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# PI output feedback for the wave PDE with second order dynamical boundary conditions

Christophe ROMAN<sup>1</sup>

**Abstract**—The paper focuses on a one-dimensional wave equation subject to second order dynamic boundary conditions with source terms on a Hilbert space. The purpose is the regulation of the wave velocity toward a given constant by the means of a Proportional-Integral action. The regulation problem is reformulated as a stability problem over an attractor which is not bounded in the state space. The asymptotic stability of which is established using La Salle invariance principle in Hilbert space. In order to proceed, this paper proves that the evolution problem associated with a maximal linear operator (or equivalently linear  $C_0$ -semigroup of contraction) gives precompact trajectories. This result is used on the considered wave equation on the state space-attractor quotient. The state space-attractor quotient is proven to be a Hilbert space.

## I. PROBLEM STATEMENT

The considered system is for all  $t \geq 0$ ,  $\forall x \in (0, 1)$ ,

$$\begin{cases} v_{tt}(x, t) = (a(x)v_x)_x(x, t) - q(x)v_t(x, t) + f(x), & (1a) \\ v_{tt}(1, t) = -a_1v_x(1, t) - q_1v_t(1, t) + U(t) + f_1, & (1b) \\ v_{tt}(0, t) = a_2v_x(0, t) - q_2v_t(0, t) + f_2, & (1c) \\ v(\cdot, 0) = v_0, \quad v_t(\cdot, 0) = v_1. & (1d) \end{cases}$$

The parameter  $a(\cdot)$  is only space dependent and stands for the elasticity of the wave equation. The parameter  $q(\cdot)$  is a viscous coefficient. The parameters  $q_1$ ,  $q_2$ ,  $a_1$ ,  $a_2$  are boundary coefficients. We assume that  $a(\cdot) \in H^1(0, 1)$ ,  $q(\cdot) \in L^2(0, 1)$ ,  $a_1$ ,  $a_2$ , and  $q_2$ , are strictly positive, and that  $q_1 \in \mathbb{R}$ . As  $q(\cdot)$  is positive, it is also named viscous damping coefficient. The source terms  $f \in L^2(0, 1)$ ,  $f_1$ ,  $f_2 \in \mathbb{R}$  are supposed unknown, therefore, their values are not used in the control law computation  $U(t)$ . The functions  $v_0$  and  $v_1$  are the initial conditions. Note that considering a weak formulation of the system (1), one can show that  $a \in L^2(0, 1)$  is sufficient to obtain well-posedness. In detail,  $v_x$  and  $a$  are in dual space, and the dual of  $L^2$  is  $L^2$  for the weak formulation. In the case of the strong formulation  $a \in H^{-1}$  the dual of  $H^1$ . The following inclusion holds  $H^1 \subset L^2 \subset H^{-1}$ .

The objective is to regulate  $v_t(\cdot, t)$  to  $v_1^{\text{ref}}$ , by the means of a proportional-integral control law using the measurement

of the velocity collocated with the actuation, i.e.,  $v_t(1, t)$ , in other words,

$$U(t) = -k(v_t(1, t) - v_1^{\text{ref}}) - k_i \int_0^t (v_t(1, t) - v_1^{\text{ref}}) dt. \quad (2)$$

This can be equivalently written as

$$\begin{cases} U(t) = -k(v_t(1, t) - v_1^{\text{ref}}) - k_i \eta_v(t), & (3a) \\ \dot{\eta}_v(t) = v_t(1, t) - v_1^{\text{ref}}, \quad \eta_v(0) = 0. & (3b) \end{cases}$$

In the literature boundary conditions of this type (1b)-(1c) can be linked to Wentzell's boundary conditions [8]. It involves a modification of the usual state space. This modification results in our case to the addition of two finite-dimensional state variables, in a similar way as in [17], [5], [7] and [2]. When the wave equation is more than a one-dimensional, the reader is referred to [8] and references therein.

The regulation problem on the one-dimensional wave equation with dynamic boundary condition has attracted the attention of many researchers in the control community. Some of the applications in mind are the crane regulation [6], [5], [7] and [2], drilling torsional vibrations [16], [18], piezoelectric control [12], and flexible structure [9]. The closer approach associated with the paper is [18] where the velocity regulation with a PI is considered. However, note that, the controlled boundary condition is not a second order dynamical one, opposite to the case here. Nevertheless, due to the boundary the authors consider in [18], they can achieve exponential stability even with small viscous anti-damping at the boundary opposite to the actuation. In the case under consideration, only viscous damping at the opposite boundary is considered, and asymptotic stability is achieved. In [2] the wave equation is subject to two dynamic boundary conditions but they focus on the position stabilization, and no viscous terms are considered.

An identification procedure has been proposed in [14] for the system (1) without source terms on experimental data. This means the considered problem can be associated with a experimental setup.

To the author knowledge, this paper is the first one to consider a one-dimensional wave equation with in-domain viscous damping subject to two dynamical boundary conditions, which achieves velocity regulation by collocated boundary output feedback. Collocated boundary output feedback means that the actuation and the measurement are at

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the same boundary, and that the control law needs only the output to be computed.

**Notations:** Given  $u : [0, 1] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ , we denote the partial derivative of  $u$  with respect to its first variable values at  $(x, t) \in [0, 1] \times \mathbb{R}^+$  by

$$u_x(x, t). \quad (4)$$

Similarly we denote the partial derivative of  $u$  with respect to its second variable at  $(x, t) \in [0, 1] \times \mathbb{R}^+$  by

$$u_t(x, t). \quad (5)$$

Consider  $I$  a space,  $L^2(I; \mathbb{R})$  denotes the class of equivalence of square-integrable functions from  $I$  to  $\mathbb{R}$ . Moreover  $L^2([0, 1]; \mathbb{R})$  is abusively denoted  $L^2(0, 1)$ . Furthermore  $H^n$  denotes the Sobolev space  $W^{n,2}$ , e.g.

$$u \in H^1 \Leftrightarrow u \in L^2, \quad u' \in L^2, \quad (6)$$

in which  $u'$  denotes the derivative of  $u$ .

## II. MAIN RESULT

Consider the following change of variable for all  $(x, t) \in [0, 1] \times [0, \infty)$

$$u(x, t) = v(x, t) - tv_1^{\text{ref}} + \frac{a(0)}{a_2}[-q_2v_1^{\text{ref}} + f_2] \int_0^x \frac{1}{a(s)} ds + \int_0^x \frac{1}{a(s)} \int_0^s [-v_1^{\text{ref}}q(\chi) + f(\chi)] d\chi ds, \quad (7)$$

$$\eta(t) = \eta_v(t) - \frac{a_1}{k_i a(1)} \int_0^1 [-v_1^{\text{ref}}q(s) + f(s)] ds - \frac{a_1 a(0)}{k_i a_2 a(1)} [-q_2v_1^{\text{ref}} + f_2] + \frac{qv_1^{\text{ref}} - f_1}{k_i}. \quad (8)$$

Direct computation gives that the variable  $u(x, t)$  is the solution of the following system:

$$\begin{cases} u_{tt}(x, t) = (au_x)_x(x, t) - q(x)u_t(x, t) & (9a) \\ u_{tt}(1, t) = -a_1u_x(1, t) - k_p u_t(1, t) - k_i \eta(t) & (9b) \\ \dot{\eta}(t) = u_t(1, t) & (9c) \\ u_{tt}(0, t) = -q_2u_t(0, t) + a_2u_x(0, t) & (9d) \\ u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1, \quad \eta(0) = \eta_0, & (9e) \end{cases}$$

where  $k_p = q_1 + k$ .

Note that  $\eta_v - \eta$  corresponds to the feedforward gain of the system (1).

The state is

$$\mathcal{X}(t) := [u(\cdot, t), u_t(\cdot, t), u_t(1, t), \eta(t), u_t(0, t)] \in \mathcal{H}. \quad (10)$$

where  $\mathcal{H}$  denotes the state space of the weak solutions,

$$\mathcal{H} := \{H^1(0, 1) \times L^2(0, 1) \times \mathbb{R}^3\}. \quad (11)$$

The state space for the strong solutions is

$$\mathcal{V} := \{H^2(0, 1) \times H^1(0, 1) \times \mathbb{R}^3\}. \quad (12)$$

The regulation objective of (1) is the stabilization of (9) with respect to the following subspace

$$\mathcal{S} := \{z \in \mathcal{H} : z_1 = d \text{ (a.e.)}, d \in \mathbb{R}, z_2 = 0 \text{ (a.e.)}, z_3 = z_4 = z_5 = 0\}. \quad (13)$$

In order to characterize the distance between an element of the state space and the set  $\mathcal{S}$  we consider the following functional

$$\Gamma(z) := \int_0^1 [z_1'(x)^2 + z_2(x)^2] dx + z_3^2 + z_4^2 + z_5^2.$$

It yields

$$\Gamma(z) = 0 \Leftrightarrow z \in \mathcal{S}. \quad (14)$$

The following result is stated.

*Theorem 2.1:* Consider the system (9). Assume that  $k_p > 0$  and  $k_i > 0$  then there exists a strictly positive constant  $M$  such that for all  $t \geq 0$

$$\Gamma(\mathcal{X}(t)) \leq M\Gamma(\mathcal{X}(0)). \quad (15)$$

In addition,

$$\Gamma(\mathcal{X}(t)) \xrightarrow[t \rightarrow \infty]{} 0. \quad (16)$$

In other words, the system is asymptotically stable with respect to the attractor  $\mathcal{S}$  defined in (13).

It follows that the velocity regulation of the system (1) associated with the control law (3) is achieved for  $k_i > 0$  and  $k > -q_1$ .

The proof of this theorem is quite technical as classical Lyapunov functionals are not sufficient to prove exponential stability (this is proven in Appendix B), and that the computation of the eigenvalues is tedious. The proof is based on Lyapunov analysis in order to prove stability, then asymptotically stability is obtained using LaSalle invariance principle.

Note that in order to use the invariance principle in infinite dimensional space, we have to establish the precompactness of the trajectories [10, Theorem 8.4.3 on Page 85]. In the sequel the following statement is proven: The trajectories of a evolution problem associated with a linear maximal monotone operator (equivalently linear  $C_0$ -semigroup of contraction) are precompact (see Theorem 3.7 below for a precise statement of this result). This allows future applications to other regulation problem.

## III. PROOF OF THEOREM 2.1

### A. Stability analysis

Consider the following Lyapunov functional

$$V(\mathcal{X}(t)) = \int_0^1 [u_t(x, t)^2 + a(x)u_x(x, t)^2] dx + \frac{a(1)}{a_1}(u_t(1, t)^2 + \eta(t)^2) + \frac{a(0)}{a_2}u_t(0, t)^2. \quad (17)$$

Deriving and using integration by parts one directly gets

$$\begin{aligned} \dot{V}(\mathcal{X}(t)) = & -2\left(\int_0^1 q(x)u_t(x,t)^2 dx\right. \\ & \left.+ k_p \frac{a(1)}{a_1} u_t(1,t)^2 + q_2 \frac{a(0)}{a_2} u_t(0,t)^2\right). \end{aligned} \quad (18)$$

### B. Convergence analysis

We start by computing the largest invariant space where  $\dot{V} = 0$  which expression is given in (18).

*Lemma 3.1:* The largest invariant space where  $\dot{V} = 0$  which is also given by the overdetermined system

$$\begin{cases} u_{tt}(x,t) = (au_x)_x(x,t) - q(x)u_t(x,t), & (19a) \\ u_{tt}(1,t) = -a_1 u_x(1,t) - k_p u_t(1,t) - k_i \eta(t), & (19b) \\ \dot{\eta}(t) = u_t(1,t), & (19c) \\ u_{tt}(0,t) = a_2 u_x(0,t) - q_2 u_t(0,t), & (19d) \\ u_t(1,t) = 0, & (19e) \\ u_t(0,t) = 0, & (19f) \\ u_t(\cdot,t) = 0 \text{ (a.e.)}, & (19g) \end{cases}$$

is  $\mathcal{S}$  defined in (13).

*Proof:* One obtains

$$u_t(0,t) = 0 \Rightarrow u_{tt}(0,t) = 0, \quad (20)$$

$$u_t(1,t) = 0 \Rightarrow u_{tt}(1,t) = 0, \quad (21)$$

$$u_t(\cdot,t) \stackrel{a.e.}{=} 0 \Rightarrow \int_0^1 u_t(x,t) dx = 0, \quad (22)$$

$$\int_0^1 u_t(x,t) dx = 0 \Rightarrow \int_0^1 u(x,t) dx = d, \quad d \in \mathbb{R}, \quad (23)$$

$$\int_0^1 u_t(x,t) dx = 0 \Rightarrow \int_0^1 u_{tt}(x,t) dx = 0. \quad (24)$$

Integrating (19a) with respect to  $x$  from 0 to 1, using (23) one gets

$$0 = a(0)u_x(0,t) - a(1)u_x(1,t). \quad (25)$$

Considering (19b) with (21) it holds

$$a_1 u_x(1,t) = k_i \eta(t). \quad (26)$$

Considering (19d) with (20) it yields

$$a_2 u_x(0,t) = 0. \quad (27)$$

Gathering last three equations one gets that  $\eta(t) = 0$ , and with (22) one concludes the proof. ■

As previously written in order to use LaSalle invariance principle, we need the precompactness of the trajectory [10, Theorem 8.4.3 on Page 85]. Indeed these properties can be used to establish that the  $\omega$ -limit set is nonempty and is an invariant set in  $\mathcal{H}$  using the theorem in [17, Theorem 3.1].

As far as we know, the establishment of the trajectory precompactness is based on the dissipativity or maximal monotonicity of the operator in  $\mathcal{H}$ , e.g. [6], [5], [7], [12], and [13]. In the case under consideration, we do not have a maximal

monotone operator in  $\mathcal{H}$ . Moreover the set  $\mathcal{S}$  defined in (13) and which represents the  $\omega$ -limit candidate is not bounded in  $\mathcal{H}$ . This means that the trajectories cannot be precompact in  $\mathcal{H}$ , and also that we cannot use the weaker result in [17, Theorem 3.1 (ii)] which requires only boundness.

A set is said to be precompact if and only if for every  $\varepsilon > 0$ , there exists a finite collection of open balls in  $\mathcal{M}$  of radius  $\varepsilon$  whose union contains  $\mathcal{M}$ . Precompactness implies boundness but the converse is not true.

Note that the notion of precompactness is completely depending on the embedding space. The idea is to study the system in another Hilbert space where the operator is maximal monotone. The space, where solutions are given, is changed in order to be able to establish the precompactness in a classical way. The next lemma concerns the aforementioned Hilbert space.

*Lemma 3.2:* Consider the quotient space  $\mathcal{H}/\mathcal{S}$ . It is a Hilbert space associated with the following scalar product,

$$\begin{aligned} \langle u, z \rangle_{\mathcal{H}/\mathcal{S}} = & \int_0^1 [u_2(x)z_2(x) + a(x)u'_1(x)z'_1(x)] dx \\ & + \frac{a(1)}{a_1} (u_3 z_3 + k_i u_4 z_4) + \frac{a(0)}{a_2} u_5 z_5. \end{aligned} \quad (28)$$

The proof of this lemma is based on the following theorems. Note that notations have been changed in order to match with this paper ones.

*Theorem 3.3 ([15, Theorem 1.41 (d)]):* Let  $\mathcal{S}$  be a closed subspace of a topological vector space  $\mathcal{H}$ . If  $\mathcal{H}$  is a Banach space so is the quotient space  $\mathcal{H}/\mathcal{S}$ .

*Theorem 3.4 ([15, Theorem 1.34 (d)]):* Suppose  $\sqrt{V(\cdot)}$  is a seminorm (subadditive and absolute homogeneous) on a vector space  $\mathcal{H}$ . Then  $\{z : \sqrt{V(z)} = 0\}$  is a subspace of  $\mathcal{H}$ .

*Theorem 3.5 ([15, Theorem 1.18 (b)]):* Let  $\Lambda$  be a linear functional on a topological vector space. Assume  $\Lambda(z) \neq 0$  for some  $z \in \mathcal{H}$ . The null space  $\text{Ker}(\Lambda(\cdot))$  is closed.

*Proof of Lemma 3.2:* First one directly gets that  $\sqrt{V}$  is a seminorm on  $\mathcal{H}$ , i.e.,

$$\sqrt{V(q+z)} \leq \sqrt{V(q)} + \sqrt{V(z)}, \quad (29)$$

$$\sqrt{V(\alpha z)} = |\alpha| \sqrt{V(z)}. \quad (30)$$

Second one gets that  $\mathcal{S} = \text{Ker}(V(\cdot)) = \text{Ker}(\sqrt{V(\cdot)}) = \text{Ker}(\langle \cdot, \cdot \rangle_{\mathcal{H}/\mathcal{S}})$ . Moreover using Theorem 3.4 one gets that  $\mathcal{S}$  is a subspace of  $\mathcal{H}$ .

Third, consider the following families of functional

$$\Lambda_1[s](z) = \int_0^s z'_1(x) dx, \quad (31)$$

$$\Lambda_2[s](z) = \int_0^s z_2(x) dx. \quad (32)$$

Both are indexed by  $s \in (0, 1]$ . Consider also the three following functional

$$\Lambda_3(z) = z_3, \quad \Lambda_4(z) = z_4, \quad \Lambda_5(z) = z_5. \quad (33)$$

It is direct that all  $\Lambda_i$  are linear. Note that by construction the intersection of the null spaces is

$$\bigcap_{\substack{i=1..2 \\ s \in (0,1]}} \text{Ker}(\Lambda_i[s](\cdot)) \bigcap_{i=3..5} \text{Ker}(\Lambda_i(\cdot)) = \mathcal{S}. \quad (34)$$

Therefore using Theorem 3.5 and the fact that any intersection (finite or infinite) of closed sets is a closed set, one obtains that  $\mathcal{S}$  is closed.

Fourth, using that  $\mathcal{S}$  is a closed subspace of  $\mathcal{H}$  and Theorem 3.3 it yields that  $\mathcal{H}/\mathcal{S}$  is a Banach space.

Finally, proving that (28) is a scalar product on  $\mathcal{H}/\mathcal{S}$  (this is direct) one concludes the proof. ■

Now let us consider the abstract evolution problem associated with the partial differential equation (9) in  $\mathcal{H}/\mathcal{S}$ , i.e.,

$$\begin{cases} \frac{d}{dt}\mathcal{X}(t) + \mathcal{A}\mathcal{X}(t) = 0, \\ \mathcal{X}(0) = \mathcal{X}_0, \end{cases} \quad (35a)$$

$$(35b)$$

in which the unbounded operator  $\mathcal{A}$  is defined as follows

$$z \in \text{Dom}(\mathcal{A}), \quad \mathcal{A}z = \begin{bmatrix} -z_2 \\ -(az'_1)' + \lambda z_2 \\ a_1 z'_1(1) + k_p z_3 + k_i z_4 \\ -z_3 \\ -a_2 z'_1(0) + q z_5 \end{bmatrix}, \quad (36)$$

with

$$\text{Dom}(\mathcal{A}) = \{z \in \mathcal{V}/\mathcal{S} : z_2(1) = z_3, z_2(0) = z_5\}. \quad (37)$$

The link between the different variable is

$$\mathcal{X}(t) = z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} u(\cdot, t) \\ u_t(\cdot, t) \\ u_t(1, t) \\ \eta(t) \\ u_t(0, t) \end{bmatrix}. \quad (38)$$

*Lemma 3.6:* The operator  $\mathcal{A}$  is maximal monotone in  $\mathcal{H}/\mathcal{S}$ .

*Proof:* One computes that

$$\langle z, \mathcal{A}z \rangle_{\mathcal{H}/\mathcal{S}} = \int_0^1 q(x) z_2(x)^2 dx + k_p \frac{a(1)}{a_1} z_3^2 + q_2 \frac{a(0)}{a_2} z_5^2 \geq 0. \quad (39)$$

This proves that the operator  $\mathcal{A}$  is monotone (see [1, Chapter 7 on Page 181]). In addition if we establish that

$$R(I + G) = \mathcal{H}, \quad (40)$$

then the operator  $\mathcal{A}$  is maximal monotone (see [1, Chapter 7 on Page 181],  $R$  stand for the range of the operator). In other words, let  $y \in \mathcal{H}/\mathcal{S}$ , we have to solve

$$z \in \mathcal{D}(\mathcal{A}), \quad z + \mathcal{A}z = y, \quad (41)$$

which means that

$$z_1 - z_2 = y_1, \quad (42)$$

$$z_2 - (az'_1)' + qz_2 = y_2, \quad (43)$$

$$z_3 + a_1 z'_1(1) + k_p z_3 + k_i z_4 = y_3, \quad (44)$$

$$z_4 - z_3 = y_4, \quad (45)$$

$$z_5 - a_0 z'_1(0) + q_2 z_5 = y_5, \quad (46)$$

using the fact that  $z \in \mathcal{D}(\mathcal{A})$  one gets

$$\begin{cases} (1+q)z_1 - (az'_1)' = (1+q)y_1 + y_2, \end{cases} \quad (47a)$$

$$\begin{cases} (1+k_p+k_i)z_1(1) + a_1 z'_1(1) = \\ (1+k_p+k_i)y_1(1) + y_3 - k_i y_4, \end{cases} \quad (47b)$$

$$(1+q_2)z_1(0) - a_0 z'_1(0) = (1+q_2)y_1(0) + y_5. \quad (47c)$$

This is a classical stationary problem (e.g. see [1]) with Robin's boundary conditions, using standard result (as done in [1, Example 6 On Page 226]) with  $(1+q)y_1 + y_2 \in L^2(0,1)$ , one gets that (47a)-(47c) has a unique solution  $z_1 \in H^2(0,1)$ . Note that it holds  $y \in \mathcal{H} \Leftrightarrow y \in \mathcal{H}/\mathcal{S}$  and  $y \in \mathcal{V} \Leftrightarrow y \in \mathcal{V}/\mathcal{S}$  in the case under consideration, in which  $\mathcal{H}$  and  $\mathcal{V}$  have been defined in (11) and (12). Now one can check that the element  $z = (z_1, z_2, z_3, z_4, z_5)$  with

$$z_1 \text{ is a solution of (47),} \quad (48a)$$

$$z_2 = z_1 - y_1, \quad (48b)$$

$$z_3 = \frac{y_3 - k_i y_4 - a(1)z'_1(1)}{1 + k_p + k_i}, \quad (48c)$$

$$z_4 = y_4 + z_3, \quad (48d)$$

$$z_5 = \frac{y_5 - a(0)z'_1(0)}{1 + q_2}, \quad (48e)$$

satisfies (42)-(46). Moreover using (47a)-(47c) on (48) one gets that  $z$  satisfying (48) is in  $\mathcal{D}(\mathcal{A})$ . This concludes the proof. ■

Now we need to establish that the trajectories are precompact.

In the cited literature (e.g., [5] or [13]) the precompactness is proven, on the one hand, establishing the canonical embedding from  $D(\mathcal{A})$ , equipped with the graph norm into  $\mathcal{H}/\mathcal{S}$  is compact, and, on the other hand, using the dissipativity of the operator  $-\mathcal{A}$ .

However, assuming the linearity of the operator, it occurs that there exists a more direct and general way gathering scattering result of the literature. This way is summarized in the following theorem.

*Theorem 3.7:*

Consider the following abstract evolution problem

$$\begin{cases} \frac{d\mathcal{X}(t)}{dt} + G\mathcal{X}(t) = 0, \\ \mathcal{X}(0) = \mathcal{X}_0 \in E. \end{cases} \quad (49a)$$

$$(49b)$$

The trajectory (under existence condition) is defined as

$$\gamma(\mathcal{X}_0) = \bigcup_{t \geq 0} T(t)\mathcal{X}_0, \quad (50)$$

in which  $T$  is the  $C_0$ -semigroup associated with  $G$ .

If  $G$  is a linear maximal monotone operator in the Hilbert  $E$  then the trajectory  $\gamma(\mathcal{X}_0)$  resulting from the abstract evolution problem associated with  $G$  is precompact in  $E$ .

The proof is given in Appendix A. Note that  $G$  maximal monotone is equivalent to  $T$  is a  $C_0$ -semi-group of contraction. It also yields also that the associated evolution problem is well-posed see [1, Chapter 7].

### C. Conclusion of the proof of Theorem 2.1

Due to Theorem 3.7 the trajectories of the system under consideration are precompact in  $\mathcal{H}/\mathcal{S}$ . We can use LaSalle invariance principle on Hilbert space in [10, Theorem 8.4.3 on Page 85]. We get that the system (9) trajectories converge toward  $S$  at infinity for any initial point. Using that the system is stable with respect to  $\mathcal{S}$ , one concludes that the system is asymptotically stable with respect to  $\mathcal{S}$ . Observing that  $\sqrt{\Gamma(\cdot)}$  and  $\sqrt{V(\cdot)}$  are equivalent seminorms, one concludes the proof.

## IV. CONCLUSION

This paper presented a proportional integral control on a one-dimensional wave equation. The wave equation was subject to two second order boundary conditions with viscous damping. The PI-control achieves the regulation problem in an asymptotic stable way. The paper also established that evolution problems associated with a linear maximal monotone operator have precompact trajectories. This result is really useful because (i) the maximal monotonicity is related to the well-posedness of the evolution problem in Hilbert space. It is therefore often either a property we have or we have to prove. (ii) The linearity is easily checked.

## APPENDIX

### A. Proof of Theorem 3.7

The proof is based on three observations:

- (i) If  $G$  is a linear maximal monotone operator and  $(G + I)^{-1}$  is compact, then  $\gamma(\mathcal{X}_0)$  is precompact.
- (ii) If  $G$  is monotone, then  $(G + I)^{-1}$  is a Lipschitz continuous function with 1 as a Lipschitz constant.
- (iii) A linear operator  $T$  on  $H$  is compact if  $\|Tz\| \rightarrow 0$  whenever  $z \rightarrow 0$ .

The assertion (i) is based on the following theorem and its proof.

*Theorem A.1 ([4, Theorem 3]):*

Let  $A$  be a (generally multivalued) accretive operator in a Banach space  $X$  such that  $R(\lambda A + I) \supset \overline{D(A)}$ , for sufficiently small  $\lambda$ , and let  $\mathcal{S}$  be the contraction semigroup on  $\overline{D(A)}$  generated by  $-A$ . Assume that  $0 \in R(A)$  and  $(\lambda A + I)^{-1}$  is compact for some  $\lambda > 0$ . Then  $\gamma(\mathcal{X}_0)$  is precompact for any  $\mathcal{X}_0 \in \overline{D(A)}$ .

Note as  $A$  is a maximal monotone operator in  $\mathcal{H}$  then  $D(A)$  is dense in  $\mathcal{H}$  which is also equal to  $R(\lambda A + I)$ ,  $\forall \lambda >$

0 see [1]. Moreover as  $A$  is linear  $0 \in R(A)$ . We recall that  $R$  stand for the range of an operator.

The observation (ii) is directly adapted from the following lemma.

*Lemma A.2 ([3, Lemma 2.1 (a)]):*

Let  $\text{Graph}(-A)$  be a dissipative set and  $\lambda > 0$ . Then  $(\lambda I + A)^{-1}$  is a Lipschitz continuous function with  $\lambda^{-1}$  as a Lipschitz constant.

The fact that  $A$  is monotone implies that the graph of  $-A$  is a dissipative set.

The last observation (iii) is a direct consequence of the following theorem.

*Theorem A.3 ([19, Theorem 1.14 on Page 11]):*

A linear operator  $T$  on  $H$  is compact if and only if  $\|Tz_n\| \rightarrow 0$  whenever  $z_n \rightarrow 0$  weakly in  $H$ .

*Proof of Theorem 3.7 :* Consider that  $G$  is a linear and monotone operator. One gets as  $G$  is linear and monotone that

$$(G + I)^{-1}0 = 0 \quad (51)$$

then using (ii) one gets that

$$\|(G + I)^{-1}z\| \leq \|z\| \quad (52)$$

Therefore  $\|(G + I)^{-1}z\| \rightarrow 0$  when  $z \rightarrow 0$ . From (iii) this gives us that  $(G + I)^{-1}$  is compact, now using (i) we get that the trajectories are precompact. ■

Note that Theorem A.1 and Lemma A.2 have been established for nonlinear/ multivalued accretive/dissipative operator/semi-groups.

### B. Equivalence of Lyapunov analysis under change of variable

The purpose of this appendix is to prove that one cannot prove exponential stability using a Lyapunov function candidate in their classical but general form for the system under consideration. First we start by establishing Lyapunov functional computations for a class of dynamics which contains (9). Then we apply the result to the dynamics under consideration.

*1) Lyapunov functional computation for a class of system:*

*Proposition 1:* Consider  $X(x, t) \in \mathbb{R}^n$  with  $n \in \mathbb{N}$ ,  $x \in [0, 1]$  and  $t \in [0, \infty)$ , whose dynamics is

$$X_t(x, t) = A_1(x)X(x, t) + A_2(x)X_x(x, t), \quad (53)$$

in which  $A_1(x) \in \mathbb{R}^{n \times n}$  and  $A_2(x) \in \mathbb{R}^{n \times n}$ . Consider also  $X_1(t) \in \mathbb{R}^m$  and  $X_3(t) \in \mathbb{R}^l$  with  $l \geq \max\{m, n\}$  and  $X_2(t) \in \mathbb{R}^p$  and  $X_4(t) \in \mathbb{R}^k$  with  $k \geq \max\{p, n\}$ , whose dynamic satisfies

$$\frac{d}{dt}X_1(t) = A_3X_3(t), \quad (54)$$

$$\frac{d}{dt}X_2(t) = A_4X_4(t), \quad (55)$$

in which  $A_3 \in \mathbb{R}^{m \times l}$  and  $A_4 \in \mathbb{R}^{p \times k}$ . Furthermore, assume that there exist  $A_5 \in \mathbb{R}^{m \times l}$ ,  $A_6 \in \mathbb{R}^{p \times k}$ ,  $A_7 \in \mathbb{R}^{n \times k}$ , and  $A_8 \in \mathbb{R}^{n \times k}$  such that it holds

$$X_1(t) = A_5 X_3(t), \quad (56)$$

$$X_2(t) = A_6 X_4(t), \quad (57)$$

$$X(1, t) = A_7 X_3(t), \quad (58)$$

$$X(0, t) = A_8 X_4(t). \quad (59)$$

Consider now the following functional

$$\begin{aligned} V(X, X_1, X_2) = & \int_0^1 X^T(x, t) P_0(x) X(x, t) dx \\ & + X_1^T(t) P_1 X_1(t) + X_2^T(t) P_2 X_2(t), \end{aligned} \quad (60)$$

in which  $P_0(x) \in \mathbb{R}^{n \times n}$ ,  $P_1 \in \mathbb{R}^{m \times m}$ , and  $P_2 \in \mathbb{R}^{p \times p}$  are symmetric positive definite matrices. If

$$A_2(x)^T P_0(x) - P_0(x) A_2(x) = 0, \quad (61)$$

then it holds

$$\begin{aligned} \dot{V} = & - \int_0^1 X^T(x, t) Q_0(x) X(x, t) dx \\ & - X_3(t)^T Q_1 X_3(t) - X_4(t)^T Q_2 X_4(t), \end{aligned} \quad (62)$$

with

$$Q_0(x) = -A_1^T(x) P_0(x) - P_0(x) A_1(x) + (P_0(x) A_2(x))_x, \quad (63)$$

$$Q_1 = -A_3^T P_1 A_5 - A_5^T P_1 A_3 - A_7^T P_0(1) A_2 A_7, \quad (64)$$

$$Q_2 = -A_4^T P_2 A_6 - A_6^T P_2 A_4 + A_8^T P_0(0) A_2 A_8. \quad (65)$$

Equation (63)-(65) are similar to Lyapunov equation in finite dimensional case, therefore, we will also refer to as it. These equations can be used to evaluate the decay rate as it is done for hyperbolic PDEs in [11] for example. Note that the dynamics under consideration is indeed related to first-order hyperbolic PDEs. However, we prove in Appendix B3 that each representation is equivalent with respect to the stability analysis (in the present case, where we consider velocity stabilization (regulation)).

The condition (61) which is also present in [11] is a sufficient condition for ensuring the computability of the Lyapunov derivative. To the author of knowledge, there does not exist Lyapunov analysis on first-order hyperbolic PDE and wave equation such that this condition is violated.

*Proof of Proposition 1:* From (60), the derivative along the state trajectory, using the dynamics of  $X$ ,  $X_1$ , and  $X_2$  in (53)-(55), is

$$\begin{aligned} \dot{V} = & \int_0^1 (X^T[A_1(x)^T P_0(x) + P_0^T(x) A_1(x)] X \\ & + X_x^T A_2(x)^T P_0(x) X + X^T P_0(x) A_2(x) X_x) dx \\ & + X_3(t)^T A_3^T P_1 X_1(t) + X_1^T(t) P_1 A_3 X_3(t) \\ & + X_4(t)^T A_4^T P_2 X_2(t) + X_2^T(t) P_2 A_4 X_4(t). \end{aligned} \quad (66)$$

Using integration by parts, and (56)-(57), one concludes the proof. ■

2) *Lyapunov stability analysis:* Consider (9) in a vectorial framework,

$$X(x, t) = \begin{bmatrix} u_t(x, t) \\ (au_x)(x, t) \end{bmatrix}, \quad X_1(t) = \begin{bmatrix} u_t(1, t) \\ \eta(t) \end{bmatrix}, \quad (67)$$

$$X_3(t) = \begin{bmatrix} u_t(1, t) \\ (au_x)(1, t) \\ \eta(t) \end{bmatrix} \quad (68)$$

$$X_2(t) = [u_t(0, t)], \quad X_4(t) = \begin{bmatrix} u_t(0, t) \\ (au_x)(0, t) \end{bmatrix}. \quad (69)$$

It yields

$$X_t(x, t) = A_1(x) X(x, t) + A_2(x) X_x(x, t), \quad (70)$$

in which

$$A_1(x) = \begin{bmatrix} -\lambda(x) & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2(x) = \begin{bmatrix} 0 & 1 \\ a(x) & 0 \end{bmatrix}. \quad (71)$$

Moreover it holds

$$\dot{X}_1(t) = A_3 X_3(t), \quad (72)$$

where

$$A_3 = \begin{bmatrix} -k_p & -\frac{a_1}{a(1)} & -k_i \\ 1 & 0 & 0 \end{bmatrix}. \quad (73)$$

Then one can also get

$$\dot{X}_0(t) = A_4 X_4(t) \quad (74)$$

in which

$$A_4 = \begin{bmatrix} -q_2 & \frac{a_2}{a(0)} \end{bmatrix} \quad (75)$$

One obtains

$$X(1, t) = A_7 X_3(t), \quad X_1(t) = A_5 X_3(t) \quad (76)$$

where

$$A_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (77)$$

Finally, it yields

$$X(0, t) = A_8 X_4(t), \quad X_2(t) = A_6 X_4(t) \quad (78)$$

in which

$$A_8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_6 = [1 \quad 0]. \quad (79)$$

Considering

$$P_0(x) = \begin{bmatrix} p_1(x) & p_2(x) \\ * & a(x) * p_1(x) \end{bmatrix}, \quad P_1 = \begin{bmatrix} p_3 & p_5 \\ * & p_4 \end{bmatrix}, \quad (80)$$

$$P_2 = p_6. \quad (81)$$

which are symmetric positive definite matrices given that

$$\forall x \in [0, 1], \quad p_1(x) > 0, \quad a(x)p_1(x)^2 - p_2(x)^2 > 0. \quad (82)$$

$$p_3 > 0, \quad p_4 > 0, \quad p_3 p_4 - p_5^2 > 0, \quad p_6 > 0 \quad (83)$$

$P_0(x)$  satisfies (61) with  $A_2(x)$  given in (71). The application of Proposition 1 gives

$$Q_0(x) = \begin{bmatrix} ap'_2 + a_x p_2 + 2qap_1 & qp_2 + ap'_1 + a_x p_1 \\ * & p_2 \end{bmatrix} (x) \quad (84)$$

$$Q_1 = \begin{bmatrix} 2(p_3 k_p - p_5) - a(1)p_2(1) & \frac{a_1}{a(1)} p_3 - a(1)p_1(1) \\ * & -p_2(1) \\ * & * \\ \dots & \dots \\ k_i p_3 - p_4 + k_p p_5 & p_5 \frac{a_1}{a(1)} \\ & 2k_i p_5 \end{bmatrix} \quad (85)$$

$$Q_2 = \begin{bmatrix} a(0)p_2(0) + 2p_6 q_2 & a(0)p_1(0) - p_6 \frac{a_2}{a(0)} \\ * & p_2(0) \end{bmatrix} \quad (86)$$

Some necessary conditions for  $Q_0(x)$ ,  $Q_1$ , and  $Q_2$  to be symmetric positive definite matrices are  $p'_2(x) > 0$ ,  $p_2(0) > 0$ , and  $p_2(1) < 0$ . It obviously does not exist for such a function. We cannot establish the exponential stability using the aforementioned Lyapunov analysis.

3) *Equivalence of change of variable*: Considering Proposition 1, if there exist some  $P_0(x)$ ,  $P_1$ , and  $P_2$  symmetric positive definite matrices such that  $Q_1(x)$ ,  $Q_2$ , and  $Q_3$  are also symmetric positive definite matrices then  $\exists \gamma$  such that  $\dot{V} \leq -\gamma V$  then exponential stability with respect to  $\ker(V(\cdot))$  yields.

The purpose of this section is to show the equivalence in terms of exponential stability analysis for bijective change of variables with respect to the Lyapunov analysis suggested in Proposition 1. In the above section, it has been shown that for the problem under consideration (9) considering classical quadratic Lyapunov functional (60) one cannot get  $Q_0(x)$ ,  $Q_1$ , and  $Q_2$  to be positive definite. This section is about proving that this holds for any reformulation of the stability analysis (for any change of variable, for example for the associated coupled first order hyperbolic PDE with ODE).

We start by a list of propositions concerning change of variables in the present context. Consider the following change of variable

$$Z(x, t) = G_0(x)X(x, t) \quad (87)$$

$$Z_1(t) = G_1 X_1(t) \quad (88)$$

$$Z_2(t) = G_2 X_2(t) \quad (89)$$

$$Z_3(t) = G_3 X_3(t) \quad (90)$$

$$Z_4(t) = G_4 X_4(t) \quad (91)$$

in which  $G_0(x) \in \mathbb{R}^{n \times n}$  is invertible for all  $x \in [0, 1]$ ,  $G_1$ ,  $G_2$ ,  $G_3$  and  $G_4$  are also invertible.

*Proposition 2*: If  $X$  is a solution of (53), then  $Z$  defined in (87) is a solution of

$$Z_t = B_1 Z + B_2 Z_x \quad (92)$$

with

$$B_1 = (G_0 A_1 - G_0 A_2 G_0^{-1} (G_0)_x) G_0^{-1} \quad (93)$$

$$= G_0 A G_0^{-1} + G_0 A_2 (G_0^{-1})_x, \quad (94)$$

$$B_2 = G_0 A_2 G_0^{-1}. \quad (95)$$

*Proof*: The proof is done by differentiating (87) with respect to  $x$  and  $t$ , and using (53). ■

*Proposition 3*: Consider (87), it holds

$$\int_0^1 X^T P_0 X dx = \int_0^1 Z^T R_0 Z dx, \quad (96)$$

with

$$P_0 = G_0^T R_0 G_0, \quad \text{or} \quad R_0 = (G_0^T)^{-1} P_0 G_0^{-1}. \quad (97)$$

*Proof*: The proof is obtained directly using (87). ■

*Proposition 4*: Consider (87) and (97), both conditions (61) and

$$B_2^T R_0 - R_0 B_2 = 0 \quad (98)$$

are equivalent.

*Proof*: Using (97), (61) is equivalent to

$$A_2^T G_0^T R_0 G_0 - G_0^T R_0 G_0 A_2 = 0. \quad (99)$$

Multiplying by  $G_0^{-1}$  at right, by  $(G_0^T)^{-1}$  at left, and then using (95) one obtains (98). ■

*Proposition 5*: If  $P$  is symmetric positive definite, and  $G$  is invertible, then  $G^T P G$  is symmetric positive definite.

*Proposition 6*:  $P$  is a symmetric positive definite matrix, we denote it by

$$P \succ 0. \quad (100)$$

The two following sentences are equivalent:

$$-A_1^T P_0 - P_0 A_1 + (P_0 A_2)_x \stackrel{s}{>} 0. \quad (101)$$

$$-B_1^T R_0 - R_0 B_1 + (R_0 B_2)_x \stackrel{s}{>} 0. \quad (102)$$

*Proof*: Expressing  $P_0$  as (97) in (101), then multiplying by  $(G_0^T)^{-1}$  at left and by  $G_0^{-1}$  at right, it holds using Proposition 5 that

$$\begin{aligned} & - (G_0 A_1 G_0^{-1})^T R_0 - R_0 G_0 A_1 G_0^{-1} \\ & + (G_0^T)^{-1} (G_0)_x^T R_0 G_0 A_2 G_0^{-1} \\ & + (R_0)_x G_0 A_2 G_0^{-1} \\ & + R_0 (G_0)_x A_2 G_0^{-1} + R_0 G_0 (A_2)_x G_0^{-1} \stackrel{s}{>} 0. \end{aligned} \quad (103)$$

From (98), using (95) one gets

$$R_0 G_0 A_2 G_0^{-1} = (G_0 A_2 G_0^{-1})^T R_0. \quad (104)$$

From (103), using (104), adding zero in the form  $R_0 G_0 A_2 (G_0^{-1})_x - R_0 G_0 A_2 (G_0^{-1})_x = 0$ , and using (93)-(95) one finally obtains (102), and concludes the proof. ■

*Proposition 7*: If  $X_1$  and  $X_2$  are solution of (56) and (57), then  $Z_1$  and  $Z_2$  defined in (88) and (89) are solution of

$$\dot{Z}_1 = B_3 Z_3(t), \quad (105)$$

$$\dot{Z}_2 = B_4 Z_4(t), \quad (106)$$

in which

$$B_3 = G_1 A_7 G_3^{-1}, \quad B_4 = G_2 A_8 G_4^{-1}. \quad (107)$$



*Proof* : The proof is straight from the definitions (88), (89) and (107). ■

*Proposition 8*: If  $X$  is a solution of (58) and (59), then  $Z$  defined by (87) is a solution of

$$Z(1, t) = B_7 Z_3(t), \quad (108)$$

$$Z(0, t) = B_8 Z_4(t), \quad (109)$$

in which

$$B_7 = G_0 A_7 G_3^{-1}, \quad B_8 = G_0 A_8 G_4^{-1}. \quad (110)$$

Moreover it holds

$$B_7^T R_0(1) B_2 B_7 = (G_3^{-1})^T A_7^T P_0(1) A_2 A_7 G_3^{-1}, \quad (111)$$

$$B_8^T R_0(0) B_2 B_8 = (G_4^{-1})^T A_8^T P_0(1) A_2 A_8 G_4^{-1}. \quad (112)$$

*Proof* : The proof is straight from the definition (87), (95) and (110). ■

Consider the following definition

$$N_0(x) = -B_1^T(x) R_0(x) - R_0(x) B_1(x) + (R_0(x) B_2(x))_x, \quad (113)$$

$$N_1 = -B_3^T R_1 B_5 - B_5^T R_1 B_3 - B_7^T R_0(1) B_2 B_7, \quad (114)$$

$$N_2 = -B_4^T R_2 B_6 - B_6^T R_2 B_4 + B_8^T R_0(0) B_2 B_8. \quad (115)$$

*Theorem A.1*: Considering (87) and Proposition 1, both following sentences are equivalent

- There exist  $P \stackrel{s}{>} 0$  such that  $Q \stackrel{s}{>} 0$ .
- There exist  $R \stackrel{s}{>} 0$  such that  $N \stackrel{s}{>} 0$ .

In the above sentence  $P$  denotes  $(P_0, P_1, P_2)$  and  $P \stackrel{s}{>} 0$  means  $P_0 \stackrel{s}{>} 0$ ,  $P_1 \stackrel{s}{>} 0$ , and  $P_2 \stackrel{s}{>} 0$ . The same idea for  $Q = (Q_0, Q_1, Q_2)$ ,  $R = (R_0, R_1, R_2)$ , and  $N = (N_0, N_1, N_2)$ . The  $N$  is the  $Q$  computed with  $B$  and  $R$  is the place of  $A$  and  $P$  it has been defined in (113)-(115).

*Proof* : The proof is established using Proposition 6, Proposition 7, and Proposition 8. The link between  $P$  and  $R$  is

$$P_0 = G_0^T R_0 G_0, \quad P_1 = G_1^T R_1 G_1, \quad P_2 = G_2^T R_2 G_2 \quad (116)$$

4) *Conclusion on the equivalence of Lyapunov analysis*: In finite-dimensional systems and also in infinite-dimensional system it is obvious that the Lyapunov analysis does not depend on the variable we choose the dynamics to be stated by. If a system is stable, it is stable for every basis of the state space. However, that quadratic Lyapunov functional kept their form for exponential stability analysis was not so obvious, but as previously shown in Appendix B it is obtained without much difficulty. Moreover the link between the weighting coefficient is not surprising (116). Nevertheless for the wave equation under consideration, this allows us to state that the system (9) cannot be proven exponentially stable using quadratic Lyapunov functional of the kind (60). ■

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