Critical points for least-squares estimation of dipolar sources in inverse problems for Poisson equation

Paul Asensio, Juliette Leblond

Critical points for least-squares estimation of dipolar sources in inverse problems for Poisson equation

Paul Asensio[1*] and Juliette Leblond[1*]

[1]Team FACTAS, Inria, 2004 Route des Lucioles, B.P. 93, Sophia Antipolis, 06902, France.

*Corresponding author(s). E-mail(s): paul.asensio@inria.fr; juliette.leblond@inria.fr;

Abstract

In this work, we study some aspects of the solvability of the minimization of a non-convex least-squares criterion involved in dipolar source recovery issues, using boundary values of a solution to a Poisson problem in a domain of dimension 3. This Poisson problem arises in particular from the quasi-static approximation of Maxwell equations with localized sources modeled as dipoles. We establish the uniqueness of the minimizer of the criterion for general geometries and the uniqueness of its critical point for the Euclidean geometry, that is when the boundary is a plane. This has consequences on the numerical approach, for the convergence of the computed solution to the global minimizer. Related inverse potential problems have applications in bio medical imaging issues pertaining to neurosciences, and in paleomagnetism issues pertaining to geosciences. There, solutions to such inverse problems are used to recover electric currents in the brain, or rock magnetizations, from measurements of the induced electric potential or magnetic field.

Keywords: Inverse problems, Poisson PDE, dipolar sources, optimization, critical points.

1 Introduction

1.1 General framework

In this work, we study the inverse problem of recovering vector-valued pointwise sources in a domain $\Omega \subset \mathbb{R}^n$ from measurement of the emitted potential on its boundary $\partial\Omega$. In particular, we investigate the uniqueness of the critical points of the least-squares criterion linked to this source recovery inverse problem. As the criterion is not convex, uniqueness of local minimizer is needed to ensure the convergence of classical optimization schemes to a global minimizer.

Such inverse problems have applications in medical imaging (electroencephalography (EEG)) as in [1] and in geosciences (rock magnetization) as in [2]. In these applications, the sources are either electric currents due to brain activity or magnetizations due to remanent rock magnetization. In both cases, the model comes from Maxwell equations in the quasi-static approximation [3, Sec. 5.3]. This approximation leads to a Poisson equation with a source term in divergence form:

$$\begin{cases} \Delta u = \nabla \cdot \mu \text{ in } \mathbb{R}^n, \\ \lim\limits_{|z|\to\infty} |u(z)| = 0. \end{cases} \tag{1}$$

for some measure $\mu \in [\mathcal{M}(\mathbb{R}^n)]^n$ with compact support included in $\Omega \subset \mathbb{R}^n$ a smooth domain and a solution $u \in \mathcal{D}'(\mathbb{R}^n)$.

In this work, we study the setting where Ω is the lower half-space and $\partial\Omega$ the horizontal plane. This configuration is more adapted to the geometry a Scanning Magnetic Microscopy used in particular paleontology studies about rock magnetization where we typically have data on a horizontal plane above the rock. For EEG, the measurements are considered on the scalp of a patient usually modeled as a sphere.

1.2 Main Problem

This inverse problem of source recovery is strongly ill-posed. A classical possible regularization is to restrict the space of sources to finite sums of a fixed number $n_d \geq 1$ of dipoles:

$$\mathcal{S}_{n_d} = \left\{ \sum_{i=1}^{n_d} p_i \delta_{x^i}, x^i \in \Omega, \, p_i \in \mathbb{R}^n, \forall i \in \{1, ..., n_d\} \right\},$$

where a dipole $p\delta_x$ is parameterized by its location $x \in \Omega$ and its moment $p \in \mathbb{R}^3$. The measure δ_x is the Dirac delta distribution supported at $x \in \Omega$ and $n_d \in \mathbb{N}$ plays the role of the regularization parameter. This model is well adapted for localized dipolar sources such as epileptic foci in EEG. This regularization is enough to guarantee the uniqueness of the solution to the inverse problem [4, Thm 1] when dealing with both Dirichlet and Neumann data. Nevertheless, because of the non-convexity of the least-squares criterion, still exists the question of the multiple local minima of the objective function hence of the numerical solvability of the problem even when the data are

noiseless and the original source term μ actually belongs to the approximation space $\mathcal{S}_{n_d}$.

In this work, we deal with the dimension 3 ($n = 3$), with a single dipole $\mu = p_0\delta_{x_0}$ ($n_d = 1$) where $p_0 \in \mathbb{R}^3$ and $x_0 \in \Omega$. In this case, (1) becomes:

$$\begin{cases} \Delta u_0 = 4\pi\nabla \cdot (p_0\delta_{x_0}) \text{ in } \mathbb{R}^3, \\ \lim\limits_{|z|\to\infty} |u_0(z)| = 0. \end{cases} \tag{2}$$

As in (1), the partial differential equation in (2) is to be understood in the distributional sense, with the Dirac delta distribution δ_{x_0} supported at $x_0 \in \Omega$ and the divergence $\nabla \cdot (p_0\delta_{x_0})$ being defined by:

$$\begin{aligned} \langle \delta_{x_0}, \varphi \rangle &= \varphi(x_0), \\ \langle \nabla \cdot (p_0\delta_{x_0}), \varphi \rangle &= -p_0 \cdot \nabla\varphi(x_0), \end{aligned}$$

for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^3)$, the set of infinitely differentiable functions with compact support in $\mathbb{R}^3$. The solution to (2) in the dual space $\mathcal{D}'(\mathbb{R}^3)$ of $\mathcal{D}(\mathbb{R}^3)$ is unique as the difference of two solutions would be a bounded harmonic function in $\mathbb{R}^3$ hence zero according to Liouville's Theorem (see [5, Thm 2.1]). This solution is given by:

$$u_0(z) = \frac{p_0 \cdot (x_0 - z)}{|x_0 - z|^3}, \ \forall z \in \mathbb{R}^3 \setminus \{x_0\}.$$

Let from now on $\Omega \subset \mathbb{R}^3$ be an open connected domain such that $\Omega^c = \mathbb{R}^3 \setminus \Omega$ contains a non-empty open set and $\partial\Omega$ be its boundary assumed to be Lipschitz. We will make use of the notation $x = (x_1, x_2, x_3)^T$ for all vector $x \in \mathbb{R}^3$ and $0_{\mathbb{R}^3} = (0, 0, 0)^T$ for the null vector of $\mathbb{R}^3$. We are interested in retrieving the location $x_0 \in \Omega$ and the moment $p_0 \in \mathbb{R}^3$ using measurements (assumed to be noiseless) of the solution u_0 on the boundary $\partial\Omega$ by minimizing the least-square criterion $J_{\partial\Omega}$:

$$\begin{aligned} J_{\partial\Omega} : \Omega \times \mathbb{R}^3 &\longrightarrow \mathbb{R}_+ \\ (x, p) &\longmapsto \int_{\partial\Omega} \left(\frac{p \cdot (x - z)}{|x - z|^3} - \frac{p_0 \cdot (x_0 - z)}{|x_0 - z|^3} \right)^2 \mathrm{d}\sigma(z), \end{aligned} \tag{3}$$

for all $(x_0, p_0) \in \Omega \times \mathbb{R}^3$ and where σ is the Lebesgue measure on $\partial\Omega$.

To ascertain that we actually find p_0 and x_0 when numerically minimizing (3), we prove that the least-squares criterion has only one critical point $(x_c, p_c) = (x_0, p_0)$ when Ω is the lower half-space in $\mathbb{R}^3$. Analogous results were given by [6] for similar problems in two-dimensional domains.

1.3 Overview

In Section 2, we state our main results. Then, Theorem 2 is proven in Section 3. In Section 4, we provide numerical illustrations. We finally provide some concluding remarks and further possible developments of the present work in Section 5.

2 Main results

A preliminary result is given by Proposition 1 for general geometric situations.

Proposition 1 *For each $(x_0, p_0) \in \Omega \times \mathbb{R}^3$ with $p_0 \neq 0_{\mathbb{R}^3}$, the criterion $J_{\partial\Omega}$ admits a unique global minimizer $(x^*, p^*) \in \Omega \times \mathbb{R}^3$. It coincides with the original source: $x^* = x_0$ and $p^* = p_0$.*

If $p_0 = 0_{\mathbb{R}^3}$, the minimizers of $J_{\partial\Omega}$ are all the $(x, 0_{\mathbb{R}^3})$ with $x \in \Omega$.

Proof As $J_{\partial\Omega}$ is non-negative and $J_{\partial\Omega}(x_0, p_0) = 0$, $J_{\partial\Omega}$ admits 0 as a global minimum. Let $(x^*, p^*) \in \Omega \times \mathbb{R}^3$ be a global minimizer of $J_{\partial\Omega}$, we have $J_{\partial\Omega}(x^*, p^*) = 0$. Let us define on Ω^c the difference:

$$\forall z \in \Omega^c,\ h(z) = \frac{p^* \cdot (x^* - z)}{|x^* - z|^3} - \frac{p_0 \cdot (x_0 - z)}{|x_0 - z|^3}.$$

such that $\int_{\partial\Omega} h^2 \mathrm{d}\sigma = J_{\partial\Omega}(x^*, p^*) = 0$. Clearly, h is a continuous bounded function in Ω^c, is harmonic in $\Omega^c \setminus \partial\Omega$ and, if Ω^c is unbounded:

$$h(z) \underset{|z|\to\infty}{\to} 0.$$

As h is 0 on $\partial\Omega$ according to the strict positivity of the integral, h is 0 on Ω^c according to [7, Vol. 1, Chap. II, Par. 4, Prop. 1 and 9]. Hence, we have:

$$\forall z \in \Omega^c,\ Q(z) = \big(p^* \cdot \big(x^* - z\big)\big)^2 |x_0 - z|^6 - (p_0 \cdot (x_0 - z))^2 |x^* - z|^6 = 0$$

As Q is a polynomial that is null on Ω^c which contains a non-empty open set in $\mathbb{R}^3$, it is null in $\mathbb{R}^3$. However, as $|z|$ goes to infinity, we have the following asymptotic expansion for Q:

$$\begin{aligned}\frac{Q(z)}{|z|^6} &\underset{|z|\to\infty}{=} -\Big[(p_0 \cdot z)^2 - (p^* \cdot z)^2\Big] \\ &+ 2\left[(p_0 \cdot x_0)(p_0 \cdot z) - (p^* \cdot x^*)(p^* \cdot z) + 3(x^* \cdot z)\frac{(p_0 \cdot z)^2}{|z|^2} - 3(x_0 \cdot z)\frac{(p^* \cdot z)^2}{|z|^2}\right] \\ &+ O(1).\end{aligned}$$

By identifying the first coefficient with 0 we see that $p^* = \pm p_0$. Then if $p_0 \neq 0_{\mathbb{R}^3}$, by identifying the second term with 0, we sequentially find that $p_0 \cdot x^* = p_0 \cdot x_0$ by looking in the direction p_0 and finally $x^* = x_0$. Furthermore, because h is null on $\partial\Omega$, we have $p^* = p_0$. Hence, if $p_0 \neq 0_{\mathbb{R}^3}$, there is a unique global minimizer to $J_{\partial\Omega}$: $(x^*, p^*) = (x_0, p_0)$. If $p_0 = 0_{\mathbb{R}^3}$, then for all $x \in \Omega$, $(x, 0_{\mathbb{R}^3})$ is a global minimizer of $J_{\partial\Omega}$. □

Remark 1 This proof could have been done in any dimension $n \geq 3$ with minimal changes. Hence, the Proposition 1 remains true for all $n \geq 3$.

The main result of this work is given by Theorem 2 below related to the Euclidean geometry. Let $\Omega = \mathbb{R}^3_-$ be the lower half-space, $\mathbb{R}^3_+$ be the upper half-space and $\partial\Omega = \Pi$ be the horizontal plane of height $z = 0$ in $\mathbb{R}^3$. Let $x_0 = (x_{01}, x_{02}, x_{03})^T$ be a point in $\mathbb{R}^3_-$ and p_0 a vector in $\mathbb{R}^3$ (the situation is summarized on Figure 1). We are interested in estimating x_0 and p_0 using measurements of u_0 on Π by solving the least-squares inverse problem:

$$(x^*, p^*) = \underset{(x,p)\in\mathbb{R}^3_-\times\mathbb{R}^3}{\arg\min} J_\Pi(x, p),$$

with the criterion J_Π being given by:

$$\begin{aligned} J_\Pi : \mathbb{R}^3_- \times \mathbb{R}^3 &\longrightarrow \mathbb{R}_+ \\ (x, p) &\longmapsto \int_\Pi \left(\frac{p \cdot (x - z)}{|x - z|^3} - \frac{p_0 \cdot (x_0 - z)}{|x_0 - z|^3} \right)^2 \mathrm{d}\sigma(z), \end{aligned} \tag{4}$$

where σ is the Lebesgue measure on Π: $\mathrm{d}\sigma(z) = \mathrm{d}z_1\mathrm{d}z_2$, $z = (z_1, z_2, 0) \in \Pi$.

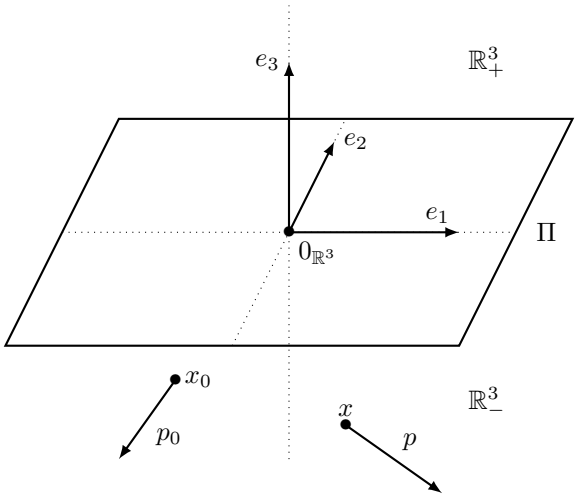

Fig. 1 Schematic geometry of the study in the half-space

Our main result is the following:

Theorem 2 *For each $(x_0, p_0) \in \mathbb{R}^3_- \times \mathbb{R}^3$ with $p_0 \neq 0_{\mathbb{R}^3}$, the criterion J_Π admits a unique critical point $(x_c, p_c) \in \mathbb{R}^3_- \times \mathbb{R}^3$ such that $\nabla_x J_\Pi(x_c, p_c) = \nabla_p J_\Pi(x_c, p_c) = 0_{\mathbb{R}^3}$. It coincides with the global minimizer: $x_c = x_0$ and $p_c = p_0$.*
If $p_0 = 0_{\mathbb{R}^3}$, the critical points are all the $(x, 0_{\mathbb{R}^3})$ with $x \in \mathbb{R}^3_-$.

This result ensures that classical optimization schemes will converge to the global minimizer, hence to the original source (and not to a local minimizer with no physical meaning).
Section 3 is devoted to the proof of Theorem 2.

3 Proof

Let us fix $(x_0, p_0) \in \mathbb{R}^3_- \times \mathbb{R}^3$. In this section, we want to compute the critical points of the criterion J_Π. To do so, in Section 3.1, we eliminate p in order to reduce the dimension of the problem which can be done linearly. In Section 3.2, we express (4) by computing certain integrals. In Section 3.3, we characterize (x_0, p_0) as being the unique critical point of (4) by first proving that any critical point $(x_c, p_c) \in \mathbb{R}^3_- \times \mathbb{R}^3$ is such that x_c and x_0 belong to the same horizontal plane, using the homogeneity of the criterion; then, we appropriately combine its derivatives to show uniqueness of a critical point with positivity arguments.

3.1 Gradient in p

By developing the square in (4), we have, for all $(x, p) \in \mathbb{R}^3_- \times \mathbb{R}^3$:

$$J_\Pi(x, p) = p^T M(x, x)p + p_0^T M(x_0, x_0)p_0 - 2p_0^T M(x, x_0)p, \tag{5}$$

where the 3×3 matrix $M(x, x_0)$ is defined by:

$$M(x, x_0) = \int_\Pi \left(\frac{x_0 - z}{|x_0 - z|^3} \right) \left(\frac{x - z}{|x - z|^3} \right)^T \mathrm{d}\sigma(z).$$

Therefore, the gradient of J_Π with respect to p is given by:

$$\nabla_p J_\Pi(x, p) = 2M(x, x)p - 2M(x, x_0)^T p_0.$$

One can see that $M(x, x)$ is symmetric and positive definite for each $x \in \mathbb{R}^3_-$, as for all $v \in \mathbb{R}^3$, $v \neq 0_{\mathbb{R}^3}$:

$$v^T M(x, x)v = \int_\Pi \left(\frac{v \cdot (x - z)}{|x - z|^3} \right)^2 \mathrm{d}\sigma(z) > 0,$$

which leads to $M(x, x)$ being invertible. Therefore, if $(x_c, p_c) \in \mathbb{R}^3_- \times \mathbb{R}^3$ is a critical point then necessarily:

$$p_c = \mathrm{p}(x_c) = M(x_c, x_c)^{-1} M(x_c, x_0)^T p_0. \tag{6}$$

One can see from (6) that in this case:

$$p_c^T M(x_c, x_c)p_c - p_0^T M(x_c, x_0)p_c = 0. \tag{7}$$

Furthermore, if $p_0 = 0_{\mathbb{R}^3}$, then the critical points of J_Π consist in the pairs $(x, 0_{\mathbb{R}^3})$ for any $x \in \mathbb{R}^3_-$. Indeed, for all $(p_c, x_c) \in \mathbb{R}^3_- \times \mathbb{R}^3$ critical point of J_Π, (7) leads to $p_c = 0_{\mathbb{R}^3}$ and as $J_\Pi(\cdot, 0_{\mathbb{R}^3})$ identically vanishes in this case, for all $x \in \mathbb{R}^3_-$, $(x, 0_{\mathbb{R}^3})$ is a critical point of J_Π. So, we will suppose in the following that:

$$p_0 \neq 0_{\mathbb{R}^3}. \tag{8}$$

3.2 Computation of the criterion

Our goal in this section is to compute the matrices $M(x, x)$ and $M(x, x_0)$ whose elements are defined for x and x_0 in $\mathbb{R}^3_-$ by:

$$M(x, x_0)_{i,j} = \int_\Pi \frac{x_j - z_j}{|x - z|^3} \frac{x_{0i} - z_i}{|x_0 - z|^3} \mathrm{d}\sigma(z), \tag{9}$$

for all $(i, j) \in \{1, 2, 3\}^2$. In order not to carry heavy notation, we will drop the dependency in x and x_0 of $M = M(x, x_0)$ and all the related derived quantities at least when there is no ambiguity.

3.2.1 Some terms computed using Clifford analysis

In order to compute the integrals in (9), we make use of Clifford analytic calculus. For an introduction to Clifford algebras, see [8]. Let $\mathrm{C}\ell_{0,3}(\mathbb{R})$ be the unital associative algebra generated on $\mathbb{R}$ by (e_1, e_2, e_3) with the relations:

$$\forall (i, j) \in \{1, 2, 3\}^2,\ e_i \odot e_j = \begin{cases} -\mathtt{1} & i = j, \\ -e_j \odot e_i & i \neq j, \end{cases}$$

where we denote by $\odot$ the product in this algebra and $\mathtt{1}$ is the multiplicative identity of the algebra. $\mathrm{C}\ell_{0,3}(\mathbb{R})$ is a 8-dimensional $\mathbb{R}$-vector space with the canonical base: $(\mathtt{1},\ e_1,\ e_2,\ e_3,\ e_1 \odot e_2,\ e_2 \odot e_3, e_3 \odot e_1,\ e_1 \odot e_2 \odot e_3)$. Elements of $\mathrm{span}(e_1, e_2, e_3)$ are called vectors and are identified with their $\mathbb{R}^3$ counterpart. The set $\mathrm{C}\ell_{0,3}(\mathbb{R})$ is a normed vector space when endowed with the Euclidean norm $|\cdot|$.

A function $f \in L^1_{loc}(\Omega, \mathrm{C}\ell_{0,3}(\mathbb{R}))$ is said to be left Clifford analytic if $D \odot f = 0$ where $D = \sum_{j=1}^3 e_j \frac{\partial}{\partial x_j}$ is the Dirac operator. Such functions actually belong to $C^\infty(\Omega, \mathrm{C}\ell_{0,3}(\mathbb{R}))$ and satisfy a Cauchy formula (see [9, Thm 1.8]). It states that for all V bounded, 3-dimensional, Lipschitz open domain in Ω, all $f \in L^1_{loc}(V, \mathrm{C}\ell_{0,3}(\mathbb{R}))$ left Clifford analytic and all $x \in \mathbb{R}^3$:

$$\frac{1}{4\pi} \int_{\partial V} \frac{x - z}{|x - z|^3} \odot \nu(z) \odot f(z) \mathrm{d}\sigma(z) = \begin{cases} f(x) & x \in V, \\ 0 & x \notin \overline{V}, \end{cases} \tag{10}$$

where $\nu(z)$ is the unitary outward vector normal to ∂V at z. This formula can be extended to unbounded domains for specific classes of functions as in Lemma 1 which is proven in Appendix A.

Lemma 1 For all $x \in \mathbb{R}^3$ and all $f \in L^1_{loc}(\mathbb{R}^3_-, \mathrm{C}\ell_{0,3}(\mathbb{R}))$ left Clifford analytic such that there exists $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ which converge to 0 at $+\infty$ and:

$$|f(z)| \leq \varphi(|z|),$$

the Cauchy formula (10) holds true on $\mathbb{R}^3_-$:

$$\frac{1}{4\pi}\int_\Pi \frac{x-z}{|x-z|^3} \odot e_3 \odot f(z)\mathrm{d}\sigma(z) = \begin{cases} f(x) & x \in \mathbb{R}^3_-, \\ 0 & x \in \mathbb{R}^3_+. \end{cases}$$

Vector-valued left Clifford analytic functions are actually gradient of harmonic functions. Hence, the prototype of such a function is, for all $x \in \mathbb{R}^3$:

$$\forall z \in \mathbb{R}^3 \setminus \{x\},\ f_x(z) = \frac{x-z}{|x-z|^3},$$

which is left Clifford analytic in $\mathbb{R}^3\backslash\{x\}$ [9, Eq. (1.9)] and satisfy the hypotheses of Lemma 1.

Let us call M_1, M_2 and M_3 the columns of $M = M(x, x_0)$ and identify them with their corresponding vectors in $\mathrm{C}\ell_{0,3}(\mathbb{R})$:

$$\begin{aligned} M_1 &= M_{1,1}\, e_1 + M_{2,1}\, e_2 + M_{3,1}\, e_3, \\ M_2 &= M_{1,2}\, e_1 + M_{2,2}\, e_2 + M_{3,2}\, e_3, \\ M_3 &= M_{1,3}\, e_1 + M_{2,3}\, e_2 + M_{3,3}\, e_3. \end{aligned}$$

Upon defining $x^+ \in \mathbb{R}^3_+$ as the symmetric of x with respect to Π and identifying x, x^+, x_0 and all $z \in \Pi$ with their corresponding vectors in $\mathrm{C}\ell_{0,3}(\mathbb{R})$, we have:

$$\begin{aligned} M_1 \odot e_1 + M_2 \odot e_2 + M_3 \odot e_3 = & \int_\Pi \frac{x_0 - z}{|x_0 - z|^3} \odot \frac{x-z}{|x-z|^3}\mathrm{d}\sigma(z) \\ = & -\int_\Pi \frac{x_0 - z}{|x_0 - z|^3} \odot e_3 \odot e_3 \odot \frac{x-z}{|x-z|^3}\mathrm{d}\sigma(z) \\ = & \int_\Pi \frac{x_0 - z}{|x_0 - z|^3} \odot e_3 \odot \frac{x^+ - z}{|x^+ - z|^3}\mathrm{d}\sigma(z) \odot e_3 \\ = & \ 4\pi \frac{x^+ - x_0}{|x^+ - x_0|^3} \odot e_3, \end{aligned} \tag{11}$$

where we used the relations:

$$\begin{aligned} e_3 \odot e_3 &= -\mathbf{1}, \\ e_3 \odot (x - z) &= -(x^+ - z) \odot e_3, \end{aligned}$$

and Lemma 1 as $z \mapsto \frac{x^+ - z}{|x^+ - z|^3}$ is left Clifford analytic in $\mathbb{R}^3_-$ since $x^+ \in \mathbb{R}^3_+$. Applying Lemma 1 again, we also have:

$$\begin{aligned}(-M_1 \odot e_1 - M_2 \odot e_2 + M_3 \odot e_3) \odot e_3 &= \int_\Pi \frac{x_0 - z}{|x_0 - z|^3} \odot e_3 \odot \frac{x - z}{|x - z|^3} \mathrm{d}\sigma(z) \\ &= 0, \end{aligned} \tag{12}$$

because $z \mapsto \frac{x-z}{|x-z|^3}$ is left Clifford analytic in $\mathbb{R}^3_+$ since $x \in \mathbb{R}^3_-$. Using these equations, we get the following relations between terms:

$$M_{1,1} + M_{2,2} = \quad M_{3,3} = -2\pi \frac{x_3 + x_{03}}{|x^+ - x_0|^3} = -2\pi \frac{x_{a3}}{|x_a|^3}, \tag{13}$$

$$M_{3,1} = -M_{1,3} = -2\pi \frac{x_1 - x_{01}}{|x^+ - x_0|^3} = \quad 2\pi \frac{x_{a1}}{|x_a|^3}, \tag{14}$$

$$M_{3,2} = -M_{2,3} = -2\pi \frac{x_2 - x_{02}}{|x^+ - x_0|^3} = \quad 2\pi \frac{x_{a2}}{|x_a|^3}, \tag{15}$$

$$M_{2,1} = \quad M_{1,2}, \tag{16}$$

where we defined $x_a \in \mathbb{R}^3_-$ by $x_a = x_0 - x^+$. Indeed, equation (13) comes from the $\mathbb{1}$ coordinate of (11) and the e_3 coordinate of (12). Equation (14) comes from the $e_3 \odot e_1$ coordinate of (11) and the e_1 coordinate of (12). Equation (15) comes from the $e_2 \odot e_3$ coordinate of (11) and the e_2 coordinate of (12). Equation (16) comes from the $e_1 \odot e_2$ coordinate of (11) or the $e_1 \odot e_2 \odot e_3$ coordinate of (12).

3.2.2 Computation of the remaining terms

We want to compute:

$$\begin{aligned} M_{1,1} &= \int_\Pi \frac{x_1 - z_1}{|x - z|^3} \frac{x_{01} - z_1}{|x_0 - z|^3} \mathrm{d}\sigma(z), \\ M_{2,2} &= \int_\Pi \frac{x_2 - z_2}{|x - z|^3} \frac{x_{02} - z_2}{|x_0 - z|^3} \mathrm{d}\sigma(z), \\ M_{2,1} = M_{1,2} &= \int_\Pi \frac{x_1 - z_1}{|x - z|^3} \frac{x_{02} - z_2}{|x_0 - z|^3} \mathrm{d}\sigma(z). \end{aligned}$$

To do so, we use the previously computed quantity $M_{3,3}$ in (13) and define:

$$\begin{aligned} m_{3,3}(x_{a1}, x_{a2}, x_{a3}) &= M_{3,3}(x, x_0) \\ &= \int_\Pi \frac{x_3}{|x - z|^3} \frac{x_{03}}{|x_0 - z|^3} \mathrm{d}\sigma(z) \\ &= -2\pi \frac{x_{a3}}{[x_{a1}^2 + x_{a2}^2 + x_{a3}^2]^{3/2}}. \end{aligned}$$

By integration and differentiation, we can transform the integrand of $M_{3,3}$ into those of $M_{1,1}$, $M_{2,2}$, $M_{2,1}$ and $M_{1,2}$. Indeed it is easily checked that for $i \in \{1,2\}$ and all $z \in \Pi$:

$$\frac{x_i - z_i}{|x-z|^3} = \int_{-\infty}^{x_3} \partial_{x_i} \frac{y}{\left[(x_1-z_1)^2 + (x_2-z_2)^2 + y^2\right]^{3/2}} \mathrm{d}y,$$
$$\frac{x_{0i} - z_i}{|x_0-z|^3} = \int_{-\infty}^{x_{03}} \partial_{x_{0i}} \frac{y}{\left[(x_{01}-z_1)^2 + (x_{02}-z_2)^2 + y^2\right]^{3/2}} \mathrm{d}y.$$

So, by applying Fubini's theorem then differentiating under the integral sign, we get for $i \in \{1,2\}$:

$$\int_{-\infty}^{x_{03}} \int_{-\infty}^{x_3} \partial_{x_i}\partial_{x_{0i}} m_{3,3}(x_{a1}, x_{a2}, y_1+y_2)\mathrm{d}y_1\mathrm{d}y_2 = M_{i,i},$$
$$\int_{-\infty}^{x_{03}} \int_{-\infty}^{x_3} \partial_{x_1}\partial_{x_{02}} m_{3,3}(x_{a1}, x_{a2}, y_1+y_2)\mathrm{d}y_1\mathrm{d}y_2 = M_{2,1} = M_{1,2}.$$

Since $m_{3,3}$ is a function of $x_{a1} = x_{01} - x_1$, $x_{a2} = x_{02} - x_2$ and $x_{a3} = x_{03} + x_3$, we can simplify the differentiations and for $i, j \in \{1,2\}$:

$$\partial_{x_i}\partial_{x_{0j}} m_{3,3} = -\partial_{x_{ai}}\partial_{x_{aj}} m_{3,3}.$$

Then, using known antiderivatives, we compute $M_{1,1}$, $M_{2,2}$, $M_{2,1}$ and $M_{1,2}$, for all $i, j \in \{1,2\}$:

$$\begin{aligned}
M_{i,j} &= 2\pi \int_{y_1=-\infty}^{x_3} \partial_{x_{ai}}\partial_{x_{aj}} \int_{y_2=-\infty}^{x_{03}} \frac{y_1+y_2}{\left[x_{a1}^2 + x_{a2}^2 + (y_1+y_2)^2\right]^{3/2}} \mathrm{d}y_1\mathrm{d}y_2 \\
&= -2\pi \int_{-\infty}^{x_{a3}} \partial_{x_{ai}}\partial_{x_{aj}} \frac{1}{\left[x_{a1}^2 + x_{a2}^2 + y^2\right]^{1/2}} \mathrm{d}y \\
&= -2\pi \int_{-\infty}^{x_{a3}} \left[-\frac{\delta_{ij}}{\left[x_{a1}^2 + x_{a2}^2 + y^2\right]^{3/2}} + 3\frac{x_{ai}x_{aj}}{\left[x_{a1}^2 + x_{a2}^2 + y^2\right]^{5/2}} \right] \mathrm{d}y \\
&= \frac{2\pi}{(x_{a1}^2 + x_{a2}^2)} \left[\delta_{ij} \frac{|x_a| + x_{a3}}{|x_a|} - x_{ai}x_{aj} \frac{2|x_a|^3 + 3x_{a3}|x_a|^2 - x_{a3}^3}{|x_a|^3 (x_{a1}^2 + x_{a2}^2)} \right]. \quad (17)
\end{aligned}$$

where δ_{ij} is the Kronecker delta symbol.

Remark 2 We can see that all the expressions $M_{i,j}$ for $(i,j) \in \{1,2,3\}^2$ are homogeneous of degree -2 in all their variables x_1, x_{01}, x_2, x_{02}, x_3 and x_{03}.

By taking the limits of the previously computed terms as x_0 goes to x, we find for that all $x \in \mathbb{R}^3_-$:

$$M(x,x) = \frac{\pi}{4x_3^2}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}.$$

One can see that $M(x,x_0)$ is also invertible for $x \neq x_0$ as for $(x_{a1}, x_{a2}) \neq (0,0)$:

$$\det(M(x,x_0)) = \frac{8\pi^3}{\left(x_{a1}^2 + x_{a2}^2\right)^2 |x_a|^4}\left(|x_a| + x_{a3}\right)^2 > 0.$$

3.3 Computation of the critical points

3.3.1 Gradient in x

For all $x \in \mathbb{R}^3_-$, let $p = \mathrm{p}(x)$ be defined as in (6) and $I_\Pi(x) = J_\Pi(x, \mathrm{p}(x))$. Then, by definition of $\mathrm{p}(x)$, the critical points of J_Π are all the $(x_c, \mathrm{p}(x_c))$ where $x_c \in \mathbb{R}^3_-$ is a critical point of I_Π. For all $x \in \mathbb{R}^3_-$, in view of (5) and (6) and because of (7), we have:

$$\begin{aligned} I_\Pi(x) &= \mathrm{p}(x)^T M(x,x)\mathrm{p}(x) + p_0^T M(x_0,x_0)p_0 - 2p_0^T M(x,x_0)\mathrm{p}(x),\\ &= -p_0^T M(x,x_0)\mathrm{p}(x) + p_0^T M(x_0,x_0)p_0, \end{aligned}$$

So, computing the gradient of I_Π, we get for all $x \in \mathbb{R}^3_-$:

$$\begin{aligned} \nabla I_\Pi(x) &= -\nabla\left(p_0^T M(x,x_0)\mathrm{p}(x)\right)\\ &= -\nabla\left(p_0^T M(x,x_0)M(x,x)^{-1}M(x,x_0)^T p_0\right), \end{aligned}$$

in view of (6). Therefore, upon defining the matrix:

$$K = K(x,x_0) = M(x,x_0)M(x,x)^{-1}M(x,x_0)^T,$$

we obtain that $(x_c, \mathrm{p}(x_c)) \in \mathbb{R}^3_- \times \mathbb{R}^3$ is a critical point of J_Π if and only if, for all $i \in \{1,2,3\}$:

$$p_0^T\left(\frac{\partial K}{\partial x_i}(x,x_0)\bigg|_{x=x_c}\right)p_0 = 0. \tag{18}$$

3.3.2 First step: use of homogeneity

One can see that for all $x \in \mathbb{R}^3_-$, $K(x,x_0)$ is symmetric positive definite as:

$$K = \left(\left(M(x,x)^{-1}\right)^{1/2} M(x,x_0)^T\right)^T\left(\left(M(x,x)^{-1}\right)^{1/2} M(x,x_0)^T\right).$$

Furthermore, since $M(x,x_0)$ is homogeneous of degree -2 (see Remark 2), $K(x,x_0)$ is also homogeneous of degree -2 in its six variables $(x_1, x_2, x_3, x_{01},$

x_{02} and x_{03}). We can therefore apply Euler's homogeneous function theorem [7, Vol. 2, App. Eq. (3.75)] and get:

$$x_1\frac{\partial K}{\partial x_1}+x_{01}\frac{\partial K}{\partial x_{01}}+x_2\frac{\partial K}{\partial x_2}+x_{02}\frac{\partial K}{\partial x_{02}}+x_3\frac{\partial K}{\partial x_3}+x_{03}\frac{\partial K}{\partial x_{03}}=-2K.$$

But, since the following relations hold true:

$$\frac{\partial K}{\partial x_{01}}=-\frac{\partial K}{\partial x_1},\quad \frac{\partial K}{\partial x_{02}}=-\frac{\partial K}{\partial x_2},\quad \frac{\partial K}{\partial x_{03}}=\frac{\partial K}{\partial x_3}-\frac{2}{x_3}K,$$

we obtain the following equation:

$$-x_3x_{a1}\frac{\partial K}{\partial x_1}-x_3x_{a2}\frac{\partial K}{\partial x_2}+x_3x_{a3}\frac{\partial K}{\partial x_3}=2(x_{03}-x_3)K.$$

Thus, for $x_c\in\mathbb{R}^3_-$ a critical point of I_Π and because of (18), $p_0^T(x_{03}-x_{c3})K(x_c,x_0)p_0=0$. But, for $x_{03}\neq x_{c3}$, $(x_{03}-x_{c3})K(x_c,x_0)$ is definite which means that $p_0=0_{\mathbb{R}^3}$ contradicting hypothesis (8). Thus, $x_c\in\mathbb{R}^3_-$ is a critical point of I_Π only if $x_{c3}=x_{03}$.

3.3.3 Second step: use of strict positivity

Let $x\in\mathbb{R}^3_-$. Let us use polar coordinates in the plane: $x_{a1}=r\cos(\theta)$ and $x_{a2}=r\sin(\theta)$ with $r\geq 0$ and $\theta\in[0,2\pi[$. Let K_r be the matrix:

$$K_r=K_r(x,x_0)=-\frac{\partial}{\partial r}K\left(\begin{pmatrix}x_{01}-r\cos(\theta)\\x_{02}-r\sin(\theta)\\x_3\end{pmatrix},x_0\right).$$

Let us prove that unless $(x_1,x_2)=(x_{01},x_{02})$ (i.e. $r=0$), $K_r(x,x_0)$ is positive definite. In what follows, we assume that $r\neq 0$. As K_r is symmetric, we have to check that its eigenvalues λ_1, λ_2 and λ_3 are positive. First, using (14), (15) and (17), one can see that the vector $(\sin(\theta),-\cos(\theta),0)^T$ is an eigenvector of $M(x,x_0)$, thus of $K(x,x_0)$ and thus of $K_r(x,x_0)$ with the eigenvalue:

$$\lambda_1=\frac{32x_3^2\pi}{r|x_a|^3}\left(\frac{x_{a3}}{|x_a|}+1\right)\frac{4r^2+3x_{a3}^2}{2|x_a|^3-x_{a3}r^2-2x_{a3}|x_a|^2}>0.$$

Furthermore, one can check that:

$$\lambda_2+\lambda_3=\mathrm{Tr}(K_r(x,x_0))-\lambda_1=\frac{32x_3^2\pi}{r^5|x_a|^6}\frac{\mathcal{A}_t^2-\mathcal{B}_t^2}{\mathcal{A}_t+\mathcal{B}_t}>0,$$

where we define the positive quantities:

$$\mathcal{A}_t^2-\mathcal{B}_t^2=r^6\left(25r^6+35r^2x_{a3}^4+50r^4x_{a3}^2+9x_{a3}^6\right)>0,$$
$$\mathcal{A}_t=13r^2x_{a3}^4+15r^4x_{a3}^2+5r^6+4x_{a3}^6>0,$$

$$\mathcal{B}_t = -x_{a3}|x_a|\left(4x_{a3}^4 + 10r^4 + 11r^2x_{a3}^2\right) > 0.$$

Similarly, we have:

$$\lambda_2\lambda_3 = \frac{\det(K_r(x,x_0))}{\lambda_1} = \frac{64x_3^4\pi^2}{r^4|x_a|^{12}}\frac{\mathcal{A}_d^2 - \mathcal{B}_d^2}{\mathcal{A}_d + \mathcal{B}_d} > 0,$$

where we define the positive quantities:

$$\begin{aligned}
\mathcal{A}_d^2 - \mathcal{B}_d^2 &= r^6\left(1024r^6 + 1457r^2x_{a3}^4 + 240x_{a3}^6 + 2240r^4x_{a3}^2\right) > 0,\\
\mathcal{A}_d &= 128r^2x_{a3}^4 + 135r^4x_{a3}^2 + 32r^6 + 24x_{a3}^6 > 0,\\
\mathcal{B}_d &= -x_{a3}|x_a|\left(24x_{a3}^4 + 80r^4 + 116r^2x_{a3}^2\right) > 0.
\end{aligned}$$

This shows that λ_1, λ_2 and λ_3 are positive for $r > 0$. The matrix $K_r(x, x_0)$ is thus positive definite if $(x_{a1}, x_{a2}) \neq (0, 0)$. Hence, as $p_0 \neq 0_{\mathbb{R}^3}$, and because $p_0^T K_r(x_c, x_0)p_0 = 0$ for any critical point $x_c \in \mathbb{R}^3_-$ of I_Π, $x \in \mathbb{R}^3_-$ is a critical point of I_Π only if $x_1 = x_{01}$ and $x_2 = x_{02}$.

3.4 Reciprocal

From Section 3.3, the only remaining possibility is $x_c = x_0$. We first remark from (6) that $\mathrm{p}(x_0) = M(x_0, x_0)^{-1}M(x_0, x_0)^T p_0 = p_0$. Furthermore, (x_0, p_0) is obviously a critical point of J_Π as it is its global minimizer (see e.g. Proposition 1). We can thus conclude that J_Π has a unique critical point: (x_0, p_0). This ends the proof of Theorem 2.

4 Numerical illustration

To illustrate the uniqueness of the critical point of I_Π, we consider $x_0 = (0, 0, -1)$ and $p_0 = (1, -5, 0.1)$ and study $I_\Pi(x)$ for $x = (x_1, x_2, -1)$ in the same horizontal plane as x_0 with $(x_1, x_2) \in [-4, 4]^2$. The quantities I_Π and ∇I_Π were computed using their expression in Section 3.3.1 together with the formulas (13), (14), (15), (16) and (17) using a Matlab code. Figure 2 represents the criterion I_Π and its variation on the horizontal plane while Figure 3 illustrates its vertical variation. Figure 4 shows the squared norm of the gradient of I_Π while Figure 5 shows the 2D gradient $(\partial_{x_1} I_\Pi, \partial_{x_2} I_\Pi)$ and a contour plot of I_Π. The behavior of J_Π with p is not represented as it is quadratic hence strictly convex.

Figures 2 and 3 illustrate the non-convexity of the criterion hence the need of this study for the convergence of optimization schemes. These two figures also illustrate the convergence of the criterion I_Π to $\|u_0\|_{L^2(\Pi)}$ as $|x| \to \infty$ or $x_3 \to 0$ and show that there is a unique global minimizer of I_Π on respectively the plane of height x_{03} and the vertical line passing through x_0. Figure 4 shows the uniqueness of the critical point of I_Π in the plane of height x_{03}. Figure 5 is

an illustration of the positivity of the radial derivative of the criterion on any horizontal plane as shown in Section 3.3.3.

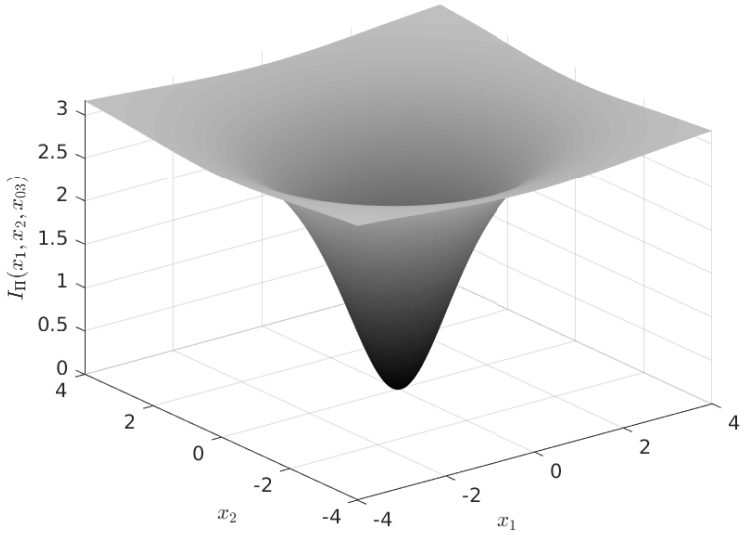

Fig. 2 Criterion I_Π on the horizontal square $[-4, 4]^2 \times \{-1\}$

5 Conclusion

In this work, we proved the uniqueness of the critical point of the least-squares criterion J_Π in the Euclidean geometry ($\partial\Omega = \Pi$). We plan to study the same problem in the spherical setting ($\Omega = \mathbb{B}$ the unit ball in $\mathbb{R}^3$ centered at $0_{\mathbb{R}^3}$ and $\partial\Omega = \mathbb{S}$) and more general geometries. This setting leads to complications in the absence of homogeneity of the functions considered. Yet, we managed to establish the uniqueness in the particular case where $x_0 = 0_{\mathbb{R}^3}$ is at the center of the sphere. Unlike in the two-dimensional case, the result does not directly generalize to other surfaces.

In other works, we showed similar results for another form of Poisson equation linked to a charge localization problem with data corresponding to the induced electric field:

$$\begin{cases} \Delta E_0 = 4\pi\nabla(q_0\delta_{x_0}) \text{ in } \mathbb{R}^3, \\ \lim\limits_{|z|\to\infty} |E_0(z)| = 0, \end{cases}$$

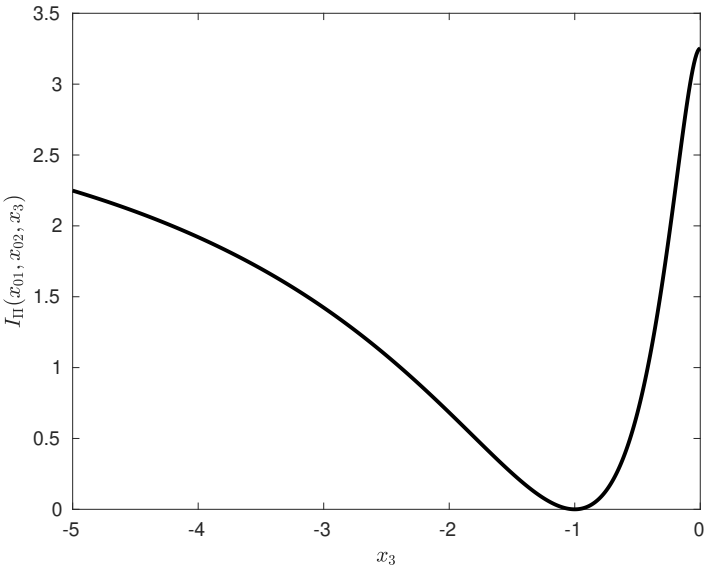

Fig. 3 Criterion I_Π on the vertical line $\{0\} \times \{0\} \times [-5, 0]$

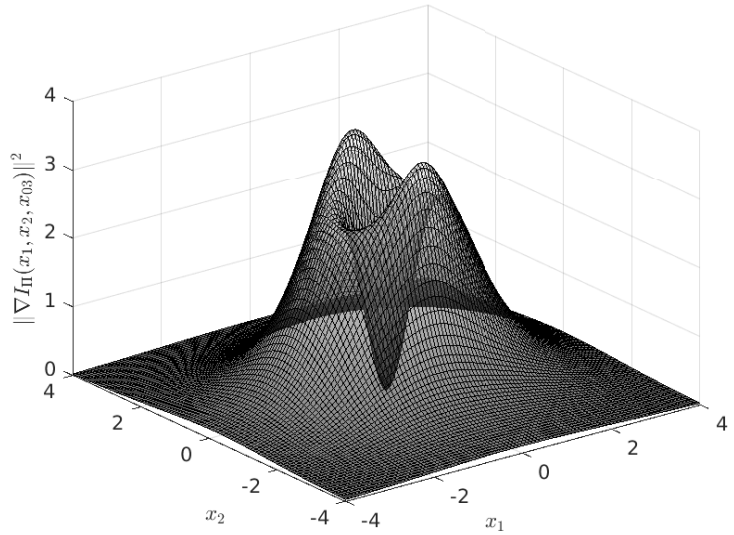

Fig. 4 Squared norm of ∇I_Π on the horizontal square $[-4, 4]^2 \times \{-1\}$

which is associated with the least-square criterion:

$$\begin{aligned} j_{\partial\Omega} : \Omega \times \mathbb{R}^3 &\longrightarrow \mathbb{R}_+ \\ (x, q) &\longmapsto \int_{\partial\Omega} \left| q \frac{x - z}{|x - z|^3} - q_0 \frac{x_0 - z}{|x_0 - z|^3} \right|^2 \mathrm{d}\sigma(z), \end{aligned} \tag{19}$$

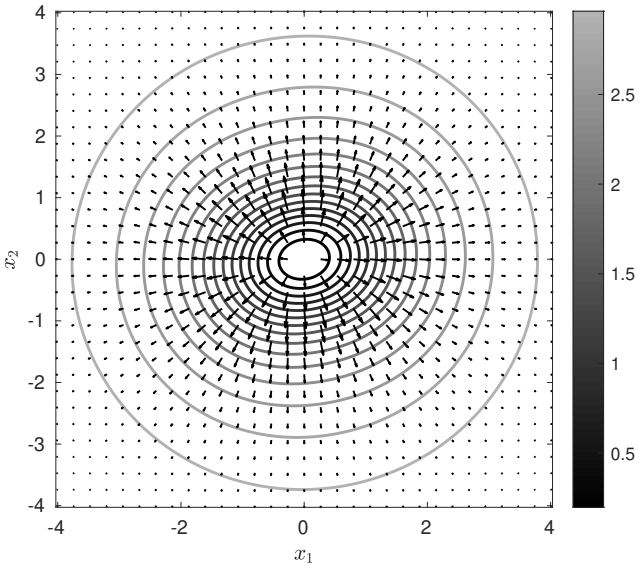

Fig. 5 2D gradient of I_Π (black arrows) and contour plots of I_Π (gray curves) on the horizontal square $[-4,4]^2 \times \{-1\}$

The study of (19) was done for two different geometries (the sphere $\mathbb{S}$ and the horizontal plane Π) and we showed the uniqueness of the critical point in both cases. Further generalizations are still expected for the criteria $J_{\partial\Omega}$ given by (3) and $j_{\partial\Omega}$ given by (19) for multiple dipolar sources and for incomplete or noisy data.

Acknowledgments. The authors would like to thank Laurent Baratchart for his contribution to this work.

Appendix A Use of Cauchy formula on the plane

This appendix aims at proving Lemma 1. Let $r > |x|$, $\mathbb{D}_r \subset \Pi$ the disk of radius r and center $0_{\mathbb{R}^3}$ and $\mathbb{S}_r^- \subset \mathbb{R}^3_-$ the lower half-sphere of radius r and center $0_{\mathbb{R}^3}$. By applying the Cauchy formula (10) to f in the half-ball of boundary $\mathbb{S}_r^- \cup \mathbb{D}_r$, we get for $r > |x|$:

$$\int_{\mathbb{D}_r \cup \mathbb{S}_r^-} \frac{x-z}{|x-z|^3} \odot \nu(z) \odot f(z) \mathrm{d}\sigma(z) = \begin{cases} f(x) & x \in \mathbb{R}^3_-, \\ 0 & x \in \mathbb{R}^3_+. \end{cases}$$

Let $\Lambda : z \mapsto \frac{x-z}{|x-z|^3} \odot \nu(z) \odot f(z)$. Clearly:

$$\lim_{r\to\infty} \int_{\mathbb{D}_r} \Lambda(z)\mathrm{d}\sigma(z) = \int_{\Pi} \Lambda(z)\mathrm{d}\sigma(z) = \int_{\Pi} \frac{x-z}{|x-z|^3} \odot e_3 \odot f(z)\mathrm{d}\sigma(z).$$

Furthermore, by change of variable and for $r > |x|$:

$$\begin{aligned}\left|\int_{\mathbb{S}_r^-} \Lambda(z)\mathrm{d}\sigma(z)\right| &= \left|\int_{\mathbb{S}_1^-} \frac{\frac{x_0}{r}-y}{\left|\frac{x_0}{r}-y\right|^3} \odot y \odot f(ry)\mathrm{d}\sigma(y)\right| \\ &\leq \int_{\mathbb{S}_1^-} \left|\frac{\frac{x_0}{r}-y}{\left|\frac{x_0}{r}-y\right|^3} \odot y \odot f(ry)\right| \mathrm{d}\sigma(y) \\ &\leq \int_{\mathbb{S}_1^-} \left|\frac{\frac{x_0}{r}-y}{\left|\frac{x_0}{r}-y\right|^3}\right| |f(ry)|\mathrm{d}\sigma(y) \\ &\leq \varphi(r) \int_{\mathbb{S}_1^-} \frac{1}{\left|\frac{x_0}{r}-y\right|^2}\mathrm{d}\sigma(y), \end{aligned} \tag{A1}$$

using the property [8, Thm 5.16] that for all $x \in \mathrm{C}\ell_{0,3}(\mathbb{R})$ and $y \in \mathbb{R}^3$ identified with its corresponding vector in $\mathrm{C}\ell_{0,3}(\mathbb{R})$:

$$|x \odot y| = |x||y|.$$

As the integral on the right-hand side of (A1) is bounded, we see that:

$$\lim_{r\to\infty} \int_{\mathbb{S}_r^-} \Lambda(z)\mathrm{d}\sigma(z) = 0.$$

This establishes Lemma 1.

References

[1] Clerc, M., Leblond, J., Marmorat, J.-P., Papadopoulo, T.: Source localization using rational approximation on plane sections. Inverse Problems **28**(5), 055018 (2012). https://doi.org/10.1088/0266-5611/28/5/055018

[2] Baratchart, L., Hardin, D.P., Lima, E.A., Saff, E.B., Weiss, B.P.: Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions. Inverse Problems **29**(1), 015004 (2012). https://doi.org/10.1088/0266-5611/29/1/015004

[3] Jackson, J.D.: Classical Electrodynamics, 3rd ed. edn. Wiley, New York, NY (1999). https://doi.org/10.1002/3527600434.eap109

[4] Badia, A.E., Ha-Duong, T.: An inverse source problem in potential analysis. Inverse Problems **16**(3), 651–663 (2000). https://doi.org/10.1088/0266-5611/16/3/308

[5] Axler, S., Bourdon, P., Ramey, W.: Harmonic Function Theory. Springer, New York (1992). https://doi.org/10.1007/b97238

[6] Baratchart, L., Abda, A., Ben Hassen, F., Leblond, J.: Recovery of pointwise sources or small inclusions in 2D domains and rational approximation. Inverse Problems **21**, 51–74 (2005). https://doi.org/10.1088/0266-5611/21/1/005

[7] Dautray, R., Lions, J.-L.: Mathematical Analysis and Numerical Methods for Science and Technology vol. 1-2. Springer, New York, NY (2012)

[8] Gilbert, J., Murray, M.: Clifford Algebras and Dirac Operators in Harmonic Analysis. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (1991)

[9] Mitrea, M.: Clifford Wavelets, Singular Integrals, and Hardy Spaces. Springer, Berlin (1994). https://doi.org/10.1007/bfb0073556