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ON THE LIMITING AMPLITUDE PRINCIPLE FOR THE WAVE EQUATION WITH VARIABLE COEFFICIENTS

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ABSTRACT. In this paper, we prove new results on the validity of the limiting amplitude principle (LAP) for the wave equation with nonconstant coefficients, not necessarily in divergence form. Under suitable assumptions on the coefficients and on the source term, we establish the LAP for space dimensions 2 and 3. This result is extended to one space dimension with an appropriate modification. We also quantify the LAP and thus provide estimates for the convergence of the time-domain solution to the frequency-domain solution. Our proofs are based on time-decay results of solutions of some auxiliary problems.

Keywords wave equation with variable coefficients, limiting amplitude principle, long-time asymptotics

Mathematics Subject Classification 35L05, 35L10, 35B10, 35B40

1. INTRODUCTION

One of the possibilities of obtaining solutions to the Helmholtz equation is by solving a corresponding time-domain wave equation. An essential ingredient in connecting time- and frequency-domain wave problems is the limiting amplitude principle (LAP). Originally proposed as one of the tools to select the unique solution of the Helmholtz equation problem in an infinite domain it has been studied in numerous works over the last 70 years.

The LAP can be crudely stated as follows: the solution to the time-dependent wave equation with time-harmonic source term converges, for large times, to the solution of the Helmholtz equation with the spatial source term and frequency

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corresponding to the original time-harmonic source term. Our main motivation for revisiting the LAP comes from numerical analysis. Helmholtz problems can be challenging to solve in practice in some situations such as when the wavenumber (frequency) parameter is large. Numerical methods have been proposed to address a classical Helmholtz problem efficiently through its reformulations in the time domain. They include the controllability method introduced in [7, 13], together with its spectral version [17] and its extensions [16, 15], the WaveHoltz method [1], the time-domain preconditioner of [26], and the front-tracking adaptive method of [2]. The numerical analysis of these methods requires a quantification of the modeling error (reformulation of the frequency-domain problem into a time-domain problem), which will add to the error due to the numerical approximation of the problem in the time domain. This motivates the study of the LAP under some new angles, with particular focus on the quantification of large-time convergence. As opposed to a direct study of the resolvent operator, our analysis is based on decay estimates for the solutions of some auxiliary problems. Since decay results are still the subject of intense investigation, an advantage of this approach is that new findings in that area directly translate into improvements in the quantification of the large-time convergence in the LAP.

Main results. In this paper, we will be concerned with the following setup. Given an angular frequency $\omega > 0$, material parameters α, β , which smoothly vary within some bounded domain, and a compactly supported source term F , we consider the following frequency-domain and time-domain problems, respectively: we consider

$$(1.1) \quad \begin{cases} -\omega^2 U(\mathbf{x}) - \beta^{-1}(\mathbf{x}) \nabla \cdot (\alpha(\mathbf{x}) \nabla U(\mathbf{x})) = F(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d, \\ \lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}|^{\frac{d-1}{2}} \left[\partial_{|\mathbf{x}|} U(\mathbf{x}) - i\omega \sqrt{\beta_0/\alpha_0} U(\mathbf{x}) \right] = 0, \end{cases}$$

and

$$(1.2) \quad \begin{cases} \partial_t^2 u(\mathbf{x}, t) - \beta^{-1}(\mathbf{x}) \nabla \cdot (\alpha(\mathbf{x}) \nabla u(\mathbf{x}, t)) = e^{-i\omega t} F(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d, t > 0, \\ u(\mathbf{x}, 0) = 0, \quad \partial_t u(\mathbf{x}, 0) = 0, & \mathbf{x} \in \mathbb{R}^d. \end{cases}$$

Our assumptions on α, β , and F are stated as follows.

Assumption 1.1. (smoothness, compactly supported derivatives & positivity of coefficients) *Let $\alpha, \beta \in C^\infty(\mathbb{R}^d)$, be real-valued functions such that $\alpha(\mathbf{x}) \geq \alpha_{\min}$, $\beta(\mathbf{x}) \geq \beta_{\min}$ for $\mathbf{x} \in \mathbb{R}^d$, and $\alpha(\mathbf{x}) \equiv \alpha_0$, $\beta(\mathbf{x}) \equiv \beta_0$ for $\mathbf{x} \in \mathbb{R}^d \setminus \Omega_{in}$, with some bounded domain $\Omega_{in} \subset \mathbb{R}^d$ and constants $\alpha_{\min}, \beta_{\min}, \alpha_0, \beta_0 > 0$.*

Assumption 1.2. (nontrapping coefficients) *Let α, β be non-trapping, i.e. such that all rays associated with the metric α/β escape to infinity [6, Sect. 1]. In other words (see e.g. [14, Def. 7.6 & Cor. 7.10]), defining $H(\mathbf{q}, \mathbf{p}) := \alpha(\mathbf{q}) |\mathbf{p}|^2 - \beta(\mathbf{q})$, given $\mathbf{q}_0, \mathbf{p}_0 \in \mathbb{R}^d$ such that $H(\mathbf{q}_0, \mathbf{p}_0) = 0$, the solution vector of the*

canonical system of differential equations with the Hamiltonian $H(\mathbf{q}, \mathbf{p})$,

$$(1.3) \quad \begin{cases} \frac{d}{dt}\mathbf{q}(t) = 2\alpha(\mathbf{q})\mathbf{p}(t), & t > 0, \\ \frac{d}{dt}\mathbf{p}(t) = -\frac{\beta(\mathbf{q})}{\alpha(\mathbf{q})}\nabla_{\mathbf{q}}\alpha(\mathbf{q}) + \nabla_{\mathbf{q}}\beta(\mathbf{q}), & t > 0, \\ \mathbf{q}(0) = \mathbf{q}_0, \quad \mathbf{p}(0) = \mathbf{p}_0, \end{cases}$$

must satisfy $|\mathbf{q}(t)| \rightarrow \infty$ as $t \rightarrow \infty$.

Assumption 1.3. (compactly supported source) Let $F \in L^2(\mathbb{R}^d)$ such that $\text{supp } F \subset \Omega_{in}$, with Ω_{in} as in Assumption 1.1.

Under Assumptions 1.1–1.3, we prove the following versions of the LAP.

Theorem 1.4. Let $d = 2, 3$. Suppose that Assumptions 1.1–1.3 are satisfied. Let $U(\mathbf{x})$ and $u(\mathbf{x}, t)$ be the solutions to (1.1) and (1.2), respectively. Then, there exists a constant $C > 0$ depending on F , α , β , and ω such that for $d = 2$:

$$\|u(\cdot, t) - e^{-i\omega t}U\|_{H^1(\Omega)} + \|\partial_t u(\cdot, t) + i\omega e^{-i\omega t}U\|_{L^2(\Omega)} \leq C \frac{1 + \log(1 + t^2)}{(1 + t^2)^{1/2}}, \quad t \geq 0,$$

for $d = 3$:

$$\|u(\cdot, t) - e^{-i\omega t}U\|_{H^1(\Omega)} + \|\partial_t u(\cdot, t) + i\omega e^{-i\omega t}U\|_{L^2(\Omega)} \leq \frac{C}{(1 + t^2)^{1/2}}, \quad t \geq 0,$$

where Ω is an arbitrary bounded domain.

Theorem 1.5. Let $d = 1$. Suppose that Assumptions 1.1 and 1.3 are satisfied. Let $U(\mathbf{x})$ and $u(\mathbf{x}, t)$ be the solutions to (1.1) and (1.2), respectively. Then, there exist constants $\Lambda > 0$ and $C > 0$ depending on F , α , β , and ω such that

$$\|u(\cdot, t) - e^{-i\omega t}U - U_\infty\|_{H^1(\Omega)} + \|\partial_t u(\cdot, t) + i\omega e^{-i\omega t}U\|_{L^2(\Omega)} \leq Ce^{-\Lambda t}, \quad t \geq 0,$$

where

$$(1.4) \quad U_\infty := \frac{1}{2i\omega\sqrt{\alpha_0\beta_0}} \int_{\Omega_{in}} F(x)\beta(x)dx,$$

and Ω is an arbitrary bounded domain.

Previous results on the LAP. Let us provide a brief overview of previous works on the LAP. The simplest version of the LAP dealing with the constant coefficient, three-dimensional wave equation has been known at least since 1948 [28, 29]. There, it is proven that this physical principle selects the unique solution of the stationary problem satisfying the Sommerfeld radiation condition.

In the seminal work [20], these results are extended to the wave equation containing a compactly supported potential term. Namely, the equation $\partial_t^2 u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) + \tilde{c}(\mathbf{x})u(\mathbf{x}, t) = q(\mathbf{x})e^{-i\omega t}$ is considered and it is assumed that the potential $\tilde{c}(\mathbf{x})$ is a continuous, compactly supported function with bounded derivative

and such that the operator $-\Delta + \tilde{c}(\mathbf{x})$ does not have discrete spectrum on $L^2(\mathbb{R}^3)$. The source term $q(\mathbf{x})$ is taken to be a bounded function with compact support. The convergence in time to the solution of a stationary problem (up to the factor $e^{i\omega t}$) is not explicitly quantified, but the proof shows that the convergence occurs with an exponential rate.

The pair of papers [22, 23] contains further important results. First of all, the conditions on $\tilde{c}(\mathbf{x})$ are weakened: Hölder continuity is sufficient (allowing also a finite number of square integrable singularities) and the support does not have to be compact. Moreover, paper [23] deals with the LAP for the variable speed wave equation $\partial_t^2 u(\mathbf{x}, t) - c^2(\mathbf{x}) \Delta u(\mathbf{x}, t) = q(\mathbf{x}) e^{-i\omega t}$. This also demonstrates that, while results for an equation with a potential term (i.e. Schrödinger / Klein-Gordon equation) do not automatically transplant to those for the variable speed wave equation, both problems are amenable to a very similar treatment involving resolvent analysis. For the case of the wave equation, it is important that the spatially varying speed tends to a constant value at infinity sufficiently fast. The source term is assumed to decay at infinity as well. In both cases of wave equations, pointwise convergence results are obtained and the convergence rate is not specified.

The paper [25] considers the wave equation with a potential and a source term both having algebraic localisation (meaning that they decay algebraically in space). An algebraic rate of pointwise convergence is established and shown to be related to the localisation of the potential and the forcing term: better localisation entails faster convergence. As in most of the works, results are presented only for the three-dimensional case.

In [19], following the work of [8] and considering also non-zero initial data, it is claimed that the assumptions of [20] were incomplete and, in addition to the absence of eigenvalues, the absence of some special solutions also needs to be assumed. More precisely, such solutions (known as zero-resonances) satisfy the homogeneous stationary wave equation with zero wavenumber, they have some weak decay and they satisfy a non-oscillation condition at infinity, but they are not in $L^2(\mathbb{R}^3)$. Zero-resonances are generally known to exist and would be an obstacle for the validity of the LAP. However, they can be ruled out by assuming that $\tilde{c} \in C^1(\mathbb{R}^3)$, $\tilde{c} \geq 0$ and, for $|x| \gg 1$, $\tilde{c}(\mathbf{x}) = \mathcal{O}(1/|\mathbf{x}|^\gamma)$ with $\gamma > 3$.

Eidus' paper [10] provides an extensive overview of the results available at that time and treats the problem in great generality. In particular, it deals with the wave equation arising from a positive second-order differential operator in divergence form $-\sum_{k,j=1}^d \partial_{x_k} (a_{kj}(\mathbf{x}) \partial_{x_j}) + \tilde{c}(\mathbf{x})$. It is assumed that \tilde{c} is real-valued and locally Hölder continuous, $a_{kj} \in C^2(\mathbb{R}^d)$ is real-valued, symmetric and coercive (i.e. satisfying, for any vector $v \in \mathbb{R}^d$, $\sum_{k,j}^d a_{kj} v_k v_j \geq a_0 |v|^2$ with some $a_0 > 0$), and $|\nabla a_{kj}(\mathbf{x})|$, $\tilde{c}(\mathbf{x})$ decay fast enough at infinity. The problem is posed in an unbounded domain of \mathbb{R}^d with a finite boundary where the zero Dirichlet boundary condition is imposed. However, it is mentioned in [10, Ch. 2, p. 21] that the

obtained results must also hold if this unbounded (exterior) domain is taken to be the whole \mathbb{R}^d . The time convergence is shown for the H^1 -norm of the solution and the L^2 -norm of its time derivative, with both norms taken over bounded sets.

As a generalization, Vainberg [30], besides geometrical features, also considers higher-order constant coefficient hypoelliptic operators in \mathbb{R}^d , whereas [18] treats dissipative wave equations with variable dissipation and potential terms.

Ramm [25] considers a general linear operator and formulates necessary and sufficient conditions for the validity of the LAP in terms of certain properties of the resolvent operator. A more general form of the LAP is formulated, involving time convergence in mean, namely, the convergence of the quantity $\frac{1}{t} \int_0^t e^{i\omega\tau} u(\mathbf{x}, \tau) d\tau$, for $t \rightarrow \infty$, to the stationary solution. This is shown to be equivalent to the validity of the limiting absorption principle.

Bloom [5] deals with the symmetric system of second order wave equations in \mathbb{R}^3 written in the form similar to ours (i.e. having variable coefficients outside and inside the divergence operator, as in (1.2)). The LAP is shown to hold and the rate of pointwise convergence is found to be algebraic, namely, under the assumptions there, it is slower than $t^{-1/2}$. It is not specified whether these results are still valid if the exterior domain is replaced by the entire space \mathbb{R}^3 .

The single wave equation of the form (1.2) is considered in [27] and the problem is posed in the whole space \mathbb{R}^3 . Similarly to [19], the necessity to rule out zero-resonances is discussed and it prompts an investigation of the resolvent in the low-frequency limit.

In this brief literature overview, we have focussed on problems posed in the whole space \mathbb{R}^d , while almost entirely omitting the geometrical issues, which are the most commonly discussed aspects in the literature, see the classical works of Morawetz (e.g. [21]) and her collaborators. More on that can be found in the introductory part of [10]. In general, geometry (finite/infinite boundary, convexity/star-shapeness of the scatterer), space dimension, coefficients and order of the differential operator all play important roles in the validity of the LAP. For example, the principle is violated if the geometry of the problem allows for the existence of trapped modes, or the differential operator (typically, of a higher order) has eigenvalues embedded into its continuous spectrum.

In the present work, we study the LAP for a problem where both material parameters α and β are allowed to be nonconstant and prove our results in spatial dimensions $d = 1, 2$ and 3 . The main result given in Theorem 1.4 does not only prove the validity of the LAP but also estimates the convergence rates. Additionally, we provide Theorem 1.5, which covers the one-dimensional case where a classical formulation of the LAP (i.e. when $U_\infty = 0$) is known not to be valid [9, Sect. 3, Thm. 6]. On a technical side, a novelty of our approach to the proof of the LAP is that it avoids the direct study of the resolvent operator and relies instead on several decay/convergence results. The main features of the present work are:

- The LAP is proven for the wave equation with nonconstant coefficients, which is not necessarily in divergence form. Besides the “classical” case $d = 3$, we also consider $d = 2$.
- The validity of the LAP is extended to the case $d = 1$ with an appropriate modification.
- The convergence in the LAP is quantified and is shown to be algebraic in time for $d = 2, 3$ and exponential for $d = 1$.

We believe that exponential and algebraic rates of the convergence for the cases $d = 1$ and $d = 2$ are generally sharp, but the rate of the decay for the case $d = 3$ can be improved.

Outline. The paper is organised as follows. In Section 2, we present time-decay estimates for the time-domain problem with suitable initial data and source term. In Section 3, we establish the LAP in the form given in Theorems 1.4 and 1.5. Time-decay estimates, which are instrumental for that, are proven in Section 4. Finally, in Section 5, we summarise the obtained results and give prospects for further work in related directions. Some technical estimates used in the proofs are deferred to the appendix.

2. TIME-DECAY RESULTS

In this section, we establish some decay-in-time results for solutions to the wave equation with sufficiently localised initial data, which are used in our proof of the LAP in Section 3 below. The proofs of these results are deferred to Section 4. More precisely, we are concerned with the solution of the Cauchy problem

$$(2.1) \quad \begin{cases} \partial_t^2 u(\mathbf{x}, t) - \beta^{-1}(\mathbf{x}) \nabla \cdot (\alpha(\mathbf{x}) \nabla u(\mathbf{x}, t)) = f(\mathbf{x}, t), & \mathbf{x} \in \mathbb{R}^d, t > 0, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \partial_t u(\mathbf{x}, 0) = u_1(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d, \end{cases}$$

and its constant-coefficient analog with zero source term:

$$(2.2) \quad \begin{cases} \partial_t^2 v(\mathbf{x}, t) - c_0^2 \Delta v(\mathbf{x}, t) = 0, & \mathbf{x} \in \mathbb{R}^d, t > 0, \\ v(\mathbf{x}, 0) = v_0(\mathbf{x}), \quad \partial_t v(\mathbf{x}, 0) = v_1(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d, \end{cases}$$

where $c_0 := \sqrt{\alpha_0/\beta_0}$. We start by considering problem (2.1) in the case of localised data and zero source term.

Proposition 2.1. *Let $d \geq 2$, $f \equiv 0$. Suppose that $u_0 \in H^3(\mathbb{R}^d)$, $u_1 \in H^2(\mathbb{R}^d)$ and α, β satisfy Assumptions 1.1 and 1.2. Additionally, the initial data are assumed to satisfy the following localisation condition:*

$$(2.3) \quad \int_{\mathbb{R}^d} (1 + |\mathbf{x}|^2)^{d+1+\epsilon} (|u_0(\mathbf{x})|^2 + |u_1(\mathbf{x})|^2) d\mathbf{x} < \infty$$

with some $\epsilon > 0$. Then, for any bounded $\Omega \subset \mathbb{R}^d$, the solution of (2.1) obeys the following decay estimate for some constant $C > 0$:

$$(2.4) \quad \|u(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t u(\cdot, t)\|_{L^2(\Omega)} \leq \frac{C}{(1+t^2)^{\frac{d-1}{2}}}, \quad t \geq 0.$$

For the case of zero initial data and a localised source term, we have the following result.

Proposition 2.2. *Let $d \geq 2$, $u_0 \equiv 0$, $u_1 \equiv 0$ and α, β satisfy Assumptions 1.1 and 1.2. Additionally, suppose that $f \in C(\mathbb{R}_+, L^2(\mathbb{R}^d))$ with $\mathbb{R}_+ := [0, \infty)$, $\bigcup_{t>0} \text{supp } f(\cdot, t) \subset \Omega_f$ for some bounded domain $\Omega_f \subset \mathbb{R}^d$, and there exist constants $C_f, p > 0$ such that*

$$(2.5) \quad \|f(\cdot, t)\|_{L^2(\mathbb{R}^d)} + \|\partial_t f(\cdot, t)\|_{L^2(\mathbb{R}^d)} \leq \frac{C_f}{(1+t^2)^{\frac{p}{2}}}, \quad t \geq 0.$$

Then, for any bounded domain $\Omega \subset \mathbb{R}^d$, the solution of (2.1) obeys the following decay estimates for $t \geq 0$ and some constant $C > 0$ depending on p and d .

For $d = 2$:

$$(2.6) \quad \|u(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t u(\cdot, t)\|_{H^1(\Omega)} \leq C \begin{cases} \frac{1 + \log(1+t^2)}{(1+t^2)^{\frac{p}{2}}}, & 0 < p \leq 1, \\ \frac{1}{(1+t^2)^{\frac{1}{2}}}, & p > 1, \end{cases}$$

For $d > 2$:

$$(2.7) \quad \|u(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t u(\cdot, t)\|_{H^1(\Omega)} \leq C \begin{cases} \frac{1}{(1+t^2)^{\frac{p}{2}}}, & 0 < p \leq 1, \\ \frac{1}{(1+t^2)^{\frac{r}{2}}}, & p > 1, \end{cases}$$

where $r := \min(d-1, p)$.

Next, we consider the wave equation (2.2) with constant coefficients and $f \equiv 0$.

Lemma 2.3. *Let $d = 2, 3$. Let $\omega > 0$, $\rho_0 > 0$, $\rho_1 > \rho_0$ be some fixed constants. Let $\mathbb{S}^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ be the $(d-1)$ -dimensional unit sphere and let $\mathbb{B}_{\rho_0} := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| < \rho_0\}$ be the ball of radius ρ_0 , both centered at $\mathbf{x} = \mathbf{0}$. Fix $\Omega \Subset \mathbb{B}_{\rho_0}$ meaning that $\Omega \subseteq \mathbb{B}_{\rho_0-\epsilon} \subset \mathbb{B}_{\rho_0}$ for some $\epsilon > 0$. We make the following assumptions on the initial conditions v_0 and v_1 .*

- For $d = 2$, we assume that

$$(2.8) \quad v_0(\mathbf{x}) = A_0(|\mathbf{x}|) Y_0\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + V_0(\mathbf{x}), \quad v_1(\mathbf{x}) = A_0(|\mathbf{x}|) Y_1\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + V_1(\mathbf{x}),$$

where $V_0 \in W^{4,\infty}(\mathbb{R}^2)$, $V_1 \in W^{3,\infty}(\mathbb{R}^2)$, $A_0 \in C^4(\mathbb{R}_+)$, $Y_0 \in C^4(\mathbb{S}^1)$, $Y_1 \in C^3(\mathbb{S}^1)$ are such that $A_0(|\mathbf{x}|) = V_0(\mathbf{x}) = V_1(\mathbf{x}) \equiv 0$ for $|\mathbf{x}| \leq \rho_0$, and that there exist a constant $C_0 > 0$ such that

$$(2.9) \quad |\mathbf{x}|^{5/2} (|V_0(\mathbf{x})| + |V_1(\mathbf{x})|) \leq C_0$$

holds true for all $\mathbf{x} \in \mathbb{R}^2$. Moreover, $A_0(\rho) = e^{i\frac{\omega}{c_0}\rho}/\rho^{3/2}$ for $\rho > \rho_1$.

- For $d = 3$, we assume that $v_0 \in W^{4,\infty}(\mathbb{R}^3)$, $v_1 \in W^{3,\infty}(\mathbb{R}^3)$ and that there exists a constant $C_0 > 0$ such that

$$(2.10) \quad |\mathbf{x}|^2 (|v_0(\mathbf{x})| + |v_1(\mathbf{x})|) \leq C_0$$

holds true for all $\mathbf{x} \in \mathbb{R}^3$.

Then, there exists a constant $C > 0$ such that, for all $\mathbf{x} \in \Omega$ and $t \geq 0$, the solution of (2.2) with the initial data as above satisfies

$$(2.11) \quad |v(\mathbf{x}, t)| + |\nabla v(\mathbf{x}, t)| + |\partial_t v(\mathbf{x}, t)| \\ + |\Delta v(\mathbf{x}, t)| + |\partial_t \nabla v(\mathbf{x}, t)| + |\partial_t \Delta v(\mathbf{x}, t)| \leq \frac{C}{(1+t^2)^{1/2}}.$$

Note that, in this result, we use smoothness and the presence of the oscillatory exponential term in the radial factor in the case $d = 2$ to deduce the $\mathcal{O}(1/t)$ decay instead of the more classical L^∞ -decay $\mathcal{O}(1/t^{1/2})$ (see e.g. [4]).

In a similar vein, we will also need an analogous result for problem (2.2) with even less standard initial conditions (namely, such that the decay at infinity of v_0, v_1 entails only that $v_0 \in W^{1,p}(\mathbb{R}^d)$, $v_1 \in L^p(\mathbb{R}^d)$ with $p > 4$). If such initial conditions have some additional structure, the solution can be shown to decay in time with the same decay rate as in Lemma 2.3.

Lemma 2.4. *Let $d = 2, 3$. Let $\omega > 0$, $\rho_0 > 0$, $\rho_1 > \rho_0$ be some fixed constants. Using the notation introduced in Lemma 2.3, suppose that $\Omega \Subset \mathbb{B}_{\rho_0}$. Assume that*

$$(2.12) \quad v_0(\mathbf{x}) = A(|\mathbf{x}|) Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right), \quad v_1(\mathbf{x}) = -c_0 \partial_{|\mathbf{x}|} v_0(\mathbf{x}) = -c_0 A'(|\mathbf{x}|) Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right),$$

where $\partial_{|\mathbf{x}|}$ denotes the derivative in the radial direction of the variable \mathbf{x} , $A \in C^4(\mathbb{R}_+)$, $Y \in C^4(\mathbb{S}^{d-1})$ such that $A(\rho) \equiv 0$ for $\rho \in [0, \rho_0]$ and $A(\rho) = e^{i\frac{\omega}{c_0}\rho}/\rho^{\frac{d-1}{2}}$ for $\rho > \rho_1$. Then, there exists a constant $C > 0$ such that, for all $\mathbf{x} \in \Omega$ and $t \geq 0$, the solution of (2.2) with the initial data (2.12) satisfies

$$(2.13) \quad |v(\mathbf{x}, t)| + |\nabla v(\mathbf{x}, t)| + |\partial_t v(\mathbf{x}, t)| \\ + |\Delta v(\mathbf{x}, t)| + |\partial_t \nabla v(\mathbf{x}, t)| + |\partial_t \Delta v(\mathbf{x}, t)| \leq \frac{C}{(1+t^2)^{1/2}}.$$

In the one-dimensional case, we have the following exponential decay result, which is proven in [3].

Proposition 2.5 (see [3]). *Let $d = 1$ and $f \equiv 0$. Suppose that $u_0 \in H^1(\mathbb{R})$, $u_1 \in L^2(\mathbb{R})$, $\text{supp } u_0, \text{supp } u_1 \subset \Omega_0$ for some bounded $\Omega_0 \subset \mathbb{R}$ and assume $\alpha, \beta, \Omega_{in}$ be as in Assumption 1.1. Then, for any bounded $\Omega \subset \mathbb{R}$, the solution of (2.1) obeys the decay estimate*

$$(2.14) \quad \|u(\cdot, t) - u_\infty\|_{H^1(\Omega)} + \|\partial_t u(\cdot, t)\|_{L^2(\Omega)} \leq C e^{-\Lambda t}, \quad t \geq 0,$$

for some explicit constants $C = C(u_0, u_1, \alpha, \beta, |\Omega_0|, |\Omega_{in}|)$, $\Lambda = \Lambda(\alpha, \beta, |\Omega_{in}|) > 0$ with $|\Omega_0|, |\Omega_{in}|$ denoting the Lebesgue measure of the sets Ω_0, Ω_{in} , respectively, and

$$(2.15) \quad u_\infty := \frac{1}{2\sqrt{\alpha_0\beta_0}} \int_{\Omega_0} u_1(x) \beta(x) dx.$$

3. PROOF OF THE LAP (THEOREMS 1.4 AND 1.5)

In this section, we prove Theorems 1.4 and 1.5 at once. Without loss of generality, we can assume that $\Omega = \Omega_{in}$, since both domains could be enlarged to their union without changing the problem. We also suppose that the origin $\mathbf{x} = \mathbf{0}$ is chosen to be inside Ω .

The proof is given in two steps. In Step 1, see Section 3.1, we transform problem (1.2) into an initial-value problem with zero source term for the difference

$$(3.1) \quad W(\mathbf{x}, t) := u(\mathbf{x}, t) - e^{-i\omega t} U(\mathbf{x}),$$

where $u(\mathbf{x}, t)$ and $U(\mathbf{x})$ solve problems (1.2) and (1.1), respectively. In Section 3.2, we observe that the problem introduced in Step 1 has poorly localised initial data, and we write some asymptotic representation, which will be useful in what follows. In Step 2, see Section 3.3, we decompose the problem in Step 1 into several subproblems. We distinguish the cases $d = 1$ and $d \geq 2$. In the former case, the arguments are more transparent and lead to the quantitative result of Theorem 1.5. The higher-dimensional case is more involved, as some of the subproblems do not have sufficiently localised initial data and thus require the more specific time-decay results given in Section 2.

3.1. Step 1: Transformation into an auxiliary homogeneous problem. By inspection, we see that $W(\mathbf{x}, t)$ defined by (3.1) satisfies

$$(3.2) \quad \begin{cases} \partial_t^2 W(\mathbf{x}, t) - \beta^{-1}(\mathbf{x}) \nabla \cdot (\alpha(\mathbf{x}) \nabla W(\mathbf{x}, t)) = 0, & \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ W(\mathbf{x}, 0) = -U(\mathbf{x}), \quad \partial_t W(\mathbf{x}, 0) = i\omega U(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d. \end{cases}$$

Completing the proofs of Theorems 1.4 and 1.5 is tantamount to showing that there exists a unique constant $U_\infty \in \mathbb{C}$ explicitly given by (1.4) and constants $\Lambda, C > 0$ depending on F, α, β, ω such that

for $d = 1$:

$$(3.3) \quad \|W(\cdot, t) - U_\infty\|_{H^1(\Omega)} + \|\partial_t W(\cdot, t)\|_{L^2(\Omega)} \leq C e^{-\Lambda t}, \quad t \geq 0,$$

for $d = 2$:

$$(3.4) \quad \|W(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t W(\cdot, t)\|_{L^2(\Omega)} \leq C \frac{1 + \log(1 + t^2)}{(1 + t^2)^{1/2}}, \quad t \geq 0,$$

for $d = 3$:

$$(3.5) \quad \|W(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t W(\cdot, t)\|_{L^2(\Omega)} \leq \frac{C}{(1 + t^2)^{1/2}}, \quad t \geq 0.$$

3.2. Slow decay of the initial data of problem (3.2). One immediate difficulty when dealing with (3.2) is that the initial data $W(\cdot, 0)$ and $\partial_t W(\cdot, 0)$ do not belong to $H^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, respectively. The slow decay of the initial conditions in (3.2) can be seen as follows. Let us rewrite (1.1) as the constant-coefficient problem

$$(3.6) \quad \begin{aligned} -\Delta U(\mathbf{x}) - \frac{\omega^2}{c_0^2} U(\mathbf{x}) &= \frac{1}{\alpha_0} [\beta(\mathbf{x}) F(\mathbf{x}) + (\beta(\mathbf{x}) - \beta_0) \omega^2 U(\mathbf{x}) + \\ &\quad + \nabla \cdot (\alpha(\mathbf{x}) \nabla U(\mathbf{x})) - \alpha_0 \Delta U(\mathbf{x})] \\ &=: F_1(\mathbf{x}), \end{aligned}$$

where we recall that $c_0^2 = \alpha_0/\beta_0$. Assumptions 1.1 and 1.3 on the coefficients and on F imply that $F_1(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathbb{R}^d \setminus \bar{\Omega}$. Moreover, since the coefficients α and β are smooth and bounded away from zero, and $F \in L^2(\mathbb{R}^d)$, standard well-posedness results (see e.g. [12, Sec. 6.3.1]) give $U \in H^2(\Omega)$, and hence $F_1 \in L^1(\Omega)$. Therefore, we can write the integral representation of the solution U in $\mathbb{R}^d \setminus \bar{\Omega}$

$$(3.7) \quad U(\mathbf{x}) = \int_{\Omega} K(\mathbf{x} - \mathbf{y}) F_1(\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^d \setminus \bar{\Omega}.$$

Here

$$(3.8) \quad K(\mathbf{x}) := \frac{i}{4} \left(\frac{\omega}{2\pi c_0} \right)^{\frac{d-2}{2}} \frac{1}{|\mathbf{x}|^{\frac{d-2}{2}}} H_{\frac{d-2}{2}}^{(1)} \left(\frac{\omega}{c_0} |\mathbf{x}| \right)$$

is the Green's function (see e.g. [11]) that satisfies the Sommerfeld radiation condition $\lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}|^{\frac{d-1}{2}} \left[\partial_{|\mathbf{x}|} K(\mathbf{x}) - i \frac{\omega}{c_0} K(\mathbf{x}) \right] = 0$ and $-\Delta K(\mathbf{x}) - \frac{\omega^2}{c_0^2} K(\mathbf{x}) = \delta(\mathbf{x})$, with

δ being the d -dimensional Dirac delta function. In (3.8), $H_p^{(1)}$ denotes the Hankel function of the first kind of order p . Since \mathbf{y} in (3.7) ranges in a bounded set and $F_1 \in L^1(\Omega)$, we employ Lemma A.1 in the appendix and deduce that

$$(3.9) \quad U(\mathbf{x}) = \mathcal{O}\left(1/|\mathbf{x}|^{(d-1)/2}\right), \quad \partial_{|\mathbf{x}|} U(\mathbf{x}) - \frac{i\omega}{c_0} U(\mathbf{x}) = \mathcal{O}\left(1/|\mathbf{x}|^{(d+1)/2}\right), \quad |\mathbf{x}| \gg 1.$$

This implies that U , and therefore $W(\cdot, 0)$ and $\partial_t W(\cdot, 0)$, do not belong to $L^2(\mathbb{R}^d)$. At the same time, this gives a precise decay rate in the Sommerfeld radiation condition when the source term F_1 in (3.6) is compactly supported.

3.3. Step 2. Time-decay by decomposition into subproblems. In order to deal with the slowly decaying initial data in (3.2) discussed in Section 3.2, we perform some auxiliary decompositions. As $\mathbf{0} \in \Omega$, we can fix R large enough and $\epsilon > 0$ such that Ω is contained in the open ball $\mathbb{B}_{R-\epsilon} \subset \mathbb{R}^d$ of radius $R-\epsilon$ and center $\mathbf{x} = \mathbf{0}$. Let $\{\eta_0, \eta_1\}$ be a smooth, radial partition of unity, i.e. $\eta_0(\mathbf{x}) = \eta_0(|\mathbf{x}|)$, $\eta_1(\mathbf{x}) = \eta_1(|\mathbf{x}|)$, and $\eta_0(|\mathbf{x}|) + \eta_1(|\mathbf{x}|) = 1$ for all $\mathbf{x} \in \mathbb{R}^d$, such that

$$\eta_0(|\mathbf{x}|) = \begin{cases} 0, & |\mathbf{x}| < R - \epsilon, \\ 1, & |\mathbf{x}| > R, \end{cases} \quad \eta_1(|\mathbf{x}|) = \begin{cases} 1, & |\mathbf{x}| < R - \epsilon, \\ 0, & |\mathbf{x}| > R. \end{cases}$$

We proceed separately with the case $d = 1$ and the cases $d = 2, 3$.

• **Case $d = 1$ (Theorem 1.5).**

Note that we have $H_{-1/2}^{(1)}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{ix}$, and hence (3.8) yields, for $d = 1$, $K(|x|) = \frac{i c_0}{2 \omega} e^{i \frac{\omega}{c_0} |x|}$. In this case, the Green function K does not decay at infinity, but the radiation conditions on K , and thus on U , are exact, i.e. for $x \notin \Omega$, we have $c_0 \partial_{|x|} U(x) = i \omega U(x)$, where $\partial_{|x|} \equiv (\text{sgn } x) \partial_x$. Therefore, we can write

$$(3.10) \quad W(x, t) = \widetilde{W}_0(x, t) + \widetilde{W}_1(x, t),$$

where $\widetilde{W}_0(x, t)$, $\widetilde{W}_1(x, t)$ solve the following initial-value problems, respectively:

$$(3.11) \quad \begin{cases} \partial_t^2 \widetilde{W}_0(x, t) - \beta^{-1}(x) \partial_x \left(\alpha(x) \partial_x \widetilde{W}_0(x, t) \right) = 0, & x \in \mathbb{R}, \quad t > 0, \\ \widetilde{W}_0(x, 0) = -\eta_0(|x|) U(x), \quad \partial_t \widetilde{W}_0(x, 0) = c_0 \partial_{|x|} \left(\eta_0(|x|) U(x) \right), & x \in \mathbb{R}, \end{cases}$$

$$(3.12) \quad \begin{cases} \partial_t^2 \widetilde{W}_1(x, t) - \beta^{-1}(x) \partial_x \left(\alpha(x) \partial_x \widetilde{W}_1(x, t) \right) = 0, & x \in \mathbb{R}, \quad t > 0, \\ \widetilde{W}_1(x, 0) = -\eta_1(|x|) U(x), \quad \partial_t \widetilde{W}_1(x, 0) = \left(c_0 \partial_{|x|} \eta_1(|x|) + i \omega \eta_1(|x|) \right) U(x), & x \in \mathbb{R}. \end{cases}$$

Observe that problem (3.11), whose initial data are supported outside $\mathbb{B}_{R-\epsilon}$, is solved by a linear combination of two reflection-free outgoing waves

$$(3.13) \quad \begin{aligned} \widetilde{W}_0(x, t) = & -H(x - c_0 t) \eta_0(x - c_0 t) U(x - c_0 t) \\ & - H(-x - c_0 t) \eta_0(-x - c_0 t) U(x + c_0 t), \end{aligned}$$

where H is the Heaviside step function. Note that the smoothness of the solution is not affected by the discontinuity of the Heaviside function due to the vanishing

of η_0 . Because of the support property of η_0 , by inspection of (3.13), we have that

$$(3.14) \quad \widetilde{W}_0(x, t) = \partial_t \widetilde{W}_0(x, t) \equiv 0, \quad x \in \Omega, \quad t > 0.$$

To deal with \widetilde{W}_1 in (3.10), we observe that the initial data of (3.12) have compact support. Hence, problem (3.12) is amenable to the application of Proposition 2.5, which yields

$$(3.15) \quad \left\| \widetilde{W}_1(\cdot, t) - U_\infty \right\|_{H^1(\Omega)} + \left\| \partial_t \widetilde{W}_1(\cdot, t) \right\|_{L^2(\Omega)} \leq C e^{-\Lambda t}, \quad t \geq 0,$$

$$(3.16) \quad \begin{aligned} U_\infty &:= \frac{1}{2\sqrt{\alpha_0\beta_0}} \int_{-R}^R \left(c_0 \partial_{|x|} \eta_1(|x|) + i\omega \eta_1(|x|) \right) U(x) \beta(x) dx \\ &= \frac{i\omega}{2\sqrt{\alpha_0\beta_0}} \int_{-R+\epsilon}^{R-\epsilon} U(x) \beta(x) dx - \frac{1}{2} [U(R-\epsilon) + U(-R+\epsilon)] \\ &= \frac{1}{2i\omega\sqrt{\alpha_0\beta_0}} \int_{-R+\epsilon}^{R-\epsilon} F(x) \beta(x) dx \end{aligned}$$

for some constants $C, \Lambda > 0$. Note that in passing from the first to the second line in (3.16), η_1 disappears upon integration by parts using that $\partial_{|x|} U(x) = i\omega/c_0 U(x)$ and $\beta(x) \equiv \beta_0$ for $x \in [-R, -R+\epsilon] \cup [R-\epsilon, R]$. The passage from the second to the third line of the equality is justified upon integration of (1.1) in x over the interval $(-R+\epsilon, R-\epsilon)$ and using again the exact radiation conditions at its endpoints.

Together with (3.14) and (3.10), estimate (3.15) implies (3.3) which completes the proof of Theorem 1.5.

• **Cases $d = 2, 3$ (Theorem 1.4).**

We perform a decomposition, which is similar to (3.10) but contains more terms that have to be treated in a more delicate fashion. Namely, we write

$$(3.17) \quad W(\mathbf{x}, t) = \sum_{k=0}^4 W_k(\mathbf{x}, t),$$

where each W_k , $k = 0, \dots, 4$, solves the homogeneous wave equation

$$\partial_t^2 W_k(\mathbf{x}, t) - \beta^{-1}(\mathbf{x}) \nabla \cdot (\alpha(\mathbf{x}) \nabla W_k(\mathbf{x}, t)) = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0,$$

subject to the following initial conditions on \mathbb{R}^d , respectively:

(3.18)

$$\begin{aligned} W_0(\mathbf{x}, 0) &= -\eta_0(|\mathbf{x}|) U_0(|\mathbf{x}|), & \partial_t W_0(\mathbf{x}, 0) &= c_0 \partial_{|\mathbf{x}|} \left(\eta_0(|\mathbf{x}|) U_0(\mathbf{x}) \right), \\ W_1(\mathbf{x}, 0) &= \eta_0(|\mathbf{x}|) (U_0(\mathbf{x}) - U(\mathbf{x})), & \partial_t W_1(\mathbf{x}, 0) &= -c_0 \eta_0(|\mathbf{x}|) \partial_{|\mathbf{x}|} \left(U_0(\mathbf{x}) - U(\mathbf{x}) \right), \\ W_2(\mathbf{x}, 0) &= 0, & \partial_t W_2(\mathbf{x}, 0) &= -c_0 \eta_0(|\mathbf{x}|) \left(\partial_{|\mathbf{x}|} U(\mathbf{x}) - \frac{i\omega}{c_0} U(\mathbf{x}) \right), \\ W_3(\mathbf{x}, 0) &= 0, & \partial_t W_3(\mathbf{x}, 0) &= -c_0 \eta_0'(|\mathbf{x}|) U_0(\mathbf{x}), \\ W_4(\mathbf{x}, 0) &= -\eta_1(|\mathbf{x}|) U(\mathbf{x}), & \partial_t W_4(\mathbf{x}, 0) &= i\omega \eta_1(|\mathbf{x}|) U(\mathbf{x}). \end{aligned}$$

Here, we have introduced U_0 , the leading term in the long-range asymptotic expansion of (3.7). More precisely, according to representation (3.7) and Lemma A.1, we have

$$(3.19) \quad U_0(\mathbf{x}) = \frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{4\pi|\mathbf{x}|^{\frac{d-1}{2}}} \left(\frac{\omega}{2\pi i c_0} \right)^{\frac{d-3}{2}} \int_{\Omega} e^{-i\frac{\omega}{c_0} \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|}} F_1(\mathbf{y}) d\mathbf{y}.$$

Furthermore, for $|\mathbf{x}| \gg 1$,

(3.20)

$$\begin{aligned} U(\mathbf{x}) - U_0(\mathbf{x}) &= \frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{4\pi|\mathbf{x}|^{\frac{d+1}{2}}} \left(\frac{\omega}{2\pi i c_0} \right)^{\frac{d-3}{2}} \int_{\Omega} e^{-i\frac{\omega}{c_0} \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|}} \left[(d-3)(d-1) \frac{ic_0}{8\omega} \right. \\ &\quad \left. + \frac{d-1}{2} \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|} + \frac{i\omega}{2c_0} \left(|\mathbf{y}|^2 - \left(\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|} \right)^2 \right) \right] F_1(\mathbf{y}) d\mathbf{y} + \mathcal{O} \left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}} \right), \end{aligned}$$

(3.21)

$$\begin{aligned} \partial_{|\mathbf{x}|} [U(\mathbf{x}) - U_0(\mathbf{x})] &= -\frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{4\pi|\mathbf{x}|^{\frac{d+1}{2}}} \left(\frac{\omega}{2\pi i c_0} \right)^{\frac{d-3}{2}} \int_{\Omega} e^{-i\frac{\omega}{c_0} \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|}} \left[\frac{(d-3)(d-1)}{8} \right. \\ &\quad \left. - \frac{d-1}{2} \frac{i\omega}{c_0} \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|} + \frac{\omega^2}{2c_0^2} \left(|\mathbf{y}|^2 - \left(\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|} \right)^2 \right) \right] F_1(\mathbf{y}) d\mathbf{y} \\ &\quad + \mathcal{O} \left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}} \right), \end{aligned}$$

(3.22)

$$\partial_{|\mathbf{x}|} U(\mathbf{x}) - \frac{i\omega}{c_0} U(\mathbf{x}) = \frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{4\pi|\mathbf{x}|^{\frac{d+1}{2}}} \left(\frac{\omega}{2\pi i c_0} \right)^{\frac{d-3}{2}} \frac{1-d}{2} \int_{\Omega} e^{-i\frac{\omega}{c_0} \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|}} F_1(\mathbf{y}) d\mathbf{y} + \mathcal{O} \left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}} \right).$$

Observe that the problems for W_3 and W_4 have compactly supported smooth initial data. Hence, Proposition 2.1 applies to give

$$(3.23) \quad \|W_k(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t W_k(\cdot, t)\|_{L^2(\Omega)} \leq \frac{C}{(1+t^2)^{\frac{d-1}{2}}}, \quad t \geq 0, \quad k = 3, 4,$$

for some constant $C > 0$.

In order to obtain similar decay estimates for W_1 and W_2 , let us introduce

$$(3.24) \quad Z_{12}(\mathbf{x}, t) := W_{12}(\mathbf{x}, t) - V_{12}(\mathbf{x}, t),$$

where $W_{12}(\mathbf{x}, t) := W_1(\mathbf{x}, t) + W_2(\mathbf{x}, t)$ and $V_{12}(\mathbf{x}, t)$ is the unique solution to the constant-coefficient problem

$$(3.25) \quad \begin{cases} \partial_t^2 V_{12}(\mathbf{x}, t) - c_0^2 \Delta V_{12}(\mathbf{x}, t) = 0, & \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ V_{12}(\mathbf{x}, 0) = -\eta_0(|\mathbf{x}|)(U_0(\mathbf{x}) - U(\mathbf{x})), & \mathbf{x} \in \mathbb{R}^d, \\ \partial_t V_{12}(\mathbf{x}, 0) = c_0 \eta_0(|\mathbf{x}|) \left(\frac{i\omega}{c_0} U(\mathbf{x}) - \partial_{|\mathbf{x}|} U_0(\mathbf{x}) \right), & \mathbf{x} \in \mathbb{R}^d. \end{cases}$$

We are going to deduce the decay of $W_{12}(\mathbf{x}, t)$ from the decay of $Z_{12}(\mathbf{x}, t)$ and $V_{12}(\mathbf{x}, t)$. First of all, since

$$\frac{i\omega}{c_0} U(\mathbf{x}) - \partial_{|\mathbf{x}|} U_0(\mathbf{x}) = \partial_{|\mathbf{x}|} [U(\mathbf{x}) - U_0(\mathbf{x})] - \left(\partial_{|\mathbf{x}|} U(\mathbf{x}) - \frac{i\omega}{c_0} U(\mathbf{x}) \right),$$

we see from (3.20)–(3.22) that the initial conditions of (3.25) satisfy the assumptions of Lemma 2.3 with

$$A_0(|\mathbf{x}|) := \eta_0(|\mathbf{x}|) \frac{e^{\frac{i\omega}{c_0}|\mathbf{x}|}}{|\mathbf{x}|^{\frac{d+1}{2}}},$$

$$Y_0\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) := \frac{1}{4\pi} \left(\frac{\omega}{2\pi i c_0}\right)^{\frac{d-3}{2}} \int_{\Omega} e^{-i\frac{\omega}{c_0} \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|}} \left[(d-3)(d-1) \frac{ic_0}{8\omega} + \frac{d-1}{2} \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|} + \frac{i\omega}{2c_0} \left(|\mathbf{y}|^2 - \left(\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|}\right)^2 \right) \right] F_1(\mathbf{y}) d\mathbf{y},$$

$$Y_1\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) := -\frac{c_0}{4\pi} \left(\frac{\omega}{2\pi i c_0}\right)^{\frac{d-3}{2}} \int_{\Omega} e^{-i\frac{\omega}{c_0} \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|}} \left[\frac{(d-7)(d-1)}{8} - \frac{d-1}{2} \frac{i\omega}{c_0} \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|} + \frac{\omega^2}{2c_0^2} \left(|\mathbf{y}|^2 - \left(\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|}\right)^2 \right) \right] F_1(\mathbf{y}) d\mathbf{y}.$$

This entails that the following decay estimate are valid uniformly in $\mathbf{x} \in \Omega$ for $t \geq 0$:

$$(3.26) \quad |V_{12}(\mathbf{x}, t)| + |\nabla V_{12}(\mathbf{x}, t)| + |\partial_t V_{12}(\mathbf{x}, t)| \\ + |\Delta V_{12}(\mathbf{x}, t)| + |\partial_t \nabla V_{12}(\mathbf{x}, t)| + |\partial_t \Delta V_{12}(\mathbf{x}, t)| \leq \frac{C}{(1+t^2)^{1/2}}.$$

with some constant $C > 0$. In particular, (3.26) implies

$$(3.27) \quad \|V_{12}(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t V_{12}(\cdot, t)\|_{L^2(\Omega)} \leq \frac{C}{(1+t^2)^{1/2}}, \quad t \geq 0.$$

In order to deal with $Z_{12}(\mathbf{x}, t)$, we note that it satisfies

$$(3.28) \quad \begin{cases} \partial_t^2 Z_{12}(\mathbf{x}, t) - \beta^{-1}(\mathbf{x}) \nabla \cdot (\alpha(\mathbf{x}) \nabla Z_{12}(\mathbf{x}, t)) = F_{12}(\mathbf{x}, t), & \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ Z_{12}(\mathbf{x}, 0) = 0, \quad \partial_t Z_{12}(\mathbf{x}, 0) = 0, & \mathbf{x} \in \mathbb{R}^d, \end{cases}$$

where

$$(3.29) \quad F_{12}(\mathbf{x}, t) := \beta^{-1}(\mathbf{x}) \nabla \alpha(\mathbf{x}) \cdot \nabla V_{12}(\mathbf{x}, t) + (\beta^{-1}(\mathbf{x}) \alpha(\mathbf{x}) - c_0^2) \Delta V_{12}(\mathbf{x}, t).$$

Estimate (3.26) entails the decay of all the terms entering (3.29) and of their time derivative, so that Proposition 2.2 is applicable with $p = 1$. This gives for $d = 2$:

$$(3.30) \quad \|Z_{12}(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t Z_{12}(\cdot, t)\|_{L^2(\Omega)} \leq C \frac{1 + \log(1+t^2)}{(1+t^2)^{1/2}}, \quad t \geq 0,$$

for $d = 3$:

$$(3.31) \quad \|Z_{12}(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t Z_{12}(\cdot, t)\|_{L^2(\Omega)} \leq \frac{C}{(1+t^2)^{1/2}}, \quad t \geq 0,$$

with some constant $C > 0$. Together with (3.27), estimates (3.30) and (3.31) thus imply

for $d = 2$:

$$(3.32) \quad \|W_{12}(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t W_{12}(\cdot, t)\|_{L^2(\Omega)} \leq C \frac{1 + \log(1+t^2)}{(1+t^2)^{1/2}}, \quad t \geq 0,$$

for $d = 3$:

$$(3.33) \quad \|W_{12}(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t W_{12}(\cdot, t)\|_{L^2(\Omega)} \leq \frac{C}{(1+t^2)^{1/2}}, \quad t \geq 0.$$

Finally, we shall deal with $W_0(\mathbf{x}, t)$. Similarly to (3.24), let us consider the quantity

$$(3.34) \quad Z_0(\mathbf{x}, t) := W_0(\mathbf{x}, t) - V_0(\mathbf{x}, t),$$

where $V_0(\mathbf{x}, t)$ is the unique solution to the constant-coefficient problem

$$(3.35) \quad \begin{cases} \partial_t^2 V_0(\mathbf{x}, t) - c_0^2 \Delta V_0(\mathbf{x}, t) = 0, & \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ V_0(\mathbf{x}, 0) = -\eta_0(|\mathbf{x}|) U_0(\mathbf{x}), \quad \partial_t V_0(\mathbf{x}, 0) = c_0 \partial_{|\mathbf{x}|}(\eta_0(|\mathbf{x}|) U_0(\mathbf{x})), & \mathbf{x} \in \mathbb{R}^d. \end{cases}$$

By setting

$$A(|\mathbf{x}|) := \eta_0(|\mathbf{x}|) \frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{|\mathbf{x}|^{\frac{d-1}{2}}}, \quad Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) := -\frac{1}{4\pi} \left(\frac{\omega}{2\pi i c_0}\right)^{\frac{d-3}{2}} \int_{\Omega} e^{-i\frac{\omega}{c_0}\left(\frac{\mathbf{x}}{|\mathbf{x}|} \cdot \mathbf{y}\right)} F_1(\mathbf{y}) d\mathbf{y},$$

it is easy to see that (3.35) satisfies the assumptions of Lemma 2.4. Therefore, we obtain that the following decay estimate are valid uniformly in $\mathbf{x} \in \Omega$ for $t \geq 0$:

$$(3.36) \quad \begin{aligned} & |V_0(\mathbf{x}, t)| + |\nabla V_0(\mathbf{x}, t)| + |\partial_t V_0(\mathbf{x}, t)| \\ & + |\Delta V_0(\mathbf{x}, t)| + |\partial_t \nabla V_0(\mathbf{x}, t)| + |\partial_t \Delta V_0(\mathbf{x}, t)| \leq \frac{C}{(1+t^2)^{1/2}}, \end{aligned}$$

with some constant $C > 0$. In particular,

$$(3.37) \quad \|V_0(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t V_0(\cdot, t)\|_{L^2(\Omega)} \leq \frac{C}{(1+t^2)^{1/2}}, \quad t \geq 0.$$

Repeating the same reasoning as before, we now deduce the decay of $W_0(\mathbf{x}, t)$ from that of $Z_0(\mathbf{x}, t)$. To do so, we observe that $Z_0(\mathbf{x}, t)$ satisfies

$$(3.38) \quad \begin{cases} \partial_t^2 Z_0(\mathbf{x}, t) - \beta^{-1}(\mathbf{x}) \nabla \cdot (\alpha(\mathbf{x}) \nabla Z_0(\mathbf{x}, t)) = F_0(\mathbf{x}, t), & \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ Z_0(\mathbf{x}, 0) = 0, \quad \partial_t Z_0(\mathbf{x}, 0) = 0, & \mathbf{x} \in \mathbb{R}^d, \end{cases}$$

where

$$(3.39) \quad F_0(\mathbf{x}, t) := \beta^{-1}(\mathbf{x}) \nabla \alpha(\mathbf{x}) \cdot \nabla V_0(\mathbf{x}, t) + (\beta^{-1}(\mathbf{x}) \alpha(\mathbf{x}) - c_0^2) \Delta V_0(\mathbf{x}, t).$$

Estimate (3.36) makes Proposition 2.2 applicable to yield for $d = 2$:

$$(3.40) \quad \|Z_0(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t Z_0(\cdot, t)\|_{L^2(\Omega)} \leq C \frac{1 + \log(1+t^2)}{(1+t^2)^{1/2}}, \quad t \geq 0,$$

for $d = 3$:

$$(3.41) \quad \|Z_0(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t Z_0(\cdot, t)\|_{L^2(\Omega)} \leq \frac{C}{(1+t^2)^{1/2}}, \quad t \geq 0,$$

with some constant $C > 0$. Therefore, recalling (3.37), we have for $d = 2$:

$$(3.42) \quad \|W_0(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t W_0(\cdot, t)\|_{L^2(\Omega)} \leq C \frac{1 + \log(1+t^2)}{(1+t^2)^{1/2}}, \quad t \geq 0,$$

for $d = 3$:

$$(3.43) \quad \|W_0(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t W_0(\cdot, t)\|_{L^2(\Omega)} \leq \frac{C}{(1+t^2)^{1/2}}, \quad t \geq 0.$$

Consequently, by combining (3.42), (3.43), (3.23), (3.32), and (3.33) with (3.17), we obtain estimates (3.4) and (3.5) hold true. This concludes the proof of Theorem 1.4.

4. PROOFS OF THE AUXILIARY TIME DECAY RESULTS

4.1. Proof of Proposition 2.1. This proof is based on an application and an extension of a decay result from [6].

Because of Assumption 1.1 on α and β (positivity and regularity), the operator $P := -\beta^{-1}(\mathbf{x}) \nabla \cdot (\alpha(\mathbf{x}) \nabla)$ is formally self-adjoint in $L^2_\beta(\mathbb{R}^d)$, the $L^2(\mathbb{R}^d)$ space endowed with the β -weighted L^2 inner product. Note that the sets $L^2_\beta(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ coincide since the weight β is bounded and uniformly bounded away from zero. Moreover, P is positive so that there exists a unique self-adjoint, positive operator B such that $B^2 = P$. With the notation $\sqrt{P} := B$ and $1/\sqrt{P} := B^{-1}$, we can formally write the solution of (2.1) with $f \equiv 0$ as

$$(4.1) \quad u(\mathbf{x}, t) = \cos\left(t\sqrt{P}\right) u_0(\mathbf{x}) + \frac{\sin\left(t\sqrt{P}\right)}{\sqrt{P}} u_1(\mathbf{x}), \quad t \geq 0.$$

Under Assumptions 1.1 and 1.2 on α and β (compactly supported derivatives and nontrapping), the following operator-norm estimates are obtained in [6, Thm. 1.5]. Namely, there exists a constant $C > 0$ such that

$$(4.2) \quad \left\| q_\nu^{-1} \frac{\sin\left(t\sqrt{P}\right)}{\sqrt{P}} q_\nu^{-1} \right\|_{L^2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq \frac{C}{(1+t^2)^{\frac{d-1}{2}}}, \quad t \geq 0,$$

$$(4.3) \quad \left\| q_\nu^{-1} \cos\left(t\sqrt{P}\right) q_\nu^{-1} \right\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq \frac{C}{(1+t^2)^{\frac{d}{2}}}, \quad t \geq 0,$$

where $q_\nu := (1 + |\mathbf{x}|^2)^{\nu/2}$ with some $\nu > d + 1$.

Set $\mu := d + 1 + \epsilon$. According to (2.3), we have $q_\mu u_0 \in H^1(\mathbb{R}^d)$ and $q_\mu u_1 \in L^2(\mathbb{R}^d)$. Then, we deduce from (4.1)–(4.3) that, for $t \geq 0$,

$$(4.4) \quad \begin{aligned} \|q_\mu^{-1} u(\cdot, t)\|_{L^2(\mathbb{R}^d)} &\leq C \left(\frac{1}{(1+t^2)^{\frac{d}{2}}} \|q_\mu u_0\|_{H^1(\mathbb{R}^d)} + \frac{1}{(1+t^2)^{\frac{d-1}{2}}} \|q_\mu u_1\|_{L^2(\mathbb{R}^d)} \right) \\ &\leq \frac{C_0}{(1+t^2)^{\frac{d-1}{2}}} \end{aligned}$$

for some constant $C_0 > 0$.

To obtain the estimate for the time derivative $\partial_t u$, we note that $w := \partial_t u$ solves $\partial_t^2 w + Pw = 0$, $w(\mathbf{x}, 0) = u_1(\mathbf{x})$, $\partial_t w(\mathbf{x}, 0) = -Pu_0(\mathbf{x})$. Hence, we have

$$w(\cdot, t) = \cos\left(t\sqrt{P}\right) u_1 - \frac{\sin\left(t\sqrt{P}\right)}{\sqrt{P}} (Pu_0).$$

Therefore, using (4.2) and (4.3), and the assumed regularity of u_0, u_1 , we estimate, for $t \geq 0$,

(4.5)

$$\begin{aligned} \|q_\mu^{-1} \partial_t u(\cdot, t)\|_{L^2(\mathbb{R}^d)} &\leq C \left(\frac{1}{(1+t^2)^{\frac{d}{2}}} \|q_\mu u_1\|_{H^1(\mathbb{R}^d)} + \frac{1}{(1+t^2)^{\frac{d-1}{2}}} \|q_\mu Pu_0\|_{L^2(\mathbb{R}^d)} \right) \\ &\leq \frac{C_1}{(1+t^2)^{\frac{d-1}{2}}} \end{aligned}$$

for some constant $C_1 > 0$.

To complete the H^1 -estimate of u , we estimate the L^2 -norm of ∇u . First, we observe that $\tilde{w} := \partial_t^2 u$ solves $\partial_t^2 \tilde{w} + P\tilde{w} = 0$, $\tilde{w}(\mathbf{x}, 0) = -Pu_0(\mathbf{x})$, $\partial_t \tilde{w}(\mathbf{x}, 0) = -Pu_1(\mathbf{x})$. Hence, as before, we have, for $t \geq 0$,

$$\tilde{w}(\cdot, t) = -\cos\left(t\sqrt{P}\right) (Pu_0) - \frac{\sin\left(t\sqrt{P}\right)}{\sqrt{P}} (Pu_1),$$

$$\|q_\mu^{-1} \partial_t^2 u(\cdot, t)\|_{L^2(\mathbb{R}^d)} \leq C \left(\frac{1}{(1+t^2)^{\frac{d}{2}}} \|q_\mu Pu_0\|_{H^1(\mathbb{R}^d)} + \frac{1}{(1+t^2)^{\frac{d-1}{2}}} \|q_\mu Pu_1\|_{L^2(\mathbb{R}^d)} \right).$$

Hence, we arrive at

$$(4.6) \quad \|q_\mu^{-1} Pu(\cdot, t)\|_{L^2(\mathbb{R}^d)} = \|q_\mu^{-1} \partial_t^2 u(\cdot, t)\|_{L^2(\mathbb{R}^d)} \leq \frac{C_2}{(1+t^2)^{\frac{d-1}{2}}}$$

for some constant $C_2 > 0$. Employing the notation $\overline{(\cdot)}$ for the complex conjugate, we consider the following inner product on $L_\beta^2(\mathbb{R}^d)$

$$(4.7) \quad \langle q_\mu^{-1} Pu, q_\mu^{-1} u \rangle_{L_\beta^2(\mathbb{R}^d)} = - \int_{\mathbb{R}^d} \nabla \cdot \left(\alpha(\mathbf{x}) \nabla u(\mathbf{x}, t) \right) \overline{u(\mathbf{x}, t)} q_\mu^{-2}(\mathbf{x}) d\mathbf{x}.$$

Inserting the identities

$$\begin{aligned} \nabla \cdot \left(\alpha(\mathbf{x}) \nabla u(\mathbf{x}, t) \right) \overline{u(\mathbf{x}, t)} q_\mu^{-2}(\mathbf{x}) &= \nabla \cdot \left(\alpha(\mathbf{x}) q_\mu^{-2}(\mathbf{x}) \overline{u(\mathbf{x}, t)} \nabla u(\mathbf{x}, t) \right) \\ &\quad - \alpha(\mathbf{x}) |\nabla u(\mathbf{x}, t)|^2 q_\mu^{-2}(\mathbf{x}) \\ &\quad - \left(\nabla u(\mathbf{x}, t) \cdot \nabla q_\mu^{-2}(\mathbf{x}) \right) \alpha(\mathbf{x}) \overline{u(\mathbf{x}, t)}, \end{aligned}$$

$$(\nabla u(\mathbf{x}, t) \cdot \nabla q_\mu^{-2}(\mathbf{x})) = -2\mu \frac{|\mathbf{x}|}{1 + |\mathbf{x}|^2} q_\mu^{-2}(\mathbf{x}) \partial_{|\mathbf{x}|} u(\mathbf{x}, t)$$

into (4.8) and integrating by parts the term with the divergence give

$$(4.8) \quad \alpha_{\min} \|q_\mu^{-1} \nabla u\|_{L^2(\mathbb{R}^d)}^2 \leq \int_{\mathbb{R}^d} \alpha(\mathbf{x}) |\nabla u(\mathbf{x}, t)|^2 q_\mu^{-2}(\mathbf{x}) d\mathbf{x} \\ \leq \left| \langle q_\mu^{-1} P u, q_\mu^{-1} u \rangle_{L_\beta^2(\mathbb{R}^d)} \right| + \mu \|\alpha\|_{L^\infty(\mathbb{R}^d)} \left| \langle q_\mu^{-1} \partial_{|\mathbf{x}|} u, q_\mu^{-1} u \rangle_{L^2(\mathbb{R}^d)} \right|.$$

Furthermore, employing the Cauchy-Schwarz inequality, we can estimate

$$\left| \langle q_\mu^{-1} P u, q_\mu^{-1} u \rangle_{L_\beta^2(\mathbb{R}^d)} \right| \leq \|\beta\|_{L^\infty(\mathbb{R}^d)} \|q_\mu^{-1} P u\|_{L^2(\mathbb{R}^d)} \|q_\mu^{-1} u\|_{L^2(\mathbb{R}^d)},$$

$$\left| \langle q_\mu^{-1} \partial_{|\mathbf{x}|} u, q_\mu^{-1} u \rangle_{L^2(\mathbb{R}^d)} \right| \leq \|q_\mu^{-1} \nabla u\|_{L^2(\mathbb{R}^d)} \|q_\mu^{-1} u\|_{L^2(\mathbb{R}^d)} \\ \leq \frac{\alpha_{\min}}{4\mu \|\alpha\|_{L^\infty(\mathbb{R}^d)}} \|q_\mu^{-1} \nabla u\|_{L^2(\mathbb{R}^d)}^2 + \frac{\mu \|\alpha\|_{L^\infty(\mathbb{R}^d)}}{\alpha_{\min}} \|q_\mu^{-1} u\|_{L^2(\mathbb{R}^d)}^2$$

Here, on the second line, we used the elementary inequality $|a| |b| \leq \frac{\delta}{2} |a|^2 + \frac{1}{2\delta} |b|^2$ valid for any $\delta > 0$. Therefore, estimate (4.8) entails

$$\frac{3}{4} \alpha_{\min} \|q_\mu^{-1} \nabla u\|_{L^2(\mathbb{R}^d)}^2 \leq \|\beta\|_{L^\infty(\mathbb{R}^d)} \|q_\mu^{-1} P u\|_{L^2(\mathbb{R}^d)} \|q_\mu^{-1} u\|_{L^2(\mathbb{R}^d)} \\ + \frac{\mu^2}{\alpha_{\min}} \|\alpha\|_{L^\infty(\mathbb{R}^d)}^2 \|q_\mu^{-1} u\|_{L^2(\mathbb{R}^d)}^2.$$

Recalling (4.4) and (4.6), this leads to

$$(4.9) \quad \|q_\mu^{-1} \nabla u\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{4}{3} \left(\frac{C_0 C_2}{\alpha_{\min}} \|\beta\|_{L^\infty(\mathbb{R}^d)} + \frac{\mu^2 C_0^2}{\alpha_{\min}^2} \|\alpha\|_{L^\infty(\mathbb{R}^d)}^2 \right) \frac{1}{(1 + t^2)^{d-1}}.$$

Finally, denoting χ_Ω the characteristic function of the set Ω , we have

$$\|u\|_{L^2(\Omega)} = \|u \chi_\Omega\|_{L^2(\mathbb{R}^d)} \leq C_{\Omega, \mu} \|q_\mu^{-1} u\|_{L^2(\mathbb{R}^d)},$$

and similarly for ∇u and $\partial_t u$. Hence, the estimates (4.4), (4.5), and (4.9) furnish (2.4). \square

4.2. Proof of Proposition 2.2. Without loss of generality, we can take $\Omega = \Omega_f$ (by enlarging both sets if necessary). Let P , \sqrt{P} , and $1/\sqrt{P}$ be defined as at

the beginning of Section 2.1. The following operator-norm estimate was obtained in [6, Thm. 1.5]:

$$(4.10) \quad \left\| \frac{\sin(t\sqrt{P})}{\sqrt{P}} \chi_\Omega \right\|_{L^2(\mathbb{R}^d) \rightarrow H^1(\Omega)} \leq \frac{C_0}{(1+t^2)^{\frac{d-1}{2}}}, \quad t \geq 0,$$

for some $C_0 > 0$, where χ_Ω denotes the characteristic function of the set Ω .

According to the Duhamel principle, the solution to (2.1) with $u_0 \equiv 0$, $u_1 \equiv 0$ can be written as

$$(4.11) \quad u(\cdot, t) = \int_0^t \frac{\sin((t-\tau)\sqrt{P})}{\sqrt{P}} f(\cdot, \tau) d\tau.$$

Using a basic Bochner integral estimate in $H^1(\Omega)$ and (4.10), we obtain, for $t > 0$,

$$(4.12) \quad \begin{aligned} \|u(\cdot, t)\|_{H^1(\Omega)} &\leq \int_0^t \left\| \frac{\sin((t-\tau)\sqrt{P})}{\sqrt{P}} f(\cdot, \tau) \right\|_{H^1(\Omega)} d\tau \\ &\leq \int_0^t \frac{C_0}{(1+(t-\tau)^2)^{\frac{d-1}{2}}} \|f(\cdot, \tau)\|_{L^2(\Omega)} d\tau, \end{aligned}$$

where in the second line we also took into account the assumption that the support of $f(\cdot, \tau)$ is contained in Ω for each $\tau > 0$.

Employing the assumed estimate (2.5) on f , namely $\|f(\cdot, \tau)\|_{L^2(\Omega)} \leq C_f / (1+\tau^2)^{p/2}$ for some constants $C_f, p > 0$ and all $\tau > 0$, and denoting $C := C_0 C_f$, we proceed to estimate

$$(4.13) \quad \begin{aligned} \|u(\cdot, t)\|_{H^1(\Omega)} &\leq \int_0^t \frac{C d\tilde{\tau}}{(1+(t-\tilde{\tau})^2)^{\frac{d-1}{2}} (1+\tilde{\tau}^2)^{\frac{p}{2}}} \\ &= \frac{C}{t^{d+p-2}} \left[\int_0^{1/2} \frac{d\tau}{(1/t^2 + (1-\tau)^2)^{\frac{d-1}{2}} (1/t^2 + \tau^2)^{\frac{p}{2}}} \right. \\ &\quad \left. + \int_{1/2}^1 \frac{d\tau}{(1/t^2 + (1-\tau)^2)^{\frac{d-1}{2}} (1/t^2 + \tau^2)^{\frac{p}{2}}} \right] \\ &\leq \frac{2^{d-1}C}{t^{p-1}} \frac{1}{(1+t^2)^{\frac{d-1}{2}}} \int_0^{1/2} \frac{d\tau}{(1/t^2 + \tau^2)^{\frac{p}{2}}} \\ &\quad + \frac{2^p C}{t^{d-2}} \frac{1}{(1+t^2)^{\frac{p}{2}}} \int_0^{1/2} \frac{d\tau}{(1/t^2 + \tau^2)^{\frac{d-1}{2}}}. \end{aligned}$$

Here we used the change of variable $\tilde{\tau} \mapsto \tau := \tilde{\tau}/t$ and employed the estimates

$$\frac{1/t^{d+p-2}}{(1/t^2 + (1-\tau)^2)^{\frac{d-1}{2}}} \leq \frac{1/t^{d+p-2}}{(1/t^2 + 1/4)^{\frac{d-1}{2}}} = \frac{2^{d-1}}{t^{p-1}} \frac{1}{(4+t^2)^{\frac{d-1}{2}}} \leq \frac{2^{d-1}}{t^{p-1}} \frac{1}{(1+t^2)^{\frac{d-1}{2}}},$$

$$0 \leq \tau \leq \frac{1}{2}, t \geq 0,$$

$$\frac{1/t^{d+p-2}}{(1/t^2 + \tau^2)^{\frac{d-1}{2}}} \leq \frac{1/t^{d+p-2}}{(1/t^2 + 1/4)^{\frac{d-1}{2}}} = \frac{2^p}{t^{d-2}} \frac{1}{(4+t^2)^{\frac{d-1}{2}}} \leq \frac{2^p}{t^{d-2}} \frac{1}{(1+t^2)^{\frac{d-1}{2}}},$$

$$\frac{1}{2} \leq \tau \leq 1, t \geq 0,$$

in the integrals over $[0, 1/2]$ and $[1/2, 1]$, respectively. In the last line of (4.13), we have also made the change of variable $\tau \mapsto 1 - \tau$. Using Lemma A.2, we continue estimate (4.13):

$$(4.14) \quad \|u(\cdot, t)\|_{H^1(\Omega)} \leq \frac{2^q C}{(1+t^2)^{\frac{d-1}{2}}} \begin{cases} C_{1,p} t^{1-p}, & 0 < p < 1, \\ \log(t + \sqrt{1+t^2}), & p = 1, \\ C_{2,p}, & p > 1, \end{cases}$$

$$+ \frac{2^q C}{(1+t^2)^{\frac{d}{2}}} \begin{cases} \log(t + \sqrt{1+t^2}), & d = 2, \\ C_{2,d-1}, & d > 2, \end{cases}$$

where $q := \max(d-1, p)$, $C_{1,s} := \frac{1}{1-s}$, $C_{2,s} := \int_0^\infty \frac{dz}{(1+z^2)^{s/2}}$. We continue by considering separately the cases $d = 2$ and $d > 2$.

Since $C_{1,p} > 1$, estimate (4.14) for $d = 2$ reads

$$(4.15) \quad \|u(\cdot, t)\|_{H^1(\Omega)} \leq 2^q C \begin{cases} C_{1,p} (1 + C_{3,p}) \frac{\log(t + \sqrt{1+t^2})}{(1+t^2)^{\frac{d}{2}}}, & 0 < p < 1, \\ 2 \frac{\log(t + \sqrt{1+t^2})}{(1+t^2)^{\frac{1}{2}}}, & p = 1, \\ (1 + C_{4,p}) \max(C_{2,p}, 1) \frac{1}{(1+t^2)^{\frac{1}{2}}}, & p > 1, \end{cases}$$

$$\leq \tilde{C}_p \begin{cases} \frac{\log(t + \sqrt{1+t^2})}{(1+t^2)^{\frac{d}{2}}}, & 0 < p \leq 1, \\ \frac{1}{(1+t^2)^{\frac{1}{2}}}, & p > 1, \end{cases}$$

where $C_{3,p} := \sup_{t \geq 0} \frac{t^{1-p}}{(1+t^2)^{\frac{1-p}{2}} \log(t + \sqrt{1+t^2})}$, $C_{4,p} := \sup_{t \geq 0} \frac{\log(t + \sqrt{1+t^2})}{(1+t^2)^{\frac{p-1}{2}}}$.

Similarly, when $d > 2$, we have

$$(4.16) \quad \begin{aligned} \|u(\cdot, t)\|_{H^1(\Omega)} &\leq 2^q C \begin{cases} (1 + C_{5,d,p}) \max(C_{1,p}, C_{2,d-1}) \frac{1}{(1+t^2)^{\frac{p}{2}}}, & 0 < p < 1, \\ (1 + C_{4,d-1}) \max(C_{2,d-1}, 1) \frac{1}{(1+t^2)^{\frac{1}{2}}}, & p = 1, \\ 2C_{2,r} \frac{1}{(1+t^2)^{\frac{r}{2}}}, & p > 1, \end{cases} \\ &\leq \tilde{C}_{p,d} \begin{cases} \frac{1}{(1+t^2)^{\frac{p}{2}}}, & 0 < p \leq 1, \\ \frac{1}{(1+t^2)^{\frac{r}{2}}}, & p > 1, \end{cases} \end{aligned}$$

where $r := \min(d-1, p)$, $C_{5,d,p} := \sup_{t \geq 0} \frac{t^{1-p}}{(1+t^2)^{\frac{d-1-p}{2}}}$.

To finish the proof, it remains to obtain the estimate for the time derivative $\partial_t u$. To this effect, we note that $w := \partial_t u$ solves $\partial_t^2 w + Pw = \partial_t f$, $w(\mathbf{x}, 0) = 0$, $\partial_t w(\mathbf{x}, 0) = f(\mathbf{x}, 0)$. Hence, we have

$$\partial_t u(\cdot, t) = w(\cdot, t) = \frac{\sin(t\sqrt{P})}{\sqrt{P}} f(\cdot, 0) + \int_0^t \frac{\sin((t-\tau)\sqrt{P})}{\sqrt{P}} \partial_t f(\cdot, \tau) d\tau,$$

and consequently, from (4.10) and (2.5), we obtain, again with $C = C_0 C_f$,

$$\|\partial_t u(\cdot, t)\|_{H^1(\Omega)} \leq \frac{C}{(1+t^2)^{\frac{d-1}{2}}} + \int_0^t \left\| \frac{\sin((t-\tau)\sqrt{P})}{\sqrt{P}} \partial_t f(\cdot, \tau) \right\|_{H^1(\Omega)} d\tau.$$

Therefore, owing to (2.5), the estimate for $\partial_t u$ can be obtained from the estimates for u given in (4.15) and (4.16) by only adding an extra term, which is the first term on the right-hand side of (4.2). Namely, we have, for $d = 2$,

$$(4.17) \quad \begin{aligned} \|\partial_t u(\cdot, t)\|_{H^1(\Omega)} &\leq \begin{cases} \max(C, \tilde{C}_p) \left[\frac{1}{(1+t^2)^{\frac{1}{2}}} + \frac{\log(t + \sqrt{1+t^2})}{(1+t^2)^{\frac{p}{2}}} \right], & 0 < p \leq 1, \\ (C + \tilde{C}_p) \frac{1}{(1+t^2)^{\frac{1}{2}}}, & p > 1, \end{cases} \\ &\leq 2 \max(C, \tilde{C}_p) \begin{cases} \frac{1 + \log(t^2 + 1)}{(1+t^2)^{\frac{p}{2}}}, & 0 < p \leq 1, \\ \frac{1}{(1+t^2)^{\frac{1}{2}}}, & p > 1. \end{cases} \end{aligned}$$

Here we used the inequalities

$$(4.18) \quad \log\left(t + \sqrt{1+t^2}\right) \leq \log 2 + \frac{1}{2} \log(t^2 + 1) \leq 1 + \log(t^2 + 1), \quad t \geq 0,$$

$$1 + \frac{1}{1 + \log(t^2 + 1)} \frac{1}{(1+t^2)^{\frac{1-p}{2}}} \leq 2, \quad t \geq 0, 0 < p \leq 1.$$

Similarly, using the inequalities

$$1 + \frac{1}{(1+t^2)^{\frac{d-1-p}{2}}} \leq 2, \quad t \geq 0, d > 2, 0 < p \leq 1,$$

$$1 + \frac{1}{(1+t^2)^{\frac{d-1-r}{2}}} \leq 2, \quad t \geq 0, r \leq d-1,$$

we have, for $d > 2$,

$$(4.19) \quad \|\partial_t u(\cdot, t)\|_{H^1(\Omega)} \leq 2 \max\left(C, \tilde{C}_{p,d}\right) \begin{cases} \frac{1}{(1+t^2)^{\frac{p}{2}}}, & 0 < p \leq 1, \\ \frac{1}{(1+t^2)^{\frac{r}{2}}}, & p > 1. \end{cases}$$

Altogether, when $d = 2$, estimates (4.17) and (4.15) together with the simplifying inequality (4.18) imply (2.6). Analogously, for $d > 2$, estimates (4.19) and (4.16) furnish (2.7). \square

4.3. Proof of Lemma 2.3. In the main body of the proof, we prove that the bound

$$(4.20) \quad |v(\mathbf{x}, t)| \leq \frac{C}{(1+t^2)^{1/2}}, \quad \mathbf{x} \in \Omega, \quad t \geq 0,$$

is valid for some constant $C > 0$, assuming that $v_0 \equiv 0$. The case $v_0 \not\equiv 0$ and the estimate of the other terms in (2.11) is discussed in the final part of this proof. We consider separately the cases $d = 2$ and $d = 3$.

- Case $d = 2$.

The solution is given by the Poisson's formula [12, Par. 2.4.1 (c)]

$$(4.21) \quad v(\mathbf{x}, t) = \frac{t}{2\pi} \int_0^1 \frac{r}{(1-r^2)^{1/2}} \int_{|\mathbf{s}|=1} v_1(\mathbf{x} + \mathbf{s}rc_0t) d\sigma_{\mathbf{s}} dr,$$

where $d\sigma_{\mathbf{s}}$ denotes the surface measure of the unit circle \mathbb{S}^1 . Introducing $\rho := |\mathbf{x} + \mathbf{s}rc_0t|$, $\phi := \frac{\mathbf{x} + \mathbf{s}rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|}$, and using (2.8), we can write

$$(4.22) \quad v(\mathbf{x}, t) = \frac{t}{2\pi} \int_0^1 \frac{r}{(1-r^2)^{1/2}} \int_{|\mathbf{s}|=1} [A_0(\rho)Y_1(\phi) + V_1(\mathbf{x} + \mathbf{s}rc_0t)] d\sigma_{\mathbf{s}} dr$$

$$=: P(\mathbf{x}, t) + Q(\mathbf{x}, t).$$

We shall prove that there exists $t_0 > 0$ such that the bounds

$$(4.23) \quad |P(\mathbf{x}, t)| \leq \frac{\tilde{C}}{t}, \quad |Q(\mathbf{x}, t)| \leq \frac{\tilde{C}_0}{t}$$

are valid uniformly in $\mathbf{x} \in \Omega$ with some constants $\tilde{C}, \tilde{C}_0 > 0$ for any $t \geq t_0$. Since it is evident from (4.21) that the solution v is bounded for any finite $t \geq 0$, (4.22) and the estimates in (4.23) imply (4.20).

Estimate of Q for $t \geq t_0$:

Let us first deal with the second term in the decomposition (4.22). We have

$$Q(\mathbf{x}, t) = \frac{t}{2\pi} \int_{a_1/t}^1 \frac{r}{(1-r^2)^{1/2}} \int_{|\mathbf{s}|=1} V_1(\mathbf{x} + \mathbf{s}rc_0t) d\sigma_{\mathbf{s}} dr.$$

Here, we observed that, since $V_1(\mathbf{x}) \equiv 0$ for $|\mathbf{x}| \leq \rho_0$, the integration range in the r variable could be reduced from $(0, 1)$ to $(a_1/t, 1)$ with

$$(4.24) \quad a_1 := \frac{1}{c_0} \inf_{|\mathbf{s}|=1, \mathbf{x} \in \Omega} \left[\sqrt{(\mathbf{x} \cdot \mathbf{s})^2 + \rho_0^2 - |\mathbf{x}|^2} - \mathbf{x} \cdot \mathbf{s} \right].$$

Since $\Omega \Subset \mathbb{B}_{\rho_0}$, we have $a_1 > 0$. We are implicitly assuming that $a_1/t \leq 1$, i.e. $t \geq a_1$. By rearranging the factors, we can write

$$(4.25) \quad Q(\mathbf{x}, t) = \frac{1}{2\pi} \int_{a_1/t}^1 \frac{(rt)^{-3/2}}{(1-r^2)} \int_{|\mathbf{s}|=1} \left(\frac{rt}{\rho} \right)^{5/2} \rho^{5/2} V_1(\mathbf{x} + \mathbf{s}rc_0t) d\sigma_{\mathbf{s}} dr.$$

For $r > a_1/t$, we have $\rho > \rho_0$. Thus, we can estimate rt/ρ in (4.25) as follows. From the triangle inequality and $\mathbf{x} \in \Omega$, $\rho > \rho_0$, we have

$$(4.26) \quad rc_0t = |\mathbf{x} + \mathbf{s}rc_0t - \mathbf{x}| \leq \rho + |\mathbf{x}| \implies \frac{rt}{\rho} = \frac{1}{c_0} \left(1 + \frac{|\mathbf{x}|}{\rho} \right) \leq \frac{2}{c_0}.$$

Moreover, assumption (2.9) with $d = 2$ implies that $\rho^{5/2} V_1(\mathbf{x} + \mathbf{s}rc_0t) \leq C_0$. Inserting this and (4.26) into (4.25) gives

$$|Q(\mathbf{x}, t)| \leq \frac{2^{5/2} C_0}{c_0^{5/2} t^{3/2}} \int_{a_1/t}^1 \frac{dr}{(1-r^2) r^{3/2}}$$

If we assume that $t \geq t_0 = 2a_1$, we obtain

$$(4.27) \quad |Q(\mathbf{x}, t)| \leq \frac{2^{5/2} C_0}{c_0^{5/2} t^{3/2}} \int_{a_1/t}^1 \frac{dr}{(1-r^2)^{1/2} r^{3/2}} = \frac{2^{5/2} C_0}{c_0^{5/2} t^{3/2}} \left(\int_{a_1/t}^{1/2} \dots + \int_{1/2}^1 \dots \right) \\ \leq \frac{2^{7/2} C_0}{\sqrt{3} c_0^{5/2} t^{3/2}} \int_{a_1/t}^{1/2} \frac{dr}{r^{3/2}} + \frac{2^{5/2} C_0}{c_0^{5/2} t^{3/2}} \int_{1/2}^1 \frac{dr}{(1-r^2)^{1/2} r^{3/2}} \leq \frac{\tilde{C}_0}{t}$$

for some constant $\tilde{C}_0 > 0$. This completes the proof of the estimate of Q in (4.23) with $t_0 = 2a_1$.

Estimate of P for $t \geq t_0$:

In order to prove the estimate of P in (4.23), let us write

(4.28)

$$\begin{aligned}
P(\mathbf{x}, t) &= \frac{t}{2\pi} \int_0^1 \frac{e^{ir\omega t}}{(1-r)^{1/2}} \int_{|\mathbf{s}|=1} \left[\frac{re^{-ir\omega t}}{(1+r)^{1/2}} A_0(\rho) Y_1(\phi) \right. \\
&\quad \left. - \left(\frac{e^{-i\omega t}}{\sqrt{2}} - \frac{e^{-i\omega t}}{\sqrt{2}} \right) A_0(|\mathbf{x} + \mathbf{s}c_0 t|) Y_1\left(\frac{\mathbf{x} + \mathbf{s}c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|}\right) \right] d\sigma_{\mathbf{s}} dr \\
&= \frac{t}{2\pi} \int_0^1 \frac{e^{ir\omega t}}{(1-r)^{1/2}} \int_{|\mathbf{s}|=1} \left[\frac{re^{-ir\omega t}}{(1+r)^{1/2}} A_0(\rho) Y_1(\phi) \right. \\
&\quad \left. - \frac{e^{-i\omega t}}{\sqrt{2}} A_0(|\mathbf{x} + \mathbf{s}c_0 t|) Y_1\left(\frac{\mathbf{x} + \mathbf{s}c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|}\right) \right] d\sigma_{\mathbf{s}} dr \\
&\quad + \frac{t}{2\sqrt{2}\pi} \int_{|\mathbf{s}|=1} A_0(|\mathbf{x} + \mathbf{s}c_0 t|) Y_1\left(\frac{\mathbf{x} + \mathbf{s}c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|}\right) d\sigma_{\mathbf{s}} \int_0^1 \frac{e^{-i(1-r)\omega t}}{(1-r)^{1/2}} dr \\
&=: P_1(\mathbf{x}, t) + P_2(\mathbf{x}, t).
\end{aligned}$$

We start with

$$(4.29) \quad P_2(\mathbf{x}, t) = \frac{1}{t^{1/2}} F_2(\mathbf{x}, t) \int_0^1 \frac{e^{-ir\omega t}}{r^{1/2}} dr,$$

where we made a change of variable $r \mapsto (1-r)$ and introduced

$$(4.30) \quad F_2(\mathbf{x}, t) := \frac{t^{3/2}}{2^{3/2}\pi} \int_{|\mathbf{s}|=1} A_0(|\mathbf{x} + \mathbf{s}c_0 t|) Y_1\left(\frac{\mathbf{x} + \mathbf{s}c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|}\right) d\sigma_{\mathbf{s}}.$$

Using the assumed form of A_0 and (4.26), we have, uniformly for $\mathbf{x} \in \Omega$, $\mathbf{s} \in \mathbb{S}^1$,

$$(4.31) \quad (rt)^{3/2} |A_0(\rho)| \leq \begin{cases} \left(\frac{\rho + \rho_0}{c_0}\right)^{3/2} \|A_0\|_{L^\infty(\mathbb{R}_+)}, & 0 \leq \rho \leq \rho_1, \\ \left(\frac{rt}{\rho}\right)^{3/2} \leq \left(\frac{2}{c_0}\right)^{3/2}, & \rho > \rho_1, \\ \leq C_1, & \rho \geq 0, \end{cases}$$

for some constant $C_1 > 0$. Thus, using (4.31) with $r = 1$ and recalling the assumptions on Y_1 , we deduce

$$(4.32) \quad \sup_{\mathbf{x} \in \Omega} \|F_2(\mathbf{x}, \cdot)\|_{L^\infty(\mathbb{R}_+)} =: C_2 < \infty.$$

Finally, employing Lemma A.3 in the appendix, we obtain from (4.29) and (4.32), for $\mathbf{x} \in \Omega$ and $t \geq t_0$,

$$(4.33) \quad |P_2(\mathbf{x}, t)| \leq \frac{\tilde{C}_2}{t}$$

with some constant $\tilde{C}_2 > 0$ and any $t_0 > 0$.

To deal with P_1 , we note that the integrand is a smooth function of r in $[0, 1)$ and it behaves like $(1-r)^{1/2}$ as $r \rightarrow 1$. Integrating by parts in the r variable with $e^{ir\omega t} dr$ as differential, both boundary terms vanish (recall also that $A_0(|\mathbf{x}|) \equiv 0$ for $\mathbf{x} \in \Omega$). We thus arrive at

$$(4.34) \quad P_1(\mathbf{x}, t) = -\frac{1}{2\pi i\omega} \int_0^1 \int_{|\mathbf{s}|=1} e^{ir\omega t} \partial_r \left(\frac{1}{(1-r)^{1/2}} \left[\frac{re^{-ir\omega t}}{(1+r)^{1/2}} A_0(\rho) Y_1(\phi) - \frac{e^{-i\omega t}}{\sqrt{2}} A_0(|\mathbf{x} + \mathbf{s}c_0 t|) Y_1\left(\frac{\mathbf{x} + \mathbf{s}c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|}\right) \right] \right) d\sigma_{\mathbf{s}} dr \\ = I_1(\mathbf{x}, t) + I_2(\mathbf{x}, t) + I_3(\mathbf{x}, t) + I_4(\mathbf{x}, t),$$

where

$$(4.35) \quad I_1(\mathbf{x}, t) := \frac{i}{2\pi\omega} \int_0^1 \int_{|\mathbf{s}|=1} \frac{r}{(1-r^2)^{1/2}} Y_1(\phi) e^{ir\omega t} \partial_r (e^{-ir\omega t} A_0(\rho)) d\sigma_{\mathbf{s}} dr,$$

$$(4.36) \quad I_2(\mathbf{x}, t) := \frac{i}{2\pi\omega} \int_0^1 \int_{|\mathbf{s}|=1} \frac{r}{(1-r^2)^{1/2}} A_0(\rho) \partial_r Y_1(\phi) d\sigma_{\mathbf{s}} dr,$$

$$(4.37) \quad I_3(\mathbf{x}, t) := \frac{i}{4\pi\omega} \int_0^1 \int_{|\mathbf{s}|=1} \frac{2+r}{(1-r)^{1/2} (1+r)^{3/2}} A_0(\rho) Y_1(\phi) d\sigma_{\mathbf{s}} dr,$$

$$(4.38) \quad I_4(\mathbf{x}, t) := \frac{i}{4\pi\omega} \int_0^1 \int_{|\mathbf{s}|=1} \frac{1}{(1-r)^{3/2}} \left[\frac{r}{(1+r)^{1/2}} A_0(\rho) Y_1(\phi) - \frac{e^{-i(1-r)\omega t}}{\sqrt{2}} A_0(|\mathbf{x} + \mathbf{s}c_0 t|) Y_1\left(\frac{\mathbf{x} + \mathbf{s}c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|}\right) \right] d\sigma_{\mathbf{s}} dr.$$

We start by estimating the term I_1 . We have

$$(4.39) \quad \partial_r A_0(\rho) = c_0 t A_0'(\rho) \frac{\mathbf{x} \cdot \mathbf{s} + rc_0 t}{|\mathbf{x} + \mathbf{s}rc_0 t|}.$$

Moreover, since $|\mathbf{x} + \mathbf{s}rc_0 t| = rc_0 t \left(1 + 2\frac{\mathbf{x} \cdot \mathbf{s}}{rc_0 t} + \frac{|\mathbf{x}|^2}{r^2 c_0^2 t^2} \right)^{1/2}$ for $|\mathbf{s}| = 1$, we have that, for $rt \gg 1$, the estimate

$$(4.40) \quad 1 - \frac{\mathbf{x} \cdot \mathbf{s} + rc_0 t}{|\mathbf{x} + \mathbf{s}rc_0 t|} = \frac{|\mathbf{x}|^2 - (\mathbf{x} \cdot \mathbf{s})^2}{2r^2 c_0^2 t^2} + \mathcal{O}\left(\frac{1}{r^3 t^3}\right) = \mathcal{O}\left(\frac{1}{r^2 t^2}\right)$$

is valid. This can be seen from the Taylor expansion of $(1+w)^{-1/2}$ around zero, with $w := 2\frac{\mathbf{x} \cdot \mathbf{s}}{rc_0 t} + \frac{|\mathbf{x}|^2}{r^2 c_0^2 t^2}$.

Then, we can write

$$\begin{aligned} \frac{r^{5/2}t^{3/2}}{c_0} e^{ir\omega t} \partial_r (e^{-ir\omega t} A_0(\rho)) &= (rt)^{5/2} \left(A_0'(\rho) - \frac{i\omega}{c_0} A_0(\rho) \right) \\ &\quad - (rt)^{5/2} A_0'(\rho) \left(1 - \frac{\mathbf{x} \cdot \mathbf{s} + rc_0 t}{|\mathbf{x} + \mathbf{s}rc_0 t|} \right), \end{aligned}$$

where both terms on the right-hand side are uniformly bounded for $rt > 0$, $\mathbf{x} \in \Omega$, $|\mathbf{s}| = 1$. This can be deduced from (4.40) using (4.26) and the estimates $\left| A_0'(\rho) - \frac{i\omega}{c_0} A_0(\rho) \right| = 3/(2\rho^{5/2})$, $|A_0'(\rho)| \leq C/\rho^{3/2}$ for $\rho > \rho_1$ and some constant $C > 0$. Therefore, we have

$$(4.41) \quad F_3(\mathbf{x}, rt) := \frac{ic_0}{2\pi\omega} \int_{|\mathbf{s}|=1} Y_1(\phi) \frac{r^{5/2}t^{3/2}}{c_0} e^{ir\omega t} \partial_r (e^{-ir\omega t} A_0(\rho)) d\sigma_{\mathbf{s}},$$

$$\sup_{\mathbf{x} \in \Omega} \|F_3(\mathbf{x}, \cdot)\|_{L^\infty(a_1, \infty)} =: C_3 < \infty.$$

Since both A_0, A_0' vanish on $[0, \rho_0]$, the integrals in r in each of (4.35)–(4.37) reduces to $(a_1/t, 1)$ (see the discussion before (4.24)). Hence we can estimate I_1 in (4.35) for $t \geq t_0 := 2a_1$ as

$$(4.42) \quad |I_1(\mathbf{x}, t)| = \frac{1}{t^{3/2}} \left| \int_{a_1/t}^1 \frac{1}{r^{3/2} (1-r^2)^{1/2}} F_3(\mathbf{x}, rt) dr \right|$$

$$\leq \frac{2C_3}{3^{1/2}t^{3/2}} \int_{a_1/t}^{1/2} \frac{dr}{r^{3/2}} + \frac{2^{3/2}C_3}{t^{3/2}} \int_{1/2}^1 \frac{dr}{(1-r)^{1/2}} \leq \frac{\tilde{C}_3}{t}$$

with some constant $\tilde{C}_3 > 0$.

In a similar but simpler fashion, we can estimate the terms I_2 and I_3 . Since

$$(4.43) \quad \partial_r Y_1(\phi) = \frac{c_0 t \mathbf{s} \cdot \nabla Y_1(\phi)}{|\mathbf{x} + \mathbf{s}rc_0 t|} - \frac{c_0 t (\mathbf{x} \cdot \mathbf{s} + rc_0 t)}{|\mathbf{x} + \mathbf{s}rc_0 t|^3} (\mathbf{x} + \mathbf{s}rc_0 t) \cdot \nabla Y_1(\phi),$$

we have

$$(4.44) \quad F_4(\mathbf{x}, rt) := \frac{ic_0 (rt)^{5/2}}{2\pi\omega} \int_{|\mathbf{s}|=1} A_0(\rho) \left[\frac{\mathbf{s} \cdot \nabla Y_1(\phi)}{|\mathbf{x} + \mathbf{s}rc_0 t|} - \frac{\mathbf{x} \cdot \mathbf{s} + rc_0 t}{|\mathbf{x} + \mathbf{s}rc_0 t|^3} (\mathbf{x} + \mathbf{s}rc_0 t) \cdot \nabla Y_1(\phi) \right] d\sigma_{\mathbf{s}},$$

$$|F_4(\mathbf{x}, rt)| \leq \frac{c_0}{\pi\omega} \int_{|\mathbf{s}|=1} \frac{rt}{\rho} (rt)^{3/2} |A_0(\rho)| |\nabla Y_1(\phi)| d\sigma_{\mathbf{s}},$$

and hence, using (4.26) and (4.31), we deduce that

$$\sup_{\mathbf{x} \in \Omega} \|F_4(\mathbf{x}, \cdot)\|_{L^\infty(a_1, \infty)} =: C_4 < \infty.$$

Similarly,

$$(4.45) \quad F_5(\mathbf{x}, rt) := \frac{i(rt)^{3/2}}{4\pi\omega} \int_{|\mathbf{s}|=1} A_0(\rho) Y_1(\phi) d\sigma_{\mathbf{s}},$$

$$|F_5(\mathbf{x}, rt)| = \frac{1}{4\pi\omega} \int_{|\mathbf{s}|=1} (rt)^{3/2} |A_0(\rho)| |Y_1(\phi)| d\sigma_{\mathbf{s}},$$

and hence, using again (4.31),

$$(4.46) \quad \sup_{\mathbf{x} \in \Omega} \|F_5(\mathbf{x}, \cdot)\|_{L^\infty(a_1, \infty)} =: C_5 < \infty.$$

Therefore, we obtain, for $t \geq t_0 = 2a_1$,

$$(4.47) \quad |I_2(\mathbf{x}, t)| = \frac{1}{t^{3/2}} \left| \int_{a_1/t}^1 \frac{1}{r^{3/2} (1-r^2)^{1/2}} F_4(\mathbf{x}, rt) dr \right| \\ \leq \frac{2C_4}{3^{1/2}t^{3/2}} \int_{a_1/t}^{1/2} \frac{dr}{r^{3/2}} + \frac{2^{3/2}C_4}{t^{3/2}} \int_{1/2}^1 \frac{dr}{(1-r)^{1/2}} \leq \frac{\tilde{C}_4}{t},$$

$$(4.48) \quad |I_3(\mathbf{x}, t)| = \frac{1}{t^{3/2}} \left| \int_{a_1/t}^1 \frac{2+r}{r^{3/2} (1-r)^{1/2} (1+r)^{3/2}} F_5(\mathbf{x}, rt) dr \right| \\ \leq \frac{5C_5}{2^{1/2}t^{3/2}} \int_{a_1/t}^{1/2} \frac{dr}{r^{3/2}} + \frac{8C_5}{3^{1/2}t^{3/2}} \int_{1/2}^1 \frac{dr}{(1-r)^{1/2}} \leq \frac{\tilde{C}_5}{t}$$

with some constants $\tilde{C}_4, \tilde{C}_5 > 0$.

To treat the term I_4 , we introduce

$$(4.49) \quad \tilde{F}_6(\mathbf{x}, r, t) := \int_{|\mathbf{s}|=1} \left[\frac{r}{(1+r)^{1/2}} A_0(\rho) Y_1(\phi) \right. \\ \left. - \frac{e^{-i(1-r)\omega t}}{\sqrt{2}} A_0(|\mathbf{x} + \mathbf{s}c_0 t|) Y_1\left(\frac{\mathbf{x} + \mathbf{s}c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|}\right) \right] d\sigma_{\mathbf{s}},$$

$$(4.50) \quad F_6(\mathbf{x}, r, t) := \frac{1}{1-r} \tilde{F}_6(\mathbf{x}, r, t).$$

Using $\tilde{F}_6(\mathbf{x}, 1, t) = 0$, we rewrite (4.50) as

$$(4.51) \quad F_6(\mathbf{x}, r, t) = -\frac{1}{1-r} \left(\tilde{F}_6(\mathbf{x}, 1, t) - \tilde{F}_6(\mathbf{x}, r, t) \right) = -\frac{1}{1-r} \int_r^1 \partial_r \tilde{F}_6(\mathbf{x}, \tau, t) d\tau.$$

Then, for $r \in (1 - a_1/t, 1)$, we can estimate

$$\begin{aligned}
|F_6(\mathbf{x}, r, t)| &\leq \left\| \partial_r \tilde{F}_6(\mathbf{x}, \cdot, t) \right\|_{L^\infty(1-a_1/t, 1)} \\
&\leq \int_{|s|=1} \left[\left(\frac{1}{2(1+r)^{3/2}} + \frac{1}{(1+r)^{1/2}} \right) |A_0(\rho)| |Y_1(\phi)| \right. \\
&\quad + \frac{1}{(1+r)^{1/2}} |\partial_r A_0(\rho)| |Y_1(\phi)| + \frac{1}{(1+r)^{1/2}} |A_0(\rho)| |\partial_r Y_1(\phi)| \\
&\quad \left. + \frac{\omega t}{\sqrt{2}} |A_0(|\mathbf{x} + \mathbf{s}c_0 t|)| \left| Y_1 \left(\frac{\mathbf{x} + \mathbf{s}c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|} \right) \right| \right] d\sigma_{\mathbf{s}}.
\end{aligned}$$

Therefore, employing (4.31) with $r = 1$, (4.39) and (4.43), taking into account again the behaviour of $A_0(\rho)$ and $A'_0(\rho)$ for $\rho > \rho_1$, we deduce that

$$(4.52) \quad \sup_{\mathbf{x} \in \Omega, r \in (1-a_1/t, 1), t > 2a_1} |F_6(\mathbf{x}, r, t) t^{1/2}| =: C_6 < \infty.$$

Moreover, for $r \in (1/2, 1 - a_1/t)$ and for $\epsilon \in (0, 1/2]$, taking into account (4.50), we have

$$\left| (1-r)^{1/2+\epsilon} t^\epsilon F_6(\mathbf{x}, r, t) \right| = \left| \tilde{F}_6(\mathbf{x}, r, t) \right| \frac{t^\epsilon}{(1-r)^{1/2-\epsilon}} \leq \left| \tilde{F}_6(\mathbf{x}, r, t) \right| \frac{t^{1/2}}{a_1^{1/2-\epsilon}}.$$

Hence, with the constants C_2 and C_5 introduced in (4.32) and (4.46), respectively, we can obtain from (4.49) that, for any $\epsilon \in (0, 1/2]$,

(4.53)

$$\begin{aligned}
\sup_{\mathbf{x} \in \Omega, r \in (1/2, 1-a_1/t), t > 2a_1} \left| (1-r)^{1/2+\epsilon} t^{1+\epsilon} F_6(\mathbf{x}, r, t) \right| &\leq \frac{1}{a_1^{1/2-\epsilon}} \left(\frac{8\pi\omega}{3^{1/2}} C_5 + 2\pi C_2 \right) \\
&=: C_7 < \infty.
\end{aligned}$$

Bounds (4.52) and (4.53) imply that, for $t \geq t_0 = 2a_1$ and $\epsilon \in (0, 1/2]$, we get

$$\begin{aligned}
(4.54) \quad |I_4(\mathbf{x}, t)| &\leq \frac{1}{4\pi\omega} \int_0^1 \frac{1}{(1-r)^{1/2}} |F_6(\mathbf{x}, r, t)| dr \\
&= \frac{1}{4\pi\omega} \left(\int_0^{1/2} \dots + \int_{1/2}^{1-a_1/t} \dots + \int_{1-a_1/t}^1 \dots \right) \\
&\leq \frac{2^{3/2}}{t^{3/2}} \left(C_5 \int_0^{1/2} \frac{dr}{r^{1/2}} + \frac{C_2}{4\omega} \right) + \frac{C_7}{4\pi\omega t^{1+\epsilon}} \int_{1/2}^{1-a_1/t} \frac{dr}{(1-r)^{1+\epsilon}} \\
&\quad + \frac{C_6}{4\pi\omega t^{1/2}} \int_{1-a_1/t}^1 \frac{dr}{(1-r)^{1/2}} \leq \frac{\tilde{C}_6}{t}
\end{aligned}$$

with some constant $\tilde{C}_6 > 0$. Here, for the interval $(0, 1/2)$, we estimated the integrand directly from (4.38) using again (4.32) and (4.46).

From estimates (4.42), (4.47), (4.48), and (4.54) of the terms I_1 , I_2 , I_3 , and I_4 , respectively, in decomposition (4.34), we obtain for P_1 the same estimate as (4.33) for P_2 , which altogether give the estimate of P in (4.23), again with $t_0 = 2a_1$. This concludes the proof of (4.20) in the case $d = 2$.

- Case $d = 3$.

The solution is given by the Kirchhoff's formula [12, Par. 2.4.1 (c)]

$$(4.55) \quad v(\mathbf{x}, t) = \frac{1}{4\pi} \int_{|\mathbf{s}|=1} t v_1(\mathbf{x} + \mathbf{s}c_0 t) d\sigma_{\mathbf{s}},$$

where $d\sigma_{\mathbf{s}}$ denotes the surface measure of the unit sphere \mathbb{S}^2 . In this case, it is immediate to see that (4.55) implies (4.20), owing to assumption (2.10).

- Conclusion of the proof.

The estimates for the solution derivatives appearing in (2.11), under the assumption made at the beginning that $v_0 \equiv 0$, follow by observation that the differentiation under the integral sign in (4.22) and (4.55) is permitted and does not yield any singular or time growing multipliers, as long as the initial datum v_1 is sufficiently smooth.

The same argument also justifies our restriction to the case $v_0 \equiv 0$. In fact, in order to extend the result to the case $v_0 \not\equiv 0$, it is enough to note that we can write $v = \tilde{w} + \partial_t w$ with \tilde{w} and w solving the problems $\partial_t^2 \tilde{w} - c_0^2 \Delta \tilde{w} = 0$, $\tilde{w}(\mathbf{x}, 0) = 0$, $\partial_t \tilde{w}(\mathbf{x}, 0) = v_1(\mathbf{x})$ and $\partial_t^2 w - c_0^2 \Delta w = 0$, $w(\mathbf{x}, 0) = 0$, $\partial_t w(\mathbf{x}, 0) = v_0(\mathbf{x})$, respectively. \square

4.4. Proof of Lemma 2.4. Similarly to Lemma 2.3, in the main body of the proof, we shall derive the estimate

$$(4.56) \quad |v(\mathbf{x}, t)| \leq \frac{C}{(1+t^2)^{1/2}}, \quad \mathbf{x} \in \Omega, \quad t \geq 0,$$

for some constant $C > 0$, and the estimate of the other terms in (2.13) is discussed at the end.

- Case $d = 2$.

The solution is given explicitly by the Poisson's formula [12, Par. 2.4.1 (c)]

$$(4.57) \quad v(\mathbf{x}, t) = \frac{1}{2\pi} \left[\int_0^1 \frac{rt}{(1-r^2)^{1/2}} \int_{|\mathbf{s}|=1} v_1(\mathbf{x} + \mathbf{s}rc_0 t) d\sigma_{\mathbf{s}} dr + \partial_t \left(\int_0^1 \frac{rt}{(1-r^2)^{1/2}} \int_{|\mathbf{s}|=1} v_0(\mathbf{x} + \mathbf{s}rc_0 t) d\sigma_{\mathbf{s}} dr \right) \right].$$

Upon insertion of (2.12) into (4.57), and setting $\rho := |\mathbf{x} + \mathbf{s}rc_0t|$, $\phi := \frac{\mathbf{x} + \mathbf{s}rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|}$, we rearrange the terms to obtain

$$(4.58) \quad v(\mathbf{x}, t) = \frac{1}{2\pi} \int_{|\mathbf{s}|=1} \int_0^1 \frac{rt}{(1-r^2)^{1/2}} [\partial_t A(\rho) - c_0 A'(\rho)] Y(\phi) dr d\sigma_{\mathbf{s}} \\ + \frac{1}{2\pi} \int_{|\mathbf{s}|=1} \int_0^1 \frac{r}{(1-r^2)^{1/2}} A(\rho) [t\partial_t Y(\phi) + Y(\phi)] dr d\sigma_{\mathbf{s}}.$$

Since $|\mathbf{s}| = 1$, we have

$$(4.59) \quad \partial_t A(\rho) = rc_0 A'(\rho) \frac{\mathbf{x} \cdot \mathbf{s} + rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|}, \quad r\partial_r A(\rho) = t\partial_t A(\rho),$$

$$(4.60) \quad \partial_t Y(\phi) = \frac{rc_0 \mathbf{s} \cdot \nabla Y(\phi)}{|\mathbf{x} + \mathbf{s}rc_0t|} - \frac{rc_0(\mathbf{x} \cdot \mathbf{s} + rc_0t)}{|\mathbf{x} + \mathbf{s}rc_0t|^3} (\mathbf{x} + \mathbf{s}rc_0t) \cdot \nabla Y(\phi), \\ r\partial_r Y(\phi) = t\partial_t Y(\phi).$$

Note that, using (4.59), we can rewrite, for $t > 0$,

$$(4.61) \quad \partial_t A(\rho) - c_0 A'(\rho) = \frac{1}{t} (r-1) \partial_r A(\rho) - \frac{1}{t} \left(\frac{|\mathbf{x} + \mathbf{s}rc_0t|}{\mathbf{x} \cdot \mathbf{s} + rc_0t} - 1 \right) \partial_r A(\rho) \\ = \frac{1}{t} (r-1) \partial_r A(\rho) + c_0 \left(\frac{\mathbf{x} \cdot \mathbf{s} + rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|} - 1 \right) A'(\rho).$$

Plugging (4.61) into (4.58), we observe that the term with $\partial_r A$ can be integrated by parts in the variable r , due to the cancellation of the singularity at $r = 1$. In doing so, both boundary terms at $r = 0$ and $r = 1$, respectively, vanish. With some simplifications that employ (4.60) and the identity

$$\left(\frac{r(1-r)}{(1-r^2)^{1/2}} \right)' + \frac{r}{(1-r^2)^{1/2}} = \frac{1}{(1-r^2)^{1/2}(1+r)},$$

we arrive at

$$(4.62) \quad v(\mathbf{x}, t) = \frac{1}{2\pi} \int_{|\mathbf{s}|=1} \int_0^1 \frac{1}{(1-r^2)^{1/2}} \left[\frac{1}{1+r} A(\rho) Y(\phi) \right. \\ \left. + rc_0t \left(\frac{\mathbf{x} \cdot \mathbf{s} + rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|} - 1 \right) A'(\rho) Y(\phi) \right] dr d\sigma_{\mathbf{s}} + Q(\mathbf{x}, t),$$

where we have set

$$Q(\mathbf{x}, t) := \frac{1}{2\pi} \int_{|\mathbf{s}|=1} \int_0^1 \frac{rc_0t}{(1-r^2)^{1/2}} \left[\frac{1}{|\mathbf{x} + \mathbf{s}rc_0t|} A(\rho) \mathbf{s} \cdot \nabla Y(\phi) \right. \\ \left. - \frac{\mathbf{x} \cdot \mathbf{s} + rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|^3} A(\rho) (\mathbf{x} + \mathbf{s}rc_0t) \cdot \nabla Y(\phi) \right] dr d\sigma_{\mathbf{s}}.$$

Let us split $Q(\mathbf{x}, t) = Q_1(\mathbf{x}, t) + Q_2(\mathbf{x}, t)$, with

(4.63)

$$Q_1(\mathbf{x}, t) := \frac{1}{2\pi} \int_{|\mathbf{s}|=1} \int_0^1 \frac{A(\rho)}{(1-r^2)^{1/2}} \times \left[\frac{(\mathbf{x} \cdot \mathbf{s})(\mathbf{x} \cdot \mathbf{s} + rc_0t)(\mathbf{x} + \mathbf{s}rc_0t) \cdot \nabla Y(\phi)}{|\mathbf{x} + \mathbf{s}rc_0t|^3} - \frac{\mathbf{x} \cdot \nabla Y(\phi)}{|\mathbf{x} + \mathbf{s}rc_0t|} \right] dr d\sigma_{\mathbf{s}},$$

(4.64)

$$Q_2(\mathbf{x}, t) := \frac{1}{2\pi} \int_{|\mathbf{s}|=1} \int_0^1 \frac{A(\rho)}{(1-r^2)^{1/2}} \times \left(1 - \frac{\mathbf{x} \cdot \mathbf{s} + rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|} \right) \left(1 + \frac{\mathbf{x} \cdot \mathbf{s} + rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|} \right) \frac{(\mathbf{x} + \mathbf{s}rc_0t) \cdot \nabla Y(\phi)}{|\mathbf{x} + \mathbf{s}rc_0t|} dr d\sigma_{\mathbf{s}}.$$

By setting

(4.65)

$$P_1(\mathbf{x}, t) := \frac{1}{2\pi} \int_0^1 \frac{1}{(1-r^2)^{1/2}} \int_{|\mathbf{s}|=1} rc_0t \left(\frac{\mathbf{x} \cdot \mathbf{s} + rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|} - 1 \right) A'(\rho) Y(\phi) d\sigma_{\mathbf{s}} dr,$$

(4.66)

$$P_2(\mathbf{x}, t) := \frac{1}{2\pi} \int_0^1 \frac{1}{(1-r^2)^{1/2} (1+r)} \int_{|\mathbf{s}|=1} A(\rho) Y(\phi) d\sigma_{\mathbf{s}} dr,$$

we can rewrite (4.62) as

(4.67)

$$v(\mathbf{x}, t) = P_1(\mathbf{x}, t) + P_2(\mathbf{x}, t) + Q_1(\mathbf{x}, t) + Q_2(\mathbf{x}, t).$$

We remark that the decomposition of Q into Q_1 and Q_2 is beneficial in the estimate for large rt . Namely, we have, for $rt \gg 1$, uniformly in $\mathbf{x} \in \Omega$, $\mathbf{s} \in \mathbb{S}^1$,

(4.68)

$$\frac{(\mathbf{x} \cdot \mathbf{s})(\mathbf{x} \cdot \mathbf{s} + rc_0t)(\mathbf{x} + \mathbf{s}rc_0t) \cdot \nabla Y(\phi)}{|\mathbf{x} + \mathbf{s}rc_0t|^3} - \frac{\mathbf{x} \cdot \nabla Y(\phi)}{|\mathbf{x} + \mathbf{s}rc_0t|} = \mathcal{O}\left(\frac{1}{rt}\right),$$

(4.69)

$$\left(1 - \frac{\mathbf{x} \cdot \mathbf{s} + rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|} \right) \left(1 + \frac{\mathbf{x} \cdot \mathbf{s} + rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|} \right) \frac{(\mathbf{x} + \mathbf{s}rc_0t) \cdot \nabla Y(\phi)}{|\mathbf{x} + \mathbf{s}rc_0t|} = \mathcal{O}\left(\frac{1}{r^2t^2}\right).$$

Note that (4.69) follows from estimate (4.40).

As in the proof of Lemma 2.3, we shall prove that there exists $t_0 > 0$ such that the bounds

(4.70)

$$|P_1(\mathbf{x}, t)| \leq \frac{\tilde{C}_0}{t}, \quad |P_2(\mathbf{x}, t)| \leq \frac{\tilde{C}_1}{t}, \quad |Q_1(\mathbf{x}, t)| \leq \frac{\tilde{C}_2}{t}, \quad |Q_2(\mathbf{x}, t)| \leq \frac{\tilde{C}_3}{t},$$

are valid uniformly in $\mathbf{x} \in \Omega$ with some constants $\tilde{C}_0, \tilde{C}_1, \tilde{C}_2, \tilde{C}_3 > 0$ for any $t \geq t_0$. Then, due to the boundedness of the solution v for any finite $t \geq 0$ (see (4.57)), the bounds in (4.70) imply (4.56).

Since $A(\rho)$, $A'(\rho)$ are different from zero only for $\rho > \rho_0$, i.e. for $rc_0t > ((\mathbf{x} \cdot \mathbf{s})^2 + \rho_0^2 - |\mathbf{x}|^2)^{1/2} - \mathbf{x} \cdot \mathbf{s}$, the integration range in the r variable in each term of (4.67) effectively reduces from $(0, 1)$ to $(a_1/t, 1)$ with $a_1 > 0$ defined in (4.24). With this argument, we are implicitly assuming that $t \geq a_1$. We will actually prove (4.70) with $t_0 := 2a_1$.

Estimate of P_1 for $t \geq t_0$:

Let us introduce

$$(4.71) \quad F_1(\mathbf{x}, rt) := \frac{c_0}{2\pi} \int_{|\mathbf{s}|=1} (rt)^{5/2} \left(\frac{\mathbf{x} \cdot \mathbf{s} + rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|} - 1 \right) A'(\rho) Y(\phi) d\sigma_{\mathbf{s}},$$

so that we can write

$$\begin{aligned} P_1(\mathbf{x}, t) &= \int_{a_1/t}^1 \frac{1}{(1-r^2)^{1/2} (rt)^{3/2}} F_1(\mathbf{x}, rt) dr = \int_{a_1/t}^{1/2} \dots + \int_{1/2}^1 \dots \\ &=: P_{1,1}(\mathbf{x}, t) + P_{1,2}(\mathbf{x}, t), \end{aligned}$$

assuming $a_1/t \leq 1/2$, i.e. $t \geq 2a_1$.

Since we consider only $\rho = |\mathbf{x} + \mathbf{s}rc_0t| > \rho_0 > 0$ (as the integrand of $F_1(\mathbf{x}, rt)$ vanishes otherwise), the denominators in (4.40) and (4.71) are bounded away from zero. Moreover, since $rc_0t = |\mathbf{s}rc_0t| \leq \rho + |\mathbf{x}| \leq \rho + \rho_0$, we have, uniformly for $\mathbf{x} \in \Omega$, $\mathbf{s} \in \mathbb{S}^1$,

$$(4.72) \quad (rt)^{1/2} |A'(\rho)| \leq \begin{cases} \left(\frac{\rho + \rho_0}{c_0} \right)^{1/2} \|A'\|_{L^\infty(\mathbb{R}_+)}, & 0 \leq \rho \leq \rho_1, \\ \left(\frac{rt}{\rho} \right)^{1/2} \left(\frac{\omega}{c_0} + \frac{1}{2\rho_1} \right) \leq \left(\frac{\rho + \rho_0}{c_0\rho} \right)^{1/2} \left(\frac{\omega}{c_0} + \frac{1}{2\rho_1} \right), & \rho > \rho_1, \\ \leq C_0, & \rho > 0, \end{cases}$$

for some constant $C_0 > 0$. This, together with (4.40), implies that

$$\sup_{\mathbf{x} \in \Omega} \|F_1(\mathbf{x}, \cdot)\|_{L^\infty(a_1, \infty)} =: C_1 < \infty$$

for some constant $C_1 > 0$. Therefore, we can estimate, for $t \geq t_0 = 2a_1$ and $\mathbf{x} \in \Omega$,

$$|P_{1,1}(\mathbf{x}, t)| \leq 2^{1/2} \frac{C_1}{t^{3/2}} \int_{a_1/t}^{1/2} \frac{dr}{r^{3/2}} < \frac{2^{3/2} C_1}{a_1^{1/2} t},$$

$$|P_{1,2}(\mathbf{x}, t)| \leq 2^{3/2} \frac{C_1}{t^{3/2}} \int_{1/2}^1 \frac{dr}{(1-r)^{1/2}} = \frac{4C_1}{t^{3/2}},$$

and thus we have the bound for P_1 in (4.70) with $t_0 = 2a_1$ and some constant $\tilde{C}_0 > 0$.

Estimate of Q_1 , Q_2 for $t \geq t_0$:

We proceed to estimate the terms Q_1, Q_2 in a similar fashion. As before, we note that all the denominators in (4.63) and (4.64) are bounded away from zero. Therefore, by setting

$$(4.73) \quad F_2(\mathbf{x}, rt) := \frac{(rt)^{3/2}}{2\pi} \int_{|\mathbf{s}|=1} \left[\frac{(\mathbf{x} \cdot \mathbf{s})(\mathbf{x} \cdot \mathbf{s} + rc_0t)(\mathbf{x} + \mathbf{s}rc_0t) \cdot \nabla Y(\phi)}{|\mathbf{x} + \mathbf{s}rc_0t|^3} - \frac{\mathbf{x} \cdot \nabla Y(\phi)}{|\mathbf{x} + \mathbf{s}rc_0t|} \right] A(\rho) d\sigma_{\mathbf{s}},$$

(4.74)

$$F_3(\mathbf{x}, rt) := \frac{(rt)^{5/2}}{2\pi} \int_{|\mathbf{s}|=1} \left(1 - \frac{(\mathbf{x} \cdot \mathbf{s} + rc_0t)^2}{|\mathbf{x} + \mathbf{s}rc_0t|^2} \right) \frac{(\mathbf{x} + \mathbf{s}rc_0t) \cdot \nabla Y(\phi)}{|\mathbf{x} + \mathbf{s}rc_0t|} A(\rho) d\sigma_{\mathbf{s}},$$

and recalling (4.68) and (4.69), we have

$$(4.75) \quad \sup_{\mathbf{x} \in \Omega} \|F_2(\mathbf{x}, \cdot)\|_{L^\infty(a_1, \infty)} =: C_2 < \infty, \quad \sup_{\mathbf{x} \in \Omega} \|F_3(\mathbf{x}, \cdot)\|_{L^\infty(a_1, \infty)} =: C_3 < \infty.$$

Consequently, we estimate as before, for $\mathbf{x} \in \Omega, t \geq t_0 = 2a_1$,

$$|Q_1(\mathbf{x}, t)| \leq 2^{1/2} \frac{C_2}{t^{3/2}} \int_{a_1/t}^{1/2} \frac{dr}{r^{3/2}} + 2^{3/2} \frac{C_2}{t^{3/2}} \int_{1/2}^1 \frac{dr}{(1-r)^{1/2}} < \frac{\tilde{C}_2}{t},$$

$$|Q_2(\mathbf{x}, t)| \leq 2^{1/2} \frac{C_3}{t^{5/2}} \int_{a_1/t}^{1/2} \frac{dr}{r^{5/2}} + 2^{5/2} \frac{C_3}{t^{5/2}} \int_{1/2}^1 \frac{dr}{(1-r)^{1/2}} < \frac{\tilde{C}_3}{t}$$

with some constants $\tilde{C}_2, \tilde{C}_3 > 0$. This proves the bounds for Q_1 and Q_2 in (4.70) again with $t_0 = 2a_1$.

Estimate of P_2 for $t \geq t_0$:

For the term P_2 , we proceed as in the estimate of the term P in Lemma 2.3. Let us rewrite (4.66) as

(4.76)

$$P_2(\mathbf{x}, t) = \frac{1}{2\pi} \int_0^1 \frac{e^{ir\omega t}}{(1-r)^{1/2}} \int_{|\mathbf{s}|=1} \left[\frac{e^{-ir\omega t}}{(1+r)^{3/2}} A(\rho) Y(\phi) - \frac{e^{-i\omega t}}{2^{3/2}} A(|\mathbf{x} + \mathbf{s}c_0t|) Y\left(\frac{\mathbf{x} + \mathbf{s}c_0t}{|\mathbf{x} + \mathbf{s}c_0t|}\right) \right] d\sigma_{\mathbf{s}} dr$$

$$+ \frac{1}{2^{5/2}\pi} \int_{|\mathbf{s}|=1} A(|\mathbf{x} + \mathbf{s}c_0t|) Y\left(\frac{\mathbf{x} + \mathbf{s}c_0t}{|\mathbf{x} + \mathbf{s}c_0t|}\right) d\sigma_{\mathbf{s}} \int_0^1 \frac{e^{-i(1-r)\omega t}}{(1-r)^{1/2}} dr$$

$$=: P_{2,1}(\mathbf{x}, t) + P_{2,2}(\mathbf{x}, t).$$

We start with

$$P_{2,2}(\mathbf{x}, t) = \frac{1}{t^{1/2}} F_4(\mathbf{x}, t) \int_0^1 \frac{e^{-ir\omega t}}{r^{1/2}} dr,$$

where we made a change of variable $r \mapsto (1 - r)$ and introduced

$$(4.77) \quad F_4(\mathbf{x}, t) := \frac{t^{1/2}}{2^{5/2}\pi} \int_{|\mathbf{s}|=1} A(|\mathbf{x} + \mathbf{s}c_0t|) Y\left(\frac{\mathbf{x} + \mathbf{s}c_0t}{|\mathbf{x} + \mathbf{s}c_0t|}\right) d\sigma_{\mathbf{s}}.$$

Similarly to (4.72), we have $t^{1/2}A(|\mathbf{x} + \mathbf{s}c_0t|) \leq C_{00}$ for some constant $C_{00} > 0$, and thus

$$(4.78) \quad \sup_{\mathbf{x} \in \Omega} \|F_4(\mathbf{x}, \cdot)\|_{L^\infty(\mathbb{R}_+)} =: C_4 < \infty.$$

Hence, employing Lemma A.3, we obtain, for $\mathbf{x} \in \Omega$ and sufficiently large $t > 0$,

$$(4.79) \quad |P_{2,2}(\mathbf{x}, t)| \leq \frac{\tilde{C}_4}{t}.$$

To deal with $P_{2,1}$, we note that the integrand is a smooth function of r in $[0, 1)$ and it behaves like $(1 - r)^{1/2}$ as $r \rightarrow 1$. Integrating by parts in the r variable with $e^{ir\omega t} dr$ as differential, both boundary terms vanish (recall also that $A(|\mathbf{x}|) \equiv 0$ for $\mathbf{x} \in \Omega$). We thus have

$$(4.80) \quad \begin{aligned} P_{2,1}(\mathbf{x}, t) &= -\frac{1}{2\pi i \omega t} \int_0^1 \int_{|\mathbf{s}|=1} e^{ir\omega t} \partial_r \left(\frac{1}{(1-r)^{1/2}} \left[\frac{e^{-ir\omega t}}{(1+r)^{3/2}} A(\rho) Y(\phi) \right. \right. \\ &\quad \left. \left. - \frac{e^{-i\omega t}}{2^{3/2}} A(|\mathbf{x} + \mathbf{s}c_0t|) Y\left(\frac{\mathbf{x} + \mathbf{s}c_0t}{|\mathbf{x} + \mathbf{s}c_0t|}\right) \right] \right) d\sigma_{\mathbf{s}} dr \\ &= I_1(\mathbf{x}, t) + I_2(\mathbf{x}, t) + I_3(\mathbf{x}, t) + I_4(\mathbf{x}, t), \end{aligned}$$

where

$$(4.81) \quad I_1(\mathbf{x}, t) := \frac{i}{2\pi \omega t} \int_0^1 \int_{|\mathbf{s}|=1} \frac{1}{(1-r)^{1/2} (1+r)^{3/2}} Y(\phi) e^{ir\omega t} \partial_r (e^{-ir\omega t} A(\rho)) d\sigma_{\mathbf{s}} dr,$$

$$(4.82) \quad I_2(\mathbf{x}, t) := \frac{i}{2\pi \omega t} \int_0^1 \int_{|\mathbf{s}|=1} \frac{1}{(1-r)^{1/2} (1+r)^{3/2}} A(\rho) \partial_r Y(\phi) d\sigma_{\mathbf{s}} dr,$$

$$(4.83) \quad I_3(\mathbf{x}, t) := -\frac{3i}{4\pi \omega t} \int_0^1 \int_{|\mathbf{s}|=1} \frac{1}{(1-r)^{1/2} (1+r)^{5/2}} A(\rho) Y(\phi) d\sigma_{\mathbf{s}} dr,$$

$$(4.84) \quad \begin{aligned} I_4(\mathbf{x}, t) &:= \frac{i}{4\pi \omega t} \int_0^1 \int_{|\mathbf{s}|=1} \frac{1}{(1-r)^{3/2}} \left[\frac{1}{(1+r)^{3/2}} A(\rho) Y(\phi) \right. \\ &\quad \left. - \frac{e^{-i(1-r)\omega t}}{2^{3/2}} A(|\mathbf{x} + \mathbf{s}c_0t|) Y\left(\frac{\mathbf{x} + \mathbf{s}c_0t}{|\mathbf{x} + \mathbf{s}c_0t|}\right) \right] d\sigma_{\mathbf{s}} dr. \end{aligned}$$

For the term I_1 , observe that, using (4.59), we have

$$\begin{aligned} \frac{r^{3/2}t^{1/2}}{c_0} e^{ir\omega t} \partial_r (e^{-ir\omega t} A(\rho)) &= (rt)^{3/2} \left(A'(\rho) - \frac{i\omega}{c_0} A(\rho) \right) \\ &\quad - (rt)^{3/2} A'(\rho) \left(1 - \frac{\mathbf{x} \cdot \mathbf{s} + rc_0 t}{|\mathbf{x} + \mathbf{s}rc_0 t|} \right), \end{aligned}$$

where both terms on the right-hand side are uniformly bounded for $rt > 0$, $\mathbf{x} \in \Omega$, $|\mathbf{s}| = 1$, due to (4.40) and the assumption on the form of A . Therefore, for

$$F_5(\mathbf{x}, rt) := \frac{ic_0}{2\pi\omega} \int_{|\mathbf{s}|=1} Y(\phi) \frac{r^{3/2}t^{1/2}}{c_0} e^{ir\omega t} \partial_r (e^{-ir\omega t} A(\rho)) d\sigma_{\mathbf{s}},$$

we have

$$\sup_{\mathbf{x} \in \Omega} \|F_5(\mathbf{x}, \cdot)\|_{L^\infty(a_1, \infty)} =: C_5 < \infty.$$

Since both A , A' vanish on $[0, \rho_0]$, the integrals in r in each of (4.81)–(4.83) reduces to $(a_1/t, 1)$ (see the discussion before (4.24)). Hence we can estimate I_1 in (4.81) for $t \geq t_0 = 2a_1$ as

$$\begin{aligned} (4.85) \quad |I_1(\mathbf{x}, t)| &= \frac{1}{t^{3/2}} \left| \int_{a_1/t}^1 \frac{1}{r^{3/2} (1-r)^{1/2} (1+r)^{3/2}} F_5(\mathbf{x}, rt) dr \right| \\ &\leq \frac{2^{1/2} C_5}{t^{3/2}} \int_{a_1/t}^{1/2} \frac{dr}{r^{3/2}} + \frac{2^{3/2} C_5}{t^{3/2}} \int_{1/2}^1 \frac{dr}{(1-r)^{1/2}} \leq \frac{\tilde{C}_5}{t} \end{aligned}$$

with some constant $\tilde{C}_5 > 0$.

In a similar but simpler fashion we can estimate the terms I_2 and I_3 . Namely, recalling (4.60), we have

$$\begin{aligned} F_6(\mathbf{x}, rt) &:= \frac{ic_0 (rt)^{3/2}}{2\pi\omega} \int_{|\mathbf{s}|=1} A(\rho) \left[\frac{\mathbf{s} \cdot \nabla Y(\phi)}{|\mathbf{x} + \mathbf{s}rc_0 t|} \right. \\ &\quad \left. - \frac{\mathbf{x} \cdot \mathbf{s} + rc_0 t}{|\mathbf{x} + \mathbf{s}rc_0 t|^3} (\mathbf{x} + \mathbf{s}rc_0 t) \cdot \nabla Y(\phi) \right] d\sigma_{\mathbf{s}}, \\ |F_6(\mathbf{x}, rt)| &\leq \frac{c_0}{\pi\omega} \int_{|\mathbf{s}|=1} \left(\frac{rt}{\rho} \right)^{3/2} \rho^{1/2} |A(\rho)| |\nabla Y(\phi)| d\sigma_{\mathbf{s}}, \end{aligned}$$

and hence we deduce that

$$(4.86) \quad \sup_{\mathbf{x} \in \Omega} \|F_6(\mathbf{x}, \cdot)\|_{L^\infty(a_1, \infty)} =: C_6 < \infty.$$

Moreover,

$$F_7(\mathbf{x}, rt) := -\frac{3i(rt)^{1/2}}{4\pi\omega} \int_{|\mathbf{s}|=1} A(\rho) Y(\phi) d\sigma_{\mathbf{s}},$$

$$|F_7(\mathbf{x}, rt)| \leq \frac{3}{4\pi\omega} \int_{|\mathbf{s}|=1} \left(\frac{rt}{\rho}\right)^{1/2} \rho^{1/2} |A(\rho)| |Y(\phi)| d\sigma_{\mathbf{s}},$$

and thus

$$(4.87) \quad \sup_{\mathbf{x} \in \Omega} \|F_7(\mathbf{x}, \cdot)\|_{L^\infty(a_1, \infty)} =: C_7 < \infty.$$

Therefore, we obtain, for $t \geq t_0 = 2a_1$,

$$(4.88) \quad |I_2(\mathbf{x}, t)| = \frac{1}{t^{3/2}} \left| \int_{a_1/t}^1 \frac{1}{r^{3/2} (1-r)^{1/2} (1+r)^{3/2}} F_6(\mathbf{x}, rt) dr \right| \\ \leq \frac{2^{1/2} C_6}{t^{3/2}} \int_{a_1/t}^{1/2} \frac{dr}{r^{3/2}} + \frac{2^{3/2} C_6}{t^{3/2}} \int_{1/2}^1 \frac{dr}{(1-r)^{1/2}} \leq \frac{\tilde{C}_6}{t},$$

$$(4.89) \quad |I_3(\mathbf{x}, t)| = \frac{1}{t^{3/2}} \left| \int_{a_1/t}^1 \frac{1}{r^{1/2} (1-r)^{1/2} (1+r)^{5/2}} F_7(\mathbf{x}, rt) dr \right| \\ \leq \frac{2^{1/2} C_7}{t^{3/2}} \left(\int_{a_1/t}^{1/2} \frac{dr}{r^{1/2}} + \int_{1/2}^1 \frac{dr}{(1-r)^{1/2}} \right) \leq \frac{\tilde{C}_7}{t^{3/2}}$$

with some constants $\tilde{C}_6, \tilde{C}_7 > 0$.

To treat the term I_4 , we introduce

$$(4.90) \quad \tilde{F}_8(\mathbf{x}, r, t) := \int_{|\mathbf{s}|=1} \left[\frac{1}{(1+r)^{3/2}} A(\rho) Y(\phi) \right. \\ \left. - \frac{e^{-i(1-r)\omega t}}{2^{3/2}} A(|\mathbf{x} + \mathbf{s}c_0 t|) Y\left(\frac{\mathbf{x} + \mathbf{s}c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|}\right) \right] d\sigma_{\mathbf{s}},$$

$$(4.91) \quad F_8(\mathbf{x}, r, t) := \frac{1}{1-r} \tilde{F}_8(\mathbf{x}, r, t).$$

Using $\tilde{F}_8(\mathbf{x}, 1, t) = 0$, (4.91) can be rewritten as

$$(4.92) \quad F_8(\mathbf{x}, r, t) = \frac{1}{1-r} \left(\tilde{F}_8(\mathbf{x}, 1, t) - \tilde{F}_8(\mathbf{x}, r, t) \right) = -\frac{1}{1-r} \int_r^1 \partial_r \tilde{F}_8(\mathbf{x}, \tau, t) d\tau.$$

From this, we estimate, for $r \in (1 - a_1/t, 1)$,

$$\begin{aligned} |F_8(\mathbf{x}, r, t)| &\leq \left\| \partial_r \tilde{F}_8(\mathbf{x}, \cdot, t) \right\|_{L^\infty(1-a_1/t, 1)} \\ &\leq \int_{|s|=1} \left[\frac{3}{2(1+r)^{5/2}} |A(\rho)| |Y(\phi)| + \frac{1}{(1+r)^{3/2}} |\partial_r A(\rho)| |Y(\phi)| \right. \\ &\quad \left. + \frac{1}{(1+r)^{3/2}} |A(\rho)| |\partial_r Y(\phi)| + \frac{\omega t}{2^{3/2}} |A(|\mathbf{x} + \mathbf{s}c_0 t|)| \left| Y\left(\frac{\mathbf{x} + \mathbf{s}c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|}\right) \right| \right] d\sigma_s. \end{aligned}$$

Therefore, employing (4.59) and (4.60), taking into account the behaviour of $A(\rho)$ and $A'(\rho)$ for $\rho > \rho_1$, we deduce that

$$(4.93) \quad \sup_{\mathbf{x} \in \Omega, r \in (1-a_1/t, 1), t > 2a_1} |F_8(\mathbf{x}, r, t) / t^{1/2}| =: C_8 < \infty.$$

Moreover, for $r \in (1/2, 1 - a_1/t)$ and $\epsilon \in (0, 1/2]$, taking into account (4.91), we have

$$\left| (1-r)^{1/2+\epsilon} t^\epsilon F_8(\mathbf{x}, r, t) \right| = \left| \tilde{F}_8(\mathbf{x}, r, t) \right| \frac{t^\epsilon}{(1-r)^{1/2-\epsilon}} \leq \left| \tilde{F}_8(\mathbf{x}, r, t) \right| \frac{t^{1/2}}{a_1^{1/2-\epsilon}}.$$

Hence, with C_4 and C_7 defined in (4.78) and (4.87), respectively, we can obtain from (4.90) that, for any $\epsilon \in (0, 1/2]$,

$$(4.94) \quad \sup_{\mathbf{x} \in \Omega, r \in (1/2, 1-a_1/t), t > 2a_1} \left| (1-r)^{1/2+\epsilon} t^\epsilon F_8(\mathbf{x}, r, t) \right| \leq \frac{1}{a_1^{1/2-\epsilon}} \left(\frac{2^{1/2} 4\pi\omega}{3} C_7 + 2\pi C_4 \right) =: C_9 < \infty.$$

Altogether (4.93) and (4.94) imply that, for $t \geq t_0 = 2a_1$ and $\epsilon \in (0, 1/2]$,

$$(4.95) \quad \begin{aligned} |I_4(\mathbf{x}, t)| &\leq \frac{1}{4\pi\omega t} \int_0^1 \frac{1}{(1-r)^{1/2}} |F_8(\mathbf{x}, r, t)| dr \\ &= \frac{1}{4\pi\omega t} \left(\int_0^{1/2} \dots + \int_{1/2}^{1-a_1/t} \dots + \int_{1-a_1/t}^1 \dots \right) \\ &\leq \frac{2^{3/2}}{t^{3/2}} \left(\frac{C_7}{3} \int_0^{1/2} \frac{dr}{r^{1/2}} + \frac{C_4}{4\omega} \right) + \frac{C_9}{4\pi\omega t^{1+\epsilon}} \int_{1/2}^{1-a_1/t} \frac{dr}{(1-r)^{1+\epsilon}} \\ &\quad + \frac{C_8}{4\pi\omega t^{1/2}} \int_{1-a_1/t}^1 \frac{dr}{(1-r)^{1/2}} \leq \frac{\tilde{C}_8}{t} \end{aligned}$$

with some constant $\tilde{C}_8 > 0$. Here, for the interval $(0, 1/2)$, we estimated the integrand directly from (4.84) using again (4.78) and (4.87).

From estimates (4.85), (4.88), (4.89), and (4.95), of I_1 , I_2 , I_3 , and I_4 , respectively, in decomposition (4.80), we obtain for $P_{2,1}$ the same estimate as (4.79) for

$P_{2,2}$. The estimate for P_2 in (4.70) with $t_0 = 2a_1$ readily follows from (4.76). This completes the proof of (4.56) in the case $d = 2$.

- Case $d = 3$.

In this case, the solution is given by the Kirchoff's formula [12, Par. 2.4.1 (c)]

$$(4.96) \quad v(\mathbf{x}, t) = \frac{t}{4\pi} \int_{|\mathbf{s}|=1} \left[v_1(\mathbf{x} + \mathbf{s}c_0t) + \partial_t v_0(\mathbf{x} + \mathbf{s}c_0t) + \frac{2}{t} v_0(\mathbf{x} + \mathbf{s}c_0t) \right] d\sigma_{\mathbf{s}}.$$

The assumed form of the initial conditions yields

$$(4.97) \quad v(\mathbf{x}, t) = \frac{t}{4\pi} \int_{|\mathbf{s}|=1} \left[\left(\frac{\mathbf{x} \cdot \mathbf{s} + c_0t}{|\mathbf{x} + \mathbf{s}c_0t|} - 1 \right) c_0 A'(\mathbf{x} + \mathbf{s}c_0t) Y \left(\frac{\mathbf{x} + \mathbf{s}c_0t}{|\mathbf{x} + \mathbf{s}c_0t|} \right) \right. \\ \left. + A(\mathbf{x} + \mathbf{s}c_0t) \partial_t Y \left(\frac{\mathbf{x} + \mathbf{s}c_0t}{|\mathbf{x} + \mathbf{s}c_0t|} \right) + \frac{2}{t} A(\mathbf{x} + \mathbf{s}c_0t) Y \left(\frac{\mathbf{x} + \mathbf{s}c_0t}{|\mathbf{x} + \mathbf{s}c_0t|} \right) \right] d\sigma_{\mathbf{s}}.$$

From (4.40), (4.60), and the fact that, for $d = 3$, both $A(\mathbf{x} + \mathbf{s}c_0t)$ and $A'(\mathbf{x} + \mathbf{s}c_0t)$ are $\mathcal{O}(1/t)$ for $t \gg 1$ uniformly for $\mathbf{x} \in \Omega$, $\mathbf{s} \in \mathbb{S}^2$, we see that each term in the integrand of (4.97) is $\mathcal{O}(1/t^2)$. The estimate (4.56) hence follows.

- Conclusion of the proof.

As in the proof of Lemma 2.3, the estimates of the solution derivatives in (2.13) follow from that of the solution since differentiation under the integral sign in (4.62) and (4.97) is permitted and no singular nor time growing multiplier arise. \square

5. CONCLUSIONS AND OUTLOOK

Motivated primarily by the development of time-domain methods for solving the Helmholtz equation with variable coefficients, we have established the rigorous proof of the LAP under physically reasonable assumptions on the coefficients of the wave equation and the source term. Under an appropriate modification, the LAP was extended to 1D. Moreover, since the speed of stabilisation towards the harmonic regime is a deciding factor for utilising time-domain approaches on practice, we have also provided rigorous estimates for this time convergence.

Our main focus was on the 1D and 2D cases for which the LAP was generally understudied previously. In these cases, exponential (for 1D) and algebraic (for 2D) convergence rates are generally sharp. In the 3D case, the works on wave equations of similar form and some of our numerical experiments (for radial data) suggest that our algebraic convergence result could be improved to the exponential one.

Another possibility to extend our results is to remove the non-trapping assumption on the coefficients. Even though the LAP is still expected to be valid, in this case, the time convergence rate would be much slower: the algebraic rate would be replaced with inversely logarithmical with some exponent.

APPENDIX .

We collect here some technical estimates needed in the proofs of Sections 3–4.

Lemma A.1. *For $\mathbf{y} \in \Omega$, $\Omega \in \mathbb{R}^d$ bounded domain, and K defined by (3.8), we have the following asymptotic expansions valid for $|\mathbf{x}| \gg 1$:*

$$(A.1) \quad K(\mathbf{x} - \mathbf{y}) = \frac{1}{4\pi} \left(\frac{\omega}{2\pi i c_0} \right)^{\frac{d-3}{2}} \frac{e^{i\frac{\omega}{c_0}(|\mathbf{x}| - \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|})}}{|\mathbf{x}|^{\frac{d-1}{2}}} \left[1 + \frac{1}{|\mathbf{x}|} \left((d-3)(d-1) \frac{i c_0}{8\omega} \right. \right. \\ \left. \left. + \frac{d-1}{2} \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|} + \frac{i\omega}{2c_0} \left(|\mathbf{y}|^2 - \left(\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|} \right)^2 \right) \right) \right] + \mathcal{O} \left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}} \right),$$

$$(A.2) \quad \partial_{|\mathbf{x}|} K(\mathbf{x} - \mathbf{y}) = \frac{1}{4\pi} \left(\frac{\omega}{2\pi i c_0} \right)^{\frac{d-3}{2}} \frac{e^{i\frac{\omega}{c_0}(|\mathbf{x}| - \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|})}}{|\mathbf{x}|^{\frac{d-1}{2}}} \left[i \frac{\omega}{c_0} - \frac{1}{|\mathbf{x}|} \left(\frac{\omega^2}{2c_0^2} \left(|\mathbf{y}|^2 - \left(\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|} \right)^2 \right) \right. \right. \\ \left. \left. - \frac{d-1}{2} \frac{i\omega}{c_0} \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|} + \frac{d^2-1}{8} \right) \right] + \mathcal{O} \left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}} \right).$$

Proof. Setting

$$(A.3) \quad \tilde{K}(\mathbf{x} - \mathbf{y}) := \frac{1}{|\mathbf{x} - \mathbf{y}|^{\frac{d-2}{2}}} H_{\frac{d-2}{2}}^{(1)} \left(\frac{\omega}{c_0} |\mathbf{x} - \mathbf{y}| \right),$$

we have

$$(A.4) \quad \partial_{|\mathbf{x}|} \tilde{K}(\mathbf{x} - \mathbf{y}) = \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla \tilde{K}(\mathbf{x} - \mathbf{y}) = \frac{|\mathbf{x}|^2 - \mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|} \frac{1}{|\mathbf{x} - \mathbf{y}|^{\frac{d}{2}}} \left[\frac{\omega}{c_0} \left(H_{\frac{d-2}{2}}^{(1)} \right)' \left(\frac{\omega}{c_0} |\mathbf{x} - \mathbf{y}| \right) \right. \\ \left. + \left(1 - \frac{d}{2} \right) \frac{1}{|\mathbf{x} - \mathbf{y}|} H_{\frac{d-2}{2}}^{(1)} \left(\frac{\omega}{c_0} |\mathbf{x} - \mathbf{y}| \right) \right].$$

Using the asymptotic behavior of $H_p^{(1)}$ for large arguments [24, Sect. 10.17(i–ii)]

$$H_p^{(1)}(x) = \left(\frac{2}{\pi} \right)^{\frac{1}{2}} e^{i(x - \frac{2p+1}{4}\pi)} \left(\frac{1}{x^{1/2}} + \frac{i(4p^2-1)}{8x^{3/2}} \right) + \mathcal{O} \left(\frac{1}{x^{5/2}} \right),$$

$$\frac{d}{dx} H_p^{(1)}(x) = \left(\frac{2}{\pi} \right)^{\frac{1}{2}} e^{i(x - \frac{2p+1}{4}\pi)} \left(\frac{i}{x^{1/2}} - \frac{4p^2+3}{8x^{3/2}} \right) + \mathcal{O} \left(\frac{1}{x^{5/2}} \right), \quad x \gg 1,$$

we can write (A.3) and (A.4), respectively, as

$$(A.5) \quad \tilde{K}(\mathbf{x} - \mathbf{y}) = \left(\frac{2c_0}{\pi\omega} \right)^{1/2} \frac{e^{-i(d-1)\pi/4}}{|\mathbf{x} - \mathbf{y}|^{\frac{d-1}{2}}} e^{i\frac{\omega}{c_0}|\mathbf{x}-\mathbf{y}|} \left[1 + \frac{ic_0}{8\omega} (d-3)(d-1) \frac{1}{|\mathbf{x} - \mathbf{y}|} \right] + \mathcal{O} \left(\frac{1}{|\mathbf{x} - \mathbf{y}|^{\frac{d+3}{2}}} \right),$$

$$(A.6) \quad \partial_{|\mathbf{x}|} \tilde{K}(\mathbf{x} - \mathbf{y}) = \left(\frac{2c_0}{\pi\omega} \right)^{1/2} \frac{|\mathbf{x}|^2 - \mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|} \frac{e^{-i(d-1)\pi/4}}{|\mathbf{x} - \mathbf{y}|^{\frac{d+1}{2}}} e^{i\frac{\omega}{c_0}|\mathbf{x}-\mathbf{y}|} \left[i\frac{\omega}{c_0} + \left(1 - \frac{d}{2} - \frac{(d-2)^2 + 3}{8} \right) \frac{1}{|\mathbf{x} - \mathbf{y}|} \right] + \mathcal{O} \left(\frac{1}{|\mathbf{x} - \mathbf{y}|^{\frac{d+5}{2}}} \right).$$

By using the identity

$$|\mathbf{x} - \mathbf{y}| = |\mathbf{x}| \left(1 - 2\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|^2} + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2} \right)^{1/2},$$

and the Taylor expansion of $(1+z)^{-\gamma/2}$ with $z := -2\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|^2} + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2}$ about $z = 0$, we obtain, for $|\mathbf{x}| \gg 1$,

$$\frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} = \frac{1}{|\mathbf{x}|^\gamma} \left(1 + \gamma \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|^2} \right) + \mathcal{O} \left(\frac{1}{|\mathbf{x}|^{\gamma+2}} \right).$$

We will use this formula with $\gamma = \frac{d-1}{2}, \frac{d+1}{2}, \frac{d+3}{2}$. Moreover, using the Taylor expansions of $(1+z_1)^{1/2}$ and $\exp(z_2)$ with $z_1 := -2\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|^2} + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2}$ and $z_2 := i\frac{\omega}{c_0}|\mathbf{x}-\mathbf{y}| \left[\left(1 - \frac{2\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|^2} + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2} \right)^{1/2} - 1 + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|^2} \right]$ about $z_1 = z_2 = 0$, we obtain

$$\begin{aligned} e^{i\frac{\omega}{c_0}|\mathbf{x}-\mathbf{y}|} &= e^{i\frac{\omega}{c_0}(|\mathbf{x}| - \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|})} e^{i\frac{\omega}{c_0}|\mathbf{x}| \left[\left(1 - \frac{2\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|^2} + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2} \right)^{1/2} - 1 + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|^2} \right]} \\ &= e^{i\frac{\omega}{c_0}(|\mathbf{x}| - \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|})} \left(1 + i\frac{\omega}{c_0}|\mathbf{x}| \left[\left(1 - \frac{2\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|^2} + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2} \right)^{1/2} - 1 + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|^2} \right] \right) + \mathcal{O} \left(\frac{1}{|\mathbf{x}|^2} \right) \\ &= e^{i\frac{\omega}{c_0}(|\mathbf{x}| - \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|})} \left(1 + i\frac{\omega}{2c_0} \frac{|\mathbf{x}|^2 |\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2}{|\mathbf{x}|^3} \right) + \mathcal{O} \left(\frac{1}{|\mathbf{x}|^2} \right). \end{aligned}$$

Therefore, we get from (A.5) and (A.6)

$$(A.7) \quad \tilde{K}(\mathbf{x} - \mathbf{y}) = \left(\frac{2c_0}{\pi\omega} \right)^{1/2} \frac{e^{-i(d-1)\pi/4}}{|\mathbf{x}|^{\frac{d-1}{2}}} e^{i\frac{\omega}{c_0}(|\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|})} \left[1 + \frac{1}{|\mathbf{x}|} \left(i\frac{c_0}{\omega} \frac{(d-3)(d-1)}{8} \right. \right. \\ \left. \left. + \frac{d-1}{2} \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|} + i\frac{\omega}{2c_0} \frac{|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}\cdot\mathbf{y})^2}{|\mathbf{x}|^2} \right) \right] + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}} \right),$$

(A.8)

$$\partial_{|\mathbf{x}|} \tilde{K}(\mathbf{x} - \mathbf{y}) = \left(\frac{2c_0}{\pi\omega} \right)^{1/2} \frac{e^{-i(d-1)\pi/4}}{|\mathbf{x}|^{\frac{d-1}{2}}} e^{i\frac{\omega}{c_0}(|\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|})} \left[i\frac{\omega}{c_0} - \frac{1}{|\mathbf{x}|} \left(\frac{\omega^2}{2c_0^2} \left(|\mathbf{y}|^2 - \left(\frac{\mathbf{x}}{|\mathbf{x}|} \cdot \mathbf{y} \right)^2 \right) \right. \right. \\ \left. \left. + i\frac{\omega}{2c_0} (1-d) \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|} + \frac{d^2-1}{8} \right) \right] + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}} \right).$$

Since $K(\mathbf{x}) = \frac{i}{4} \left(\frac{\omega}{2\pi c_0} \right)^{\frac{d-2}{2}} \tilde{K}(\mathbf{x})$, estimates (A.7) and (A.8) imply (A.1) and (A.2). \square

The next result is elementary but it is convenient to present it separately.

Lemma A.2. *Let $a, b > 0$, and define*

$$(A.9) \quad J := \int_0^1 \frac{dx}{(x^2 + a^2)^{b/2}} \geq 0.$$

Then, we have

$$(A.10) \quad J \leq \begin{cases} C_{1,b}, & b < 1, \\ \log\left(\frac{1}{a} + \sqrt{1 + \frac{1}{a^2}}\right), & b = 1, \\ C_{2,b} \frac{1}{a^{b-1}}, & b > 1, \end{cases}$$

where $C_{1,b} := \frac{1}{1-b}$, $C_{2,b} := \int_0^\infty \frac{dx}{(1+x^2)^{b/2}}$.

Proof. After the change of variable $x \mapsto z := x/a$, we have

$$J = \frac{1}{a^{b-1}} \int_0^{1/a} \frac{dz}{(z^2 + 1)^{b/2}}.$$

Using

$$\int_0^{1/a} \frac{dz}{(z^2 + 1)^{b/2}} \leq \int_0^{1/a} \frac{dz}{z^b} = \frac{a^{b-1}}{1-b}$$

when $b < 1$,

$$\int_0^{1/a} \frac{dz}{(z^2 + 1)^{b/2}} \leq \int_0^\infty \frac{dz}{(z^2 + 1)^{b/2}} =: C_{2,b}$$

when $b > 1$, and

$$J = \int_0^{1/a} \frac{dz}{(z^2 + 1)^{1/2}} = \log \left(\frac{1}{a} + \sqrt{1 + \frac{1}{a^2}} \right)$$

when $b = 1$, the estimate (A.10) follows immediately. \square

The following estimate for a simple singular oscillatory integral is certainly not new. However, since it is not easy to find a precise reference for it, we provide it here for the sake of completeness.

Lemma A.3. *Let $a > 0$. For $t \gg 1$, we have the following estimate*

$$(A.11) \quad \int_0^a \frac{e^{-ixt}}{x^{1/2}} dx = \left(\frac{\pi}{2t} \right)^{1/2} (1 - i) + \mathcal{O} \left(\frac{1}{t} \right).$$

Proof. Making a change of variable $x \mapsto z(x) := \sqrt{xt}$, we have

$$(A.12) \quad I(t) := \int_0^a \frac{e^{-ixt}}{x^{1/2}} dx = \frac{2}{\sqrt{t}} \int_0^{\sqrt{at}} e^{-iz^2} dz.$$

Since the integrand in (A.12) is analytic, we can invoke the Cauchy theorem to deform the integration contour in the complex plane. In particular, we choose the new contour $\Gamma_1 \cup \Gamma_2$ that consists of two parts: the straight line segment Γ_1 and the circular arc Γ_2 . This contour is traversed counterclockwise with Γ_1 and Γ_2 defined, respectively, as

$$\begin{aligned} \Gamma_1 &:= \{z \in \mathbb{C} : z = re^{-i\pi/4}, r \in (0, R)\}, \\ \Gamma_2 &:= \left\{ z \in \mathbb{C} : z = Re^{i\phi}, \phi \in \left(-\frac{\pi}{4}, 0 \right) \right\}, \end{aligned}$$

where for the sake of brevity, we have set $R := \sqrt{at}$. In other words, we can write

$$(A.13) \quad \begin{aligned} I(t) &= \frac{2}{\sqrt{t}} \left(\int_{\Gamma_1} e^{-iz^2} dz + \int_{\Gamma_2} e^{-iz^2} dz \right) \\ &= \frac{2(1-i)}{\sqrt{2t}} \int_0^R e^{-r^2} dr + \frac{2iR}{\sqrt{t}} \int_{-\pi/4}^0 \exp(-iR^2 e^{2i\phi} + i\phi) d\phi. \end{aligned}$$

Note that, for $R = \sqrt{at} \gg 1$, we have

$$(A.14) \quad \begin{aligned} \int_0^R e^{-r^2} dr &= \int_0^\infty e^{-r^2} dr - \int_R^\infty e^{-r^2} dr = \frac{\sqrt{\pi}}{2} - e^{-R^2} \int_0^\infty e^{-r^2 - 2Rr} dr \\ &= \frac{\sqrt{\pi}}{2} + \mathcal{O}(e^{-at}), \end{aligned}$$

$$\begin{aligned}
\text{(A.15)} \quad \int_{-\frac{\pi}{4}}^0 \exp(-iR^2 e^{2i\phi} + i\phi) d\phi &= \frac{1}{2i} \int_{\phi=-\frac{\pi}{4}}^{\phi=0} \left(\frac{1}{e^{2i\phi}}\right)^{1/2} \exp(-iR^2 e^{2i\phi}) d(e^{2i\phi}) \\
&= \frac{1}{2R^2} \left(e^{-iR^2} - \frac{1+i}{\sqrt{2}} e^{-R^2} \right) \\
&\quad + \frac{1}{4R^2} \int_{\phi=-\frac{\pi}{4}}^{\phi=0} \left(\frac{1}{e^{2i\phi}}\right)^{3/2} \exp(-iR^2 e^{2i\phi}) d(e^{2i\phi}) \\
&= \frac{1}{2R^2} \left(e^{-iR^2} - \frac{1+i}{\sqrt{2}} e^{-R^2} \right) \\
&\quad + \frac{i}{2R^2} \int_{-\frac{\pi}{4}}^0 \exp(-iR^2 e^{2i\phi} - i\phi) d\phi \\
&= \mathcal{O}\left(\frac{1}{R^2}\right) = \mathcal{O}\left(\frac{1}{t}\right)
\end{aligned}$$

Inserting (A.14) and (A.15) into (A.13) furnishes the claimed estimate (A.11). \square

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