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DISJOINT AND COMMON HYPERCYCLIC ALGEBRAS

FRÉDÉRIC BAYART, FERNANDO COSTA JÚNIOR, AND DIMITRIS PAPATHANASIOU

ABSTRACT. Over time the concept of hypercyclicity has been explored in different manners and contexts, gaining new forms and applications. In particular when the space has an adjacent structure, we can always look for sets of hypercyclic vectors compatible with that framework. In this paper we deal with hypercyclic operators acting on Fréchet sequence algebras and give criteria for the existence of common and disjoint hypercyclic algebras.

1. INTRODUCTION

This paper goes over some structures of hypercyclic vectors in a linear dynamical system (X, T) , that is, a couple of a topological vector space X and a continuous linear operator T on X . We say that a vector $x \in X$ is hypercyclic for T when its orbit $Orb(x, T) := \{x, Tx, T^2x, \dots\}$ is dense in X . Since its first appearance in the '80s with the thesis of Kitai (see [10]), this concept has been the object of multiple interesting generalizations and modifications which yielded to many new related concepts (frequent hypercyclicity, common hypercyclicity, disjoint hypercyclicity, etc.).

When the underlying space has a richer structure, it is natural to investigate the existence of sets of hypercyclic vectors maintaining that framework. For instance, when X is an F -algebra, that is, a metrizable and complete topological algebra, and denoting by $HC(T)$ the set of all hypercyclic vectors for T , it is natural to ask whether $HC(T) \cup \{0\}$ contains a non-trivial algebra. This is what we call a hypercyclic algebra. The first result in this vein was done independently by Shkarin in [11] and by Bayart and Matheron in [3]. They have shown that the derivation operator $D : f \mapsto f'$, acting on the Fréchet algebra $H(\mathbb{C})$ of entire functions endowed with the pointwise multiplication, supports a hypercyclic algebra.

In this paper we are interested in the existence of algebraic structures within two concepts related to hypercyclicity, namely disjoint and common hypercyclicity. We will study them in the context of weighted backward shifts on Fréchet sequence algebras. Precisely, we assume that X is a subspace of the space $\omega = \mathbb{C}^{\mathbb{N}_0}$ of all complex sequences, whose topology is induced by a non-decreasing sequence of seminorms $(\|\cdot\|_q)_{q \geq 1}$ and that X is endowed

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with a product \cdot such that, for all $x, y \in X$, all $q \geq 1$,

$$\|x \cdot y\|_q \leq \|x\|_q \times \|y\|_q.$$

There are two natural products on a Fréchet sequence space: the coordinatewise product and the convolution or Cauchy product. They are defined as follows. Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be sequences in ω . Their coordinatewise product is defined as

$$(a_n)_n \cdot (b_n)_n = (a_n b_n)_n.$$

Their convolution product is defined as

$$(a_n)_n \cdot (b_n)_n = (c_n)_n,$$

where $c_n = \sum_{k=0}^n a_k b_{n-k}$ for all $n \geq 0$. It is clear that ℓ_p and c_0 are Fréchet sequence algebras for the coordinatewise product, and that ℓ_1 is also a Fréchet sequence algebra for the convolution product. Endowing $H(\mathbb{C})$ with

$$\left\| \sum_{n \geq 0} a_n z^n \right\|_q = \sum_{n \geq 0} |a_n| q^n$$

and ω with

$$\|(x_n)\|_q = \sum_{n=0}^q |x_n|,$$

we also obtain that $H(\mathbb{C})$ and ω are Fréchet sequence algebras for both products (on $H(\mathbb{C})$, the Cauchy product of f and g is nothing else but the product of the two functions f and g).

Our results will cover both the coordinatewise and the Cauchy product. Before going further, we mention that they behave very differently. The main point is that the coordinatewise product preserves the support, whereas the Cauchy product mixes it. In particular, if x and y are two vectors with disjoint supports, then $(x + y)^m = x^m + y^m$ for the coordinatewise product whereas this is not at all the case for the Cauchy product. This explains why the latter case is considerably more difficult to handle.

As announced above, we will work with weighted backward shifts, one of the favourite class of examples in linear dynamics. Given a sequence of nonzero complex numbers $w = (w_n)_{n \in \mathbb{N}}$, the (unilateral) weighted backward shift B_w with weight w is defined by

$$B_w(x_0, x_1, \dots) = (w_1 x_1, w_2 x_2, \dots).$$

The weight w will be called *admissible* (for X) if B_w is a bounded operator on X . It is known that, provided the canonical basis (e_n) is a Schauder basis of X , B_w is hypercyclic if and only if there exists a sequence (n_k) such that for all $l \in \mathbb{N}$, $((w_{l+1} \cdots w_{n_k+l})^{-1} e_{n_k+l})$ goes to zero.

1.1. Disjoint hypercyclic algebras. The notion of disjointness was introduced in linear dynamics in [5] and [6]; we shall limit ourselves to the case of two operators. We say that two operators T_1 and T_2 acting on the same F -space X are disjointly hypercyclic if there exists a vector $x \in X$, called a disjointly hypercyclic vector for T_1 and T_2 , such that $(T_1^n x, T_2^n x)_{n \geq 0}$ is dense in X^2 . Many examples or counterexamples of disjointly hypercyclic operators can be found in the literature. In particular, in [6] and in [7], our favourite examples, weighted backward shifts, are studied in depth: in [6], it is characterized when two shifts $B_1^{r_1}$ and $B_2^{r_2}$ are disjointly hypercyclic on ℓ_p when they are not raised to the same power (namely $r_1 < r_2$), whereas in [7] such a characterization is given when $r_1 = r_2 = 1$. From the proofs given in [6] and in [7] we see that the two cases are very different, as it will become clearer later on in this paper.

Assume now that X is an algebra. We say that T_1 and T_2 support a disjoint hypercyclic algebra if there exists a nontrivial subalgebra A of X such that any non zero element of A is a disjointly hypercyclic vector for T_1 and T_2 . We will study whether two backward shifts acting on a Fréchet sequence algebra X support a disjoint hypercyclic algebra or not. When X is endowed with the coordinatewise product, mixing the arguments of [4] with that of [6, 7], we will get a complete characterization of the existence of a disjoint hypercyclic algebra for $B_1^{r_1}$ and $B_2^{r_2}$ (to avoid cumbersome statements, we will focus on the cases $r_1 = r_2 = 1$ and $r_1 = 1 < r_2 = 2$). Despite the problem of the support, we will also succeed to provide such a characterization when X is endowed with the convolution product and $r_1 = r_2 = 1$ (see the forthcoming Theorems 2.5, 2.10, 2.19). The most difficult case is that of the Cauchy product and two shifts raised to different powers. We shall concentrate ourselves on the multiples of B and D . In [6], it is shown that λB and μB^2 are disjointly hypercyclic on ℓ_1 if and only if $1 < \lambda < \mu$ and that, for all $\lambda, \mu > 0$, λD and μD^2 are disjointly hypercyclic on $H(\mathbb{C})$. Under these assumptions we will even get a disjointly hypercyclic algebra.

Theorem 1.1. *Let $\lambda, \mu > 0$.*

- a) λB and μB^2 support a disjoint hypercyclic algebra on ℓ_1 (endowed with the convolution product) if and only if $1 < \lambda < \mu$.
- b) λD and μD^2 support a disjoint hypercyclic algebra on $H(\mathbb{C})$.

1.2. Common hypercyclic algebras. If $(T_\lambda)_{\lambda \in \Lambda}$ is a family of hypercyclic operators acting on the same F -space X , it is natural to ask whether it admits a common hypercyclic vector, namely if $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is non-empty. This is trivial if Λ is countable since each $HC(T_\lambda)$ is a residual set. However, this becomes already an issue for the first natural example which comes in mind, that is the multiples of the unweighted backward shift B . It was proved by Abakumov and Gordon in [1] that indeed $\bigcap_{\lambda > 1} HC(\lambda B) \neq \emptyset$ where B is seen acting on any ℓ_p -space or on c_0 . Further examples were given in subsequent papers (for instance in [8], [2])

Suppose now that we have a family $(T_\lambda)_{\lambda \in \Lambda}$ of operators acting on the same F -algebra, each one supporting a hypercyclic algebra. It is natural to ask whether $\bigcap_{\lambda \in \Lambda} HC(T_\lambda) \cup \{0\}$ contains a nontrivial algebra, which will be called a *common hypercyclic algebra*. Even when

Λ has two elements, this is unclear. In this paper, we concentrate ourselves on the case where (T_λ) is a family of weighted shifts acting on X .

Firstly, regarding Fréchet sequence algebras endowed with the coordinatewise product, we have been able to get a general statement covering all the cases (and even more!) known for the existence of a common hypercyclic vector. As a corollary of our work, we have the following statement.

- Theorem 1.2.** (i) *Let $X = \ell_p$, $1 \leq p < +\infty$, or $X = c_0$ endowed with the coordinatewise product. Then the family $(\lambda B)_{\lambda > 1}$ admits a common hypercyclic algebra.*
- (ii) *Let $X = \ell_p$, $1 \leq p < +\infty$, or $X = c_0$ endowed with the coordinatewise product. For $\lambda > 0$, define $w(\lambda) = (1 + \lambda/n)_{n \geq 1}$. Then the family $(B_{w(\lambda)})_{\lambda > 0}$ admits a common hypercyclic algebra.*
- (iii) *Let $X = H(\mathbb{C})$ endowed with the coordinatewise product. Then the family $(\lambda D)_{\lambda > 0}$ admits a common hypercyclic algebra.*

The statement of (ii) is particularly interesting. Indeed, for this family of weights, the existence of a single common hypercyclic vector was only known for $\lambda > 1/p$.

Again, it is much more difficult to handle Fréchet sequence algebras endowed with the Cauchy product. Nevertheless, we will be able to give a general sufficient condition which implies the following result.

- Theorem 1.3.** (i) *Let $X = \ell_1$ endowed with the Cauchy product. Then $(\lambda B)_{\lambda > 1}$ admits a common hypercyclic algebra.*
- (ii) *Let $X = H(\mathbb{C})$ endowed with the Cauchy product. Then $(\lambda D)_{\lambda > 0}$ admits a common hypercyclic algebra.*

1.3. Organization of the paper. All the criteria of this paper are based on a Baire argument. This idea first appeared in [3, Remark 8.28] and is a sort of Birkhoff transitivity theorem for hypercyclic algebras. Here we adapt this criterion to produce either common or disjoint hypercyclic algebras. Each section begins with its own particular criterion followed by more practical criteria for the particular case of Fréchet sequence algebras and then we apply these results on some classical examples.

1.4. Notations. The symbol \mathbb{N} will stand for the set of positive integers, whereas $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We shall denote by $\mathcal{P}_f(A)$ the set of finite subsets of a given set A .

For $x = \sum_{n=0}^{+\infty} x_n e_n \in \omega$, the support of x is equal to $\text{supp}(x) = \{n \in \mathbb{N}_0 : x_n \neq 0\}$. The notation c_{00} will denote the set of sequences in ω with finite support.

For $u \in X^d$ and $\alpha \in \mathbb{N}_0^d$, u^α will mean $u_1^{\alpha_1} \cdots u_d^{\alpha_d}$. If z is any complex number and $m \in \mathbb{N}$, $z^{1/m}$ will denote any m th root of z .

When working on a Fréchet space $(X, \|\cdot\|_p)$, it is often convenient to endow X with an F -norm $\|\cdot\|$ defining the topology of X (see [9, Section 2.1]). Such an F -norm can be defined by the formula

$$\|x\| = \sum_{p=1}^{+\infty} \frac{1}{2^p} \min(1, \|x\|_p).$$

In particular, an F -norm satisfies the triangle inequality and the inequality

$$(1) \quad \forall \lambda \in \mathbb{C}, \forall x \in X, \|\lambda x\| \leq (|\lambda| + 1)\|x\|,$$

a property which replaces the positive homogeneity of the norm.

We finally recall some results on unconditional convergence in Fréchet spaces (see for instance [9, Appendix A]). A series $\sum_{n=0}^{+\infty} x_n$ in a Fréchet space X is called unconditionally convergent if for any bijection $\pi : \mathbb{N}_0 \rightarrow \mathbb{N}_0$, the series $\sum_{n=0}^{+\infty} x_{\pi(n)}$ is convergent. This amounts to saying that, for any $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that, whenever $\sup_n |\alpha_n| \leq 1$, the series $\sum_{n=0}^{+\infty} \alpha_n x_n$ converges and

$$\left\| \sum_{n=N}^{+\infty} \alpha_n x_n \right\| < \varepsilon.$$

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2. DISJOINT HYPERCYCLIC ALGEBRAS

2.1. How to get a disjoint hypercyclic algebra. As described in the introduction, we adapt Birkhoff's transitivity theorem to get a criterion ensuring disjoint hypercyclicity. We may observe the similarity with [6, Proposition 2.3].

Proposition 2.1. *Let T_1 and T_2 be two operators acting on the same F -algebra X . Assume that for each $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, there exists $m_0 \in I$ such that, for all U, V_1 and V_2 nonempty open subsets of X , for all W neighbourhood of zero, there exist $u \in U$ and $N \in \mathbb{N}$ such that*

$$\begin{cases} T_1^N(u^n) \in W, T_2^N(u^n) \in W \text{ for all } n \in I, n \neq m_0 \\ T_1^N(u^{m_0}) \in V_1, T_2^N(u^{m_0}) \in V_2. \end{cases}$$

Then T_1 and T_2 admit a disjoint hypercyclic algebra.

Proof. For $V, V' \subset X$ open and nonempty, for W a neighbourhood of zero, for $I \subset \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, let us define

$$\begin{aligned} \mathcal{A}(I, V, V', W) = \left\{ u \in X : \exists N \in \mathbb{N}, T_1^N(u^n) \in W \text{ for all } n \in I \setminus \{m_0(I)\} \right. \\ \left. T_2^N(u^n) \in W \text{ for all } n \in I \setminus \{m_0(I)\} \right. \\ \left. T_1^N(u^{m_0(I)}) \in V, T_2^N(u^{m_0(I)}) \in V' \right\} \end{aligned}$$

Fixing (V_k) a basis of open subsets of X and (W_l) a basis of neighbourhoods of 0, one can verify that each set $\mathcal{A}(I, V_k, V_{k'}, W_l)$ is open and dense, hence $\bigcap_{I,k,k',l} \mathcal{A}(I, V_k, V_{k'}, W_l)$ is non-empty and we can argue exactly like in [4, Theorem 2.1] to prove that any vector in this intersection generates a disjoint hypercyclic algebra for T_1 and T_2 . \square

Similarly to [4], for the Cauchy product we will apply this proposition for $m_0 = \max I$ and for the coordinatewise product we will do the opposite choice $m_0 = \min I$. The corresponding corollaries are the following.

Corollary 2.2. *Let T_1 and T_2 be two operators acting on the same F -algebra X . Assume that for all $m \geq 1$, for all U, V_1 and V_2 nonempty open subsets of X , for all W neighbourhood of zero, there exist $u \in U$ and $N \in \mathbb{N}$ such that*

$$\begin{cases} T_1^N(u^n) \in W, T_2^N(u^n) \in W \text{ for all } n < m \\ T_1^N(u^m) \in V_1, T_2^N(u^m) \in V_2. \end{cases}$$

Then T_1 and T_2 admit a disjoint hypercyclic algebra.

Corollary 2.3. *Let T_1 and T_2 be two operators acting on the same F -algebra X . Assume that for all positive integers $m_0 < m_1$, for all U, V_1 and V_2 nonempty open subsets of X , for all W neighbourhood of zero, there exist $u \in U$ and $N \in \mathbb{N}$ such that*

$$\begin{cases} T_1^N(u^n) \in W, T_2^N(u^n) \in W \text{ for all } n \in \{m_0 + 1, \dots, m_1\} \\ T_1^N(u^{m_0}) \in V_1, T_2^N(u^{m_0}) \in V_2. \end{cases}$$

Then T_1 and T_2 admit a disjoint hypercyclic algebra.

2.2. Disjoint hypercyclic algebras for backward shifts - coordinatewise product.

We now show how these general results can be applied to get disjoint hypercyclic algebras for backward shifts operators. We first begin with Fréchet sequence algebras endowed with the coordinatewise product. We get complete characterizations either for the case where the shifts are raised to different powers as for the case where the powers are the same.

We recall that a Fréchet space $(X, (\|\cdot\|_q))$ admits a continuous norm if there exists a norm $\|\cdot\| : X \rightarrow \mathbb{R}$ that is continuous for the topology of X , namely there exists $q \in \mathbb{N}$ and $C > 0$ with $\|x\| \leq C\|x\|_q$ for all $x \in X$. In particular, for any q large enough, $\|\cdot\|_q$ itself is a norm. The following lemma is taken from [4, Lemma 4.1].

Lemma 2.4. *Let X be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Then the sequence (e_n) is bounded below.*

This follows from the simple observation that, for all $q \geq 0$, $0 < \|e_n\|_q = \|e_n \cdot e_n\|_q \leq \|e_n\|_q^2$, which implies $\|e_n\|_q \geq 1$.

2.2.1. *Backward shifts raised to different powers.* We first state a characterization of the existence of a disjoint hypercyclic algebra for B_1 and B_2^2 when B_1 and B_2 are two weighted shifts on the Fréchet sequence algebra X .

Theorem 2.5. *Let X be a Fréchet sequence algebra for the coordinatewise product with a continuous norm and such that (e_n) is a Schauder basis. Let $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ be two bounded weighted shifts on X . The following assertions are equivalent.*

- (i) B_1 and B_2^2 support a disjoint hypercyclic algebra.
- (ii) For all $p \in \mathbb{N}$, for all $m \geq 1$, there exists a sequence of integers (n_k) such that, for all $l = 0, \dots, p$,

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ (w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{-1/m} e_{2n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \end{aligned}$$

$$\frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0.$$

(iii) There exists a sequence (n_k) such that, for all $l \geq 0$, for all $\gamma > 0$,

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-\gamma} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ (w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{-\gamma} e_{2n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

Proof. (i) \implies (ii). Let $m \geq 1$ and let $x \in X$, $x = \sum_{n=0}^{+\infty} x_n e_n$, be such that x^m is a disjoint hypercyclic vector for B_1 and B_2^2 . Let $p \geq 1$ and let (n_k) be a sequence of integers such that $(B_j^{j n_k}(x^m))_k$ goes to $e_0 + \cdots + e_p$, $j = 1, 2$. Let us see what we obtain for $j = 1$. Since convergence in X implies coordinatewise convergence, for all $l = 0, \dots, p$, $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)} x_{n_k+l}^m)$ converges to 1. Hence the sequences $((w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m} x_{n_k+l})$ are bounded below. Writing

$$(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m} e_{n_k+l} = \frac{1}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m} x_{n_k+l}} x_{n_k+l} e_{n_k+l}$$

we get that $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0$. Similarly we obtain for all $l = 0, \dots, p$,

$$\begin{aligned} (w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{-1/m} e_{2n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ (2) \quad w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)} x_{2n_k+l}^m &\xrightarrow{k \rightarrow +\infty} 1. \end{aligned}$$

Moreover, write

$$B_1^{n_k}(x^m) = z + \sum_{l=0}^p w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)} x_{2n_k+l}^m e_{n_k+l} + z',$$

with $\text{supp}(z) \subset [0, n_k)$ and $\text{supp}(z') \subset (n_k + p, +\infty)$. Again, since convergence in X implies pointwise convergence, we get

$$w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)} x_{2n_k+l}^m e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0 \text{ for all } l = 0, \dots, p$$

which in turn, in view of (2), gives

$$\frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0 \text{ for all } l = 0, \dots, p.$$

(ii) \implies (iii). This follows from a diagonal argument.

(iii) \implies (i). We intend to apply Corollary 2.3. Let $m_0 < m_1$ be two positive integers, let U, V_1, V_2 be nonempty open subsets of X and let W be a neighbourhood of 0. Let $x \in U$

with finite support, let $y_i = \sum_{l=0}^p y_{i,l} e_l$ belonging to V_i , $i = 1, 2$. We set

$$u = x + \sum_{l=0}^p \frac{y_{1,l}^{1/m_0}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m_0}} e_{n_k+l} + \sum_{l=0}^p \frac{y_{2,l}^{1/m_0}}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{1/m_0}} e_{2n_k+l},$$

which belongs to U for all large values of k . We recall that, since the sequence (e_n) is bounded below, for all $l \geq 0$, the sequences $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})$ and $(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})$ tend to $+\infty$. Let $n \in \{m_0, \dots, m_1\}$ and let us compute $B_2^{2n_k}(u^n)$:

$$B_2^{2n_k}(u^n) = \sum_{l=0}^p \frac{y_{2,l}^{n/m_0}}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{\frac{n}{m_0}-1}} e_l.$$

The assumptions tell us that $B_2^{2n_k}(u^{m_0}) = y_2$ while $B_2^{2n_k}(u^n)$ tends to 0 if $n > m_0$. On the other hand,

$$B_1^{n_k}(u^n) = \sum_{l=0}^p \frac{y_{1,l}^{n/m_0}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{\frac{n}{m_0}-1}} e_l + \sum_{l=0}^p y_{2,l}^{n/m_0} \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{\frac{n}{m_0}}} e_{n_k+l}.$$

Writing

$$\frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{\frac{n}{m_0}}} e_{n_k+l} = \frac{1}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{\frac{n}{m_0}-1}} \times \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l},$$

we get that the second (finite) sum tends to zero for all $n \geq m_0$, whereas the first sum can be handled exactly as for B_2 . \square

Remark 2.6. If we are just interested in disjointly hypercyclic operators, our proof can be easily modified to give the following extension of [6, Theorem 4.1] : let X be a Fréchet sequence space such that (e_n) is a Schauder basis for X . Let $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ be two bounded weighted shifts on X . Then B_1 and B_2^2 are disjointly hypercyclic if and only if there exists a sequence (n_k) such that, for all $l \geq 0$,

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ (w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{-1} e_{2n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

We point out that we do not need that (e_n) is bounded below. This last assumption was only useful when $n > m_0$, where we used it to deduce that $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})$ and $(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})$ tend to $+\infty$. For the existence of a disjointly hypercyclic vector, we always have $n = m_0 = 1$.

This remark also leads us to an interesting example of a couple of disjointly hypercyclic operators not having a disjoint hypercyclic algebra.

Example 2.7. Let B_1 and B_2^2 be weighted shifts induced respectively by the weight sequences $w_n^{(1)} = \frac{n^2}{(n-1)^2}$ with $w_1^{(1)} = 1$ and $w_n^{(2)} = 2$, $n \geq 1$, both acting on the weighted $\ell_1(v)$ with $v_n = n$ for all $n \geq 1$. By weighted $\ell_1(v)$ we mean the Banach space

$$\ell_1(v) = \{(x_n)_n : \sum_{n=1}^{\infty} |x_n| v_n < +\infty\}$$

equipped with the weighted ℓ_1 -norm

$$\|(x_n)_n\| = \sum_{n=1}^{\infty} |x_n| v_n.$$

For this example, all conditions of the previous theorem are satisfied for $m = 1$ (which grants a disjointly hypercyclic vector) but, for $m = 2$ and for any sequence $(n_k)_k$, we have that

$$\|(w_1^{(1)} \cdots w_{n_k}^{(1)})^{-1/2} e_{n_k}\| = \frac{n_k}{(n_k^2)^{1/2}} = 1$$

does not go to 0 as $k \rightarrow +\infty$.

Remark 2.8. In particular, if the sequence (e_n) is bounded (it is already bounded below), the conditions of Theorem 2.5 are equivalent to say that there exists a sequence (n_k) such that, for all $l = 0, \dots, p$,

$$\begin{aligned} w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)} &\xrightarrow{k \rightarrow +\infty} +\infty, \\ w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)} &\xrightarrow{k \rightarrow +\infty} +\infty, \\ \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} &\xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

In turn, this amounts to saying that B_1 and B_2^2 are disjointly hypercyclic. Hence, all the examples given in [6] support a disjoint hypercyclic algebra.

It is plain that this result can be extended for a finite list $B_{w^{(1)}}^{n_1}, \dots, B_{w^{(r)}}^{n_r}$ raised to different powers n_1, \dots, n_r . The statement of precise conditions are left to the reader.

We close this subsection by addressing the case of ω which is the prototypical example of a Fréchet space without a continuous norm. The following result resembles [4, Theorem 4.8].

Theorem 2.9. *If $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ are weighted backward shifts on ω endowed with the coordinatewise product, then B_1 and B_2^2 support a disjoint hypercyclic algebra.*

Proof. For V_1 and V_2 non-empty open subsets of ω , $I \subset \mathbb{N}$ finite and non-empty and $s > 0$, let us define

$$\begin{aligned} E(I, s) = \left\{ P \in \mathbb{C}[z] : |\hat{P}(\min I)| \geq 1/s, |\hat{P}(\max I)| \geq 1/s, \right. \\ \left. |\hat{P}(n)| \leq s \text{ for all } n \in \mathbb{N}, \right. \end{aligned}$$

$$\hat{P}(n) = 0 \text{ when } n \notin I \}$$

$$\mathcal{A}(I, s, V_1, V_2) = \left\{ u \in \omega : \forall P \in E(I, s), \exists N \geq 1, B_i^{iN}(P(u)) \in V_i, i = 1, 2 \right\}.$$

By a Baire category argument, it is enough to prove that each set $\mathcal{A}(I, s, V_1, V_2)$ is dense and open. The last property follows easily from the compactness of $E(I, s)$. Thus, let us fix I, s, V_1 and V_2 and let us prove that $\mathcal{A}(I, s, V_1, V_2)$ is dense. We set $m_0 = \min(I)$ and $m_1 = \max(I)$. Let U be a non-empty open subset of ω . Let $p \in \mathbb{N}_0$, $u_0, \dots, u_p, v_{i,0}, \dots, v_{i,p} \in \mathbb{C}$, for $i = 1, 2$, and $\varepsilon > 0$ be such that, for all x, y_1 and y_2 in ω ,

$$|x_l - u_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in U,$$

$$|y_{i,l} - v_{i,l}| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } y_i \in V_i, i = 1, 2.$$

Let us first look at the sequence $(w_1^{(1)} \cdots w_n^{(1)})$. Three possibilities (which are not mutually exclusive) can occur:

- either $(w_1^{(1)} \cdots w_n^{(1)})$ is bounded and bounded below;
- or it admits a subsequence going to zero;
- or it admits a subsequence going to $+\infty$.

Thus, we get the existence of a subsequence $(w_1^{(1)} \cdots w_{n_k}^{(1)})$ going to $a_{1,0} \in [0, +\infty]$. We then do the same with $(w_2^{(1)} \cdots w_{n_{k+1}}^{(1)})$ and so on. By successive extractions, we get the existence of a sequence of integers (n_k) (we can assume that $n_{k+1} - 2n_k > p$ and $n_k > p$ for all k) and of $a_{i,0}, \dots, a_{i,p} \in [0, +\infty]$, $i = 1, 2$, such that, for all $l = 0, \dots, p$, and $i = 1, 2$, $(w_{l+1}^{(i)} \cdots w_{n_k+l}^{(i)})$ tends to $a_{i,l}$. We set $A_{i,1} = \{l \in \{0, \dots, p\} : a_{i,l} = +\infty\}$, $A_{i,2} = \{l \in \{0, \dots, p\} : a_{i,l} = 0\}$ and $A_{i,3} = \{l \in \{0, \dots, p\} : a_{i,l} \in (0, +\infty)\}$.

We fix now $(\alpha(k))$, $(\beta(k))$ two sequences of non-zero complex numbers and $(z_i(k))$, $i = 1, 2$, sequences in \mathbb{C}^{p+1} such that $(\alpha(k), \beta(k), z_1(k), z_2(k))$ is dense in $\mathbb{C}^{2(p+2)}$. We set

$$x = u + \sum_{k=0}^{+\infty} y_1(k) + \sum_{k=0}^{+\infty} y_2(k)$$

where, for $l = 0, \dots, p$ and $i = 1, 2$,

$$y_{i, in_k+l}(k) = \begin{cases} \frac{v_{i,l}^{1/m_0}}{\alpha(k)^{1/m_0} (w_{l+1}^{(i)} \cdots w_{n_k+l}^{(i)})^{1/m_0}} & \text{provided } l \in A_{i,1}, \\ \frac{v_{i,l}^{1/m_1}}{\beta(k)^{1/m_1} (w_{l+1}^{(i)} \cdots w_{n_k+l}^{(i)})^{1/m_1}} & \text{provided } l \in A_{i,2}, \\ z_{i,l}(k) & \text{provided } l \in A_{i,3} \end{cases}$$

and $y_{i,j}(k) = 0$ if $j \neq in_k, \dots, in_k + p$.

We claim that $x \in U \cap \mathcal{A}(I, s, V_1, V_2)$. The definition of ε and p ensure that $x \in U$. Let $P \in E(I, s)$. There exists an increasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\alpha(\phi(k)) \rightarrow \hat{P}(m_0)$, $\beta(\phi(k)) \rightarrow \hat{P}(m_1)$ and $a_{i,l}P(z_{i,l}(\phi(k))) \rightarrow v_{i,l}$ for all $l \in A_{i,3}$ and $i = 1, 2$. We claim that

$(B_i^{in_{\phi(k)}}(P(x)))$ belongs to V_i , $i = 1, 2$, provided k is large enough. It suffices to prove that for $l = 0, \dots, p$ and $i = 1, 2$, the l -th coordinate of $B_i^{in_{\phi(k)}}(P(x))$ tends to $v_{i,l}$. Assume first that $l \in A_{i,1}$. This l -th coordinate is equal to

$$w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)} P \left(\frac{v_{i,l}^{1/m_0}}{\alpha(\phi(k))^{1/m_0} (w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)})^{1/m_0}} \right).$$

Now, since $w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)}$ tends to $+\infty$, and $m_0 = \min(I)$,

$$w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)} P \left(\frac{v_{i,l}^{1/m_0}}{\alpha(\phi(k))^{1/m_0} (w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)})^{1/m_0}} \right) = \hat{P}(m_0) \frac{v_{i,l}}{\alpha(\phi(k))} + o(1)$$

and this tends to $v_{i,l}$. When $l \in A_{i,2}$, the proof is similar since now, because $w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)}$ tends to 0, and $m_1 = \max(I)$,

$$w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)} P \left(\frac{v_{i,l}^{1/m_1}}{\beta(\phi(k))^{1/m_1} (w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)})^{1/m_1}} \right) = \hat{P}(m_1) \frac{v_{i,l}}{\beta(\phi(k))} + o(1)$$

and this also goes to $v_{i,l}$. Finally, when $l \in A_{i,3}$, the l -th coordinate of $B_i^{in_{\phi(k)}}(P(x))$ is equal to $w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)} P(z_{i,l}(\phi(k)))$ which tends again to $v_{i,l}$. \square

2.2.2. Backward shifts raised to the same power. We now do the same for two shifts raised to the same power.

Theorem 2.10. *Let X be a Fréchet sequence algebra for the coordinatewise product with a continuous norm and such that (e_n) is a Schauder basis. Let $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ be two bounded weighted shifts on X . The following assertions are equivalent.*

- (i) B_1 and B_2 support a disjoint hypercyclic algebra.
- (ii) For all $p \in \mathbb{N}$, for all $m \geq 1$, there exists a sequence (n_k) such that

$$(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0, \text{ for all } l = 0, \dots, p,$$

$$\left\{ \left(\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \right)_{l \geq 0} : k \geq 0 \right\} \text{ is dense in } \omega.$$

- (iii) There exists a sequence (n_k) such that, for all $\gamma > 0$, for all $l \geq 0$,

$$(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-\gamma} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0$$

$$\left\{ \left(\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \right)_{l \geq 0} : k \geq 0 \right\} \text{ is dense in } \omega.$$

Proof. (i) \implies (ii). Let $m \geq 1$ and let $x \in X$, $x = \sum_{n=0}^{+\infty} x_n e_n$, be such that x^m is a disjointly hypercyclic vector for B_1 and B_2 . Let also $(\lambda^{(k)})$ be a dense sequence in ω of vectors with finite support. We may assume that $\text{supp}(\lambda^{(k)}) \subset [0, k]$ and that, for any fixed k_0 , there are infinitely many k such that $\lambda^{(k)} = \lambda^{(k_0)}$. We then choose an increasing sequence of integers (n_k) such that, for all $k \geq 1$,

$$(3) \quad \left\| B_1^{n_k}(x^m) - \sum_{l=0}^k e_l \right\| < \frac{1}{k},$$

$$(4) \quad \left\| B_2^{n_k}(x^m) - \sum_{l=0}^k \lambda_l^{(k)} e_l \right\| < \frac{1}{k}.$$

As we have already observed, (3) implies that, for all $l \geq 0$,

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)} x_{n_k+l}^m &\xrightarrow{k \rightarrow +\infty} 1. \end{aligned}$$

Let us now fix U a nonempty open subset of ω and $k_0 \in \mathbb{N}$ be such that $\lambda^{(k_0)} \in U$. Let $(n_{\phi(k)})$ be a subsequence of (n_k) such that $\lambda^{(\phi(k))} = \lambda^{(k_0)}$ for all k . By (4), we know that

$$w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)} x_{n_{\phi(k)}+l}^m \xrightarrow{k \rightarrow +\infty} \lambda_l^{(k_0)} \text{ for all } l \geq 0.$$

Combining the two last properties, we find that

$$\frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}} \xrightarrow{k \rightarrow +\infty} \lambda_l^{(k_0)} \text{ for all } l \geq 0.$$

Hence, for k large enough,

$$\left(\frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}} \right)_{l \geq 0} \in U.$$

(ii) \implies (iii). This follows from a diagonal argument.

(iii) \implies (i). Let $m_0 < m_1$ be two positive integers, let U, V_1, V_2 be nonempty open subsets of X and let W be a neighbourhood of 0. Let $x \in U$ with finite support, let $y_i = \sum_{l=0}^p y_{i,l} e_l$ belonging to V_i , $i = 1, 2$. Without loss of generality, we may assume that $y_{1,l} \neq 0$ for all $l = 0, \dots, p$. We then consider a sequence (n_k) of integers such that, for all $l = 0, \dots, p$,

$$(5) \quad \begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m_0} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} &\xrightarrow{k \rightarrow +\infty} \frac{y_{2,l}}{y_{1,l}}. \end{aligned}$$

Observe that $w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}$ goes to $+\infty$ for all $l \geq 0$. We finally set

$$u = x + \sum_{l=0}^p \frac{y_{1,l}^{1/m_0}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m_0}} e_{n_k+l}.$$

Clearly, u belongs to U (provided k is large enough), $B_1^{n_k}(u^{m_0}) = y_1$ and $B_1^{n_k}(u^n)$ goes to 0 for all $n > m_0$. Now, let us write

$$B_2^{n_k}(u^n) = \sum_{l=0}^p y_{1,l}^{n/m_0} \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{n/m_0}} e_l.$$

If $n = m_0$, then (5) tells us immediately that $B_2^{n_k}(u^m)$ tends to y_2 . For $n > m_0$,

$$y_{1,l}^{n/m_0} \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{n/m_0}} \sim_{k \rightarrow +\infty} \frac{y_{1,l}^{m_0-1} y_{2,l}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{m_0-1}}.$$

Since $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})$ tends to $+\infty$, this implies that $B_2^{n_k}(u^n)$ belongs to W for k large enough, so that we may apply Corollary 2.3 to prove that B_1 and B_2 support a disjoint hypercyclic algebra. \square

Remark 2.11. If we are just interested in disjointly hypercyclic operators, our proof can be easily modified to give the following result : let X be a Fréchet sequence space such that (e_n) is a Schauder basis for X . Let $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ be two bounded weighted shifts on X . Then B_1 and B_2 are disjointly hypercyclic if and only if there exists a sequence (n_k) such that $((w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1} e_{n_k+l})$ tends to zero for all $l \geq 0$ and the set

$$\left\{ \left(\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \right)_{l \geq 0} : k \geq 0 \right\}$$

is dense in ω .

Remark 2.12. In particular, if the sequence (e_n) is bounded, the conditions of the above theorem are all equivalent to say that B_1 and B_2 are disjointly hypercyclic. Hence, all the examples given in [7] support a disjoint hypercyclic algebra.

Finally, we address the case of ω .

Theorem 2.13. *If $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ are weighted backward shifts on ω endowed with the coordinatewise product, then the following are equivalent.*

- (i) B_1 and B_2 are disjointly hypercyclic
- (ii) B_1 and B_2 support a disjoint hypercyclic algebra.
- (iii) The set

$$\left\{ \left(\frac{w_{l+1}^{(2)} \cdots w_{n+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n+l}^{(1)}} \right)_{l \geq 0} : n \geq 0 \right\}$$

is dense in ω .

Proof. The implication $(ii) \Rightarrow (i)$ is immediate. Let us first show that $(i) \Rightarrow (iii)$. Let U be a non-empty open subset of ω , $p \in \mathbb{N}_0$, $u_0, \dots, u_p \in \mathbb{C}$ and $\varepsilon > 0$ be such that, for all $y \in \omega$

$$|y_l - u_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } y \in U.$$

If x is a disjoint hypercyclic vector for B_1 and B_2 , there is a subsequence (n_k) such that

$$B_1^{n_k} x \rightarrow \sum_{l=0}^p e_l$$

and

$$B_2^{n_k} x \rightarrow \sum_{l=0}^p u_l e_l.$$

That means that for all $l = 0, \dots, p$,

$$w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)} x_{n_k+l} \rightarrow 1$$

and

$$w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)} x_{n_k+l} \rightarrow u_l$$

from which we immediately get that

$$\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \rightarrow u_l.$$

Next, we show that $(iii) \Rightarrow (ii)$. For V_1 and V_2 non-empty open subsets of ω , $I \subset \mathbb{N}$ finite and non-empty and $s > 0$, let us define

$$E(I, s) = \left\{ P \in \mathbb{C}[z] : |\hat{P}(\min I)| \geq 1/s, |\hat{P}(\max I)| \geq 1/s, \right. \\ \left. |\hat{P}(n)| \leq s \text{ for all } n \in \mathbb{N}, \right. \\ \left. \hat{P}(n) = 0 \text{ when } n \notin I \right\}$$

$$\mathcal{A}(I, s, V_1, V_2) = \left\{ u \in \omega : \forall P \in E(I, s), \exists N \geq 1, B_i^N(P(u)) \in V_i, i = 1, 2 \right\}.$$

By a Baire category argument, it is enough to prove that each set $\mathcal{A}(I, s, V_1, V_2)$ is dense and open. The last property follows easily from the compactness of $E(I, s)$. Thus, let us fix I, s, V_1 and V_2 and let us prove that $\mathcal{A}(I, s, V_1, V_2)$ is dense. We set $m_0 = \min(I)$ and $m_1 = \max(I)$. Let U be a non-empty open subset of ω . Let $p \in \mathbb{N}_0$, $u_0, \dots, u_p, v_{i,0}, \dots, v_{i,p} \in \mathbb{C}$, for $i = 1, 2$ (we may assume that $v_{i,l} \neq 0$ for all $l = 0, \dots, p$), and $\varepsilon > 0$ be such that, for all x, y_1 and y_2 in ω ,

$$|x_l - u_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in U,$$

$$|y_{i,l} - v_{i,l}| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } y_i \in V_i, i = 1, 2.$$

Let by assumption (n_k) be a subsequence such that

$$\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \rightarrow \frac{v_{2,l}}{v_{1,l}}, \quad l = 0, \dots, p.$$

By passing to a subsequence we may assume that $n_0 > p$, that $n_{k+1} - n_k > p$ for all k , and that there are $a_0, \dots, a_p \in [0, +\infty]$ such that, for all $l = 0, \dots, p$, $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})$ tends to a_l . We set $A_1 = \{l \in \{0, \dots, p\} : a_l = +\infty\}$, $A_2 = \{l \in \{0, \dots, p\} : a_l = 0\}$ and $A_3 = \{l \in \{0, \dots, p\} : a_l \in (0, +\infty)\}$.

We fix now $(\alpha(k))$, $(\beta(k))$ two sequences of non-zero complex numbers and $(z(k))$ a sequence in \mathbb{C}^{p+1} such that $(\alpha(k), \beta(k), z(k))$ is dense in \mathbb{C}^{p+3} . We set

$$x = u + \sum_{k=0}^{+\infty} y(k)$$

where, for $l = 0, \dots, p$,

$$y_{n_k+l}(k) = \begin{cases} \frac{v_{1,l}^{1/m_0}}{\alpha(k)^{1/m_0} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m_0}} & \text{provided } l \in A_1, \\ \frac{v_{1,l}^{1/m_1}}{\beta(k)^{1/m_1} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m_1}} & \text{provided } l \in A_2, \\ z_l(k) & \text{provided } l \in A_3 \end{cases}$$

and $y_j(k) = 0$ if $j \neq n_k, \dots, n_k + p$.

We claim that $x \in U \cap \mathcal{A}(I, s, V_1, V_2)$. The definition of ε and p ensure that $x \in U$. Let $P \in E(I, s)$. There exists an increasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\alpha(\phi(k)) \rightarrow \hat{P}(m_0)$, $\beta(\phi(k)) \rightarrow \hat{P}(m_1)$ and $a_l P(z_l(\phi(k))) \rightarrow v_{1,l}$ for all $l \in A_3$. We claim that $(B_i^{n_{\phi(k)}}(P(x)))$ belongs to V_i , $i = 1, 2$, provided k is large enough. It suffices to prove that for $l = 0, \dots, p$ and $i = 1, 2$, the l -th coordinate of $B_i^{n_{\phi(k)}}(P(x))$ tends to $v_{i,l}$. The details for the case $i = 1$ are identical as in [4, Theorem 4.8] hence we will present the case $i = 2$. Assume first that $l \in A_1$. The l -th coordinate of $B_2^{n_{\phi(k)}}(P(x))$ is equal to

$$w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)} P \left(\frac{v_{1,l}^{1/m_0}}{\alpha(\phi(k))^{1/m_0} (w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)})^{1/m_0}} \right).$$

Now, since $w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}$ tends to $+\infty$, and $m_0 = \min(I)$, we have for $m > m_0$,

$$\frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{(w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)})^{m/m_0}} = \frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}} (w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)})^{1-\frac{m}{m_0}} \rightarrow 0.$$

Hence,

$$\begin{aligned} & w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)} P \left(\frac{v_{1,l}^{1/m_0}}{\alpha(\phi(k))^{1/m_0} (w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)})^{1/m_0}} \right) = \\ & \frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}} \hat{P}(m_0) \frac{v_{1,l}}{\alpha(\phi(k))} + o(1) \end{aligned}$$

and this tends to $v_{2,l}$. When $l \in A_2$, the proof is similar. Finally, when $l \in A_3$, the l -th coordinate of $B_2^{n_{\phi(k)}}(P(x))$ is equal to

$$w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)} P(z_l(\phi(k))) = \frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}} w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)} P(z_l(\phi(k)))$$

which tends again to $v_{2,l}$. □

2.3. Disjoint hypercyclic algebras for backward shifts - convolution product.

2.3.1. Backward shifts raised to different powers. We now turn to the case of Fréchet sequence algebras endowed with the Cauchy product, and we first study the existence of a disjoint hypercyclic algebra for two backward shifts raised to different powers. This situation seems more complicated and we choose to present only two significant examples. The first one is the multiples of B and B^2 on ℓ_1 . It is known by [6, Cor 4.2] that λB and μB^2 are disjointly hypercyclic if and only if $1 < \lambda < \mu$. We show that under this condition, they even support a disjoint hypercyclic algebra. The proof will be done in two steps. We first handle the case $\lambda < \mu^{1/2}$. The proof uses a shifting factor which is placed after the term to approximate.

Theorem 2.14. *Let $1 < \lambda < \mu^{1/2}$. Then λB and μB^2 , acting on ℓ_1 endowed with the Cauchy product, support a disjoint hypercyclic algebra.*

Proof. We shall prove that the assumptions of Corollary 2.2 are satisfied. Let $m \geq 1$, let U, V_1, V_2 be nonempty open sets and let W be a neighbourhood of 0. Let $p \geq 1$, $x \in U$, $y_1 \in V_1$, $y_2 \in V_2$ with $\text{supp}(x), \text{supp}(y_1), \text{supp}(y_2) \subset [0, p]$ and let us write $y_i = \sum_{l=0}^p y_{i,l} e_l$, $i = 1, 2$. Let finally $\delta > 0$ be such that $B(y_i, 2\delta) \subset V_i$. For σ a large integer, we set

$$\begin{aligned} N &= m\sigma - 3p \\ d_{1,l} &= \frac{y_{1,l}}{m\lambda^{N/m}\delta^{(m-1)/m}}, & \varepsilon_1 &= \frac{\delta^{1/m}}{\lambda^{N/m}} \\ d_{2,l} &= \frac{y_{2,l}}{m\mu^{N/m}\delta^{(m-1)/m}}, & \varepsilon_2 &= \frac{\delta^{1/m}}{\mu^{N/m}} \\ z_1 &= \sum_{l=0}^p d_{1,l} e_{\sigma-3p+l} + \varepsilon_1 e_\sigma \end{aligned}$$

$$z_2 = \sum_{l=0}^p d_{2,l} e_{2\sigma-6p+l} + \varepsilon_2 e_{2\sigma}$$

$$u = x + z_1 + z_2.$$

The terms $\varepsilon_1 e_\sigma$ and $\varepsilon_2 e_{2\sigma}$ are shifting terms which will be used to build an appropriate approximating term when we will take the m -th power, despite the mixing of the supports.

It is easy to check that, provided σ , hence N , is large enough, then $u \in U$. We shall prove that we also have

$$\begin{cases} (\lambda B)^N(u^m) \in V_1, (\lambda B)^N(u^n) \in W \text{ for } n < m, \\ (\mu B^2)^N(u^m) \in V_2, (\mu B^2)^N(u^n) \in W \text{ for } n < m. \end{cases}$$

The two last properties are easier to prove: there is no more difficulty to prove them than to prove that μB^2 admits a hypercyclic algebra. Indeed, if $n < m$, then $\text{supp}(u^n) \subset [0, 2n\sigma]$ with $2n\sigma < 2N$ whence $(\mu B^2)^N(u^n) = 0$. On the other hand,

$$u^m = z + \sum_{l=0}^p \frac{y_{2,l}}{\mu^N} e_{2m\sigma-6p+l} + \frac{\delta}{\mu^N} e_{2m\sigma}$$

with $\max(\text{supp}(z)) < 2N$, hence

$$(\mu B^2)^N(u^m) = y_2 + \delta e_{6p} \in V_2.$$

The proofs of the first two properties are more difficult: the powers of z_2 are mixed with that of $x + z_1$. Such a mixing term does not always disappear when you apply B^N . It is also not always smaller than λ^{-N} . But taking into account the strong condition $\mu > \lambda^2$, we will show that it keeps smaller than λ^{-N} if its support exceeds $[0, N]$. We start by writing

$$\begin{aligned} u^m &= (x + z_1)^m + \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma \geq 1}} \binom{m}{\alpha, \beta, \gamma} x^\alpha z_1^\beta z_2^\gamma \\ &= z + \sum_{l=0}^p \frac{y_{1,l}}{\lambda^N} e_{m\sigma-3p+l} + \frac{\delta}{\lambda^N} e_{m\sigma} + \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma \geq 1}} \binom{m}{\alpha, \beta, \gamma} x^\alpha z_1^\beta z_2^\gamma \end{aligned}$$

where

$$\binom{m}{\alpha, \beta, \gamma} = \frac{m!}{\alpha! \beta! \gamma!}$$

is the multinomial coefficient and $\max(\text{supp}(z)) < N$. For the first three terms, we proceed as before to observe that, provided σ is large enough,

$$(\lambda B)^N \left(z + \sum_{l=0}^p \frac{y_{1,l}}{\lambda^N} e_{m\sigma-3p+l} + \frac{\delta}{\lambda^N} e_{m\sigma} \right) = y_1 + \delta e_{3p}.$$

Let us now look at $(\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma)$ with $\gamma \geq 1$ and $\alpha + \beta + \gamma = m$. Expanding the product and using the behaviour of the terms $d_{i,j}$ and ε_i , $x^\alpha z_1^\beta z_2^\gamma$ is a sum of a finite number of

terms $c_j e_j$ (this finite number does not depend on σ), with

$$j \leq p\alpha + \beta\sigma + 2\gamma\sigma$$

$$|c_j| \leq C_{\alpha,\beta,\gamma} \frac{1}{\lambda^{\beta N/m} \mu^{\gamma N/m}}$$

where the involved constant $C_{\alpha,\beta,\gamma}$ does not depend on σ (of course, it also depends on x , y_1 , y_2 , δ and m). We then distinguish two cases. Assume first that $\beta + 2\gamma < m$. Then, provided σ is large enough, $p\alpha + \beta\sigma + 2\gamma\sigma < N$, hence $(\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma) = 0$. On the contrary, if $\beta + 2\gamma \geq m$, then

$$\left\| (\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma) \right\| \leq C_{\alpha,\beta,\gamma} \frac{\lambda^N}{\lambda^{\beta N/m} \mu^{\gamma N/m}}$$

and the right handside goes to 0 since $\mu > \lambda^2$, hence $\mu^\gamma \lambda^\beta > \lambda^m$. This shows that, provided σ is large enough, $(\lambda B)^N(u^m) \in V_1$. Finally, for $n < m$, it is easy to check that

$$(\lambda B)^N(u^n) = \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma \geq 1}} \binom{n}{\alpha, \beta, \gamma} (\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma).$$

We use exactly the same arguments to prove that

- $(\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma) = 0$ provided $\beta + 2\gamma \leq m$;
- $(\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma)$ tends to zero as σ tends to $+\infty$ provided $\beta + 2\gamma > m$.

Hence, $(\lambda B)^N(u^n)$ belongs to W , which finishes the proof of Theorem 2.14. \square

We turn to the case $\lambda \geq \mu^{1/2}$. The proof follows a similar scheme, but we now put the shifting term before the approximating term.

Theorem 2.15. *Let $1 < \lambda < \mu \leq \lambda^2$. Then λB and μB^2 , acting on ℓ^1 endowed with the Cauchy product, support a disjoint hypercyclic algebra.*

Proof. Again we shall prove that the assumptions of Corollary 2.2 are satisfied. Thus, let $m \geq 1$, let U, V_1, V_2 be nonempty open subsets of ℓ_1 and let W be a neighbourhood of 0. Let $p \geq 1$, $x \in U$, $y_1 \in V_1$, $y_2 \in V_2$ with $\text{supp}(x)$, $\text{supp}(y_1)$ and $\text{supp}(y_2) \subset [0, p]$ and let us write $y_i = \sum_{l=0}^p y_{i,l} e_l$, $i = 1, 2$. Let $\sigma > 2p$ and let $\varepsilon > 0$ be such that the ball $B(x, 2\varepsilon)$ is contained in U . For N a large integer, we define

$$z_1 = \frac{1}{m\varepsilon^{m-1}\lambda^N} \sum_{l=0}^p y_{1,l} e_{N-(m-1)\sigma+l}$$

$$z_2 = \frac{1}{m\varepsilon^{m-1}\mu^N} \sum_{l=0}^p y_{2,l} e_{2N-(m-1)\sigma+l}$$

$$u = x + \varepsilon e_\sigma + z_1 + z_2.$$

The definition of ε ensures that, for N large enough, u belongs to U . Let us show that the other assumptions of Corollary 2.2 are satisfied. First, consider $n < m$ and write

$$u^n = w_1 + w_2$$

where

$$w_1 = \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma \leq 1}} \binom{n}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma$$

$$w_2 = \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma > 1}} \binom{n}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma + \sum_{\substack{\alpha+\beta+\gamma+\delta=n \\ \delta \geq 1}} \binom{n}{\alpha, \beta, \gamma, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma z_2^\delta.$$

The support of w_1 is contained in $[0, N - (m-1)\sigma + p + (n-1)\sigma] \subset [0, N]$ since $\sigma > p$ and $n < m$. Thus $(\lambda B)^N w_1^n = 0$. Moreover, it is easy to check that

$$\|w_2\| \leq C \max\left(\frac{1}{\lambda^{2N}}, \frac{1}{\mu^N}\right)$$

where the constant C depends on $x, p, \varepsilon, y_1, y_2$ but not on N . Since $1 < \lambda < \mu$, the norm of $(\lambda B)^N w_2$ can be adjusted to be arbitrarily small, so that $(\lambda B)^N u^n$ belongs to W . In the same vein, write

$$u^n = w'_1 + w'_2$$

where

$$w'_1 = \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma \leq 2}} \binom{n}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma + \sum_{\substack{\alpha+\beta+\delta=n \\ \delta=1}} \binom{n}{\alpha, \beta, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_2^\delta$$

$$w'_2 = \sum_{(\alpha, \beta, \gamma, \delta) \in E_n} \binom{n}{\alpha, \beta, \gamma, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma z_2^\delta$$

with

$$E_n = \left\{ (\alpha, \beta, \gamma, \delta) \in \mathbb{N}_0^4 : \alpha + \beta + \gamma + \delta = n, (\gamma \geq 3 \text{ and } \delta = 0) \right. \\ \left. \text{or } (\delta \geq 2 \text{ and } \gamma = 0) \text{ or } (\gamma \geq 1 \text{ and } \delta \geq 1) \right\}.$$

As before, the support of w'_1 is contained in $[0, 2N)$, so that $(\mu B^2)^N(w'_1) = 0$. Since

$$\|w'_2\| \leq C \max\left(\frac{1}{\lambda^{3N}}, \frac{1}{\mu^{2N}}, \frac{1}{\lambda^N \mu^N}\right)$$

and $\mu \leq \lambda^2 < \lambda^3$, we get that $(\mu B^2)^N(w'_2)$ belongs to W provided that N is large enough. Let us now inspect u^m . We first write it

$$u^m = \tilde{w}_1 + \frac{1}{\lambda^N} \sum_{l=0}^p y_{1,l} e_{N+l} + \tilde{w}_2$$

where

$$\tilde{w}_1 = \sum_{\alpha+\beta+\gamma \in F_m} \binom{m}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma$$

$$\tilde{w}_2 = \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma>1}} \binom{m}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma + \sum_{\substack{\alpha+\beta+\gamma+\delta=m \\ \delta \geq 1}} \binom{m}{\alpha, \beta, \gamma, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma z_2^\delta$$

with $F_m = \{(\alpha, \beta, \gamma) \in \mathbb{N}_0^3 : \alpha + \beta + \gamma = m, \gamma \leq 1\} \setminus \{(0, m-1, 1)\}$. The same proof shows that $\text{supp}(\tilde{w}_1) \subset [0, N - (m-1)\sigma + p + (m-2)\sigma + p] \subset [0, N]$ and that $(\lambda B)^N(\tilde{w}_2)$ tends to 0 as N tends to $+\infty$. Hence, $(\lambda B)^N(u^m) \in V_1$ for N large enough. We also write

$$\begin{aligned} u^m &= \tilde{w}'_1 + \frac{1}{\mu^N} \sum_{l=0}^p y_{2,l} e_{2N+l} + \tilde{w}'_2 \\ \tilde{w}'_1 &= \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma \leq 2}} \binom{m}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma + \sum_{(\alpha, \beta, \delta) \in F'_m} \binom{m}{\alpha, \beta, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_2^\delta \\ w'_2 &= \sum_{(\alpha, \beta, \gamma, \delta) \in E_m} \binom{m}{\alpha, \beta, \gamma, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma z_2^\delta \\ F'_m &= \{(\alpha, \beta, \delta) \in F_m : \delta = 1\}. \end{aligned}$$

Looking at the support of \tilde{w}'_1 and at the norm of \tilde{w}'_2 , we show again that $(\mu B^2)^N(u^m)$ belongs to V_2 for N large enough, showing that the assumptions of Corollary 2.2 are satisfied. \square

We end up this section by proving the existence of a disjoint hypercyclic algebra for the multiples of D and D^2 . The proof follows the scheme of that of Theorem 2.14, but we need a more careful analysis.

Theorem 2.16. *Let $\lambda, \mu > 0$. Then λD and μD^2 support a disjoint hypercyclic algebra on $H(\mathbb{C})$ endowed with the convolution product.*

Proof. The scheme of the proof is similar to that of Theorem 2.14; nevertheless, we will need to be more careful when we analyze $(\lambda D)^N(x^\alpha z_1^\beta z_2^\gamma)$ for $\beta + 2\gamma = m$. We fix $m \geq 1$, U, V_1, V_2 and W as above. Let $q \geq 0$ and $x \in U$ with support in $[0, q]$. Let now $p \geq 1$ be such that $2p > qm$ and such that there exists $y_i \in V_i$ with $\text{supp}(y_i) \subset [0, p]$, $i = 1, 2$. We again write $y_i = \sum_{l=0}^p y_{i,l} e_i$ and we consider $r \geq 1$ and $\delta > 0$ such that $B_{\|\cdot\|_r}(y_i, 2\delta) \subset V_i$. We then consider $\omega > 1/m$ satisfying the following property: for all integers $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma \leq m$ and $\beta + 2\gamma > m$, we have

$$(6) \quad \beta + 2\gamma(m - m(m-1)\omega) > m$$

(observe that when $\omega = 1/m$, then the left-hand side is equal to $\beta + 2\gamma$, which is greater than m , and that there is only a finite number of constraints to satisfy). For σ a large integer, we set

$$\begin{aligned} N &= m\sigma - 3p, \\ d_{1,l} &= \frac{l! y_{1,l}}{m \varepsilon_1^{m-1} \lambda^N (N+l)!}, \quad \varepsilon_1 = \frac{((3p)!)^{1/m} \delta^{1/m}}{(3p)^{r/m} \lambda^{N/m} [(N+3p)!]^{1/m}}, \end{aligned}$$

$$\begin{aligned}
d_{2,l} &= \frac{l!y_{2,l}}{m\varepsilon_2^{m-1}\mu^N(2N+l)!}, & \varepsilon_2 &= \frac{\delta^{1/m}}{\mu^{N/m}[(2N+6p)!]^\omega}, \\
z_1 &= \sum_{l=0}^p d_{1,l}e_{\sigma-3p+l} + \varepsilon_1 e_\sigma, \\
z_2 &= \sum_{l=0}^p d_{2,l}e_{2\sigma-6p+l} + \varepsilon_2 e_{2\sigma}, \\
u &= x + z_1 + z_2.
\end{aligned}$$

The first part of the proof of Theorem 2.14 carries on to our new context. First, we observe that, using Stirling's formula, we now have the crude estimates, for some $C > 0$ and some $\kappa > 0$,

$$\begin{aligned}
|d_{1,l}| &\leq \frac{C\kappa^\sigma}{\sigma^\sigma} \\
|d_{2,l}| &\leq \frac{C\kappa^\sigma}{\sigma^{2(m-m(m-1)\omega)\sigma}} \\
\varepsilon_1 &\leq \frac{C\kappa^\sigma}{\sigma^\sigma} \\
\varepsilon_2 &\leq \frac{C\kappa^\sigma}{\sigma^{2m\omega\sigma}} \leq \frac{C\kappa^\sigma}{\sigma^{2(m-m(m-1)\omega)\sigma}}
\end{aligned}$$

(the very last inequality is a consequence of $\omega > 1/m$). Hence, we still have $u \in U$ provided σ is large enough. Looking at the support of u^n shows that $(\mu D^2)^N(u^n) = 0$ for all $n < m$, whereas the values of ε_2^m and $m\varepsilon_2^{m-1}d_{2,l}$ ensure that

$$(\mu D^2)^N(u^m) = y_2 + \frac{\delta}{((2N+6p)!)^{m\omega-1}(6p)!} e_{6p}.$$

Since $\omega > 1/m$, we clearly have $\|(\mu D^2)^N(u^m) - y_2\|_r < 2\delta$ provided N is large enough.

Regarding $(\lambda D)^N$, the support of each element and the definitions of $d_{1,l}$, ε_1 ensure that

$$\begin{aligned}
(\lambda D)^N(u^m) &= y_1 + \frac{\delta}{(3p)^r} e_{3p} + \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma \geq 1, \beta+2\gamma \geq m}} \binom{m}{\alpha, \beta, \gamma} (\lambda D)^N(x^\alpha z_1^\beta z_2^\gamma) \\
(\lambda D)^N(u^n) &= \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma \geq 1, \beta+2\gamma \geq m}} \binom{n}{\alpha, \beta, \gamma} (\lambda D)^N(x^\alpha z_1^\beta z_2^\gamma) \text{ for } n < m.
\end{aligned}$$

Hence, it remains to show that $(\lambda D)^N(x^\alpha z_1^\beta z_2^\gamma)$ tends to zero as σ tends to $+\infty$ when $\alpha + \beta + \gamma \leq m$, $\gamma \geq 1$ and $\beta + 2\gamma \geq m$. We first assume that $\beta + 2\gamma > m$. The product $x^\alpha z_1^\beta z_2^\gamma$ may be written as a finite sum (whose number of terms do not depend on σ) $\sum_j c_j e_j$ where

$$j \leq q\alpha + \beta\sigma + 2\gamma\sigma$$

$$|c_j| \leq \frac{C_{\alpha,\beta,\gamma} \kappa_{\alpha,\beta,\gamma}^\sigma}{\sigma^{\beta\sigma+2\gamma(m-(m-1)\omega)\sigma}}.$$

Thus, using $\|(\lambda D)^N e_j\|_r \leq \lambda^N j^r j^N$, we get that for some different constants $C_{\alpha,\beta,\gamma}$ and $\kappa_{\alpha,\beta,\gamma}$,

$$\left\| (\lambda D)^N (x^\alpha z_1^\beta z_2^\gamma) \right\|_r \leq \frac{C_{\alpha,\beta,\gamma} \kappa_{\alpha,\beta,\gamma}^\sigma \sigma^{m\sigma}}{\sigma^{\beta\sigma+2\gamma(m-(m-1)\omega)\sigma}}.$$

The choice of ω (sufficiently close to $1/m$, see (6)) guarantees that this goes to 0 as σ tends to $+\infty$. Finally, assume that $\beta + 2\gamma = m$ and define

$$w_2 = \sum_{l=0}^p d_{2,l} e_{2\sigma-6p+l}$$

so that

$$\begin{aligned} x^\alpha z_1^\beta z_2^\gamma &= x^\alpha z_1^\beta (w_2 + \varepsilon_2 e_{2\sigma})^\gamma \\ &= \sum_{k=0}^{\gamma} \binom{\gamma}{k} \varepsilon_2^{\gamma-k} x^\alpha z_1^\beta w_2^k e_{2(\gamma-k)\sigma}. \end{aligned}$$

Assume first that $k \neq 0$. Then

$$\begin{aligned} \max(\text{supp}(x^\alpha z_1^\beta w_2^k e_{2(\gamma-k)\sigma})) &\leq q\alpha + \beta\sigma + 2\gamma\sigma - 5p \\ &\leq qm + m\sigma - 5p \\ &< N \end{aligned}$$

since $qm < 2p$. This implies that

$$(\lambda D)^N (x^\alpha z_1^\beta w_2^k e_{2(\gamma-k)\sigma}) = 0$$

and it only remains to handle the term $\varepsilon_2^\gamma x^\alpha z_1^\beta e_{2\gamma\sigma}$. As before, it is equal to a finite sum $\sum_j c_j e_j$ with $j \leq q\alpha + (\beta + 2\gamma)\sigma$ but now we can be slightly more precise on $|c_j|$:

$$|c_j| \leq \frac{C_{\alpha,\beta,\gamma} \kappa_{\alpha,\beta,\gamma}^\sigma}{\sigma^{\beta\sigma+2\gamma m\omega\sigma}}.$$

Therefore,

$$\left\| (\lambda D)^N (x \varepsilon_2^\gamma x^\alpha z_1^\beta e_{2\gamma\sigma}) \right\|_r \leq \frac{C_{\alpha,\beta,\gamma} \kappa_{\alpha,\beta,\gamma}^\sigma \sigma^{m\sigma}}{\sigma^{\beta\sigma+2\gamma m\omega\sigma}}.$$

Now, since $\omega > 1/m$ and $\beta + 2\gamma = m$, this tends to 0 as σ goes to $+\infty$. This finishes the proof of Theorem 2.16. \square

Theorem 2.9 has the following analogue when we are dealing with the convolution product.

Theorem 2.17. *If $B_1 = B_{w(1)}$ and $B_2 = B_{w(2)}$ are weighted backward shifts on ω endowed with the Cauchy product, then B_1 and B_2^2 support a disjoint hypercyclic algebra.*

Proof. We will apply Corollary 2.2. Let $m \geq 1$, U, V_1, V_2 be nonempty open subsets of ω , and W be a zero neighbourhood. Let also $p \in \mathbb{N}_0$, $u_0, \dots, u_p, v_{i,0}, \dots, v_{i,p} \in \mathbb{C}$, for $i = 1, 2$, and $\varepsilon > 0$ be such that, for all $x \in \omega$,

$$\begin{aligned} |x_l - u_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in U, \\ |x_l - v_{i,l}| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in V_i, i = 1, 2 \text{ and} \\ |x_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in W. \end{aligned}$$

We set

$$x = \sum_{l=0}^p u_l e_l + e_{2p} + z_1 + z_2$$

where

$$z_1 = \sum_{l=0}^p \frac{v_{1,l}}{w_{l+1}^{(1)} \cdots w_{6pm+l}^{(1)}} e_{2p(2m+1)+l}$$

and

$$z_2 = \sum_{l=0}^p \frac{v_{2,l}}{w_{l+1}^{(2)} \cdots w_{12pm+l}^{(2)}} e_{2p(5m+1)+l}.$$

It is immediate that $x \in U$. If $N = 6pm$, it is tedious but straightforward to check that the support of $B_1^N(x^n)$, with $n \leq m$, intersects $[0, p]$ only when $n = m$. Specifically, for $l = 0, \dots, p$, the l -th coordinate of $B_1^N(x^m)$ is the l -th coordinate of $B_1^N(e_{2p}^{m-1} z_1)$ which equals to $v_{1,l}$. That means that $B_1^N(x^n) \in W$, for $n < m$ and $B_1^N(x^m) \in V_1$. Similarly, it follows that $B_2^{2N}(x^n) \in W$, for $n < m$ and $B_2^{2N}(x^m) \in V_2$ which concludes the proof. \square

2.3.2. Backward shifts raised to the same power. We conclude this section by studying the existence of disjoint hypercyclic algebras for two backward shifts (raised to the same power). We will need something similar to the converse inequality of the continuity of the product under the F -norm. This idea is captured by the notion of a regular Fréchet sequence algebra which first appeared in [4].

Definition 2.18. A Fréchet sequence algebra $(X, (\|\cdot\|_q))$ is said to be *regular* when it satisfies the following properties:

- (1) X admits a continuous norm;
- (2) (e_i) is a Schauder basis for X ;
- (3) for any $r \geq 1$, there exist $q \geq 1$ and $C > 0$ such that, for all $n, k \geq 0$,

$$\|e_n\|_r \cdot \|e_k\|_r \leq C \|e_{n+k}\|_q.$$

We were able to get a general result saying that two disjoint hypercyclic backward shifts support a disjoint hypercyclic algebra, a statement simpler than that regarding the coordinatewise product.

Theorem 2.19. *Let X be a regular Fréchet sequence algebra for the Cauchy product and let $B_1 = B_{w^{(1)}}$, $B_2 = B_{w^{(2)}}$ be two bounded weighted shifts on X . The following assertions are equivalent.*

- (i) B_1 and B_2 are disjointly hypercyclic.
- (ii) B_1 and B_2 support a disjoint hypercyclic algebra.
- (iii) There exists a sequence (n_k) such that $((w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1} e_{n_k+l})$ tends to zero for all $l \geq 0$ and the set

$$\left\{ \left(\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \right)_{l \geq 0} : k \geq 0 \right\}$$

is dense in ω .

For the proof of Theorem 2.19 we will use property (3) from the definition of regularity through the following lemma from [4, Corollary 4.14] which works as some kind of index managing tool.

Lemma 2.20. *Let X be a regular Fréchet sequence algebra for the Cauchy product and let (w_n) be an admissible weight sequence on X . Then, for all $m \geq 1$, for all $N \geq 1$, for all $r \geq 1$, and for all $\rho \geq 0$, there exist $C > 0$ and $q \geq 1$ such that, for all $n \geq mN$, for all $s \in \{1, \dots, N\}$,*

$$(w_{n-s+1+\rho})^{m-1} \cdots (w_{n-1+\rho})^{m-1} (w_{n+\rho})^{m-1} \|e_{n-ms+m\rho}\|_r \leq C \|e_{n-s+\rho}\|_q.$$

Proof of Theorem 2.19. The implication (ii) \implies (i) is trivial. Let us prove the other ones.

(iii) \implies (ii). Let $m \geq 1$, let U, V_1, V_2 be nonempty open subsets of X and let W be a neighbourhood of 0. Let $p \geq 1$, $x \in U$, $y_1 \in V_1$, $y_2 \in V_2$ with $\text{supp}(x), \text{supp}(y_1), \text{supp}(y_2) \subset [0, p]$ and let us write $y_i = \sum_{l=0}^p y_{i,l} e_l$. Without loss of generality, we may assume $y_{1,l} \neq 0$ for all $l = 0, \dots, p$. Let finally B be a ball for the seminorm $\|\cdot\|_r$ with radius δ such that $B \subset W$, $x + B \subset U$ and $y_i + B \subset V_i$, $i = 1, 2$. We consider (n_k) a sequence of integers such that

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1} e_{n_k+l} &\text{ tends to 0 for all } l = 0, \dots, p \\ \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} &\text{ tends to } \frac{y_{2,l}}{y_{1,l}} \text{ for all } l = 0, \dots, p \\ \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} &\text{ tends to 1 for all } l = 3p, \dots, 3p + (m-1). \end{aligned}$$

We can assume without loss of generality $n_1 > (m-1)3p$. We then consider, for each $k \geq 1$, the integer J_k defined by

$$n_k + 3p \leq mJ_k < n_k + 3p + m$$

and we set $N_k = J_k - (mJ_k - n_k) \geq 1$ so that $3p \leq J_k - N_k < 3p + m$. We finally define

$$\varepsilon = \max_{0 \leq l \leq p} \left(\frac{\|e_{N_k+l}\|_r}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{\frac{1}{2(m-1)}} \times \min \left(\frac{1}{\|e_{J_k}\|_r}, \frac{1}{(w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{1/m}} \right)^{\frac{1}{2}},$$

$$d_l = \frac{y_{1,l}}{m\varepsilon^{m-1}w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}},$$

$$u = x + \sum_{l=0}^p d_l e_{N_k+l} + \varepsilon e_{J_k}.$$

Let us assume for a while that the following facts are true:

- (7) $\varepsilon \|e_{J_k}\|_r \rightarrow 0$ as $k \rightarrow +\infty$,
- (8) $|d_l| \cdot \|e_{N_k+l}\|_r \rightarrow 0$ as $k \rightarrow +\infty$, for all $l = 0, \dots, p$,
- (9) $\varepsilon^m w_1^{(1)} \cdots w_{mJ_k}^{(1)} \rightarrow 0$ as $k \rightarrow +\infty$.

Then it is clear that u belongs to U for a large integer k and that $B_i^{n_k}(u^n) = 0$ for $i = 1, 2$ and $n < m$, since $\max(\text{supp}(u^n)) \leq (m-1)J_k = n_k - N_k < n_k$. We can also write

$$u^m = z + \sum_{l=0}^p \frac{y_{1,l}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} e_{N_k+(m-1)J_k+l} + \varepsilon^m e_{mJ_k},$$

with

$$\max(\text{supp}(z)) \leq (m-2)J_k + 2N_k + 2p < N_k + (m-1)J_k = n_k.$$

Hence,

$$B_i^{n_k}(u^m) = \sum_{l=0}^p \frac{w_{l+1}^{(i)} \cdots w_{n_k+l}^{(i)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} y_{1,l} e_l + \varepsilon^m w_{mJ_k-n_k+1}^{(i)} \cdots w_{mJ_k}^{(i)} e_{mJ_k-n_k}.$$

When $i = 1$, the above sum is equal to y_1 ; when $i = 2$, it goes to y_2 . Hence, we have to prove that, for $i = 1, 2$,

$$\varepsilon^m w_{mJ_k-n_k+1}^{(i)} \cdots w_{mJ_k}^{(i)} e_{mJ_k-n_k} \xrightarrow{k \rightarrow +\infty} 0.$$

Since $(mJ_k - n_k)$ is a bounded sequence, this amounts to saying that

$$\varepsilon^m w_{mJ_k-n_k+1}^{(i)} \cdots w_{mJ_k}^{(i)} \xrightarrow{k \rightarrow +\infty} 0.$$

For $i = 1$, this is (9). For $i = 2$, we just observe that $mJ_k - n_k \in \{3p, \dots, 3p + m - 1\}$ and we write

$$\frac{w_{mJ_k-n_k+1}^{(2)} \cdots w_{mJ_k}^{(2)}}{w_{mJ_k-n_k+1}^{(1)} \cdots w_{mJ_k}^{(1)}} = \frac{w_{mJ_k-n_k+1}^{(2)} \cdots w_{n_k+(mJ_k-n_k)}^{(2)}}{w_{mJ_k-n_k+1}^{(1)} \cdots w_{n_k+(mJ_k-n_k)}^{(1)}} \xrightarrow{k \rightarrow +\infty} 1.$$

It remains to prove (7), (8), (9).

Let us first prove (7). By property (3) from Definition 2.18 and an easy induction on m , there exist $q \geq 1$ and $C > 0$ (depending on r and m) such that, for all $k \geq 1$ and all $l \in \{0, \dots, p\}$,

$$\frac{\|e_{N_k+l}\|_r^{\frac{1}{2(m-1)}} \cdot \|e_{J_k}\|_r}{\|e_{J_k}\|_r^{\frac{1}{2}}} = (\|e_{N_k+l}\|_r \cdot \|e_{J_k}\|_r^{m-1})^{\frac{1}{2(m-1)}} \\ \leq C \|e_{n_k+l}\|_q^{\frac{1}{2(m-1)}}.$$

Hence,

$$\varepsilon \|e_{J_k}\|_r \leq C \max_{0 \leq l \leq p} \left(\frac{\|e_{n_k+l}\|_q}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{\frac{1}{2(m-1)}}$$

and this goes to zero as k tends to $+\infty$.

Regarding (8), we first write

$$|d_l| \cdot \|e_{N_k+l}\|_r \leq C \left(\frac{\|e_{N_k+l}\|_r}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{1/2} \times \max \left(\|e_{J_k}\|_r, (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{1/m} \right)^{(m-1)/2}.$$

If the maximum is attained for $\|e_{J_k}\|_r$, then using

$$\begin{aligned} \|e_{N_k+l}\|_r \cdot \|e_{J_k}\|_r^{m-1} &\leq C \|e_{(m-1)J_k+N_k+l}\|_q \\ &\leq C \|e_{n_k+l}\|_q, \end{aligned}$$

we get

$$|d_l| \cdot \|e_{N_k+l}\|_r \leq C \left(\frac{\|e_{n_k+l}\|_q}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{1/2}$$

and this goes to 0. Otherwise, we apply Lemma 2.20 with $n = mJ_k$ and $s = mJ_k - n_k - l$ to get

$$\begin{aligned} (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{m-1} \|e_{N_k+l}\|_r^m &\leq C (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{m-1} \|e_{mN_k+ml}\|_q \\ &= C (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{m-1} \|e_{mJ_k-ms}\|_q \\ &\leq C' (w_1^{(1)} \cdots w_{mJ_k-s}^{(1)})^{m-1} \|e_{mJ_k-s}\|_{q'} \\ &= C' (w_1^{(1)} \cdots w_{n_k+l}^{(1)})^{m-1} \|e_{n_k+l}\|_{q'} \end{aligned}$$

whence

$$|d_j| \cdot \|e_{N_k+l}\|_r \leq C'' \left(\frac{\|e_{n_k+l}\|_{q'}}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{1/2m},$$

which again goes to 0.

Finally, let us prove (9). Once more we apply Lemma 2.20 just as above and get, for all $l = 0, \dots, p$,

$$\begin{aligned} &\left(\frac{\|e_{N_k+l}\|_r}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{\frac{m}{2(m-1)}} (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{\frac{1}{2}} \\ &= \frac{\left(\|e_{N_k+l}\|_r^m (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{m-1} \right)^{\frac{1}{2(m-1)}}}{(w_1^{(1)} \cdots w_{n_k+l}^{(1)})^{\frac{m}{2(m-1)}}} \\ &\leq C_1 \frac{\left(\|e_{n_k+l}\|_{q'} (w_1^{(1)} \cdots w_{n_k+l}^{(1)})^{m-1} \right)^{\frac{1}{2(m-1)}}}{(w_1^{(1)} \cdots w_{n_k+l}^{(1)})^{\frac{m}{2(m-1)}}} \end{aligned}$$

$$= C_1 \left(\frac{\|e_{n_k+l}\|_{q'}}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{\frac{1}{2(m-1)}},$$

which achieves the proof of (9) since the last term goes to zero.

(i) \implies (iii). The proof is exactly the same as the proof of (i) \implies (ii) in Theorem 2.10, restricted to the case $m = 1$. \square

The situation in ω with the convolution product is easier as the next analogue of Theorem 2.13 shows.

Theorem 2.21. *If $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ are weighted backward shifts on ω endowed with the Cauchy product, then the following are equivalent.*

- (i) B_1 and B_2 are disjointly hypercyclic
- (ii) B_1 and B_2 support a disjoint hypercyclic algebra.
- (iii) The set

$$\left\{ \left(\frac{w_{l+1}^{(2)} \cdots w_{n+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n+l}^{(1)}} \right)_{l \geq 0} : n \geq 0 \right\}$$

is dense in ω .

Proof. The implication (ii) \implies (i) is immediate and the equivalence (i) \Leftrightarrow (iii) follows by Theorem 2.13. Hence, we establish that (iii) \implies (ii). We will apply Corollary 2.2. Let $m \geq 1$, U, V_1, V_2 be nonempty open subsets of ω , and W be a zero neighbourhood. Let also $p \in \mathbb{N}_0$, $u_0, \dots, u_p, v_{i,0}, \dots, v_{i,p} \in \mathbb{C}$, for $i = 1, 2$, and $\varepsilon > 0$ be such that, for all $x \in \omega$,

$$|x_l - u_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in U,$$

$$|x_l - v_{i,l}| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in V_i, i = 1, 2 \text{ and}$$

$$|x_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in W.$$

We may also assume that $v_{1,l} \neq 0$ for $l = 0, \dots, p$. Let (n_k) be a subsequence such that

$$\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \rightarrow \frac{v_{2,l}}{v_{1,l}}, \quad \text{for } l = 0, \dots, p.$$

We then consider, for each $k \geq 1$, the integer J_k defined by

$$n_k + 3p \leq mJ_k < n_k + 3p + m$$

and we set $N_k = J_k - (mJ_k - n_k) \geq 1$ so that $3p \leq J_k - N_k < 3p + m$. Since evidently, $N_k \sim \frac{n_k}{m}$ we may assume that $N_k > p$, for all $k \geq 1$. We now set for $k \geq 1$,

$$x = \sum_{l=0}^p u_l e_l + z + e_{J_k},$$

where

$$z = z(k) = \sum_{l=0}^p \frac{v_{1,l}}{m \cdot w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} e_{N_k+l}.$$

We clearly have that $x \in U$. For $i = 1, 2$ and $n \leq m$, arguing as in the proof of Theorem 2.19, we get that the support of $B_i^{n_k}(x^n)$ intersects $[0, p]$ only when $n = m$. Specifically, the l -th coordinate of $B_i^{n_k}(x^m)$ is the l -th coordinate of $B_i^{n_k}(me_{J_k}^{m-1}z)$ which equals to

$$\frac{w_{l+1}^{(i)} \cdots w_{n_k+l}^{(i)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} v_{1,l}$$

and tends to $v_{i,l}$ as $k \rightarrow \infty$. Hence, for k large, we have that $B_i^{n_k}(x^n) \in W$, if $n < m$, and $B_i^{n_k}(x^m) \in V_i$, which concludes the proof. \square

3. COMMON HYPERCYCLIC ALGEBRAS

3.1. How to get a common hypercyclic algebra. We begin this section by a parametrized version of [4, Proposition 2.3]. Let us fix a topological space Λ which is a countable union of compact sets.

Proposition 3.1. *Let Λ be a countable union of compact sets and let $(T_\lambda)_{\lambda \in \Lambda}$ be a family of operators in $\mathcal{L}(X)$ such that the map $(\lambda, x) \mapsto T_\lambda(x)$ is continuous from $\Lambda \times X$ into X . Assume that, for all compact sets $K \subset \Lambda$, for all $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, there exists $m_0 \in I$ such that, for all U, V non-empty open subsets of X , for all neighborhood W of 0, one can find $u \in U$ such that, for all $\lambda \in K$, there exists $N \in \mathbb{N}$ satisfying*

$$\begin{cases} T_\lambda^N(u^m) \in W \text{ for } m \in I \setminus \{m_0\} \\ T_\lambda^N(u^{m_0}) \in V. \end{cases}$$

Then the set of points generating a common hypercyclic algebra for $(T_\lambda)_{\lambda \in \Lambda}$ is residual in X .

Proof. By the Baire Category theorem, it is enough to show the result assuming that $\Lambda = K$ is a compact set. For $V \subset X$ open and non-empty, W a neighborhood of 0, $I \subset \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, define

$$\mathcal{A}(I, V, W) = \left\{ u \in X : \forall \lambda \in \Lambda, \exists N \in \mathbb{N}, T_\lambda^N(u^m) \in W \text{ for } m \in I \setminus \{m_0(I)\} \text{ and } T_\lambda^N(u^{m_0(I)}) \in V \right\}$$

(here, $m_0(I)$ is uniquely defined by I using the assumptions of the lemma). The assumption tells us that each set $\mathcal{A}(I, V, W)$ is dense. The compactness of Λ together with the continuity of $(\lambda, x) \mapsto T_\lambda(x)$ ensure that these sets are also open. Fix now (V_k) a basis of open sets of X and (W_l) a basis of open neighborhoods of 0 and let us consider $u \in \bigcap_{I,k,l} \mathcal{A}(I, V_k, W_l)$. Let $P \in \mathbb{C}[z]$ be non-constant, $P(z) = \sum_{m \in I} \hat{P}(m)z^m$ with $\hat{P}(m) \neq 0$ for $m \in I$ and $P(0) = 0$. Let V be any non-empty open subset of X and let k, l be such that $\hat{P}(m_0(I))V_k + (\sum_{m \neq m_0(I)} |\hat{P}(m)|)W_l \subset V$. Since $u \in \mathcal{A}(I, V_k, W_l)$, for all $\lambda \in \Lambda$, there exists $N \in \mathbb{N}$ such that

$$T_\lambda^N(P(u)) = \hat{P}(m_0(I))T_\lambda^N(u^{m_0(I)}) + \sum_{m \neq m_0(I)} \hat{P}(m)T_\lambda^N(u^m) \in V.$$

Hence, $P(u)$ belongs to $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ and the proof is complete. \square

3.2. Common hypercyclic algebras for a family of backward shifts - coordinatewise product. We now specialize our study to backward shifts. Thus we fix X a Fréchet sequence algebra under the coordinatewise product with a continuous norm in which $\text{span}(e_i)$ is dense. Let $(w(\lambda))_{\lambda \in \Lambda}$ be a family of admissible weights. For simplicity, we will denote by T_λ the operator $B_{w(\lambda)}$ and by S_{λ^r} the forward shift defined by $S_{\lambda^r} e_i = \frac{1}{w_{i+1}^r(\lambda)} e_{i+1}$. For $x = (x_n) \in \omega$, we will also denote $x^{1/m} = (x_n^{1/m})$.

Proposition 3.1 will be interpreted under the following form.

Lemma 3.2. *Assume that the map $(\lambda, x) \mapsto T_\lambda(x)$ is continuous. Assume that, for all $m_0 \leq m_1 \in \mathbb{N}$, for all $v \in X$ with finite support, for all $M > 0$, for all \mathcal{O} neighborhood of 0, one can find*

- parameters $\lambda_1, \dots, \lambda_q$,
- sets of parameters $\Lambda_1, \dots, \Lambda_q \subset \Lambda$ with $\Lambda \subset \bigcup_{i=1}^q \Lambda_i$,
- integers $N_1, \dots, N_q \in \mathbb{N}$ with $N_1 > M$ and $N_{i+1} - N_i > M$ for all i ,

such that

- (i) $\sum_{i=1}^q S_{\lambda_i^{1/m_0}}^{N_i}(v^{1/m_0}) \in \mathcal{O}$;
- (ii) $\sum_{j=i+1}^q T_\lambda^{N_j} \left(\left(S_{\lambda_j^{1/m_0}}^{N_j} v^{1/m_0} \right)^m \right) \in \mathcal{O}$ for all $m \in [m_0, m_1]$, all $i \in \{1, \dots, q\}$, all $\lambda \in \Lambda_i$;
- (iii) $T_\lambda^{N_i} \left(\left(S_{\lambda_i^{1/m_0}}^{N_i} v^{1/m_0} \right)^m \right) \in \mathcal{O}$ for all $m \in (m_0, m_1]$, all $i \in \{1, \dots, q\}$, all $\lambda \in \Lambda_i$;
- (iv) $T_\lambda^{N_i} \left(\left(S_{\lambda_i^{1/m_0}}^{N_i} v^{1/m_0} \right)^{m_0} \right) - v \in \mathcal{O}$ for all $i \in \{1, \dots, q\}$, all $\lambda \in \Lambda_i$.

Then the set of points generating a common hypercyclic algebra for $(T_\lambda)_{\lambda \in \Lambda}$ is residual in X .

Proof. We apply Proposition 3.1. Let $I \subset \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$. We set $m_0 = \min(I)$, $m_1 = \max(I)$. Let U, V, W be three non-empty open subsets of X with $0 \in W$. Let $x \in U$, $v \in V$ with finite support and let \mathcal{O} be a neighborhood of zero such that $x + \mathcal{O} \subset U$, $v + \mathcal{O} + \mathcal{O} \subset V$ and $\mathcal{O} + \mathcal{O} \subset W$.

Let M be bigger than any integer in the support of x and v . The assumptions of the lemma give us parameters $\lambda_1, \dots, \lambda_q$, sets of parameters $\Lambda_1, \dots, \Lambda_q$ and integers N_1, \dots, N_q . We set

$$u = x + \sum_{i=1}^q S_{\lambda_i^{1/m_0}}^{N_i}(v^{1/m_0})$$

so that assumption (i) says that u belongs to U . Fix $\lambda \in \Lambda$ and let $i \in \{1, \dots, q\}$ be such that $\lambda \in \Lambda_i$. We intend to prove that $T_\lambda^{N_i}(u^m) \in W$ for $m \in I \setminus \{m_0\}$ and $T_\lambda^{N_i}(u^{m_0}) \in V$.

Since we are working with the coordinatewise product and by the choice of M , we have

$$u^m = x^m + \sum_{i=1}^q \left(S_{\lambda_i^{1/m_0}}^{N_i} (v^{1/m_0}) \right)^m$$

and

$$T_\lambda^{N_i}(u^m) = T_\lambda^{N_i} \left(\left(S_{\lambda_i^{1/m_0}}^{N_i} (v^{1/m_0}) \right)^m \right) + \sum_{j=i+1}^q T_\lambda^{N_j} \left(\left(S_{\lambda_j^{1/m_0}}^{N_j} (v^{1/m_0}) \right)^m \right).$$

Assumptions (ii), (iii) and (iv) and the choice of v and \mathcal{O} allow us to conclude that $T_\lambda^N(u^m) \in W$ for $m \in I \setminus \{m_0\}$ and $T_\lambda^N(u^{m_0}) \in V$. \square

We now give a more concrete application. We first need a definition.

Definition 3.3. Let $(x_{k,\alpha})_{k \geq 1, \alpha \in I}$ be a family of vectors of X . The series $\sum_{k \geq 1} x_{k,\alpha}$, $\alpha \in I$, are said to be *uniformly unconditionally convergent* if, for any neighborhood \mathcal{O} of 0, there exists $K \geq 1$ such that, for all $\alpha \in I$, for all sequences $(\omega_k) \subset [-1, 1]^{\mathbb{N}}$,

$$\sum_{k \geq K} \omega_k x_{k,\alpha} \in \mathcal{O}.$$

Implicitly, in the previous definition, we assume that each series $\sum_k \omega_k x_{k,\alpha}$ is convergent. Of course, if I is finite and each series $\sum_k x_{k,\alpha}$ is unconditionally convergent, then they are uniformly unconditionally convergent.

Theorem 3.4. Let $\Lambda = [a, b]$ be a compact interval and let $(w(\lambda))_{\lambda \in \Lambda}$ be a family of admissible weight sequences. Assume that

- (a) (e_i) is an unconditional basis of X ;
- (b) for each n , the function $\log(w_n)$ is non-decreasing and Lipschitz on Λ , with Lipschitz constant less than or equal to $L_n > 0$.
- (c) for all $p > 0$, for all $m_0 \geq 1$, there exists an increasing sequence of integers (n_k) such that

- the series $\sum_{k \geq 1} \frac{1}{(w_{l+1}(a) \cdots w_{l+n_k}(a))^{1/m_0}} e_{l+n_k}$ are unconditionally convergent for all $l = 0, \dots, p$;
- the series $\sum_{k \geq 1} \frac{1}{w_{l+1}(a) \cdots w_{l+n_{k+i}-n_i}(a)} e_{l+n_{k+i}-n_i}$, $i \geq 1$, $l = 0, \dots, p$ are uniformly unconditionally convergent;
- for all $k \geq 1$ and all $l = 0, \dots, p$, $w_{l+1}(a) \cdots w_{l+n_k}(a) \geq 1$;
- $\sum_{i > 1} 1/C_{n_i} = +\infty$, where $C_n = \sum_{k=1}^n L_k$.

Then the set of points generating a common hypercyclic algebra for $(T_\lambda)_{\lambda \in \Lambda}$ is residual in X .

Proof. Let us first prove that $(\lambda, x) \mapsto T_\lambda(x)$ is continuous. We first observe that the family $(T_\lambda)_{\lambda \in \Lambda}$ is pointwise bounded. Indeed, for all $x \in X$, since all functions (w_n) are non-decreasing, it follows from the unconditional convergence of $\sum_{n \geq 1} w_n(b) x_n e_{n-1}$ that the family $(T_\lambda(x))_{\lambda \in \Lambda}$ is bounded. Hence, by the Banach-Steinhaus theorem, the family

$(T_\lambda)_{\lambda \in \Lambda}$ is equicontinuous. Now let us fix some $(\mu, y) \in \Lambda \times X$. For all $(\lambda, x) \in \Lambda \times X$, we write

$$T_\lambda(x) - T_\mu(y) = T_\lambda(x - y) + T_\lambda(y) - T_\mu(y).$$

The equicontinuity of $(T_\lambda)_{\lambda \in \Lambda}$ ensures that, fixing $\varepsilon > 0$, there exists $\delta > 0$ such that, provided $\|x - y\| < \delta$, then $\|T_\lambda(x - y)\| < \varepsilon$ for all $\lambda \in \Lambda$. On the other hand, we write

$$T_\lambda(y) - T_\mu(y) = \sum_{n \geq 1} (w_n(\lambda) - w_n(\mu)) y_n e_{n-1}.$$

Again, the unconditional convergence of $\sum_{n \geq 1} w_n(b) y_n e_{n-1}$ implies that there exists $N \geq 1$ such that, for all $\lambda \in \Lambda$,

$$\left\| \sum_{n > N} (w_n(\lambda) - w_n(\mu)) y_n e_{n-1} \right\| < \varepsilon.$$

Finally, we observe that the continuity of each function w_n implies, provided $|\lambda - \mu|$ is small enough,

$$\left\| \sum_{n=1}^N (w_n(\lambda) - w_n(\mu)) y_n e_{n-1} \right\| < \varepsilon.$$

We now show that the assumptions of Lemma 3.2 are satisfied. Let $v = \sum_{l=0}^p v_l e_l \in X$ with finite support, $m_0 \leq m_1$, $M > 0$ and \mathcal{O} be a neighborhood of 0. By linearity (which is not destroyed by taking powers since we are using the coordinatewise product), it is enough to verify that the assumptions are satisfied for $v = e_l$. Let (n_k) be the sequence given by the assumptions of the theorem, and let $K \geq 1$ be such that, for all $i \geq 0$, for all sequences $(\omega_k) \subset [-1, 1]^{\mathbb{N}}$,

$$(10) \quad \sum_{k \geq K} \frac{\omega_k}{w_{l+1}(a) \cdots w_{l+n_i+k-n_i}(a)} e_{l+n_i+k-n_i} \in \mathcal{O}$$

$$(11) \quad \sum_{k \geq K} \frac{\omega_k}{(w_{l+1}(a) \cdots w_{l+n_k}(a))^{1/m_0}} e_{l+n_k} \in \mathcal{O}.$$

Without loss of generality, we may assume that $K \geq M$. We then set $N_i = n_{K_i}$. Since the sequence (C_n) is nondecreasing, we still have

$$\sum_{i \geq 1} \frac{1}{C_{N_i+p}} = +\infty.$$

Let $\tau > 0$ be a small real number (a precised condition on τ will be fixed later) and define $\lambda_1 = a$, $\lambda_{i+1} = \lambda_i + \frac{\tau}{C_{N_i+p}}$. Let q be the first integer such that $\lambda_{q+1} \geq b$ and define $\Lambda_i = [\lambda_i, \lambda_{i+1})$ for $i = 1, \dots, q-1$, $\Lambda_q = [\lambda_q, b]$. Regarding (i) of Lemma 3.2, we write

$$(12) \quad \sum_{i=1}^q S_{\lambda_i}^{N_i/m_0}(e_l) = \sum_{i=1}^q \frac{1}{(w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i))^{1/m_0}} e_{l+N_i}.$$

Since each (w_n) is nondecreasing, using (11), we get that (i) is true.

Let now $i \in \{1, \dots, q\}$, $\lambda \in \Lambda_i$ and $m \in [m_0, m_1]$. Then

$$\sum_{j=i+1}^q T_\lambda^{N_i} \left(\left(S_{\lambda_j^{1/m_0}}^{N_j}(e_l) \right)^m \right) = \sum_{j=i+1}^q \frac{w_{l+N_j-N_i+1}(\lambda) \cdots w_{l+N_j}(\lambda)}{(w_{l+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j))^{m/m_0}} e_{l+N_j-N_i}$$

Now, since

$$\begin{aligned} (w_{l+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j))^{m/m_0} &\geq w_{l+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j) \\ &\geq w_{l+1}(a) \cdots w_{l+N_j-N_i}(a) w_{l+N_j-N_i+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j) \end{aligned}$$

and

$$w_{l+N_j-N_i+1}(\lambda) \cdots w_{l+N_j}(\lambda) \leq w_{l+N_j-N_i+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j)$$

we get

$$\sum_{j=i+1}^q T_\lambda^{N_i} \left(\left(S_{\lambda_j^{1/m_0}}^{N_j}(e_l) \right)^m \right) = \sum_{j=i+1}^q \frac{\alpha_{i,j}(\lambda)}{w_{l+1}(a) \cdots w_{l+N_j-N_i}(a)} e_{l+N_j-N_i}$$

where $\alpha_{i,j}(\lambda) \in [0, 1]$. Hence, by (10), (ii) of Lemma 3.2 is satisfied. Finally, we have for $i \in \{1, \dots, q\}$ and $\lambda \in \Lambda_i$,

$$T_\lambda^{N_i} \left(\left(S_{\lambda_i^{1/m_0}}^{N_i}(e_l) \right)^m \right) = \frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{(w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i))^{m/m_0}} e_l.$$

If $m = m_0$, we just write

$$\begin{aligned} \left| \frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i)} - 1 \right| &\leq \left| \exp \left(\sum_{k=1}^{N_i} (\log w_{l+k}(\lambda) - \log w_{l+k}(\lambda_i)) \right) - 1 \right| \\ &\leq \left| \exp \left(\sum_{k=1}^{N_i} L_{l+k} |\lambda - \lambda_i| \right) - 1 \right| \\ &\leq |\exp(C_{N_i+p} |\lambda - \lambda_i|) - 1| \\ &\leq |\exp(\tau) - 1|. \end{aligned}$$

Hence, provided τ is small enough, (iv) is satisfied and we may also ensure that

$$\left| \frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i)} \right| \leq 2.$$

Hence, if $m > m_0$, using again the monotonicity of each w_n , for all $i \in \{1, \dots, q\}$ and all $\lambda \in \Lambda_i$,

$$\frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{(w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i))^{m/m_0}} \leq \frac{2}{(w_{l+1}(a) \cdots w_{l+N_i}(a))^{1/m_0}}.$$

Applying a last time (11) (with only one nonzero ω_k now), we get that (iii) is satisfied, which closes the proof of Theorem 3.4. \square

Remark 3.5. If we are only interested in the existence of a common hypercyclic vector, we have a similar statement by verifying the assumptions of Theorem 3.4 only for $m_0 = 1$.

Before giving specific examples, let us explain the dependance between the behaviour of (C_n) and the choice of (n_k) . We work on $X = \ell^1$ and we assume first that $w_n(\lambda) = \lambda$ for all $n \geq 1$, $\lambda > 1$. In that case, $C_n \sim n$ and the sequence n_i cannot grow too fast in order to ensure the divergence of $\sum_i 1/C_{n_i}$. It is then natural to set $n_i = Ni$ for some N , and this will be enough to ensure the uniform unconditional convergence of the series involved in Theorem 3.4, because the product $w_1(\lambda) \cdots w_n(\lambda) = \lambda^n$ grows very fast.

Assume now that $w_n(\lambda) = 1 + \lambda/n$. In that case, $L_n = 1/n$ and C_n behaves like $\log n$. This means that we may choose a sequence (n_i) going very fast to $+\infty$, like $n_i = 2^i$. This will be necessary to ensure uniform unconditional convergence, since now $w_1(\lambda) \cdots w_n(\lambda) \sim n^\lambda$, which grows much slowly, especially if we allow λ to be close to 0.

Let us proceed with the details. We first give a result which should be thought as a version for hypercyclic algebras of [2, Corollary 4.10].

Theorem 3.6. *Let $\Lambda \subset \mathbb{R}$ be an interval and let $(w(\lambda))_{\lambda \in \Lambda}$ be a family of admissible weight sequences. Assume that*

- (a) (e_i) is an unconditional basis of X ;
- (b) all functions $\log(w_n)$ are non-decreasing and are Lipschitz on compact sets with uniformly bounded Lipschitz constants;
- (c) for all $m \geq 1$ and all $\lambda \in \Lambda$, the series $\sum_n (w_1(\lambda) \cdots w_n(\lambda))^{-1/m} e_n$ converges.

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda) \cup \{0\}$ contains a non-trivial algebra.

Proof. We may assume that $\Lambda = [a, b]$ is a compact interval. Let $C > 0$ be such that all functions $\log(w_n)$ are C -Lipschitz. We then set $n_k = k$, $k \geq 0$, and observe that, for all $i \geq 0$,

$$\sum_{k \geq 1} \frac{1}{(w_{l+1}(a) \cdots w_{l+n_{k+i}-n_i})^{1/m}} e_{l+n_{k+i}-n_i} = \sum_{k \geq 1} \frac{1}{(w_{l+1}(a) \cdots w_{l+k}(a))^{1/m}} e_{l+k},$$

which shows that the assumptions on unconditional uniform convergence are satisfied. \square

Corollary 3.7. *Let X be a Fréchet sequence algebra for the coordinatewise product with a continuous norm and such that (e_i) is an unconditional basis of X . Let $w = (w_n)$ be an admissible weight sequence and define*

$$\lambda_w = \inf \left\{ \lambda > 0 : \sum_n \lambda^{-n/m} (w_1 \cdots w_n)^{-1/m} e_n \text{ converges for all } m > 0 \right\}.$$

Then $\bigcap_{\lambda > \lambda_w} HC(\lambda B_w) \cup \{0\}$ contains a non-trivial algebra.

This corollary includes the case of the families $(\lambda B)_{\lambda > 1}$ on ℓ_p or c_0 and $(\lambda D)_{\lambda > 0}$ on $H(\mathbb{C})$. For families $(\lambda B)_{\lambda > 0}$ on ω the same result holds with an even easier proof, although this space does not admit a continuous norm.

Theorem 3.8. *The family $(\lambda B)_{\lambda > 0}$ admits a common hypercyclic algebra on ω for the coordinatewise product.*

Proof. It is enough to prove that the sets of vectors generating a common hypercyclic algebra for $(\lambda B)_{0 < \lambda < 1}$ and $(\lambda B)_{\lambda > 1}$ on ω with the coordinatewise product are G_δ dense. Since we know that B itself also has a G_δ dense set of vectors generating a hypercyclic algebra, we can gather the three cases $0 < \lambda < 1$, $\lambda = 1$ and $\lambda > 1$ by a Baire argument (a countable intersection of comeager sets is comeager).

The proof for the family $(\lambda B)_{\lambda > 1}$ imitates that of Theorem 3.4 and is left to the reader. Let us consider the family $(\lambda B)_{0 < \lambda < 1}$. Here we need another version of Lemma 3.2 obtained by an application of Proposition 3.1 with $m = \max(I)$. The statement is the same but with m_1 in the place of m_0 . Let T_λ be λB and S_λ be $\frac{1}{\lambda}F$, where F is the unweighted forward shift. Let $\Lambda = [a, b] \subset (0, 1)$, let $m_0 \leq m_1$ in \mathbb{N} and let $v \in \omega$. Let $M > 0$ and finally let \mathcal{O} be an open neighborhood of 0 in ω . There exist $\varepsilon > 0$ and $p \in \mathbb{N}$ such that

$$|x_l| < \varepsilon \text{ for } l = 0, \dots, p \implies (x_n)_n \in \mathcal{O}.$$

We define $K_v = \max_{l=0, \dots, p} |v_l|$ and fix $\tau > 0$ satisfying, for all $m \in [m_0, m_1]$,

$$|\exp(\tau) - 1| K_v^{m/m_1} < \varepsilon.$$

We also fix a big integer $N > \max(p, M)$ (more conditions will be given later) and we define $N_i = iN$ for all $i \geq 1$, $\lambda_1 = a$ and $\lambda_i = \lambda_{i-1} + \frac{\tau}{N_i}$ for all $i > 1$. Since $\sum \frac{\tau}{N_i} = +\infty$, there exists $q \in \mathbb{N}$ such that $\lambda_{q+1} > b$. Let $\Lambda_i = [\lambda_i, \lambda_{i+1})$ for $i = 1, \dots, q$.

By construction we have $\Lambda \subset \bigcup_{i=1}^q \Lambda_i$, $N_1 > M$ and $N_{i+1} - N_i > M$. Also, since $N > p$, we get immediately that

$$\sum_{i=1}^q S_{\lambda_i^{1/m_1}}^{N_i} v^{1/m_1} \in \mathcal{O}$$

and, for all $m \in [m_0, m_1]$ and all $\lambda \in \Lambda_i$, $i = 1, \dots, q$,

$$\sum_{j=i+1}^q T_\lambda^{N_j} \left(S_{\lambda_j^{1/m_1}}^{N_j} v^{1/m_1} \right)^m \in \mathcal{O}.$$

Now, let $m \in [m_0, m_1]$ and $\lambda \in \Lambda_i$ for some $i = 1, \dots, q$. We write

$$T_\lambda^{N_i} \left(S_{\lambda_i^{1/m_1}}^{N_i} v^{1/m_1} \right)^m = \left(\frac{\lambda}{\lambda_i^{m/m_1}} \right)^{N_i} v^{m/m_1}.$$

If $m = m_1$, then for all $l = 0, \dots, p$,

$$\begin{aligned} \left| \left(\frac{\lambda}{\lambda_i^{m/m_1}} \right)^{N_i} v_l^{m/m_1} - v_l \right| &= \left| \left(\left(\frac{\lambda}{\lambda_i} \right)^{N_i} - 1 \right) |v_l| \right| \\ &\leq \left(\exp(N_i |\lambda - \lambda_i|) - 1 \right) K_v \\ &\leq \left(\exp\left(N_i \frac{\tau}{N_i}\right) - 1 \right) K_v \\ &\leq (\exp(\tau) - 1) K_v \\ &< \varepsilon, \end{aligned}$$

that is, $T_\lambda^{N_i} \left(S_{\lambda_i^{1/m_1}}^{N_i} v^{1/m_1} \right)^m - v \in \mathcal{O}$. Moreover, if $\tau > 0$ is small enough, we have

$$\left(\frac{\lambda}{\lambda_i} \right)^{N_i} \leq 2.$$

If $m \in [m_0, m_1)$, then for all $l = 0, \dots, p$,

$$\begin{aligned} \left| \left(\frac{\lambda}{\lambda_i^{m/m_1}} \right)^{N_i} v_l^{m/m_1} \right| &= \left(\frac{\lambda}{\lambda_i^{m/m_1}} \right)^{N_i} |v_l^{m/m_1}| \\ &\leq \left(\frac{\lambda}{\lambda_i} \right)^{N_i} (\lambda_i^{1-\frac{m}{m_1}})^{N_i} |K_v^{m/m_1}| \\ &\leq 2b^{N_i(1-\frac{m}{m_1})} |K_v^{m/m_1}|. \end{aligned}$$

Hence, provided N is big enough, this is less than ε since $0 < b < 1$ and $1 - \frac{m}{m_1} \geq \frac{1}{m_1}$. \square

We can also get examples of families which are not multiples of a single operator.

Example 3.9. Let $X = c_0$ and consider $w_n(\lambda) = 1 + \frac{\lambda}{n}$, $\lambda > 0$. Then $\bigcap_{\lambda > 0} HC(B_{w(\lambda)}) \cup \{0\}$ contains a non-trivial algebra.

To include the ℓ_p -case, we need to change the choice of (n_k) .

Example 3.10. Let $X = \ell_p$, $1 \leq p < \infty$ and consider $w_n(\lambda) = 1 + \frac{\lambda}{n}$, $\lambda > 0$. Then $\bigcap_{\lambda > 0} HC(B_{w(\lambda)}) \cup \{0\}$ contains a non-trivial algebra.

Proof. Again, we need only to consider the case $\Lambda = [a, b]$ with $a > 0$. We apply Theorem 3.4 with $L_k = 1/k$ and $n_k = 2^k$. The uniform unconditional convergence of the involved series is ensured by the inequalities

$$w_1(a) \cdots w_n(a) \geq \exp(c \log n) = n^c \text{ for some } c > 0$$

and

$$\begin{aligned} &\left\| \sum_{k \geq K} \frac{1}{(w_{l+1}(a) \cdots w_{l+n_{k+i}-n_i}(a))^{1/m} e^{l+n_{k+i}-n_i}} \right\| \\ &\leq \sum_{k \geq K} \frac{1}{(w_{l+1}(a) \cdots w_{l+2^{k+i}-2^i}(a))^{1/m}} \\ &\leq C_l \sum_{k \geq K} \frac{1}{2^{ic/m} (2^k - 1)^{c/m}} \\ &\leq C_l \sum_{k \geq K} \frac{1}{(2^k - 1)^{c/m}}. \end{aligned}$$

\square

As mentioned in the introduction, this last result is interesting even for the existence of a single hypercyclic vector, which was only known for $\lambda > 1/p$. The novelty here is that, since the weight varies very slowly, we may take a sequence (n_k) increasing much more quickly to $+\infty$ without reducing the size of the intervals I_k such that, for all $\lambda, \mu \in I_k$, $T_\lambda^{m_k} S_\mu^{m_k} e_0$ is close to e_0 . Allowing (n_k) to be large helps to ensure the smallness of the sum appearing in (12), even if the product $w_1(\lambda) \cdots w_n(\lambda)$ is not very large.

3.3. Common hypercyclic algebras for a family of backward shifts - Cauchy product. In this subsection we are dealing with the question of the existence of common hypercyclic algebras, for families of weighted backward shifts when the underlying Fréchet sequence algebra is endowed with the Cauchy product. We will establish a general criterion for the existence of such algebras which encompasses the case of the multiples of the backward shift on ℓ_1 and of the multiples of D on $H(\mathbb{C})$ as particular cases. Again, our main tool will be Proposition 3.1. Nevertheless, fixing $[a, b] \subset \Lambda$, we will not be able to devise a partition of each $[a, b] \subset \Lambda$ as in the proof of Theorem 3.4. We will only succeed to do this for intervals $[a, \kappa a]$, where κ will be independent of a . To come back to the assumptions of Proposition 3.1, we will need an auxiliary lemma. For a family of admissible weighted sequences $(w(\lambda))$ we will denote as usual by $T_\lambda = B_{w(\lambda)}$ the corresponding weighted backward shift.

Lemma 3.11. *Let X be a Fréchet sequence algebra under the Cauchy product in which $\text{span}(e_i)$ is dense and let $\Lambda \subset \mathbb{R}$ be an interval. Let $(w(\lambda))_{\lambda \in \Lambda}$ be a family of admissible weights and assume that the map $(\lambda, x) \mapsto T_\lambda x$ is continuous. Suppose that for all $[a_0, b_0] \subset \Lambda$, all $m \geq 1$ and all non-empty open set $V \subset X$, there exists $\kappa := \kappa(m, V, a_0, b_0) > 1$ such that, for all $a \in [a_0, b_0]$, for all \mathcal{O} open neighborhood of zero and for all $M_0 \in \mathbb{N}$, one can find $u \in X$ and $M_1 \in \mathbb{N}$ satisfying*

- (i) u has finite support and $u \in \mathcal{O}$;
- (ii) for each $\lambda \in [a, \kappa a] \cap [a_0, b_0]$ there is $N \leq M_1$ such that, for all $x \in X$ with $\text{supp}(x) \subset [0, M_0]$, $(T_\lambda)^N(u+x)^n = 0$ for $n \leq m-1$ and $(T_\lambda)^N(u+x)^m \in V$.

Then $(T_\lambda)_{\lambda \in \Lambda}$ admits a common hypercyclic algebra.

Proof. We are going to apply Proposition 3.1. So let $K = [a_0, b_0] \subset \Lambda$, let $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, set $m = \max(I)$ and let $U, V, W \subset X$ be non-empty open subsets, with $0 \in W$. We begin by considering κ given by the hypothesis. Let s be the first positive integer satisfying $a_0 \kappa^s \geq b$ and define $a_i = \kappa^i a_0$, for $i = 1, \dots, s-1$, and $a_s = b_0$. We will use the assumptions of the lemma to construct a sequence of pairs $(u(1), N_1), \dots, (u(s), N_s)$ satisfying, for all $i = 1, \dots, s$,

- $u(i) \in U$;
- for all $\lambda \in [a_0, a_i]$, there is $N \leq N_i$ such that $(T_\lambda)^N(u(i)^n) \in W$, for $n < m$, and $(T_\lambda)^N(u(i)^m) \in V$.

This construction being done, it is clear that $u := u(s)$ is the desired point we are looking for in order to apply Proposition 3.1.

We fix $u(0) \in U$ with finite support and we let $M(0) = \max(\text{supp}(u(0)))$ and \mathcal{O}_0 such that $u(0) + \mathcal{O}_0 \subset U$. For $a = a_0$, $\mathcal{O} = \mathcal{O}_0$ and $M_0 = M(0)$ in the lemma we find $u \in X$

and $M_1 =: N_1$ such that u has finite support, $u \in \mathcal{O}_0$ and, for each $\lambda \in [a_0, a_1]$ there is $N \leq N_1$ such that, for all $x \in X$ with $\text{supp}(x) \subset [0, M_0]$, $(T_\lambda)^N(u+x)^n = 0$ for $n \leq m-1$ and $(T_\lambda)^N(u+x)^m \in V$. In particular, for $x = u(0)$ we have $u(1) := u(0) + u \in U$ and, for all $\lambda \in [a_0, a_1]$, there is $N \leq N_1$ such that $(T_\lambda)^N(u(1)^n) = 0 \in W$, for $n < m$, and $(T_\lambda)^N(u(1)^m) \in V$.

Assume that $u(i)$ has been defined and let us define $u(i+1)$. Instead of applying directly the lemma we need to adjust things so that the corresponding interval is not just $[a_i, a_{i+1}]$ but $[a_0, a_{i+1}]$. By the inductive properties of $u(i)$, by the continuity of $(\lambda, x) \mapsto T_\lambda x$ and of the product of X and by the compactness of $[a_0, a_i]$, we can find \mathcal{O}_i an open neighborhood of zero that

- $u(i) + \mathcal{O}_i \subset U$;
- for all $y \in u(i) + \mathcal{O}_i$ and all $\lambda \in [a_0, a_i]$, there is $N \leq N_i$ such that $(T_\lambda)^N(y^n) \in W$, for $n < m$, and $(T_\lambda)^N(y^m) \in V$.

Now we apply the lemma with $a = a_i$, $M_0 = \max(\text{supp}(u(i)))$ and $\mathcal{O} = \mathcal{O}_i$ and we find $u \in X$ and M_1 such that u has finite support, $u \in \mathcal{O}_i$ and, for each $\lambda \in [a_i, a_{i+1}]$ there is $N \leq M_1$ such that, for all $x \in X$ with $\text{supp}(x) \subset [0, M_0]$, $(T_\lambda)^N(u+x)^n = 0$ for $n \leq m-1$ and $(T_\lambda)^N(u+x)^m \in V$. We set $N_{i+1} = \max(M_1, N_i)$ and $u(i+1) = u(i) + u$. Applying the previous result with $x = u(i)$ when $\lambda \in [a_i, a_{i+1}]$ or using that $u(i+1) \in u(i) + \mathcal{O}_i$ when $\lambda \in [a_0, a_{i+1}]$, we finally get that for all $\lambda \in [a_0, a_{i+1}]$, there is $N \leq N_{i+1}$ such that $(T_\lambda)^N(u(i+1)^n) = 0 \in W$, for $n < m$, and $(T_\lambda)^N(u(i+1)^m) \in V$. \square

Now we are ready to prove the main theorem of this subsection.

Theorem 3.12. *Let $\Lambda \subset \mathbb{R}$ be an interval, let X be a regular Fréchet sequence algebra under the Cauchy product and let $(w(\lambda))_{\lambda \in \Lambda}$ be a family of admissible weighted sequences such that all functions $\log(w_n)$ are non-decreasing and Lipschitz on compact sets with uniformly bounded Lipschitz constants. Suppose that (e_n) is an unconditional basis of X and that*

(a) *for all $\lambda \in \Lambda$,*

$$\sum_{n=1}^{\infty} \frac{1}{w_1(\lambda) \cdots w_n(\lambda)} e_n \in X;$$

(b) *for all $m \in \mathbb{N}$ and all $[a_0, b_0] \subset \Lambda$ there exist $c \in (0, 1)$ and $\kappa_0 > 1$ such that*

$$(13) \quad \lim_{N \rightarrow \infty} \sum_{n=cN}^N \frac{[w_1(\kappa_0 a) \cdots w_{mN}(\kappa_0 a)]^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)N+n}(a)} e_n = 0, \text{ for all } a \in [a_0, b_0].$$

Then $(T_\lambda)_{\lambda \in \Lambda}$ admits a common hypercyclic algebra.

Proof. First we observe that an application of the Banach-Steinhaus theorem as in the proof of Theorem 3.4 yields that the map $(\lambda, x) \mapsto T_\lambda x$ is continuous. Given $[a_0, b_0] \subset \Lambda$, $m \geq 1$ and $V \subset X$ open and non-empty, from condition (b) there exist $c \in (0, 1) \cap \mathbb{Q}$ and $\kappa_0 > 1$ such that (13) holds. Since the functions $\lambda \mapsto w_n(\lambda)$ are nondecreasing and by unconditionality, (13) holds for all $\kappa \in (1, \kappa_0)$. Fix $d \in (c, \frac{c+1}{2}) \cap \mathbb{Q}$, take $y = \sum_{j=0}^p y_j e_j \in V$ and find $\rho \geq 1$ and $\eta > 0$ so that $y + B \subset V$, where B is the ball for the seminorm $\|\cdot\|_\rho$

centered at the origin and with radius 3η . By unconditionality of the basis (e_n) , there exist $\rho_0 \geq 1$ and $C_{\rho, \rho_0} > 0$ such that, for all $x = \sum_n x_n e_n$ and all $\theta \in \ell_\infty$,

$$(14) \quad \left\| \sum_{n=0}^{+\infty} \theta_n x_n e_n \right\|_\rho \leq C_{\rho, \rho_0} \|\theta\|_\infty \left\| \sum_{n=0}^{+\infty} x_n e_n \right\|_{\rho_0}.$$

Take $q > p$ big enough so that

$$(15) \quad \left\| \sum_{n \geq q} \frac{z_n}{w_1(a_0) \cdots w_n(a_0)} e_n \right\|_\rho < \eta,$$

for all $z \in c_{00}$ with $\|z\|_\infty \leq \|y\|_\infty (\max(1, w_1(b_0), \dots, w_p(b_0)))^p$ (this is possible by condition (a)). We also fix $M > 0$ so that all the functions $\log(w_n)$ are M -Lipschitz on $[a_0, b_0]$. By continuity of exp we can find $\gamma > 0$ so that

$$(16) \quad |\exp(x) - 1| \leq \frac{\eta}{C_{\rho, \rho_0} \|y\|_{\rho_0}}, \text{ for all } x \in \mathbb{R} \text{ with } |x| \leq \gamma.$$

We then choose $\tau > 0$ and $\kappa \in (1, \kappa_0)$ such that

$$(17) \quad \tau < \frac{\gamma}{M}, \quad (\kappa - 1)b_0 \leq \tau \frac{d - c}{(m - 1 + d)q}.$$

Now let $a \in [a_0, b_0]$, \mathcal{O} an open neighborhood of zero and $M_0 \in \mathbb{N}$ be arbitrary. There exist $\sigma \geq 1$ and $\delta > 0$ such that $B_{\|\cdot\|_\sigma}(0, \delta) \subset \mathcal{O}$. Take $r > 0$ large enough such that $cr > M_0$ and $cr, dr \in \mathbb{N}$. Later on we will request more conditions on the size of r . For $j = cr, \dots, dr$, let λ_j be defined inductively by $\lambda_{cr} = a$ and $\lambda_{j+1} = \lambda_j + \frac{\tau}{(m-1)qr+qj}$. We notice that

$$\begin{aligned} \lambda_{dr} &= a + \sum_{j=cr}^{dr-1} \frac{\tau}{(m-1)qr+qj} \\ &\geq a + \tau \frac{d-c}{(m-1+d)q} \\ &\geq \kappa a. \end{aligned}$$

Define

$$u := u(r) = \sum_{j=cr}^{dr-1} \sum_{l=0}^p d_{j,l} e_{qj+l} + \varepsilon e_{qr},$$

where

$$\begin{aligned} d_{j,l} &:= \frac{y_l}{m\varepsilon^{m-1} w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})}, \\ \varepsilon &= \left(\frac{1}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} \right)^{\frac{1}{m}}. \end{aligned}$$

Let also $M_1 := mqr$. We shall show that $\|u\|_\sigma < \delta$ if r is big enough. For the double sum we have

$$\left\| \sum_{j=cr}^{dr-1} \sum_{l=0}^p d_{j,l} e_{qj+l} \right\|_\sigma = \left\| \sum_{j=cr}^{dr-1} \sum_{l=0}^p \frac{y_l (w_1(\kappa a) \cdots w_{mqr}(\kappa a))^{\frac{m-1}{m}}}{m w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})} e_{qj+l} \right\|_\sigma.$$

Now, we write the quotient of weights as

$$\frac{w_1(a) \cdots w_{(m-1)qr+qj+l}(a)}{w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})} \times \frac{(w_1(\kappa a) \cdots w_{mqr}(\kappa a))^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)qr+qj+l}(a)}.$$

Using unconditionality of the basis (e_n) as in (14), there exist $\sigma_0 \geq \sigma$ and $C_{\sigma, \sigma_0} > 0$ depending only on σ such that

$$\begin{aligned} & \left\| \sum_{j=cr}^{dr-1} \sum_{l=0}^p d_{j,l} e_{qj+l} \right\|_{\sigma_0} \\ & \leq C_{\sigma, \sigma_0} \left\| \sum_{j=cr}^{dr-1} \sum_{l=0}^p \frac{y_l w_1(a) \cdots w_l(a)}{m} \frac{(w_1(\kappa a) \cdots w_{mqr}(\kappa a))^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)qr+qj+l}(a)} e_{qj+l} \right\|_{\sigma_0} \\ & \leq C_{\sigma, \sigma_0} \left\| \sum_{n=cqr}^{qr} z_n \frac{(w_1(\kappa a) \cdots w_{mqr}(\kappa a))^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)qr+n}(a)} e_n \right\|_{\sigma_0}, \end{aligned}$$

for some eventually null sequence $z = (z_n)$ satisfying

$$\|z\|_\infty \leq \|y\|_\infty (\max(1, w_1(b_0), \dots, w_p(b_0))^p) / m.$$

Therefore, by assumption (b) and the unconditionality of (e_n) , we conclude that the sum $\sum_{j=cr}^{dr-1} \sum_{l=0}^p d_{j,l} e_{qj+l}$ converges to 0 when $r \rightarrow +\infty$. On the other hand, for the term εe_{qr} , since X is regular, we find $\sigma_1 \geq \sigma$ and $C_{\sigma, \sigma_1} > 0$ depending only on σ and m such that

$$\begin{aligned} \|\varepsilon e_{qr}\|_\sigma &= \left\| \frac{1}{[w_1(\kappa a) \cdots w_{mqr}(\kappa a)]^{\frac{1}{m}}} e_{qr} \right\|_\sigma \\ &= \left[\frac{1}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} \|e_{qr}\|_\sigma^m \right]^{\frac{1}{m}} \\ &\leq C_{\sigma, \sigma_1} \left[\frac{1}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} \|e_{mqr}\|_{\sigma_1} \right]^{\frac{1}{m}} \\ &\leq C_{\sigma, \sigma_1} \left\| \frac{1}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} e_{mqr} \right\|_{\sigma_1}^{\frac{1}{m}}, \end{aligned}$$

and this converges to 0 as $r \rightarrow +\infty$. This shows that condition (i) of Lemma 3.11 is satisfied if r is big enough.

Now, taking $\lambda \in [a, \kappa a]$, there exists $k \in \{cr, \dots, dr-1\}$ such that $\lambda \in [\lambda_k, \lambda_{k+1}]$. We choose $N = (m-1)qr + qk \leq M_1$ and take $x \in X$ with $\text{supp}(x) \subset [0, M_0]$. Clearly

we have $\max(\text{supp}((u+x)^n)) \leq (m-1)qr < N$ for all $n = 1, \dots, m-1$, which implies $(T_\lambda)^N(u+x)^n = 0$. On the other hand,

$$(u+x)^m = x' + \sum_{j=cr}^{dr-1} \sum_{l=0}^p m\varepsilon^{m-1} d_{j,l} e^{(m-1)qr+qj+l} + \varepsilon^m e_{mqr},$$

with $\text{supp}(x') \subset [0, (m-2)qr+2(dqr+p)] \cup [0, (m-1)qr+M_0]$. It follows that $\max(\text{supp}(x')) < N$ if r is big enough. We also have that $(m-1)qr+qj+p < N$ if $j < k$. Hence,

$$\begin{aligned} & (T_\lambda)^N(u+x)^m \\ &= \sum_{l=0}^p m\varepsilon^{m-1} d_{k,l} w_{(m-1)qr+qk-N+l+1}(\lambda) \cdots w_{(m-1)qr+qk+l}(\lambda) e^{(m-1)qr+qk-N+l} \\ & \quad + \sum_{j=k+1}^{dr-1} \sum_{l=0}^p m\varepsilon^{m-1} d_{j,l} w_{(m-1)qr+qj-N+l+1}(\lambda) \cdots w_{(m-1)qr+qj+l}(\lambda) e^{(m-1)qr+qj-N+l} \\ & \quad + \varepsilon^m w_{mqr-N+1}(\lambda) \cdots w_{mqr}(\lambda) e_{mqr-N} \\ &=: P_1 + P_2 + P_3, \end{aligned}$$

where, after substituting $d_{j,l}$, N and ε by their values,

$$\begin{aligned} P_1 &= \sum_{l=0}^p \frac{w_{l+1}(\lambda) \cdots w_{(m-1)qr+qk+l}(\lambda)}{w_{l+1}(\lambda_{k+1}) \cdots w_{(m-1)qr+qk+l}(\lambda_{k+1})} y_l e_l \\ P_2 &= \sum_{j=k+1}^{dr-1} \sum_{l=0}^p \frac{w_{(j-k)q+l+1}(\lambda) \cdots w_{(m-1)qr+qj+l}(\lambda)}{w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})} y_l e^{(j-k)q+l} \\ P_3 &= \frac{w_{(r-k)q+1}(\lambda) \cdots w_{mqr}(\lambda)}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} e^{(r-k)q}. \end{aligned}$$

From the definition of $\lambda_{cr}, \dots, \lambda_{dr}$, by the Lipschitz condition on the functions $\log w_n$ and by (17), we get

$$\begin{aligned} & \left| \sum_{j=1}^{(m-1)qr+qk} \log(w_{l+j}(\lambda)) - \log(w_{l+j}(\lambda_{k+1})) \right| \\ & \leq \sum_{j=1}^{(m-1)qr+qk} |\log(w_{l+j}(\lambda)) - \log(w_{l+j}(\lambda_{k+1}))| \\ & \leq \sum_{j=1}^{(m-1)qr+qk} M(\lambda_{k+1} - \lambda_k) \\ & = ((m-1)qr + qk) M \frac{\tau}{(m-1)qr + qk} \\ & < \gamma. \end{aligned}$$

Hence, by (16) we have

$$\begin{aligned} & \left| \frac{w_{l+1}(\lambda) \cdots w_{(m-1)qr+qk+l}(\lambda)}{w_{l+1}(\lambda_{k+1}) \cdots w_{(m-1)qr+qk+l}(\lambda_{k+1})} - 1 \right| \\ &= \left| \exp \left(\sum_{j=1}^{(m-1)qr+qk} \log(w_{l+j}(\lambda)) - \log(w_{l+j}(\lambda_{k+1})) \right) - 1 \right| \\ &\leq \frac{\eta}{C_{\rho, \rho_0} \|y\|_{\rho_0}}. \end{aligned}$$

Writing

$$P_1 - y = \sum_{l=0}^p \theta_l y_l e_l \text{ with } |\theta_l| \leq \frac{\eta}{C_{\rho, \rho_0} \|y\|_{\rho_0}},$$

we get by (14) that $\|P_1 - y\|_{\rho} \leq \eta$. Furthermore, let us write

$$\begin{aligned} & \frac{w_{(j-k)q+l+1}(\lambda) \cdots w_{(m-1)qr+qj+l}(\lambda)}{w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})} \\ &= \frac{w_1(\lambda_{j+1}) \cdots w_l(\lambda_{j+1})}{w_1(a_0) \cdots w_{(j-k)q+l}(a_0)} \prod_{s=1}^{(j-k)q+l} \frac{w_s(a_0)}{w_s(\lambda_{j+1})} \prod_{s=(j-k)q+l+1}^{(m-1)qr+qj+l} \frac{w_s(\lambda)}{w_s(\lambda_{j+1})}. \end{aligned}$$

Then we get since the functions (w_n) are nondecreasing,

$$P_2 = \sum_{n \geq q} \frac{z_n}{w_1(a_0) \cdots w_n(a_0)} e_n$$

for some eventually null sequence (z_n) with

$$\|z\|_{\infty} \leq \|y\|_{\infty} (\max(1, w_1(b_0), \dots, w_p(b_0)))^p.$$

By (15) we conclude that $\|P_2\|_{\rho} < \eta$. Finally,

$$\begin{aligned} \frac{w_{(r-k)q+1}(\lambda) \cdots w_{mqr}(\lambda)}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} &\leq \frac{w_{(r-k)q+1}(\kappa a) \cdots w_{mqr}(\kappa a)}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} \\ &= \frac{1}{w_1(\kappa a) \cdots w_{(r-k)q}(\kappa a)} \\ &\leq \frac{1}{w_1(a_0) \cdots w_{(r-k)q}(a_0)}, \end{aligned}$$

hence $\|P_3\|_{\rho} < \eta$ if r is big enough. With this we conclude that

$$\|(T_{\lambda})^N(u+x)^m - y\|_{\rho} < 3\eta,$$

which shows that $(T_{\lambda})^N(u+x)^m \in V$ and completes the proof. \square

We now apply Theorem 3.12 to multiples of the backward shift and of the derivative operator.

Example 3.13. On ℓ_1 endowed with the Cauchy product, $\bigcap_{\lambda>1} HC(\lambda B) \cup \{0\}$ contains a nontrivial algebra.

Proof. We apply Theorem 3.12 with $X = \ell_1$, $\Lambda = (1, \infty)$ and $w_n(\lambda) = \lambda$. We have that the functions $\log(w_n)$ are continuously first differentiable hence uniformly Lipschitz on compact subsets of Λ . Condition (a) trivially holds and, for (b), we have that

$$\left\| \sum_{n=cN}^{\infty} \frac{[w_1(\kappa a) \cdots w_{mN}(\kappa a)]^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)N+n}(a)} e_n \right\| = \frac{a}{a-1} \times \frac{\kappa^{(m-1)N}}{a^{\lceil cN \rceil}}$$

which, for $m = 1$, tends to 0 as N goes to $+\infty$ for any $c \in (0, 1)$, $\kappa > 1$ and $a > 1$, and for $m > 1$ it goes to 0 for any $c \in (0, 1)$ and $1 < \kappa < a^{\frac{c}{m-1}}$. \square

Example 3.14. On $H(\mathbb{C})$ endowed with the Cauchy product, $\bigcap_{\lambda>0} HC(\lambda D) \cup \{0\}$ contains a non-trivial algebra.

Proof. We apply Theorem 3.12 with $X = H(\mathbb{C})$, $\Lambda = (0, \infty)$ and $w_n(\lambda) = \lambda n$. Again the functions $\log(w_n)$ are uniformly Lipschitz on compact subsets of Λ and condition (a) is clearly satisfied. For (b) we fix $m \geq 1$, $a > 0$, $\kappa > 1$ and $r \geq 1$ and we have

$$\begin{aligned} & \left\| \sum_{n \geq cN} \frac{(w_1(\kappa a) \cdots w_{mN}(\kappa a))^{(m-1)/m}}{w_1(a) \cdots w_{(m-1)N+n}(a)} e_n \right\|_r \\ &= \kappa^{(m-1)N} \sum_{n \geq cN} \frac{(mN)!^{(m-1)/m}}{((m-1)N+n)!} \left(\frac{r}{a}\right)^n \\ &= \left(\frac{\kappa a}{r}\right)^{(m-1)N} (mN)!^{(m-1)/m} \sum_{n \geq cN} \frac{(r/a)^{n+(m-1)N}}{(n+(m-1)N)!} \\ &\leq C \left(\frac{\kappa a}{r}\right)^{(m-1)N} (mN)!^{(m-1)/m} \times \frac{(r/a)^{(m-1+c)N}}{((m-1+c)N)!}. \end{aligned}$$

Since for all $\varepsilon > 0$, Stirling's formula implies

$$\begin{aligned} (mN)!^{(m-1)/m} &\leq C_\varepsilon N^{(m-1+\varepsilon)N} \\ ((m-1+c)N)! &\geq C_\varepsilon N^{(m-1+c-\varepsilon)N}, \end{aligned}$$

choosing $\varepsilon < 2c$, it follows that for all $c \in (0, 1)$, all $\kappa > 1$, all $a > 0$, all $m \geq 1$ and all $r \geq 1$, we have

$$\left(\frac{\kappa a}{r}\right)^{(m-1)N} (mN)!^{(m-1)/m} \times \frac{(r/a)^{(m-1+c)N}}{((m-1+c)N)!} \xrightarrow{N \rightarrow +\infty} 0.$$

Hence, assumption (b) is satisfied. \square

Remark 3.15. We can make use of the same ideas and prove that $(\lambda B_w)_{\lambda>0}$ admits a common hypercyclic algebra on ω with the Cauchy product. In fact, the separating term allows us to push all undesirable parcels to the right and that is all we need when the

topology of ω comes into play. One can even simplify the definitions of εe_σ and $d_{j,l}$ as their sizes are not important as everything will vanish by the support.

Question 3.16. Let $X = \ell_1$, $\Lambda = (0, +\infty)$ and $w_n(\lambda) = 1 + \lambda/n$. In the previous section, we have shown that $\bigcap_{\lambda \in \Lambda} HC(B_{w(\lambda)})$ is not empty. Does it contain (except 0) a nontrivial algebra for the Cauchy product?

We cannot apply Theorem 3.12 for this family of shifts since (b) is not satisfied. Indeed, using standard tools of calculus, it can be shown that, for all $n \in \mathbb{N}$, all $a \in \Lambda$, all $c \in (0, 1)$ and all $\kappa > 1$, there exists $C > 0$ such that, for all $N \geq 1$,

$$\left\| \sum_{n=cN}^N \frac{[w_1(\kappa a) \cdots w_{mN}(\kappa a)]^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)N+n}(a)} e_n \right\|_1 \geq CN^{a(\frac{\kappa(m-1)}{m}-1)+1}.$$

For $a = m = 2$, it is impossible to find $\kappa > 1$ such that the right hand-side goes to 0.

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