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A combinatorial description of stability for toric vector bundles

Lucie Devey

Abstract

The aim of this paper is to give a necessary and sufficient condition for the stability of a toric vector bundle in the combinatorial terms of its parliament of polytopes, a generalization of Newton polytopes for toric vector bundles by Di Rocco, Jabbusch and Smith.

We also define subparliaments of polytopes and identify them with parliaments of equivariant subbundles.

Introduction

While the geometry of line bundles on smooth projective varieties is relatively well understood, some notions, such as positivity, become vastly more complicated in higher rank, despite significant recent work on vector bundles. The notions of stable base loci, Kodaira maps and Iitaka fibrations can be extended to vector bundles (see [MU19]). Furthermore, positivity properties can be detected by looking at some asymptotic base loci that can be defined on the variety itself, rather than on the projectivization of the given vector bundle (see [BKK⁺15]).

Using the Harder—Narasimhan filtration (resp. the Jordan—Holder filtration), any vector bundle can be built up from semistable (resp. stable) bundles. The importance of knowing whether a given vector bundle is (semi)stable is thus fundamental, especially for the construction of moduli spaces.

The aim of this paper is to study slope-stability of toric vector bundles, since in toric geometry, most complicated notions turn out to have a combinatorial description.

In [DJS18], Sandra Di Rocco, Kelly Jabbusch and Gregory G. Smith generalized the construction of the Newton polytope of a line bundle, associating to any equivariant vector bundle \mathcal{E} its parliament of polytopes $PP_{\mathcal{E}}$: a collection of convex polytopes $(P_e)_{e \in G(\mathcal{E})}$ indexed by the elements in the ground set of a matroid $M(\mathcal{E})$ associated to \mathcal{E} representable in a vector space $E \cong \mathbb{C}^r$.

The wealth of information about a toric vector bundle \mathcal{E} contained in its parliament is astounding. For instance, lattice points in the parliament of polytopes for \mathcal{E} correspond to a torus-equivariant generating set for the space of global sections of \mathcal{E} ([DJS18] Proposition 1.1). More precisely, a lattice point in the polytope P_e of $PP_{\mathcal{E}}$ corresponds to a global section s , taking value e at the identity of the torus $T \subset X$. In addition, we can also recover some positivity properties of \mathcal{E} , such as global generatedness (see [DJS18] Theorem 1.2), ampleness (see [DJS18] Corollary 6.7), or bigness (see [Nø20] Theorem 7.5). Parliaments of polytopes have been used to find counterexamples as well. For instance, Example 5.3 of [DJS18] exhibits a toric vector bundle on \mathbb{P}^2 that is ample but not globally generated, and Example 6.4 of [DJS18] exhibits a toric vector bundle on \mathbb{P}^2 that is ample and globally generated

but not very ample. They have also enable one to find a criterion for a projectivized toric vector bundle to be a Mori dream space (see [Nø20] Theorem 4.18).

Our main result is an algorithm for the stability of toric vector bundles in terms of their parliaments of polytopes. Indeed, we reduce the problem of stability to verifying that the equivariant saturated subsheaves that come from flats (closed subset of a matroid) of $M(\mathcal{E})$ have slope less than the slope of \mathcal{E} . Additionally, we define, for any such equivariant saturated subsheaf \mathcal{F} of \mathcal{E} (and thus for $\mathcal{F} = \mathcal{E}$), a polytope called the average polytope $P_{\mathcal{F}}$ (see Definition 2.15) which can be visualized on the parliament of \mathcal{E} . It is the Newton polytope of $c_1(\mathcal{F})/\text{rk}(\mathcal{F})$. As such, the average polytope of \mathcal{F} embodies the slopes of \mathcal{F} with respect to every polarization α of X . In Definition 2.26, we define a total order $<_{\alpha}$ between polytopes, such that comparing average polytopes $P_{\mathcal{F}}$ and $P_{\mathcal{F}'}$ corresponds to comparing the α -slopes of their respective sheaves \mathcal{F} and \mathcal{F}' . It enables us to state our main theorem.

Theorem (2.27). *Let X be a smooth complete toric variety. A toric vector bundle \mathcal{E} on X is α -(semi)stable if and only if for any flat f of the matroid $M(\mathcal{E})$, we have*

$$P_{\mathcal{F}} <_{\alpha} P_{\mathcal{E}} \quad (\text{resp. } P_{\mathcal{F}} \leq_{\alpha} P_{\mathcal{E}}),$$

for the total order $<_{\alpha}$ between polytopes defined in Definition 2.26 and where \mathcal{F} is the equivariant saturated sheaf corresponding to the linear subspace $\langle f \rangle \subset E$.

In particular, in [DDK20], the authors would like a computer program to check (semi)stability of any equivariant torsion free sheaf with respect to any polarization. This Theorem is an answer to their hope.

In another part of the article, we prove that any equivariant subbundle \mathcal{F} of a toric vector bundle \mathcal{E} corresponds to a flat of $M(\mathcal{E})$ compatible with the Klyachko filtrations of \mathcal{E} . We describe the parliament of the equivariant subbundles.

Proposition (3.3). *The parliament of an equivariant subbundle \mathcal{F} of an equivariant bundle \mathcal{E} is a subparliament of $PP_{\mathcal{E}}$ or, in other words, a subset of $PP_{\mathcal{E}}$ containing only polytopes with indices in a compatible flat of the matroid $M(\mathcal{E})$.*

The last part of this article consists of exploring the geometric information that can be reconstructed from the parliament of polytopes of a toric vector bundle. In particular, we translate results of Payne (see [Pay08] Proposition 3.4, Corollary 3.6) and Klyachko (see [Kly90] Corollary 1.2.4) in terms of parliaments of polytopes.

Proposition (5.5). *The data of the parliament of polytopes of a globally generated equivariant vector bundle \mathcal{E} , up to translation of each direct component and quotiented by $GL_r(\mathbb{C})$, enables us to reconstruct the isomorphism class of the vector bundle \mathcal{E} .*

We start in Section 1 by recalling facts about matroids, giving the construction of parliaments of polytopes from [DJS18] and fixing notation.

Section 2 furnishes the definition of average polytopes that allows us to visualize the slope of an equivariant saturated sheaf. It also contains the most important result of the article (Theorem 2.27): a combinatorial description of slope-stability of equivariant vector bundles using parliaments of polytopes.

In Section 3, given the parliament of polytopes of a toric vector bundle, we describe the parliaments of its equivariant subbundles.

In Section 4, we treat the stability of the restriction of an equivariant bundle \mathcal{E} to a torus invariant curve in terms of the parliament $PP_{\mathcal{E}}$. We also give examples of α -(semi)stable equivariant vector bundle with non (semi)stable restrictions to torus invariant curves.

In Section 5, we discuss the definition of parliament of polytopes and state what data is encoded in the parliament of an equivariant bundle.

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1 Parliaments of polytopes and matroid terminology

Parliaments of polytopes were introduced in [DJS18] in order to give explicit polyhedral interpretations of properties to equivariant vector bundles.

The parliament of polytopes $PP_{\mathcal{E}}$ of a given rank r -equivariant vector bundle \mathcal{E} over a smooth complete toric variety X is composed of at least r polytopes linked with some combinatorial data: the polytopes are labeled by elements of a representable rank r -matroid. We first recall some matroid terminology.

1.1 Matroid terminology

Matroids are a generalization of the notion of linear independence in vector spaces.

Definition 1.1. *A matroid M is the data of a finite set G and a collection \mathcal{B} , of subsets of G , called bases, satisfying the following properties:*

B1: \mathcal{B} is nonempty;

B2: (basis exchange property) If $A, B \in \mathcal{B}$ are distinct and $a \in A \setminus B$, then there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

We call G the ground set and \mathcal{B} the set of bases of the matroid $M = (G, \mathcal{B})$.

Definition 1.2. *An isomorphism of matroids $\varphi : M_1 = (G_1, \mathcal{B}_1) \rightarrow M_2 = (G_2, \mathcal{B}_2)$ is a bijection from G_1 to G_2 such that*

$$A \in \mathcal{B}_1 \Leftrightarrow \varphi(A) \in \mathcal{B}_2 .$$

Definition 1.3. *A representable matroid of rank r is a matroid isomorphic to*

$$M = (G, \mathcal{B}) ,$$

where G is a finite subset of some dimension r -vector space E and \mathcal{B} is the set of bases of E formed by vectors of G .

Definition 1.4. *A flat of a matroid (G, \mathcal{B}) represented in E , is a subset $f \subset G \subset E$ such that*

$$\langle f \rangle \cap G = f ,$$

where $\langle f \rangle \subset E$ is the subspace spanned by the vectors in f .

1.2 Parliaments of polytopes

Consider a smooth complete toric variety X of dimension d with fan Σ and a toric vector bundle \mathcal{E} on X of rank r with fiber at the identity of the torus E . Let us denote by $\{v_0, \dots, v_n\}$ the set of vectors generating the rays ρ_i of Σ . (See [CLS11] for fundamentals on toric varieties.)

The construction of the parliament of polytopes of \mathcal{E} requires the Klyachko classification.

Theorem 1.5 (Theorem 0.1.1 of [Kly90]). *There is an equivalence of categories between the category of rank r -equivariant vector bundle \mathcal{E} on X and the category of $n+1$ compatible decreasing \mathbb{Z} -filtrations $(E^i(j))_{\rho_i \in \Sigma(1)}$*

$$E^i(j) = \begin{cases} E & \text{if } j \leq A_1^i \\ H_i & \text{if } A_1^i < j \leq A_2^i \\ \dots & \\ \langle u_i \rangle & \text{if } A_{r-1}^i < j \leq A_r^i \\ \{0\} & \text{if } A_r^i < j \end{cases} .$$

of a dimension r - \mathbb{C} -vector space E , where the compatibility condition is the existence of decompositions of E by 1-dimensional vector spaces L_u^σ indexed by $\mathbf{u}(\sigma) \in M^r$ as follows.

$$\forall \sigma \in \Sigma(d), \exists (L_u^\sigma)_{u \in \mathbf{u}(\sigma)} \text{ s.t. } E = \bigoplus_{u \in \mathbf{u}(\sigma)} L_u^\sigma \text{ and } \forall \rho_i \preceq \sigma, E^i(j) = \sum_{\langle u, v_i \rangle \geq j} L_u^\sigma . \quad (\text{CC})$$

Definition 1.6. *The r points in $\mathbf{u}(\sigma)$ are called the associated characters of σ .*

Remark 1.7. The compatibility condition (CC) implies the existence, for every maximal cone $\sigma \in \Sigma(d)$, of a basis

$$B_\sigma = \{L_u^\sigma \mid u \in \mathbf{u}(\sigma)\} ,$$

called a compatible basis. Nevertheless, given a maximal cone, a compatible basis may not be unique (see Example 4.4 of [DJS18]) although $\mathbf{u}(\sigma)$ always is.

Remark 1.8. An important feature is that for any maximal cone $\sigma \in \Sigma(d)$, the equivariant bundle \mathcal{E} splits equivariantly on U_σ as

$$\mathcal{E}|_{U_\sigma} \simeq \bigoplus_{u \in \mathbf{u}(\sigma)} \mathcal{O}_X(\text{div}(u))|_{U_\sigma} .$$

The parliament of polytopes of a vector bundle \mathcal{E} is a collection of polytopes

$$\{(P_e, e) \mid e \in G(\mathcal{E}) \subseteq E\} ,$$

where $G(\mathcal{E})$ is the ground set of a matroid $M(\mathcal{E})$ defined in the following manner. Consider all matroids containing

$$L(\mathcal{E}) = \left\{ \bigcap_{i \in \{0, \dots, n\}} E^i(j_i) \mid (j_i)_{i \in \{0, \dots, n\}} \in \mathbb{Z}^{n+1} \right\}$$

as a meet-subsemilattice (as a partially ordered subset which has a meet i.e. a

greatest lower bound). Among such matroids we select the matroids such that the number of elements in the ground set is minimal and among those matroids, only one up to isomorphism has a minimal number of circuits (minimal dependant sets): call this matroid $M(\mathcal{E})$. See Proposition 3.1 [DJS18]. Moreover, the ground set $G(\mathcal{E})$ is computable by Algorithm 3.2 of [DJS18].

Algorithm 3.2 Computing $G(\mathcal{E})$

```

 $r \leftarrow$  the dimension of the largest linear subspace of  $L(\mathcal{E})$ 
 $G \leftarrow \emptyset$ 
for  $k = 1$  to  $r$  do
  for  $k$ -dimensional linear subspace  $V \in L(\mathcal{E})$  do
     $G' \leftarrow G \cap V$ 
    if  $\langle G' \rangle \subsetneq V$  then
       $G \leftarrow G \sqcup CB_V$ ,  $CB_V$  is a basis of a complement to  $\langle G' \rangle$  in  $V$ 
    end if
  end for
end for
return  $G$ 

```

Additionally, if G_E^1 and G_E^2 are ground sets resulting from Algorithm 3.2 applied to $L(\mathcal{E})$ then there exists an isomorphism φ of matroids sending G_E^1 to G_E^2 and that we call of type (\star) :

Definition 1.9. An isomorphism of matroids $\varphi : M(G_E^1) \rightarrow M(G_E^2)$ of type (\star) is a bijection

$$\varphi : G_E^1 \rightarrow G_E^2$$

which respects Algorithm 3.2 for \mathcal{E} , that is to say which sends CB_V^1 to CB_V^2 at Step V of Algorithm 3.2.

Indeed, in the proof of Proposition 3.1 [DJS18], \mathcal{L}' is unique but the unicity of the free expansion of \mathcal{L}' holds only up to isomorphism of matroids (Di Rocco, Jabbusch and Smith used Proposition 10.2.3 [Whi08]).

We use the following notation for the ground set $G(\mathcal{E})$ of $M(\mathcal{E})$

$$G(\mathcal{E}) = \{e_0, \dots, e_l\} \quad \text{for some } l \geq r .$$

Definition 1.10. The parliament of polytopes of \mathcal{E} is composed of the set of indexed polytopes $(P_e)_{e \in G(\mathcal{E})}$ defined as

$$P_e := \{m \in \mathbb{R}^d \mid \forall \rho_i \in \Sigma(1), \langle m, v_i \rangle \leq \max \{j \mid e \in E^i(j)\} \} ,$$

where the ground set $G(\mathcal{E})$ is seen modulo isomorphism of matroids of type (\star) .

Remark 1.11. A parliament is a set of at least r indexed polytopes, all coming from $n \times r$ hyperplanes

$$(H_{i,k} = (\langle \cdot, v_i \rangle = A_k^i))_{\rho_i \in \Sigma, k \in \{1, \dots, r\}} .$$

A natural question is: given such a set of indexed polytopes constructed from a family of decreasing separated-exhaustive \mathbb{Z} -filtrations $((E^i(j))_{j \in \mathbb{Z}})_{\rho_i \in \Sigma(1)}$, how can we see the compatibility condition (CC)?

For any fixed maximal cone σ , there should exist r associated characters $u \in \mathbf{u}(\sigma)$ at the intersection of some hyperplanes

$$(H_{i,k_{i,u}})_{\rho_i \in \sigma(1)} \quad \text{with } k_{i,u} \in \{1, \dots, r\} .$$

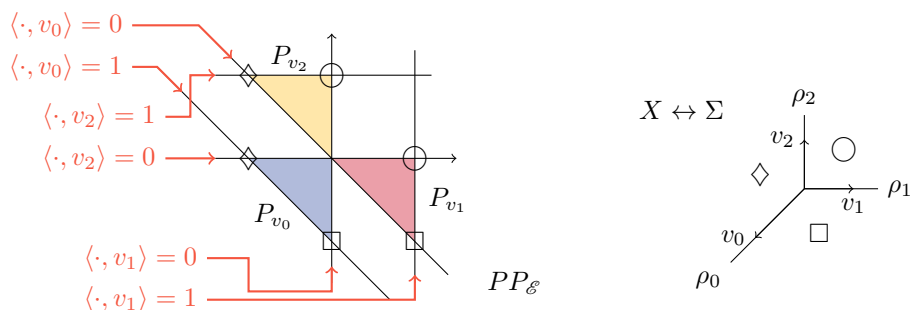
These hyperplanes should be part of those defining some polytope $P_{L_u^\sigma}$ in $PP_{\mathcal{E}}$. And in the construction of the r associated characters of $\mathbf{u}(\sigma)$, each hyperplane should be used exactly once :

$$\forall i \in \sigma(1), \quad \{k_{i,u} \mid u \in \mathbf{u}(\sigma)\} = \{1, \dots, r\} .$$

Example 1.12. Consider $X = \mathbb{P}^2$ and its tangent bundle $\mathcal{E} = \mathcal{T}_X$. The \mathbb{Z} -filtrations are

$$\text{for } i = 0, 1, 2, \quad E^i(j) = \begin{cases} \mathbb{C}^2 & \text{if } j < 0 \\ \langle v_i \rangle & \text{if } 0 \leq j < 1 \\ \{0\} & \text{otherwise} \end{cases} .$$

The ground set is $G(\mathcal{E}) = \{v_0, v_1, v_2\} = \Sigma(1)$ and the parliament $PP_{\mathcal{E}}$ is the following.



We associate to each maximal cone σ a symbol (say \square) and we represent the associated characters $\mathbf{u}(\sigma)$ by r symbols \square on the parliament. The compatibility condition (CC) is verified. \diamond

2 Stability of equivariant vector bundles

In this section, we propose a necessary and sufficient condition for the (semi)stability of a toric vector bundle \mathcal{E} in terms of its parliament of polytopes $PP_{\mathcal{E}}$. This condition can be either seen as an algorithm to check stability or as a visual property on the parliament $PP_{\mathcal{E}}$.

2.1 Definition

We start by the definition of slope for equivariant coherent sheaves on smooth complete toric varieties. It leads us to the definition of stability for toric vector bundles. The slope of an equivariant coherent sheaf \mathcal{E} depends on the choice of a polarization (an ample divisor H or more generally a movable curve α positive for a certain volume form) of the variety X that we leave aside for now and on which we go back in the next subsection.

Definition 2.1. *Let X be a smooth complete toric variety of dimension n and let α be a polarization of X . The slope of any equivariant coherent sheaf \mathcal{E} with respect to α is*

$$\mu_\alpha(\mathcal{E}) := \frac{c_1(\mathcal{E}) \cdot \alpha}{\text{rk}(\mathcal{E})} \in \mathbb{Q} ,$$

where c_1 is the first Chern class of \mathcal{E} . The dependence on α is often omitted.

Definition 2.2. An equivariant bundle \mathcal{E} is α -semistable if for any nonzero subsheaf $\mathcal{F} \subseteq \mathcal{E}$, the respective slopes satisfy the inequality

$$\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E}) .$$

It is α -stable if, in addition, for any nonzero subsheaf $\mathcal{F} \subsetneq \mathcal{E}$, the strict inequality $\mu_\alpha(\mathcal{F}) < \mu_\alpha(\mathcal{E})$ holds.

An equivariant bundle \mathcal{E} is α -polystable if it is a direct sum of α -stable bundles.

Remark 2.3. As explained in Remark 2.4 of [HNS22], to prove that some equivariant bundle \mathcal{E} is α -semistable, it is sufficient to verify that $\mu_\alpha(\mathcal{F}) < \mu_\alpha(\mathcal{E})$ holds for every equivariant saturated subsheaves \mathcal{F} .

We will need the following combinatorial description of equivariant saturated subsheaves of \mathcal{E} .

Theorem 2.4 (Proposition 2.3 of [HNS22] or Corollary 0.0.2. [DDK21]). *Let \mathcal{E} be a rank r -equivariant vector bundle on a smooth complete toric variety X . Via the Klyachko classification, \mathcal{E} corresponds to a \mathbb{Z} -filtration $(E^i(j))_{j \in \mathbb{Z}}$ of $E \cong \mathbb{C}^r$ for each ray $\rho_i \in \Sigma(1)$.*

The equivariant saturated subsheaves \mathcal{F} of \mathcal{E} are then in one-to-one correspondence with the subfiltrations

$$(F^i(j) = E^i(j) \cap F)_{j \in \mathbb{Z}} \quad \text{of} \quad (E^i(j))_{j \in \mathbb{Z}} ,$$

for some vector subspace $F \subseteq E$.

Remark 2.5. In the previous theorem, the subspace $F \subset E$ is enough to define \mathcal{F} as an equivariant saturated subsheaf of \mathcal{E} . Nevertheless, the equivariant saturated subsheaves \mathcal{F} of \mathcal{E} do not exactly correspond to the flats of $M(\mathcal{E})$. It might be that

$$\langle G(\mathcal{E}) \cap F \rangle \subsetneq F .$$

We prove in a next subsection that we may restrict the stability problem to verifying that equivariant saturated subsheaves coming from flats of $M(\mathcal{E})$ have slope $\mu_\alpha(\mathcal{F})$ smaller than the slope $\mu_\alpha(\mathcal{E})$ of \mathcal{E} .

For simplicity, we extend the definition of parliaments of toric bundles (Definition 1.10) to equivariant saturated subsheaves of toric bundles.

Definition 2.6. *Let \mathcal{E} be a toric vector bundle on X and let \mathcal{F} be an equivariant saturated subsheaf of \mathcal{E} . The parliament of polytopes of \mathcal{F} is composed of the set of indexed polytopes $(P_g)_{g \in G(\mathcal{F})}$ defined as*

$$P_g := \{ m \in \mathbb{R}^d \mid \forall \rho_i \in \Sigma(1), \langle m, v_i \rangle \leq \max \{ j \mid g \in F^i(j) \} \} ,$$

where the ground set $G(\mathcal{F})$ is seen modulo isomorphism of matroids of type (\star) .

2.2 Polarization

Usually, the slope of a toric vector bundle on X is defined with respect to some ample divisor H called a polarization of X . More precisely, it involves the self-intersection product H^{d-1} .

Proposition 2.7. *There exists a one-to-one correspondence:*

$$\{ \text{polarized toric varieties} \} \rightarrow \{ \text{lattice polytopes} \}$$

that associate to an ample divisor H on a toric variety X its Newton polytope P_D .

Proof. See Paragraph 2.1.2 of [Ale15]. \square

Remark 2.8. The toric variety X being fixed, the set of polarizations of X can be extended from ample divisors H to movable divisors L on X . This way the movable divisor L polarizes X as well as the blow up of X making L ample.

Besides, consider a movable curve class α and require it to be positive along a spanning set of rays. It comes from Theorem 3.12, Lemma 4.1 and Theorem 4.2 of [LX19] that α can be written as

$$\alpha = \langle L_\alpha^{d-1} \rangle$$

where L_α is a unique big and movable divisor and $\langle \rangle$ is the Boucksom positive product¹, which identifies with the intersection product on ample classes. It justifies that in the definition of slope, H^{d-1} may be replaced by a movable curve α positive along a spanning set of rays.

The idea of generalizing slopes to movable curves came to Greb, Kebekus and Peternell (see [GKP16]).

Definition 2.9. Let X be a smooth complete toric variety with fan Σ .

A polarization on X is a movable curve α positive along a spanning set of rays of Σ .

By Minkowski's theorem for polytopes (see [Sch93] Section 7), the data of a polarization α is equivalent to the data of its weights, defined as follows.

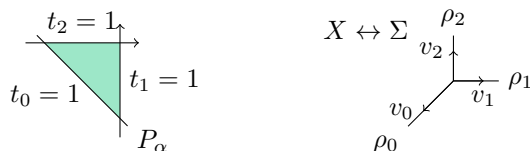
Definition 2.10. By Remark 2.8, a movable curve α on X , positive along a spanning set of rays, can be written as the positive self-intersection of a unique big and movable divisor L_α

$$\alpha = \langle L_\alpha^{d-1} \rangle .$$

Let us denote P_{L_α} the Newton polytope of L_α and write f_i for the volume of the face of P_{L_α} which has external normal vector v_i . We call the weights of α the numbers

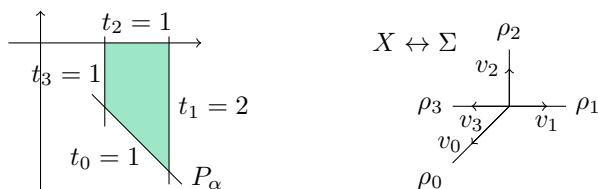
$$\text{for } \rho_i \in \Sigma(1), \quad t_i = f_i \frac{(d-1)!}{\|v_i\|} \geq 0 .$$

Example 2.11. On \mathbb{P}^2 , the polarization $\alpha = D_0 = \langle D_0^{2-1} \rangle$ has weights $(1, 1, 1)$.



\diamond

Example 2.12. The polarization $\alpha = 2D_1 - D_3$ on $\text{Bl}_{[0:1:0]}\mathbb{P}^2$ has weights $(1, 2, 1, 1)$.



\diamond

¹developed in [BDPP13] and constructed algebraically in [BFJ09]

2.3 Slope and weights

In this subsection, we consider a rank l -equivariant saturated subsheaf \mathcal{F} of a toric vector bundle \mathcal{E} on a toric variety X defined by the filtrations

$$F^i(j) = \begin{cases} F & \text{if } j \leq A_1^i \\ \cdots & \\ \langle u_i \rangle & \text{if } A_{l-1}^i < j \leq A_l^i \\ \{0\} & \text{if } A_l^i < j. \end{cases} .$$

We reformulate the slope of \mathcal{F} . In Proposition 3.1 of [Pay08], Payne expressed the Chern classes of any equivariant vector bundle. Kool extended this result, computing the first Chern class of any equivariant coherent sheaf (see Corollary 3.18 of [Koo11]). We may reformulate it as follows.

Theorem 2.13. *The first Chern class of \mathcal{F} is given by*

$$c_1(\mathcal{F}) = \sum_{\rho_i \in \Sigma(1)} \left(\sum_{k=1}^l A_k^i \right) D_i .$$

Proposition 2.14. *If α has weights $(t_i)_{\rho_i \in \Sigma(1)}$ (see Definition 2.10), then the slope of \mathcal{F} is*

$$\mu_\alpha(\mathcal{F}) = \sum_i \left(\frac{1}{r} \sum_{k=1}^l A_k^i \right) t_i .$$

Proof. In Section 4 of [LX19], Lehmann and Xiao show that the numbers t_i are the intersection numbers $D_i \cdot \alpha$. We then have that

$$\text{rk}(\mathcal{F})\mu_\alpha(\mathcal{F}) = c_1(\mathcal{F}) \cdot \alpha = \sum_i \left(\sum_{k=1}^l A_k^i \right) D_i \cdot \alpha = \sum_i \left(\sum_{k=1}^l A_k^i \right) t_i . \quad \square$$

2.4 The average polytope

In this subsection, we consider a non-zero equivariant saturated subsheaf \mathcal{F} of a toric vector bundle \mathcal{E} on a toric variety X . By Theorem 2.4, it is defined by filtrations

$$F^i(j) = \begin{cases} F & \text{if } j \leq A_1^i \\ \cdots & \\ \langle u_i \rangle & \text{if } A_{l-1}^i < j \leq A_l^i \\ \{0\} & \text{if } A_l^i < j. \end{cases}$$

where $F^i(j) = E^i(j) \cap F$, the $(E^i(j))_{\rho_i \in \Sigma(1), j \in \mathbb{Z}}$ being the defining filtrations of \mathcal{E} . The slope of \mathcal{F} may be visualized using the notion of average polytope.

Definition 2.15. *We define the average polytope $P_{\mathcal{F}}$ of \mathcal{F} to be the Newton polytope associated to the divisor*

$$\frac{c_1(\mathcal{F})}{r} = \sum_i \left(\frac{1}{r} \sum_{k=1}^l A_k^i \right) D_i .$$

The average polytope $P_{\mathcal{F}}$ of \mathcal{F} is a visualization of the slopes of \mathcal{F} . In the following proposition, we recover, for any polarization α of X , the α -slope of \mathcal{F} , only by means of the average polytope $P_{\mathcal{F}}$.

Proposition 2.16. *The α -slope of an equivariant saturated subsheaf $\mathcal{F} \subset \mathcal{E}$ with average polytope $P_{\mathcal{F}} = P_D$ where $D = \sum_{\rho_i \in \Sigma(1)} a_i D_i$ is*

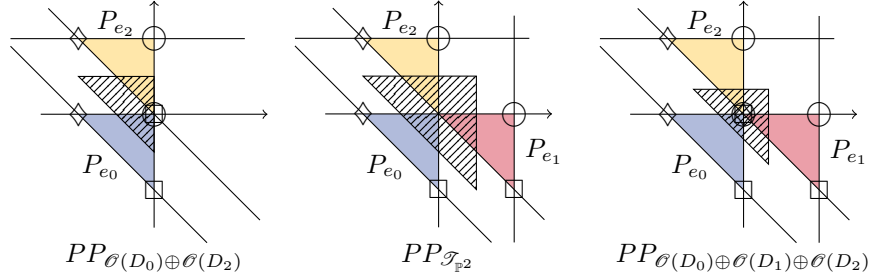
$$\mu_{\alpha}(\mathcal{F}) = \sum_{\rho_i \in \Sigma(1)} a_i t_i .$$

Proof. See Proposition 2.14 and Definition 2.15. □

Example 2.17. The average polytopes of

$$\mathcal{F} = \mathcal{O}(D_0) \oplus \mathcal{O}(D_2), \quad \mathcal{F} = \mathcal{T}_{\mathbb{P}^2} \quad \text{and} \quad \mathcal{F} = \mathcal{O}(D_0) \oplus \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$$

are the hashed polytopes in the following pictures



The difference between the two last pictures is the following. The hyperplanes forming a '+' in the middle have different multiplicities (1 or 2). Indeed, the tangent bundle $\mathcal{T}_{\mathbb{P}^2}$ has rank $l = 2$ whereas the splitting bundle $\mathcal{O}(D_0) \oplus \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$ has rank $l = 3$. The slopes with respect to the polarization α with weights $(1, 1, 1)$ (see Example 2.11) are

$$\begin{cases} \mu_{\alpha}(\mathcal{O}(D_0) \oplus \mathcal{O}(D_2)) = \frac{1}{2} + 0 + \frac{1}{2} = 1 ; \\ \mu_{\alpha}(\mathcal{T}_{\mathbb{P}^2}) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} ; \\ \mu_{\alpha}(\mathcal{O}(D_0) \oplus \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 . \end{cases}$$

◇

Remark 2.18. Consider two equivariant saturated subsheaves \mathcal{F}_1 and \mathcal{F}_2 with average polytopes P_1 and P_2 satisfying $P_2 = \gamma P_1 + v$, where γ is a coefficient of dilatation and v a vector of translation. For any polarization α , we have the following relation between their slopes

$$\mu_{\alpha}(\mathcal{F}_1) = \gamma \times \mu_{\alpha}(\mathcal{F}_2) .$$

Indeed, let us denote by $\sum_{\rho_i \in \Sigma(1)} a_i D_i$ and $\sum_{\rho_i \in \Sigma(1)} b_i D_i$ the divisors associated to P_1 and P_2 . We then have that $a_i = \gamma b_i + \langle v, v_i \rangle$ for all $\rho_i \in \Sigma(1)$ so that

$$\mu_{\alpha}(\mathcal{F}_1) = \gamma \times \mu_{\alpha}(\mathcal{F}_2) + \left(\sum_{\rho_i \in \Sigma(1)} \langle v, v_i \rangle D_i \right) \cdot \alpha .$$

By the exact sequence of Theorem 4.1.3 [CLS11], we have $\sum_{\rho_i \in \Sigma(1)} \langle v, v_i \rangle D_i \equiv 0$. As a consequence, we may look at the average polytope as a polytope modulo translation.

2.5 Reduction to flats of $M(\mathcal{E})$

In this subsection, we fix a polarization α on X . Every slope will be computed with respect to α . By the following Proposition we intend to prove that the (α) -stability

of toric vector bundle \mathcal{E} depends solely on the slope of its equivariant subsheaves given by flats of $M(\mathcal{E})$.

Proposition 2.19. *Let \mathcal{E} be a toric vector bundle on X and consider its parliament of polytopes*

$$PP_{\mathcal{E}} = \{P_e \mid e \in G(\mathcal{E})\} .$$

For any rank l -equivariant saturated subsheaf \mathcal{F}_0 of \mathcal{E} , there exists a flat $G(\mathcal{E}) \cap F_1$ of $M(\mathcal{E})$ defining an equivariant saturated subsheaf \mathcal{F}_1 having slope $\mu(\mathcal{F}_1)$ greater than $\mu(\mathcal{F}_0)$.

The following proof highly requires Subsection 1.2.

Proof of the proposition. By Theorem 2.4, the defining filtrations of \mathcal{F}_0 are given by

$$(E^i(j) \cap F_0)_{j \in \mathbb{Z}} ,$$

where $(E^i(j))_{j \in \mathbb{Z}}$ are the Klyachko filtrations of \mathcal{E} and F_0 is a subspace of E .

First, we may consider $G(\mathcal{E})$ to be a ground set obtained by taking CB_V to be composed of a maximal number of elements in $G(\mathcal{F}_0)$ at each step of Algorithm 3.2 for

$$L(\mathcal{E}) = \left\{ V_{(j_1, \dots, j_n)} = \bigcap_{i=1}^n E^i(j_i) \mid (j_1, \dots, j_n) \in \mathbb{Z}^n \right\} .$$

Indeed if G_2 is another possible output of Algorithm 3.2 then there exists an isomorphism $\varphi : M(G(\mathcal{E})) \rightarrow M(G_2)$ of matroid of type (\star) (see Subsection 1.2). A flat $f = G(\mathcal{E}) \cap F$ is sent to a flat $f' = G_2 \cap F' = \varphi(G \cap F)$ for some vector subspace $F' \subset E$. Additionally if we denote by \mathcal{F} and \mathcal{F}' the associated equivariant saturated subsheaves, then the isomorphism φ respecting the steps of Algorithm 3.2, we have that

$$\mu(\mathcal{F}) = \mu(\mathcal{F}') .$$

This choice of ground set allows us to have that $G(\mathcal{E}) \cap F_0 \subseteq G(\mathcal{F}_0)$.

Now, we may distinguish between cases between $G(\mathcal{E}) \cap F_0 = G(\mathcal{F}_0)$ and $G(\mathcal{E}) \cap F_0 \neq G(\mathcal{F}_0)$. In the first case, $G(\mathcal{F}_0)$ is a flat of $G(\mathcal{E})$ and by considering $F_1 = F_0$, we directly obtain the conclusion. We may hence assume that $G(\mathcal{E}) \cap F_0 \neq G(\mathcal{F}_0)$.

Second, we will need a series of lemmas. We use the notations of Subsection 1.2 and denote by $CB_{V \cap F_0}$ the basis appended to G_{F_0} at Step $V \cap F_0$ of Algorithm 3.2 for \mathcal{F}_0 .

Lemma 2.20. *Consider an index $g \in G(\mathcal{F}_0)$ of $P_g \in PP_{\mathcal{F}_0}$. We have that*

$$g \in CB_{V \cap F_0} \Leftrightarrow V^g = V , \quad \text{where} \quad V^g = \bigcap_{i=1}^n \text{span}(h \in G(\mathcal{E}) \mid P_h \geq_i P_g) ,$$

and $P_h \geq_i P_g$ means that the hyperplanes defining respectively P_h and P_g in the direction v_i are of the form

$$\langle \cdot, v_i \rangle = j_h \quad \text{and} \quad \langle \cdot, v_i \rangle = j_g , \quad \text{with} \quad j_h \geq j_g .$$

As a consequence, we have that

$$CB_{V^g \cap F_0} = \{g' \mid P_g = P_{g'}\} .$$

Proof. By definition of the parliament $PP_{\mathcal{F}_0}$, if $g \in G(\mathcal{F}_0)$ then the hyperplane $(\langle \cdot, v_i \rangle = j_g)$ of $M_{\mathbb{R}}$ is defining P_g if and only if $g \in E^i(j_g) \setminus E^i(j_g - 1)$.

Now consider $V = \bigcap_{i=1}^n E^i(j_g)$ such that $g \in CB_{V \cap F_0}$. The elements h with $P_h \geq_i P_g$ for all $i \in \{1, \dots, n\}$ are those arising at a step $V' \subseteq V = \bigcap_{i=1}^n E^i(j_g)$ and thus, by construction of the ground set $G(\mathcal{E})$, are exactly the elements generating V .

Definition 2.21. We say that $F \in G(l, E)$ is a specialization of F_0 with respect to the matroid $M(\mathcal{E})$ if for every $V \in L(\mathcal{E})$, we have that

$$\dim(F \cap V) \geq \dim(F_0 \cap V) .$$

Lemma 2.22. If $G(\mathcal{E}) \cap F_0 \neq G(\mathcal{F}_0)$ then F_0 has a specialization F such that

$$G(\mathcal{E}) \cap F \supsetneq G(\mathcal{E}) \cap F_0 .$$

Proof. Consider the first element $g \in G(\mathcal{F}_0)$, in the order given by Algorithm 3.2 for \mathcal{F}_0 , not belonging to $G(\mathcal{E})$. It means that there exists some elements $g_1, \dots, g_s \in G(\mathcal{E})$ not all in $G(\mathcal{F}_0)$ (say $g_1 \notin G(\mathcal{F}_0)$) generating g such that V^{g_1}, \dots, V^{g_s} arise before V^g in Algorithm 3.2 for \mathcal{E} . Remark that we may only consider g_i 's in V^g (the others are not useful to generate g). First, the space $V^{g_i} \cap V^g$, appearing before V^{g_i} and containing g_i , is equal to V^{g_i} . We thus have that $V^{g_1}, \dots, V^{g_s} \subset V^g$.

Let H be any complementary space to $\langle g, g_1 \rangle$ in E containing $\langle G(\mathcal{E}) \cap F_0 \rangle$ (it is possible by the assumption $G(\mathcal{E}) \cap F_0 \neq G(\mathcal{F}_0)$), let ρ be the rotation of E fixing H and sending g to g_1 and let finally $F = \rho(F_0)$ be the image of F_0 by ρ .

We want to prove that for every $V \in L(\mathcal{E})$, we have that

$$\dim(F \cap V) \geq \dim(F_0 \cap V) .$$

For any $V \in L(\mathcal{E})$, let us distinguish three cases:

- if $V \subset H$ then $\dim(F \cap V) = \dim(F_0 \cap V)$.
- if $V \cap \langle g, g_1 \rangle = \langle g, g_1 \rangle$ then $\dim(F \cap V) = \dim(F_0 \cap V)$.
- if $V \cap \langle g, g_1 \rangle = \langle h \rangle$ then $\dim(F \cap V) = \dim(F_0 \cap V) + \varepsilon$,
with $\varepsilon = \begin{cases} 0 & \text{if } h \notin \{g, g_1\} \\ -1 & \text{if } h = g \\ 1 & \text{if } h = g_1 \end{cases}$

Consider $V \in L(\mathcal{E})$. If $F_0 \cap V$ appears before $F_0 \cap V^g$ in Algorithm 3.2 for \mathcal{F}_0 ,

$$\text{then } F_0 \cap V \subset \langle G(\mathcal{E}) \cap F_0 \rangle \subset H \quad \text{so that } \dim(F \cap V) = \dim(F_0 \cap V) .$$

Assume now that $F_0 \cap V$ appears after $F_0 \cap V^g$ in Algorithm 3.2 for F_0 .

If $g \in V \cap F_0$ then $F_0 \cap V^g \subset F_0 \cap V$ (the space $F_0 \cap V \cap V^g$ appearing before $F_0 \cap V^g$ and containing g , is equal to $F_0 \cap V^g$). As $V^{g_1} \subset V^g$, we are in the second case and

$$\dim(F \cap V) = \dim(F_0 \cap V) .$$

If $g \notin V \cap F_0$ then V cannot correspond to the third case with $h = g$ and

$$\dim(F \cap V) \geq \dim(F_0 \cap V) .$$

It remains to mention that

$$G(\mathcal{E}) \cap F \supsetneq (G(\mathcal{E}) \cap F_0) \cup \{g_1\} \supsetneq G(\mathcal{E}) \cap F_0 .$$

Lemma 2.23. If F_0 is a specialization of F then the slope of \mathcal{F}_0 is greater than the slope of \mathcal{F}

$$\mu(\mathcal{F}_0) \geq \mu(\mathcal{F}) .$$

Proof. If we fix a ray $\rho_i \in \Sigma(1)$ then for any $j \in \mathbb{Z}$ we have that

$$\dim(F \cap E^i(j)) \geq \dim(F_0 \cap E^i(j)) .$$

Another way of saying it is that for all $k \in \{1, \dots, l\}$, if A_k^i and B_k^i are the remaining gaps after intersecting the filtration $(E^i(j))_{j \in \mathbb{Z}}$ respectively with F_0 and F then we have that $A_k^i \geq B_k^i$. We conclude by Proposition 2.14.

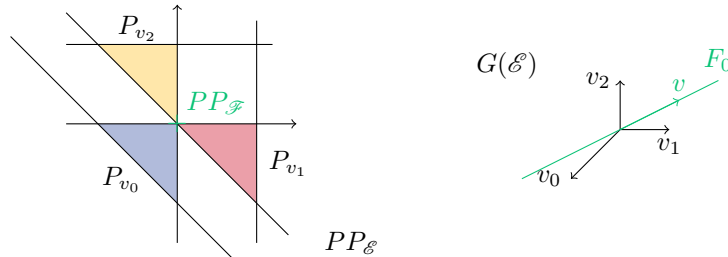
Finally, let us end the proof of Proposition 2.19. We consider a specialization F of F_0 . If $G(\mathcal{E}) \cap F = G(\mathcal{F})$ then we define $F_1 = F$ and it is finished. Otherwise, we consider a specialization of F as in Lemma 2.22. We continue recursively until we obtain F_1 satisfying $G(\mathcal{E}) \cap F_1 = G(\mathcal{F}_1)$. By Lemma 2.23, we have the inequality

$$\mu(\mathcal{F}_1) \geq \mu(\mathcal{F}_0) .$$

□

Remark 2.24. This is a real improvement because finding every flat on $M(\mathcal{E})$ is algorithmic: take the closure of any subset of the finite set $G(\mathcal{E})$.

Example 2.25. Consider the tangent bundle $\mathcal{E} = \mathcal{T}_{\mathbb{P}^2}$ on \mathbb{P}^2 (see Example 1.12). An equivariant saturated subsheaf $\mathcal{F}_0 \subset \mathcal{E}$ of rank 1 corresponds to a subspace $F_0 = \langle v \rangle$. If $v \notin G(\mathcal{E})$ then $G(\mathcal{F}_0) = \{v\}$ and the parliament of \mathcal{F} is composed of exactly one polytope $P_v = \{0\}$. Here, taking a specialization of F_0 corresponds to rotate F_0 to $\langle v_0 \rangle$, $\langle v_1 \rangle$ or $\langle v_2 \rangle$.



The parliament of polytopes of this specialization would be composed either of P_{v_0} , P_{v_1} or P_{v_2} . Visually, the average polytope is indeed bigger after specialization. ◊

2.6 Result

In this subsection, we state our necessary and sufficient condition for stability of toric vector bundles. It involves comparing polytopes by means of the following order.

Definition 2.26. Let X be a smooth complete toric variety and α a polarization with weights

$$t_i := D_i \cdot \alpha = f_i \frac{(d-1)!}{\|v_i\|} \quad \text{for every ray } \rho_i \in \Sigma .$$

We define a total order on Newton polytopes of X by saying

$$P_1 <_{\alpha} P_2 \quad \iff \quad \sum_i a_1^i t_i < \sum_i a_2^i t_i$$

where for $j \in \{1, 2\}$, P_j is the Newton polytope of the divisor $\sum_i a_j^i D_i$.

Theorem 2.27. *Let X be a smooth complete toric variety. A toric vector bundle \mathcal{E} on X is α -(semi)stable if and only if, for any non-zero flat $f \subsetneq G(\mathcal{E})$ of $M(\mathcal{E})$, we have*

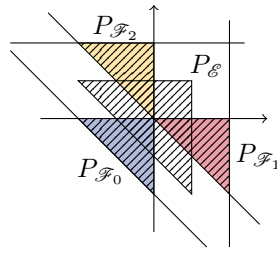
$$P_{\mathcal{F}} <_{\alpha} P_{\mathcal{E}} \quad (\text{resp. } P_{\mathcal{F}} \leq_{\alpha} P_{\mathcal{E}}) ,$$

where \mathcal{F} is the equivariant saturated sheaf corresponding by Theorem 2.4 to the linear subspace $\langle f \rangle \subset E$.

Example 2.28. Consider the tangent bundle $\mathcal{E} = \mathcal{T}_{\mathbb{P}^2}$ on \mathbb{P}^2 . The non-trivial flats of $G(\mathcal{E}) = \{v_0, v_1, v_2\}$ are

$$f_0 = \{v_0\}, f_1 = \{v_1\} \text{ and } f_2 = \{v_2\}.$$

They correspond to the three non-trivial equivariant saturated subsheaves \mathcal{F}_0 , \mathcal{F}_1 and \mathcal{F}_2 of \mathcal{E} . Let us look at the average polytopes:



By Theorem 2.27, the fact that $P_{\mathcal{F}_0}, P_{\mathcal{F}_1}, P_{\mathcal{F}_2} <_{\alpha} P_{\mathcal{E}}$ proves that \mathcal{E} is α -stable. \diamond

Corollary 2.29. *Consider any smooth complete variety X of dimension d and its tangent bundle on $\mathcal{E} = \mathcal{T}_X$. Assume that we have*

$$\text{for all } \rho_i \in \Sigma(1), -\rho_i \notin \Sigma(1) .$$

Then a necessary condition for \mathcal{E} to be semistable with respect to a polarization

$$\alpha = \langle L^{d-1} \rangle$$

is that α corresponds to $(t_i)_{\rho_i \in \Sigma(1)}$ such that for all ray $\rho_{i_0} \in \Sigma(1)$

$$t_{i_0} \leq \frac{\sum_{\rho_i \in \Sigma(1)} t_i}{d} .$$

If X has dimension 2, this condition is also sufficient.

Proof. The Klyachko classification of \mathcal{E} is given by

$$\text{for } \rho_i \in \Sigma(1), E^i(j) = \begin{cases} \mathbb{C}^d & \text{if } j < 0 \\ \langle v_i \rangle & \text{if } 0 \leq j < 1 \\ \{0\} & \text{otherwise} \end{cases} .$$

The ground set of the matroid $M(\mathcal{E})$ associated to \mathcal{E} is given by

$$G(\mathcal{E}) = \{v_i \mid \rho_i \in \Sigma(1)\} .$$

The parliament $PP_{\mathcal{E}}$ is composed of the polytopes

$$P_{v_i} = (\langle \cdot, v_i \rangle \leq 1) \cap \bigcap_{\rho_j \neq \rho_i \in \Sigma(1)} (\langle \cdot, v_j \rangle \leq 0) .$$

Each subset $\{v_i\} \subset G(\mathcal{E})$ is a flat of $M(\mathcal{E})$ and its associated equivariant saturated subsheaf $\mathcal{F}_i \subset \mathcal{E}$ has slope

$$\mu(\mathcal{F}_i) = t_i \quad \text{while} \quad \mu(\mathcal{E}) = \frac{\sum t_i}{d} .$$

If X has dimension 2, there exists no other flats. \square

Remark 2.30. In particular, we recover the stability of the tangent bundle of \mathbb{P}^d . Indeed, the Picard number of \mathbb{P}^d being 1 and P_{D_0} having $f_i/||v_i|| = 1$ (see Definition 2.10), every polarization corresponds to weights $t_i = a$ for some $a \in \mathbb{R}_+$.

3 Subparliaments of polytopes

In this section, we first define subparliaments of polytopes and identify them with parliaments of equivariant subbundles. Second, we give an example of toric subbundle which is not a direct factor.

3.1 Subparliaments as parliaments of toric subbundles

Let us state our definition of subparliament of a parliament of polytopes.

Definition 3.1. *Let $f = G(\mathcal{E}) \cap F$ be a flat of the matroid $M(\mathcal{E})$ such that there exists a compatible basis $(B_\sigma)_{\sigma \in \Sigma(d)}$ for \mathcal{E} satisfying*

$$\text{for all } \sigma \in \Sigma(d), \quad \#B_\sigma \cap f = \dim F .$$

Then we call f a compatible flat, and we call the subset of polytopes of $PP_{\mathcal{E}}$ indexed by elements of $f = F \cap G(\mathcal{E})$

$$\{(P_e, e) \mid e \in f\}$$

a subparliament of the parliament of polytopes \mathcal{E} .

Equivariant subbundles have been combinatorially described in [KD19]. We translate their results in terms of parliaments of polytopes.

Proposition 3.2. *Via the Klyachko classification, a rank r -equivariant vector bundle \mathcal{E} on X corresponds to a \mathbb{Z} -filtration $(E^i(j))_{j \in \mathbb{Z}}$ of $E \cong \mathbb{C}^r$ for each $\rho_i \in \Sigma(1)$.*

The equivariant subbundles \mathcal{F} of \mathcal{E} are then in one-to-one correspondence with the subfiltrations

$$(F^i(j) = E^i(j) \cap F)_{j \in \mathbb{Z}} \quad \text{of} \quad (E^i(j))_{j \in \mathbb{Z}} ,$$

for some vector subspace $F \subseteq E$ such that there exist ground sets $G(\mathcal{F})$ and $G(\mathcal{E})$ resulting from Algorithm 3.2 for $L(\mathcal{F})$ and $L(\mathcal{E})$ with

$$G(\mathcal{E}) \cap F = G(\mathcal{F})$$

and such that there exist compatible bases $(B_\sigma)_{\sigma \in \Sigma(d)}$ for \mathcal{E} which, restricted to F , satisfy the compatibility condition (CC) for \mathcal{F} .

Proof. By Proposition 4.1.1 of [KD19], the equivariant subbundles \mathcal{F} of \mathcal{E} are then in one-to-one correspondence with the subfiltrations

$$(F^i(j) = E^i(j) \cap F)_{j \in \mathbb{Z}} \quad \text{of} \quad (E^i(j))_{j \in \mathbb{Z}} ,$$

for some vector subspace $F \subseteq E$ such that $\{F, \{E^i(j)\}_{\rho_i \in \sigma(1)}\}$ of E forms a distributive lattice.

Let us show that there exist ground sets $G(\mathcal{F}), G(\mathcal{E})$ obtained by Algorithm 3.2 for $L(\mathcal{F}), L(\mathcal{E})$ with

$$G(\mathcal{E}) \cap F = G(\mathcal{F}) .$$

Consider the ground set $G(\mathcal{E})$ obtained by taking a maximal number of vectors in F in Algorithm 3.2. If $V_1, V_2 \in L(\mathcal{E})$, let us say that $V_1 < V_2$ if V_1 appears before V_2 in Algorithm 3.2. We denote by G_V the intersection of V with the temporary ground set at the beginning of Step $V \in L(\mathcal{E})$. During Step V , we need to add a basis CB_V of any complement of $\langle G_V \rangle$ in V to the ground set. We choose this basis to be composed of a maximal number of vectors in F so that we have

$$\begin{aligned} V \cap F &= (\langle G_V \rangle + \langle CB_V \rangle) \cap F \\ &= \left(\sum_{\substack{V' = V_1 \cap V \\ V_1 < V}} V' + \langle CB_V \rangle \right) \cap F \\ &= \left(\sum_{\substack{V' = V_1 \cap V \\ V_1 < V}} V' \right) \cap F + \langle CB_V \cap F \rangle \\ &= \sum_{\substack{V' = V_1 \cap V \\ V_1 < V}} (V' \cap F) + \langle CB_V \cap F \rangle . \end{aligned}$$

The last step comes from $\{F, \{E^i(j)\}_{\rho_i \in \sigma(1)}\}$ forming a distributive lattice of E . By induction on the $V \in L(\mathcal{E})$ appearing in Algorithm 3.2, $G_V \cap F$ generates every $(V_1 \cap V) \cap F$ for $V_1 < V$. We finally obtain that $G(\mathcal{E}) \cap F$ generates every $V \cap F$ for $V \in L(\mathcal{E})$.

Now if we denote by $G_{V \cap F}^{\mathcal{F}}$ the intersection of $V \cap F$ with the temporary ground set at the beginning of Step $V \cap F \in L(\mathcal{F})$ of Algorithm 3.2 (applied to \mathcal{F}), then

$$V \cap F = \langle G_{V \cap F}^{\mathcal{F}} \rangle + \langle CB_{V \cap F}^{\mathcal{F}} \rangle = \sum_{\substack{V' \cap F = (V_1 \cap F) \cap (V \cap F) \\ (V_1 \cap F) < (V \cap F)}} (V' \cap F) + \langle CB_{V \cap F}^{\mathcal{F}} \rangle .$$

Moreover, since $V_1 < V \Rightarrow (V_1 \cap F < V \cap F)$ and $(V_1 \cap F < V \cap F) \Rightarrow (V_1 \cap V < V)$, we have

$$\begin{aligned} &\{V' \cap F \mid V' = V_1 \cap V \text{ and } V_1 < V\} \\ &= \{V' \cap F \mid V' \cap F = (V_1 \cap F) \cap (V \cap F) \text{ and } (V_1 \cap F) < (V \cap F)\} . \end{aligned}$$

We may take $CB_{V \cap F}^{\mathcal{F}}$ to be $CB_V \cap F$, and we finally obtain $G(\mathcal{F}) = G(\mathcal{E}) \cap F$.

Moreover, for each cone $\sigma \in \Sigma(d)$, any compatible basis B_σ^F of F extends to a compatible basis B_σ of E . Indeed, to construct a compatible basis B_σ (resp. B_σ^F), we apply Algorithm 3.2 to

$$L_\sigma(\mathcal{E}) = \left\{ \bigcap_{\rho_i \subset \sigma} E^i(j_i) \mid (j_i)_{\rho_i \subset \sigma} \right\} \quad (\text{resp. } L_\sigma(\mathcal{F}) = \{V \cap F \mid V \in L_\sigma(\mathcal{E})\}) ,$$

taking CB_V included in G (resp. taking $CB_{V \cap F}^{\mathcal{F}}$ included in $G \cap F$). Extending the

bases B_σ^F is possible because G generates each $V \in L_\sigma(\mathcal{E}) \subset L(\mathcal{E})$ as well as $G \cap F$ generates each $V \cap F \in L_\sigma(\mathcal{F}) \subset L(\mathcal{F})$. \square

Theorem 3.3. *Let \mathcal{E} be a toric vector bundle and let \mathcal{F} be the equivariant subbundle of \mathcal{E} corresponding by Klyachko's theorem to the filtrations $(E^i(j) \cap F)_{j \in \mathbb{Z}}$ for each $\rho_i \in \Sigma(1)$.*

Then, for any choice of ground set $G(\mathcal{E})$, the parliament of \mathcal{F} is a subparliament

$$PP_{\mathcal{F}} = \{(P_e, e) \mid e \in f'\} \subset PP_{\mathcal{E}},$$

where $f' = G(\mathcal{E}) \cap F'$ is some compatible flat with $\dim(F') = \dim(F)$.

Proof. First, assume that the ground set $G(\mathcal{E})$ for the parliament of \mathcal{E} is obtained via Algorithm 3.2 by taking at each step a maximal number of elements in F . Then by Proposition 3.2, $G(\mathcal{F}) = G(\mathcal{E}) \cap F$ is a possible output for Algorithm 3.2 applied to $L(\mathcal{F})$. Now if $e \in G(\mathcal{F}) = G(\mathcal{E}) \cap F$ then the polytope $P_e \in PP_{\mathcal{F}}$ and

$$\begin{aligned} P_e &= \{m \mid \forall i, \langle m, v_i \rangle \leq \max \{j \mid e \in E^i(j) \cap F\}\} \\ &= \{m \mid \forall i, \langle m, v_i \rangle \leq \max \{j \mid e \in E^i(j)\}\} \end{aligned}$$

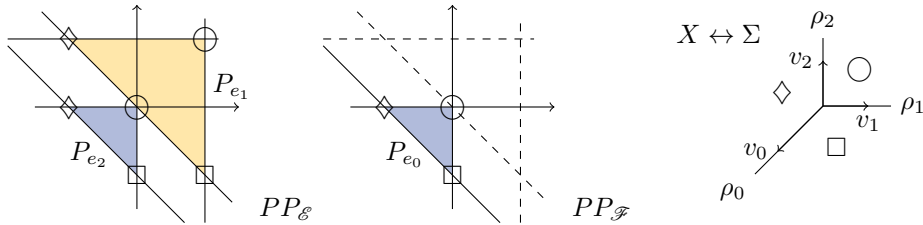
is the same as the one in the parliament of \mathcal{E} .

Second, if G_E is any other ground set obtained by Algorithm 3.2 then there is an isomorphism $\varphi : M(G(\mathcal{E})) \rightarrow M(G_E)$ of matroids compatible with Algorithm 3.2 (see Subsection 1.2). Hence, the flat $G(\mathcal{F}) = G(\mathcal{E}) \cap F$ (generating F) is sent to a flat $G_E \cap F'$ (generating F') for some vector subspace $F' \subset E$ of same dimension than F . As a consequence, the parliament of \mathcal{F} is the subparliament of \mathcal{E} corresponding to the compatible flat $G_E \cap F'$. \square

Example 3.4. Consider the toric variety $X = \mathbb{P}^2$ with T -invariant divisors D_0, D_1, D_2 and the splitting, equivariant, rank 2-vector bundle $\mathcal{E} = \mathcal{O}(D_0) \oplus \mathcal{O}(D_1 + D_2)$ on X . The corresponding \mathbb{Z} -filtrations of $E = \mathbb{C}^2$ are of the form

$$E^i(j) = \begin{cases} \mathbb{C}^2 & \text{if } j \leq 0 \\ \langle e_1 \rangle & \text{if } 0 < j \leq 1 \quad (\text{for } i = 1, 2) \\ \{0\} & \text{otherwise} \end{cases} \quad \text{and} \quad E^0(j) = \begin{cases} \mathbb{C}^2 & \text{if } j \leq 0 \\ \langle e_2 \rangle & \text{if } 0 < j \leq 1 \\ \{0\} & \text{otherwise} \end{cases}$$

where $\langle e_1 \rangle$ and $\langle e_2 \rangle$ are different lines in \mathbb{C}^2 . The ground set is $G(\mathcal{E}) = \{e_1, e_2\}$. The toric subbundle $\mathcal{F} = \mathcal{O}(D_0) \subset \mathcal{E}$ corresponds to the flat $f = \{e_2\} \subset G(\mathcal{E})$.

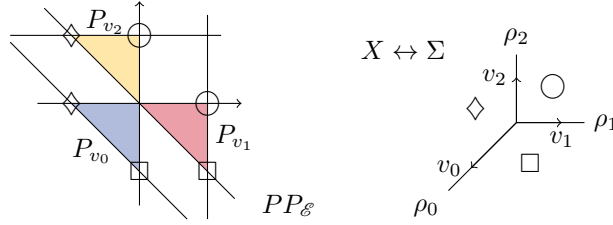


The compatibility condition (CC) for \mathcal{E} is satisfied on taking

$$B_\sigma = \{L_u^\sigma \mid u \in \mathbf{u}(\sigma)\} = \{e_1, e_2\} \quad \text{for all } \sigma \in \Sigma(d).$$

We see that for all $\sigma \in \Sigma(d)$, $B_\sigma \cap f = \{e_2\}$ generates $F = \langle f \rangle$, the flat $f = \{e_2\}$ is thus compatible. The parliament $PP_{\mathcal{F}}$ is a subparliament of $PP_{\mathcal{E}}$. \diamond

Example 3.5. Let us come back to Example 1.12 with $X = \mathbb{P}^2$ and its tangent bundle $\mathcal{E} = \mathcal{T}_{\mathbb{P}^2}$.



There exists no non-trivial equivariant subbundle \mathcal{F} of \mathcal{E} because the non-trivial flats are $f_0 = \{v_0\}$, $f_1 = \{v_1\}$ and $f_2 = \{v_2\}$ and are not compatible. \diamond

3.2 Subbundle that is not a direct factor

There exists equivariant vector bundles \mathcal{E} with a non-trivial equivariant subbundle \mathcal{F} that we cannot factorize in a direct sum of \mathcal{E} .

Example 3.6. Consider the equivariant vector bundle \mathcal{E} on \mathbb{P}^2 defined by the filtrations

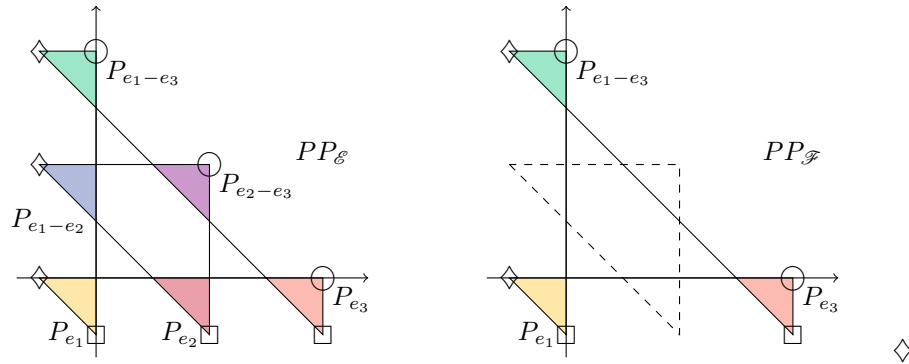
$$E^0(j) = \begin{cases} \mathbb{C}^3 & \text{if } j < -3 \\ \langle e_1, e_2 \rangle & \text{if } -3 \leq j < -1 \\ \langle e_1 \rangle & \text{if } -1 \leq j < 1 \\ \{0\} & \text{otherwise} \end{cases}, \quad E^1(j) = \begin{cases} \mathbb{C}^3 & \text{if } j < 0 \\ \langle e_2, e_3 \rangle & \text{if } 0 \leq j < 2 \\ \langle e_3 \rangle & \text{if } 2 \leq j < 4 \\ \{0\} & \text{otherwise} \end{cases}$$

and $E^2(j) = \begin{cases} \mathbb{C}^3 & \text{if } j < 0 \\ \langle e_1 - e_3, e_1 - e_2 \rangle & \text{if } 0 \leq j < 2 \\ \langle e_1 - e_3 \rangle & \text{if } 2 \leq j < 4 \\ \{0\} & \text{otherwise} \end{cases}$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. The ground set of $M(\mathcal{E})$ is

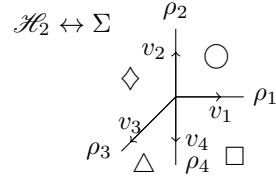
$$G(\mathcal{E}) = \{e_1, e_2, e_3, e_1 - e_2, e_1 - e_3, e_2 - e_3\}.$$

Taking $F = \langle e_2 \rangle^\perp$, we obtain an equivariant subbundle \mathcal{F} .

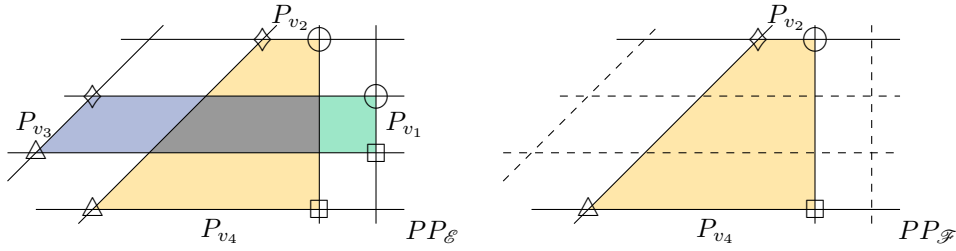


Example 3.7. The following more simple example was pointed out to me by Bivas Khan. Consider the tangent bundle $\mathcal{E} = \mathcal{T}_{\mathcal{H}_2}$ of the Hirzebruch surface, the rays of

the fan are $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, 1)$, $v_4 = (0, -1)$.



The linear space $F = \langle v_2 \rangle$ defines a subbundle \mathcal{F} and \mathcal{E} cannot be written as $\mathcal{F} \oplus \mathcal{G}$. Here we will see that the obstruction to being a direct sum comes only from the matroid (and not from the form of the polytopes). For the picture we tensorize \mathcal{E} and \mathcal{F} by the line bundle $\mathcal{O}(D_3 + D_4)$ which does not change the stability.



Remark that we could have taken the same filtrations on any complete smooth 2-dimensional toric variety having a fan with 4 rays for instance $X = \mathbb{P}^1 \times \mathbb{P}^1$. \diamond

In particular, these examples give rise to non-trivial Harder–Narasimhan filtrations

$$0 \subseteq \mathcal{F} \subseteq \mathcal{E}.$$

4 Stability of restrictions to invariant curves

It is often useful to compare the α -(semi)stability of a toric vector bundle \mathcal{E} to the (semi)stability of its restrictions $\mathcal{E}|_C$ to any torus invariant curve.

Remark 4.1. Theorem 2.5 from [BLG16] furnishes a sufficient condition to deduce from the semistability of a toric vector bundle \mathcal{E} , the semistability of its restriction $\mathcal{E}|_C$ to any invariant curves C : the characteristic class of \mathcal{E} is 0

$$\Delta(\mathcal{E}) = c_2(\mathcal{E}) - \frac{r-1}{2r}c_1(\mathcal{E})^2 = 0.$$

In this section, we explain how to see the (semi)stability of the restrictions $\mathcal{E}|_C$ of a vector bundle \mathcal{E} to a torus invariant curve given the parliament $PP_{\mathcal{E}}$.

In [DJS18] Subsection 3.1, Di Rocco, Jabbusch and Smith recover the parliament of restrictions $\mathcal{E}|_C$ to torus invariant curves from the parliament of polytopes of \mathcal{E} .

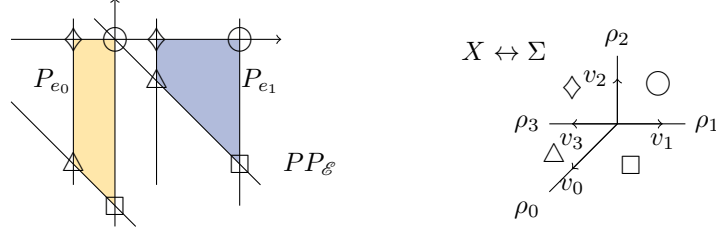
Lemma 4.2. *By the cone-orbit correspondence, torus invariant curves correspond to a cone $\tau \in \Sigma(d-1)$. Since X is complete, there are two maximal cones σ and σ' in $\Sigma(d)$ containing τ .*

The parliament of polytopes of $\mathcal{E}|_C$ is composed of the projection on τ^\perp of the line segments parallel to τ^\perp joining the associated characters in $\mathbf{u}(\sigma)$ and $\mathbf{u}(\sigma')$ (renormalized by $1/\|u_\tau\|$).

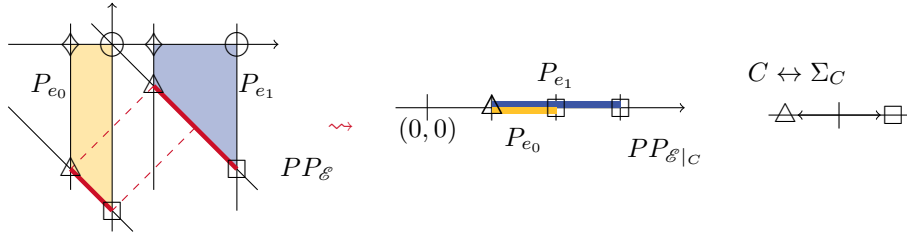
Example 4.3. Consider $X = \text{Bl}_{[0:1:0]}(\mathbb{P}^2)$ and let \mathcal{E} be the equivariant vector bundle

$$\mathcal{E} = \mathcal{O}(4D_0 + D_3) \oplus \mathcal{O}(3D_1 - D_3)$$

on X . The parliament of polytopes of \mathcal{E} is



The parliament of \mathcal{E} restricted to the torus invariant curve D_0 is obtained the following way.



The bundle $\mathcal{E}|_{D_0}$ would be stable if the segments P_{e_0} and P_{e_1} were of the same length. In this example, there is not any torus invariant curve C such that $\mathcal{E}|_C$ is semistable.

A contrario, taking the movable curve $\alpha = 2D_1 - D_3$, the corresponding $(t_i)_{\rho_i \in \Sigma(1)}$ are $t_0 = 1$, $t_1 = 2$, $t_2 = 1$ and $t_3 = 1$ (as in Example 2.12) and \mathcal{E} is α -polystable. Indeed, $\mu(\mathcal{L}_0) = 4t_0 + t_3 = 5$ is equal to $\mu(\mathcal{L}_1) = 3t_1 - t_3 = 5$. \diamond

Example 4.4. For any choice of movable curve α , the tangent bundle $\mathcal{T}_{\mathbb{P}^2}$ of \mathbb{P}^2 is an example of α -stable equivariant vector bundle \mathcal{E} with a non stable restriction to a torus invariant curve. \diamond

5 Alternative definition of parliaments

We propose definitions for the parliament of polytopes of a toric vector bundle \mathcal{E} . For each definition, the data of the parliament of polytopes of some globally generated equivariant vector bundle \mathcal{E} corresponds to some isomorphism class of \mathcal{E} (corresponding to the title of the subsection).

The naive definition for the parliament of polytopes of a toric vector bundle \mathcal{E} would be

$$PP_{\mathcal{E}} := \{P_e \mid e \in G(\mathcal{E})\}$$

which does not keep trace of the label of each polytope. This definition do not even allow us to distinguish a rank 2- from a rank 3-vector bundle on \mathbb{P}^2 : the tangent bundle $\mathcal{T}_{\mathbb{P}^2}$ and the splitting bundle $\mathcal{O}_{\mathbb{P}^2}(D_0) \oplus \mathcal{O}_{\mathbb{P}^2}(D_1) \oplus \mathcal{O}_{\mathbb{P}^2}(D_2)$ have the same parliament of polytopes.

5.1 Equivariant isomorphism class of framed equivariant vector bundle

The definition we gave in Definition 1.10 for the parliament of polytopes of a toric vector bundle \mathcal{E} was $PP_{\mathcal{E}} := \{(P_e, e) \mid e \in G(\mathcal{E})\}$ where $G(\mathcal{E})$ is defined modulo isomorphism of type (\star) .

Definition 5.1. *A framed equivariant vector bundle is a toric vector bundle with a choice of isomorphism $E \cong \mathbb{C}^r$.*

A morphism of framed equivariant vector bundle is a morphism of equivariant vector bundle compatible with the framing.

Proposition 5.2. *The data of the parliament of polytopes*

$$PP_{\mathcal{E}} := \{(P_e, e) \mid e \in G(\mathcal{E})\} / (\star)$$

of a globally generated equivariant vector bundle \mathcal{E} is equivalent to knowing the equivariant isomorphism class of the framed equivariant vector bundle \mathcal{E} .

Proof. In [Pay08] Proposition 3.4, the equivariant classes of a framed equivariant vector bundle is uniquely determined by the following

$$\left(\{\mathbf{u}(\sigma)\}_{\sigma \in \Sigma(d)}, \quad \{Fl(\rho)\}_{\rho \in \Sigma(1)} \right),$$

where $Fl(\rho_i)$ is the flag appearing in the filtration $(E^i(j))_j$. From the parliament we recover the flag

$$Fl(\rho_i) : \{0\} \subset E^i(A_r^i) \subset \dots \subset E^i(A_1^i) = E$$

by considering all different vector spaces

$$E_{i,j} = \sum_{\substack{\langle u, v_i \rangle \geq j, \\ u \in P_e \text{ for some } e \in G(\mathcal{E})}} e.$$

Indeed, if $e \in E_{i,j}$ then there exists $u \in P_e$ such that $\langle u, v_i \rangle \geq j$ and by definition of P_e , we obtain that $e \in E^i(j)$. And conversely, by Theorem 1.2 of [DJS18], as \mathcal{E} is taken globally generated, any $u \in \mathbf{u}(\sigma)$ belongs to some polytope and by (CC), we have $E^i(j) \subseteq E_{i,j}$. \square

5.2 Equivariant isomorphism class of equivariant vector bundle

We may not want to deal with framings anymore. Let us thus quotient by the action of $GL_r(\mathbb{C})$ on \mathbb{C}^r .

Proposition 5.3. *The data of the parliament of polytopes*

$$PP_{\mathcal{E}} := \{(P_e, e) \mid e \in G(\mathcal{E})\} / (\star) / GL_r(\mathbb{C})$$

of a globally generated equivariant vector bundle \mathcal{E} is equivalent to knowing the equivariant isomorphism class of the equivariant vector bundle \mathcal{E} .

Remark 5.4. Here morphisms do not have to preserve the framings anymore.

Proof of the proposition. It follows from Proposition 5.2 and Corollary 3.6 of [Pay08], the equivariant classes of a toric vector bundle is uniquely determined by

$$\left(\{\mathbf{u}(\sigma)\}_{\sigma \in \Sigma(d)}, \quad \mathcal{O}_{\{Fl(\rho)\}_{\rho \in \Sigma(1)}} \right),$$

where $\mathcal{O}_{\{Fl(\rho)\}_{\rho \in \Sigma(1)}}$ is the $GL_r(\mathbb{C})$ -orbit of the flag given by the filtration $(E^i(j))_j$.

5.3 Isomorphism class of equivariant vector bundle

Finally, we may want an object which represents the isomorphism class of \mathcal{E} and not its equivariant isomorphism class. For that we need to quotient by the group T of compositions of translations for each direct component.

Proposition 5.5. *The data of the parliament of polytopes*

$$PP_{\mathcal{E}} := \{(P_e, e) \mid e \in G(\mathcal{E})\} / (\star) / GL_r(\mathbb{C}) / T,$$

of a globally generated equivariant vector bundle \mathcal{E} is equivalent to knowing the isomorphism class of \mathcal{E} .

Proof. It is known that any line bundle $\mathcal{O}_X(D)$ on toric variety is isomorphic to an equivariant line bundle $\mathcal{O}_X(D_T) = \mathcal{O}_X(D) \otimes \chi^u$ for a unique $u \in M$.

In fact this result has been generalized by Klyachko in ([Kly90] Corollary 1.2.4). If two equivariant vector bundles \mathcal{E} and \mathcal{F} are isomorphic then there exists characters χ_1, \dots, χ_m such that

$$\mathcal{E}_i \otimes \chi_i \text{ and } \mathcal{F}_i \text{ are equivariantly isomorphic,}$$

where $\mathcal{E} = \mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_m$ and $\mathcal{F} = \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_m$ are some direct decompositions of \mathcal{E} and \mathcal{F} .

As a level of parliaments, tensoring by a character χ^u corresponds to translating the parliament of $u \in M$. \square

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