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Anton Arnold, Sjoerd Geevers, Ilaria Perugia, Dmitry Ponomarev

ON THE EXPONENTIAL TIME-DECAY FOR THE ONE-DIMENSIONAL WAVE EQUATION WITH VARIABLE COEFFICIENTS

ANTON ARNOLD[1*], SJOERD GEEVERS[2*], ILARIA PERUGIA[2*], DMITRY PONOMAREV[1,3*]

[1] *Institute of Analysis and Scientific Computing, Vienna University of Technology*
Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria
[2] *Faculty of Mathematics, University of Vienna*
Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
[3] *St. Petersburg Department of V. A. Steklov Mathematical Institute, RAS,*
Fontanka 27, 191023 St. Petersburg, Russia

Abstract. We consider the initial-value problem for the one-dimensional, time-dependent wave equation with positive, Lipschitz continuous coefficients, which are constant outside a bounded region. Under the assumption of compact support of the initial data, we prove that the local energy decays exponentially fast in time, and provide the explicit constant to which the solution converges for large times. We give explicit estimates of the rate of this exponential decay by two different techniques. The first one is based on the definition of a modified, weighted local energy, with suitably constructed weights. The second one is based on the integral formulation of the problem and, under a more restrictive assumption on the variation of the coefficients, allows us to obtain improved decay rates.

Keywords wave equation with variable coefficients, local energy decay, long-time asymptotics

Mathematics Subject Classification 35L05, 35L10, 35B40

1. Introduction

The paper is dedicated to the investigation of the time-decay of the solution of the following initial-value problem

$$\begin{cases} \beta(x)\,\partial_t^2 u(x,t) - \partial_x\left(\alpha(x)\,\partial_x u(x,t)\right) = 0, & x\in\mathbb{R},\ t>0,\\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), & x\in\mathbb{R}, \end{cases} \tag{1}$$

where the coefficients α, β are positive, Lipschitz, and constant outside a bounded domain, and the initial data u_0, u_1 are compactly supported. Studying such time-decay is important for practical applications and it shall be used in our follow-up paper [2].

Most relevant works, such as the recent ones [10] and [3], are focussed on proving the decay in time of the solution derivatives in L^2-norm taken over bounded sets. Such local energy decay is usually obtained in spatial dimensions $d \geq 2$. Another recent paper [4] contains a time-decay result in the one-dimensional case under certain restrictions on α, β and localisation of the initial data u_0, u_1.

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It is the purpose of the present work to improve such L^2-estimates of the solution derivatives by showing that, in case of compactly supported initial data, the decay of the solution in time is exponential rather than algebraic. We also prove that, in this case, an auxiliary condition on α, β used in [4] can be avoided. Moreover, we generalise the result to prove the decay not only of derivatives but also of the solution function itself. Namely, we conclude that the solution converges exponentially fast in time to a constant, which is explicitly computable in terms of α, β and u_1.

Let us denote $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ and start with a well-posedness result for problem (1).

Proposition 1.1. *Let $\alpha \in W^{1,\infty}(\mathbb{R})$ and $\beta \in L^\infty(\mathbb{R})$ with $\alpha(x) \geq \alpha_{\min} > 0$ and $\beta(x) \geq \beta_{\min} > 0$ for $x \in \mathbb{R}$. Furthermore, let $u_0 \in H^1(\mathbb{R})$ and $u_1 \in L^2(\mathbb{R})$. Then, problem (1) has a unique solution $u \in C^1\left(\mathbb{R}_+, L^2(\mathbb{R})\right) \cap C\left(\mathbb{R}_+, H^1(\mathbb{R})\right)$, with the continuous dependence on the initial data, according to the following estimate*

$$\|u(\cdot,t)\|_{H^1(\mathbb{R})} + \|\partial_t u(\cdot,t)\|_{L^2(\mathbb{R})} \leq C\left(\|u_0\|_{H^1(\mathbb{R})} + \|u_1\|_{L^2(\mathbb{R})}\right), \quad 0 \leq t \leq T, \tag{2}$$

valid for any $T > 0$, with some constant $C = C(T, \alpha, \beta) > 0$.
Moreover, if $u_0 \in H^2(\mathbb{R})$ and $u_1 \in H^1(\mathbb{R})$, then we have $u \in C^2\left(\mathbb{R}_+, L^2(\mathbb{R})\right) \cap C^1\left(\mathbb{R}_+, H^1(\mathbb{R})\right) \cap C\left(\mathbb{R}_+, H^2(\mathbb{R})\right)$.

For the long-time analysis, we shall now make more restrictive assumptions.

Assumption 1.2. *Let α, $\beta \in W^{1,\infty}(\mathbb{R})$ be real-valued functions such that $\alpha(x) \geq \alpha_{\min}$, $\beta(x) \geq \beta_{\min}$ for $x \in \mathbb{R}$, and $\alpha(x) \equiv \alpha_0$, $\beta(x) \equiv \beta_0$ for $x \in \mathbb{R} \backslash \Omega_{in}$, with some open bounded interval $\Omega_{in} \subset \mathbb{R}$ and constants $\alpha_{\min}$, $\beta_{\min}$, α_0, $\beta_0 > 0$.*

Assumption 1.3. *Suppose that $u_0 \in H^1(\mathbb{R})$, $u_1 \in L^2(\mathbb{R})$ are real-valued functions with the supports $\operatorname{supp} u_0$, $\operatorname{supp} u_1 \subset \Omega_0$ for some open bounded interval $\Omega_0 \subset \mathbb{R}$.*

Our main results are given by the following two theorems.

Theorem 1.4. *Assume that α, β, Ω_{in} are as in Assumption 1.2, and u_0, u_1, Ω_0 are as in Assumption 1.3. Then, for any open bounded interval $\Omega \subset \mathbb{R}$, the solution of (1) obeys the decay estimate*

$$\|u(\cdot,t) - u_\infty\|_{H^1(\Omega)} + \|\partial_t u(\cdot,t)\|_{L^2(\Omega)} \leq Ce^{-\Lambda t}, \quad t \geq 0, \tag{3}$$

for some constant $C = C(u_0, u_1, \alpha, \beta, |\Omega_0|, |\Omega_{in}|) > 0$, with $|\Omega_0|$, $|\Omega_{in}|$ denoting the Lebesgue measure of the sets Ω_0, Ω_{in}, respectively, and

$$u_\infty := \frac{1}{2(\alpha_0\beta_0)^{1/2}} \int_{\Omega_0} u_1(x)\beta(x)\,dx, \quad \Lambda := \frac{\gamma_0}{2t_0} e^{-2\gamma_0}\left(1 - 4\gamma_0 e^{-4\gamma_0}\right)^{1/2}, \tag{4}$$

$$\gamma_0 := \frac{t_0}{4}\left\| \frac{\alpha'}{(\alpha\beta)^{1/2}} + \left(\frac{\alpha}{\beta^3}\right)^{1/2} \beta' \right\|_{L^\infty(\Omega_{in})}, \quad t_0 := \int_{\Omega_{in}} \left(\frac{\beta(x)}{\alpha(x)}\right)^{1/2} dx. \tag{5}$$

We remark that the steady state u_∞ given in (4) generalises the one resulting from the d'Alembert formula for constant α and β, and for u_0, u_1 with compact support:

$$u_\infty = \frac{1}{2}\sqrt{\frac{\beta_0}{\alpha_0}} \int_{\mathbb{R}} u_1(x)\,dx.$$

Note that even though the decay (3) is exponential, the decay rate Λ might be be rather small in case of significantly varying coefficients α, β or large Ω_{in}. In particular, the estimate for the decay rate given by the second equation of (4) is exponentially small for large values of γ_0. Therefore, an alternative estimate is provided by the following theorem, which employs

a different idea and yields a better decay rate, but in return requires an assumption on the variation of α and β.

Theorem 1.5. *Assume that α, β, Ω_{in} are as in Assumption 1.2, u_0, u_1, Ω_0 are as in Assumption 1.3, and t_0 is defined in* (5)*. Moreover, suppose that*

$$b_0 := \|\alpha\|_{L^\infty(\Omega_{in})}^{3/2} \|\beta\|_{L^\infty(\Omega_{in})}^{1/2} \left\| \left(\frac{1}{(\alpha\beta)^{1/2}} \right)' \right\|_{L^2(\Omega_{in})}^2 < \frac{1}{t_0}. \tag{6}$$

Then, for the solution of (1)*, estimate* (3) *holds true, and the decay rate is given by*

$$\Lambda = \frac{1}{2t_0} \left| \log (b_0 t_0) \right|. \tag{7}$$

Remark 1.6. *Theorems 1.4 and 1.5 are valid also for complex-valued u_0, u_1 in Assumption 1.3. Indeed, due to the linearity of problem* (1) *and the real-valuedness of α, β, both theorems could be applied separately to the real and imaginary parts of the initial data.*

Theorems 1.4 and 1.5 are to be compared with the following recent result from [4].

Proposition 1.7. [4, Thm. 1.2 (for $d = 1$)] *Assume that $\alpha(x) \equiv \alpha_0$ and that β satisfies Assumption 1.2. Moreover, suppose that β is such that*

$$\eta := \|\beta\|_{L^\infty(\Omega_{in})}^{1/2} \left\| \left(\frac{1}{\beta^{1/2}} \right)' \right\|_{L^\infty(\Omega_{in})} |\Omega_{in}| < 1. \tag{8}$$

Then, for any bounded $\Omega \subset \mathbb{R}$, the solution of (1) *satisfies*

$$\|\partial_x u(\cdot,t)\|_{L^2(\Omega)} + \|\partial_t u(\cdot,t)\|_{L^2(\Omega)} \leq \frac{C}{t^{(1-\eta)/2}}, \quad t \geq t_1, \tag{9}$$

for some constant $C > 0$ and sufficiently large $t_1 > 0$.

Note that, when the initial data are compactly supported, the algebraic decay in (9) appears to be sub-optimal compared to both Theorem 1.4 and Theorem 1.5, which show the exponential decay of the solution derivatives. As observed in [4, Rem. 1.2], the smallness assumption (8) on η measuring the relative perturbation of the coefficients is essential in the proof there, but it is not needed in our proof of Theorem 1.4.

For the wave equation (1) with *constant* coefficients and compactly supported initial data, the solution converges on all bounded domains Ω to the constant u_∞ in *finite time*. However, for variable coefficients, convergence in finite time does not hold and exponential decay is the generic scenario: as an example, consider (1) with $\alpha(x) \equiv \alpha_0$ in $\mathbb{R}$ and β Lipschitz with $\beta(x) = \beta_1$ on a closed subinterval of some open set Ω_{in}, and $\beta(x) = \beta_0$ in $\mathbb{R} \setminus \Omega_{in}$. A localised, travelling wave packet starting within Ω_{in} gets partly reflected and partly transmitted. Hence, its local norm $\|u(\cdot,t) - u_\infty\|_{H^1(\Omega_{in})}$ decays by a constant factor at each of these reflections, giving rise to an exponential decay. On the other hand, we note that, for non-compactly supported initial data, the decay of the derivatives of the solution in time is generally related to the decay of the initial data at infinity. This can be easily seen already for the constant-coefficient case: the solution furnished explicitly by the d'Alembert formula decays to some u_∞ only algebraically fast if the assumed decay at infinity of u_0 or u_1 is algebraic.

The following Sections 2, 3, and 4 are devoted to the proof of Proposition 1.1, Theorem 1.4, and Theorem 1.5, respectively. Some conclusions are drawn in Section 5, and technical lemmas used in the proof of Theorem 1.4 are deferred to the Appendix.

2. Proof of Proposition 1.1

One standard approach to prove the well-posedness of problem (1) is to construct the solution u, for coefficients α and β sufficiently regular, by a Galerkin approximation (e.g. see [5, Sect. 7.2], or [9, Par. 11.2]). Here, we shall, however, use semigroup theory tools, since they require lower regularity assumptions on α and β. Although this approach is quite standard (e.g. see [8, Par. 2], [11, Par. 3], or [9, Sect. 12.3.2]), we still outline it, adapting to our particular setting.

With the notation

$$\phi(x,t) := (u(x,t), v(x,t))^T, \qquad v(x,t) := \partial_t u(x,t),$$

$$A = A(x) := i \begin{pmatrix} 0 & 1 \\ \frac{1}{\beta(x)} \partial_x (\alpha(x) \partial_x \cdot) - \frac{1}{\beta(x)} & 0 \end{pmatrix}, \qquad C = C(x) := \begin{pmatrix} 0 & 0 \\ \frac{1}{\beta(x)} & 0 \end{pmatrix},$$

problem (1) is equivalent to the first order linear system

$$\begin{cases} \partial_t \phi(x,t) = (-iA(x) + C(x))\phi(x,t), & x \in \mathbb{R}, \quad t > 0, \\ \phi(x,0) = \phi_0(x) := (u_0(x), u_1(x))^T, & x \in \mathbb{R}. \end{cases} \tag{10}$$

We split the proof into 2 steps.

Step 1. Functional setting. We set $L^2_\beta(\mathbb{R}) := L^2(\mathbb{R}; \beta(x)\,dx)$. Due to the assumption β, $1/\beta \in L^\infty(\mathbb{R})$, the space $L^2_\beta(\mathbb{R})$ actually consists of the same functions as $L^2(\mathbb{R})$. We define the operator $B^2 := -\frac{1}{\beta(x)} \partial_x (\alpha(x) \partial_x \cdot) + \frac{1}{\beta(x)}$ in $L^2_\beta(\mathbb{R})$. Since α, β are assumed to be positive and bounded away from zero, the operator B^2 is strictly positive. We claim that this operator is self-adjoint with its (maximal) domain being $\operatorname{Dom} B^2 = H^2(\mathbb{R})$. Indeed, the symmetry of the operator B^2 is evident, and $w \in H^2(\mathbb{R})$ implies $B^2 w \in L^2_\beta(\mathbb{R})$. By [12, Lem 2.3], it remains to verify its surjectivity on $L^2_\beta(\mathbb{R})$ or, equivalently, on $L^2(\mathbb{R})$. The latter follows from the fact that, for $\alpha \in W^{1,\infty}(\mathbb{R})$, the operator $\widetilde{B}^2 := -\partial_x (\alpha(x) \partial_x \cdot) + 1$ from $H^2(\mathbb{R})$ into $L^2(\mathbb{R})$ is strictly positive, and hence invertible.

Next, we define in $L^2_\beta(\mathbb{R})$ the positive square root $B := \sqrt{B^2}$, whose (maximal) domain is $\operatorname{Dom} B = H^1(\mathbb{R})$. In fact, for any $w \in H^1(\mathbb{R})$, we have

$$\begin{aligned} \|Bw\|^2_{L^2_\beta(\mathbb{R})} = \langle Bw, Bw \rangle_{L^2_\beta(\mathbb{R})} &= \int_{\mathbb{R}} \left(B^2 w\right)(x)\, w(x)\, \beta(x)\, dx \\ &= \int_{\mathbb{R}} \left[\alpha(x) (w'(x))^2 + w^2(x)\right] dx, \end{aligned}$$

which, due to positivity and boundedness of α, shows the equivalence between $\|B \cdot\|_{L^2_\beta(\mathbb{R})}$ and $\|\cdot\|_{H^1(\mathbb{R})}$.

With this preparation, we can introduce the Hilbert space $X := H^1(\mathbb{R}) \times L^2(\mathbb{R})$ equipped with the inner product $\langle \phi, \widetilde{\phi} \rangle_X := \langle Bu, B\widetilde{u} \rangle_{L^2_\beta(\mathbb{R})} + \langle v, \widetilde{v} \rangle_{L^2_\beta(\mathbb{R})}$. Note that we have $\phi_0 \in X$.

Step 2. Evolution semigroup. Observe that the matrix operator A is symmetric in X, since for any ϕ, $\widetilde{\phi} \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$, we have

$$\begin{aligned} \left\langle A\phi, \widetilde{\phi} \right\rangle_X &= i \langle Bv, B\widetilde{u} \rangle_{L^2_\beta(\mathbb{R})} + i \left\langle -B^2 u, \widetilde{v} \right\rangle_{L^2_\beta(\mathbb{R})} \\ &= i \left\langle v, B^2 \widetilde{u} \right\rangle_{L^2_\beta(\mathbb{R})} - i \langle Bu, B\tilde{v} \rangle_{L^2_\beta(\mathbb{R})} = \left\langle \phi, A\widetilde{\phi} \right\rangle_X. \end{aligned}$$

In fact, A is self-adjoint in X with (maximal) domain $\operatorname{Dom} A = H^2(\mathbb{R}) \times H^1(\mathbb{R})$. The latter statement follows from the maximality of $\operatorname{Dom} B^2 = H^2(\mathbb{R})$.

By Stone's theorem [7, Ch. 1 Thm. 10.8], the operator $-iA$ generates a C_0-semigroup of unitary operators on X, and the operator C is a bounded perturbation. Therefore, using results

on the perturbation of semigroups [7, Sect. 3.1 Thm. 1.1], the operator $-iA+C$, which defines problem (10), generates a C_0-semigroup of bounded operators on X. Consequently, we have the following two solution concepts for problem (10), and thus for problem (1), depending on the regularity of $\phi_0 = (u_0, u_1)^T$ (see [7, p. 105] for this discussion):

i) For $\phi_0 \in X$, problem (10) has a unique mild solution, i.e. $\phi \in C(\mathbb{R}_+, X)$, and hence $u \in C\left(\mathbb{R}_+, H^1(\mathbb{R})\right) \cap C^1\left(\mathbb{R}_+, L^2(\mathbb{R})\right)$, $v = \partial_t u \in C\left(\mathbb{R}_+, L^2(\mathbb{R})\right)$

ii) For $\phi_0 \in \mathrm{Dom}\, A$, problem (10) has a unique classical solution (in the semigroup sense), i.e. $\phi \in C^1(\mathbb{R}_+, X) \cap C(\mathbb{R}_+, \mathrm{Dom}\, A)$, and hence $u \in C\left(\mathbb{R}_+, H^2(\mathbb{R})\right) \cap C^1\left(\mathbb{R}_+, H^1(\mathbb{R})\right)$, $v = \partial_t u \in C^1\left(\mathbb{R}_+, L^2(\mathbb{R})\right)$.

Finally, estimate (2) is an automatic consequence of the semigroup approach due to [7, Sect. 1.2 Thm. 2.2]. □

3. Proof of Theorem 1.4

Without loss of generality, let us take $\Omega = \Omega_{in} = \Omega_0 = (0, x_0)$ for some $x_0 > 0$. This is possible since the decay in the region of interest Ω can be deduced by enlarging it (if necessary) to an interval containing both Ω_0 and Ω_{in}; moreover, as the problem is posed on the entire real line, we can choose the origin at the left end of the resulting interval.

While the result of this theorem is stated for the mild solution

$$u \in C^1\left(\mathbb{R}_+, L^2(\mathbb{R})\right) \cap C\left(\mathbb{R}_+, H^1(\mathbb{R})\right), \tag{11}$$

its proof requires the regularity of classical solutions established in Proposition 1.1. Hence we approximate the initial data $(u_0,\, u_1)^T \in X := H^1(\mathbb{R}) \times L^2(\mathbb{R})$ by a sequence of functions $(u_0^n,\, u_1^n)_{n\in\mathbb{N}}^T \subset H^2(\mathbb{R}) \times H^1(\mathbb{R})$ with support in Ω_0 and converging in X to $(u_0,\, u_1)^T$. Due to Proposition 1.1, each approximate initial condition $(u_0^n,\, u_1^n)$ gives rise to a classical solution

$$u^n \in C^2\left(\mathbb{R}_+, L^2(\mathbb{R})\right) \cap C^1\left(\mathbb{R}_+, H^1(\mathbb{R})\right) \cap C\left(\mathbb{R}_+, H^2(\mathbb{R})\right). \tag{12}$$

By the linearity of problem (1) and bound (2), these classical solutions converge uniformly to the mild solution u on any finite time interval $[0, T]$:

$$u^n \overset{n\to\infty}{\longrightarrow} u \quad \text{in } C^1([0,T], L^2(\mathbb{R})) \cap C([0,T], H^1(\mathbb{R})). \tag{13}$$

We recall that the notions of classical and mild solutions are discussed in [7, p. 105].

The proof is performed in 3 steps. In Step 1, for any fixed $n \in \mathbb{N}$, we consider problem (1) for initial data $(u_0^n,\, u_1^n) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ and classical solution u^n as in (12). We transform (1) into an auxiliary problem, for which we construct a weighted, local energy functional that would admit an exponential decay under certain conditions on weight functions. In Step 2, we construct these weight functions such that the rate of decay of the energy functional can be estimated explicitly. In Step 3, we use the local energy decay and the convergence in (13) to prove that the mild solution u of (1), for large times, converges to a constant uniformly on $[0, x_0]$. Moreover, we identify this constant as u_∞ defined in (4).

Step 1. Auxiliary local energy functional. As anticipated, here and in Step 2, we establish the exponential-in-time decay of any classical solution u^n, i.e. for fixed index $n \in \mathbb{N}$.

Let us perform the change of variable

$$x \mapsto y(x) := \frac{1}{t_0}\int_0^x \sqrt{\beta(\xi)/\alpha(\xi)}\, d\xi, \quad x(s) = y^{-1}(s), \tag{14}$$

$$t \mapsto \tau(t) := t/t_0, \quad t(\tau) = t_0\tau, \quad t_0 := \int_0^{x_0} \sqrt{\beta(\xi)/\alpha(\xi)}\, d\xi, \tag{15}$$

and denote $v(y,\tau) := u^n(x(y), t_0\tau)$, $v_0(y) := u_0^n(x(y))$, $v_1(y) := t_0\, u_1^n(x(y))$,

$$\rho(y) := \frac{1}{\sqrt{\alpha(x(y))\,\beta(x(y))}}, \quad \gamma(y) := \frac{\rho'(y)}{\rho(y)}, \quad \rho_0 := \rho(0) = \rho(1) = \frac{1}{\sqrt{\alpha_0\beta_0}}. \tag{16}$$

Due to Assumption 1.2, the map $x \mapsto y(x)$ is $W^{2,\infty}(\mathbb{R})$, and hence v inherits the regularity of u^n given by (12):

$$v \in C^2\left(\mathbb{R}_+, L^2(\mathbb{R})\right) \cap C^1\left(\mathbb{R}_+, H^1(\mathbb{R})\right) \cap C\left(\mathbb{R}_+, H^2(\mathbb{R})\right). \tag{17}$$

We thus arrive at a problem equivalent to (1):

$$\begin{cases} \partial_\tau^2 v(y,\tau) - \partial_y^2 v(y,\tau) + \gamma(y)\,\partial_y v(y,\tau) = 0, & y \in \mathbb{R}, \quad \tau > 0, \\ v(y,0) = v_0(y), \quad \partial_\tau v(y,0) = v_1(y), & y \in \mathbb{R}, \end{cases} \tag{18}$$

where $\gamma(y) \equiv 0$ for $y \in \mathbb{R}\backslash(0,1)$ and supp v_0, supp $v_1 \subseteq (0,1)$.

Motivated by the form of the global energy introduced in [6] for the wave equation in (18), we consider the local energy functional

$$\begin{aligned} \mathcal{E}_{loc}(\tau) :=& \frac{1}{2}\int_0^1 \left[(\partial_\tau v(y,\tau))^2 + (\partial_y v(y,\tau))^2\right] \frac{dy}{\rho(y)} \\ =& \frac{1}{4}\int_0^1 \left[(\partial_\tau v(y,\tau) + \partial_y v(y,\tau))^2 + (\partial_\tau v(y,\tau) - \partial_y v(y,\tau))^2\right] \frac{dy}{\rho(y)} > 0. \end{aligned} \tag{19}$$

This quantity is decaying since, by direct calculations employing integration by parts, using (16) and the wave equation in (18), we have

$$\begin{aligned} \mathcal{E}_{loc}'(\tau) &= \int_0^1 \left[\partial_\tau v(y,\tau)\,\partial_\tau^2 v(y,\tau) + \partial_y v(y,\tau)\,\partial_y\partial_\tau v(y,\tau)\right] \frac{dy}{\rho(y)} \\ &= \int_0^1 \left[\partial_\tau v(y,\tau)\left(\partial_\tau^2 v(y,\tau) - \partial_y^2 v(y,\tau) + \frac{\rho'(y)}{\rho(y)}\partial_y v(y,\tau)\right)\right] \frac{dy}{\rho(y)} \\ &\quad + \left[\partial_y v(1,\tau)\,\partial_\tau v(1,\tau) - \partial_y v(0,\tau)\,\partial_\tau v(0,\tau)\right] \frac{1}{\rho_0} \\ &= \left[\partial_y v(1,\tau)\,\partial_\tau v(1,\tau) - \partial_y v(0,\tau)\,\partial_\tau v(0,\tau)\right] \frac{1}{\rho_0} \le 0. \end{aligned} \tag{20}$$

Note that we have used here the regularity of a classical solution. Moreover, in order to deduce the sign in the last line, we have used the exact outflow boundary conditions (well-defined for any $\tau \ge 0$ due to (17)):

$$(\partial_y - \partial_\tau)\, v(y,\tau)\big|_{y=0} = 0 = (\partial_y + \partial_\tau)\, v(y,\tau)\big|_{y=1}, \tag{21}$$

which are due to the support properties of the functions v_0, v_1 and γ.

We note, in passing, that the global energy, defined in the same way as (19) but with the integration range replaced by $\mathbb{R}$, is conserved.

Even though the conventional local energy functional (19) can (and, in the proof of Theorem 1.5, will) be used to show the exponential decay of the solution derivatives under some quantitative restriction on the coefficients α and β, here, we proceed with an alternative strategy that does not require such a restriction. To this effect, we consider now a modified local energy functional in the spirit of [1, Sec. 3]:

$$E_{loc}(\tau) := \frac{1}{2}\int_0^1 \left[\phi_1(y)\,(\partial_\tau v(y,\tau) + \partial_y v(y,\tau))^2 + \phi_2(y)\,(\partial_\tau v(y,\tau) - \partial_y v(y,\tau))^2\right] \frac{dy}{\rho(y)}, \tag{22}$$

where $\phi_1(y)$, $\phi_2(y)$ are some strictly positive weight functions on $[0,1]$, which are yet to be chosen.

Similarly to (20), we differentiate under the integral sign to obtain (suppressing the arguments of all functions for the sake of brevity)

(23)
$$\begin{aligned}E'_{loc}(\tau) =& \int_0^1 \left[\phi_1\left(\partial_\tau v+\partial_y v\right)\left(\partial_\tau^2 v+\partial_y\partial_\tau v\right)+\phi_2\left(\partial_\tau v-\partial_y v\right)\left(\partial_\tau^2 v-\partial_y\partial_\tau v\right)\right]\frac{dy}{\rho}\\ =& \int_0^1 \left[\phi_1\left(\partial_\tau v+\partial_y v\right)\left(\partial_y^2 v-\gamma\partial_y v+\partial_y\partial_\tau v\right)+\phi_2\left(\partial_\tau v-\partial_y v\right)\left(\partial_y^2 v-\gamma\partial_y v-\partial_y\partial_\tau v\right)\right]\frac{dy}{\rho}\\ =& \frac{1}{2}\int_0^1 \left[\phi_1\partial_y\frac{\left(\partial_\tau v+\partial_y v\right)^2}{\rho}-\phi_2\partial_y\frac{\left(\partial_\tau v-\partial_y v\right)^2}{\rho}+\left(\phi_1-\phi_2\right)\gamma\frac{\left(\partial_\tau v\right)^2-\left(\partial_y v\right)^2}{\rho}\right]dy,\end{aligned}$$

where we used the wave equation in (18) to eliminate second-order time derivatives, and the definition of γ in (16). Integrating (23) by parts and employing boundary conditions (21), we obtain

$$\begin{aligned}E'_{loc}(\tau) =& \frac{1}{2}\int_0^1 \Big[-\phi_1'(y)\left(\partial_\tau v(y,\tau)+\partial_y v(y,\tau)\right)^2+\phi_2'(y)\left(\partial_\tau v(y,\tau)-\partial_y v(y,\tau)\right)^2\\ &+\left(\phi_1(y)-\phi_2(y)\right)\gamma(y)\left(\left(\partial_\tau v(y,\tau)\right)^2-\left(\partial_y v(y,\tau)\right)^2\right)\Big]\frac{dy}{\rho(y)}\\ &-\left[\phi_2(1)\left(\partial_y v(1,\tau)\right)^2+\phi_1(0)\left(\partial_y v(0,\tau)\right)^2\right]\frac{2}{\rho_0}.\end{aligned}$$

We estimate

$$\begin{aligned}\left(\phi_1-\phi_2\right)\gamma\left(\left(\partial_\tau v\right)^2-\left(\partial_y v\right)^2\right) &= \left(\phi_1-\phi_2\right)\gamma\left(\partial_\tau v+\partial_y v\right)\left(\partial_\tau v-\partial_y v\right)\\ &\le \frac{1}{2}\left|\phi_1-\phi_2\right|\left|\gamma\right|\left(\left(\partial_\tau v+\partial_y v\right)^2+\left(\partial_\tau v-\partial_y v\right)^2\right),\end{aligned}$$

and therefore, since $\rho>0$, we arrive at

$$\begin{aligned}E'_{loc}(\tau) \le& -\frac{1}{2}\int_0^1 \left[\left(\phi_1'(y)-\frac{|\gamma(y)|}{2}\left|\phi_1(y)-\phi_2(y)\right|\right)\left(\partial_\tau v(y,\tau)+\partial_y v(y,\tau)\right)^2\right.\\ &\left.+\left(-\phi_2'(y)-\frac{|\gamma(y)|}{2}\left|\phi_1(y)-\phi_2(y)\right|\right)\left(\partial_\tau v(y,\tau)-\partial_y v(y,\tau)\right)^2\right]\frac{dy}{\rho(y)}.\end{aligned}\tag{24}$$

We aim to obtain an estimate of the form $E'_{loc}(\tau)\le-\lambda_0 E_{loc}(\tau)$ with some $\lambda_0>0$, and explore the possibility of choosing the weight functions $\phi_1(y)$, $\phi_2(y)$ that would lead to such an estimate. In [1] it is shown in a more general setting that such weights do exist theoretically. Here, we present an explicit construction of ϕ_1, ϕ_2, which will permit us to obtain an explicit estimate of the decay rate of the solution.

Step 2. Construction of the weights ϕ_1, ϕ_2. Motivated by (24), we shall now construct functions $\phi_1(y)$, $\phi_2(y)>0$ in (22) that satisfy, on $[0,1]$, the following differential inequalities

$$\begin{cases}\phi_1'(y)-\frac{|\gamma(y)|}{2}\left|\phi_1(y)-\phi_2(y)\right|\ge\lambda\phi_1(y), & y\in(0,1),\\ -\phi_2'(y)-\frac{|\gamma(y)|}{2}\left|\phi_1(y)-\phi_2(y)\right|\ge\lambda\phi_2(y), & y\in(0,1),\\ \phi_1(0)=\phi_2(0)=1,\end{cases}\tag{25}$$

for some constant $\lambda > 0$ to be chosen. To this end, we consider functions $\varphi_1(y)$, $\varphi_2(y) > 0$ solving the linear ODE system

$$\begin{cases} \varphi_1'(y) - \gamma_0(\varphi_1(y) - \varphi_2(y)) = \lambda\varphi_1(y), & y \in (0,1), \\ -\varphi_2'(y) - \gamma_0(\varphi_1(y) - \varphi_2(y)) = \lambda\varphi_2(y), & y \in (0,1), \\ \varphi_1(0) = \varphi_2(0) = 1, \end{cases} \tag{26}$$

where $\gamma_0 := \frac{1}{2}\|\gamma\|_{L^\infty(0,1)}$.

It is easy to see that any solution of (26) automatically satisfies (25), provided that $\varphi_1(y)$, $\varphi_2(y) > 0$ for all $y \in [0,1]$. Indeed, it suffices to verify that

$$0 \leq \varphi_1(y) - \varphi_2(y) = |\varphi_1(y) - \varphi_2(y)|, \quad y \in [0,1]. \tag{27}$$

Summing both ODEs of (26), multiplying by $e^{-2\gamma_0 y}$, and integrating on $(0,y)$, we obtain

$$\varphi_1(y) - \varphi_2(y) = \lambda \int_0^y e^{2\gamma_0(y-\xi)} [\varphi_1(\xi) + \varphi_2(\xi)]\, d\xi, \quad y \in [0,1],$$

and hence positivity of the integrand directly implies (27).

Note that it is possible to consider more general versions of (25) (and hence (26)), with equations involving different parameters λ_1, λ_2 on the right-hand sides, and initial conditions given by a positive constant different from 1. However, it can be shown that no advantage could be gained from such generalisations.

We shall now estimate the largest possible value $\lambda > 0$ (or, more precisely, its supremum) such that $\varphi_1, \varphi_2 > 0$ in $[0,1]$. According to the first equation in (26) and (27), $\varphi_1(y)$ is a monotonically increasing function of y for $y \in (0,1)$ and thus $\varphi_1(0) = 1$ implies $\varphi_1(y) > 1$ for $y \in (0,1]$. Similarly, it follows from the second equation of (26) that $\varphi_2(y)$ decreases monotonically in $[0,1]$ starting from the value $\varphi_2(0) = 1$. Therefore, to guarantee the positivity of $\varphi_2(y)$ in $[0,1]$, it suffices to require that $\varphi_2(1) > 0$. We can write the solution of (26) explicitly:

$$\varphi_1(y) = \frac{e^{\gamma_0 y}}{\sqrt{\lambda^2+\gamma_0^2}} \left[\sqrt{\lambda^2+\gamma_0^2} \cosh\left(\sqrt{\lambda^2+\gamma_0^2}\, y\right) + (\lambda - \gamma_0) \sinh\left(\sqrt{\lambda^2+\gamma_0^2}\, y\right) \right], \tag{28}$$

$$\varphi_2(y) = \frac{e^{\gamma_0 y}}{\sqrt{\lambda^2+\gamma_0^2}} \left[\sqrt{\lambda^2+\gamma_0^2} \cosh\left(\sqrt{\lambda^2+\gamma_0^2}\, y\right) - (\lambda + \gamma_0) \sinh\left(\sqrt{\lambda^2+\gamma_0^2}\, y\right) \right], \tag{29}$$

Then, the above requirement $\varphi_2(1) > 0$ is equivalent to

$$\tanh\left(\sqrt{\gamma_0^2+\lambda^2}\right) < \frac{\sqrt{\gamma_0^2+\lambda^2}}{\gamma_0+\lambda}. \tag{30}$$

To determine the interval of admissible values of λ satisfying (30), we consider the corresponding equality

$$\tanh\left(\sqrt{\gamma_0^2+\lambda_*^2}\right) = \frac{\sqrt{\gamma_0^2+\lambda_*^2}}{\gamma_0+\lambda_*}, \tag{31}$$

which implicitly defines a function $\lambda_*(\gamma_0)$ on $\mathbb{R}_+$. The proof of the unique solvability of (31) for λ_*, as well as the fact that (30) then holds for all $\lambda \in (0, \lambda_*)$, is deferred to Lemma A.1 in the Appendix (inequality (30) is trivially satisfied for $\lambda = 0$). The function $\lambda_*(\gamma_0)$ is illustrated in Figure 1. We see that $\lambda_*(\gamma_0)$ is monotonically decreasing with $\lambda_* \to 0$ as $\gamma_0 \to \infty$ (this can also be verified analytically by implicit differentiation of (31)).

While the parameter choice $\lambda = \lambda_*$ is not admissible for the construction of the weight functions (since $\varphi_2(1) = 0$ if $\lambda = \lambda_*$), any value $\lambda \in (0, \lambda_*)$ is admissible, and therefore guarantees positivity of φ_1, φ_2.

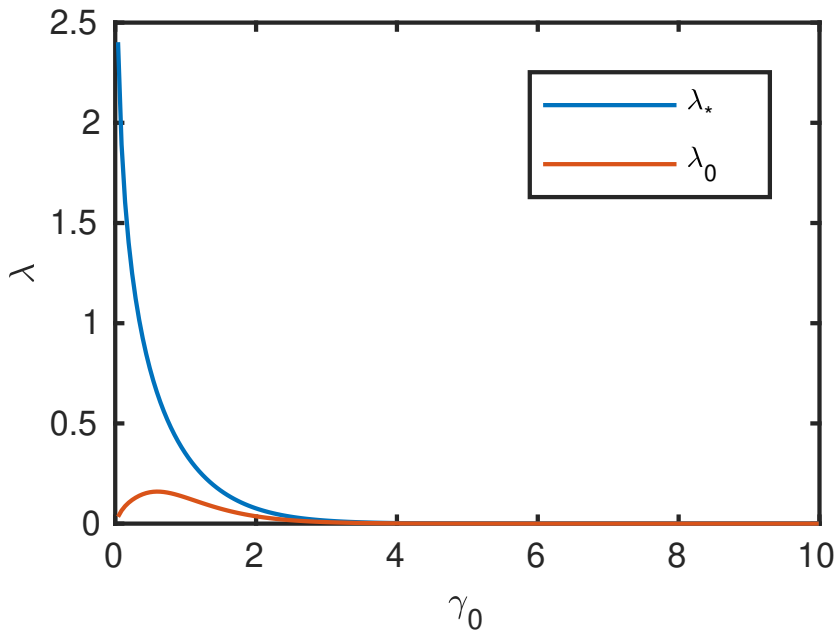

FIGURE 1. Blue: function $\lambda_*(\gamma_0)$, implicitly defined by (31). Red: function $\lambda_0(\gamma_0)$, defined in (32).

Since the function $\lambda_*(\gamma_0)$ is not defined explicitly, we give an example of an explicit lower bound that eventually yields the explicit (but suboptimal) decay rate stated in (4). The function

$$\lambda_0(\gamma_0) := \gamma_0 e^{-2\gamma_0}\sqrt{1-4\gamma_0 e^{-4\gamma_0}} \tag{32}$$

satisfies (30) and $\lambda_0(\gamma_0) < \lambda_*(\gamma_0)$ for any $\gamma_0 > 0$, and it is depicted in Figure 1. The proof of the inequality $\lambda_0(\gamma_0) < \lambda_*(\gamma_0)$ is given in Lemma A.3 in the Appendix.

Therefore, we can take $\lambda = \lambda_0 := \lambda_0(\gamma_0)$, which satisfies inequality (30). Hence the weights $\phi_1(y) := \varphi_1(y) > 0$, $\phi_2(y) := \varphi_2(y) > 0$ satisfy (25). Consequently, we obtain from (24) and (22)

$$E'_{loc}(\tau) \leq -\lambda_0 E_{loc}(\tau), \quad \tau > 0 \quad \Longrightarrow \quad E_{loc}(\tau) \leq E_{loc}(0)\, e^{-\lambda_0 \tau}, \quad \tau \geq 0. \tag{33}$$

Step 3. Convergence of the solution and identification of the constant u_∞. The local energy functional (22) can be rewritten in the original variables (x,t), and we emphasise here that it is a functional of the approximate classical solution u^n:

$$\widetilde{E}_{loc}[u^n](t) := E_{loc}\left(\frac{t}{t_0}\right) = \frac{t_0}{2}\int_0^{x_0}\left[\widetilde{\varphi}_1(x)\left(\partial_t u^n(x,t) + \sqrt{\frac{\alpha(x)}{\beta(x)}}\partial_x u^n(x,t)\right)^2 + \widetilde{\varphi}_2(x)\left(\partial_t u^n(x,t) - \sqrt{\frac{\alpha(x)}{\beta(x)}}\partial_x u^n(x,t)\right)^2\right]\beta(x)\,dx,$$

where $\widetilde{\varphi}_1(x) := \varphi_1(y(x))$, $\widetilde{\varphi}_1(x) := \varphi_2(y(x))$ are given by (28)–(29), with $\lambda = \lambda_0$ and the mapping $x \mapsto y(x)$ given by (14). Then, from (33), we obtain

$$\widetilde{E}_{loc}[u^n](t) \leq \widetilde{E}_{loc}[u^n](0)\, e^{-\frac{\lambda_0}{t_0}t}, \quad t \geq 0,$$

where the decay rate λ_0/t_0 is independent of the index n. Due to the convergence (13), we can pass to the limit $n \to \infty$ and obtain the same inequality for the mild solution u:

$$\widetilde{E}_{loc}[u](t) \leq \widetilde{E}_{loc}[u](0)\, e^{-\frac{\lambda_0}{t_0}t}, \quad t \geq 0, \tag{34}$$

and, due to the regularity of u in (11), we have $\widetilde{E}_{loc}[u] \in C(\mathbb{R}_+)$.

Setting

$$C_0 := \min\left\{\inf_{y\in(0,1)}\varphi_1(y), \inf_{y\in(0,1)}\varphi_2(y)\right\} = \min\{\varphi_1(0), \varphi_2(1)\} = \varphi_2(1) > 0,$$

we have

$$\widetilde{E}_{loc}[u](t) \geq C_0 t_0 \int_0^{x_0} \left[\beta(x)\left(\partial_t u(x,t)\right)^2 + \alpha(x)\left(\partial_x u(x,t)\right)^2\right] dx,$$

and thus deduce that

$$\|\partial_t u(\cdot,t)\|^2_{L^2(0,x_0)} \leq \frac{\|1/\beta\|_{L^\infty(0,x_0)}}{C_0 t_0}\widetilde{E}_{loc}[u](t), \quad \|\partial_x u(\cdot,t)\|^2_{L^2(0,x_0)} \leq \frac{\|1/\alpha\|_{L^\infty(0,x_0)}}{C_0 t_0}\widetilde{E}_{loc}[u](t). \tag{35}$$

This, together with (34), yields the exponential decay of $\|\partial_t u\|_{L^2(0,x_0)}$ and $\|\partial_x u\|_{L^2(0,x_0)}$ claimed in (3). In order to complete the proof, it remains to identify u_∞, and prove that, for all $t \geq 0$, $u(\cdot,t)$ converges in $H^1(0,x_0)$ to u_∞.

Using the regularity of u given by (11), we have, for $x \in (0,x_0)$, $t > 0$,

$$u(x,t) = u(0,t) + \int_0^x \partial_x u(s,t)\, ds = u(x_0,t) - \int_x^{x_0} \partial_x u(s,t)\, ds,$$

which implies

$$\|u(\cdot,t) - u(0,t)\|^2_{L^2(0,x_0)}, \|u(\cdot,t) - u(x_0,t)\|^2_{L^2(0,x_0)} \leq x_0^2 \|\partial_x u(\cdot,t)\|^2_{L^2(0,x_0)}. \tag{36}$$

Let us now integrate the wave equation in (1) over the rectangle $(0,x_0)\times(0,t)$, $t > 0$. Using the outflow boundary conditions (11)

$$\left(\partial_x - \sqrt{\frac{\beta_0}{\alpha_0}}\partial_t\right) u(x,t)\Big|_{x=0} = 0, \quad \left(\partial_x + \sqrt{\frac{\beta_0}{\alpha_0}}\partial_t\right) u(x,t)\Big|_{x=x_0} = 0,$$

and the compact support of u_0, u_1, we obtain

$$\int_0^{x_0} \beta(s)\,\partial_t u(s,t)\, ds - \int_0^{x_0} \beta(s)\, u_1(s)\, ds = -\sqrt{\alpha_0\beta_0}\,(u(0,t) + u(x_0,t)). \tag{37}$$

Due to the weak regularity of the mild solution u, the same identity actually first needs to be derived for the classical solutions u^n from (12), and then (37) follows by passing to the limit $n \to \infty$. From (37), we deduce

$$\begin{aligned} u(x,t) - \frac{1}{2\sqrt{\alpha_0\beta_0}}\int_0^{x_0} \beta(s)\, u_1(s)\, ds =& \frac{1}{2}\left[u(x,t) - u(0,t)\right] + \frac{1}{2}\left[u(x,t) - u(x_0,t)\right] \\ &- \frac{1}{2\sqrt{\alpha_0\beta_0}}\int_0^{x_0} \beta(s)\,\partial_t u(s,t)\, ds \end{aligned} \tag{38}$$

for all $x \in (0,x_0)$ and all $t \geq 0$. For the first two terms on the right-hand side of (38), we have

$$\|u(x,t) - u(0,t)\|^2_{H^1(0,x_0)} = \|u(x,t) - u(0,t)\|^2_{L^2(0,x_0)} + \|\partial_x u(x,t)\|^2_{L^2(0,x_0)},$$

$$\|u(x,t)-u(x_0,t)\|^2_{H^1(0,x_0)}=\|u(x,t)-u(x_0,t)\|^2_{L^2(0,x_0)}+\|\partial_x u(x,t)\|^2_{L^2(0,x_0)}.$$

Moreover, since the integral term on the right-hand side of (38) is independent of x, we also have

$$\begin{aligned}\left\|\frac{1}{2\sqrt{\alpha_0\beta_0}}\int_0^{x_0}\beta(s)\,\partial_t u(s,t)\,ds\right\|_{H^1(0,x_0)}&=\left\|\frac{1}{2\sqrt{\alpha_0\beta_0}}\int_0^{x_0}\beta(s)\,\partial_t u(s,t)\,ds\right\|_{L^2(0,x_0)}\\&\le\frac{1}{2}\sqrt{\frac{x_0}{\alpha_0\beta_0}}\,\|\beta\|_{L^2(0,x_0)}\,\|\partial_t u\|_{L^2(0,x_0)}.\end{aligned}$$

Therefore, taking the $H^1(0,x_0)$ norm of the identity in (38), with u_∞ defined as in (4), we deduce from (36) that

$$\begin{aligned}\|u(\cdot,t)-u_\infty\|_{H^1(0,x_0)}\le&\frac{1}{2}\|u(x,t)-u(0,t)\|_{H^1(0,x_0)}+\frac{1}{2}\|u(x,t)-u(x_0,t)\|_{H^1(0,x_0)}\\&+\frac{1}{2}\sqrt{\frac{x_0}{\alpha_0\beta_0}}\,\|\beta\|_{L^2(0,x_0)}\,\|\partial_t u\|_{L^2(0,x_0)}\\\le&\sqrt{1+x_0^2}\,\|\partial_x u\|_{L^2(0,x_0)}+\frac{1}{2}\sqrt{\frac{x_0}{\alpha_0\beta_0}}\,\|\beta\|_{L^2(0,x_0)}\,\|\partial_t u\|_{L^2(0,x_0)}.\end{aligned}$$

Finally, employing (35), we obtain

$$\begin{aligned}\|u(\cdot,t)-u_\infty\|_{H^1(0,x_0)}\le&\frac{1}{2\,(C_0t_0)^{1/2}}\left[2\left(\left(1+x_0^2\right)\|1/\alpha\|_{L^\infty(0,x_0)}\right)^{1/2}\right.\\&\left.+\left(\frac{x_0}{\alpha_0\beta_0}\,\|1/\beta\|_{L^\infty(0,x_0)}\right)^{1/2}\|\beta\|_{L^2(0,x_0)}\right]\widetilde{E}_{loc}[u]^{1/2}(t)\end{aligned}$$

which, together with (34), leads to the bound of $\|u(\cdot,t)-u_\infty\|_{H^1(0,x_0)}$ in (3). From (32) and (34), we find the claimed decay rate $\Lambda=\frac{\gamma_0}{2\,t_0}e^{-2\gamma_0}\left(1-4\gamma_0e^{-4\gamma_0}\right)^{1/2}$.

Since $\|u(\cdot,t)-u_\infty\|_{H^1(0,x_0)}$ decays to zero, u_∞ is uniquely defined as in (4). This concludes the proof. □

In the proof of Theorem 1.4, modified weight functions could also have been obtained by imposing the initial condition for (25) at an internal point y_0 in $(0,1)$ instead of at $y_0=0$, and then solving the two corresponding linear ODEs forward and backward. Potentially, this may lead to an increased maximal decay rate, as compared to that we have deduced from (31).

We also remark that the decay rate given by $\lambda_0(\gamma_0)$ is exponentially small for large γ_0. This is not an artefact of the approximation of $\lambda_*(\gamma_0)$ by $\lambda_0(\gamma_0)$ (see Figure 1), but is rather characteristic to the weight function approach in the modified local energy.

4. Proof of Theorem 1.5

As in the proof of Theorem 1.4 in Section 3, we assume, without loss of generality, that $\Omega=\Omega_{in}=\Omega_0=(0,x_0)$ for some $x_0>0$.

To prove the claim, we shall modify Step 1 of the proof of Theorem 1.4. Namely, instead of working with the modified local energy functional (22), we shall focus on its conventional counterpart (19) and estimate the quantities that enter it. Since we shall not differentiate the energy functional in time, we work here directly with the mild solution u. We consider again equation (18) in the variables (y,τ) given by (14)–(15). By treating the third term on the left-hand side as a (known) source term, it results in a wave equation with constant coefficients, to

which we apply the d'Alembert formula:

(39)
$$v(y,\tau)=\frac{1}{2}\left[v_0(y+\tau)+v_0(y-\tau)\right]+\frac{1}{2}\int_{y-\tau}^{y+\tau}v_1(z)\,dz-\frac{1}{2}\int_0^\tau\int_{y-s}^{y+s}\gamma(z)\,\partial_y v(z,\tau-s)\,dzds.$$

Therefore, we obtain

$$\begin{aligned}\partial_y v(y,\tau)=&\frac{1}{2}\left[v_0'(y+\tau)+v_0'(y-\tau)\right]+\frac{1}{2}\left[v_1(y+\tau)-v_1(y-\tau)\right]\\&+\frac{1}{2}\int_0^\tau\left[\gamma(y-s)\,\partial_y v(y-s,\tau-s)-\gamma(y+s)\,\partial_y v(y+s,\tau-s)\right]ds,\end{aligned}\tag{40}$$

$$\begin{aligned}\partial_\tau v(y,\tau)=&\frac{1}{2}\left[v_0'(y+\tau)-v_0'(y-\tau)\right]+\frac{1}{2}\left[v_1(y+\tau)+v_1(y-\tau)\right]\\&-\frac{1}{2}\int_0^\tau\left[\gamma(y-s)\,\partial_y v(y-s,\tau-s)+\gamma(y+s)\,\partial_y v(y+s,\tau-s)\right]ds.\end{aligned}\tag{41}$$

Identity (41) becomes evident by making a change of variable $s\mapsto\tau-s$ in the double-integral term in (39) before and after the differentiation with respect to τ:

$$\begin{aligned}&\partial_\tau\int_0^\tau\int_{y-s}^{y+s}\gamma(z)\,\partial_y v(z,\tau-s)\,dzds=\partial_\tau\int_0^\tau\int_{y-\tau+s}^{y+\tau-s}\gamma(z)\,\partial_y v(z,s)\,dzds\\&=\int_0^\tau\left[\gamma(y+\tau-s)\,\partial_y v(y+\tau-s,s)+\gamma(y-\tau+s)\,\partial_y v(y-\tau+s,s)\right]ds\\&=\int_0^\tau\left[\gamma(y+s)\,\partial_y v(y+s,\tau-s)+\gamma(y-s)\,\partial_y v(y-s,\tau-s)\right]ds.\end{aligned}$$

Note that, for $\tau\geq 1$, the terms in (40) and (41) that do not contain the integral vanish for all y in $(0,1)$. To estimate $\mathcal{E}_{loc}$ for $\tau\geq 1$, we insert (40) and (41) in the definition of $\mathcal{E}_{loc}(\tau)$ in (19) and use that $\operatorname{supp}\gamma\subset(0,1)$:

$$\begin{aligned}\mathcal{E}_{loc}(\tau)=&\frac{1}{4}\int_0^1\left[\left(\int_y^1\gamma(r)\,\partial_y v(r,\tau+y-r)\,dr\right)^2\right.\\&\left.+\left(\int_0^y\gamma(r)\,\partial_y v(r,\tau-y+r)\,dr\right)^2\right]\frac{dy}{\rho(y)}\\\leq&\frac{1}{4}\int_0^1\left[\int_y^1\rho(\tilde r)\,\gamma^2(\tilde r)\,d\tilde r\int_y^1\left(\partial_y v(r,\tau+y-r)\right)^2\frac{dr}{\rho(r)}\right.\\&\left.+\int_0^y\rho(\tilde r)\,\gamma^2(\tilde r)\,d\tilde r\int_0^y\left(\partial_y v(r,\tau-y+r)\right)^2\frac{dr}{\rho(r)}\right]\frac{dy}{\rho(y)}.\end{aligned}\tag{42}$$

Denoting

$$b_0:=\|\alpha\|_{L^\infty(0,x_0)}^{3/2}\|\beta\|_{L^\infty(0,x_0)}^{1/2}\left\|\left(1/(\alpha\beta)^{1/2}\right)'\right\|_{L^2(0,x_0)}^2,$$

as in (6), we have

$$\max\left\{\sup_{y\in(0,1)}\frac{1}{\rho(y)}\int_0^y\rho(s)\,\gamma^2(s)\,ds,\ \sup_{y\in(0,1)}\frac{1}{\rho(y)}\int_y^1\rho(s)\,\gamma^2(s)\,ds\right\}\leq b_0t_0.$$

Therefore, we continue with (42), and estimate

$$\mathcal{E}_{loc}(\tau) \leq \frac{b_0 t_0}{4}\left[\int_0^1\int_0^y \left(\partial_y v(r,\tau-y+r)\right)^2 \frac{1}{\rho(r)}drdy + \int_0^1\int_y^1 \left(\partial_y v(r,\tau+y-r)\right)^2 \frac{1}{\rho(r)}drdy\right]. \tag{43}$$

In the two terms on the right-hand side of (43), the integration involves the values of $\partial_y v$ on two right-angled triangles with vertices $\{(0,\tau-1),(0,\tau),(1,\tau)\}$ and $\{(1,\tau-1),(0,\tau),(1,\tau)\}$. These triangles are parametrised by the sets of lines parallel to their hypotenuses, which are the two characteristic families of the wave equation in (18). We now change the integration variables to have parametrisation of these triangles by the sets of horizontal lines. Consequently, for the first integral, denoting $\sigma := \tau - y + r$, we have that $\sigma \in (\tau-1,\tau)$ and $r \in (0,\sigma-\tau+1)$. Similarly, for the second integral, denoting $\sigma := \tau + y - r$, we have that $\sigma \in (\tau-1,\tau)$ and $r \in (\tau-\sigma,1)$. Therefore, rewriting the integrals in (43) as integrals in the variables σ and r, we obtain

$$\begin{aligned}&\int_0^1\int_0^y \left(\partial_y v(r,\tau-y+r)\right)^2 \frac{1}{\rho(r)}drdy + \int_0^1\int_y^1 \left(\partial_y v(r,\tau+y-r)\right)^2 \frac{1}{\rho(r)}drdy\\ &\quad = \int_{\tau-1}^{\tau}\int_0^{\sigma-\tau+1} \left(\partial_y v(r,\sigma)\right)^2 \frac{1}{\rho(r)}drd\sigma + \int_{\tau-1}^{\tau}\int_{\tau-\sigma}^1 \left(\partial_y v(r,\sigma)\right)^2 \frac{1}{\rho(r)}drd\sigma.\end{aligned}$$

With this identity, taking into account the positivity of the integrands, we obtain from (43) that

$$\mathcal{E}_{loc}(\tau) \leq \frac{b_0 t_0}{2}\int_{\tau-1}^{\tau}\int_0^1 \left(\partial_y v(r,\sigma)\right)^2 \frac{1}{\rho(r)}drd\sigma \leq b_0 t_0 \int_{\tau-1}^{\tau} \mathcal{E}_{loc}(\sigma)\,d\sigma. \tag{44}$$

Since $\mathcal{E}_{loc}(\tau)$ is a positive non-increasing function (recall (20)), it follows from (44) that

$$\mathcal{E}_{loc}(\tau) \leq b_0 t_0\, \mathcal{E}_{loc}(\tau-1), \quad \tau \geq 1.$$

Iterating this inequality $\lfloor\tau\rfloor$ times (where $\lfloor X\rfloor$ denotes the integer part of X), and using the assumption $b_0 t_0 < 1$, we obtain, for $\tau \geq 1$,

$$\begin{aligned}\mathcal{E}_{loc}(\tau) &\leq \mathcal{E}_{loc}(\lfloor\tau\rfloor) \leq (b_0t_0)^2\mathcal{E}_{loc}(\lfloor\tau\rfloor-2) \leq \ldots \leq (b_0t_0)^{\lfloor\tau\rfloor}\mathcal{E}_{loc}(0)\\ &= \mathcal{E}_{loc}(0)\, e^{\lfloor\tau\rfloor\log(b_0t_0)} \leq \mathcal{E}_{loc}(0)\, e^{(\tau-1)\log(b_0t_0)} = \mathcal{E}_{loc}(0)\,\frac{e^{\tau\log(b_0t_0)}}{b_0t_0},\end{aligned}$$

which, due to $\log(b_0t_0) < 0$, shows an exponential decay in τ.

When scaling back to the original variables (x,t) we obtain

$$\tilde{\mathcal{E}}_{loc}(t) := \mathcal{E}_{loc}\left(\frac{t}{t_0}\right) \leq \tilde{\mathcal{E}}_{loc}(0)\,\frac{e^{\frac{t}{t_0}\log(b_0t_0)}}{b_0t_0}, \quad t \geq t_0.$$

Step 3 of the proof of Theorem 1.4 can then be repeated (with a slight modification of the multiplicative constants), yielding the exponential convergence (with rate (7)) of the solution to the constant u_∞.

We note that, even though the last bound was proved for $t \geq t_0$, it can be extended to $t \geq 0$ merely at the expense of enlarging the constant in front of the time-decaying exponential factor. This is due to the well-posedness result for the wave equation on a finite time-interval, see Proposition 1.1 and, in particular, (2). □

5. Conclusion

For the homogeneous wave equation in $\mathbb{R}$ with compactly supported initial data, in the case of constant coefficients, the solution converges to the steady state in finite time. However, exponential convergence is the generic scenario in the case of variable coefficients.

We proved two different results for this convergence in local energy. The first one employs a multiplier technique, constructing appropriate weight functions to define a modified local energy that satisfies a first order differential inequality. This allowed us to prove the exponential decay. Using the solution balance over space-time rectangles, we were able to identify the constant steady state. The second result improves the first one, under an additional assumption on the variation of the coefficients. After rescaling the equation, its integral form allowed us to estimate the local energy by iterating in time, leading to an exponential decay with improved rates.

Our results complement those of algebraic time-decay presented in [4], where the initial data were assumed to be sufficiently localised but not compactly supported. In that case, exponential decay is generally not expected.

Appendix .

Lemma A.1. *For any fixed $\gamma_0 > 0$, the equation*

$$\tanh\left(\sqrt{\gamma_0^2+\lambda^2}\right) = \frac{\sqrt{\gamma_0^2+\lambda^2}}{\gamma_0+\lambda}, \quad \lambda > 0, \tag{45}$$

has a unique solution $\lambda = \lambda_*$. Moreover, for $\lambda > 0$, the strict inequality (30) *is satisfied if and only if $\lambda \in (0, \lambda_*)$.*

Proof. Since the right-hand side of (45), for $\lambda > 0$, is smaller than 1, equation (45) is equivalent to

$$f_{\gamma_0}(\tilde\lambda) := \gamma_0\sqrt{1+\tilde\lambda^2} - \operatorname{artanh}\left(\frac{\sqrt{1+\tilde\lambda^2}}{1+\tilde\lambda}\right) = 0, \tag{46}$$

where $\tilde\lambda := \lambda/\gamma_0$. We have

$$f'_{\gamma_0}(\tilde\lambda) = \frac{2\gamma_0\tilde\lambda^2 - \tilde\lambda + 1}{2\tilde\lambda\sqrt{1+\tilde\lambda^2}}, \qquad \lim_{\tilde\lambda\to+\infty} f'_{\gamma_0}(\tilde\lambda) = \gamma_0 > 0.$$

Hence $\lim_{\tilde\lambda\to+\infty} f_{\gamma_0}(\tilde\lambda) = +\infty$ and, moreover, $\lim_{\tilde\lambda\to 0^+} f_{\gamma_0}(\tilde\lambda) = -\infty$.

Case 1: For $\gamma_0 \geq 1/8$, we have $f'_{\gamma_0}(\tilde\lambda) \geq 0$ for any $\tilde\lambda > 0$. Hence, f_{γ_0} is strictly increasing and f_{γ_0} has a unique zero at some $\tilde\lambda_* > 0$ and $f_{\gamma_0}(\tilde\lambda) < 0$ for $\tilde\lambda \in (0, \tilde\lambda_*)$.
Case 2: For $\gamma_0 \in (0, 1/8)$, we consider $f_{\gamma_0}(\tilde\lambda_1)$, where $\tilde\lambda_1 := \frac{1-\sqrt{1-8\gamma_0}}{4\gamma_0}$ is the first zero of f'_{γ_0}. Define $g(\gamma_0) := \frac{1-\sqrt{1-8\gamma_0}}{4\gamma_0}$. From $g'(\gamma_0) > 0$ for all $\gamma_0 \in (0, 1/8)$, $\lim_{\gamma_0\to 0^+} g(\gamma_0) = 1$, and $g(1/8) = 2$, we deduce that g is strictly increasing and $\tilde\lambda_1 = g(\gamma_0) \in (1, 2)$ for all $\gamma_0 \in (0, 1/8)$. Since $\operatorname{artanh}\left(\frac{\sqrt{1+\tilde\lambda^2}}{1+\tilde\lambda}\right)$ is an increasing function of $\tilde\lambda$ in $(1, 2)$, we obtain from (46) that $f_{\gamma_0}(\tilde\lambda_1) \leq \frac{\sqrt{1+4}}{8} - \operatorname{artanh}\left(\frac{\sqrt{2}}{2}\right) \simeq -0.6019$. This shows that $f_{\gamma_0}(\tilde\lambda_1) < 0$ for $\gamma_0 \in (0, 1/8)$. Hence, f_{γ_0} also has a unique zero $\tilde\lambda_* > \tilde\lambda_2 > 0$ in this case, where $\tilde\lambda_2 := \frac{1+\sqrt{1-8\gamma_0}}{4\gamma_0}$ is the second zero of f'_{γ_0}.

This proves unique solvability of equation (45). Moreover, the reasoning above also shows that, for any $\gamma_0 > 0$, there is a unique $\lambda_* = \lambda_*(\gamma_0) > 0$ such that, for $\lambda > 0$, the following

equivalences hold true:

$$\lambda \in (0, \lambda_*) \iff f_{\gamma_0}(\lambda/\gamma_0) < 0 \iff \sqrt{\gamma_0^2+\lambda^2} - \operatorname{artanh}\left(\frac{\sqrt{\gamma_0^2+\lambda^2}}{\gamma_0^2+\lambda}\right) < 0$$
$$\iff \tanh\left(\sqrt{\gamma_0^2+\lambda^2}\right) < \frac{\sqrt{\gamma_0^2+\lambda^2}}{\gamma_0+\lambda}.$$

Here, the last equivalence is due to the strict monotonicity of the tanh function. The second part of the statement thus follows. □

Lemma A.2. *For any $a > 0$ and*

$$1 < q < \sqrt{1 + \frac{e^{4a} + 4a - e^{2a}\sqrt{e^{4a}+8a}}{8a^2}}, \tag{47}$$

we have

$$\frac{q}{\tanh(aq)} > 1 + \sqrt{q^2-1}. \tag{48}$$

Proof. The proof is split into two steps.
Step 1. We shall first analyse for which values $q_0 \geq 1$ the following auxiliary inequality holds:

$$(1+2a)\,e^{-2a} - 2ae^{-2a}q_0^2 \geq \sqrt{q_0^2-1}. \tag{49}$$

Using $Q := q_0^2 \geq 1$ we rewrite (49) as

$$1 + 2a - 2aQ \geq e^{2a}\sqrt{Q-1}. \tag{50}$$

This is equivalent to

$$4a^2Q^2 - \left(4a(1+2a) + e^{4a}\right)Q + (1+2a)^2 + e^{4a} \geq 0, \tag{51}$$

as long as both sides of (50) are nonnegative, i.e. for $Q \in \left[1, \frac{1+2a}{2a}\right]$.

The left-hand side of (51) is a quadratic expression in Q which is positive for $Q \in \mathbb{R}\setminus[Q_1, Q_2]$, with

$$Q_1 := Q_0 - \frac{e^{2a}\sqrt{e^{4a}+8a}}{8a^2}, \quad Q_2 := Q_0 + \frac{e^{2a}\sqrt{e^{4a}+8a}}{8a^2},$$
$$Q_0 := 1 + \frac{e^{4a}+4a}{8a^2}.$$

It can be checked easily that $Q_1 \in \left(1, \frac{1+2a}{2a}\right)$, and therefore any $Q \in [1, Q_1]$ satisfies (50). Consequently, (49) is satisfied if and only if $q_0 \in \left[1, \sqrt{Q_1}\right]$.

Step 2. We observe that since $e^{-bx} \geq e^{-b}(1+b-bx)$ for any b, $x \in \mathbb{R}$ (by convexity of e^{-bx}), we have (with $b := 2a > 0$, $x := q_0 > 1$)

$$\begin{aligned}\frac{q_0}{\tanh(aq_0)} &= \frac{q_0\left(e^{aq_0}+e^{-aq_0}\right)}{e^{aq_0}-e^{-aq_0}} > q_0\left(1+e^{-2aq_0}\right)\\ &\geq q_0\left[1+e^{-2a}(1+2a-2aq_0)\right]\\ &\geq 1 + (1+2a)\,e^{-2a} - 2ae^{-2a}q_0^2.\end{aligned} \tag{52}$$

By assumption (47), we have $q \in \left(1, \sqrt{Q_1}\right)$. Therefore, (52) can be combined with (49), to yield (48). □

Lemma A.3. *The function*

$$\lambda_0(\gamma_0) := \gamma_0 e^{-2\gamma_0}\sqrt{1-4\gamma_0 e^{-4\gamma_0}}, \quad \gamma_0 > 0 \tag{53}$$

satisfies $\lambda_0(\gamma_0) < \lambda_*(\gamma_0)$*, where* $\lambda_*(\gamma_0)$ *is defined via* (31)*. Moreover, each* $\lambda = \lambda_0(\gamma_0)$ *satisfies inequality* (30)*.*

Proof. To show that $\lambda{:=}\lambda_0(\gamma_0)$ satisfies (30) we first derive the following auxiliary inequality:

$$\begin{aligned}\sqrt{1+(\lambda/\gamma_0)^2} &= \sqrt{1+e^{-4\gamma_0}\left(1-4\gamma_0 e^{-4\gamma_0}\right)} \\ &< \sqrt{1+\frac{e^{4\gamma_0}+4\gamma_0-e^{2\gamma_0}\sqrt{e^{4\gamma_0}+8\gamma_0}}{8\gamma_0^2}},\end{aligned}$$

where we employed definition (53) and used the estimate $\sqrt{1+x} < 1+x/2-x^2/8+x^3/16$, with $x = 8\gamma_0 e^{-4\gamma_0} > 0$, which follows from the Taylor expansion of the square root.

The above inequality forms exactly the assumption (47) with $q := \sqrt{1+(\lambda/\gamma_0)^2} > 1$ and $a := \gamma_0$. The assertion of Lemma A.2 is equivalent to the validity of (30).

Finally, Lemma A.1 yields $\lambda_0(\gamma_0) < \lambda_*(\gamma_0)$, which completes the present proof. □

References

[1] A. Arnold, J. Carrillo, and M. Tidriri. Large-time behavior of discrete kinetic equations with non-symmetric interactions. *Math. Models Methods Appl. Sci*, 12(11):1555–1564, 2002.

[2] A. Arnold, S. Geevers, I. Perugia, and D. Ponomarev. On the limiting amplitude principle for the wave equation with variable coefficients. arXiv:2202.10105, 2022.

[3] J.-M. Bouclet and N. Burq. Sharp resolvent and time-decay estimates for dispersive equations on asymptotically Euclidean backgrounds. *Duke Mathematical Journal*, 170(11):2575 – 2629, 2021.

[4] R. Charao and R. Ikehata. A note on decay rates of the local energy for wave equations with Lipschitz wavespeeds. *J. Math. Anal. Appl.*, 483(2):1–14, 2020.

[5] L. Evans. *Partial Differential Equations (2nd edition)*. AMS, 2010.

[6] J. Lewis. The heterogeneous string: coupled helices in Hilbert space. *Quart. Appl. Math.*, 38(4):461–467, 1981.

[7] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, 1983.

[8] M. Reed. *Abstract Non-linear Wave Equations*. Springer-Verlag, 1976.

[9] M. Renardy and R. Rogers. *An Introduction to Partial Differential Equations (2nd edition)*. Springer, 2004.

[10] J. Shapiro. Local energy decay for Lipschitz wavespeeds. *Comm. Partial Differential Equations*, 43(5):839–858, 2018.

[11] W. A. Strauss. *Nonlinear wave equations*. American Mathematical Society, 1990.

[12] G. Teschl. *Mathematical methods in quantum mechanics*. American Mathematical Society, 2009.

Email address: anton.arnold@asc.tuwien.ac.at
Email address: sjoerd.geevers@univie.ac.at
Email address: ilaria.perugia@univie.ac.at
Email address: dmitry.ponomarev@asc.tuwien.ac.at