ . We say that  $M$  is represented in  $E$ .*

**Definition 1.4.** A flat of a matroid  $(G, \mathcal{B})$  represented in  $E$ , is a subset  $f \subset G \subset E$  such that

$$\langle f \rangle \cap G = f ,$$

where  $\langle f \rangle \subset E$  is the subspace spanned by the vectors in  $f$ .

## 1.2 Parliaments of polytopes

The construction of parliaments of polytopes of  $\mathcal{E}$  requires the Klyachko classification.

**Theorem 1.5** (Theorem 0.1.1 of [Kly89]). *There is an equivalence of categories between the category of rank  $r$ -equivariant vector bundles  $\mathcal{E}$  on  $X$  and the category of  $n$  compatible decreasing  $\mathbb{Z}$ -filtrations  $(E^i(j))_{\rho_i \in \Sigma(1)}$*

$$E^i(j) = \begin{cases} E & \text{if } j \leq A_1^i \\ H_i & \text{if } A_1^i < j \leq A_2^i \\ \dots & \\ \langle u_i \rangle & \text{if } A_{r-1}^i < j \leq A_r^i \\ \{0\} & \text{if } A_r^i < j \end{cases} .$$

of a dimension  $r$   $\mathbb{C}$ -vector space  $E$  together with a compatibility condition as follows. There exist decompositions of  $E$  into 1-dimensional vector spaces  $L_u^\sigma$  satisfying

$$\forall \sigma \in \Sigma(d), \exists (L_u^\sigma)_{u \in \mathbf{u}(\sigma)} \text{ s.t. } E = \bigoplus_{u \in \mathbf{u}(\sigma)} L_u^\sigma \text{ and } \forall \rho_i \preceq \sigma, E^i(j) = \sum_{\langle u, v_i \rangle \geq j} L_u^\sigma . \quad (\text{CC})$$

*Remark 1.6.* There is an equivalence of categories between the category of equivariant reflexive sheaves  $\mathcal{F}$  on  $X$  of rank  $r$ , and the category of  $n$  compatible decreasing  $\mathbb{Z}$ -filtrations of a dimension  $r$   $\mathbb{C}$ -vector space  $F$  where we do not impose the compatible condition (see Theorem 5.19 [Per04]).

**Definition 1.7.** *The  $r$  points in  $\mathbf{u}(\sigma)$  are called the associated characters of  $\sigma$ .*

*Remark 1.8.* The compatibility condition (CC) implies the existence, for every maximal cone  $\sigma \in \Sigma(d)$ , of a basis  $B_\sigma$ , given by a generator of  $L_u^\sigma$  for each  $u \in \mathbf{u}(\sigma)$ . We call  $B_\sigma$  a compatible basis. Nevertheless, given a maximal cone, a compatible basis may not be unique (see Example 4.4 of [DJS14]) although  $\mathbf{u}(\sigma)$  always is.

*Remark 1.9.* An important feature is that for any maximal cone  $\sigma \in \Sigma(d)$ , the equivariant bundle  $\mathcal{E}$  splits equivariantly on  $U_\sigma$  as

$$\mathcal{E}|_{U_\sigma} \simeq \bigoplus_{u \in \mathbf{u}(\sigma)} \mathcal{O}_X(\text{div}(u))|_{U_\sigma} .$$

A parliament of polytopes of a vector bundle  $\mathcal{E}$  is a certain collection of polytopes

$$\{(P_e, e) \mid e \in G(\mathcal{E}) \subseteq E\} ,$$

where  $G(\mathcal{E})$  is the ground set of a matroid  $M(\mathcal{E})$  defined in the following manner. The matroids, called  $M(\mathcal{E})$ , associated to  $\mathcal{E}$  can be seen as minimal matroids generating

$$L(\mathcal{E}) = \left\{ \bigcap_{i \in \{0, \dots, n\}} E^i(j_i) \mid (j_i)_{i \in \{0, \dots, n\}} \in \mathbb{Z}^{n+1} \right\}$$

as a meet-subsemilattice (as a partially ordered subset which has a meet i.e. a greatest lower bound). Despite Proposition 3.1 [DJS14], this matroid is not unique<sup>1</sup>. However, the ground sets  $G(\mathcal{E})$  generating the matroids  $M(\mathcal{E})$  are the possible outputs of Algorithm 3.2 of [DJS14].

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**Algorithm 3.2** Computing  $G(\mathcal{E})$ 


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 $r \leftarrow$  the dimension of the largest linear subspace of  $L(\mathcal{E})$ 
 $G \leftarrow \emptyset$ 
for  $k = 1$  to  $r$  do
  for  $k$ -dimensional linear subspace  $V \in L(\mathcal{E})$  do
     $G' \leftarrow G \cap V$ 
    if  $\text{Span}(G') \subsetneq V$  then
       $G \leftarrow G \sqcup CB_V$ ,  $CB_V$  is a basis of a complement to  $\text{Span}(G')$  in  $V$ 
    end if
  end for
end for
return  $G$ 

```

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Additionally, if  $G_E^1$  and  $G_E^2$  are ground sets resulting from Algorithm 3.2 applied to  $L(\mathcal{E})$  then the same number of elements is added at Step  $V$ . Indeed,  $CB_V^1, CB_V^2$  are bases of a complement to  $\text{Span}(G'^1), \text{Span}(G'^2)$ , which have same dimension

$$\dim(\text{Span}(G'^\varepsilon)) = \dim\left(\text{Span}(V' \cap V)_{V' \cap V \subsetneq V}\right) \quad \forall \varepsilon \in \{1, 2\}$$

in  $V$ . The equality holds because, as  $L(\mathcal{E})$  is stable by intersection, the vector spaces  $V' \cap V$  that are different from  $V$ , are steps of the algorithm and are already generated by  $G'^\varepsilon$  at Step  $V$ . Therefore, there exists a map  $\varphi$  sending  $G_E^1$  to  $G_E^2$ , we call it of type  $(\star)$ .

**Definition 1.10.** A map  $\varphi : M(G_E^1) \rightarrow M(G_E^2)$  of type  $(\star)$  is a bijection

$$\varphi : G_E^1 \rightarrow G_E^2$$

which respects Algorithm 3.2 for  $\mathcal{E}$ , that is to say which sends  $CB_V^1$  to  $CB_V^2$  at Step  $V$  of Algorithm 3.2.

We use the following notation for a ground set  $G(\mathcal{E})$  of  $M(\mathcal{E})$

$$G(\mathcal{E}) = \{e_0, \dots, e_l\} \quad \text{for some } l \geq r.$$

**Definition 1.11.** Any output  $G(\mathcal{E})$  from Algorithm 3.2 gives rise to a parliament of polytopes of  $\mathcal{E}$ . These are the sets of indexed polytopes  $(P_e)_{e \in G(\mathcal{E})}$  defined as

$$P_e := \{m \in M_{\mathbb{R}} \mid \forall \rho_i \in \Sigma(1), \langle m, v_i \rangle \leq \max\{j \mid e \in E^i(j)\}\},$$

where  $G(\mathcal{E})$  is seen modulo isomorphism of matroids of type  $(\star)$ .

*Example 1.12.* To a fixed toric vector bundle, we associated a finite number of parliaments of polytopes, corresponding to each isomorphism class of matroids obtained by Algorithm 3.2. Here we give some counterexamples of the uniqueness of  $M(\mathcal{E})$ . Consider the canonical basis  $\{e_1, e_2, e_3\}$  of  $\mathbb{C}^3$ .

---

<sup>1</sup>see Example 1.12 for counterexamples

For the first counterexample, let  $X$  be the projective plane  $\mathbb{P}^2$  and let  $\mathcal{E}$  be the toric bundle of rank 3 defined by the following filtrations

$$E^i(j) = \begin{cases} \mathbb{C}^3 & \text{if } j \leq 0 \\ V_i & \text{if } 0 < j \leq 1 \\ \{0\} & \text{if } 1 < j \end{cases} ,$$

where  $V_0 = \langle e_1 \rangle$ ,  $V_1 = \langle e_1 + e_2 \rangle$  and  $V_2 = \langle e_2, e_3 \rangle$ . The meet-subsemilattice is

$$L_1(\mathcal{E}) = \{ \{0\}, \langle e_1 \rangle, \langle e_1 + e_2 \rangle, \langle e_2, e_3 \rangle, \mathbb{C}^3 \} .$$

Two non-isomorphic matroids are given by  $G_1 = \{e_1, e_1 + e_2, e_2, e_3\}$  and  $G_2 = \{e_1, e_1 + e_2, e_2 + e_3, e_3\}$ . Here the non-uniqueness comes from  $L_1(\mathcal{E})$  non being stable by sums.

A second example can be found on  $\mathbb{P}^2$ , let  $\mathcal{E}$  be defined by the following filtrations

$$E^i(j) = \begin{cases} \mathbb{C}^3 & \text{if } j \leq 0 \\ V_i & \text{if } 0 < j \leq 1 \\ \langle e_1 \rangle & \text{if } 1 < j \leq 2 \\ \{0\} & \text{if } 2 < j \end{cases} ,$$

where  $V_0 = \langle e_1, e_2 \rangle$ ,  $V_1 = \langle e_1, e_3 \rangle$  and  $V_2 = \langle e_1, e_2 + e_3 \rangle$ . The meet-subsemilattice is

$$L_2(\mathcal{E}) = \{ \{0\}, \langle e_1 \rangle, \langle e_1, e_2 \rangle, \langle e_1, e_3 \rangle, \langle e_1, e_2 + e_3 \rangle, \mathbb{C}^3 \} .$$

Two non-isomorphic matroids are given by  $G_1 = \{e_1, e_2, e_2 + e_3, e_3\}$  and  $G_2 = \{e_1, e_1 + e_2, e_2 + e_3, e_3\}$ . Here the non-uniqueness comes from  $L_2(\mathcal{E}) \cup \{G_2\}$  non being stable by iterated sums and intersections.  $\diamond$

By Remark 1.6, an equivariant reflexive sheaf is defined by filtrations as toric vector bundles. We extend the definition of parliaments of toric bundles (Definition 1.11) to preparliaments of equivariant saturated subsheaves of toric bundles. A preparliament that satisfies the compatibility condition (CC) will be a parliament, and the corresponding equivariant saturated subsheaf a toric bundle. We keep the notation  $PP_{\mathcal{F}}$  for preparliaments.

**Proposition 1.13.** *A preparliament is a set of indexed polytopes constructed from a family of decreasing separated-exhaustive  $\mathbb{Z}$ -filtrations  $((E^i(j))_{j \in \mathbb{Z}})_{\rho_i \in \Sigma(1)}$ . Let us denote the  $n \times r$  hyperplanes used in the construction of the polytopes by*

$$(H_{i,k} = (\langle \cdot, v_i \rangle = A_k^i))_{\rho_i \in \Sigma(1), k \in \{1, \dots, r\}} .$$

*A parliament is a preparliament such that for any fixed maximal cone  $\sigma$ , there exists  $r$  associated characters  $u \in \mathbf{u}(\sigma)$  at the intersection of some hyperplanes*

$$(H_{i,k_{i,u}})_{\rho_i \in \sigma(1)} \quad \text{with } k_{i,u} \in \{1, \dots, r\} ,$$

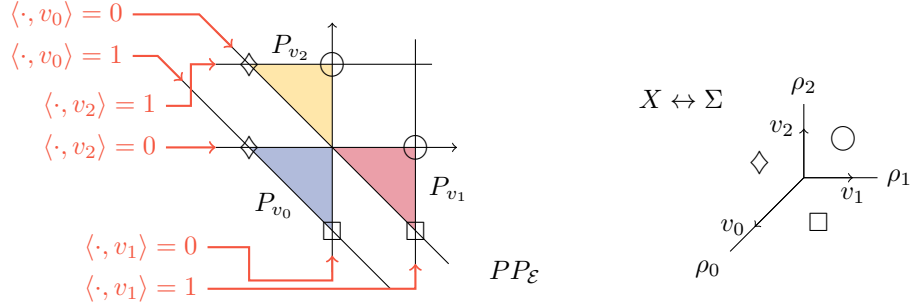
*such that these hyperplanes are part of those defining some polytope  $P_{L_u^\sigma}$  in  $PP_{\mathcal{E}}$  and in the construction of the  $r$  associated characters of  $\mathbf{u}(\sigma)$ , each hyperplane should be used exactly once :*

$$\forall i \in \sigma(1), \quad \{k_{i,u} \mid u \in \mathbf{u}(\sigma)\} = \{1, \dots, r\} .$$

*Example 1.14.* Consider  $X = \mathbb{P}^2$  and its tangent bundle  $\mathcal{E} = \mathcal{T}_X$ . The  $\mathbb{Z}$ -filtrations are

$$\text{for } i = 0, 1, 2, E^i(j) = \begin{cases} \mathbb{C}^2 & \text{if } j < 0 \\ \langle v_i \rangle & \text{if } 0 \leq j < 1 \\ \{0\} & \text{otherwise} \end{cases} .$$

The ground set is  $G(\mathcal{E}) = \{v_0, v_1, v_2\} = \Sigma(1)$  and the parliament  $PP_{\mathcal{E}}$  is the following.



We associate to each maximal cone  $\sigma$  a symbol (say  $\square$ ) and we represent the associated characters  $\mathbf{u}(\sigma)$  by  $r$  symbols  $\square$  on the parliament. The compatibility condition (CC) is verified.  $\diamond$

*Remark 1.15.* For any toric vector bundle  $\mathcal{E}$ , we can define a virtual parliament of polytope  $P\tilde{P}_{\mathcal{E}}$ . This way, we are keeping trace of the defining hyperplanes even if  $\mathcal{E}$  does not have any positivity property.

## 2 Stability of equivariant vector bundles

Most positivity properties of a toric vector bundle can be detected on its parliaments of polytopes. We may expect other properties such as stability to be visualisable on the parliaments. This notion is important especially in the spirit of constructing moduli spaces of toric vector bundles. The problem of classification of vector bundles, or of construction of moduli spaces of stable vector bundles, is still completely open, even in the toric case. In this section, we work on finding a necessary and sufficient condition for the (semi)stability of a toric vector bundle  $\mathcal{E}$  in terms of its parliament of polytopes  $PP_{\mathcal{E}}$ . This condition can be either seen as an algorithm to check stability or as a visual property on the parliament  $PP_{\mathcal{E}}$ .

### 2.1 Definitions

We start by the definition of slope for equivariant coherent sheaves on smooth complete toric varieties. It leads us to the definition of stability for toric vector bundles. The slope of an equivariant coherent sheaf  $\mathcal{E}$  depends on the choice of a polarization (an ample divisor  $H$  or more generally a movable curve  $\alpha$ ) of the variety  $X$  that we leave aside for now and on which we go back in the next subsection.

**Definition 2.1.** *Let  $X$  be a smooth complete toric variety of dimension  $d$  and let  $\alpha$  be a polarization of  $X$ . The slope of any equivariant coherent sheaf  $\mathcal{E}$  with respect to  $\alpha$  is*

$$\mu_{\alpha}(\mathcal{E}) := \frac{c_1(\mathcal{E}) \cdot \alpha}{\text{rk}(\mathcal{E})} \in \mathbb{Q} ,$$



where  $c_1$  is the first Chern class of  $\mathcal{E}$ . The dependence on  $\alpha$  is often omitted.

**Definition 2.2.** An equivariant bundle  $\mathcal{E}$  is  $\alpha$ -semistable if for any nonzero subsheaf  $\mathcal{F} \subseteq \mathcal{E}$ , the respective slopes satisfy the inequality

$$\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E}) .$$

It is  $\alpha$ -stable if, in addition, for any nonzero subsheaf  $\mathcal{F} \subsetneq \mathcal{E}$ , the strict inequality  $\mu_\alpha(\mathcal{F}) < \mu_\alpha(\mathcal{E})$  holds.

An equivariant bundle  $\mathcal{E}$  is  $\alpha$ -polystable if it is a direct sum of  $\alpha$ -stable bundles.

*Remark 2.3.* As explained in Remark 2.4 of [HNS19], to prove that some equivariant bundle  $\mathcal{E}$  is  $\alpha$ -semistable, it is sufficient to verify that  $\mu_\alpha(\mathcal{F}) < \mu_\alpha(\mathcal{E})$  holds for every equivariant saturated subsheaf  $\mathcal{F}$ .

We will need the following combinatorial description of equivariant saturated subsheaves of  $\mathcal{E}$ .

**Theorem 2.4** (Proposition 2.3 of [HNS19] or Corollary 0.0.2. [DDK20]). *Let  $\mathcal{E}$  be a rank  $r$ -equivariant vector bundle on a smooth complete toric variety  $X$ . Via the Klyachko classification,  $\mathcal{E}$  corresponds to a  $\mathbb{Z}$ -filtration  $(E^i(j))_{j \in \mathbb{Z}}$  of  $E \cong \mathbb{C}^r$  for each ray  $\rho_i \in \Sigma(1)$ . The equivariant saturated subsheaves  $\mathcal{F}$  of  $\mathcal{E}$  are then in one-to-one correspondence with the subfiltrations*

$$(F^i(j) = E^i(j) \cap F)_{j \in \mathbb{Z}} \quad \text{of} \quad (E^i(j))_{j \in \mathbb{Z}} ,$$

for some vector subspace  $F \subseteq E$ .

**Definition 2.5.** Let  $\mathcal{E}$  be a toric vector bundle on  $X$  and let  $\mathcal{F}$  be an equivariant saturated subsheaf of  $\mathcal{E}$ . A preparliament of polytopes of  $\mathcal{F}$  is composed of a set of indexed polytopes  $(P_g)_{g \in G(\mathcal{F})}$  defined as

$$P_g := \{ m \in \mathbb{R}^d \mid \forall \rho_i \in \Sigma(1), \langle m, v_i \rangle \leq \max \{ j \mid g \in F^i(j) \} \} ,$$

where the ground set  $G(\mathcal{F})$  is obtained by Algorithm 3.2 and is seen modulo isomorphism of matroids of type  $(\star)$ .

## 2.2 Polarization

Usually, the slope of a toric vector bundle on  $X$  is defined with respect to some ample divisor  $H$  called a polarization of  $X$ . More precisely, it involves the self-intersection product  $H^{d-1}$ .

In [GKP14]), in order to do birational geometry, Greb, Kebekus and Peternell generalize slopes to movable curves on any complex, projective manifold  $X$ .

Let us explain it in the toric case. The set of polarizations of  $X$  can be extended from ample divisors  $H$  to movable divisors  $L$  on  $X$ . This way the movable divisor  $L$  polarizes  $X$  as well as the blow up of  $X$  making  $L$  ample. Now, consider a movable curve class  $\alpha$  and require it to be positive along a spanning set of rays. It follows from Theorem 3.12, Lemma 4.1 and Theorem 4.2 of [LX16] that  $\alpha$  can be written as

$$\alpha = \langle L_\alpha^{d-1} \rangle$$

where  $L_\alpha$  is a unique big and movable divisor and  $\langle \rangle$  is the Boucksom positive product<sup>2</sup>. To compute the positive intersection product of big toric divisor classes  $L_1, \dots, L_p$ , we

<sup>2</sup>developed in [BDPP04] and constructed algebraically in [BFJ06]

consider a higher birational model that makes  $L_1, \dots, L_p$  nef when removing some positive linear combination of exceptional divisors  $D_i$ , then  $\langle L_1, \dots, L_p \rangle$  is defined as

$$(L_1 - D_1) \cdot \dots \cdot (L_p - D_p).$$

As such the positive intersection product restricts to the intersection product on ample classes. It justifies that in the following definition of slope,  $H^{d-1}$  may be replaced by a movable curve  $\alpha$  positive along a spanning set of rays (that is to say  $\alpha \cdot D_i > 0$  for rays' generators  $v_i \in \Sigma(1)$  generating  $N_{\mathbb{R}}$ ).

**Definition 2.6.** Let  $X$  be a smooth complete toric variety with fan  $\Sigma$ . A polarization on  $X$  is a movable curve  $\alpha$  positive along a spanning set of rays of  $\Sigma$ .

By Minkowski's theorem for polytopes (see [Sch93] Section 7), the data of a polarization  $\alpha$  is equivalent to the data of its weights, defined as follows.

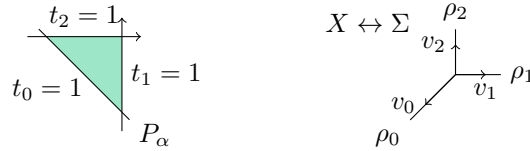
**Definition 2.7.** A movable curve  $\alpha$  on  $X$ , positive along a spanning set of rays, can be written as the positive self-intersection of a unique big and movable divisor  $L_\alpha$

$$\alpha = \langle L_\alpha^{d-1} \rangle.$$

Let us denote  $P_{L_\alpha}$  the moment polytope of  $L_\alpha$  and write  $f_i$  for the volume of the face of  $P_{L_\alpha}$  which has external normal vector  $v_i$ . We call the weights of  $\alpha$  the numbers

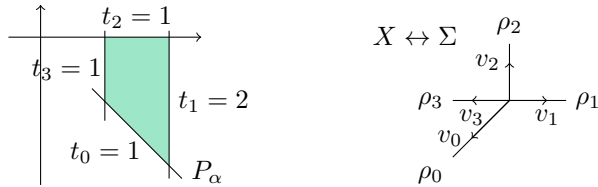
$$\text{for } \rho_i \in \Sigma(1), \quad t_i = f_i \frac{(d-1)!}{\|v_i\|} \geq 0.$$

*Example 2.8.* On  $\mathbb{P}^2$ , the polarization  $\alpha = D_0 = \langle D_0^{2-1} \rangle$  has weights  $(1, 1, 1)$ .



◇

*Example 2.9.* The polarization  $\alpha = 2D_1 - D_3$  on  $\text{Bl}_{[0:1:0]} \mathbb{P}^2$  has weights  $(1, 2, 1, 1)$ .



◇

### 2.3 Slope and weights

In this section, we consider a rank  $l$ -equivariant saturated subsheaf  $\mathcal{F}$  of a toric vector bundle  $\mathcal{E}$  on a toric variety  $X$  defined by the filtrations

$$F^i(j) = \begin{cases} F & \text{if } j \leq A_1^i \\ \dots & \\ \langle u_i \rangle & \text{if } A_{l-1}^i < j \leq A_l^i \\ \{0\} & \text{if } A_l^i < j. \end{cases}$$

We reformulate the slope of  $\mathcal{F}$ . In his original paper [Kly89], in Remark 3.2.4, Klyachko expressed the Chern classes of any equivariant vector bundle<sup>3</sup>. Kool extended this result, computing the first Chern class of any equivariant coherent sheaf (see Corollary 3.18 of [Koo08]). We may reformulate it as follows.

**Theorem 2.10.** *The first Chern class of  $\mathcal{F}$  is given by*

$$c_1(\mathcal{F}) = \sum_{\rho_i \in \Sigma(1)} \left( \sum_{k=1}^l A_k^i \right) D_i .$$

**Proposition 2.11.** *Let  $\mathcal{F}$  be a rank  $l$ -equivariant saturated subsheaf of  $\mathcal{E}$  as before. If  $\alpha$  has weights  $(t_i)_{\rho_i \in \Sigma(1)}$  (see Definition 2.7), then the slope of  $\mathcal{F}$  is*

$$\mu_\alpha(\mathcal{F}) = \sum_i \left( \frac{1}{l} \sum_{k=1}^l A_k^i \right) t_i .$$

*Proof.* In Section 4 of [LX16], Lehmann and Xiao show that the numbers  $t_i$  are the intersection numbers  $D_i \cdot \alpha$ . We then have that

$$\text{rk}(\mathcal{F})\mu_\alpha(\mathcal{F}) = c_1(\mathcal{F}) \cdot \alpha = \sum_i \left( \sum_{k=1}^l A_k^i \right) D_i \cdot \alpha = \sum_i \left( \sum_{k=1}^l A_k^i \right) t_i . \quad \square$$

## 2.4 The average polytope

In this section, we consider a nonzero equivariant saturated subsheaf  $\mathcal{F}$  of a toric vector bundle  $\mathcal{E}$  on a toric variety  $X$ . By Theorem 2.4, it is defined by filtrations

$$F^i(j) = \begin{cases} F & \text{if } j \leq A_1^i \\ \cdots & \\ \langle u_i \rangle & \text{if } A_{l-1}^i < j \leq A_l^i \\ \{0\} & \text{if } A_l^i < j. \end{cases}$$

where  $F^i(j) = E^i(j) \cap F$ , the  $(E^i(j))_{\rho_i \in \Sigma(1), j \in \mathbb{Z}}$  being the defining filtrations of  $\mathcal{E}$ . The slope of  $\mathcal{F}$  may be visualized using the notion of average polytope.

**Definition 2.12.** *We define the average polytope  $P_{\mathcal{F}}$  of  $\mathcal{F}$  to be the moment polytope associated to the divisor*

$$\frac{c_1(\mathcal{F})}{l} = \sum_i \left( \frac{1}{l} \sum_{k=1}^l A_k^i \right) D_i .$$

*Notation 2.13.* We may also use the notation  $P_{PP_{\mathcal{F}}}$  or  $P_{\mathcal{F}}$  if it appears to be more convenient.

The average polytope  $P_{\mathcal{F}}$  of  $\mathcal{F}$  is a visualization of the slopes of  $\mathcal{F}$ . In the following proposition, we recover, for any polarization  $\alpha$  of  $X$ , the  $\alpha$ -slope of  $\mathcal{F}$ , only by means of the average polytope  $P_{\mathcal{F}}$ .

**Proposition 2.14.** *The  $\alpha$ -slope of an equivariant saturated subsheaf  $\mathcal{F} \subset \mathcal{E}$  with average polytope  $P_{\mathcal{F}} = P_D$  where  $D = \sum_{\rho_i \in \Sigma(1)} a_i D_i$  is*

$$\mu_\alpha(\mathcal{F}) = \sum_{\rho_i \in \Sigma(1)} a_i t_i ,$$

<sup>3</sup>See also Proposition 3.1 of [Pay07]

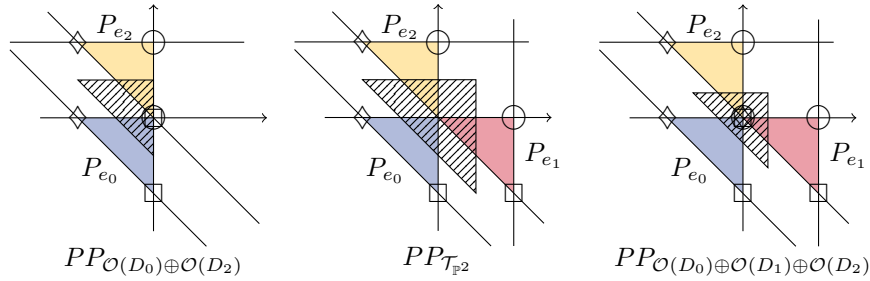
where  $\alpha$  has weights  $(t_i)_{\rho_i \in \Sigma(1)}$ .

*Proof.* See Proposition 2.11 and Definition 2.12.  $\square$

*Example 2.15.* The average polytopes of

$$\mathcal{F} = \mathcal{O}(D_0) \oplus \mathcal{O}(D_2), \quad \mathcal{F} = \mathcal{T}_{\mathbb{P}^2} \quad \text{and} \quad \mathcal{F} = \mathcal{O}(D_0) \oplus \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$$

are the hashed polytopes in the following pictures



The difference between the two last pictures is the following. The hyperplanes forming a '+' in the middle have different multiplicities (1 or 2). Indeed, the tangent bundle  $\mathcal{T}_{\mathbb{P}^2}$  has rank  $l = 2$  whereas the splitting bundle  $\mathcal{O}(D_0) \oplus \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$  has rank  $l = 3$ . The slopes with respect to the polarization  $\alpha$  with weights  $(1, 1, 1)$  (see Example 2.8) are

$$\begin{cases} \mu_\alpha(\mathcal{O}(D_0) \oplus \mathcal{O}(D_2)) = \frac{1}{2} + 0 + \frac{1}{2} = 1; \\ \mu_\alpha(\mathcal{T}_{\mathbb{P}^2}) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}; \\ \mu_\alpha(\mathcal{O}(D_0) \oplus \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1. \end{cases}$$

$\diamond$

*Remark 2.16.* Consider two equivariant saturated subsheaves  $\mathcal{F}_1$  and  $\mathcal{F}_2$  with average polytopes  $P_1$  and  $P_2$  satisfying  $P_2 = \gamma P_1 + v$ , where  $\gamma$  is a coefficient of dilatation and  $v$  a vector of translation. For any polarization  $\alpha$ , we have the following relation between their slopes

$$\mu_\alpha(\mathcal{F}_1) = \gamma \times \mu_\alpha(\mathcal{F}_2).$$

Indeed, let us denote by  $\sum_{\rho_i \in \Sigma(1)} a_i D_i$  and  $\sum_{\rho_i \in \Sigma(1)} b_i D_i$  the divisors associated to  $P_1$  and  $P_2$ . We then have that  $a_i = \gamma b_i + \langle v, v_i \rangle$  for all  $\rho_i \in \Sigma(1)$  so that

$$\mu_\alpha(\mathcal{F}_1) = \gamma \times \mu_\alpha(\mathcal{F}_2) + \left( \sum_{\rho_i \in \Sigma(1)} \langle v, v_i \rangle D_i \right) \cdot \alpha.$$

By the exact sequence of Theorem 4.1.3 [CLS11], we have

$$\sum_{\rho_i \in \Sigma(1)} \langle v, v_i \rangle D_i \equiv 0.$$

As a consequence, we may look at the average polytope as a polytope modulo translation.

## 2.5 Result

In this subsection, we work on a necessary and sufficient condition for stability of toric vector bundles. It involves comparing average polytopes by means of the following order.

**Definition 2.17.** Let  $X$  be a smooth complete toric variety and  $\alpha$  a polarization with weights

$$t_i := D_i \cdot \alpha = f_i \frac{(d-1)!}{\|v_i\|} \quad \text{for every ray } \rho_i \in \Sigma .$$

We define a total order on moment polytopes of  $X$  by saying

$$P_1 <_\alpha P_2 \quad \iff \quad \sum_i a_1^i t_i < \sum_i a_2^i t_i$$

where for  $j \in \{1, 2\}$ ,  $P_j$  is the moment polytope of the divisor  $\sum_i a_j^i D_i$ .

Using Definition 2.17, Theorem 2.4 and Remark 2.3, we may reformulate the definition of stability (Definition 2.2) of toric vector bundles into the following proposition.

**Proposition 2.18.** A toric vector bundle  $\mathcal{E}$  is  $\alpha$ -stable (resp.  $\alpha$ -semistable) if and only if for any nonzero subspace  $F \subsetneq E$ , we have

$$P_F <_\alpha P_E \quad (\text{resp. } P_F \leq_\alpha P_E) ,$$

where  $P_F$  is the moment polytope of the preparliament  $PP_{\mathcal{F}}$  obtained by considering the parliament  $PP_{\mathcal{E}}$  and keeping only the hyperplanes corresponding to the filtrations  $(E^i(j) \cap F)_{i,j}$  of  $\mathcal{F}$ .

The following result is a first step towards an algorithm to verify stability.

**Theorem 2.19.** Let  $\mathcal{E}$  be any toric vector bundle on a toric variety  $X$ . There exists a finite matroid  $M(\mathcal{E})^S$  representable in  $E$  such that  $\mathcal{E}$  is  $\alpha$ -stable (resp.  $\alpha$ -semistable) if and only if

$$P_F <_\alpha P_E \quad (\text{resp. } P_F \leq_\alpha P_E) ,$$

for any space  $\{0\} \subsetneq F = \langle f \rangle \subsetneq E$  given by a flat  $f$  of  $M(\mathcal{E})^S$ .

*Proof.* Recall that to define parliaments of polytopes, we introduced, from the filtrations of  $\mathcal{E}$ , a semilattice

$$L(\mathcal{E}) = \left\{ \bigcap_{i \in \{0, \dots, n\}} E^i(j_i) \mid (j_i)_{i \in \{0, \dots, n\}} \in \mathbb{Z}^{n+1} \right\} .$$

Consider an integer  $l \in \{1, \dots, r-1\}$  and let us define the noncontinuous function

$$\varphi_l : G(l, r) \rightarrow \mathbb{Z}^{L(\mathcal{E})} , \quad F \mapsto (\dim(F \cap V))_{V \in L(\mathcal{E})} .$$

The  $\alpha$ -slope of a rank  $l$ -saturated subsheaf  $\mathcal{F}$  corresponding to a vector space  $F$  depends only on the image of  $F$  by  $\varphi_l$ . As a consequence, to ensure the (semi)stability of  $\mathcal{E}$  we only need to compare the slope of one representative  $F \in G(l, r)$  for each image  $\varphi_l(F) \in \mathbb{Z}^{L(\mathcal{E})}$  with the slope of  $\mathcal{E}$ . Since the image  $\text{Im}(\varphi_l) \subseteq \{0, \dots, r\}^{L(\mathcal{E})}$  is finite, we have the finite family of subspaces checking stability. We obtain a ground set of a finite matroid computing stability by taking a basis of each member of the family.

*Remark 2.20.* To be more precise, we may define a partial order on  $G(l, r)$  by

$$F_1 <_{\mathcal{E}} F_2 \quad \iff \quad \varphi_l(F_1)_V < \varphi_l(F_2)_V \quad \text{for all } V \in L(\mathcal{E}) .$$

The toric bundle  $\mathcal{E}$  is (semi)stable if and only if for all  $l \in \{1, \dots, r-1\}$ , for all maximal element of  $\text{Im}(\varphi_l)$  with representative  $F$ , we have that

$$P_F <_\alpha P_E \quad (\text{resp. } P_F \leq_\alpha P_E).$$

*Remark 2.21.* The matroid  $M(\mathcal{E})^S$  does not depend on  $\alpha$ .

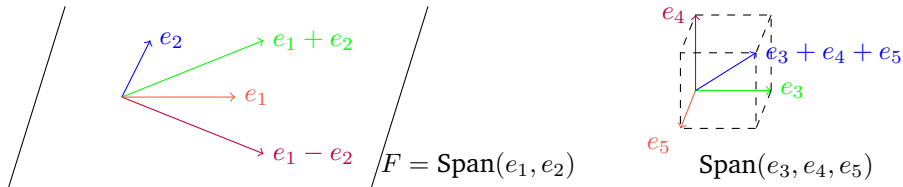
*Example 2.22.* In the next subsections, we prove that  $M(\mathcal{E})^S$  can be taken to be the matroid  $M(\mathcal{E})$  of any parliament of  $\mathcal{E}$ , if  $\mathcal{E}$  has rank 1, 2 or 3 or if  $\mathcal{E} = \mathcal{T}_X$  is the tangent bundle of a smooth complete toric variety. It is not the case in general.

Indeed, we claim that the flats of a matroid  $M(\mathcal{E})$  are not enough to know that  $\mathcal{E}$  is not (semi)stable. Consider  $X = \mathbb{P}^1 \times \mathbb{P}^1$  and the filtrations

$$\text{for } \rho_i \in \Sigma(1), E^i(j) = \begin{cases} \mathbb{C}^5 & \text{if } j < 0 \\ V_i & \text{if } 0 \leq j < 1 \\ \{0\} & \text{otherwise} \end{cases},$$

where the vector spaces  $V_i$  are defined as

$$V_1 = \langle e_1 + e_2, e_3 \rangle, V_2 = \langle e_1 - e_2, e_4 \rangle, V_3 = \langle e_1, e_5 \rangle \text{ and } V_4 = \langle e_2, e_3 + e_4 + e_5 \rangle.$$



A matroid that can be obtained by Algorithm 3.2 is given by the ground set

$$G = \{e_1 + e_2 + e_3, e_1 + e_2 - e_3, e_1 - e_2 + 2e_4, e_1 - e_2 - 2e_4, e_1 + 3e_5, e_1 - 3e_5, e_2 + 4e_3 + 4e_4 + 4e_5, e_2 - 4e_3 - 4e_4 - 4e_5\}.$$

Consider the vector space  $F = \langle e_1, e_2 \rangle$ . The slope of  $\mathcal{E}$  and  $\mathcal{F}$  are

$$\mu(\mathcal{E}) = \frac{2+2+2+2}{5} = \frac{8}{5} \quad \text{and} \quad \mu(\mathcal{F}) = \frac{1+1+1+1}{2} = 2.$$

Consequently  $\mathcal{E}$  is unstable, but it cannot be seen by looking at the flats of  $G$ . Indeed, we have that

- A flat  $f$  of dimension  $l = 1, 2$  or  $3$  satisfies

$$\text{Span}(V_i \cap f) = V_i \cap \text{Span}(f)$$

and its associated saturated subsheaf  $\mathcal{F}$  has thus slope  $\mu(\mathcal{F}) = l/l = 1$ .

- A flat of dimension 4 either

$$\text{Span}(V_i \cap f) = V_i \cap \text{Span}(f)$$

and its associated saturated subsheaf has slope  $4/4 = 1$ , or is of the form  $V_i + V_j = \langle e_1, e_2, e'_1, e'_2 \rangle$  with  $e'_1, e'_2$  in  $G$  and its associated saturated subsheaf  $\mathcal{F}$  has slope  $\mu(\mathcal{F}) = \frac{6}{4} = \frac{3}{2}$ .

◇

*Remark 2.23.* We may enhance Example 2.22 by proving that all the flats of all the matroids  $M(\mathcal{E})$  of a parliament are not enough to check the (semi)stability of  $\mathcal{E}$ . Indeed, we may force the ground set to be exactly the set  $G$  of Example 2.22. It suffices to incorporate 1-dimensional spaces  $V = \langle g \rangle$  for any  $g \in G$  into the filtrations (take for instance  $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ).

## 2.6 Case of rank less or equal to 3

The purpose of the following theorem is to prove that if  $\mathcal{E}$  has rank less or equal to 3, then the matroid  $M(\mathcal{E})$  of any parliament of  $\mathcal{E}$  checks stability.

**Theorem 2.24.** *Let  $X$  be a smooth complete toric variety, let  $\mathcal{E}$  be a toric vector bundle on  $X$ , of rank less or equal to 3, and let  $(PP_{\mathcal{E}}, G(\mathcal{E}))$  be any of its parliament. Then  $\mathcal{E}$  is  $\alpha$ -(semi)stable if and only if for any nonzero flat  $f \subsetneq G(\mathcal{E})$ , we have*

$$P_{\mathcal{F}} <_{\alpha} P_{G(\mathcal{E})} \quad (\text{resp. } P_{\mathcal{F}} \leq_{\alpha} P_{G(\mathcal{E})}) ,$$

where  $\mathcal{F}$  is the equivariant saturated sheaf corresponding by Theorem 2.4 to the linear subspace  $\langle f \rangle \subset E$ .

Before proving the theorem, let us start by a lemma.

**Lemma 2.25.** *Consider two vector spaces  $F$  and  $F_0$  of the same dimension  $l$ . Assume that for every ray  $\rho_i \in \Sigma(1)$ , for any  $j \in \mathbb{Z}$ , we have that*

$$\dim(F \cap E^i(j)) \geq \dim(F_0 \cap E^i(j)) .$$

Then the slope of  $\mathcal{F}$  is greater than or equal to the slope of  $\mathcal{F}_0$

$$\mu(\mathcal{F}) \geq \mu(\mathcal{F}_0) .$$

*Proof.* If  $A_k^i$  and  $B_k^i$  are the remaining gaps after intersecting the filtration  $(E^i(j))_{j \in \mathbb{Z}}$  respectively with  $F_0$  and  $F$  then for all  $k \in \{1, \dots, l\}$ ,  $\rho_i \in \Sigma(1)$ ,

$$\begin{cases} A_k^i = \min \{j \mid \dim(F_0 \cap E^i(j)) \leq l - k + 1\} \\ B_k^i = \min \{j \mid \dim(F \cap E^i(j)) \leq l - k + 1\} \end{cases}$$

satisfy  $A_k^i \geq B_k^i$ . We conclude by Proposition 2.11.  $\square$

We may now start the proof of Theorem 2.24.

*Proof.* By Lemma 2.25, it is enough to show that for any vector space  $\{0\} \subsetneq F_0 \subsetneq E$ , there exists a flat  $f \subsetneq G(\mathcal{E})$  with  $F = \langle f \rangle$  of the same dimension than  $F_0$  and satisfying

$$\dim(F \cap V) \geq \dim(F_0 \cap V) \quad \text{for all } V \in L(\mathcal{E}) .$$

In the case where  $\dim(F_0) = 1$ , take  $F$  generated by a vector of  $G(\mathcal{E})$  in  $\bigcap_{V \text{ st } F_0 \cap V \neq \{0\}} V$ . It is possible because  $\bigcap_{F_0 \cap V \neq \{0\}} V$  is nonzero ( $F_0$  being included into it) and  $G(\mathcal{E})$  has been constructed such that  $\langle G(\mathcal{E}) \cap \bigcap_{V \text{ st } F_0 \cap V \neq \{0\}} V \rangle = \bigcap_{V \text{ st } F_0 \cap V \neq \{0\}} V$ . It concludes the proof for bundles of rank 2.

It remains the case where  $\dim(F_0) = 2$  and  $\text{rk}(\mathcal{E}) = 3$ . Let us denote by

$$V_1^1, \dots, V_a^1, V_1^2, \dots, V_b^2$$

the vector spaces of  $L(\mathcal{E})$  such that  $\dim(F_0 \cap V_k^\varepsilon) = \varepsilon$ . Remark that  $b$  is either equal to 1 (and  $V_1^2 = E$ ) or 2 (and  $\{V_1^2, V_2^2\} = \{F, E\}$ ). If  $b = 2$  then there exists a flat  $f \subseteq G(\mathcal{E})$  such that  $\langle f \rangle = F_0$  and we are done. Assume thus that  $b = 1$  and let us distinguish cases.

- If there exists  $k_1, k_2 \in \{1, \dots, a\}$  such that  $V_{k_1}^1, V_{k_2}^1$  of dimension 1, then  $F_0$  is also generated by a flat

$$f = \{e_1, e_2\}$$

of  $G(\mathcal{E})$ , where  $V_{k_1}^1 = \langle e_1 \rangle$ ,  $V_{k_2}^1 = \langle e_2 \rangle$ . We can take  $F = F_0$ .

- If there exists exactly one  $V_{k_1}^1$  of dimension 1, then take a generator  $e_1 \in G(\mathcal{E})$  of  $V_{k_1}^1$  and any other vector  $e_2 \in G(\mathcal{E})$ . We define  $F$  as  $F = \langle e_1, e_2 \rangle$ . As for all  $k \neq k_1$ , we have that  $\dim(V_k^1) \geq 2$ ,  $\dim(F) = 2$  and both  $V_k^1$  and  $F$  are included in  $E$  of dimension 3, it holds that

$$\dim(F \cap V_k^1) \geq 1 = \dim(F_0 \cap V_k^1).$$

- If for all  $k \in \{1, \dots, a\}$ , we have  $\dim(V_k^1) \geq 2$  then, by our last argument, any dimension 2-space  $F = \langle f \rangle$  generated by a flat of  $G(\mathcal{E})$  satisfies

$$\dim(F \cap V_k^1) \geq 1 = \dim(F_0 \cap V_k^1) \quad \text{for all } k \in \{1, \dots, a\}.$$

□

## 2.7 Case of tangent bundles

Let  $X$  be a smooth complete toric variety and let  $\mathcal{E} = \mathcal{T}_X$  be the tangent bundle on  $X$ . In this section, we prove that  $M(\mathcal{E})$  is a matroid that check stability. The Klyachko classification of  $\mathcal{E}$  is given by

$$\text{for } \rho_i \in \Sigma(1), E^i(j) = \begin{cases} \mathbb{C}^d & \text{if } j < 0 \\ \langle v_i \rangle & \text{if } 0 \leq j < 1 \\ \{0\} & \text{otherwise} \end{cases}.$$

The only matroid  $M(\mathcal{E})$  associated to  $\mathcal{E}$  has ground set given by

$$G(\mathcal{E}) = \{v_i \mid \rho_i \in \Sigma(1)\}.$$

The parliament  $PP_{\mathcal{E}}$  is composed of the polytopes

$$P_{v_i} = (\langle \cdot, v_i \rangle \leq 1) \cap \bigcap_{\rho_j \neq \rho_i \in \Sigma(1)} (\langle \cdot, v_j \rangle \leq 0).$$

**Proposition 2.26.** *A tangent bundle  $\mathcal{E} = \mathcal{T}_X$  is  $\alpha$ -(semi)stable if and only if for any nonzero flat  $f \subsetneq G(\mathcal{E})$ , we have*

$$P_{\mathcal{F}} <_{\alpha} P_{G(\mathcal{E})} \quad (\text{resp. } P_{\mathcal{F}} \leq_{\alpha} P_{G(\mathcal{E})}),$$

where  $\mathcal{F}$  is the equivariant saturated sheaf corresponding by Theorem 2.4 to the linear subspace  $\langle f \rangle \subset E$ .



*Proof.* The vector spaces in

$$L(\mathcal{E}) = \left\{ \bigcap_{i \in \{0, \dots, n\}} E^i(j_i) \mid (j_i)_{i \in \{0, \dots, n\}} \in \mathbb{Z}^{n+1} \right\}$$

are of dimension either 0, 1 or  $d$ . Hence, any saturated subsheaf  $\mathcal{F}_0 \subset \mathcal{E}$  corresponds to a subspace  $F_0 \subset \mathbb{C}^d$  having intersection with elements of  $L(\mathcal{E})$  either

$$F_0 \cap V = \{0\}, \quad F_0 \cap V = V \text{ and } \dim(V) = 1, \quad \text{or } F_0 \cap V = F_0.$$

Each  $V \in L(\mathcal{E})$  of dimension 1 has a generator  $g_V$  in  $G(\mathcal{E})$ . Let us consider a flat  $f$  of  $G(\mathcal{E})$  obtained by considering the generators  $g_V \in G(\mathcal{E})$  of the vector spaces  $V \in L(\mathcal{E})$  of dimension 1 included in  $F_0$ , and by adjoining elements of  $G$  such that the vector space  $F := \langle f \rangle$  satisfies

$$\dim(F) = \dim(F_0).$$

By Lemma 2.25, we have the inequality

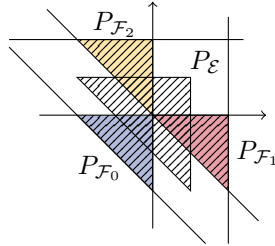
$$\mu(\mathcal{F}) \geq \mu(\mathcal{F}_0),$$

and the flats of  $G(\mathcal{E})$  are enough to check the (semi)stability of  $\mathcal{E}$ .  $\square$

*Example 2.27.* Consider the tangent bundle  $\mathcal{E} = \mathcal{T}_{\mathbb{P}^2}$  on  $\mathbb{P}^2$ . The nontrivial flats of  $G(\mathcal{E}) = \{v_0, v_1, v_2\}$  are

$$f_0 = \{v_0\}, f_1 = \{v_1\} \text{ and } f_2 = \{v_2\}.$$

They correspond to the three nontrivial equivariant saturated subsheaves  $\mathcal{F}_0, \mathcal{F}_1$  and  $\mathcal{F}_2$  of  $\mathcal{E}$ . Let us look at the average polytopes:



By Theorem 2.24, the fact that  $P_{\mathcal{F}_0}, P_{\mathcal{F}_1}, P_{\mathcal{F}_2} <_{\alpha} P_{\mathcal{E}}$  proves that  $\mathcal{E}$  is  $\alpha$ -stable.  $\diamond$

**Corollary 2.28.** Consider any smooth complete variety  $X$  of dimension  $d$  and its tangent bundle on  $\mathcal{E} = \mathcal{T}_X$ . Assume that we have

$$\text{for all } \rho_i \in \Sigma(1), \quad -\rho_i \notin \Sigma(1).$$

Then a necessary condition for  $\mathcal{E}$  to be semistable with respect to a polarization

$$\alpha = \langle L^{d-1} \rangle$$

is that  $\alpha$  corresponds to  $(t_i)_{\rho_i \in \Sigma(1)}$  such that for all ray  $\rho_{i_0} \in \Sigma(1)$

$$t_{i_0} \leq \frac{\sum_{\rho_i \in \Sigma(1)} t_i}{d}.$$

If  $X$  has dimension 2, this condition is also sufficient.

*Proof.* Each subset  $\{v_i\} \subset G(\mathcal{E})$  is a flat of  $M(\mathcal{E})$  and its associated equivariant saturated subsheaf  $\mathcal{F}_i \subset \mathcal{E}$  has slope

$$\mu(\mathcal{F}_i) = t_i \quad \text{while} \quad \mu(\mathcal{E}) = \frac{\sum t_i}{d}.$$

If  $X$  has dimension 2, there exists no other flats.  $\square$

*Remark 2.29.* In particular, we recover the stability of the tangent bundle of  $\mathbb{P}^d$ . Indeed, the Picard number of  $\mathbb{P}^d$  being 1 and  $P_{D_0}$  having  $f_i/\|v_i\| = 1$  (see Definition 2.7), every polarization corresponds to weights  $t_i = a$  for some  $a \in \mathbb{R}_+$ .

### 3 Parliaments of equivariant subbundles

In this section, we first define subparliaments of polytopes and identify them with parliaments of equivariant subbundles. Second, we give an example of toric subbundle which is not a direct factor.

#### 3.1 Subparliaments as parliaments of toric subbundles

Let us state our definition of subparliament of a parliament of polytopes.

**Definition 3.1.** Let  $f = G(\mathcal{E}) \cap F$  be a flat of the matroid  $M(\mathcal{E})$  such that there exists a compatible basis  $(B_\sigma)_{\sigma \in \Sigma(d)}$  for  $\mathcal{E}$  satisfying

$$\text{for all } \sigma \in \Sigma(d), \quad \#B_\sigma \cap f = \dim F.$$

Then we call  $f$  a compatible flat, and we call the subset of polytopes of  $PP_{\mathcal{E}}$  indexed by elements of  $f = F \cap G(\mathcal{E})$

$$\{(P_e, e) \mid e \in f\}$$

a subparliament of the parliament of polytopes  $\mathcal{E}$ .

Equivariant subbundles have been combinatorially described in [KD18]. We translate their results in terms of parliaments of polytopes.

**Proposition 3.2.** Via the Klyachko classification, a rank  $r$ -equivariant vector bundle  $\mathcal{E}$  on  $X$  corresponds to a  $\mathbb{Z}$ -filtration  $(E^i(j))_{j \in \mathbb{Z}}$  of  $E \cong \mathbb{C}^r$  for each  $\rho_i \in \Sigma(1)$ . The equivariant subbundles  $\mathcal{F}$  of  $\mathcal{E}$  are then in one-to-one correspondence with the subfiltrations

$$(F^i(j) = E^i(j) \cap F)_{j \in \mathbb{Z}} \quad \text{of} \quad (E^i(j))_{j \in \mathbb{Z}},$$

for some vector subspace  $F \subseteq E$  such that there exist ground sets  $G(\mathcal{F})$  and  $G(\mathcal{E})$  resulting from Algorithm 3.2 for  $L(\mathcal{F})$  and  $L(\mathcal{E})$  with

$$G(\mathcal{E}) \cap F = G(\mathcal{F})$$

and such that there exist compatible bases  $(B_\sigma)_{\sigma \in \Sigma(d)}$  for  $\mathcal{E}$  which, restricted to  $F$ , satisfy the compatibility condition (CC) for  $\mathcal{F}$ .

*Proof.* By Proposition 4.1.1 of [KD18], the equivariant subbundles  $\mathcal{F}$  of  $\mathcal{E}$  are then in one-to-one correspondence with the subfiltrations

$$(F^i(j) = E^i(j) \cap F)_{j \in \mathbb{Z}} \quad \text{of} \quad (E^i(j))_{j \in \mathbb{Z}},$$

for some vector subspace  $F \subseteq E$  such that  $\{F, \{E^i(j)\}_{\rho_i \in \sigma(1)}\}$  of  $E$  forms a distributive lattice.

Let us show that there exist ground sets  $G(\mathcal{F})$ ,  $G(\mathcal{E})$  obtained by Algorithm 3.2 for  $L(\mathcal{F})$ ,  $L(\mathcal{G})$  with

$$G(\mathcal{E}) \cap F = G(\mathcal{F}).$$

Consider the ground set  $G(\mathcal{E})$  obtained by taking a maximal number of vectors in  $F$  in Algorithm 3.2. If  $V_1, V_2 \in L(\mathcal{E})$ , let us say that  $V_1 < V_2$  if  $V_1$  appears before  $V_2$  in Algorithm 3.2. We denote by  $G_V$  the intersection of  $V$  with the temporary ground set at the beginning of Step  $V \in L(\mathcal{E})$ . During Step  $V$ , we need to add a basis  $CB_V$  of any complement of  $\langle G_V \rangle$  in  $V$  to the ground set. We choose this basis to be composed of a maximal number of vectors in  $F$  so that we have

$$\begin{aligned} V \cap F &= (\langle G_V \rangle + \langle CB_V \rangle) \cap F \\ &= \left( \sum_{\substack{V' = V_1 \cap V \\ V_1 < V}} V' + \langle CB_V \rangle \right) \cap F \\ &= \left( \sum_{\substack{V' = V_1 \cap V \\ V_1 < V}} V' \right) \cap F + \langle CB_V \cap F \rangle \\ &= \sum_{\substack{V' = V_1 \cap V \\ V_1 < V}} (V' \cap F) + \langle CB_V \cap F \rangle. \end{aligned}$$

The last step comes from  $\{F, \{E^i(j)\}_{\rho_i \in \sigma(1)}\}$  forming a distributive lattice of  $E$ . By induction on the  $V \in L(\mathcal{E})$  appearing in Algorithm 3.2,  $G_V \cap F$  generates every  $(V_1 \cap V) \cap F$  for  $V_1 < V$ . We finally obtain that  $G(\mathcal{E}) \cap F$  generates every  $V \cap F$  for  $V \in L(\mathcal{E})$ .

Now if we denote by  $G_{V \cap F}^{\mathcal{F}}$  the intersection of  $V \cap F$  with the temporary ground set at the beginning of Step  $V \cap F \in L(\mathcal{F})$  of Algorithm 3.2 (applied to  $\mathcal{F}$ ), then

$$V \cap F = \langle G_{V \cap F}^{\mathcal{F}} \rangle + \langle CB_{V \cap F}^{\mathcal{F}} \rangle = \sum_{\substack{V' \cap F = (V_1 \cap F) \cap (V \cap F) \\ (V_1 \cap F) < (V \cap F)}} (V' \cap F) + \langle CB_{V \cap F}^{\mathcal{F}} \rangle.$$

Moreover, since  $V_1 < V \Rightarrow (V_1 \cap F < V \cap F)$  and  $(V_1 \cap F < V \cap F) \Rightarrow (V_1 \cap V < V)$ , we have

$$\begin{aligned} &\{V' \cap F \mid V' = V_1 \cap V \text{ and } V_1 < V\} \\ &= \{V' \cap F \mid V' \cap F = (V_1 \cap F) \cap (V \cap F) \text{ and } (V_1 \cap F) < (V \cap F)\}. \end{aligned}$$

We may take  $CB_{V \cap F}^{\mathcal{F}}$  to be  $CB_V \cap F$ , and we finally obtain  $G(\mathcal{F}) = G(\mathcal{E}) \cap F$ .

Moreover, for each cone  $\sigma \in \Sigma(d)$ , any compatible basis  $B_\sigma^F$  of  $F$  extends to a compatible basis  $B_\sigma$  of  $E$ . Indeed, to construct a compatible basis  $B_\sigma$  (resp.  $B_\sigma^F$ ), we apply Algorithm 3.2 to

$$L_\sigma(\mathcal{E}) = \left\{ \bigcap_{\rho_i \in \sigma} E^i(j_i) \mid (j_i)_{\rho_i \in \sigma} \right\} \quad (\text{resp. } L_\sigma(\mathcal{F}) = \{V \cap F \mid V \in L_\sigma(\mathcal{E})\}),$$

taking  $CB_V$  included in  $G(\mathcal{E})$  (resp. taking  $CB_{V \cap F}^{\mathcal{F}}$  included in  $G(\mathcal{E}) \cap F$ ). Extending the bases  $B_\sigma^F$  is possible because  $G(\mathcal{E})$  generates each  $V \in L_\sigma(\mathcal{E}) \subset L(\mathcal{E})$  as well as  $G(\mathcal{F}) = G(\mathcal{E}) \cap F$  generates each  $V \cap F \in L_\sigma(\mathcal{F}) \subset L(\mathcal{F})$ .  $\square$

**Definition 3.3.** Consider a toric vector bundle  $\mathcal{E}$  and a parliament of polytopes  $PP_{\mathcal{E}}$  of  $\mathcal{E}$ . A subparliament of polytopes of  $PP_{\mathcal{E}}$  is a set of the form

$$PP_{\mathcal{F}} = \{(P_e, e) \mid e \in G(\mathcal{E}) \cap F\} \subset PP_{\mathcal{E}},$$

for some vector space  $F$ , which is a parliament meaning that it satisfies the compatibility condition (see Proposition 1.13).

**Theorem 3.4.** Let  $\mathcal{E}$  be a toric vector bundle and let  $\mathcal{F}$  be the equivariant subbundle of  $\mathcal{E}$  corresponding by Klyachko's theorem to the filtrations  $(E^i(j) \cap F)_{j \in \mathbb{Z}}$  for each  $\rho_i \in \Sigma(1)$ . Then, there exists a parliament  $PP_{\mathcal{E}}$  of  $\mathcal{E}$  such that a parliament of  $\mathcal{F}$  is a subparliament of  $PP_{\mathcal{E}}$

$$PP_{\mathcal{F}} = \{(P_e, e) \mid e \in G(\mathcal{E}) \cap F\} \subset PP_{\mathcal{E}}.$$

*Proof.* Consider a ground set  $G(\mathcal{E})$  for the parliament of  $\mathcal{E}$  obtained via Algorithm 3.2 by taking at each step a maximal number of elements in  $F$ . Then by Proposition 3.2,  $G(\mathcal{F}) = G(\mathcal{E}) \cap F$  is a possible output for Algorithm 3.2 applied to  $L(\mathcal{F})$ . Now if  $e \in G(\mathcal{F}) = G(\mathcal{E}) \cap F$  then the polytope  $P_e \in PP_{\mathcal{F}}$  and

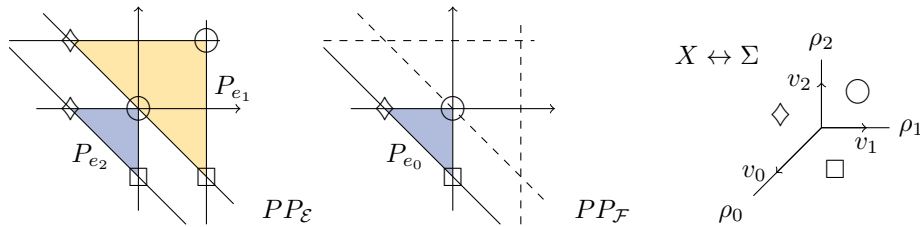
$$\begin{aligned} P_e &= \{m \mid \forall i, \langle m, v_i \rangle \leq \max \{j \mid e \in E^i(j) \cap F\}\} \\ &= \{m \mid \forall i, \langle m, v_i \rangle \leq \max \{j \mid e \in E^i(j)\}\} \end{aligned}$$

is the same as the one in the parliament of  $\mathcal{E}$ .  $\square$

*Example 3.5.* Consider the toric variety  $X = \mathbb{P}^2$  with its  $T$ -invariant divisors  $D_0, D_1$  and  $D_2$  and the splitting, equivariant, rank 2-vector bundle  $\mathcal{E} = \mathcal{O}(D_0) \oplus \mathcal{O}(D_1 + D_2)$  on  $X$ . The corresponding  $\mathbb{Z}$ -filtrations of  $E = \mathbb{C}^2$  are of the form

$$E^i(j) = \begin{cases} \mathbb{C}^2 & \text{if } j \leq 0 \\ \langle e_1 \rangle & \text{if } 0 < j \leq 1 \\ \{0\} & \text{otherwise} \end{cases} \quad (\text{for } i = 1, 2) \quad \text{and} \quad E^0(j) = \begin{cases} \mathbb{C}^2 & \text{if } j \leq 0 \\ \langle e_2 \rangle & \text{if } 0 < j \leq 1 \\ \{0\} & \text{otherwise} \end{cases}$$

where  $\langle e_1 \rangle$  and  $\langle e_2 \rangle$  are different lines in  $\mathbb{C}^2$ . The ground set is  $G(\mathcal{E}) = \{e_1, e_2\}$ . The toric subbundle  $\mathcal{F} = \mathcal{O}(D_0) \subset \mathcal{E}$  corresponds to the flat  $f = \{e_2\} \subset G(\mathcal{E})$ .

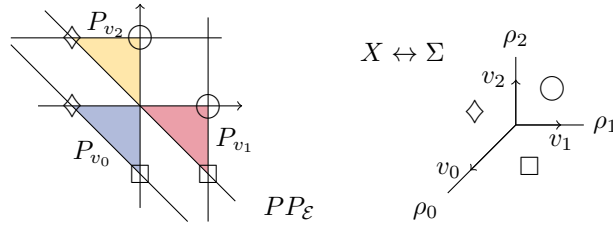


The compatibility condition (CC) for  $\mathcal{E}$  is satisfied on taking

$$B_\sigma = \{L_u^\sigma \mid u \in \mathbf{u}(\sigma)\} = \{e_1, e_2\} \quad \text{for all } \sigma \in \Sigma(d).$$

We see that for all  $\sigma \in \Sigma(d)$ ,  $B_\sigma \cap f = \{e_2\}$  generates  $F = \langle f \rangle$ , the flat  $f = \{e_2\}$  is thus compatible. The parliament  $PP_{\mathcal{F}}$  is a subparliament of  $PP_{\mathcal{E}}$ .  $\diamond$

*Example 3.6.* Let us come back to Example 1.14 with  $X = \mathbb{P}^2$  and its tangent bundle  $\mathcal{E} = \mathcal{T}_{\mathbb{P}^2}$ .



There exists no nontrivial equivariant subbundle  $\mathcal{F}$  of  $\mathcal{E}$  because the nontrivial flats are  $f_0 = \{v_0\}$ ,  $f_1 = \{v_1\}$  and  $f_2 = \{v_2\}$  and are not compatible.  $\diamond$

**Corollary 3.7.** *The subparliaments of the parliaments of  $\mathcal{E}$  are the parliaments of the equivariant subbundles of  $\mathcal{E}$ .*

### 3.2 Subbundle that is not a direct factor

There exists equivariant vector bundles  $\mathcal{E}$  with a nontrivial equivariant subbundle  $\mathcal{F}$  that we cannot factorize in a direct sum of  $\mathcal{E}$ .

*Example 3.8.* Consider the equivariant vector bundle  $\mathcal{E}$  on  $\mathbb{P}^2$  defined by the filtrations

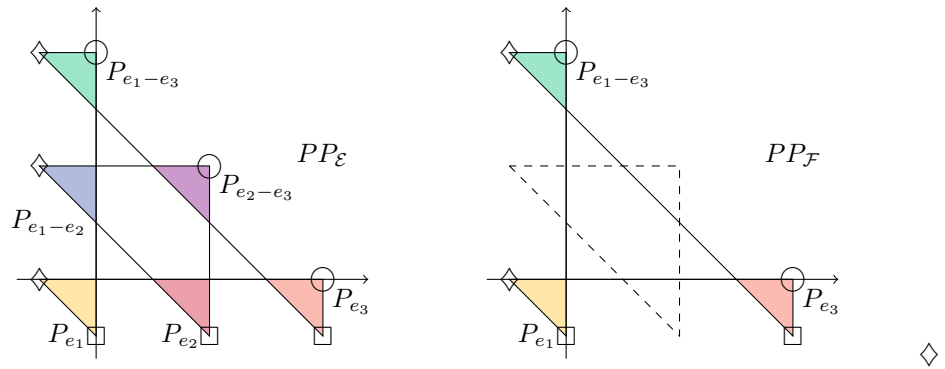
$$E^0(j) = \begin{cases} \mathbb{C}^3 & \text{if } j < -3 \\ \langle e_1, e_2 \rangle & \text{if } -3 \leq j < -1 \\ \langle e_1 \rangle & \text{if } -1 \leq j < 1 \\ \{0\} & \text{otherwise} \end{cases}, \quad E^1(j) = \begin{cases} \mathbb{C}^3 & \text{if } j < 0 \\ \langle e_2, e_3 \rangle & \text{if } 0 \leq j < 2 \\ \langle e_3 \rangle & \text{if } 2 \leq j < 4 \\ \{0\} & \text{otherwise} \end{cases}$$

$$\text{and } E^2(j) = \begin{cases} \mathbb{C}^3 & \text{if } j < 0 \\ \langle e_1 - e_3, e_1 - e_2 \rangle & \text{if } 0 \leq j < 2 \\ \langle e_1 - e_3 \rangle & \text{if } 2 \leq j < 4 \\ \{0\} & \text{otherwise} \end{cases}$$

where  $e_1 = (1, 0, 0)$ ,  $e_2 = (0, 1, 0)$ ,  $e_3 = (0, 0, 1)$ . The unique possible ground set of  $M(\mathcal{E})$  is

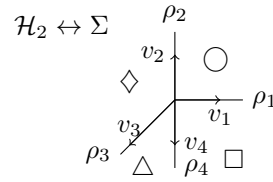
$$G(\mathcal{E}) = \{e_1, e_2, e_3, e_1 - e_2, e_1 - e_3, e_2 - e_3\}.$$

Taking  $F = \langle e_2 \rangle^\perp$ , we obtain an equivariant subbundle  $\mathcal{F}$ .

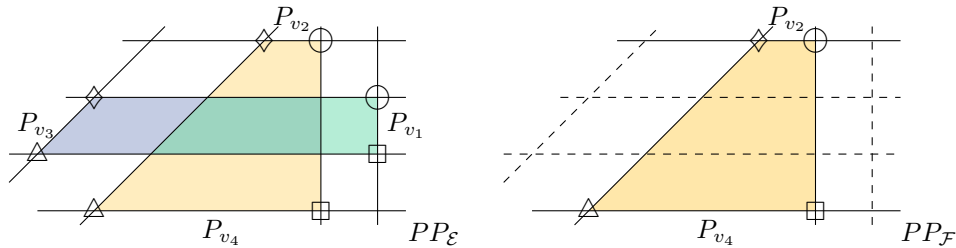


*Example 3.9.* The following more simple example was pointed out to me by Bivas Khan. Consider the tangent bundle  $\mathcal{E} = \mathcal{T}_{\mathcal{H}_2}$  of the Hirzebruch surface, the rays of the fan are

$$v_1 = (1, 0), v_2 = (0, 1), v_3 = (-1, 1), v_4 = (0, -1).$$



The linear space  $F = \langle v_2 \rangle$  defines a subbundle  $\mathcal{F}$  and  $\mathcal{E}$  cannot be written as  $\mathcal{F} \oplus \mathcal{G}$ . Here we will see that the obstruction to being a direct sum comes only from the matroid (and not from the form of the polytopes). For the picture we tensorize  $\mathcal{E}$  and  $\mathcal{F}$  by the line bundle  $\mathcal{O}(D_3 + D_4)$  which does not change the stability.



Remark that we could have taken the same filtrations on any complete smooth toric surface having a fan with 4 rays for instance  $X = \mathbb{P}^1 \times \mathbb{P}^1$ .  $\diamond$

In particular, these examples give rise to nontrivial Harder–Narasimhan filtrations

$$0 \subseteq \mathcal{F} \subseteq \mathcal{E}.$$

## 4 Stability of restrictions to invariant curves

The  $\alpha$ -semistability of the restrictions  $\mathcal{E}|_C$  of a toric vector bundle  $\mathcal{E}$  to a torus invariant curve is much easier to view on the parliament of  $\mathcal{E}$ . Moreover, it is often useful to compare the  $\alpha$ -semistability of a toric vector bundle  $\mathcal{E}$  to the semistability of its restrictions  $\mathcal{E}|_C$  to any torus invariant curve.

*Remark 4.1.* Theorem 2.5 from [BLG13] furnishes a sufficient condition to deduce from the semistability of a toric vector bundle  $\mathcal{E}$ , the semistability of its restriction  $\mathcal{E}|_C$  to any invariant curves  $C$ : the characteristic class of  $\mathcal{E}$  is 0

$$\Delta(\mathcal{E}) = c_2(\mathcal{E}) - \frac{r-1}{2r} c_1(\mathcal{E})^2 = 0.$$

In this section, we explain how to see the (semi)stability of the restrictions  $\mathcal{E}|_C$  of a vector bundle  $\mathcal{E}$  to a torus invariant curve  $C$ , given a parliament  $PP_{\mathcal{E}}$ . In [DJS14] Subsection 3.1, Di Rocco, Jabbusch and Smith recover the parliaments of restrictions  $\mathcal{E}|_C$  to torus invariant curves from the parliaments of polytopes of  $\mathcal{E}$ .

**Proposition 4.2.** *By the cone-orbit correspondence, torus invariant curves correspond to a cone  $\tau \in \Sigma(d-1)$ . Since  $X$  is complete, there are two maximal cones  $\sigma$  and  $\sigma'$  in  $\Sigma(d)$  containing  $\tau$ .*

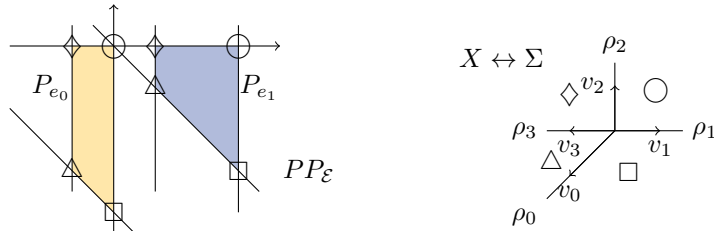
*If  $PP_{\mathcal{E}}$  is a parliament of  $\mathcal{E}$ , then a parliament of polytopes of  $\mathcal{E}|_C$  is composed of the projection on  $\tau^\perp$  of the line segments parallel to  $\tau^\perp$  joining the associated characters in  $\mathbf{u}(\sigma)$  and*

$\mathbf{u}(\sigma)$  (renormalized by  $1/||u_\tau||$ ).

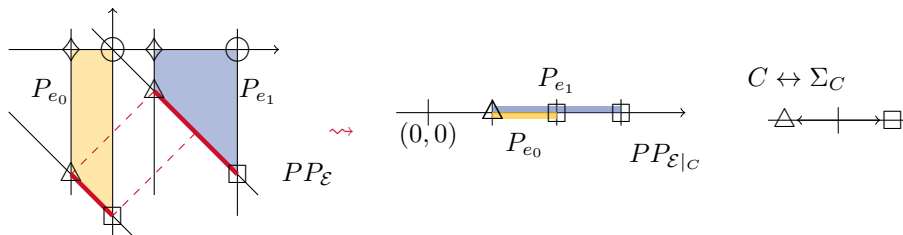
*Example 4.3.* Consider  $X = \text{Bl}_{[0:1:0]}(\mathbb{P}^2)$  and let  $\mathcal{E}$  be the equivariant vector bundle

$$\mathcal{E} = \mathcal{O}_X(4D_0 + D_3) \oplus \mathcal{O}_X(3D_1 - D_3)$$

on  $X$ . The unique parliament of polytopes of  $\mathcal{E}$  is



The parliament of  $\mathcal{E}$  restricted to the torus invariant curve  $D_0$  is obtained the following way.



The semistability of such restriction  $\mathcal{E}|_C$  should be easier to understand as torus invariant curves  $C$  are isomorphic to  $\mathbb{P}^1$ . Moreover, in 1957, Grothendieck classified vector bundles on  $\mathbb{P}^1$ .

**Theorem 4.4** ([Gro57] Theorem 2.1). *Every holomorphic vector bundle  $\mathcal{E}$  on  $\mathbb{P}^1$  is holomorphically isomorphic to a direct sum of line bundles:*

$$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(d_r).$$

The problem of semistability is then reduced to verify if  $d_1 = \dots = d_r$ . This can be achieved by looking at the parliaments of polytopes of  $\mathcal{E}$ . The operation on parliaments corresponding to restricting the toric bundle to a invariant curve is explained in Proposition 4.2. We obtain the following result.

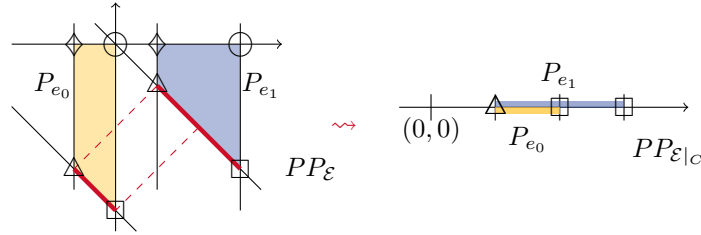
**Proposition 4.5.** *Let  $C$  be a torus invariant curve on  $X$ .  $C$  corresponds to a wall  $\tau \in \Sigma(d-1)$  between two maximal cones  $\sigma, \sigma' \in \Sigma(d)$ . A toric vector bundle  $\mathcal{E}$  is semistable if and only if the normalized lattice distance between associated characters  $u \in \mathbf{u}(\sigma)$  and  $u' \in \mathbf{u}(\sigma')$  in the one-dimensional lattice  $(\tau^\perp + u) \cap M$  is the same for any pair  $u, u'$ .*

The example which follows concerns a semistable vector bundle such that its restriction to an invariant curve is not semistable.

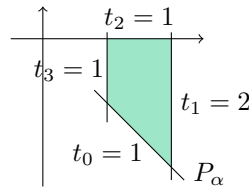
*Example 4.6.* Consider  $X = \text{Bl}_{[0:1:0]}(\mathbb{P}^2)$  and let  $\mathcal{E}$  be the equivariant vector bundle

$$\mathcal{E} = \mathcal{O}_X(4D_0 + D_3) \oplus \mathcal{O}_X(3D_1 - D_3)$$

on  $X$ . The parliament  $\mathcal{E}_{D_0}$  of  $\mathcal{E}$  restricted to the torus invariant curve  $D_0$  is obtained the following way.



The bundle  $\mathcal{E}|_{D_0}$  would be stable if the segments  $P_{e_0}$  and  $P_{e_1}$  were of the same length. In this example, there is not any torus invariant curve  $C$  such that  $\mathcal{E}|_C$  is semistable. A contrario, taking the movable curve  $\alpha = 2D_1 - D_3$ , as in Example 2.9, with its corresponding  $(t_i)_{\rho_i \in \Sigma(1)}$



gives  $\mathcal{E}$  is  $\alpha$ -polystable:  $\mu(\mathcal{L}_0) = 4t_0 + t_3 = 5$  is equal to  $\mu(\mathcal{L}_1) = 3t_1 - t_3 = 5$ .  $\diamond$

*Example 4.7.* For any choice of movable curve  $\alpha$ , the tangent bundle  $\mathcal{T}_{\mathbb{P}^2}$  of  $\mathbb{P}^2$  is an example of  $\alpha$ -stable equivariant vector bundle  $\mathcal{E}$  with a non stable restriction to a torus invariant curve.  $\diamond$

## 5 Alternative definition of parliaments

We propose definitions for the parliaments of polytopes of a toric vector bundle  $\mathcal{E}$ . For each definition, the data of a parliament of polytopes of some globally generated equivariant vector bundle  $\mathcal{E}$  corresponds to some isomorphism class of  $\mathcal{E}$  (corresponding to the title of the subsection).

*Remark 5.1.* If we consider virtual polytopes as proposed in Remark 1.15, the condition of global generateness becomes superfluous.

The naive definition for a parliament of polytopes of a toric vector bundle  $\mathcal{E}$  would be

$$PP_{\mathcal{E}} := \{P_e \mid e \in G(\mathcal{E})\}$$

which does not keep trace of the label of each polytope. This definition do not even allow us to distinguish a rank 2- from a rank 3-vector bundle on  $\mathbb{P}^2$ : the tangent bundle  $\mathcal{T}_{\mathbb{P}^2}$  and the splitting bundle  $\mathcal{O}_{\mathbb{P}^2}(D_0) \oplus \mathcal{O}_{\mathbb{P}^2}(D_1) \oplus \mathcal{O}_{\mathbb{P}^2}(D_2)$  have the same parliament of polytopes.

### 5.1 Equivariant isomorphism class of framed toric vector bundle

The definition we gave in Definition 1.11 for a parliament of polytopes of a toric vector bundle  $\mathcal{E}$  was  $PP_{\mathcal{E}} := \{(P_e, e) \mid e \in G(\mathcal{E})\}$  where  $G(\mathcal{E})$  is defined modulo isomorphism of type  $(\star)$ .



**Definition 5.2.** A framed equivariant vector bundle is a toric vector bundle with a choice of isomorphism  $E \cong \mathbb{C}^r$ .

A morphism of framed equivariant vector bundle is a morphism of equivariant vector bundle compatible with the framing.

**Proposition 5.3.** The data of a parliament of polytopes

$$PP_{\mathcal{E}} := \{(P_e, e) \mid e \in G(\mathcal{E})\} / (\star)$$

of a globally generated equivariant vector bundle  $\mathcal{E}$  is equivalent to knowing the equivariant isomorphism class of the framed equivariant vector bundle  $\mathcal{E}$ .

*Proof.* In [Pay07] Proposition 3.4, the equivariant classes of a framed equivariant vector bundle is uniquely determined by the following

$$\left( \{\mathbf{u}(\sigma)\}_{\sigma \in \Sigma(d)}, \quad \{Fl(\rho)\}_{\rho \in \Sigma(1)} \right),$$

where  $Fl(\rho_i)$  is the flag appearing in the filtration  $(E^i(j))_j$ . From the parliament we recover the flag

$$Fl(\rho_i) : \{0\} \subset E^i(A_r^i) \subset \dots \subset E^i(A_1^i) = E$$

by considering all different vector spaces

$$E_{i,j} = \sum_{\substack{\langle u, v_i \rangle \geq j, \\ u \in P_e \text{ for some } e \in G(\mathcal{E})}} e.$$

Indeed, if  $e \in E_{i,j}$  then there exists  $u \in P_e$  such that  $\langle u, v_i \rangle \geq j$  and by definition of  $P_e$ , we obtain that  $e \in E^i(j)$ . And conversely, by Theorem 1.2 of [DJS14], as  $\mathcal{E}$  is taken globally generated, any  $u \in \mathbf{u}(\sigma)$  belongs to some polytope and by (CC), we have  $E^i(j) \subseteq E_{i,j}$ .  $\square$

## 5.2 Equivariant isomorphism class of toric vector bundle

We may not want to deal with framings anymore. Let us thus quotient by the action of  $GL_r(\mathbb{C})$  on  $\mathbb{C}^r$ .

**Proposition 5.4.** The data of a parliament of polytopes

$$PP_{\mathcal{E}} := \{(P_e, e) \mid e \in G(\mathcal{E})\} / (\star) / GL_r(\mathbb{C})$$

of a globally generated equivariant vector bundle  $\mathcal{E}$  is equivalent to knowing the equivariant isomorphism class of the equivariant vector bundle  $\mathcal{E}$ .

*Remark 5.5.* Here morphisms do not have to preserve the framings anymore.

*Proof of the proposition.* It follows from [Pay07] Corollary 3.6 that the equivariant classes of a toric vector bundle is uniquely determined by

$$\left( \{\mathbf{u}(\sigma)\}_{\sigma \in \Sigma(d)}, \quad \mathcal{O}_{\{Fl(\rho)\}_{\rho \in \Sigma(1)}} \right),$$

where  $\mathcal{O}_{\{Fl(\rho)\}_{\rho \in \Sigma(1)}}$  is the  $GL_r(\mathbb{C})$ -orbit of the flag given by the filtration  $(E^i(j))_j$ . We conclude by Proposition 5.3.

### 5.3 Isomorphism class of toric vector bundle

Finally, we may want an object which represents the isomorphism class of  $\mathcal{E}$  and not its equivariant isomorphism class. For that we need to quotient by the group  $T$  of compositions of translations for each direct component.

**Proposition 5.6.** *The data of a parliament of polytopes*

$$PP_{\mathcal{E}} := \{(P_e, e) \mid e \in G(\mathcal{E})\} / (\star) / GL_r(\mathbb{C}) / T ,$$

*of a globally generated equivariant vector bundle  $\mathcal{E}$  is equivalent to knowing the isomorphism class of  $\mathcal{E}$ .*

*Proof.* It is known that any line bundle  $\mathcal{O}_X(D)$  on toric variety is isomorphic to an equivariant line bundle  $\mathcal{O}_X(D_T) = \mathcal{O}_X(D) \otimes \chi^u$  for a unique  $u \in M$ .

In fact this result has been generalized by Klyachko in ([Kly89] Corollary 1.2.4). If two equivariant vector bundles  $\mathcal{E}$  and  $\mathcal{F}$  are isomorphic then there exists characters  $\chi_1, \dots, \chi_m$  such that

$$\mathcal{E}_i \otimes \chi_i \text{ and } \mathcal{F}_i \text{ are equivariantly isomorphic,}$$

where  $\mathcal{E} = \mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_m$  and  $\mathcal{F} = \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_m$  are some direct decompositions of  $\mathcal{E}$  and  $\mathcal{F}$ .

As a level of parliaments, tensoring by a character  $\chi^u$  corresponds to translating the parliament of  $u \in M$ . □

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