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Stability of the global weak axisymmetric solution to the quantum Euler system with vorticity in dimension $d = 2$

Boris Haspot* and Marc-Antoine Vassenet †

Abstract

We consider the stability of the global weak solution of the Quantum Euler system in two space dimensions. More precisely, we establish the compactness of global finite energy weak solution for large initial data provided that the initial data are axisymmetric. The main novelty is that the initial velocity is not necessary irrotational, our main argument is based on a generalization of the Madelung transform which enables to prove new Kato estimates on the irrotational part of the velocity.

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1 Introduction

The quantum Euler system that we are considering is described by the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \rho \partial_t u + \rho u \cdot \nabla u + \nabla P(\rho) = \kappa \rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right), & (x, t) \in \mathbb{R}^d \times \mathbb{R}_+. \\ (\rho, u)|_{t=0} = (\rho_0, u_0), x \in \mathbb{R}^d. \end{cases} \quad (1.1)$$

Here $u = u(t, x) \in \mathbb{R}^d$ with $d \geq 2$ stands for the velocity field, $\rho = \rho(t, x) \in \mathbb{R}^+$ is the density, $\kappa \geq 0$ is the capillary coefficient and P the pressure. This system appears as a relevant model in various areas of physics: superfluids theory (see e.g. [20, 13]), weakly interacting Bose gases (see e.g. [15]), quantum semiconductors (see e.g. [14])... We would like also to point out that the above system belongs to a wider class of fluid dynamic equations, the so called Euler-Korteweg systems modeling in particular liquid-vapor mixture. In order to describe the behavior of the mixture at the interface between the fluid and the vapor, the system is endowed with an internal capillarity $\kappa(\rho)$ and it reads as follows:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (x, t) \in \mathbb{R}^d \times I \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \operatorname{div} K, & (x, t) \in \mathbb{R}^d \times I \\ (\rho, u)|_{t=0} = (\rho_0, u_0), & x \in \mathbb{R}^d. \end{cases} \quad (1.2)$$

The general Korteweg tensor can be written as follows:

$$\operatorname{div} K = \operatorname{div} \left(\rho \kappa(\rho) \Delta \rho + \frac{1}{2} (\kappa(\rho) + \rho \kappa'(\rho)) |\nabla \rho|^2 \operatorname{Id} - \kappa(\rho) \nabla \rho \otimes \nabla \rho \right). \quad (1.3)$$

The capillary coefficient κ is a smooth function $\mathbb{R}^+ \rightarrow \mathbb{R}^{+*}$. We can notice that the quantum Euler system (1.1) corresponds to the Euler-Korteweg system (1.2) with $\kappa(\rho) = \frac{\kappa}{2\rho}$ with $\kappa > 0$.

One can recall now the physical energy bounds associated to the Euler-Korteweg system. Let $\bar{\rho} > 0$ be a constant reference density, and let Π be defined by:

$$\Pi(s) = s \left(\int_{\bar{\rho}}^s \frac{P(z)}{z^2} dz - \frac{P(\bar{\rho})}{\bar{\rho}} \right),$$

so that $P(s) = s\Pi'(s) - \Pi(s)$, $\Pi'(\bar{\rho}) = 0$ and:

$$\partial_t \Pi(\rho) + \operatorname{div}(u \Pi(\rho)) + P(\rho) \operatorname{div}(u) = 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^d).$$

Notice that Π is convex as far as P is non decreasing (since $P'(s) = s\Pi''(s)$), which is the case for γ -type pressure laws. Multiplying the equation of momentum conservation in the system (1.2) by u and integrating by parts over \mathbb{R}^d , we

obtain at least formally:

$$\begin{aligned} & \int_{\mathbb{R}^d} \left(\frac{1}{2} \rho |u|^2 + (\Pi(\rho) - \Pi(\bar{\rho})) + \frac{1}{2} \kappa(\rho) |\nabla \rho|^2 \right) dx \\ & \leq \int_{\mathbb{R}^d} \left(\frac{1}{2} \rho_0 |u_0|^2 + (\Pi(\rho_0) - \Pi(\bar{\rho})) + \frac{\kappa(\rho_0)}{2} |\nabla \rho_0|^2 \right) dx. \end{aligned} \quad (1.4)$$

It follows that assuming that the initial total energy is finite:

$$\mathcal{E}_0 = \int_{\mathbb{R}^d} \left(\rho_0 \frac{|u_0|^2}{2} + (\Pi(\rho_0) - \Pi(\bar{\rho})) + \frac{\kappa(\rho_0)}{2} |\nabla \rho_0|^2 \right) dx < +\infty,$$

then we have the a priori following bounds:

$$\begin{aligned} \Pi(\rho) - \Pi(\bar{\rho}), \quad \rho |u|^2 & \in L^\infty(0, \infty, L^1(\mathbb{R}^d)), \\ \text{and } \kappa(\rho) |\nabla \rho|^2 & \in L^\infty(0, \infty, L^1(\mathbb{R}^d)). \end{aligned} \quad (1.5)$$

Before stating our main result, we propose for the convenience of the reader a short review on the results concerning both the existence of strong solution for the Euler-Korteweg system and the existence of global weak solution with large initial data. The system (1.2) can be recasted as a quasilinear degenerate Schrödinger equation by using the classical effective velocity $v = u + i \sqrt{\frac{\kappa(\rho)}{\rho}} \nabla \rho$, the momentum equation takes then the following form:

$$\partial_t v + u \cdot \nabla v + i(\nabla v) \cdot w + i \nabla(a(\rho) \operatorname{div} v) + \nabla F(\rho) = 0, \quad (1.6)$$

with $F'(\rho) = \frac{P'(\rho)}{\rho}$, $w = \sqrt{\frac{\kappa(\rho)}{\rho}} \nabla \rho$ and $a(\rho) = \sqrt{\rho \kappa(\rho)}$. In other words we can expect dispersive estimates on the solutions of the Euler Korteweg system provided that we consider an irrotational velocity, indeed in this framework $\nabla(a(\rho) \operatorname{div} v)$ is strongly elliptic. By working on the equation (1.6) Benzoni, Danchin and Descombes in [8] have proved the local existence for the Euler Korteweg system of strong solutions for large data provided that the initial data $(\rho_0 - 1, u_0)$ belong to $H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$ with $s > \frac{d}{2} + 1$ and the initial density is far away from the vacuum $\rho_0 \geq c > 0$. The proof is based on tricky energy estimates in H^s Sobolev space combined with a gauge method in order to overcome some difficulties related to a loss of derivative on some convective terms, however in [8] the authors do not exhibit any dispersive estimates on the solution. As mentioned before, it seems reasonable to think that the existence of global strong solution can only be deduced by obtaining dispersive estimates on the solution. In particular in [7, 6] the authors prove the existence of global strong solution which scatter in dimension $d \geq 3$, for small initial data with irrotational flow by using the theory of non space time resonance. It is relevant to mention that the case $d \leq 2$ seems less favorable, indeed in [4] Audiard showed in dimension $d = 2$ the existence of travelling wave of arbitrary small energy (see [9] for the one-dimensional case). In particular since these solutions do not scatter, it may prevent some result of global strong scattering solution

for small initial data for $d \leq 2$ (See also [5] for the existence of multi-soliton of the Euler-Korteweg system in dimension one).

The question of the existence of global weak solution with large initial data for the Euler-Korteweg system remains actually largely open in dimension $d \geq 1$, we would like to explain what are the main difficulties related to the proof of such result. Classically the method for showing the existence of global weak solution consist in constructing in a first step a sequence of global approximate regular solution $(\rho_n, u_n)_{n \in \mathbb{N}}$ (via a Fadeo-Galerkin argument) satisfying uniformly in n the energy estimate (1.4). The second step is to prove that we can extract a subsequence $(\rho_{\varphi(n)}, u_{\varphi(n)})_{n \in \mathbb{N}}$ which converges weakly to a global weak solution (ρ, u) of the Euler-Korteweg system (1.1). It corresponds to the notion of stability of the global weak solution, however it requires in general to have enough compactness informations on the sequel $(\rho_n, u_n)_{n \in \mathbb{N}}$ in order to pass to the limit in the nonlinear terms involved in the system (1.2). One can then easily notice that in the case of the Euler-Korteweg system the main difficulty consists in passing to the limit in the quadratic terms $\kappa(\rho_n)\nabla\rho_n \otimes \nabla\rho_n$ and $\rho_n u_n \otimes u_n$ which are only bounded uniformly in n in $L^\infty(L^1(\mathbb{R}^d))$. It implies that these terms converges up to extraction only to some measure ν and ν_1 , the natural question which arises is to prove that $\nu = \kappa(\rho)\nabla\rho \otimes \nabla\rho$ and $\nu_1 = \rho u \otimes u$ where ρ and $\sqrt{\rho}u$ are strong limit of the sequences $(\rho_{\varphi(n)})_{n \in \mathbb{N}}$ and $(\sqrt{\rho_{\varphi(n)}}u_{\varphi(n)})_{n \in \mathbb{N}}$ in appropriate functional space.

In the case of the quantum Euler system (1.1), Antonelli and Marcati in [1, 2] have obtained the existence of global weak solution in dimension $d = 2, 3$ for "well-prepared" initial data $(\rho_0, \sqrt{\rho_0}u_0)$ in the sense that there exists some wave function $\psi_0 \in H^1$ such that $\rho_0 = |\psi_0|^2$ and $\rho_0 u_0 = \text{Im}(\bar{\psi}_0 \nabla \psi_0)$. It is important to point out that these conditions impose that the initial velocity is irrotational. Indeed in this case there exists a correspondance between the system (1.1) and the nonlinear Schrödinger equation via the notion of polar factorization where we have $\rho = |\psi|^2$ and $\rho u = \text{Im}(\bar{\Psi} \nabla \Psi)$ (see [1, 2]) with Ψ solution of the following (NLS) equation:

$$i\partial_t \Psi + \frac{1}{2}\Delta \Psi = g(|\Psi|^2)\Psi, \quad \Psi(0, \cdot) = \Psi_0. \quad (\text{NLS})$$

It is of course reminiscent of the so called *Madelung transform* which ensures that when $u = \nabla\phi$ is irrotational we have:

$$(\rho, \nabla\phi) \mapsto \Psi := \sqrt{\rho}e^{i\phi}. \quad (1.7)$$

In [1, 2] the authors begin by constructing a sequence of exact solutions $(\rho_n, \sqrt{\rho_n}u_n)_{n \in \mathbb{N}}$ of the system (1.1) using polar factorization associated to Ψ_n solution of the equation (NLS) provided that the sequence $(\Psi_{0,n})_{n \in \mathbb{N}}$ of initial data belong to H^s Sobolev space with $s > 0$ sufficiently large (in addition this sequence converges strongly to Ψ_0 in H^1). We should however point out that this procedure must be realized carefully since ψ_n can vanish, and so that $\sqrt{\rho_n}u_n = \text{Im}(\bar{\psi}_n \nabla \psi_n / |\psi_n|)$ is only defined for $|\psi_n| \neq 0$ and $\sqrt{\rho_n}u_n = 0$ otherwise using a Sard theorem. The key point of the proof is now the strong L^2_{loc} convergence

of the nonlinear terms $\sqrt{\rho_n}u_n \otimes \sqrt{\rho_n}u_n$ and $|\nabla\sqrt{\rho_n}|^2$, this is a consequence of standard stability estimates (based on Strichartz inequalities) which claim that Ψ_n converges to Ψ in $L_T^\infty(H^1)$ and of the fact that via the formula of polar factorization we have $|\nabla\Psi_n|^2 = |\nabla\sqrt{\rho_n}|^2 + |\sqrt{\rho_n}u_n|^2$. It suffices then now to prove the strong L^2 convergence by combining weak L^2 convergence and convergence of the L^2 norms $\|\sqrt{\rho_n}u_n\|_{L^2}$ and $\|\nabla\sqrt{\rho_n}\|_{L^2}$. Concerning the existence of global weak solution in one dimension for the system (1.1), we mention also the work of Antonelli et al [3] where the authors give in particular some sufficient and necessary condition for an initial data $(\rho_0, \sqrt{\rho_0}u_0)$ to admit an associated wave function Ψ_0 in H^1 and even in H^2 . It allows then to construct more regular global weak solution $(\rho, \sqrt{\rho}u)$ provided that the initial associated wave function is in H^2 , in addition they obtain also dispersive estimates on these solutions by adapting standard dispersive estimates issue of the NLS equation.

To finish this short review, we mention also the global existence of dissipative solutions for the Euler-Korteweg system (1.2) which are even weaker than the weak solutions (see [10]). We also would like to refer to [11] for a very interesting survey on the well-posedness of the Euler-Korteweg system and Gross-Pitaevskii equations.

In the sequel, we aim at solving the problem of the stability of the global weak solution of the quantum Euler system (1.1) in dimension $d = 2$ for an initial velocity which is not necessary irrotational. In other words, if we assume that there exists a sequence of global smooth approximate solutions $(\rho_n, u_n)_{n \in \mathbb{N}}$ of system (1.1) satisfying uniformly in n the bounds (1.5) issue of the energy estimate (1.4), we wish to prove that up to a subsequence the sequel $(\rho_n, u_n)_{n \in \mathbb{N}}$ converges weakly to a global weak solution (ρ, u) of the system (1.1).

Remark 1.1. *Naturally a result of stability of weak solutions does not imply the global existence of weak solutions. Indeed, it remains to prove the global existence of approximate solutions satisfying uniformly in n all the required energy estimates. This work is in general very delicate.*

Dividing now the momentum equation of (1.1) by ρ , applying the operator curl in dimension $d = 2$ (with $\text{curl}u = \partial_1u_2 - \partial_2u_1$) and using the mass equation, we get at least formally the following equation:

$$\partial_t \left(\frac{\text{curl}u}{\rho} \right) + u \cdot \nabla \left(\frac{\text{curl}u}{\rho} \right) = 0. \quad (1.8)$$

This remarkable relation on $\frac{\text{curl}u}{\rho}$ has been identified in particular by Serre in [21] for the compressible Euler system in dimension $d = 2$. It implies at least formally that we can bound the $L^\infty(\mathbb{R}^2)$ and $L^1(\mathbb{R}^2)$ norm of respectively $\frac{\text{curl}u}{\rho}$ and $\text{curl}u$ all along the time by using a maximum principle. That is why in the sequel we shall work with initial data verifying:

$$\frac{1}{\rho_0} \in L^\infty(\mathbb{R}^2) \quad \text{and} \quad \text{curl}u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2). \quad (1.9)$$

In addition the initial data have finite energy, it implies from the estimate (1.4)

that:

$$\nabla\sqrt{\rho_0} \in L^2(\mathbb{R}^2), \sqrt{\rho_0}u_0 \in L^2(\mathbb{R}^2) \text{ and } (\rho_0 - 1) \in L^2_\gamma(\mathbb{R}^2), \quad (1.10)$$

$L^2_\gamma(\mathbb{R}^2)$ is here an Orlicz space (we refer to the Section 2.1 for the definition). We assume now that there exists a sequence $(\rho_n, u_n)_{n \in \mathbb{N}}$ of global regular solutions of (1.1) satisfying the following properties:

$$\begin{aligned} \|(\rho_n - 1, u_n)\|_{C(\mathbb{R}^+, H^s)} &\leq C_n^s, \text{ with } C_n^s \xrightarrow{n \rightarrow +\infty} +\infty, \forall s \geq 0, \\ \left\| \frac{1}{\rho_n} \right\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)} &\leq C_n, \text{ with } C_n \xrightarrow{n \rightarrow +\infty} +\infty. \end{aligned} \quad (1.11)$$

We expect also that $(\rho_n, u_n)_{n \in \mathbb{N}}$ verify for $C > 0$ independently on n and any $n \in \mathbb{N}$ the energy estimates (1.4) and the maximum principle issue of the equation (1.8):

$$\begin{aligned} \|\nabla\sqrt{\rho_n}\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))} &\leq C, \|\rho_n - 1\|_{L^\infty(\mathbb{R}^+, L^2_\gamma(\mathbb{R}^2))} \leq C, \\ \|\sqrt{\rho_n}u_n\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))} &\leq C, \|\operatorname{curl}u_n\|_{L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^2))} \leq C, \\ \left\| \frac{\operatorname{curl}u_n}{\rho_n} \right\|_{L^\infty(\mathbb{R}^+, L^\infty(\mathbb{R}^2))} &\leq C. \end{aligned} \quad (1.12)$$

Furthermore the initial data $(\rho_n^0, u_n^0)_{n \in \mathbb{N}}$ verify for $C > 0$ independent on n :

$$\begin{cases} \|\nabla\sqrt{\rho_n^0}\|_{L^2(\mathbb{R}^2)} \leq C, & \|\sqrt{\rho_n^0}u_n^0\|_{L^2(\mathbb{R}^2)} \leq C, & \|(\rho_n^0 - 1)\|_{L^2_\gamma(\mathbb{R}^2)} \leq C, \\ \|\operatorname{curl}u_n^0\|_{L^1(\mathbb{R}^2)} \leq C, & \|\operatorname{curl}u_n^0\|_{L^\infty(\mathbb{R}^2)} \leq C, \end{cases} \quad (1.13)$$

and

$$\begin{aligned} \|(\rho_n^0 - 1, u_n^0)\|_{H^s(\mathbb{R}^2)} &\leq C_n^s, \text{ with } C_n^s \xrightarrow{n \rightarrow +\infty} +\infty, \forall s \geq 0, \\ \left\| \frac{1}{\rho_n^0} \right\|_{L^\infty(\mathbb{R}^2)} &\leq C. \end{aligned} \quad (1.14)$$

To finish we assume that :

$$\begin{cases} \rho_n^0 \xrightarrow{n \rightarrow +\infty} \rho_0 \text{ strongly in } L^1_{loc}(\mathbb{R}^2), \\ \rho_n^0 u_n^0 \xrightarrow{n \rightarrow +\infty} \rho_0 u_0 \text{ strongly in } L^1_{loc}(\mathbb{R}^2), \end{cases} \quad (1.15)$$

and

$$\begin{cases} \rho_n^0 - 1 \xrightarrow{n \rightarrow +\infty} \rho_0 - 1 \text{ strongly in } L^2_\gamma(\mathbb{R}^2), \\ \nabla\sqrt{\rho_n^0} \xrightarrow{n \rightarrow +\infty} \nabla\sqrt{\rho_0} \text{ strongly in } L^2(\mathbb{R}^2), \\ \sqrt{\rho_n^0}u_n^0 \xrightarrow{n \rightarrow +\infty} \sqrt{\rho_0}u_0 \text{ strongly in } L^2(\mathbb{R}^2). \end{cases} \quad (1.16)$$

Remark 1.2. *It is important to mention again that the question of the existence of global solution with large initial data for the Quantum Euler system in dimension $d = 2$ remains actually an open problem. That is why the sequence of global solution $(\rho_n, u_n)_{n \in \mathbb{N}}$ must be understood as a sequence of global regular solution of an approximate system of the Quantum Euler system which satisfies uniformly in n all the estimates issue of the energy inequality (1.4). A classical method for getting such sequence of solution is to use a Fadeo-Galerkine method, we will explain however why in our framework this type of construction seems particularly delicate.*

In order to distinguish in a simple way the irrotational and the rotational part of the convective term $u \cdot \nabla u$ in (1.1) we are going to consider axisymmetric initial data, more precisely the initial data satisfy the following relation ¹:

$$\begin{cases} \rho_0(x) = \rho_0(|x|), \\ u_0(x) = \nabla \theta_0(|x|) + \nabla^\perp (\theta_1)_0(|x|), \end{cases} \quad (1.17)$$

with ρ_0, θ_0 and $(\theta_1)_0$ radial functions. In the sequel, we assume now that the initial data satisfy (1.17), (1.9) and (1.10). Since we are interested in looking at solutions of the form :

$$\rho(t, x) = \rho_1(t, |x|), \quad u(t, x) = \nabla \theta(t, |x|) + \nabla^\perp \theta_1(t, |x|) = u^1(t, x) + u^2(t, x), \quad (1.18)$$

with $x \in \mathbb{R}^2 \setminus \{0\}$, we can then rewrite the system (1.1) at least formally as follows:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u^1) = 0, \\ \rho \partial_t u^1 + \rho u^1 \cdot \nabla u^1 + \rho u^2 \cdot \nabla u^2 + \nabla P(\rho) = \kappa \rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right), \\ \rho \partial_t u^2 + \rho u^1 \cdot \nabla u^2 + \rho u^2 \cdot \nabla u^1 = 0. \end{cases} \quad (1.19)$$

Due to the assumption (1.18) on the initial data, we will assume that the initial data $(\rho_n^0, u_n^0)_{n \in \mathbb{N}}$ are also axisymmetric and verify:

$$\begin{cases} \rho_n^0(x) = \rho_n(0, |x|), \\ u_n^0(x) = \nabla \theta_0^n(|x|) + \nabla^\perp (\theta_1)_0^n(|x|). \end{cases} \quad (1.20)$$

We can observe now that the rotational part of the velocity u^1 satisfies essentially the momentum equation of the Euler-Korteweg system (1.2) except that this equation is coupling with the convective term $\rho u^2 \cdot \nabla u^2$. It is then tempting at this level to consider this term as a source term and to apply a Madelung transformation to the equation on u^1 by considering the new unknown $\Psi_1 = \sqrt{\rho} e^{ic\theta_1}$ with $c \in \mathbb{R}$ a constant well chosen. We will see later that this is possible and that the unknown Ψ_1 satisfies the following equation:

$$\partial_t \Psi_1 + ic_1 \Delta \Psi_1 + ic_2 \Psi_1 (F(|\Psi_1|^2) - F(1) + G) = 0, \quad (1.21)$$

¹We will note in dimension $d = 2$, $x^\perp = (-x_2, x_1)^t$ for $x = (x_1, x_2)^t$ and $\nabla^\perp = (-\partial_2, \partial_1)$.

with $c_1, c_2 \in \mathbb{R}$, F a regular function depending on the pressure P and G a source term depending on the convective term $\rho u^2 \cdot \nabla u^2$.

Our main feature now consists in proving that up to a subsequence the sequence $(\rho_n, u_n)_{n \in \mathbb{N}}$ converges strongly to a global weak solution (ρ, u) of the Quantum Euler system. As we mentioned previously the main difficulty consists in proving that the nonlinear terms $\rho_n u_n \otimes u_n$ and $\nabla \sqrt{\rho_n} \otimes \nabla \sqrt{\rho_n}$ converge in the sense of the distribution respectively to $\rho u \otimes u$ and $\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}$ where ρ and u are strong limit of the sequences $(\rho_n)_{n \in \mathbb{N}}$ and $(u_n)_{n \in \mathbb{N}}$. Since $\text{curl} u_n$ should satisfy the equation (1.8), it is natural to think that $\text{curl} u_n$ verifies the estimates (1.12). It implies in particular that the rotational part $\mathbb{P}u_n$ must belong to Sobolev spaces $W^{1,p}(\mathbb{R}^2)$ with $p > 2$ via Sobolev embedding, using now compact embedding it is sufficient to show that the term $\rho_n \mathbb{P}u_n \otimes \mathbb{P}u_n$ converges strongly to $\rho \mathbb{P}u \otimes \mathbb{P}u$. The term the most delicate to treat is then the convective term $\rho_n \mathbb{Q}u_n \otimes \mathbb{Q}u_n$. At this level the use of axisymmetric solution is crucial since it enables to write a momentum equation on $\mathbb{Q}u_n$ as in the system (1.19) where u^1 takes the place of $\mathbb{Q}u_n$. We can then extend the so called Madelung transform by considering the unknown $\Psi_n = \sqrt{\rho_n} e^{ic\theta_n}$ with $\mathbb{Q}u_n = \nabla \theta_n$ and $c \in \mathbb{R}$. Since Ψ_n satisfies a nonlinear Schrödinger equation, we can use the Kato local smoothing effects on

$$\nabla \Psi_n = \left(\nabla \sqrt{\rho_n} - \frac{i}{2c_2} (\mathbb{Q}u_n) \sqrt{\rho_n} \right) e^{ic\theta_n},$$

provided that we estimate G_n in (1.21) in a suitable way. Then we use now the Aubin-Lions lemma to prove that the sequence $\rho_n \mathbb{Q}u_n \otimes \mathbb{Q}u_n$ converges to $\rho \mathbb{Q}u \otimes \mathbb{Q}u$. We recall now what is a global weak solution for the Quantum Euler system.

Definition 1.1. *We say that $(\rho, \Lambda, \Lambda_1)$ with $\rho \geq 0$ is said to be a global weak solution of the Quantum Euler system (1.1) with initial data (ρ_0, u_0) with $\Lambda_0 = \sqrt{\rho_0} \mathbb{Q}u_0$, $(\Lambda_1)_0 = \mathbb{P}u_0$ satisfying (1.9), (1.17), and (1.10) if:*

(1) **Integrability conditions:**

$$\begin{aligned} \rho - 1 &\in L^\infty(\mathbb{R}^+, L^2_2(\mathbb{R}^2)), \quad \nabla \sqrt{\rho} \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2)), \\ \Lambda &\in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2)), \quad \sqrt{\rho} \Lambda_1 \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2)). \end{aligned} \quad (1.22)$$

(2) **Kato's effect:** For any $T > 0$ we have:

$$\nabla \sqrt{\rho} \in L^2_T(H^{\frac{1}{2}}_{loc}), \quad \Lambda \in L^2_T(H^{\frac{1}{2}}_{loc}). \quad (1.23)$$

(3) **Mass equation:**

$$-\int_{\mathbb{R}^2} \rho_0 \varphi(0, \cdot) dx + \int_{\mathbb{R}_+ \times \mathbb{R}^2} \rho \partial_t \varphi dt dx + \int_{\mathbb{R}_+ \times \mathbb{R}^2} \sqrt{\rho} \Lambda \cdot \nabla \varphi dt dx = 0, \quad (1.24)$$

for any $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$.

(4) **Momentum equation:**

$$\begin{aligned}
& - \int_{\mathbb{R}^2} \rho_0 u_0 \cdot \varphi(0, \cdot) dx + \int_{\mathbb{R}_+ \times \mathbb{R}^2} (\rho \Lambda_1 + \sqrt{\rho} \Lambda) \cdot \partial_t \varphi dx dt \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}^2} (\Lambda \otimes \Lambda + \sqrt{\rho} \Lambda_1 \otimes \Lambda + \Lambda \otimes \sqrt{\rho} \Lambda_1 + \rho \Lambda_1 \otimes \Lambda_1) : \nabla \varphi dx dt \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}^2} P(\rho) \operatorname{div}(\varphi) dx dt - \frac{\kappa}{2} \int_{\mathbb{R}_+ \times \mathbb{R}^2} \nabla \rho \cdot \nabla (\operatorname{div}(\varphi)) dx dt \\
& - 2\kappa \int_{\mathbb{R}_+ \times \mathbb{R}^2} (\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}) : \nabla \varphi dx dt = 0,
\end{aligned} \tag{1.25}$$

holds for any test function $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)^2$.

(5) **Energy inequality:** if

$$E(t) = \int_{\mathbb{R}^2} \left(\frac{1}{2} \rho |u|^2(t, x) + \kappa |\nabla \sqrt{\rho}(t, x)|^2 + j_\gamma(\rho(t, x)) \right) dx,$$

then the following energy inequality is satisfied for any $t > 0$:

$$E(t) \leq E(0).$$

Remark 1.3. It is important to point out that heuristically Λ takes the role of $\sqrt{\rho} \mathbb{Q}u$ whereas Λ_1 is the rotational part $\mathbb{P}u$.

We can now state our main result.

Theorem 1. Let (ρ_0, u_0) satisfying (1.9) (1.10), (1.17) and under the hypotheses (1.13), (1.14), (1.15), (1.16), (1.20) the sequence $(\rho_n, u_n)_{n \in \mathbb{N}}$ of global regular solution of the system (1.1) satisfying (1.11), (1.12) converges in the sense of distribution, up to a subsequence, to a global weak solution (ρ, u) of system (1.1) in the sense specified in Definition 1.1, for initial data

$$\rho(0, \cdot) = \rho_0, \Lambda(0, \cdot) = \sqrt{\rho_0} \mathbb{Q}u_0 \text{ and } (\Lambda_1)_0 = \mathbb{P}u_0.$$

Remark 1.4. This result of stability of global weak solution generalizes the works of Antonelli-Marcati [1, 2] to the case of initial data not necessarily irrotational.

Organization of the article First of all in section 2.1 we present the a priori estimates. In section 2.2 we show that the geometry of the solutions is conserved all along the time. In section 2.3 we use an extended version of the Madelung transform in order to use in 2.4 Kato's local smoothing effect on the rotational part of the solution. In section 2.5 we pass to the limit in the different nonlinear terms of the equations by using Aubin Lions lemma. Finally, in the last section we prove the Theorem 1.

2 Proof of the Theorem 1

We assume now that $(\rho_n, u_n)_{n \in \mathbb{N}}$ is a sequence of global regular solution of the system (1.1) satisfying (1.11), (1.12) with initial data verifying (1.13), (1.14), (1.15), (1.16) and (1.20). Our aim is to prove that this sequence converges up to a subsequence to a global solution (ρ, u) of the Quantum Euler system with initial data (ρ_0, u_0) satisfying the assumption of Theorem 1. We are going to start by proving uniform estimate.

2.1 Uniform Estimate

This section is devoted to the proof of the following propositions which give uniform estimates in n on the sequence $(\rho_n, u_n)_{n \in \mathbb{N}}$.

Proposition 2.1. *There exist some positive constant $C > 0$, not depending on n , such that for all $t \geq 0$*

$$\|\nabla \sqrt{\rho_n}(t, \cdot)\|_{L^2} \leq C, \quad \|\sqrt{\rho_n} u_n(t, \cdot)\|_{L^2} \leq C, \quad (2.1)$$

$$\|\rho_n(t, \cdot) - 1\|_{L^2_\gamma} \leq C, \quad (2.2)$$

$$\|\operatorname{curl} u_n(t, \cdot)\|_{L^1} \leq C, \quad \left\| \frac{\operatorname{curl} u_n}{\rho_n}(t, \cdot) \right\|_{L^\infty} \leq C. \quad (2.3)$$

Next we deduce the following Corollary.

Corollary 2.1. *There exists some constant $C_p > 0$ depending only on p , such that for all $t \geq 0$*

$$\|\sqrt{\rho_n}(t, \cdot) - 1\|_{H^1} \leq C, \quad (2.4)$$

$$\|\rho_n(t, \cdot) - 1\|_{L^p} \leq C_p, \quad \forall p \in [2, +\infty[, \quad (2.5)$$

$$\|\operatorname{curl} u_n(t, \cdot)\|_{L^p} \leq C_p, \quad \forall p \in [1, +\infty[, \quad (2.6)$$

$$\|\nabla \mathbb{P} u_n(t, \cdot)\|_{L^p} \leq C_p, \quad \forall p \in]1, +\infty[, \quad (2.7)$$

$$\|\mathbb{P} u_n(t, \cdot)\|_{W^{1,p}} \leq C_p, \quad \forall p \in]2, +\infty[. \quad (2.8)$$

Proof of Proposition 2.1 and Corollary 2.1: Multiplying the momentum equation of (1.1) by u_n and integrating by parts on $(0, t) \times \mathbb{R}^2$ with $t > 0$ we obtain easily:

$$\begin{aligned} & \int_{\mathbb{R}^2} \left(\frac{1}{2} \rho_n |u_n|^2(t, x) + \kappa |\nabla \sqrt{\rho_n}(t, x)|^2 \right) dx + \int_0^t \int_{\mathbb{R}^2} \nabla P(\rho_n)(s, x) \cdot u_n(s, x) ds dx \\ &= \kappa \int_{\mathbb{R}^2} \left(\frac{1}{2} \rho_n |u_n|^2(0, x) + \kappa |\nabla \sqrt{\rho_n}(0, x)|^2 \right) dx. \end{aligned} \quad (2.9)$$

Using now the same computation as in [19], we can show that for a pressure $P(\rho) = \rho^\gamma$ we get:

$$\int_{\mathbb{R}^2} \nabla P(\rho_n(t, x)) \cdot u_n(t, x) dx = \frac{1}{\gamma - 1} \int_{\mathbb{R}^2} \partial_t (\rho_n^\gamma - 1 - \gamma(\rho_n - 1))(t, x) dx. \quad (2.10)$$

In the sequel we denote by $j_\gamma(\rho) = \frac{1}{\gamma-1}(\rho^\gamma - 1 - \gamma(\rho - 1))$, from the convexity of the function $x \rightarrow x^\gamma$ we can observe that $j_\gamma(\rho) \geq 0$ for $\rho \geq 0$. Combining (2.9) and (2.10) we have:

$$\begin{aligned} \int_{\mathbb{R}^2} \left(\frac{1}{2} \rho_n |u_n|^2(t, x) + \kappa |\nabla \sqrt{\rho_n}(t, x)|^2 + j_\gamma(\rho_n(t, x)) \right) dx \\ = \int_{\mathbb{R}^2} \left(\frac{1}{2} \rho_n |u_n|^2(0, x) + \kappa |\nabla \sqrt{\rho_n}(0, x)|^2 + j_\gamma(\rho_n(0, x)) \right) dx. \end{aligned} \quad (2.11)$$

According to [19] it is well known that the L^1 norm of $j_\gamma(\rho_n)$ is equivalent to $L_2^{\gamma-2}$ norm of $(\rho_n - 1)$, we deduce then from (2.11) and (1.13) that there exists $C > 0$ such that we have $\forall t \in \mathbb{R}$ and any $n \in \mathbb{N}$:

$$\begin{aligned} \|\nabla \sqrt{\rho_n}(t, \cdot)\|_{L^2(\mathbb{R}^2)} \leq C, \quad \|\sqrt{\rho_n} u_n(t, \cdot)\|_{L^2(\mathbb{R}^2)} \leq C, \\ \|(\rho_n - 1)(t, \cdot)\|_{L_2^{\gamma-2}(\mathbb{R}^2)} \leq C. \end{aligned} \quad (2.12)$$

For $\gamma \geq 2$ according to (2.12) it yields that $(\rho_n - 1)1_{\{|\rho_n - 1| \geq 1\}}$ is uniformly bounded in n in $L^\infty(\mathbb{R}^+, L^\gamma(\mathbb{R}^2))$ and that $|\{|\rho_n(t, \cdot) - 1| \geq 1\}|$ is uniformly bounded in n in $L^\infty(\mathbb{R}^+)$. It implies in particular that $(\rho_n - 1)1_{\{|\rho_n - 1| \geq 1\}}$ is uniformly bounded in n in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))$, using now the fact that $(\rho_n - 1)1_{\{|\rho_n - 1| \leq 1\}}$ is in $L^\infty(\mathbb{R}^+, L^2)$ from (2.12) and the definition of Orlicz space we deduce that there exist $C > 0$ such that for any $t \in \mathbb{R}^+$ and any $n \in \mathbb{N}$:

$$\int_{\mathbb{R}^2} (\rho_n(t, x) - 1)^2 dx \leq C. \quad (2.13)$$

Writing $\sqrt{\rho_n} - 1 = \frac{\rho_n - 1}{\sqrt{\rho_n} + 1}$, we have using (2.13) and (2.12) for any $t \in \mathbb{R}^+$ and any $n \in \mathbb{N}$:

$$\|\sqrt{\rho_n}(t, \cdot) - 1\|_{H^1(\mathbb{R}^2)} \leq C. \quad (2.14)$$

For $1 < \gamma < 2$, we have also:

$$\|\sqrt{\rho_n} - 1\|_{L^2(\mathbb{R}^2)} \leq C. \quad (2.15)$$

Indeed from the definition of Orlicz space and (2.12), we know that $\rho_n - 1 = f_n + g_n$ with f_n and g_n are respectively uniformly bounded in $L^\infty(\mathbb{R}^+, L^\gamma(\mathbb{R}^2))$ and $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))$, now since $\sqrt{\rho_n} - 1 = \frac{\rho_n - 1}{\sqrt{\rho_n} + 1}$ it implies in a similar way that $\sqrt{\rho_n} - 1 = f'_n + g'_n$ with f'_n and g'_n respectively uniformly bounded in $L^\infty(\mathbb{R}^+, L^\gamma(\mathbb{R}^2))$ and $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))$. From the Plancherel Theorem it suffices

²We refer to [19] for the definition of L_2^γ .

to bound the $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))$ norm of $\widehat{\sqrt{\rho_n} - 1} = \widehat{f'_n} + \widehat{g'_n}$ to prove that $\sqrt{\rho_n} - 1$ is uniformly bounded in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))$. Using Riesz-Thorin, Plancherel Theorem and (2.12) we have for $C, C_1, C_2 > 0$ independent on n and $t > 0$

$$\begin{aligned} \|\widehat{f'_n}(t, \cdot) \mathbb{1}_{\{|\cdot| \leq 1\}}\|_{L^2(\mathbb{R}^2)} &\leq C, \quad \|f'_n(t, \cdot)\|_{L^\gamma(\mathbb{R}^2)} \leq C_1, \quad \|\rho_n(t, \cdot) - 1\|_{L^2_2} \leq C_2, \\ \|\widehat{g'_n}(t, \cdot) \mathbb{1}_{\{|\cdot| \leq 1\}}\|_{L^2(\mathbb{R}^2)} &\leq C, \quad \|g'_n(t, \cdot)\|_{L^2(\mathbb{R}^2)} \leq C_1. \end{aligned} \quad (2.16)$$

Moreover using (2.12) and Plancherel Theorem there exists $C > 0$ such that for any $n \in \mathbb{N}$ and $t > 0$:

$$\|\mathcal{F}(\sqrt{\rho_n(t, \cdot)} - 1) \mathbb{1}_{\{|\cdot| \geq 1\}}\|_{L^2(\mathbb{R}^2)} \leq \|\nabla \sqrt{\rho_n(t, \cdot)}\|_{L^2(\mathbb{R}^2)} \leq C, \quad (2.17)$$

where $\mathcal{F}(f) = \hat{f}$. Using (2.16) and (2.17), we obtain when $1 < \gamma < 2$ that for any $n \in \mathbb{N}$ and $t > 0$:

$$\|(\sqrt{\rho_n} - 1)(t, \cdot)\|_{H^1(\mathbb{R}^2)} \leq C. \quad (2.18)$$

Combining (2.18) and (2.14), we have for all $\gamma > 1$

$$\|\sqrt{\rho_n} - 1\|_{L^\infty(\mathbb{R}^+, H^1(\mathbb{R}^2))} \leq C. \quad (2.19)$$

Finally using Sobolev embedding, we have

$$\|\sqrt{\rho_n} - 1\|_{L^\infty(\mathbb{R}^+, L^p(\mathbb{R}^2))} \leq C_p, \quad \forall p \geq 2. \quad (2.20)$$

Now since there exist $M > 0$ independent on n such that

$$|\rho_n - 1| \mathbb{1}_{|\rho_n - 1| \leq 1} \leq M|\sqrt{\rho_n} - 1|, \quad |\rho_n - 1| \mathbb{1}_{|\rho_n - 1| \geq 1} \leq M|\sqrt{\rho_n} - 1|^2,$$

then using (2.20), we have:

$$\|\rho_n - 1\|_{L^\infty(\mathbb{R}^+, L^p(\mathbb{R}^2))} \leq C_p, \quad \forall p \geq 2. \quad (2.21)$$

We wish now to prove the estimates on $\text{curl} u_n$ in Proposition 2.1 and in Corollary 2.1.

Curl estimate: We have seen that $\text{curl} u_n$ satisfies the two following equations:

$$\partial_t \text{curl} u_n + \text{div}(\text{curl} u_n u_n) = 0, \quad (2.22)$$

and:

$$\partial_t \left(\frac{\text{curl} u_n}{\rho_n} \right) + u_n \cdot \nabla \left(\frac{\text{curl} u_n}{\rho_n} \right) = 0. \quad (2.23)$$

According to the transport equation (2.23) and the conditions on the initial data 1.13 we obtain using classical maximum principle that:

$$\left\| \frac{\text{curl} u_n}{\rho_n} \right\|_{L^\infty(\mathbb{R}^+, L^\infty(\mathbb{R}^2))} \leq C. \quad (2.24)$$

Now using (2.22) we have

$$\partial_t \operatorname{curl} u_n + u_n \cdot \nabla \operatorname{curl} u_n + \operatorname{curl} u_n \operatorname{div} u_n = 0, \quad (2.25)$$

Setting the function $\varphi_k(x) = \frac{x^2}{\sqrt{x^2 + \frac{1}{k}}}$ with $k \in \mathbb{N}^*$ which approximate the absolute value and multiplying (2.23) by $\varphi'_k(\operatorname{curl} u_n)$ we obtain:

$$\partial_t \varphi_k(\operatorname{curl} u_n) + u_n \cdot \nabla \varphi_k(\operatorname{curl} u_n) + \varphi'_k(\operatorname{curl} u_n) \operatorname{div} u_n \operatorname{curl} u_n = 0. \quad (2.26)$$

Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfy

$$f(s) = \begin{cases} 1, & s \in [0, \frac{1}{2}] \\ \text{non-negative polynomial,} & s \in [\frac{1}{2}, 1] \\ 0, & s \in [1, \infty[\end{cases}$$

such that $f \in C^2$. We observe that

$$\|f\|_\infty + \|f'\|_\infty < \infty,$$

and we define, for any $R > 0$, $f_R(x) = f\left(\frac{|x|}{R}\right)$. Multiplying (2.26) by f_R we obtain

$$\partial_t \varphi_k(\operatorname{curl} u_n) f_R + u_n \cdot \nabla \varphi_k(\operatorname{curl} u_n) f_R + \varphi'_k(\operatorname{curl} u_n) \operatorname{div} u_n \operatorname{curl} u_n f_R = 0. \quad (2.27)$$

Integrating this equation we have :

$$\int_{\mathbb{R}^2} (\partial_t \varphi_k(\operatorname{curl} u_n) f_R + u_n \cdot \nabla \varphi_k(\operatorname{curl} u_n) f_R + \varphi'_k(\operatorname{curl} u_n) \operatorname{div} u_n \operatorname{curl} u_n f_R) dx = 0.$$

After integrating by parts and using (1.11) we have

$$\int_{\mathbb{R}^2} u_n \cdot \nabla \varphi_k(\operatorname{curl} u_n) f_R dx = - \int_{\mathbb{R}^2} \operatorname{div} u_n \varphi_k(\operatorname{curl} u_n) f_R dx - \int_{\mathbb{R}^2} u_n \cdot \nabla f_R \varphi_k(\operatorname{curl} u_n) dx.$$

So we have

$$\begin{aligned} \int_{\mathbb{R}^2} \partial_t \varphi_k(\operatorname{curl} u_n) f_R dx + \int_{\mathbb{R}^2} \operatorname{div} u_n f_R (-\varphi_k(\operatorname{curl} u_n) + \varphi'_k(\operatorname{curl} u_n) \operatorname{curl} u_n) dx \\ - \int_{\mathbb{R}^2} u_n \cdot \nabla f_R \varphi_k(\operatorname{curl} u_n) dx = 0. \end{aligned} \quad (2.28)$$

We have then obtained by integrating the previous estimate on $(0, t)$:

$$\begin{aligned} \int_{\mathbb{R}^2} \left(\frac{(\operatorname{curl} u_n)^2}{\sqrt{(\operatorname{curl} u_n)^2 + \frac{1}{k}}} \right) (t, x) f_R(x) dx + \frac{1}{k} \int_0^t \int_{\mathbb{R}^2} \operatorname{div} u_n f_R \frac{\operatorname{curl}^2 u_n}{(\operatorname{curl}^2 u_n + \frac{1}{k})^{\frac{3}{2}}} dx dt \\ - \int_0^t \int_{\mathbb{R}^2} u_n \cdot \nabla f_R \left(\frac{(\operatorname{curl} u_n)^2}{\sqrt{(\operatorname{curl} u_n)^2 + \frac{1}{k}}} \right) dx dt = \int_{\mathbb{R}^2} \left(\frac{(\operatorname{curl} u_n)^2}{\sqrt{(\operatorname{curl} u_n)^2 + \frac{1}{k}}} \right) (0, x) f_R(x) dx. \end{aligned} \quad (2.29)$$

Using Lebesgue theorem and since f_R is in $L^1(\mathbb{R}^2)$, we observe that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^2} \left(\frac{(\operatorname{curl} u_n)^2}{\sqrt{(\operatorname{curl} u_n)^2 + \frac{1}{k}}} \right) (t, x) f_R(x) dx = \int_{\mathbb{R}^2} |\operatorname{curl} u_n|(t, x) f_R(x) dx, \quad (2.30)$$

Now since $\frac{1}{k} \frac{\operatorname{curl}^2 u_n}{(\operatorname{curl}^2 u_n + \frac{1}{k})^{\frac{3}{2}}} \leq \frac{1}{\sqrt{k}}$ and $\operatorname{div} u_n f_R$ is in $L^1([0, t] \times \mathbb{R}^2)$ using (1.11) and the fact that f_R is in $L^1(\mathbb{R}^2)$, we deduce that:

$$\frac{1}{k} \int_0^t \int_{\mathbb{R}^2} \operatorname{div} u_n f_R \frac{\operatorname{curl}^2 u_n}{(\operatorname{curl}^2 u_n + \frac{1}{k})^{\frac{3}{2}}} dx dt \xrightarrow{k \rightarrow +\infty} 0. \quad (2.31)$$

Similarly since $\frac{|\operatorname{curl} u_n|}{\sqrt{(\operatorname{curl} u_n)^2 + \frac{1}{k}}} \leq 1$ and $u_n \cdot \nabla f_R \operatorname{curl} u_n \in L^1([0, t] \times \mathbb{R}^2)$ from (1.11) and because $\nabla f_R \in L^1(\mathbb{R}^2)$, we have by Lebesgue theorem

$$\int_0^t \int_{\mathbb{R}^2} u_n \cdot \nabla f_R \left(\frac{(\operatorname{curl} u_n)^2}{\sqrt{(\operatorname{curl} u_n)^2 + \frac{1}{k}}} \right) dx dt \xrightarrow{k \rightarrow +\infty} \int_0^t \int_{\mathbb{R}^2} u_n \cdot \nabla f_R |\operatorname{curl} u_n| dx dt. \quad (2.32)$$

Since $|\operatorname{curl} u_n|(0, \cdot) f_R(\cdot) \in L^1(\mathbb{R}^2)$ from Lebesgue Theorem, we have

$$\int_{\mathbb{R}^2} \left(\frac{(\operatorname{curl} u_n)^2}{\sqrt{(\operatorname{curl} u_n)^2 + \frac{1}{k}}} \right) (0, x) f_R(x) dx \xrightarrow{k \rightarrow +\infty} \int_{\mathbb{R}^2} |\operatorname{curl} u_n|(0, x) f_R(x) dx. \quad (2.33)$$

Thus passing to the limit when $k \rightarrow \infty$ in (2.29) we have

$$\begin{aligned} \int_{\mathbb{R}^2} |\operatorname{curl} u_n|(t, x) f_R(x) dx &= \int_{\mathbb{R}^2} |\operatorname{curl} u_n|(0, x) f_R(x) dx \\ &+ \int_0^t \int_{\mathbb{R}^2} u_n \cdot \nabla f_R |\operatorname{curl} u_n| dx dt. \end{aligned} \quad (2.34)$$

Then since there exists a constant $C > 0$ independent on R such that $\|\nabla f_R\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{R}$ and from (1.11) we deduce that:

$$\lim_{R \rightarrow +\infty} \int_0^t \int_{\mathbb{R}^2} u_n \cdot \nabla f_R |\operatorname{curl} u_n| dx dt = 0. \quad (2.35)$$

Passing to the limit when $R \rightarrow \infty$ in (2.34) and using the Fatou Lemma and the fact that $|\operatorname{curl} u_n|(0, \cdot) \in L^1(\mathbb{R}^2)$ we obtain:

$$\int_{\mathbb{R}^2} |\operatorname{curl} u_n|(t, x) dx \leq \int_{\mathbb{R}^2} |\operatorname{curl} u_n|(0, x) dx.$$

We obtain using (1.13) that there exists $C > 0$ such that for any $t > 0$ and $n \in \mathbb{N}$:

$$\int_{\mathbb{R}^2} |(\operatorname{curl} u_n)|(t, \cdot) dx \leq C.$$

So it implies that:

$$\|\operatorname{curl}u_n\|_{L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^2))} \leq C. \quad (2.36)$$

Now, we can prove that for any $t > 0$ we have:

$$\forall p \geq 1, \|\operatorname{curl}u_n(t, \cdot)\|_{L^p} \leq C_p. \quad (2.37)$$

Indeed we have for $p \geq 1$

$$\begin{aligned} \int_{\mathbb{R}^2} |\operatorname{curl}u_n(t, x)|^p dx &\leq \int_{\mathbb{R}^2} |\operatorname{curl}u_n(t, x)| 1_{\{|\operatorname{curl}u_n(t, x)| \leq 1\}} dx \\ &\quad + \int_{\{|\operatorname{curl}u_n(t, y)| \geq 1\}} \frac{|\operatorname{curl}u_n(t, x)|^p}{\rho_n(t, x)^{p-\frac{1}{2}}} \rho_n(t, x)^{p-\frac{1}{2}} dx. \end{aligned} \quad (2.38)$$

From (2.24), (2.36) and by interpolation it yields that $\frac{\operatorname{curl}u_n}{\rho_n^{\frac{p-1}{p}}}$ is uniformly bounded in $L^\infty(\mathbb{R}^+, L^p(\mathbb{R}^2))$ for any $p \geq 1$. It implies that there exists $C > 0$ such that for any $t > 0$ we have:

$$\left\| \frac{|\operatorname{curl}u_n|^p}{\rho_n^{p-\frac{1}{2}}} \right\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))} \leq C. \quad (2.39)$$

Now we observe again that for $p > \frac{3}{2}$ there exists $M_p \in \mathbb{R}^+$ independent on n such that :

$$|\rho_n^{p-\frac{1}{2}} - 1| \leq M_p |\rho_n - 1| + M_p |\rho_n - 1|^{p-\frac{1}{2}}. \quad (2.40)$$

Combining (2.40) and (2.21), we deduce that for $p > \frac{3}{2}$, $(\rho_n^{p-\frac{1}{2}} - 1)_{n \in \mathbb{N}}$ is uniformly bounded in n in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))$. From (2.39) and the fact that $|\{|\operatorname{curl}u_n(t, \cdot)| \geq 1\}|$ is uniformly bounded in n in $L^\infty(\mathbb{R}^+)$ using (2.36), it implies that for $p > \frac{3}{2}$ and any $t > 0$:

$$\begin{aligned} &\int_{\{|\operatorname{curl}u_n(t, y)| \geq 1\}} \frac{|\operatorname{curl}u_n(t, x)|^p}{\rho_n(t, x)^{p-\frac{1}{2}}} \rho_n(t, x)^{p-\frac{1}{2}} dx \\ &\leq \left\| \frac{|\operatorname{curl}u_n|^p}{\rho_n^{p-\frac{1}{2}}} \right\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))} \|\rho_n^{p-\frac{1}{2}} - 1\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))} \\ &\quad + \int_{\{|\operatorname{curl}u_n(t, y)| \geq 1\}} \frac{|\operatorname{curl}u_n(t, x)|^p}{\rho_n(t, x)^{p-\frac{1}{2}}} dx \\ &\leq C. \end{aligned} \quad (2.41)$$

So we have, using (2.36), (2.38), (2.41) and by interpolation for any $t > 0$

$$\forall p \geq 1, \|\operatorname{curl}u_n(\cdot, \cdot)\|_{L^\infty(L^p)} \leq C_p. \quad (2.42)$$

Moreover, we have

$$\nabla \mathbb{P}u_n = \nabla \Delta^{-1} \nabla^\perp \operatorname{curl}u_n, \quad (2.43)$$

then using (2.42) and the continuity of the Riez transform in L^p for $p > 1$ we have:

$$\|\nabla \mathbb{P}u_n\|_{L^\infty(\mathbb{R}^+, L^p(\mathbb{R}^2))} \leq C_p, \quad \forall p \in]1, +\infty[. \quad (2.44)$$

Finally using Sobolev embedding we deduce from (2.44) that:

$$\|\mathbb{P}u_n\|_{L^\infty(\mathbb{R}^+, W^{1,p})} \leq C_p, \quad \forall p \in]2, +\infty[. \quad (2.45)$$

2.2 Symmetry of the solutions

We wish now to prove that for any $n \in \mathbb{N}$, (ρ_n, u_n) have the form

$$\begin{cases} \rho_n(t, x) = \rho_n(t, |x|), \\ u_n(t, x) = \nabla \theta_n(t, |x|) + \nabla^\perp \theta_{1,n}(t, |x|), \quad t \in \mathbb{R}^+. \end{cases} \quad (2.46)$$

We can verify that for any rotation matrix A satisfying $AA^t = A^tA = I$ and $\det A = 1$, we have since (ρ_n, u_n) satisfies (1.1):

$$\begin{cases} \partial_t \rho_n(t, A^t x) + \operatorname{div}(\rho_n(t, A^t x) Au_n(t, A^t x)) = 0, \\ \rho_n(t, A^t x) \partial_t Au_n(t, A^t x) + \rho_n(t, A^t x) (Au_n(t, A^t x)) \cdot \nabla Au_n(t, A^t x) \\ + \nabla P(\rho_n(t, A^t x)) = \kappa \rho_n(t, A^t x) \nabla \left(\frac{\Delta \sqrt{\rho_n(t, A^t x)}}{\sqrt{\rho_n(t, A^t x)}} \right). \end{cases} \quad (2.47)$$

From (1.20), we can observe that:

$$\begin{cases} \rho_n(0, x) = \rho_n(0, A^t x), \\ u_n(0, x) = Au_n(0, A^t x). \end{cases} \quad (2.48)$$

then since (ρ_n, u_n) are strong solution of an approximate system of (1.1) we get by uniqueness that for all $t \in \mathbb{R}_+$ we have:

$$\begin{cases} \rho_n(t, x) = \rho_n(t, A^t x), \\ u_n(t, x) = Au_n(t, A^t x). \end{cases} \quad (2.49)$$

Remark 2.1. *We recall that for the Euler-Korteweg system we have uniqueness of the solution (ρ_n, u_n) in $C([0, T]; H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^2) \times H^{s-2}(\mathbb{R}^2))$ for any $T > 0$ and with $s > \frac{d}{2} + 1$ (see [8]).*

From (2.49) we deduce that ρ_n is radial or that for any $t \geq 0$:

$$\rho_n(t, x) = \rho_n(t, |x|). \quad (2.50)$$

Let $x \in \mathbb{R}^2 \setminus \{0\}$ we write $u_n(t, x)$ in the basis $(\frac{x}{|x|}, \frac{x^\perp}{|x|})$ and we have:

$$u_n(t, x) = f^{1,n}(t, x) \frac{x}{|x|} + f^{2,n}(t, x) \frac{x^\perp}{|x|}.$$

Then using (2.49), we have (see [16] section 3)

$$f^{1,n}(t,x)\frac{x}{|x|} + f^{2,n}(t,x)\frac{x^\perp}{|x|} = f^{1,n}(t,A^t x)\frac{AA^t x}{|A^t x|} + f^{2,n}(t,A^t x)\frac{A^t Ax^\perp}{|A^t x|}.$$

Since A is a rotation we deduce that $f^{1,n}(t,x) = f^{1,n}(t,A^t x)$ and $f^{2,n}(t,x) = f^{2,n}(t,A^t x)$. So, we have

$$u_n(t,x) = f^{1,n}(t,|x|)\frac{x}{|x|} + f^{2,n}(t,|x|)\frac{x^\perp}{|x|}. \quad (2.51)$$

Since for any $n \in \mathbb{N}$, $u_n(t, \cdot)$ is continuous for any $t \geq 0$, we can prove that $u_n(t, 0) = 0$. Indeed it suffices to consider the sequences $x_k = {}^t(\frac{1}{k}, 0)$ and $y_k = {}^t(-\frac{1}{k}, 0)$ for $k \in \mathbb{N}^*$, and using (2.51), we deduce that:

$$\begin{aligned} (f^{1,n}(t, \frac{1}{k}), f^{2,n}(t, \frac{1}{k})) &\rightarrow_{k \rightarrow +\infty} u_n(t, 0) \\ \text{and } (f^{1,n}(t, \frac{1}{k}), -f^{2,n}(t, \frac{1}{k})) &\rightarrow_{k \rightarrow +\infty} u_n(t, 0). \end{aligned}$$

We deduce that $u_n(t, 0) = 0$ for any $n \in \mathbb{N}$ and $t \geq 0$. It implies also that $f^{1,n}(t, \cdot)$ and $f^{2,n}(t, \cdot)$ have zero for limit when x goes to 0 and in addition we can prove that $f^{1,n}(t, \cdot)$ and $f^{2,n}(t, \cdot)$ are continuous for any $r > 0$ (to see this we simple use the following formula

$$f^{1,n}(t,r) = \langle u_n(t,x), \frac{x}{|x|} \rangle, \quad f^{2,n}(t,r) = \langle u_n(t,x), \frac{x^\perp}{|x|} \rangle \quad (2.52)$$

for $x \in \mathbb{R}^2 \setminus \{0\}$ such that $|x| = r$). We have finally prove that $f^{1,n}(t, \cdot)$ and $f^{2,n}(t, \cdot)$ are continuous on \mathbb{R}^+ . We have then proved that (ρ_n, u_n) have the form (2.46) for $t \in \mathbb{R}_+$ with:

$$\theta_n(t,r) = \int_0^r f^{1,n}(t,r')dr' \quad \text{and} \quad \theta_{1,n}(t,r) = \int_0^r f^{2,n}(t,r')dr'. \quad (2.53)$$

In addition since $f^{1,n}(t,|x|) = \langle u_n(t,x), \frac{x}{|x|} \rangle = \psi_n(t,x)$ for $x \in \mathbb{R}^2 \setminus \{0\}$ is C^∞ on $\mathbb{R}^2 \setminus \{0\}$ by using the fact that $u_n(t, \cdot)$ is C^∞ on \mathbb{R}^2 , we deduce that $f^{1,n}$ is C^∞ on \mathbb{R}^{+*} . We have in particular for $r > 0$:

$$(f^{1,n})'(t,r) = \lim_{h \rightarrow 0} \frac{\psi_n(t, (r+h)e_1) - \psi_n(t, re_1)}{h} = \partial_1 \psi_n(t, re_1).$$

We deduce then that $\theta_n(t, \cdot)$ and $\theta_{1,n}(t, \cdot)$ are C^∞ on \mathbb{R}^{+*} for any $t \geq 0$.

2.3 New Madelung Transform

In this section we forget the subscript n to simplify the notations. We set $c_2 = \sqrt{\frac{\kappa}{2}}$ and considering as in [7, 6, 8] the following effective velocity $z = u + ic_2 \nabla \ln \rho$ we can rewrite the momentum equation in (1.1) as follows

$$\rho \partial_t z + \rho u \cdot \nabla z + \nabla P(\rho) + ic_2 \rho \nabla \operatorname{div} z + ic_2 \nabla \rho \cdot {}^t \nabla z = 0.$$

Dividing by ρ (which has sense from the estimate (1.11)) we obtain with $F'(\rho) = \frac{F'(\rho)}{\rho}$:

$$\partial_t z + u \cdot \nabla z + \nabla F(\rho) + ic_2 \nabla \operatorname{div} z + z \cdot {}^t \nabla z - u \cdot {}^t \nabla z = 0.$$

Let $z^2 = (z_1)^2 + (z_2)^2$ with $z = (z_1, z_2)^t$ and $\operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$ we can rewrite the previous equation as follows:

$$\partial_t z + u^\perp \operatorname{curl} u + \frac{1}{2} \nabla z^2 + \nabla F(\rho) + ic_2 \nabla \operatorname{div} z = 0. \quad (2.54)$$

We wish now to apply the operator \mathbb{Q} to the equation (2.54), to do this we need to know what is $\mathbb{Q}(u^\perp \operatorname{curl} u)$. We are going to take advantage of the fact that the solutions are axisymmetric, indeed using (2.46) we have for any $t \geq 0$ and $x \in \mathbb{R}^2 \setminus \{0\}$:

$$\begin{cases} \rho(t, x) = \rho(t, |x|), \\ u(t, x) = \nabla \theta(t, |x|) + \nabla^\perp \theta_1(t, |x|), \\ z(t, x) = \nabla \psi(t, |x|) + \nabla^\perp \theta_1(t, |x|), \end{cases}$$

with $\psi(t, |x|) = \theta(t, |x|) + ic_2 \ln \rho(t, |x|)$. We can observe now that:

$$u^\perp \operatorname{curl} u = (\nabla^\perp \theta - \nabla \theta_1) \Delta \theta_1.$$

Using again (2.46) we have for any $t \geq 0$, $x \in \mathbb{R}^2 \setminus \{0\}$:

$$\mathbb{Q}(u^\perp \operatorname{curl} u) = -\Delta \theta_1 \nabla \theta_1 = \operatorname{curl} u (\mathbb{P}u)^\perp = \nabla \theta_2(t, |x|). \quad (2.55)$$

Indeed we have simply used the fact that $\Delta \theta_1 \theta_1'$ admits a primitive on \mathbb{R}^{+*} . In particular we have:

$$\theta_2 = (\Delta)^{-1} \operatorname{div}(\operatorname{curl} u (\mathbb{P}u)^\perp). \quad (2.56)$$

Moreover, we have using the fact that ψ and θ_1 are radial, we have for $x \in \mathbb{R}^2 \setminus \{0\}$:

$$\begin{aligned} z^2 &= (\nabla \psi)^2 + (\nabla \theta_1)^2 + 2\partial_2 \psi \partial_1 \theta_1 - 2\partial_1 \psi \partial_2 \theta_1 \\ &= (\nabla \psi)^2 + (\nabla \theta_1)^2. \end{aligned} \quad (2.57)$$

Applying \mathbb{Q} to the equation (2.54) and using (2.55), (2.57), we have in $\mathbb{R}^2 \setminus \{0\}$:

$$\nabla \left(\partial_t \psi + \theta_2 + \frac{1}{2} (\nabla \psi)^2 + \frac{1}{2} (\nabla \theta_1)^2 + F(\rho) - F(1) + ic_2 \Delta \psi \right) = 0. \quad (2.58)$$

Thus, there exists a function B , such that for any $t \geq 0$ we have:

$$\left(\partial_t \psi + \theta_2 + \frac{1}{2} (\nabla \psi)^2 + \frac{1}{2} (\nabla \theta_1)^2 + F(\rho) - F(1) + ic_2 \Delta \psi \right)(t, \cdot) = B(t). \quad (2.59)$$

At this level since ρ , $\mathbb{P}u$ and $\mathbb{Q}u$ are regular solutions which satisfy:

$$\lim_{|x| \rightarrow +\infty} \rho(t, x) = 1, \quad \lim_{|x| \rightarrow +\infty} \partial^\alpha \mathbb{P}u(t, x) = \lim_{|x| \rightarrow +\infty} \partial^\alpha \mathbb{Q}u(t, x) = \lim_{|x| \rightarrow +\infty} \partial^\beta \rho(t, x) = 0,$$

for any $\alpha \in \mathbb{N}^3$ and $\beta \in \mathbb{N}^3$ with $|\beta| \geq 1$ where the first derivative of ∂^{α_1} , ∂^{β_1} are derivative on time. To see this it suffices to observe that $Qu(t, x) = \langle u(t, x), \frac{x}{|x|} \rangle, \frac{x}{|x|}$ when $x \neq 0$ and to use the mass and the momentum equation of (1.1). In particular it implies that:

$$\begin{aligned} \lim_{|x| \rightarrow +\infty} \left(\frac{1}{2}(\nabla\theta_1)^2 + \frac{1}{2}(\nabla\psi)^2 + F(\rho) - F(1) + ic_2\Delta\psi \right)(t, x) &= 0, \\ \lim_{|x| \rightarrow +\infty} \operatorname{Im}(\partial_t\psi + \theta_2)(t, x) &= \lim_{|x| \rightarrow +\infty} \operatorname{Im}(\partial_t\psi)(t, x) = \lim_{|x| \rightarrow +\infty} c_2 \ln \rho(t, |x|) = 0. \end{aligned}$$

We deduce then that $B(t)$ is a real. Let us observe that B is continuous. Indeed with (1.14) we have that $(\rho_n, u_n - 1)$ is in $C([0, T], C^k(\mathbb{R}^2))$, for all $k \in \mathbb{N}$ and $\frac{1}{\rho_n}$ is bounded for any $t \geq 0$. Thus $\partial_t\psi(\cdot, x)$ is continuous for all $x \in \mathbb{R}^2 \setminus \{0\}$ using (2.53), the momentum and the mass equation. Similarly $(\nabla\psi(\cdot, x))^2 + F(\rho(\cdot, x)) - F(1) + ic_2\Delta\psi(\cdot, x)$ is continuous for all $x \in \mathbb{R}^2 \setminus \{0\}$. Using (2.52) for $x = (1, 0)^t$ we see that $\frac{1}{2}(\nabla\theta_1)^2(\cdot, (1, 0)^t)$ is continuous. Since θ_1 is in $C^0(\mathbb{R}, C^k)$ and since θ_2 is a primitive of $\Delta\theta_1\theta_1'$ we have $\theta_2(\cdot, x)$ is continuous for all $x \in \mathbb{R}^2 \setminus \{0\}$. Finally B is continuous and we can set $\psi_1(t) = \psi(t) - a(t)$ with $a'(t) = B(t)$, then for $x \in \mathbb{R}^2 \setminus \{0\}$

$$\partial_t\psi_1 + \theta_2 + \frac{1}{2}(\nabla\psi_1)^2 + \frac{1}{2}(\nabla\theta_1)^2 + F(\rho) - F(1) + ic_2\Delta\psi_1 = 0. \quad (2.60)$$

Using Cole-Hopf Transform, we obtain, multiplying the previous equation by $f'(\psi_1)$ with f a function that we will chose later:

$$\begin{aligned} \partial_t f(\psi_1) + ic_2\Delta f(\psi_1) + \left(\frac{1}{2}f'(\psi_1) - ic_2f''(\psi_1) \right) (\nabla\psi_1)^2 + \frac{1}{2}f'(\psi_1)(\nabla\theta_1)^2 \\ + (F(\rho) - F(1) + \theta_2)f'(\psi_1) = 0. \end{aligned}$$

We set now $w = f(\psi_1) = e^{-\frac{i}{2c_2}\psi_1} = e^{-\frac{i}{2c_2}\theta + \frac{1}{2}\ln\rho + \frac{i}{2c_2}a(t)} = \sqrt{\rho}e^{-\frac{i}{2c_2}\theta + \frac{i}{2c_2}a}$, and we obtain:

$$\partial_t w + ic_2\Delta w - \frac{i}{4c_2}w|\nabla\theta_1|^2 - (F(|w|^2) - F(1) + \theta_2)\frac{i}{2c_2}w = 0.$$

Finally we have a modified Gross-Pitaevskii equation using that $\nabla^\perp\theta_1 = \mathbb{P}u$ for $x \in \mathbb{R}^2 \setminus \{0\}$:

$$\partial_t w + ic_2\Delta w - \frac{i}{2c_2}w(F(|w|^2) - F(1) + \frac{1}{2}|\mathbb{P}u|^2 + \theta_2) = 0, \quad (2.61)$$

with

$$\theta_2 = (\Delta)^{-1}\operatorname{div}(\operatorname{curl}u(\mathbb{P}u)^\perp).$$

2.4 Kato's local smoothing effect

In this section we are going to use Kato's local smoothing effect. It will be the key point to prove strong convergence of the sequence $(\nabla\sqrt{\rho_n}, \sqrt{\rho_n}u_n)_{n \in \mathbb{N}}$ up to a subsequence in $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^2)$. More precisely, we prove the following Proposition.

Proposition 2.2. *let $\chi \in C_c^\infty(\mathbb{R}^2)$ and $T > 0$ then there exists a constant C_T not depending on n , such that $\forall n \in \mathbb{N}$ and for any $j \in \{1, 2\}$:*

$$\|\chi \partial_j w_n\|_{L_T^2(H^{1/2})} \leq C_T. \quad (2.62)$$

First we need to prove the following lemma:

Lemma 2.1. *Let $\chi \in C_c^\infty(\mathbb{R}^2)$, there exists $C > 0$ such that for all $f \in L^2(\mathbb{R}^2)$*

$$\|\chi e^{it\Delta} f\|_{L_T^2(H^{\frac{1}{2}})} \leq C \|f\|_{L^2}. \quad (2.63)$$

Moreover for any $T > 0$ then there exists $C_T > 0$ such that for all $f \in L_T^{p'}(L^q(\mathbb{R}^2))$ with (p, q) an admissible Strichartz pair (i.e $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and $2 \leq q < +\infty$) we have:

$$\left\| \int_0^t \chi e^{i(t-s)\Delta} f(s) ds \right\|_{L_T^2(H^{1/2})} \leq C_T \|f\|_{L_T^{p'}(L^q(\mathbb{R}^2))}. \quad (2.64)$$

Proof of lemma 2.1. Let $\chi \in C_c^\infty(\mathbb{R}^2)$ and $T > 0$ using [18] corollary 4.2 there exists $C > 0$ such that

$$\int_0^\infty \int_{B(0,R)} |D_x^{1/2} e^{it\Delta} f|^2 dx dt \leq CR \|f\|_{L^2(\mathbb{R}^2)}^2, \forall f \in L^2(\mathbb{R}^2), \quad (2.65)$$

with $\mathcal{F}D_x^{\frac{1}{2}} f(\xi) = |\xi|^{\frac{1}{2}} \mathcal{F}f(\xi)$. Thus

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |\chi D_x^{1/2} e^{it\Delta} f|^2 dx dt \leq C \|f\|_{L^2(\mathbb{R}^2)}^2, \forall f \in L^2(\mathbb{R}^2). \quad (2.66)$$

Then we have from (2.65) and Strichartz estimates

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} \chi D_x^{1/2} e^{i(t-s)\Delta} f(s) ds \right|^2 dx dt &\leq \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left| \chi D_x^{1/2} e^{it\Delta} \int_{\mathbb{R}} e^{-is\Delta} f(s) ds \right|^2 dx dt \\ &\leq C \left\| \int_{\mathbb{R}} e^{-is\Delta} f(s) ds \right\|_{L^2}^2 \quad \text{from (2.65)} \\ &\leq C_1 \|f\|_{L^{p'}(\mathbb{R}, L^q(\mathbb{R}^2))}^2, \end{aligned} \quad (2.67)$$

with (p, q) an admissible Strichartz pair i.e $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and $2 \leq q < +\infty$.

Theorem 2 (Christ-Kiselev, [12]). *Consider a bounded operator*

$$T : L^p(\mathbb{R}; B_1) \rightarrow L^q(\mathbb{R}; B_2),$$

given by a locally integrable kernel $K(t, s)$ i.e.

$$Tf(t) = \int_{\mathbb{R}} K(t, s) f(s) ds. \quad (2.68)$$

$K(t, s)$ with values in bounded operators from B_1 to B_2 , where B_1 and B_2 are Banach spaces. Suppose that $p < q$. Then the operator

$$\int_{-\infty}^t K(t, s)f(s),$$

is bounded from $L^p(\mathbb{R}; B_1)$ to $L^q(\mathbb{R}; B_2)$.

Let $K(t, s) = \chi D_x^{\frac{1}{2}} e^{i(t-s)\Delta}$ and we take $p = p'$, $B_1 = L^{q'}$, $q = 2$, $B_2 = L^2$. In the proof of Theorem 2, the fact that $K(t, s)$ is a bounded operator from B_1 to B_2 is only a technical point in order to ensure that the quantities which are estimates are defined and finite. If we take $f \in C_c^\infty(\mathbb{R} \times \mathbb{R}^2)$ we can apply Christ-Kiselev's proof as it is and we obtain

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \left| \int_{-\infty}^t \chi D_x^{1/2} e^{i(t-s)\Delta} f(s) ds \right|^2 dx dt \leq C \|f\|_{L^{p'}(\mathbb{R}, L^{q'}(\mathbb{R}^2))}^2.$$

Let $f = g \mathbf{1}_{[0, T]}$ we obtain then for $t \in [0, T]$

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left| \int_0^t \chi D_x^{1/2} e^{i(t-s)\Delta} g(s) ds \right|^2 dx dt &\leq \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left| \int_{-\infty}^t \chi D_x^{1/2} e^{i(t-s)\Delta} f(s) ds \right|^2 dx dt \\ &\leq C \|f\|_{L^{p'}(\mathbb{R}, L^{q'}(\mathbb{R}^2))}^2 \\ &\leq C \|g\|_{L_T^{p'}(L^{q'})}^2, \end{aligned}$$

with (p, q) an admissible Strichartz pair (i.e. $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and $2 \leq q < +\infty$). Thus for all $f \in L_T^{p'}(L^{q'}(\mathbb{R}^2))$ and $T > 0$ with (p, q) an admissible Strichartz pair (i.e. $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and $2 \leq q < +\infty$) we have for $t \in [0, T]$:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \left| \int_0^t \chi D_x^{1/2} e^{i(t-s)\Delta} f(s) ds \right|^2 dx dt \leq C \|f\|_{L_T^{p'}(L^{q'}(\mathbb{R}^2))}^2. \quad (2.69)$$

Now we compute

$$\left| \mathcal{F}(\chi D_x^{\frac{1}{2}} f - D_x^{\frac{1}{2}}(\chi f)) \right|(\xi) = \left| \int_{\mathbb{R}^2} (\hat{\chi}(\eta)(|\xi - \eta|^{\frac{1}{2}} - |\xi|^{\frac{1}{2}}) \hat{f}(\xi - \eta) d\eta \right|.$$

Observing that $||\xi - \eta|^{\frac{1}{2}} - |\xi|^{\frac{1}{2}}| \leq 2 + |\eta|$ (indeed when $|\xi - \eta| \geq 1$ or $|\xi| \geq 1$ we have $||\xi - \eta|^{\frac{1}{2}} - |\xi|^{\frac{1}{2}}| = \frac{||\xi - \eta| - |\xi||}{|\xi - \eta|^{\frac{1}{2}} + |\xi|^{\frac{1}{2}}} \leq |\eta|$) and combining with Hölder's inequality and Plancherel theorem we have

$$\|\mathcal{F}(\chi D_x^{\frac{1}{2}} f - D_x^{\frac{1}{2}} \chi f)\|_{L^2(\mathbb{R}^2)} \leq \|\hat{\chi}(\eta)(2 + |\eta|)\|_{L^1(\mathbb{R}^2)} \|f\|_{L^2(\mathbb{R}^2)} \leq C \|f\|_{L^2(\mathbb{R}^2)}.$$

Thus

$$\|\chi D_x^{\frac{1}{2}} f - D_x^{\frac{1}{2}} \chi f\|_{L^2(\mathbb{R}^2)} \leq C \|f\|_{L^2(\mathbb{R}^2)}. \quad (2.70)$$

Thus using (2.66) , (2.70) and Strichartz estimate we deduce that there exists $C > 0$ depending on χ and T such that:

$$\|\chi e^{it\Delta} f\|_{L_T^2(\dot{H}^{\frac{1}{2}})} \leq C \|f\|_{L^2}. \quad (2.71)$$

Finally we have

$$\begin{aligned} & \left\| \int_0^t \chi e^{i(t-s)\Delta} f(s) ds \right\|_{L_T^2(\dot{H}^{1/2})}^2 \leq 2 \int_0^T \int_{\mathbb{R}^2} \left| \int_0^t \chi D_x^{1/2} e^{i(t-s)\Delta} f(s) ds \right|^2 dx dt \\ & + 2 \int_0^T \int_{\mathbb{R}^2} \left| \int_0^t (\chi D_x^{\frac{1}{2}} f - D_x^{\frac{1}{2}} \chi) e^{i(t-s)\Delta} f(s) ds \right|^2 dx dt. \end{aligned} \quad (2.72)$$

Using (2.70) and Strichartz estimates we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^2} \left| \int_0^t (\chi D_x^{\frac{1}{2}} - D_x^{\frac{1}{2}} \chi) e^{i(t-s)\Delta} f(s) ds \right|^2 dx dt & \leq C \int_0^T \int_{\mathbb{R}^2} \left| \int_0^t e^{i(t-s)\Delta} f(s) ds \right|^2 dx dt \\ & \leq CT \|f\|_{L_T^{p'}(L^{q'}(\mathbb{R}^2))}^2. \end{aligned}$$

Thus using (2.69) and (2.72) we have

$$\left\| \int_0^t \chi e^{i(t-s)\Delta} f(s) ds \right\|_{L_T^2(\dot{H}^{1/2})} \leq C \|f\|_{L_T^{p'}(L^{q'}(\mathbb{R}^2))},$$

with C depending on χ and T . Moreover using Strichartz again

$$\begin{aligned} \left\| \int_0^t \chi e^{i(t-s)\Delta} f(s) ds \right\|_{L_T^2(L^2(\mathbb{R}^2))} & = \left(\int_0^T \int_{\mathbb{R}^2} \left| \int_0^t \chi e^{i(t-s)\Delta} f(s) ds \right|^2 dx dt \right)^{\frac{1}{2}} \\ & \leq C\sqrt{T} \|f\|_{L_T^{p'}(L^{q'}(\mathbb{R}^2))}. \end{aligned}$$

Thus we have obtained (2.64) by density of $C_c^\infty(\mathbb{R} \times \mathbb{R}^2)$ in $L_T^{p'}(L^{q'}(\mathbb{R}^2))$ for a Strichartz pair (p, q) such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and $2 \leq q < +\infty$. This completes the proof of lemma 2.1. \square

We must also prove the following lemma.

Lemma 2.2. *There exists $C > 0$, not depending on n , such that for $j \in \{1, 2\}$*

$$\|\sqrt{\rho_n} \mathbb{P} u_n\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))}^2 + \|\sqrt{\rho_n} \mathbb{Q} u_n\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))}^2 \leq C, \quad (2.73)$$

$$\|\partial_j w_n\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))} \leq C. \quad (2.74)$$

Furthermore for any $p \geq 2$ there exists $C_p > 0$ such that

$$\|F(\rho_n) - F(1)\|_{L^\infty(\mathbb{R}^+, L^p(\mathbb{R}^2))} \leq C. \quad (2.75)$$

Proof. For (2.73) we have from (2.46):

$$\begin{aligned} \|\sqrt{\rho_n}u_n\|_{L^2}^2 &= \|\sqrt{\rho_n}\nabla\theta^n + \sqrt{\rho_n}\nabla^\perp\theta_1^n\|_{L^2}^2 = \|\sqrt{\rho_n}\nabla\theta^n\|_{L^2}^2 + \|\sqrt{\rho_n}\nabla^\perp\theta_1^n\|_{L^2}^2 \\ &\quad + 2\int_{\mathbb{R}^2}\langle\sqrt{\rho_n}\nabla\theta^n, \sqrt{\rho_n}\nabla^\perp\theta_1^n\rangle dx. \end{aligned} \quad (2.76)$$

Using again (2.46) we can observe that $\sqrt{\rho_n}\nabla\theta^n$ and $\sqrt{\rho_n}\nabla^\perp\theta_1^n$ are orthogonal on $\mathbb{R}^2 \setminus \{0\}$ then it implies that:

$$\|\sqrt{\rho_n}u_n\|_{L^2}^2 = \|\sqrt{\rho_n}\mathbb{P}u_n\|_{L^2}^2 + \|\sqrt{\rho_n}\mathbb{Q}u_n\|_{L^2}^2. \quad (2.77)$$

Finally using (2.1) we have (2.73).

We recall now that:

$$w_n = \sqrt{\rho_n}e^{-\frac{i}{2c_2}\theta_n + \frac{i}{2c_2}a_n}, \quad (2.78)$$

and

$$\partial_j w_n = \partial_j \sqrt{\rho_n} e^{-\frac{i}{2c_2}\theta_n + \frac{i}{2c_2}a_n} - \frac{i}{(2c_2)} \partial_j \theta_n \sqrt{\rho_n} e^{-\frac{i}{2c_2}\theta_n + \frac{i}{2c_2}a_n},$$

so since $\partial_j \theta_n = (\mathbb{Q}u_n)_j$ we have:

$$\partial_j w_n = \partial_j \sqrt{\rho_n} e^{-\frac{i}{2c_2}\theta_n + \frac{i}{2c_2}a_n} - \frac{i}{(2c_2)} (\mathbb{Q}u_n)_j \sqrt{\rho_n} e^{-\frac{i}{2c_2}\theta_n + \frac{i}{2c_2}a_n}.$$

Using the previous formula, estimate (2.74) follows from (2.1) and (2.73). For the estimate (2.75) we recall that $F(\rho) = \frac{\gamma}{\gamma-1}\rho^{\gamma-1}$ so that:

$$\begin{aligned} \int_{\mathbb{R}^2} |F(\rho_n) - F(1)|^p dx &= \int_{\mathbb{R}^2} |F(\rho_n) - F(1)|^p \mathbf{1}_{\{|\rho_n-1| < \frac{1}{2}\}} dx \\ &\quad + \int_{\mathbb{R}^2} |F(\rho_n) - F(1)|^p \mathbf{1}_{\{|\rho_n-1| \geq \frac{1}{2}\}} dx. \end{aligned}$$

There exists now $C_p > 0$ such that:

$$|F(\rho_n) - F(1)|^p \mathbf{1}_{\{|\rho_n-1| < \frac{1}{2}\}} \leq C_p |\rho_n - 1|^p,$$

and

$$|F(\rho_n) - F(1)|^p \mathbf{1}_{\{|\rho_n-1| \geq \frac{1}{2}\}} \leq C_p |\rho_n - 1|^{p(\gamma-1)} \mathbf{1}_{\{|\rho_n-1| \geq \frac{1}{2}\}}.$$

Estimate (2.75) follows from (2.5). This ends the proof of Lemma 2.2. \square

Proof of Proposition 2.2: Using (2.61) we have

$$\begin{aligned} \partial_t \partial_j w_n + ic_2 \Delta \partial_j w_n &= \frac{i}{2c_2} \partial_j w_n (F(|w_n|^2) - F(1)) + \frac{1}{2} |\mathbb{P}u_n|^2 + (\Delta)^{-1} \operatorname{div}(\operatorname{curl}u_n (\mathbb{P}u_n)^\perp) \\ &\quad + \frac{i}{2c_2} w_n (2F'(|w_n|^2)) (\operatorname{Re}w_n \partial_j \operatorname{Re}w_n + \operatorname{Im}w_n \partial_j \operatorname{Im}w_n) \\ &\quad + \partial_j \mathbb{P}u_n \cdot \mathbb{P}u_n + \partial_j (\Delta)^{-1} \operatorname{div}(\operatorname{curl}u_n (\mathbb{P}u_n)^\perp). \end{aligned} \quad (2.79)$$

Then by Duhamel principle, using the (2.79) we can express $\chi\partial_j w_n(t)$ for $t \geq 0$ as follows with $\chi \in C_c^\infty(\mathbb{R}^2)$:

$$\begin{aligned} \chi\partial_j w_n(t) = & \chi e^{-itc_2\Delta}\partial_j w_n(0) + \int_0^t \chi e^{-ic_2(t-t')\Delta} \left[\frac{i}{2c_2}\partial_j w_n(F(|w_n|^2) - F(1) + \frac{1}{2}|\mathbb{P}u_n|^2 + \theta_2^n) \right. \\ & + \frac{i}{2c_2}w_n(2F'(|w_n|^2)(\text{Re}w_n\partial_j\text{Re}w_n + \text{Im}w_n\partial_j\text{Im}w_n) + \partial_j\mathbb{P}u_n \cdot \mathbb{P}u_n + \\ & \left. \partial_j(\Delta)^{-1}\text{div}(\text{curl}u_n(\mathbb{P}u_n)^\perp) \right](t') dt'. \end{aligned} \quad (2.80)$$

The proof of (2.62) consists now in applying to the expression (2.80) the Proposition 2.1, to do this we will need the uniform estimates obtained in the Section 2.1. First of all, using (2.63) and (2.74), there exists $C > 0$ independent on n such that:

$$\begin{aligned} \|\chi e^{-ic_2t\Delta}\partial_j w_n(0)\|_{L_T^2(H^{1/2})} & \leq C\|\partial_j w_n(0)\|_{L^2} \\ & \leq C. \end{aligned} \quad (2.81)$$

Then we obtain similarly for $C_T > 0$ large enough:

$$\left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} \frac{i}{2c_2}\partial_j w_n(F(|w_n|^2) - F(1))dt' \right\|_{L_T^2(H^{1/2})} \leq C_T. \quad (2.82)$$

Indeed using the admissible Strichartz pairs (4, 4), Proposition 2.1 and (2.74), (2.75) we have

$$\begin{aligned} & \left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} \frac{i}{2c_2}\partial_j w_n(F(|w_n|^2) - F(1))dt' \right\|_{L_T^2(H^{1/2})} \\ & \leq C\|\partial_j w_n(F(|w_n|^2) - F(1))\|_{L_T^{4/3}(L^{4/3})} \\ & \leq CT^{3/4}\|\partial_j w_n(F(\rho_n) - F(1))\|_{L_T^\infty(L^{4/3})} \\ & \leq T^{3/4}C\|\partial_j w_n\|_{L_T^\infty(L^2)}\|F(\rho_n) - F(1)\|_{L_T^\infty(L^4)} \\ & \leq C_T. \end{aligned}$$

Next, we observe that we have

$$\left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} \frac{i}{2c_2}\partial_j w_n\left(\frac{1}{2}|\mathbb{P}u_n|^2\right)dt' \right\|_{L_T^2(H^{1/2})} \leq C_T. \quad (2.83)$$

Again using proposition 2.1 with the admissible Strichartz pairs (4, 4), estimates (2.8) and (2.74) we have

$$\begin{aligned} \left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} \frac{i}{2c_2}\partial_j w_n\left(\frac{1}{2}|\mathbb{P}u_n|^2\right)dt' \right\|_{L_T^2(H^{1/2})} & \leq C\|\partial_j w_n|\mathbb{P}u_n|^2\|_{L_T^{4/3}(L^{4/3})} \\ & \leq CT^{3/4}\|\partial_j w_n\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))}\|\mathbb{P}u_n\|_{L^\infty(\mathbb{R}^+, L^4(\mathbb{R}^2))} \\ & \leq C_T. \end{aligned}$$

Now we want to show that:

$$\left\| \int_0^t \chi e^{i(t-t')\Delta} \frac{i}{2c_2} \partial_j w_n(\theta_2^n) dt' \right\|_{L_T^2(H^{1/2})} \leq C_T, \quad (2.84)$$

with $\theta_2^n = (\Delta)^{-1} \operatorname{div}(\operatorname{curl} u_n (Pu_n)^\perp)$. We have, in view of (2.8) and (2.6),

$$\|\operatorname{curl} u_n(t, \cdot) (Pu_n)^\perp(t, \cdot)\|_{L^\infty(\mathbb{R}^+, L^{4/3}(\mathbb{R}^2))} \leq C.$$

By continuity of the Riesz operator in $L^{4/3}(\mathbb{R}^2)$, we deduce that:

$$\|\nabla(\Delta)^{-1} \operatorname{div}(\operatorname{curl} u_n(t, \cdot) (Pu_n)^\perp(t, \cdot))\|_{L^\infty(\mathbb{R}^+, L^{4/3}(\mathbb{R}^2))} \leq C.$$

By Sobolev embedding we get that:

$$\|(\Delta)^{-1} \operatorname{div}(\operatorname{curl} u_n (Pu_n)^\perp)\|_{L^\infty(\mathbb{R}^+, L^4(\mathbb{R}^2))} \leq C. \quad (2.85)$$

Using proposition 2.1 with the admissible Strichartz pairs (4, 4), (2.85) and (2.74), we have:

$$\begin{aligned} \left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} \frac{i}{2c_2} \partial_j w_n(\theta_2^n) dt' \right\|_{L_T^2(H^{1/2})} &\leq C \|\partial_j w_n \theta_2^n\|_{L_T^{4/3}(L^{4/3})} \\ &\leq T^{3/4} C \|\partial_j w_n\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))} \|\theta_2^n\|_{L^\infty(\mathbb{R}^+, L^4(\mathbb{R}^2))} \\ &\leq C_T, \end{aligned}$$

which gives (2.84). Next, we claim that:

$$\begin{aligned} \left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} w_n F'(|w_n|^2) (\operatorname{Re} w_n \partial_j \operatorname{Re} w_n + \operatorname{Im} w_n \partial_j \operatorname{Im} w_n) dt' \right\|_{L_T^2(H^{1/2})} \\ \leq C_T. \end{aligned} \quad (2.86)$$

We first observe that

$$\begin{aligned} &w_n F'(|w_n|^2) (\operatorname{Re} w_n \partial_j \operatorname{Re} w_n) \\ &= (\rho_n F'(\rho_n) - F'(1)) e^{-\frac{i}{2c_2} \theta_n + \frac{i}{2c_2} a_n} \cos\left(-\frac{1}{2c_2} \theta_n + \frac{1}{2c_2} a_n\right) \partial_j \operatorname{Re} w_n \\ &\quad + e^{-\frac{i}{2c_2} \theta_n + \frac{i}{2c_2} a_n} \cos\left(-\frac{1}{2c_2} \theta_n + \frac{1}{2c_2} a_n\right) F'(1) \partial_j \operatorname{Re} w_n. \end{aligned} \quad (2.87)$$

In view of (2.74), we have

$$\left\| e^{-\frac{i}{2c_2} \theta_n + \frac{i}{2c_2} a_n} \cos\left(-\frac{1}{2c_2} \theta_n + \frac{1}{2c_2} a_n\right) F'(1) \partial_j \operatorname{Re} w_n \right\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))} \leq C. \quad (2.88)$$

Invoking proposition (2.1) with Strichartz pair $(\infty, 2)$, we deduce from (2.88) that

$$\begin{aligned}
& \left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} e^{-\frac{i}{2c_2}\theta_n + \frac{i}{2c_2}a_n} \cos\left(-\frac{1}{2c_2}\theta_n + \frac{1}{2c_2}a_n\right) 2F'(1)\partial_j \text{Re}w_n dt' \right\|_{L_T^2(H^{1/2})} \\
& \leq C \left\| e^{-\frac{i}{2c_2}\theta_n + \frac{i}{2c_2}a_n} \cos\left(-\frac{1}{2c_2}\theta_n + \frac{1}{2c_2}a_n\right) 2F'(1)\partial_j \text{Re}w_n \right\|_{L_T^1(L^2)} \\
& \leq CT \|\partial_j \text{Re}w_n\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))} \\
& \leq C_T.
\end{aligned} \tag{2.89}$$

We observe now that

$$(\rho_n F'(\rho_n) - F'(1)) = (\gamma - 1)(F(\rho_n) - F(1)), \tag{2.90}$$

so that, in view of (2.75) and (2.74) we get:

$$\|(\rho_n F'(\rho_n) - F'(1)) e^{-\frac{i}{2c_2}\theta_n + \frac{i}{2c_2}a_n} \cos\left(-\frac{1}{2c_2}\theta_n + \frac{1}{2c_2}a_n\right) \partial_j \text{Re}w_n\|_{L^\infty(\mathbb{R}^+, L^{3/2}(\mathbb{R}^2))} \leq C. \tag{2.91}$$

Invoking Proposition 2.1 with Strichartz pair $(6, 3)$, we deduce from (2.91) that

$$\begin{aligned}
& \left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} (\rho_n F'(\rho_n) - F'(1)) e^{-\frac{i}{2c_2}(\theta_n - a_n)} \cos\left(-\frac{1}{2c_2}(\theta_n - a_n)\right) \partial_j \text{Re}w_n dt' \right\|_{L_T^2(H^{1/2})} \\
& \leq C \|(\rho_n F'(\rho_n) - F'(1)) e^{-\frac{i}{2c_2}\theta_n + \frac{i}{2c_2}a_n} \cos\left(-\frac{1}{2c_2}\theta_n + \frac{1}{2c_2}a_n\right) \partial_j \text{Re}w_n\|_{L_T^{6/5}(L^{3/2})} \\
& \leq C_T.
\end{aligned} \tag{2.92}$$

We deduce from (2.87), (2.89) and (2.92) that

$$\left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} 2w_2 F'(|w_n|^2) (\text{Re}w_n \partial_j \text{Re}w_n) dt' \right\|_{L_T^2(H^{1/2})} \leq C_T. \tag{2.93}$$

Similarly we have

$$\left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} 2w_n F'(|w_n|^2) \text{Im}w_n \partial_j \text{Im}w_n dt' \right\|_{L_T^2(H^{1/2})} \leq C_T. \tag{2.94}$$

So that (2.86) holds. We now claim that

$$\left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} w_n \partial_j \mathbb{P}u_n \cdot \mathbb{P}u_n dt' \right\|_{L_T^2(H^{1/2})} \leq C_T. \tag{2.95}$$

Indeed applying Proposition 2.1 with Strichartz pair $(4, 4)$, in view of estimates (2.7) and (2.73), we have

$$\begin{aligned}
& \left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} w_n \partial_j \mathbb{P}u_n \cdot \mathbb{P}u_n dt' \right\|_{L_T^2(H^{1/2})} \leq C \|w_n \partial_j \mathbb{P}u_n \cdot \mathbb{P}u_n\|_{L_T^{4/3}(L^{4/3})} \\
& \leq CT^{3/4} \|w_n \partial_j \mathbb{P}u_n \cdot \mathbb{P}u_n\|_{L^\infty(\mathbb{R}^+, L^{4/3}(\mathbb{R}^2))} \\
& \leq CT^{3/4} \|w_n \mathbb{P}u_n\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))} \|\partial_j \mathbb{P}u_n\|_{L^\infty(\mathbb{R}^+, L^4(\mathbb{R}^2))} \\
& \leq CT,
\end{aligned}$$

which is estimate (2.86). Now let us demonstrate that for $j \in \{1, 2\}$:

$$\left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} w_n \partial_j (\Delta)^{-1} \operatorname{div}(\operatorname{curl}u_n (\mathbb{P}u_n)^\perp) dt' \right\|_{L_T^2(H^{1/2})} \leq CT. \quad (2.96)$$

We first observe that

$$\begin{aligned}
w_n \partial_j (\Delta)^{-1} \operatorname{div}(\operatorname{curl}u_n (\mathbb{P}u_n)^\perp) &= e^{-\frac{i\theta_n}{2c_2} + \frac{ia_n}{2c_2}} \partial_j (\Delta)^{-1} \operatorname{div}(\operatorname{curl}u_n (\mathbb{P}u_n)^\perp) \\
&\quad + (\sqrt{\rho_n} - 1) e^{-\frac{i\theta_n}{2c_2} + \frac{ia_n}{2c_2}} \partial_j (\Delta)^{-1} \operatorname{div}(\operatorname{curl}u_n (\mathbb{P}u_n)^\perp).
\end{aligned}$$

For thesecond term in view of Proposition 2.1 with Strichartz pair $(4, 4)$ and via the continuity of the Riesz operator in L^4 , estimates (2.8), (2.4) and (2.6) give:

$$\begin{aligned}
& \left\| \int_0^t \chi e^{-ic_2(t-t')\Delta} \chi (\sqrt{\rho_n} - 1) e^{-\frac{i(\theta_n - a_n)}{2c_2}} \partial_j (\Delta)^{-1} \operatorname{div}(\operatorname{curl}u_n (\mathbb{P}u_n)^\perp) dt' \right\|_{L_T^2(H^{1/2})} \\
& \leq C \|(\sqrt{\rho_n} - 1) e^{-\frac{i(\theta_n - a_n)}{2c_2}} \partial_j (\Delta)^{-1} \operatorname{div}(\operatorname{curl}u_n (\mathbb{P}u_n)^\perp)\|_{L_T^{4/3}(L^{4/3})} \\
& \leq CT^{3/4} \|(\sqrt{\rho_n} - 1)\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))} \|\operatorname{curl}u_n (\mathbb{P}u_n)^\perp\|_{L^\infty(\mathbb{R}^+, L^4(\mathbb{R}^2))} \\
& \leq CT.
\end{aligned}$$

We operate in a similar manner for $e^{-\frac{i(\theta_n - a_n)}{2c_2}} (\partial_j (\Delta)^{-1} \operatorname{div}(\operatorname{curl}u_n (\mathbb{P}u_n)^\perp))$, and we obtain (2.96). Finally, in view of (2.81), (2.82), (2.83), (2.84), (2.86), (2.95), (2.96), we have obtained the desired estimates (2.62). This complete the proof of proposition 2.2. \square

2.5 Strong convergence of the sequence $(\nabla \sqrt{\rho_n}, \sqrt{\rho_n} \mathbb{Q}u_n)_{n \in \mathbb{N}}$

First of all, we wish to prove the strong convergence up to a subsequence of the sequence $(\rho_n)_{n \in \mathbb{N}}$ to a limit ρ in $L^\infty(L_{loc}^q(\mathbb{R}^2))$ for any $q \geq 1$. The key argument will be the use of the Aubin Lions Lemma that we recall briefly.

Aubin-Lions lemma Let X_0 , X and X_1 be three Banach spaces. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1 . For $1 \leq p, q \leq \infty$, let

$$W = \{u \in L^p([0, T]; X_0) \mid \partial_t u \in L^q([0, T]; X_1)\}.$$

- If $p < \infty$ then the embedding of W into $L^p([0, T]; X)$ is compact.
- If $p = +\infty$ and $q > 1$, then the embedding of W into $C([0, T], X)$ is compact.

Proposition 2.3. *There exist $\rho \in L^\infty_{loc}(\mathbb{R}_+, L^q_{loc}(\mathbb{R}^2))$ for any $1 \leq q < \infty$, such that, up to a subsequence,*

$$\rho_n \rightarrow \rho \text{ in } L^\infty_{loc}(\mathbb{R}_+, L^q_{loc}(\mathbb{R}^2)), \text{ as } n \rightarrow \infty. \quad (2.97)$$

Proof. Using Hölder inequality, (2.20), (2.21) and (2.12) we have for all $1 \leq p < 2$ and any compact K

$$\begin{aligned} \|\nabla \rho_n\|_{L^\infty(\mathbb{R}^+, L^p(K))} &\leq 2\|\sqrt{\rho_n}\|_{L^\infty(\mathbb{R}^+, L^q(K))} \|\nabla \sqrt{\rho_n}\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))} \\ &\leq C_{p,K}, \end{aligned} \quad (2.98)$$

where $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. We deduce immediately from (2.98) and (2.5) that for any $1 \leq p < 2$:

$$\|\rho_n\|_{L^\infty(W^{1,p}(K))} \leq C_{p,K}. \quad (2.99)$$

In order to apply Aubin-Lions Lemma, we need a information on the time derivative of ρ_n . Observing that

$$\rho_n u_n = \sqrt{\rho_n} u_n (\sqrt{\rho_n} - 1) + \sqrt{\rho_n} u_n,$$

we obtain, using (2.12) and (2.20) that for any $1 \leq p < 2$:

$$\|\rho_n u_n\|_{L^\infty(\mathbb{R}^+, L^p(K))} \leq C_{p,K}.$$

We deduce via the mass equation that:

$$\|\partial_t(\rho_n)\|_{L^\infty(\mathbb{R}^+, W^{-1,p}(K))} \leq C_{p,K}.$$

We fix now $p \in]1, 2[$, using that $W^{1,p}(K)$ is compactly embedded in $L^p(K)$ and $L^p(K)$ is continuously embedded in $W^{-1,p}(K)$, we deduce using Aubin Lions Lemma and the diagonal extraction procedure that up to a subsequence we have for all $T > 0$ and for all $R > 0$:

$$\rho_n \rightarrow \rho \text{ in } L^\infty_T(L^p(B(0, R))), \text{ as } n \rightarrow \infty. \quad (2.100)$$

From (2.100), (2.5) and by interpolation we can prove that (2.97) is verified for any $p \in [1, +\infty[$. \square

Corollary 2.2. *Again up to a subsequence, we have:*

$$P(\rho_n) \rightarrow P(\rho) \text{ in } L^q_{loc}(\mathbb{R}_+ \times \mathbb{R}^n), \text{ as } n \rightarrow \infty, \quad (2.101)$$

for any $1 \leq q < \infty$ and:

$$\sqrt{\rho_n} \rightarrow \sqrt{\rho} \text{ in } L^2_{loc}(\mathbb{R}_+ \times \mathbb{R}^n), \text{ as } n \rightarrow \infty. \quad (2.102)$$

Proof. In view of proposition 2.3, we have, up to a subsequence

$$\rho_n \rightarrow \rho \text{ in } L_{loc}^q(\mathbb{R}_+ \times \mathbb{R}^n), \text{ as } n \rightarrow \infty, \quad (2.103)$$

for any $1 \leq q < \infty$. Since we have:

$$|P(\rho_n) - P(\rho)| = |\rho_n^\gamma - \rho^\gamma| \leq \gamma |\rho_n - \rho| (|\rho_n|^{\gamma-1} + |\rho|^{\gamma-1}),$$

we deduce from (2.103) and (2.5) that (2.101) is satisfied. Let $\eta > 0$, we have

$$\begin{aligned} |\sqrt{\rho_n} - \sqrt{\rho}|^2 \mathbf{1}_{\{|\sqrt{\rho_n} + \sqrt{\rho}| \geq \eta\}} &\leq \frac{|\rho_n - \rho|^2}{|\sqrt{\rho_n} + \sqrt{\rho}|^2} \mathbf{1}_{\{|\sqrt{\rho_n} + \sqrt{\rho}| \geq \eta\}} \\ &\leq \frac{1}{\eta^2} |\rho_n - \rho|^2, \end{aligned}$$

and since $\rho_n \geq 0$ and $\rho \geq 0$ (indeed up to a subsequence ρ_n converges almost everywhere to ρ)

$$|\sqrt{\rho_n} - \sqrt{\rho}|^2 \mathbf{1}_{\{|\sqrt{\rho_n} + \sqrt{\rho}| \leq \eta\}} \leq \eta^2.$$

Then, we deduce that for any compact set Ω in $\mathbb{R}^+ \times \mathbb{R}^2$ we have for any $\eta > 0$:

$$\|\sqrt{\rho_n} - \sqrt{\rho}\|_{L^2(\Omega)}^2 \leq \frac{1}{\eta^2} \|\rho_n - \rho\|_{L^2(\Omega)}^2 + \eta^2 |\Omega|. \quad (2.104)$$

Combining (2.104) and (2.103) completes the proof. \square

We are now going to prove the main part of this section on the strong convergence of the sequence $(\nabla \sqrt{\rho_n}, \sqrt{\rho_n} \mathbb{Q} u_n)_{n \in \mathbb{N}}$ in $L_{loc}^2(\mathbb{R}^+ \times \mathbb{R}^2)$. Using the Proposition (2.2) and the estimate (2.74), we have obtained that for any $\chi \in C_c^\infty(\mathbb{R}^2)$:

$$\|\chi \partial_j w_n\|_{L_T^2(H^{1/2})} \leq C. \quad (2.105)$$

We have:

$$\begin{aligned} \partial_t(\chi \partial_j w_n) + i c_2 \chi \Delta \partial_j w_n &= \chi \left(\frac{i}{2c} \partial_j w_n (F(|w_n|^2) - F(1) + \frac{1}{2} |\mathbb{P} u_n|^2 + \theta_2^n) \right. \\ &\quad + w_n (2F'(|w_n|^2) (\text{Re} w_n \partial_j \text{Re} w_n + \text{Im} w_n \partial_j \text{Im} w_n) \\ &\quad \left. + \partial_j \mathbb{P} u_n \cdot \mathbb{P} u_n + \partial_j (\Delta)^{-1} \text{div}(\text{curl} u_n (\mathbb{P} u_n)^\perp) \right). \end{aligned} \quad (2.106)$$

We have seen in the Section 2.4 that:

$$\begin{aligned} &\left\| \chi \left(\frac{i}{2c} \partial_j w_n (F(|w_n|^2) - F(1) + \frac{1}{2} |\mathbb{P} u_n|^2 + \theta_2^n) \right. \right. \\ &\quad + w_n (2F'(|w_n|^2) (\text{Re} w_n \partial_j \text{Re} w_n + \text{Im} w_n \partial_j \text{Im} w_n) \\ &\quad \left. \left. + \partial_j \mathbb{P} u_n \cdot \mathbb{P} u_n + \partial_j (\Delta)^{-1} \text{div}(\text{curl} u_n (\mathbb{P} u_n)^\perp) \right) \right\|_{L_T^\infty(L^{4/3}(\mathbb{R}^2))} \leq C_T. \end{aligned} \quad (2.107)$$

So, we have using Sobolev embedding:

$$\begin{aligned} & \left\| \chi \left(\frac{i}{2c} \partial_j w_n (F(|w_n|^2) - F(1) + \frac{1}{2} |\mathbb{P}u_n|^2 + \theta_2^n) \right. \right. \\ & \quad + w_n (2F'(|w_n|^2) (\operatorname{Re} w_n \partial_j \operatorname{Re} w_n + \operatorname{Im} w_n \partial_j \operatorname{Im} w_n)) \\ & \quad \left. \left. + \partial_j \mathbb{P}u_n \cdot \mathbb{P}u_n + \partial_j (\Delta)^{-1} \operatorname{div}(\operatorname{curl} u_n (\mathbb{P}u_n)^\perp) \right\|_{L_T^\infty(H^{-\frac{1}{2}})} \leq C_T. \end{aligned} \quad (2.108)$$

Using now that the following embedding are continuous $H^{-\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow H^{-2}(\mathbb{R}^2)$, we get:

$$\begin{aligned} & \left\| \chi \left(\frac{i}{2c} \partial_j w_n (F(|w_n|^2) - F(1) + \frac{1}{2} |\mathbb{P}u_n|^2 + \theta_2^n) \right. \right. \\ & \quad + w_n (2F'(|w_n|^2) (\operatorname{Re} w_n \partial_j \operatorname{Re} w_n + \operatorname{Im} w_n \partial_j \operatorname{Im} w_n)) \\ & \quad \left. \left. + \partial_j \mathbb{P}u_n \cdot \mathbb{P}u_n + \partial_j (\Delta)^{-1} \operatorname{div}(\operatorname{curl} u_n (\mathbb{P}u_n)^\perp) \right\|_{L_T^2(H^{-2})} \leq C. \end{aligned} \quad (2.109)$$

Moreover using the fact that multiplication by a function of $\mathcal{S}(\mathbb{R}^d)$ is a continuous map from $H^s(\mathbb{R}^d)$ into itself and using (2.62) we have

$$\|ic_2 \chi \Delta \partial_j w_n\|_{L_T^2(H^{-2}(\mathbb{R}^2))} \leq C. \quad (2.110)$$

So we have using (2.109) and (2.110):

$$\|\partial_t(\chi \partial_j w_n)\|_{L_T^2(H^{-2}(\mathbb{R}^2))} \leq C_T. \quad (2.111)$$

Let K a compact, we define by $H_K^{\frac{1}{2}}$ the distributions which belongs to $H^{\frac{1}{2}}$ and are supported in K . We know that $H_K^{1/2}(\mathbb{R}^2)$ is compactly embedded in $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ is continuously embedded in $H^{-2}(\mathbb{R}^2)$ then from the Aubin Lions Lemma and applying the diagonal extraction procedure we deduce the following proposition.

Proposition 2.4. *Up to a subsequence there exists some map $f^j \in L_{loc}^2(\mathbb{R}^+ \times \mathbb{R}^2)$ with $j \in \{1, 2\}$ such that for all $T > 0$ and all $R > 0$ we have:*

$$\partial_j w_n \rightarrow f^j \text{ in } L_T^2(L^2(B(0, R))), \text{ as } n \rightarrow \infty. \quad (2.112)$$

We recall now that

$$\nabla w_n = \nabla \sqrt{\rho_n} e^{-\frac{i}{2c_2} \theta_n + \frac{i}{2c_2} a_n} - \frac{i}{2c_2} \nabla \theta_n \sqrt{\rho_n} e^{-\frac{i}{2c_2} \theta_n + \frac{i}{2c_2} a_n},$$

where $\partial_j \theta_n = (\mathbb{Q}u_n)_j$. It implies that:

$$\nabla w_n = (\nabla \sqrt{\rho_n} - \frac{i}{2c_2} (\mathbb{Q}u_n) \sqrt{\rho_n}) \phi_n,$$

with $\phi_n = e^{-\frac{i}{2c_2}\theta_n + \frac{i}{2c_2}a_n}$. In other word, we have:

$$\begin{aligned} \|\phi_n\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)} &\leq 1, \\ \nabla\sqrt{\rho_n} &= \operatorname{Re}(\overline{\phi_n}\nabla w_n) \quad \text{and} \quad \sqrt{\rho_n}\mathbb{Q}u_n = \operatorname{Im}(\overline{\phi_n}\nabla w_n), \\ |\nabla\sqrt{\rho_n}|^2 + \frac{1}{4c_2^2}|\sqrt{\rho_n}\mathbb{Q}u_n|^2 &= |\nabla w_n|^2. \end{aligned} \quad (2.113)$$

Since $(\phi_n)_{n \in \mathbb{N}}$ is bounded by 1, it converges up to extraction in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$ weak * to some function ϕ such that $\|\phi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)} \leq 1$. In addition according to proposition 2.4, we know that ∇w_n converges strongly to $f = (f^1, f^2)^t$ in $L^2(K)$ for any compact K of $\mathbb{R}^+ \times \mathbb{R}^2$, this implies that:

$$\overline{\phi_n}\nabla w_n \rightharpoonup \overline{\phi}f \quad \text{in} \quad L^2(K). \quad (2.114)$$

From (2.102) and (2.113) we deduce up to a subsequence and for any compact K of $\mathbb{R}^+ \times \mathbb{R}^2$ that:

$$\nabla\sqrt{\rho} = \operatorname{Re}(\overline{\phi}f) \quad \text{and} \quad \nabla\sqrt{\rho_n} \rightharpoonup \nabla\sqrt{\rho} \quad \text{in} \quad L^2(K). \quad (2.115)$$

Similarly up to an extraction we have for any K compact set of $\mathbb{R}^+ \times \mathbb{R}^2$:

$$\frac{-1}{2c_2}\sqrt{\rho_n}\mathbb{Q}u_n \rightharpoonup \frac{-1}{2c_2}\Lambda = \operatorname{Im}(\overline{\phi}f) \quad \text{in} \quad L^2(K). \quad (2.116)$$

We have seen that up to a subsequence $\sqrt{\rho_n}\mathbb{Q}u_n$ and $\nabla\sqrt{\rho_n}$ converge weakly in $L^2(K)$ for any compact K of $\mathbb{R}^+ \times \mathbb{R}^2$ to some function Λ and $\nabla\sqrt{\rho}$. We wish now to prove that these convergence are strong, it suffices then to show that:

$$\|\nabla\sqrt{\rho_n}\|_{L^2(K)} \xrightarrow{n \rightarrow +\infty} \|\nabla\sqrt{\rho}\|_{L^2(K)} \quad \text{and} \quad \|\sqrt{\rho_n}\mathbb{Q}u_n\|_{L^2(K)} \xrightarrow{n \rightarrow +\infty} \|\Lambda\|_{L^2(K)}. \quad (2.117)$$

The weak convergence implies that:

$$\|\nabla\sqrt{\rho}\|_{L^2(K)} \leq \liminf_{n \rightarrow \infty} \|\nabla\sqrt{\rho_n}\|_{L^2(K)} \quad \text{and} \quad \|\Lambda\|_{L^2(K)} \leq \liminf_{n \rightarrow \infty} \|\sqrt{\rho_n}\mathbb{Q}u_n\|_{L^2(K)}. \quad (2.118)$$

From Proposition (2.4) and (2.113) we have:

$$\begin{aligned} \liminf_{n \rightarrow \infty} \|\nabla\sqrt{\rho_n}\|_{L^2(K)}^2 + \frac{1}{4c_2^2} \liminf_{n \rightarrow \infty} \|\rho_n\mathbb{Q}u_n\|_{L^2(K)}^2 \\ \leq \lim_{n \rightarrow \infty} (\|\nabla\sqrt{\rho_n}\|_{L^2(K)}^2 + \frac{1}{4c_2^2} \|\sqrt{\rho_n}\mathbb{Q}u_n\|_{L^2(K)}^2) = \lim_{n \rightarrow \infty} \|\nabla w_n\|_{L^2(K)}^2 = \|f\|_{L^2(K)}^2. \end{aligned} \quad (2.119)$$

It remains now to prove that:

$$\|f\|_{L^2(K)}^2 = \int_K |\phi|^2 (|\nabla\sqrt{\rho}|^2 + \frac{1}{4c_2^2}|\Lambda|^2) dt dx = \|\nabla\sqrt{\rho}\|_{L^2(K)}^2 + \frac{1}{4c_2^2} \|\Lambda\|_{L^2(K)}^2. \quad (2.120)$$

To do this we will use the Aubin-Lions lemma on χw_n . First of all using (2.105) and (2.20) we have

$$\|\chi w_n\|_{L_T^2(L^2)} \leq C. \quad (2.121)$$

Combining with (2.20) we have

$$\|\nabla(\chi w_n)\|_{L_T^2(H^1)} \leq C.$$

We have using (2.61)

$$\partial_t \chi w_n + i c_2 \chi \Delta w_n - \frac{i}{2c_2} \chi w_n (F(|w_n|^2) - F(1) + \frac{1}{2} |\mathbb{P}u_n|^2 + \theta_2^n) = 0. \quad (2.122)$$

Thus using Hölder's inequality and (2.75), (2.8), (2.85) we have

$$\left\| \frac{i}{2c_2} \chi w_n (F(|w_n|^2) - F(1) + \frac{1}{2} |\mathbb{P}u_n|^2 + \theta_2^n) \right\|_{L_T^2(L^2)} \leq C.$$

Using $\chi \Delta w_n = \Delta(\chi w_n) - \Delta(\chi)w_n - 2\nabla\chi \cdot \nabla w_n$ and (2.20), (2.62), (2.121) we have

$$\|\partial_t(\chi w_n)\|_{L_T^2(H^{-2})} \leq C.$$

For any compact K , we know that $H_K^1(\mathbb{R}^2)$ is compactly embedded in $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ is continuously embedded in $H^{-2}(\mathbb{R}^2)$ then from the Aubin Lions Lemma and applying the diagonal extraction procedure we deduce the following proposition.

Proposition 2.5. *Up to a subsequence there exists some map $w \in L_{loc}^2(\mathbb{R}^+ \times \mathbb{R}^2)$ such that for all $T > 0$ and all $R > 0$ we have:*

$$w_n \rightarrow w \text{ in } L_T^2(L^2(B(0, R))), \text{ as } n \rightarrow \infty. \quad (2.123)$$

From propositions 2.5 and 2.4 we deduce that $\nabla w = f$. In addition up to a subsequence we get

$$w_n \rightarrow w \text{ a.e. as } n \rightarrow \infty$$

We have using (1.11) that $|w_n(t, x)| > 0$ everywhere thus

$$\phi_n(t, x) = \frac{w_n(t, x)}{|w_n(t, x)|}.$$

Then

$$w_n(t, x) \mathbb{1}_{\{|w(t, x)| > 0\}} \rightarrow w(t, x) \mathbb{1}_{\{|w(t, x)| > 0\}} \text{ a.e., as } n \rightarrow \infty.$$

Then

$$\begin{aligned} \phi_n(t, x) \mathbb{1}_{\{|w(t, x)| > 0\}} &= \frac{w_n(t, x)}{|w_n(t, x)|} \mathbb{1}_{\{|w(t, x)| > 0\}} \\ &\xrightarrow{n \rightarrow +\infty} \frac{w(t, x)}{|w(t, x)|} \mathbb{1}_{\{|w(t, x)| > 0\}} \text{ a.e.} \end{aligned}$$

Thus for any $\varphi \in L^1(\mathbb{R}_+ \times \mathbb{R}^2)$ using Lebesgue's theorem and the definition of the L^∞ weak * limit we have

$$\begin{aligned} \int_{\mathbb{R} \times \mathbb{R}^2} \phi_n \varphi &= \int_{\mathbb{R} \times \mathbb{R}^2} \phi_n \varphi \\ &= \int_{\mathbb{R} \times \mathbb{R}^2} \phi_n \varphi \mathbb{1}_{\{|w(t,x)|>0\}} + \int_{\mathbb{R} \times \mathbb{R}^2} \phi_n \varphi \mathbb{1}_{\{|w(t,x)|=0\}} \\ &\xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R} \times \mathbb{R}^2} \frac{w(t,x)}{|w(t,x)|} \mathbb{1}_{\{|w(t,x)|>0\}} \varphi + \int_{\mathbb{R} \times \mathbb{R}^2} \phi \mathbb{1}_{\{|w(t,x)|=0\}} \varphi. \end{aligned}$$

Then we have

$$\phi(t,x) = \frac{w(t,x)}{|w(t,x)|} \mathbb{1}_{\{|w(t,x)|>0\}} + \phi(t,x) \mathbb{1}_{\{|w(t,x)|=0\}}.$$

Recall (see e.g. Theorem 6.19 in [17]) that

$$\nabla w = 0 \text{ a.e. on } w^{-1}(\{0\}),$$

whenever w is locally in $W^{1,1}$. Thus we have

$$\begin{aligned} |\overline{\phi(t,x)} f(t,x)| &= |\phi(t,x)| |\nabla w(t,x)| \\ &= \left| \frac{w(t,x)}{|w(t,x)|} \mathbb{1}_{\{|w(t,x)|>0\}} \right| |\nabla w(t,x)| \\ &= \mathbb{1}_{\{|w(t,x)|>0\}} |\nabla w(t,x)| \quad \text{a.e.} \\ &= |\nabla w(t,x)| \quad \text{a.e.} \\ &= |f(t,x)| \quad \text{a.e..} \end{aligned}$$

Thus we can prove (2.120). Indeed we have

$$\begin{aligned} \|f\|_{L^2(K)}^2 &= \|\overline{\phi} f\|_{L^2(K)}^2 = \int_K (|\nabla \sqrt{\rho}|^2 + \frac{1}{4c_2^2} |\Lambda|^2) dt dx \\ &= \|\nabla \sqrt{\rho}\|_{L^2(K)}^2 + \frac{1}{4c_2^2} \|\Lambda\|_{L^2(K)}^2. \end{aligned}$$

Combining now (2.119), (2.120) and (2.118) we deduce (2.117). We have finally proved that:

$$\sqrt{\rho_n}(Qu_n) \rightarrow \Lambda \text{ in } L_{loc}^2(\mathbb{R}_+ \times \mathbb{R}^n), \text{ as } n \rightarrow \infty, \quad (2.124)$$

and

$$\nabla \sqrt{\rho_n} \rightarrow \nabla \sqrt{\rho} \text{ in } L_{loc}^2(\mathbb{R}_+ \times \mathbb{R}^n), \text{ as } n \rightarrow \infty. \quad (2.125)$$

Proposition 2.6. *There exists $\Lambda_1 \in L_{loc}^\infty(\mathbb{R}_+, L_{loc}^q(\mathbb{R}^2))$ for all $1 \leq q < \infty$, such that, up to a subsequence,*

$$\mathbb{P}u_n \rightarrow \Lambda_1 \text{ in } L_{loc}^\infty(\mathbb{R}_+, L_{loc}^q(\mathbb{R}^2)), \text{ as } n \rightarrow \infty. \quad (2.126)$$

Proof. For any compact K in \mathbb{R}^2 we deduce from (2.8) that for any $p \in [1, 2[$:

$$\|\mathbb{P}u_n\|_{L^\infty(W^{1,p}(K))} \leq C_{p,K}. \quad (2.127)$$

We have, using (2.25)

$$\partial_t \operatorname{curl} u_n = -\operatorname{div}(u_n \operatorname{curl}(u_n)). \quad (2.128)$$

Observing that

$$\operatorname{curl}(u_n)u_n = \frac{\operatorname{curl}(u_n)}{\rho_n} \sqrt{\rho_n} u_n (\sqrt{\rho_n} - 1) + \frac{\operatorname{curl} u_n}{\rho_n} \sqrt{\rho_n} u_n.$$

We have using Hölder's inequality and (2.12), (2.24), (2.4) that:

$$\left\| \frac{\operatorname{curl}(u_n)}{\rho_n} \sqrt{\rho_n} u_n (\sqrt{\rho_n} - 1) \right\|_{L^\infty(L^p(\mathbb{R}^2))} \leq C_p, \quad (\forall p \in [1, 2]),$$

and

$$\left\| \frac{\operatorname{curl}(u_n)}{\rho_n} \sqrt{\rho_n} u_n \right\|_{L^\infty(L^2(\mathbb{R}^2))} \leq C.$$

Then using the continuity of the Riesz operator in L^p with $1 < p < +\infty$ and the identity

$$\partial_t \mathbb{P}u_n = -(\Delta)^{-1} \nabla^\perp \operatorname{div}(u_n \operatorname{curl} u_n).$$

We deduce that $(\partial_t \mathbb{P}u_n)_{n \in \mathbb{N}}$ is uniformly bounded in $L^\infty(\mathbb{R}^+, L^p(K))$ for $p \in [1, 2[$. So using that $W^{1,p}(K)$ is compactly embedded in $L^p(K)$ and applying Aubin Lions Lemma and the diagonal extraction procedure we deduce the Proposition 2.6 for $q = p$. The other case are obtained by interpolation. \square

2.6 Convergence of the sequence $(\rho_n, u_n)_{n \in \mathbb{N}}$ to a global weak solution

We have for all $n \in \mathbb{N}$

$$\begin{aligned} & \int_{\mathbb{R}^2} (\rho_n \varphi)(0, \cdot) dx + \int_{\mathbb{R}_+ \times \mathbb{R}^2} \rho_n \partial_t \varphi dt dx \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^2} (\rho_n P u_n + \sqrt{\rho_n} (\sqrt{\rho_n} Q u_n)) \cdot \nabla \varphi dt dx = 0, \end{aligned} \quad (2.129)$$

$$\begin{aligned} & \int_{\mathbb{R}^2} ((\rho_n P u_n + \sqrt{\rho_n} (\sqrt{\rho_n} Q u_n) \varphi)(0, \cdot) dx + \int_{\mathbb{R}_+ \times \mathbb{R}^2} (\rho_n P u_n + \sqrt{\rho_n} (\sqrt{\rho_n} Q u_n)) \partial_t \varphi dx dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^2} (\sqrt{\rho_n} Q u_n \otimes \sqrt{\rho_n} Q u_n + 2\sqrt{\rho_n} P u_n \otimes \sqrt{\rho_n} Q u_n + \rho_n P u_n \otimes P u_n) : \nabla \varphi dx dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^2} P(\rho_n) \operatorname{div}(\varphi) dx dt = -\frac{\kappa}{2} \int_{\mathbb{R}_+ \times \mathbb{R}^2} \nabla \rho_n \cdot \nabla(\operatorname{div}(\varphi)) dx dt \\ & - 2\kappa \int_{\mathbb{R}_+ \times \mathbb{R}^2} (\nabla \sqrt{\rho_n} \otimes \nabla \sqrt{\rho_n}) : \nabla \varphi dx dt. \end{aligned} \quad (2.130)$$

Then passing to the limit in n we have

$$-\int_{\mathbb{R}^2} (\rho^0 \varphi^0)(\cdot) dx + \int_{\mathbb{R}_+ \times \mathbb{R}^2} \rho \partial_t \varphi dt dx + \int_{\mathbb{R}_+ \times \mathbb{R}^2} (\rho \Lambda_1 + \sqrt{\rho} \Lambda) \cdot \nabla \varphi dt dx = 0, \quad (2.131)$$

and

$$\begin{aligned} & \int_{\mathbb{R}^2} ((\rho^0 P u^0 + \rho^0 Q u^0) \varphi^0)(\cdot) dx + \int_{\mathbb{R}_+ \times \mathbb{R}^2} (\rho \Lambda_1 + \sqrt{\rho} \Lambda) \partial_t \varphi \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^2} (\Lambda \otimes \Lambda + 2\sqrt{\rho} P u \otimes \Lambda + \rho \Lambda_1 \otimes \Lambda_1) : \nabla \varphi dx dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^2} P(\rho) \operatorname{div}(\varphi) dx dt = -\frac{\kappa}{2} \int_{\mathbb{R}_+ \times \mathbb{R}^2} \nabla \rho \cdot \nabla(\operatorname{div}(\varphi)) dx dt \\ & - 2\kappa \int_{\mathbb{R}_+ \times \mathbb{R}^2} (\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}) : \nabla \varphi dx dt. \end{aligned} \quad (2.132)$$

Moreover we have using (1.16)

$$\begin{aligned} & \int_{\mathbb{R}^2} \left(\frac{1}{2} \rho_n^0 |u_n^0|^2(x) + \kappa |\nabla \sqrt{\rho_n^0}(x)|^2 + j_\gamma(\rho_n^0(x)) \right) dx \\ & \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^2} \left(\frac{1}{2} \rho^0 |u^0|^2(x) + \kappa |\nabla \sqrt{\rho^0}(x)|^2 + j_\gamma(\rho^0(x)) \right) dx. \end{aligned}$$

Using the convergences almost everywhere obtained previously, we have

$$\rho_n |u_n|^2 + \kappa |\nabla \sqrt{\rho_n}|^2 + j_\gamma(\rho_n) \xrightarrow{n \rightarrow \infty} \rho |\Lambda_1|^2 + |\Lambda|^2 + \kappa |\nabla \sqrt{\rho}(x)|^2 + j_\gamma(\rho(x)) \quad a.e.$$

Thus passing to the limit inf in (2.11) using (2.125), and using Fatou's lemma we have

$$E(t) \leq E(0). \quad (2.133)$$

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