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The group configuration theorem for generically stable types

Paul Wang¹

Introduction

In his thesis [Hru86], Ehud Hrushovski proved a group configuration theorem, building a definable group from combinatorial data, in a stable setting. The aim of this paper is to generalize the theorem, using only hypotheses on the type of the configuration, without assuming tameness of the theory.

First, we shall introduce generically stable types, and state some of their known properties. Then, we will define some notions of genericity in definable groups, and show a couple of results regarding groups with generically stable generics. Having done that, we shall state and prove a group configuration theorem (Theorem 2.16) for generically stable types. The proofs will be similar to the stable case, although a bit trickier.

From now on, for any theory T , we shall work inside T^{eq} to ensure elimination of imaginaries. We let acl , or acl^{eq} , denote the algebraic closure in T^{eq} . We use similar notations dcl and dcl^{eq} , for the definable closure. Moreover, if T is complete, we let \mathbb{U} denote a sufficiently saturated and sufficiently homogeneous model of T . A subset A of a model M is *small*, with respect to M , if M is $|A|^+$ -saturated and $|A|^+$ -strongly homogeneous. Note that we might consider models $M \subset \mathbb{U}$ and sets $A \subset M$ such that A is small with respect to M , and M itself is small with respect to \mathbb{U} . Then, we might call a complete type over M a *global* type, using the term in a loose way. By default, the sets of parameters we consider are small with respect to \mathbb{U} . If a, b are small tuples, we may write ab or $a \hat{\ } b$ for the concatenation.

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1 Generically stable types

1.1 Forking, invariant and definable types

We fix a complete theory T in a language \mathcal{L} .

Definition 1.1. Let $A \subseteq \mathbb{U}$ be a set of parameters and $\phi(x, y)$ be a formula over A . Let b be a tuple.

1. The formula $\phi(x, b)$ divides over A if there is an A -indiscernible sequence $(b_i)_{i < \omega}$, with $b_0 = b$, such that the partial type $\{\phi(x, b_i) \mid i < \omega\}$ is inconsistent.
2. The formula $\phi(x, b)$ forks over A if $\phi(x, b)$ implies a finite disjunction of formulas, possibly with additional parameters, all of which divide over A .

For any natural number k , a partial type π is k -inconsistent if, for any choice of pairwise non-equivalent formulas ϕ_1, \dots, ϕ_k in π , the conjunction $\bigwedge_i \phi_i$ is not satisfiable.

Then, by compactness and indiscernibility, one may replace "inconsistent" with " k -inconsistent for some k " in the above definition of dividing.

Definition 1.2. 1. A partial type $\pi(x)$ divides (resp. forks) over a set A if there exists a formula $\phi(x)$ (possibly with parameters outside of A) such that $\pi(x) \models \phi(x)$ and $\phi(x)$ divides (resp. forks) over A .

2. Let a, b be tuples, and C be a set. The tuple a is independent from b over C , which we denote $a \perp_C b$, if $tp(a/Cb)$ does not fork over C .
3. Let $p \in S(A)$. The type p is extensible (resp. stationary) if, for any $B \supseteq A$, there exists a (resp. a unique) $q \in S(B)$ such that $q|_A = p$ and q does not fork over A . Given a stationary type $p \in S(A)$ and $B \supseteq A$, we let $p|_B$ denote the unique nonforking extension.

Definition 1.3. Let $(a_i)_{i \in I}$ be a family of elements. Let A be a set of parameters. We say that (a_i) is an independent family over A if, for all $i \in I$, we have $a_i \perp_A (a_j)_{j \in I, j \neq i}$.

Remark 1.4. To simplify notations, if A is a small set of parameters, we write $|A|$ instead of $|A| + |\mathcal{L}| + \aleph_0$.

Definition 1.5. Let M be a model of T , let $p \in S(M)$, and $A \subseteq M$.

1. We say that p is A -definable if, for all formulas without parameters $\phi(x, y)$, there exists a formula $d_p x \phi(x, y)$ with parameters in A such that, for all $b \in M$, we have $M \models d_p x \phi(x, b)$ if and only if $\phi(x, b) \in p(x)$. We say that d_p is "the" defining scheme for p .
2. We say that p is definable if it is M -definable.
3. If p is definable, the canonical basis of p is the smallest dcl-closed set $A \subseteq M$ such that p is A -definable. By elimination of imaginaries, this set is well-defined.

4. In the case where M is $|A|^+$ -saturated, we say that p is A -invariant if, for any formula without parameters $\phi(x, y)$, for all $b_1, b_2 \in M$, if $b_1 \equiv_A b_2$, then $\phi(x, b_1) \in p(x)$ if and only if $\phi(x, b_2) \in p(x)$. In other words, the formula $\phi(x, b)$ being in the type p depends only on the type of b over A .
5. If $B \supseteq A$ is a set of parameters (not necessarily a model), and $q \in S(B)$, we say that q is A -invariant (resp. A -definable, resp. definable) if it admits an A -invariant (resp. A -definable, resp. B -definable) extension q_1 to an $|A|^+$ -saturated model $N \supseteq B$.

Fact 1.6. *Let A be a small subset of a model M . Let $p \in S(M)$ be A -invariant. Then, for any model $N \supseteq M$, the type p has a unique extension $q \in S(N)$ which is A -invariant. We write $q = p|_N$.*

Moreover, if p is A -definable, then q is A -definable, using the same defining scheme as p .

Remark 1.7. Thanks to this fact, if $p \in S(M)$ is A -invariant, where M is $|A|^+$ -saturated, and if $B \supseteq A$, we can write $p|_B \in S(B)$ for the restriction to B of $p|_N$, where N is a model containing MB . By uniqueness, the type $p|_B$ is well-defined, for it does not depend on the choice of the model N .

We consider it useful to view invariant types as families of types, or as type-constructing processes, and to identify two invariant types if they admit a common invariant extension to a big enough model.

Definition 1.8. Let A be a small subset of a model M . Let $p, q \in S(M)$, where p is A -invariant. Let us assume that p is in the variable x , and q in the variable y . We define the tensor product $p \otimes q \in S(M)$:

If $\phi(x, y, z)$ is a formula without parameters, and if $c \in M$, then $\phi(x, y, c) \in p \otimes q$ if and only if, for some (equivalently, for all) element $b \in M$ realizing $q|_{Ac}$, we have $\phi(x, b, c) \in p$. This tensor product is well-defined, and is a complete type over M . Indeed, we can check that the realizations of $p \otimes q$ are exactly the tuples of the form ab , where b realizes q , and a realizes $p|_{Mb}$. Note that if q is also A -invariant, then $p \otimes q$ is A -invariant. If p and q are A -definable, then so is $p \otimes q$.

Definition 1.9. Let A be a small subset of a model M . Let $p \in S(M)$ be an A -invariant type. Let α be an ordinal. A sequence $(a_i)_{i < \alpha}$ is a *Morley sequence* of p over A if, for all $i < \alpha$, the element a_i realizes the type $p|_{A \cup (a_j)_{j < i}}$.

Fact 1.10. *Let $A \subseteq B \subseteq C$ be small sets of parameters, and a, b, c be small tuples.*

1. *Let $p \in S(C)$. Assume that p does not fork over A . Then p does not fork over B and $p|_B$ does not fork over A .*
2. *Let $p \in S(B)$ be a type which does not fork over A . Then, there exists $q \in S(C)$ extending p such that q does not fork over A .*
3. *Assume that $a \perp_A B$ and $b \perp_{Aa} B$. Then $ab \perp_A B$.*

4. Assume $\text{acl}(A) \subseteq B$. Then :
 - (a) $\text{tp}(a/B)$ forks over A if and only if $\text{tp}(a/B)$ forks over $\text{acl}(A)$.
 - (b) $a \downarrow_A b$ if and only if $\text{acl}(Aa) \downarrow_A \text{acl}(Ab)$.
5. Assume $a \in \text{acl}(Ab) \cap \text{acl}(Ac)$ and $b \downarrow_A c$. Then $a \in \text{acl}(A)$.
6. Let $M \supset A$ be an $|A|^+$ -saturated and $|A|^+$ -strongly homogeneous model. Let $p \in S(M)$ be an A -invariant type. Then p does not fork over A .

Lemma 1.11. *Let a be an element, $C \subseteq M$, where M is a model, such that $\text{tp}(a/M)$ is definable over C . Then, for all $b \in \text{acl}(Ca)$, the type $\text{tp}(b/M)$ is definable over $\text{acl}(C)$. Similarly, if $b \in \text{dcl}(Ca)$, then $\text{tp}(b/M)$ is definable over C .*

Proof. Let us prove the first point, the second one being easier. If $b \in \text{acl}(Ca)$, let $\phi(x, y)$ be a formula over C such that $\models \phi(a, b)$ and $\models \forall x \exists^{\leq k} y \phi(x, y)$. Let $\psi(y, z)$ be a formula without parameters. Let us consider the following C -definable binary relation : $z_1 E z_2$ if and only if $\models d_p x [\forall y \phi(x, y) \rightarrow [\psi(y, z_1) \leftrightarrow \psi(y, z_2)]]$. Then, for $z_1, z_2 \in M$, if $z_1 E z_2$, then $\models \psi(b, z_1) \leftrightarrow \psi(b, z_2)$. Indeed, a realizes $p|_{Cz_1z_2}$.

Let us now show that E is a finite equivalence relation. Reflexivity and symmetry are clear. Let us prove transitivity. Let $z_1 E z_2$ and $z_2 E z_3$. Let α realize $p|_{Cz_1z_2z_3}$. Let β be such that $\models \phi(\alpha, \beta)$. Then, we have $\models [\psi(\beta, z_1) \leftrightarrow \psi(\beta, z_2)] \wedge [\psi(\beta, z_2) \leftrightarrow \psi(\beta, z_3)]$. So $\models [\psi(\beta, z_1) \leftrightarrow \psi(\beta, z_3)]$. Since $\alpha \models p|_{Cz_1z_3}$, we have indeed $z_1 E z_3$.

Finally, we shall prove that E has only finitely many classes. As E is C -definable, and $C \subseteq M$, it is enough to prove that $E(M)$ has only finitely many classes. Since a realizes $p|_M$, we know that, for $z_1, z_2 \in M$, we have $z_1 E z_2$ if and only if $\models \forall y (\phi(a, y) \rightarrow [\psi(y, z_1) \leftrightarrow \psi(y, z_2)])$. But $\models \exists^{\leq k} y \phi(a, y)$, therefore $E(M)$ has at most 2^k classes. So E has at most 2^k classes, so it is a finite equivalence relation. \square

1.2 General properties of generically stable types

The definition of generically stable types below is from [PT11] (Definition 2.1). Most of the properties in this subsection come from [GOU13] (Appendix A), from [PT11] (Proposition 2.1) and [ACP14] (Fact 1.9, Lemma 2.1 and Theorem 2.2).

- Definition 1.12.**
1. Let $(a_i)_{i \in I}$ be a sequence of elements of the same sort. Let B be a set. The mean of the types of the a_i over B is a partial (possibly complete) type, containing the formulas $\phi(x, b)$ over B such that, for cofinitely many indices $i \in I$, we have $\models \phi(a_i, b)$.
 2. Let A be a set, let p be a global type, i.e. $p \in S(M)$, where $M \supset A$ is a big enough model. The type p is generically stable over A if p is A -invariant and, for any ordinal $\alpha \geq \omega$, for any Morley sequence $(a_i)_{i < \alpha}$ of p over A , the mean of the types of the a_i over M is a complete type over M .

Remark 1.13. The property for infinite ordinals in the definition above is equivalent to that for countably infinite ordinals. Indeed, a mean over an infinite index set is always a consistent partial type. If it is not complete, there exists a formula witnessing incompleteness. Then, countably many indices are enough to witness incompleteness of the mean for this formula.

Thus, if the model M is sufficiently saturated, it is enough to check the property for Morley sequences made of elements of M .

Proposition 1.14. (see Proposition 2.1 in [PT11])

Let $p \in S(M)$ be a global type, generically stable over a small set $A \subset M$. Then :

1. For any infinite Morley sequence $(a_i)_i$ of p over A , the mean of the types of the a_i is the type p itself.
2. For any $\phi(x, y)$ over A , there exists a natural number n_ϕ such that, for any infinite Morley sequence $(a_i)_i$ of p over A , for any b , we have $\phi(x, b) \in p$ if and only if the set of indices i such that $\models \neg\phi(a_i, b)$ contains at most n_ϕ elements.
3. The global type p is definable over A .
4. Any Morley sequence of p over A is an indiscernible set over A .
5. If $B \subset M$ is a small set such that p is B -invariant, then p is generically stable over B .
6. The type $p|_A$ has a unique nonforking global extension, which is p .

Definition 1.15. Because of these properties, we may call a type $p \in S(A)$ generically stable if p has a (necessarily unique) nonforking global extension q , which is generically stable over A . Note that if q is a global type which does not fork over A , then $q|_A$ is generically stable, in the above sense, if and only if q is generically stable over A .

If $B \supseteq A$, we may also say that a type $q \in S(B)$ is generically stable over A if $q|_A$ is generically stable in the above sense, and q does not fork over A .

Fact 1.16. Let $p \in S(A)$ be a generically stable type, in the sense of Definition 1.15. Let B be a set of parameters containing A . Let $q \in S(B)$ be the unique nonforking extension of p . Then, q is still generically stable in the sense of Definition 1.15, and its global nonforking extension coincides with that of p .

Proposition 1.17. *Transitivity*

Let $p \in S(M)$ be a global type generically stable over A . Let B, C be sets of parameters such that $A \subseteq B \subseteq C \subset M$. Let $a \in M$ be a realization of $p|_A$, such that $a \downarrow_A B$ and $a \downarrow_B C$. Then $a \downarrow_A C$.

Proof. Using stationarity and Fact 1.16 above, one can check that $tp(a/C) = p|_C$. \square

Proposition 1.18. *Symmetry* ([GOU13], Theorem A.2)

Let $p \in S(M)$ be generically stable over A . Let $q \in S(A)$ be a type which does not fork over A . Let a, b be such that $a \models p|_A$ and $b \models q$. Then $a \downarrow_A b$ if and only if $b \downarrow_A a$.

The following lemma will be used repeatedly throughout the proof of the group configuration theorem.

Lemma 1.19. *Swap*

Let A be a set of parameters. Let b, c, d be elements such that the types $tp(b/A)$, $tp(c/A)$, and $tp(d/A)$ are generically stable, in the sense of Definition 1.15.

1. If $c \downarrow_A d$ and $b \downarrow_A cd$, then $bc \downarrow_A d$.
2. If $b \downarrow_A c$ and $bc \downarrow_A d$, then $b \downarrow_A cd$.

Proof. The first point is a consequence of Fact 1.10 (3), and holds in general, and the second point is a consequence of the first one and symmetry. \square

Proposition 1.20. Let $p = tp(a/M)$ be a type generically stable over A , where $A \subseteq M$.

1. If $b \in dcl(Aa)$, then the type $tp(b/M)$ is generically stable over A .
2. If $b \in acl(Aa)$, then the type $tp(b/M)$ is generically stable over $acl(A)$.

Proof. Note that the hypothesis implies $a \downarrow_A M$. Then, up to picking a bigger model $N \supseteq M$ such that $a \downarrow_A N$, we may assume that M is sufficiently saturated. Let us prove the first point. Let $b \in dcl(Aa)$. Let f a map A -definable such that $f(a) = b$. We know that $tp(a/M)$ is definable over A . So $tp(b/M)$ is definable over A , so A -invariant. Let $q = tp(b/M)$. Thus, it remains to show that the property of the Morley sequences of Definition 1.12 holds. Let α be a countable ordinal. Let $(a_i)_{i < \alpha}$ be a Morley sequence of p over A , made of elements of M . Then, we will show that $(f(a_i))_i$ is a Morley sequence of q over A .

For all $i < \alpha$, the element a_i realizes the type $p|_{A \cup (a_j)_{j < i}}$. In other words, $a_i \equiv_{A \cup (a_j)_{j < i}} a$. Since f is A -definable, we deduce that $f(a_i) \equiv_{A \cup (a_j)_{j < i}} f(a)$. A fortiori, as $f(a) = b$, we have $f(a_i) \equiv_{A \cup (f(a_j))_{j < i}} b$. So $f(a_i)$ realizes the type $q|_{A \cup (f(a_j))_{j < i}}$. So $(f(a_i))_i$ is a Morley sequence of q over A , made of elements of M . Then, the property of the mean of the types of the $f(a_i)$ is a consequence of that for the types of the a_i . So we have found a Morley sequence made of elements of M with this property; by homogeneity it holds for all Morley sequences of q .

Let us then prove the second point. Let $b \in acl(Aa)$. By the first point, it suffices to prove that $tp(ab/M)$ is generically stable over $acl(A)$. Let $c = ab$. By Lemma 1.11, we know that $q = tp(c/M)$ is definable over $acl(A)$, so is a fortiori $acl(A)$ -invariant.

By construction, c is algebraic over Ma . Let $\phi(y, x, m)$ be a formula with parameters in M such that $\phi(y, a, m)$ isolates the type of c over Ma . We will show that q is generically stable over $acl(A) \cup m$. Let $(c_i)_i$ be a Morley sequence of q over $acl(A) \cup m$. By contradiction, assume that there exists a formula $\psi(y, d)$ with parameters in $acl(A) \cup md$, where $d \in M$, such that the set of indices i such that $\models \psi(c_i, d)$ is infinite and co-infinite. We write $c_i = a_i b_i$. Then $(a_i)_i$ is a Morley sequence of p over $acl(A) \cup m$. A fortiori, it is a Morley sequence over A . We also know that, for all i , we have $M \models \phi(c_i, a_i, m)$. Thus, there exist infinitely many i such that $M \models \exists y \phi(y, a_i, m) \wedge \psi(y, d)$. Similarly,

there exist infinitely many i such that $M \models \exists y \phi(y, a_i, m) \wedge \neg \psi(y, d)$. But $(a_i)_i$ is a Morley sequence of p , which is generically stable. So $p(x) \models \exists y \phi(y, x, m) \wedge \psi(y, d)$ and $p(x) \models \exists y \phi(y, x, m) \wedge \neg \psi(y, d)$. So $\models \exists y \exists z \phi(y, a, m) \wedge \phi(z, a, m) \wedge \psi(y, d) \wedge \neg \psi(z, d)$. However, the formula $\phi(y, a, m)$ isolates the type of c over Ma . This is a contradiction. So $q = tp(c/M)$ is generically stable over $acl(A) \cup m$. Now, recall that q is definable, so invariant, over $acl(A)$. So q is generically stable over $acl(A)$, as required. \square

1.3 Strong germs

The aim of this subsection is to prove that the germs of a definable map at a generically stable type are strong. We follow the same proof as [ACP14], Theorem 2.2.

Proposition 1.21. *Let $p \in S(M)$ be an A -definable type, where $A \subseteq M$. Let a be a realization of p , let $c \in M$ and f_c be an Ac -definable map such that $p(x) \models x \in \text{dom}(f_c)$. Then, the canonical basis of $tp(af_c(a)/M)$, which we write $Cb(af_c(a)/M)$, is equal to the definable closure of the set $Cb(a/M) \cup [f_c]$.*

Proof. To simplify notations, let $C = Cb(af_c(a)/M)$. Let us first prove the following: $C \subseteq dcl(Cb(a/M) \cup [f_c])$. Let $b = f_c(a)$. It suffices to show that $tp(ab/M)$ is definable over $A[f_c]$. We know that $q = tp(ab/M)$ is definable over Ac . Let us show that q is invariant over $A[f_c]$, which will be enough to conclude. Let $\sigma \in \text{Aut}(M/A[f_c])$. Let us show that $\sigma(q) = q$. Let $d := \sigma(c)$. By hypothesis on σ , we then have $[f_d] = [f_{\sigma(c)}] = \sigma([f_c]) = [f_c]$, i.e. $[f_c] = [f_d]$. Thus, $p(x) \models f_c(x) = f_d(x)$.

Besides, we have $p(x) \cup \{f_c(x) = y\} \models q(x, y)$. Therefore $\sigma(p)(x) \cup \{f_{\sigma(c)}(x) = y\} \models \sigma(q)(x, y)$. But p is A -invariant, and we proved that $p(x) \models f_c(x) = f_{\sigma(c)}(x)$. So $p(x) \cup \{f_c(x) = y\} \models \sigma(q)(x, y)$. Since we also know that $p(x) \cup \{f_c(x) = y\} \models q(x, y)$, we deduce that $\sigma(q)(x, y) = q(x, y)$. Thus, the type $tp(ab/M)$ is definable over $A[f_c]$, as desired.

Let us now prove the converse inclusion. It is clear that $Cb(a/M) \subseteq C$. It remains to show that $[f_c] \in C = Cb(af_c(a)/M)$. By definition of C , the type $q(x, y) := tp(af_c(a)/M)(x, y)$ is C -definable. We want to show that $[f_c] \in dcl(C)$.

We use compactness. Let d be such that $d \equiv_C c$. It suffices to show that $[f_d] = [f_c]$. We know, by choice of the type q , that $q(x, y) \models f_c(x) = y$. In other words, $\models d_q x y f_c(x) = y$. Note that this formula is over C , since q is C -definable. By choice of d , we have $d \equiv_C c$. Thus $\models d_q x y f_d(x) = y$. So, if q' is the C -definable extension of q to Md , we have $q'(x, y) \models f_d(x) = y \wedge f_c(x) = y$. Finally, $q'(x, y) \models f_d(x) = f_c(x)$, so $p(x)|_{Md} \models f_d(x) = f_c(x)$, i.e. $[f_c] = [f_d]$, as desired. \square

Fact 1.22. *Let p be an A -definable type. Then, the equivalence relation on X defined by $y_1 \sim y_2$ if and only if $p|_{Ay_1y_2} \models f_{y_1}(x) = f_{y_2}(x)$ is A -definable, since the type $p|_{Ay_1y_2}$ is definable by the good defining scheme of p . We shall let $[f_y]_p$, or $[f_y]$ if the context is clear, denote the class of the element y for this equivalence relation. We call this class the germ of the function f_y at the type p .*

Proposition 1.23. ([ACP14], Lemma 2.1) *Let $q(x, y) \in S(M)$ be a type generically stable over $A \subseteq M$. Let a, b be a realization of $q(x, y)$.*

1. If $b \in \text{acl}(Ma)$, then $b \in \text{acl}(Aa)$.
2. If $b \in \text{dcl}(Ma)$, then $b \in \text{dcl}(Aa)$.

Corollary 1.24. *Strong germs ([ACP14], Theorem 2.2)*

Let $p \in S(M)$ be a type generically stable over a set $A \subseteq M$, f_c an A -definable map (with a distinguished parameter $c \in M$) such that the domain of f_c contains the realizations of $p|_{Ac}$. Then, for $a \models p|_{Ac}$, we have $f_c(a) \in \text{dcl}(Aa[f_c])$. In fact, there exists an $A \cup [f_c]$ -definable map $F_{[f_c]}$ such that $p(x)|_{Ac} \models f_c(x) = F_{[f_c]}(x)$.

Lemma 1.25. Let $p \in S(A)$ be a generically stable type, h_b an A -definable map (with a distinguished parameter b) such that $p(x) \models x \in \text{dom}(h_b)$. Let $(e_i)_{i \in I}$ be a Morley sequence of p over A , where I is infinite. Then, the germ of h_b at p is definable over $(e_i, h_b(e_i))_{i \in I} \cup A$.

Proof. By compactness, it suffices to prove that, if $b' \equiv_F b$, then $[h_{b'}] = [h_b]$, where $F = (e_i, h_b(e_i))_{i \in I} \cup A$. By generic stability of p , since we know that $\models h_b(e_i) = h_{b'}(e_i)$ for all $i \in I$, and that I is infinite, we deduce that $p|_{Abb'}(x) \models h_b(x) = h_{b'}(x)$. \square

1.4 Commutativity

Proposition 1.26. Let $p, q \in S(M)$ be A -invariant types, where A is a small set contained in M . Assume that p is generically stable over A . Then, $p(x) \otimes q(y) = q(y) \otimes p(x)$, this equality being between A -invariant types.

Proof. Let (b, a) realize $q(y) \otimes p(x)|_M$. Let us show that (a, b) realizes $p(x) \otimes q(y)|_M$. We already know that $b \models q|_{Ma}$ and $a \models p|_M$. So, we know that $b \perp_M a$, that $tp(a/M) = p$ is generically stable over M , and $tp(b/M)$ is extensible. So, by symmetry, $a \perp_M b$. But $tp(a/M) = p$ is stationary. So $tp(a/Mb) = p|_{Mb}$, thus (a, b) realizes $p(x) \otimes q(y)|_M$. \square

Definition 1.27. If $p = tp(a/A)$ is a complete type, and h is an A -definable map whose domain contains a , we let h_*p , or $h(p)$, denote the type $h_*p = tp(h(a)/A)$. It is called the image of p under h . Note that this does not depend on the choice of the realization a .

Remark 1.28. In the definition above, if p is A -invariant (resp. A -definable, resp. generically stable over A), then so is h_*p , and we have $h_*(p|_B) = (h_*p)|_B$ for all $B \supseteq A$.

Definition 1.29. Let $p(x) \in S(M)$ be a global A -invariant type, where A is a small set, and let \mathcal{F} be a set of global invariant types. We say that p commutes with \mathcal{F} if, for all global invariant types $q(y)$ in \mathcal{F} , we have $p(x) \otimes q(y) = q(y) \otimes p(x)$, this being an equality of global invariant types.

Corollary 1.30. Let p be a global B -invariant type, h a B -definable map such that $p|_B(x) \models x \in \text{dom}(h)$, and \mathcal{F} a family of global invariant types. If p commutes with \mathcal{F} , then h_*p commutes with \mathcal{F} .

Proof. Let $q(y)$ be an element of \mathcal{F} . By hypothesis on \mathcal{F} , there exists a small set $C \supseteq B$ such that $q(y)$ is C -invariant. Let us show that $h_*p(x) \otimes q(y) = q(y) \otimes h_*p(x)$. Let $D \supseteq C$, and let (k, b) realize $h_*p(x) \otimes q(y)|_D$.

Then, k realizes $h_*p|_{Db}$. So, by Remark 1.28 applied to Db , there exists a realizing $p|_{Db}$ such that $h(a) = k$. So (a, b) realizes $p \otimes q|_D$. Since p commutes with \mathcal{F} , (b, a) realizes $q \otimes p|_D$, so $b \models q|_{Da}$, a fortiori $b \models q|_{Dh(a)} = q|_{Dk}$. Therefore, (b, k) realizes $q \otimes h_*p|_D$. We have proved that h_*p commutes with \mathcal{F} . \square

Remark 1.31. These notions give us some form of symmetry for tensor products of generically stable types; see Lemma 2.28 for an example of how this symmetry is used. However, we do not know if the class of generically stable types is closed under tensor products, outside well-behaved theories, e.g. NIP.

2 The group configuration theorem

2.1 Genericity and algebraic quadrangles

Here, we define a notion of genericity for definable types concentrating on a group G or a G -space X . We then define algebraic quadrangles, aka group configurations, and explain how to build such using generic types. Few of the results are new, except maybe Propositions 2.7 and 2.10 in the case of G -spaces, which is well-known for stable theories. For more results, and a more general framework allowing definable *partial* types to be generic, see Section 3 in [HR19].

Definition 2.1. Let G be a type-definable group, and X a type-definable space on which G acts definably. Assume everything is defined over some set A . Let \cdot denote both the group law of G , and the action of G on X .

Let $B \supseteq A$, and $p, q \in S(B)$ be definable types concentrating on X . We define $Stab_\phi(p, q)$ by the formula $\forall y [d_q x \phi(g \cdot x, y) \leftrightarrow d_p x \phi(x, y)]$. We then define $Stab(p, q)$ as the intersection of the $Stab_\phi(p, q)$ with G . In the case where $p, q \in G$, we define the right stabilizer $Stab^r(p)$ by considering the right action by translations, and similarly for $Stab^r(p, q)$. Finally, if $p = q$, we write $Stab(p)$ instead of $Stab(p, p)$.

Remark 2.2. Let $p, q \in S(B)$ be as in the definition above. Let M be a model over which everything is defined. Then $Stab(p, q)(M)$ is precisely the set of elements $g \in G(M)$ such that $g \cdot p|_M = q|_M$. Also, $Stab(p)$ is a type-definable subgroup of G .

Definition 2.3. Let G be a type-definable group acting definably on a type-definable space X . Let M be a sufficiently saturated model over which everything is defined.

1. Let $H \leq G$ be a relatively type-definable subgroup. We say that H is of bounded index in G if the cardinality of G/H is bounded, i.e., does not grow beyond a fixed cardinal, regardless of the size of the model.
2. Let $p \in S(M)$ be a definable type. We say that p is d-generic (for “definable generic”) in the G -space X if $p(x) \models “x \in X”$, and $Stab(p)$ is of bounded index in G . Letting

G act definably and regularly on itself by translations, we can also speak of generic types in G .

3. We say that the space X is connected if it has a generic type over M whose stabilizer is G itself. It is generically stable if it has a generically stable generic. The group G is connected (resp. generically stable) if it is connected (resp. generically stable) for the left regular action by translations.

Proposition 2.4. *Let G be a type-definable group with a definable generic type. The following are equivalent :*

1. *The group G is connected.*
2. *The group G has no type-definable proper subgroup of bounded index.*

If these hold, then, for any definable generic type p , we have $\text{Stab}(p) = G$.

Proof. The implication 2. \implies 1. is straightforward: by definition, the stabilizer of any generic type is of bounded index. Let us prove 1. \implies 2. Let p be a generic type for G , whose stabilizer is G itself. Let $H \leq G$ be a type-definable subgroup of bounded index. Then, if M is a sufficiently saturated model containing all the parameters involved, it represents every coset of H . Then, $p|_M$ concentrates on a coset of H . So, the stabilizer $\text{Stab}(p)$ is contained in a conjugate of H . As $\text{Stab}(p) = G$, we deduce that $H = G$, as desired. \square

Lemma 2.5. ([PT11] Lemma 2.1)

Let G be a type-definable group, defined over a set of parameters A . Let $p \in S(A)$ be a generic type for G , generically stable in the sense of Definition 1.15, such that $\text{Stab}(p) = G$.

1. *Let $B \supseteq A$ and a realizing $p|_B$. Then, the element a^{-1} realizes $p|_B$. In other words, we have $p^{-1} = p$.*
2. *Let $g \in G$ and a realizing $p|_{Ag}$. Then, the element $a \cdot g$ realizes $p|_{Ag}$. In other words, the right stabilizer of p is also equal to the whole group G .*
3. *The type p is the unique generic type of the group G .*

Proof. Let α, β realize $(p \otimes p)|_B$. Then, by assumption, we know that $\beta^{-1} \cdot \alpha$ realizes $p|_B$. Then, by Proposition 1.26, we know that $\alpha\beta \equiv_B \beta\alpha$. So $\alpha^{-1} \cdot \beta$ realizes $p|_B$. Then, since $p|_B$ is a complete type, we have shown that, for all elements c realizing $p|_B$, the element c^{-1} realizes $p|_B$, as desired.

Let us then prove the second point. If g is in G and a realizes $p|_{Ag}$, then, by the first point, we know that a^{-1} realizes $p|_{Ag}$. So, by hypothesis on the stabilizer of p , $g^{-1} \cdot a^{-1}$ realizes $p|_{Ag}$. Then, again by the first point, the element $a \cdot g = (g^{-1} \cdot a^{-1})^{-1}$ realizes $p|_{Ag}$, as stated.

Let us now prove the last point. Let $q \in S(B)$ be a definable generic type for G . Let M be a sufficiently saturated model containing A and B . Then, by Proposition 2.4, we have $G = \text{Stab}(q)$. Now, let (a, b) realize $(p \otimes q)|_M$. Then, by the second point, we know that $a \cdot b$ realizes $p|_{Mb}$. On the other hand, since $\text{Stab}(q) = G$, we know that $a \cdot b$ realizes $q|_{Ma}$. So $q|_M = p|_M$. So p and q are equal as definable types. \square

Corollary 2.6. ([HR19], Lemma 3.9) *Let G be a generically stable type-definable group. Then, the connected component G^0 of G , i.e. the smallest type-definable subgroup of bounded index, exists. It is the stabilizer of any generic type of G , and it has a unique generic type.*

Proof. Let us first prove the existence of the connected component G^0 . Let $p \in S(M)$ be a generically stable generic type for G , where M is sufficiently saturated. Then, as $H = \text{Stab}(p)$ is of bounded index, every left coset and every right coset of $\text{Stab}(p)$ is represented in M . So p concentrates on some right coset of H . Now, let q be the image of $p \otimes p$ under the map $(a, b) \mapsto a \cdot b^{-1}$. Then, q concentrates on H . Also, by symmetry for generically stable types, it is easy to check that $\text{Stab}(q) \geq \text{Stab}(p) = H$. Thus, q concentrates on $H_1 := \text{Stab}(q)$. So, by Proposition 2.4, the type-definable group H_1 has no proper type-definable subgroup of bounded index. Moreover, as $H \leq H_1 \leq G$, we know that H_1 is of bounded index in G . Hence, H_1 is indeed the smallest type-definable subgroup of bounded index of G . This implies that H_1 is normal, and even invariant under definable automorphisms, in G . Also, by Lemma 2.5, the unique generic type of H_1 is the type q .

Now, let p_1 be some generic of G . By definition, $\text{Stab}(p_1)$ is of bounded index, so it contains H_1 . On the other hand, the proof of Proposition 2.4 shows that $\text{Stab}(p_1)$ is contained in some conjugate of H_1 . Since the latter is normal, we have in fact $\text{Stab}(p_1) = H_1$, as desired. \square

Proposition 2.7. *Let G be a generically stable type-definable group acting definably and transitively on a type-definable space X .*

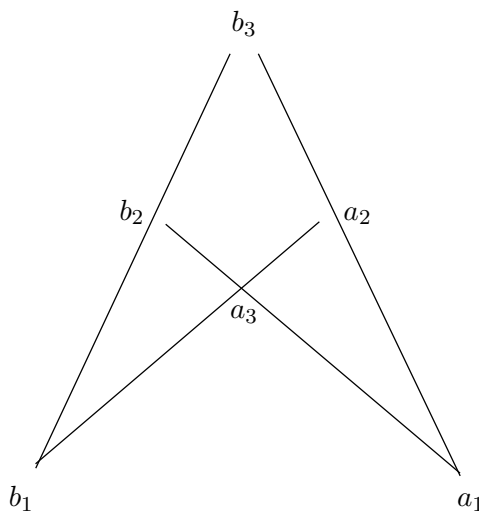
1. *If G is connected, then X has a unique generic type, whose stabilizer is G .*
2. *In general, the generics of X are left translates of each other.*

Proof. Let us show that the second point follows from the first. We know that the connected component G^0 of G exists: it is the stabilizer of any generic of G . Then, we consider the action of G^0 on X . By the first point, each G^0 -orbit contains a unique generic type, whose stabilizer is G^0 . Note that, since G^0 is normal and of bounded index in G and the action is transitive, there are only boundedly many G^0 -orbits. Now, let M be a sufficiently saturated model over which everything is defined. So M contains a point in each G^0 -orbit. Let $q_1, q_2 \in S(M)$ be two generic types of X . Let $x_1, x_2 \in X(M)$ be in the G^0 -orbits of (the realizations of) q_1 and q_2 respectively, and let $g \in G(M)$ be such that $g \cdot x_1 = x_2$. We shall prove that g sends the type q_1 to q_2 . Since G^0 is normal, we have $g \cdot G^0 = G^0 \cdot g$, so $g \cdot q_1$ concentrates on the same G^0 -orbit as q_2 , and is still generic over M . So, by the first point, we have $g \cdot q_1 = q_2$, as desired.

Let us now prove the first point. Assume that G is connected. By Lemma 2.5, let p be the unique generic type of G . Let M be a big enough model containing all the parameters involved, and let $x_0 \in X(M)$. Let $f : G \rightarrow X$ be the definable map $g \mapsto g \cdot x_0$. By transitivity of the action, it is onto. Let q be the definable type $f_*(p)$. It is easy to check that $\text{Stab}(q)(M) = G(M)$, because $\text{Stab}(p) = G$. So, since M is sufficiently saturated, we have $\text{Stab}(q) = G$, so q is generic in X .

Now, let q_1 be another generic type in X . Without loss of generality, we may assume that q_1 is M -definable. We want to show that $q_1 = q$. Let $x_1 \models q_1|_M$, and $g \models p|_{Mx_1}$. So, we have $g \perp_M x_1$. Then, by extensibility, there exists $g_1 \in G$ such that $f(g_1) = x_1$ and $g \perp_M g_1$. Then, by Lemma 2.5, we have $g \cdot g_1 \models p|_{Mg_1}$, so $f(g \cdot g_1) \models f_*(p)|_{Mg_1} = q|_{Mg_1}$. In particular, we have $f(g \cdot g_1) = g \cdot x_1 \models q|_M$. On the other hand, since q_1 is generic in X , and G is connected, we have $\text{Stab}(q_1) = G$. Moreover, by Proposition 1.26, we have $(x_1, g) \models (q_1 \otimes p)|_M$. So $g \cdot x_1 \models q_1|_{Mg}$. Therefore, $g \cdot x_1$ realizes both $q|_M$ and $q_1|_M$. So $q = q_1$, as desired. \square

Definition 2.8. Let A be a set of parameters. A *regular group configuration* over A is a tuple $(a_1, a_2, a_3, b_1, b_2, b_3)$ of elements satisfying the following properties :



1. If x, y, z are three non-colinear points in the diagram above, then the triplet (x, y, z) is an independent family over A .
2. If x, y, z are three colinear points in the diagram above, then $x \in \text{acl}(Ayz)$.

A *definable group configuration* over A is a tuple of elements $(a_1, a_2, a_3, b_1, b_2, b_3)$ satisfying the following properties :

1. The type $tp(a_1 a_2 a_3 b_1 b_2 b_3 / A)$ is definable.
2. If x, y, z are three non-colinear points in the diagram above, then the types $tp(xy / Az)$ and $tp(x / Ayz)$ are A -definable.

3. The equalities $acl(Ab_1b_2) = acl(Ab_1b_3) = acl(Ab_2b_3)$ hold.
4. For all natural numbers i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$, the elements a_j and a_k are interalgebraic over Ab_i , and the element b_i is interalgebraic over A with the canonical basis $Cb(a_ja_k/acl(Ab_i))$.

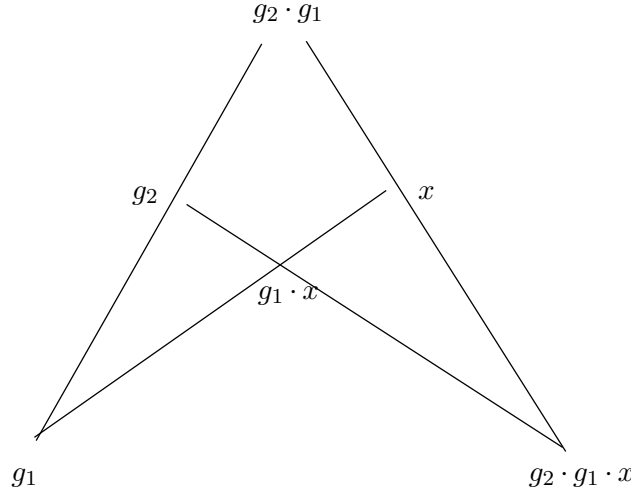
A *generically stable* (resp. *generically stable regular*) *group configuration* over A is a definable (resp. regular) group configuration $(a_1, a_2, a_3, b_1, b_2, b_3)$ over A , such that the type $tp(a_1a_2a_3b_1b_2b_3/A)$ is generically stable, in the sense of Definition 1.15.

We also use the name “algebraic quadrangle”, instead of “group configuration”.

We might call “quadrangle” a 6-tuple of elements which has not been proven to be an algebraic quadrangle (yet).

Remark 2.9. In the case of a generically stable 6-tuple, independence over M of the non-colinear triplets can be checked more easily, using Lemma 1.19. For instance, the set $\{a_1, a_2, a_3\}$ being independent over M is equivalent to having $a_1 \downarrow_M a_2$ and $a_3 \downarrow_M a_1a_2$.

Proposition 2.10. *Let G be a type-definable connected group acting definably on a type-definable connected space X . Assume that the action is free (resp. faithful) and transitive. Let p, q be the generics of G and X respectively. Assume that p and q are generically stable. Let (g_1, g_2, x) be a triplet realizing $p^{\otimes 2} \otimes q|_M$, where M is a sufficiently saturated model over which everything is defined. Then, the following family is an algebraic quadrangle (resp. a partial algebraic quadrangle) over M :*



Such a quadrangle is called a “group configuration for (G, X) ”. Note that this quadrangle is generically stable over A if and only if the tensor product $p^{\otimes 2} \otimes q$ is generically stable over A .

Proof. The algebraicity relations are clear in the case of a free action. In fact, let us deal with the more subtle case of a faithful transitive action.

First, note that the type $tp(g_1, g_2, x/M)$ is definable, since it is the tensor product $p \otimes p \otimes q$. Then, by Lemma 1.11, the type over M of the sextuple is definable. This is the first point of the definition. The third point is easy to check, since $b_1 = g_1$, $b_2 = g_2$ and $b_3 = g_2 \cdot g_1$.

Let us now prove the second point. First, note that $tp(g_1 \cdot x/M) = tp(g_2 \cdot g_1 \cdot x/M) = tp(x/M) = q$, because $Stab(q) = G$, and $x \models q|_{Mg_1g_2}$. So, to prove the second point, we may use stationarity and symmetry for generically stable types. Let us prove that $(g_2 \cdot g_1, g_1 \cdot x, g_2 \cdot g_1 \cdot x)$ realizes the tensor product $p \otimes q \otimes q|_M$, the other cases being similar. By saturation of M , let $x_0 \in X(M)$, and let $f : G \rightarrow X$ be the definable map $g \mapsto g \cdot x_0$. By transitivity of the action, this map is onto. Also, the proof of Proposition 2.7 shows that $f_*(p) = q$.

We will prove that $g_2 \cdot g_1 \cdot x \downarrow_M g_2 \cdot g_1$ (1), and then show $g_1 \cdot x \downarrow_M g_2 \cdot g_1 \hat{\ } g_2 \cdot g_1 \cdot x$ (2). We know that $x \models q|_{Mg_1g_2}$. By genericity of q , we then have $g_2 \cdot g_1 \cdot x \models q|_{Mg_1g_2}$, so $g_2 \cdot g_1 \cdot x \downarrow_M g_2 \cdot g_1$, which is (1). Let us now prove (2). We know that $x \models q|_{Mg_1g_2}$, so there exists $g \models p|_{Mg_1g_2}$ such that $f(g) = x$. By genericity, we have $g_1 \downarrow_M g_2 \cdot g_1$. Then, using Lemma 1.19 and symmetry, we deduce from $g \downarrow_M g_1 \hat{\ } g_2 \cdot g_1$ that $g_1 \downarrow_M g_2 \cdot g_1 \hat{\ } g$. Then, since $Stab^r(p) = G$, we have $g_1 \cdot g \models p|_{Mg_2 \cdot g_1 \hat{\ } g}$, i.e. $g_1 \cdot g \downarrow_M g_2 \cdot g_1 \hat{\ } g$, so $f(g_1 \cdot g) \downarrow_M g_2 \cdot g_1 \hat{\ } f(g_2 \cdot g_1 \cdot g)$. In other words, we have $g_1 \cdot x \downarrow_M g_2 \cdot g_1 \hat{\ } g_2 \cdot g_1 \cdot x$, which is (2).

Finally, let us prove the fourth point of the definition of a partial algebraic quadrangle. The first part is easy, only the statements dealing with the canonical bases require some justification. Their proof is the same as in [Pil96] (Chapter 5, Remark 4.1). By Proposition 1.21, to prove that, say g_1 is interalgebraic over M with $Cb(x, g_1 \cdot x/acl(Mg_1))$, it suffices to prove that g_1 is interalgebraic over M with its germ at q . Let us show that, in fact, for any $h \in G$, the element h is definable over $M \hat{\ } [h]$.

Note that, since $Stab(q) = G$, composition of germs at q of elements of G is well-defined, and yields a type-definable group of germs Γ , such that $g \in G \mapsto [g] \in \Gamma$ is an A -definable group homomorphism. Moreover, by the strong germ property (Corollary 1.24), the generic action of G on q induces a generic action of Γ on q . So, it suffices to prove that, if $g \in G$ is such that $[g] = 1$, then $g = 1$.

Let g be such an element. Let $a \in X$ be arbitrary. We want to show that $g \cdot a = a$, which will imply that $g = 1$. Let $b \models q|_{Mg}$. By transitivity, let $h \in G$ be such that $h \cdot b = a$. Let $\gamma \models p|_{Mg \hat{\ } h \hat{\ } a \hat{\ } b}$. Then, $\gamma \cdot h$ and $\gamma \cdot g \cdot h$ are generic over Mb . So, by symmetry for generically stable types, we have $b \models q|_{M\gamma \cdot h}$ and $b \models q|_{M\gamma \cdot g \cdot h}$. Now, since germs can be composed, and $[g] = 1$, we have $[\gamma \cdot g \cdot h] = [\gamma \cdot h]$. Also, by the strong germ property, since $b \models q|_{M\gamma \cdot h}$ and $b \models q|_{M\gamma \cdot g \cdot h}$, we have $[\gamma \cdot h] \cdot b = (\gamma \cdot h) \cdot b$ and $[\gamma \cdot g \cdot h] \cdot b = (\gamma \cdot g \cdot h) \cdot b$. As $[\gamma \cdot g \cdot h] = [\gamma \cdot h]$, we deduce that $\gamma \cdot g \cdot h \cdot b = \gamma \cdot h \cdot b$, i.e. $\gamma \cdot g \cdot a = \gamma \cdot a$. So $g \cdot a = a$. As a was arbitrary, faithfulness implies that $g = 1$, as desired. \square

Proposition 2.11. *Any generically stable regular group configuration over A is a generically stable group configuration over A .*

Proof. Let $(a_1, a_2, a_3, b_1, b_2, b_3)$ be a generically stable regular group configuration over A . Let us show that it is a generically stable group configuration over A . Looking at the definitions, the only properties to check are those regarding the canonical bases

$Cb(a_j a_k / acl(Ab_i))$, for i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$. By symmetry of the context, it suffices to show that b_1 and $Cb(a_2 a_3 / acl(Ab_1))$ are interalgebraic over A .

First, let N be a sufficiently saturated model containing Ab_1 , such that $a_2 \downarrow_{Ab_1} N$, which exists by extensibility of $tp(a_2 / Ab_1)$. Then, $tp(a_2 / N)$ is generically stable over Ab_1 . Then, by Proposition 1.20, the type $tp(a_2 a_3 / N)$ is generically stable over $acl(Ab_1)$, and stationarity yields the equality: $Cb(a_2 a_3 / acl(Ab_1)) = Cb(a_2 a_3 / N)$.

On the one hand, it is clear that $Cb(a_2 a_3 / acl(Ab_1)) \subseteq acl(Ab_1)$. Conversely, let us show that b_1 is algebraic over $A \cup Cb(a_2 a_3 / acl(Ab_1))$. Let $C = A \cup Cb(a_2 a_3 / acl(Ab_1))$. By assumption, we know that $b_1 \in acl(A a_2 a_3) \subseteq acl(C a_2 a_3)$. Moreover, the type $tp(a_2 a_3 / N)$ does not fork over C . Thus, since $b_1 \in N$, we have $a_2 a_3 \downarrow_C b_1$. Finally, since $b_1 \in acl(C a_2 a_3)$, we can apply Proposition 1.10, which implies that $b_1 \in acl(C)$, as desired. \square

Definition 2.12. Equivalent quadrangles

Let A be a set of parameters. Let $(a_1, a_2, a_3, b_1, b_2, b_3)$ and $(a'_1, a'_2, a'_3, b'_1, b'_2, b'_3)$ be quadrangles. We say that these quadrangles are *equivalent over A* , or *interalgebraic over A* if, for $i = 1, 2, 3$, we have $acl(A a_i) = acl(A a'_i)$ and $acl(A b_i) = acl(A b'_i)$.

Remark 2.13. If a quadrangle is equivalent over A to a generically stable (regular) algebraic quadrangle over A , then it is itself a generically stable (regular) algebraic quadrangle over $acl(A)$.

Indeed, Proposition 1.20 implies that generic stability is preserved. Moreover, the algebraicity relations are preserved, and, in the case of generically stable regular quadrangles, so are the independence relations, thanks to Proposition 1.10. In the case of generically stable algebraic quadrangles, stationarity yields the second point of the definition. Then, Lemma 1.11 implies the required properties for the canonical bases.

Definition 2.14. If $p(x, y)$ is a complete type in several variables, where x, y are tuples of variables, we let p_x denote the restriction of p to the tuple of variables x . Then, p_x is a complete type in the tuple of variables x .

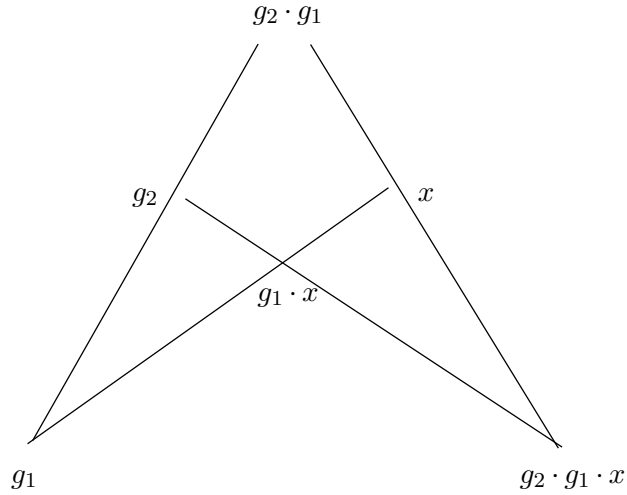
Proposition 2.15. Let $(a_1, a_2, a_3, b_1, b_2, b_3)$ be an algebraic quadrangle generically stable over a set A , and $A_0 \subseteq A$. Assume that the type $tp(a_1 a_2 a_3 b_1 b_2 b_3 / A)$ is generically stable over A_0 . Then $(a_1, a_2, a_3, b_1, b_2, b_3)$ is an algebraic quadrangle generically stable over A_0 .

Proof. The interalgebraicity relations are proved using Proposition 1.23. Let us prove one of the independence relations, say $a_1 a_2 \downarrow_{A_0} b_1$. We know that $a_1 a_2 \downarrow_A b_1$, that $a_1 a_2 \downarrow_{A_0} A$ and that $tp(a_1 a_2 / A_0)$ is generically stable. So, by Proposition 1.17, we have $a_1 a_2 \downarrow_{A_0} Ab_1$, which implies that $a_1 a_2 \downarrow_{A_0} b_1$, as desired. \square

2.2 Statement of the theorem

Theorem 2.16. Let M be a $|\mathcal{L}|^+$ -saturated model. Let $(a_1, a_2, a_3, b_1, b_2, b_3)$ be a generically stable algebraic quadrangle over M . Let $p_0(x_1, x_2, x_3, y_1, y_2, y_3) = tp(a_1, a_2, a_3, b_1, b_2, b_3 / M)$. Let $C \subset M$ be the canonical base of the type p_0 . Then, there exists a type-definable group G acting transitively, faithfully and definably on a type-definable set X , elements $b''_1, b''_2, b''_3, b'_3 \in M$, and elements $g_1, g_2 \in G$, $x_1 \in X$, generic over M , such that :

1. The tuple $b_1'' b_2'' b_3'' b_3'$ realizes $((p_0)_{y_1, y_2, y_3} \otimes (p_0)_{y_3})|_C$.
2. The group G is connected and type-definable over $\text{acl}(C \cup b_1'' b_2'' b_3'' b_3')$. The space X is connected and type-definable over $\text{acl}(C \cup b_1'' b_2'' b_3'' b_3')$.
3. The following is an algebraic quadrangle over M which is equivalent, over M , to the quadrangle $(a_1, a_2, a_3, b_1, b_2, b_3)$:



Moreover, if p_0 is a generically stable regular algebraic quadrangle over M , then the action of G on X can be assumed to be regular, i.e. transitive and free, in addition to being faithful.

Note that, in particular, G and X are definable over M , which is enough information if one does not need to control parameters.

The following proof is adapted from that of Elisabeth Bouscaren [Bou87], with ideas from [Pil96] for the general case of a faithful transitive action.

2.3 Replacing algebraicity with definability

The goal of this subsection is the following proposition, which enables us, at the cost of enlarging the base, to replace some of the algebraicity in the quadrangle by definability, while keeping a generically stable type.

Proposition 2.17. *Under the hypotheses of Theorem 2.16, there exist elements $b_1'', b_2'', b_3'', b_3' \in M$ and a quadrangle $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ equivalent over M to $(a_1, a_2, a_3, b_1, b_2, b_3)$, such that*

1. The tuple $b_1'' b_2'' b_3'' b_3'$ realizes $((p_0)_{y_1, y_2, y_3} \otimes (p_0)_{y_3})|_C$.
2. The type $\text{tp}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3/M)$ is generically stable over $\text{acl}(C b_1'' b_2'' b_3'' b_3')$.

3. We have $\alpha_1 \in dcl(C_1\alpha_2\beta_3)$, $\alpha_2 \in dcl(C_1\alpha_3\beta_1)$, and $\alpha_3 \in dcl(C_1\alpha_1\beta_2) \cap dcl(C_1\alpha_2\beta_1)$.

Remark 2.18. Note that, if we were under the stronger hypotheses of Theorem 2.16, the proposition would yield a quadrangle equivalent over M to a regular algebraic quadrangle, so still a regular algebraic quadrangle over M , generically stable over $acl(Cb''_1b''_2b''_3b'_3)$. Then, by Proposition 2.15, the new quadrangle would be a generically stable regular algebraic quadrangle over $acl(Cb''_1b''_2b''_3b'_3)$.

Lemma 2.19. *Under the hypotheses of Theorem 2.16, there exists a'_1 interalgebraic with a_1 over C , b'_3 and a'_2 algebraic over Mb_2 and Ma_3 respectively, and $b'_1 \in M$ realizing $tp(b_1/C)$, such that $a'_1 \in dcl(C'b_2b'_3a_3a'_2)$, and $tp(a'_1, a_2, a_3a'_2, b_1, b_2b'_3, b_3/M)$ is generically stable over $acl(Cb'_1)$, where $C' = acl(C)$.*

In other words, by adding elements to the tuples b_2 and a_3 , replacing a_1 with a'_1 , and C with $acl(Cb'_1)$, we may assume that $a_1 \in dcl(Mb_2a_3)$.

Proof. By Proposition 1.23, since $a_1 \in acl(Mb_2a_3) \cap acl(Ma_2b_3)$, we have in fact $a_1 \in acl(Cb_2a_3) \cap acl(Ca_2b_3)$.

Let a'_1 be the code for the finite set of conjugates of a_1 over $Ma_2a_3b_2b_3$. Then, $a_1 \in acl(a'_1)$. On the other hand, a_1 is algebraic over Ca_2b_3 and over Ca_3b_2 . So a'_1 is algebraic over Ca_2b_3 and over Ca_3b_2 , as it is the code for some subset of the conjugates of a_1 over Ca_2b_3 and Ca_3b_2 . Also, we know that $acl(Ca_2b_3) = acl(Ca_1b_3)$ and $acl(Ca_3b_2) = acl(Ca_1b_2)$. So $a'_1 \in acl(Ca_1b_3) \cap acl(Ca_1b_2)$. Moreover, $b_2 \downarrow_{Ca_1} b_3$. Thus, we may apply Proposition 1.10 to deduce that the element a'_1 is algebraic over Ca_1 .

Then, since a'_1 is algebraic over Ca_1 , by Proposition 1.20, the type $tp(a'_1a_2a_3b_1b_2b_3/M)$ is generically stable over C' . Moreover, by construction of a'_1 , we notice that $a'_1 \in dcl(Ma_2a_3b_2b_3)$. So, by Proposition 1.23, we have in fact $a'_1 \in dcl(C'a_2a_3b_2b_3)$.

By saturation of M , let $b'_1 \in M$ realize $tp(b_1/C')$. We know that the type $tp(b_2a_3a'_1/M)$ is the unique nonforking extension of $tp(b_2a_3a'_1/C')$. Since $b'_1 \in M$, we deduce, by monotonicity, $b_2a_3a'_1 \downarrow_{C'} b'_1$.

On the other hand, we have, by hypothesis on the configuration, $b_1 \downarrow_{C'} b_2a_3a'_1$. By Proposition 1.18, we then have $b_2a_3a'_1 \downarrow_{C'} b_1$. So, by stationarity, we have $b'_1 \equiv_{C'b_2a_3a'_1} b_1$.

Let b'_3, a'_2 be such that $b'_3a'_2b'_1b_2a_3a'_1 \equiv_{C'} b_3a_2b_1b_2a_3a'_1$.

Again, Proposition 1.23 implies that $b_3 \in acl(C'b_1b_2)$. Therefore, we also have $b'_3 \in acl(C'b'_1b_2) \subseteq acl(Mb_2)$. Similarly, $a'_2 \in acl(C'b'_1a_3) \subseteq acl(Ma_3)$. Finally, since $a'_1 \in dcl(C'a_2a_3b_2b_3)$, elementary equivalence over C' implies that $a'_1 \in dcl(C'a'_2a_3b_2b'_3)$.

To show that $q := tp(a'_1, a_2, a_3a'_2, b_1, b_2b'_3, b_3/M)$ is generically stable over $acl(Cb'_1)$, we will apply Proposition 1.20. Let $p = tp(a'_1a_2a_3b_1b_2b_3/M)$. This type is generically stable over C' , so over $acl(Cb'_1)$ as well. Moreover, we have shown that $a'_2b'_3 \in acl(C'b'_1a'_1a_2a_3b_1b_2b_3)$. So, by Proposition 1.20, the type $tp(a'_2b'_3a'_1a_2a_3b_1b_2b_3/M)$ is generically stable over $acl(Cb'_1)$. Thus, the configuration $(a'_1, a_2, a_3b_2, b_1, b_2b'_3, b_3)$ has the required properties. \square

Similarly, up to replacing a_2 , enlarging b_1, a_3 and the "base of generic stability", we may assume that $a_2 \in dcl(C'b_1a_3)$.

To be accurate, we construct a type over M which is generically stable over $\text{acl}(Cb'_1\beta_2)$, where $\beta_2 = b''_2b''_3 \in M$ realizes the type $(p_0)_{y_2, y_3}|_{Cb'_1}$. Indeed, if we follow a similar procedure, the changes to the original configuration have to be taken into account, namely, the extension of b_2 (and a_3). Thus, the type of the new configuration is generically stable over $\text{acl}(C \cup b''_1b''_2b''_3)$, where the three elements b''_i are independent over C , are in M , and where b''_i realizes $(p_0)_{y_i}|_C$, for $1 \leq i \leq 3$. For ease of notation, we let D denote $\text{acl}(Cb''_1b''_2b''_3)$.

Note that, during these two steps, the element b_3 of the configuration has not been modified.

Lemma 2.20. *Let M be a sufficiently saturated and homogeneous model. Let $(a_1, a_2, a_3, b_1, b_2, b_3)$ be a generically stable partial algebraic quadrangle over M , and $C' \subset M$ be small such that $a_1 \in \text{dcl}(C'a_2b_3)$, $a_2 \in \text{dcl}(C'a_3b_1)$, and $\text{tp}(a_1a_2a_3b_1b_2b_3/M)$ is generically stable over C' . Then, there exists an element $b'_3 \in M$ and a generically stable algebraic quadrangle $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ equivalent to $(a_1, a_2, a_3, b_1, b_2, b_3)$ over M , such that*

1. *The type of b'_3 over C' is $\text{tp}(b_3/C')$.*
2. *We have $\alpha_1 \in \text{dcl}(C_1\alpha_2\beta_3)$, $\alpha_2 \in \text{dcl}(C_1\alpha_3\beta_1)$, and $\alpha_3 \in \text{dcl}(C_1\alpha_1\beta_2) \cap \text{dcl}(C_1\alpha_2\beta_1)$.*
3. *The type $\text{tp}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3/M)$ is generically stable over C_1 .*

Here, $C_1 := \text{acl}(C'b'_3)$.

Proof. Let us start by noting that Proposition 1.23 implies that $a_3 \in \text{acl}(C'b_1b_2a_1a_2)$. Just as before, let a'_3 be the code of the finite set of conjugates of a_3 over $Mb_1b_2a_1a_2$. Then, a'_3 codes a subset of the finite set of conjugates of a_3 over $C'b_1b_2a_1a_2$. So $a'_3 \in \text{acl}(C'b_1b_2a_1a_2)$.

On the other hand, as before, a'_3 is interalgebraic with a_3 over C' . Moreover, if f_{b_1} is a map definable over Mb_1 such that $f_{b_1}(a_3) = a_2$, then, for all a in the finite set coded by a'_3 , we have $f_{b_1}(a) = a_2$. So $a_2 \in \text{dcl}(Ma'_3b_1)$. Similarly, $a_1 \in \text{dcl}(Ma'_3b_2)$. Finally, by construction, $a'_3 \in \text{dcl}(Mb_1b_2a_1a_2)$.

Since $a'_3 \in \text{acl}(C'b_1b_2a_1a_2)$, and the type $\text{tp}(a_1a_2b_1b_2b_3/M)$ is generically stable over C' , we deduce from Proposition 1.20 that the type $\text{tp}(a_1a_2a'_3b_1b_2b_3/M)$ is still a type generically stable over C' . Then, by Proposition 1.23, and by the relations of definability proved above, we have $a_1 \in \text{dcl}(C'a'_3b_2)$, $a_2 \in \text{dcl}(C'a'_3b_1)$ and $a'_3 \in \text{dcl}(C'b_1b_2a_1a_2)$.

Let $b'_3 \in M$ such that $b'_3 \equiv_{C'} b_3$. Then, by stationarity, we have $b'_3 \equiv_{C'b_2a'_3a_1} b_3$ and $b'_3 \equiv_{C'b_1a'_3a_2} b_3$. Let b'_1, b'_2, a'_1, a'_2 be elements such that $b'_1a'_2b'_3 \equiv_{C'b_2a'_3a_1} b_1a_2b_3$ and $b'_2a'_1b'_3 \equiv_{C'b_1a'_3a_2} b_2a_1b_3$.

Then, by elementary equivalence, the following hold :

1. $a'_2 \in \text{acl}(C'a_1b'_3)$
2. $a'_1 \in \text{acl}(C'a_2b'_3)$
3. $b'_2 \in \text{acl}(C'b_1b'_3)$

4. $b'_1 \in \text{acl}(C'b_2b'_3)$

Then, since $b'_3C' \subseteq M$, we have $a'_2 \in \text{acl}(Ma_1)$, $a'_1 \in \text{acl}(Ma_2)$, $b'_2 \in \text{acl}(Mb_1)$ and $b'_1 \in \text{acl}(Mb_2)$.

Therefore, since $\text{acl}(C'a'_3) = \text{acl}(C'a_3)$, the quadrangle defined by $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) := (a_1a'_2, a_2a'_1, a'_3, b_1b'_2, b_2b'_1, b_3)$ is equivalent to the quadrangle $(a_1, a_2, a_3, b_1, b_2, b_3)$ over M . By Remark 2.13, the new quadrangle is indeed a generically stable partial algebraic quadrangle over M .

Moreover, we have seen above that $\text{tp}(a_1a_2a'_3b_1b_2b_3/M)$ is a type generically stable over C' , thus a fortiori over $\text{acl}(C'b'_3)$. By the algebraicity above, we may then apply Proposition 1.20, to deduce that the type over M of the tuple $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ is generically stable over $\text{acl}(C'b'_3)$.

Thus, it remains to check that the required definability relations hold.

1. We know that $a_1 \in \text{dcl}(C'a'_3b_2)$ and $a'_2 \in \text{dcl}(C'a'_3b'_1)$. So $a_1a'_2 \in \text{dcl}(C'a'_3b_2b'_1)$. In other words, $\alpha_1 \in \text{dcl}(C'\alpha_3\beta_2)$.
2. Similarly, $a_2 \in \text{dcl}(C'a'_3b_1)$ and $a'_1 \in \text{dcl}(C'a'_3b'_2)$, so $a_2a'_1 \in \text{dcl}(C'a'_3b_1b'_2)$, i.e. $\alpha_2 \in \text{dcl}(C'\alpha_3\beta_1)$.
3. We have seen above that $a'_3 \in \text{dcl}(C'a_1a_2b_1b_2)$. Then, by construction of the a'_i, b'_i for $i = 1, 2$, we have $a'_3 \in \text{dcl}(C'a_1a'_2b_2b'_1) \cap \text{dcl}(C'a_2a'_1b_1b'_2)$, i.e. $\alpha_3 \in \text{dcl}(C'\alpha_1\beta_2) \cap \text{dcl}(C'\alpha_2\beta_1)$.

Thus, the new quadrangle has the required properties. \square

Thus, we have proved the proposition. From now on, we write C instead of C_1 , and we rename the elements of the new configuration $a_1a_2a_3b_1b_2b_3$, so that the type $\text{tp}(a_1a_2a_3b_1b_2b_3/M)$ is generically stable over C . To control the parameters defining the group G , one needs to remember that the initial set C was replaced by the set $\text{acl}(Cb''_1b''_2b''_3)$, and that $b'_3 \equiv_{\text{acl}(Cb''_1b''_2b''_3)} b_3$, so $b'_3 \downarrow_C b''_1b''_2b''_3$.

2.4 Germs

The aim of this subsection is to build an appropriate group from germs of definable bijections. From now on, let f_{b_1} and g_{b_2} be definable bijections sending respectively a_2 to a_3 and a_3 to a_1 , where f and g are C -definable families of definable bijections. Let $h_{b_1b_2}$ be the composite $g_{b_2} \circ f_{b_1}$.

Then, the independence hypotheses on the configuration imply that the germs of f_{b_1} and $h_{b_1b_2}$ are well-defined at $\text{tp}(a_2/C)$, and that the germ of g_{b_2} is well-defined at $\text{tp}(a_3/C)$. Moreover, we will show that the germs of these functions can be composed.

Proposition 2.21. *Let $A, D \supseteq C$ be sets of parameters. Let $\beta_1, \beta_2, \alpha_1, \alpha_2, \alpha_3$ be realizations of $\text{tp}(b_1/M)|_D$, $\text{tp}(b_2/M)|_D$, $\text{tp}(a_1/M)|_D$, $\text{tp}(a_2/M)|_D$ and $\text{tp}(a_3/M)|_D$ respectively.*

1. *If $\alpha_2 \downarrow_D A\beta_1$, then $f_{\beta_1}(\alpha_2) \downarrow_D A\beta_1$.*

2. If $\alpha_3 \downarrow_D A\beta_2$, then $g_{\beta_2}(\alpha_3) \downarrow_D A\beta_2$.
3. If $\alpha_3 \downarrow_D A\beta_1$, then $(f_{\beta_1})^{-1}(\alpha_3) \downarrow_D A\beta_1$.
4. If $\alpha_1 \downarrow_D A\beta_2$, then $(g_{\beta_2})^{-1}(\alpha_1) \downarrow_D A\beta_2$.

Proof. Let us prove the first implication, the other ones being similar. Assume that $\alpha_2 \downarrow_D A\beta_1$. We know, by hypothesis, that $\alpha_2 \downarrow_C D$. Moreover, the type $tp(\alpha_2/C)$ is generically stable. So, by transitivity, $\alpha_2 \downarrow_C AD\beta_1$. Then, $\alpha_2 \downarrow_{C\beta_1} AD$, so $f_{\beta_1}(\alpha_2) \downarrow_{C\beta_1} AD$ (*).

On the other hand, since $\alpha_2 \downarrow_C AD\beta_1$, we have $\alpha_2 \downarrow_C \beta_1$. So, by stationarity, $\alpha_2\beta_1 \equiv_C a_2b_1$, so $f_{\beta_1}(\alpha_2)\alpha_2\beta_1 \equiv_C f_{\beta_1}(a_2)a_2b_1 = a_3a_2b_1$. Since $a_3 \downarrow_C b_1$, we deduce that $f_{\beta_1}(\alpha_2) \downarrow_C \beta_1$.

Recall that the type $tp(f_{\beta_1}(\alpha_2)/C)$ is generically stable. So, by (*) and transitivity, we have $f_{\beta_1}(\alpha_2) \downarrow_C \beta_1AD$. So, by monotonicity, we have indeed $f_{\beta_1}(\alpha_2) \downarrow_D \beta_1A$. \square

Corollary 2.22. *Let β_1, β'_1 be realizations of $tp(b_1/C)$, β_2, β'_2 be realizations of $tp(b_2/C)$.*

Then, the following germs are well-defined, i.e. only depend on the germs of the functions involved : $[g_{\beta_2}]^{-1} \circ [g_{\beta'_2}]$, $[g_{\beta_2}] \circ [f_{\beta_1}]$, $[f_{\beta_1}]^{-1} \circ [f_{\beta'_1}]$.

Proof. We shall use Proposition 2.21, taking realizations of the generically stable types involved, independent from all the parameters that appear.

Let us prove for instance that the germ of $(g_{\beta_2})^{-1} \circ g_{\beta'_2}$ only depends on the germs $[g_{\beta_2}]$ and $[g_{\beta'_2}]$. Let b_2^1, b_2^2 be realizations of $tp(b_2/C)$ such that $[g_{\beta_2}] = [g_{b_2^1}]$ and $[g_{\beta'_2}] = [g_{b_2^2}]$. Let us show that the germs of the functions $(g_{\beta_2})^{-1} \circ g_{\beta'_2}$ and $(g_{b_2^1})^{-1} \circ g_{b_2^2}$ are equal.

Let α_3 realize the type of a_3 over C , such that $\alpha_3 \downarrow_C b_2^1 b_2^2 \beta_2 \beta'_2$. Then, since $[g_{\beta'_2}] = [g_{b_2^2}]$, we have $g_{\beta'_2}(\alpha_3) = g_{b_2^2}(\alpha_3)$. Moreover, by Proposition 2.21 applied to $D = C$ and $A = b_2^1 \beta_2$, we know that $g_{b_2^2}(\alpha_3) \downarrow_C b_2^1 \beta_2$. Since we have assumed that $[g_{\beta_2}] = [g_{b_2^1}]$, we can deduce that $(g_{b_2^1}^{-1} \circ g_{b_2^2})(\alpha_3) = (g_{\beta_2}^{-1} \circ g_{\beta'_2})(\alpha_3) = (g_{\beta_2}^{-1} \circ g_{\beta'_2})(\alpha_3)$. As $\alpha_3 \downarrow_C b_2^1 b_2^2 \beta_2 \beta'_2$, we have shown that the germs of the composites $(g_{\beta_2})^{-1} \circ g_{\beta'_2}$ and $(g_{b_2^1})^{-1} \circ g_{b_2^2}$ are equal. \square

Definition 2.23. Let F , resp. G , be the set of the germs of functions of the form f_{β_1} , resp. g_{β_2} , where β_1 realizes $tp(b_1/C)$, resp. β_2 realizes $tp(b_2/C)$. Then, by Fact 1.22, these sets are type-definable.

Remark 2.24. The partial types over C defining F and G are complete generically stable types. Moreover, since the type $r = tp(a_1 a_2 a_3 b_1 b_2 b_3 / M)$ is generically stable over C , it commutes with itself. So, by Corollary 1.30, the definable types F and G commute with r . Then, applying this corollary again, we deduce, by associativity, that any tensor product whose factors are among F , G , or r , is commutative. In other words, the family $\{F, G, r\}$ is commutative.

Recall that $h_{b_1 b_2}$ denotes the composite $g_{b_2} \circ f_{b_1}$.

Lemma 2.25. *The germ of the definable map $h_{b_1 b_2}$ is interalgebraic over C with b_3 .*

Proof. We know that we have a partial algebraic quadrangle over C . Thus, the element b_3 is interalgebraic over C with the canonical basis of $tp(a_1a_2/acl(Cb_3))$. Moreover, we have $a_1a_2 \downarrow_{acl(Cb_3)} b_1b_2$, because $a_1 \downarrow_C b_1b_2b_3$ and $a_2 \in acl(Ca_1b_3)$. Thus, we have the following equalities: $Cb(a_1a_2/acl(Cb_3)) = Cb(a_1a_2/acl(Cb_1b_2b_3)) = Cb(a_1a_2/acl(Cb_1b_2))$. Since $Cb(a_1/acl(Cb_1b_2)) = Cb(a_1/C) \subseteq C$, Proposition 1.21 implies that $Cb(a_1a_2/acl(Cb_1b_2))$ is interalgebraic over C with $[h_{b_1b_2}]$. This concludes the proof. \square

Definition 2.26. Let K be the set of germs of the form $f^{-1} \circ f'$ where $(f, f') \models F \otimes F|_C$. Then, the set K is defined by a complete C -definable type, also denoted as K . Indeed, the type K is the image of the tensor product $F \otimes F$ under the definable map which composes a germ with the inverse of another germ. This map is well-defined, by Corollary 2.22.

Similarly, let L be the set of germs of the form $f' \circ f^{-1}$ where $(f, f') \models F \otimes F|_C$. Finally, let Γ be the C -type-definable set of germs of the form $k \circ k'$ where $k, k' \in K$. Here, K is a shorthand for $K|_C$.

Note that, thanks to the strong germs property (Corollary 1.24), the elements on F , K and Γ act generically on the definable type $tp(a_2/C)$, and those of G and L on the type $tp(a_3/C)$.

Remark 2.27. For ease of notation, let us write $p = tp(b_1/C)$, $q = tp(b_2/C)$.

1. Since $F \otimes F$ is a complete type, the image of $p \otimes p$ under the C -definable map $(b, b') \mapsto ([f_b], [f_{b'}])$ is equal to the type $F \otimes F$. More generally, any finite tensor product whose factors are F and G is the image under the appropriate function of the tensor product of corresponding factors p and q . This follows from the definitions.
2. Therefore, the type K is the image of the type $p \otimes p$ under the C -definable map $(b, b') \mapsto [f_b^{-1} \circ f_{b'}]$. Similarly, the type L is the image of $p \otimes p$ by the function $(b, b') \mapsto [f_{b'} \circ f_b^{-1}]$.
3. By Remark 2.24, we know that $F \otimes F$ commutes with F , G and r , where r is the type $tp(a_1a_2a_3b_1b_2b_3/M)$. Then, by Corollary 1.30, the type K commutes with F , G and r . Again, we deduce that F , G , K , and r are in a commutative family, then that the family $\{F, G, K, L, r\}$ is commutative.
4. Since F commutes with itself, the inverse of a germ k realizing $K|_D$ is still a realization of $K|_D$, for all $D \supseteq C$. Similarly for L .

Lemma 2.28. *Let D be a small set containing C . Let $f, f' \in F$ be such that $(f, f') \models F \otimes F|_D$. Then $f^{-1} \circ f' \models K|_{Df}$ and $f^{-1} \circ f' \models K|_{Df'}$. On the other hand, $f' \circ f^{-1} \models L|_{Df}$ and $f' \circ f^{-1} \models L|_{Df'}$.*

Proof. Let us start by proving the statements about $f^{-1} \circ f'$. By the first point of Remark 2.27, we can apply Remark 1.28 to the case where $q = tp(b_1/M)^{\otimes 2}$ and $h : (b, b') \mapsto [f_b^{-1} \circ f_{b'}]$. We then find β_1, β'_1 such that $([f_{\beta_1}], [f_{\beta'_1}]) = (f, f')$ and $(\beta_1, \beta'_1) \models q|_D$.

Besides, the definition of K implies that $f^{-1} \circ f' \vDash K|_D$, since we assumed that $(f, f') \vDash F \otimes F|_D$. Recall that we want to show that $(f^{-1} \circ f', f) \vDash K \otimes F|_D$ and $(f^{-1} \circ f', f') \vDash K \otimes F|_D$. By the third point of Remark 2.27, K and F commute, so it is equivalent to prove that $(f, f^{-1} \circ f') \vDash F \otimes K|_D$ and $(f', f^{-1} \circ f') \vDash F \otimes K|_D$. By stationarity of F , as we have seen above that $f^{-1} \circ f' \vDash K|_D$, it suffices to show that $f \downarrow_C D \cup f^{-1} \circ f'$ and $f' \downarrow_C D \cup f^{-1} \circ f'$. By symmetry of the hypotheses on f and f' , it is enough to prove that $f \downarrow_C D \cup f^{-1} \circ f'$.

Let $g = [g_{\beta_2}] \in G$, where $\beta_2 \vDash tp(b_2/M)|_{D\beta_1\beta'_1}$. By stationarity, we know that $tp(b_1b_2/M) = tp(b_1/M) \otimes tp(b_2/M)$. Since M is a model, we know that $tp(b_1b_2/M)|_E = (tp(b_1/M) \otimes tp(b_2/M))|_E$ for all $E \supseteq C$. Then, $\beta_2\beta_1$ and $\beta_2\beta'_1$ realize the type $tp(b_2b_1/M)|_D$. Let β_3, β'_3 be such that $\beta_3\beta_2\beta_1$ and $\beta'_3\beta_2\beta'_1$ realize $tp(b_3b_2b_1/M)|_D$. Then, by Lemma 2.25, we have $[g_{\beta_2}] \circ [f_{\beta_1}] \in acl(C\beta_3)$ and $[g_{\beta_2}] \circ [f_{\beta'_1}] \in acl(C\beta'_3)$. Thus $f^{-1} \circ f' = ([g_{\beta_2}] \circ [f_{\beta_1}])^{-1} \circ [g_{\beta_2}] \circ [f_{\beta'_1}] \in acl(C\beta_3\beta'_3)$.

If we manage to prove that $\beta_1 \downarrow_C D\beta_3\beta'_3$, we can then deduce by Propositions 1.10 and 1.10 that $f \downarrow_C D \cup f^{-1} \circ f'$, which will finish the proof. By transitivity, it suffices to show that $\beta_1 \downarrow_D \beta_3\beta'_3$.

By construction, we have $\beta_2\beta_1\beta'_1 \vDash (tp(b_2/M) \otimes q)|_D$, so $\beta_1 \downarrow_D \beta_2\beta'_1$, so $\beta_1 \downarrow_D \beta_2\beta'_3$. By choice of β'_3 , we have $\beta_2 \downarrow_D \beta'_3$. So, by Lemma 1.19, we have $\beta_1\beta_2 \downarrow_D \beta'_3$, so $\beta_1\beta_3 \downarrow_D \beta'_3$. We also know that $\beta_1 \downarrow_D \beta_3$. So, again by Lemma 1.19, we can conclude that $\beta_1 \downarrow_D \beta_3\beta'_3$, as desired.

The proof for $f' \circ f^{-1}$ is similar, one only needs to “swap the roles of b_2 and b_3 ”, the context being symmetric enough, thanks to Lemma 2.25. \square

Corollary 2.29. 1. *The C -definable type K is generically stable.*

2. *The type-definable set Γ is the set of germs of the form $f^{-1} \circ f'$, where f, f' realize $F|_C$.*

Proof. 1. Let f realize $F|_C$. Then, by Lemma 2.28, there exists a Cf -definable bijection $F|_{Cf} \simeq K|_{Cf}$. Thus, by Proposition 1.20, the type $K|_{Cf}$ is generically stable, in the sense of 1.15. However, this type is definable over C , so $K|_C$ is generically stable, as stated.

2. Let γ be an element of Γ . By definition, there exist k_1, k_2 realizing $K|_C$ such that $\gamma = k_1 \circ k_2$. Let $f \vDash F|_{Ck_1k_2}$. Then, by Lemma 2.28 applied to k_1, k_2^{-1} , there exist $f_1, f_2 \in F$ such that $k_1 = f_1^{-1} \circ f$ and $k_2 = f^{-1} \circ f_2$. Then, we compute that $\gamma = f_1^{-1} \circ f_2$, as desired. \square

The following lemma will be used to prove transitivity of the action of the group Γ , and regularity in the case of a regular algebraic quadrangle. Note that

Lemma 2.30. *Let α_2, α'_2 be realizations of $tp(a_2/C)$ such that $\alpha_2 \downarrow_C \alpha'_2$. Then, there exists a germ $k \in K$ such that $k \vDash K|_{C\alpha_2} \cup K|_{C\alpha'_2}$ and $k(\alpha_2) = \alpha'_2$, where $k(\alpha_2)$ is well-defined.*

Moreover, under the stronger hypotheses of Theorem 2.16, i.e. if we started from a generically stable regular algebraic quadrangle, there exist only finitely many germs k in K such that $k \vDash K|_{C\alpha_2}$ and $k(\alpha_2) = \alpha'_2$.

Proof. First, recall that if $k \in K$ realizes $K|_{C\alpha_2}$, then the couple (k, α_2) realizes $K \otimes F|_C$. So, by symmetry (see the third point of Remark 2.27), α_2 realizes $F|_{Ck}$. By the strong germs property, the element $k(\alpha_2)$ is then well-defined.

Let $p = tp(b_1/M)$ and $t = tp(a_2/M)$. By stationarity of t , the hypothesis is equivalent to $\alpha_2\alpha'_2$ realizing $t \otimes t|_C$.

Let us prove existence : Let $\alpha_3 \equiv_C a_3$ be such that $\alpha_3 \perp_C \alpha_2\alpha'_2$, and let $\beta_1\beta'_1$ realize $p \otimes p|_{C\alpha_3}$. Then, by definition, $\beta_1 \vDash p|_{C\beta'_1\alpha_3} (*)$. So $\beta_1 \perp_{C\alpha_3} \beta'_1$, so $f_{\beta_1}^{-1}(\alpha_3) \perp_{C\alpha_3} f_{\beta'_1}^{-1}(\alpha_3)$. Also, $\beta_1\alpha_3 \equiv_C b_1a_3$. Moreover, in the configuration $a_1a_2a_3b_1b_2b_3$, we have $f_{b_1}^{-1}(a_3) = a_2$, and $a_2 \perp_C a_3$. So $f_{\beta_1}^{-1}(\alpha_3) \perp_C \alpha_3$, and similarly $f_{\beta'_1}^{-1}(\alpha_3) \perp_C \alpha_3$. So, by transitivity, $f_{\beta_1}^{-1}(\alpha_3) \perp_C f_{\beta'_1}^{-1}(\alpha_3)\alpha_3$. In other words, $f_{\beta_1}^{-1}(\alpha_3)f_{\beta'_1}^{-1}(\alpha_3)$ realizes the type $t \otimes t|_{C\alpha_3}$. By choice of α_3 , the tuple $\alpha_2\alpha'_2$ also realizes the type $t \otimes t|_{C\alpha_3}$. So, up to changing β_1 and β'_1 , we may assume that $f_{\beta_1}^{-1}(\alpha_3) = \alpha_2$ and $f_{\beta'_1}^{-1}(\alpha_3) = \alpha'_2$, without changing the fact that $\beta_1\beta'_1$ realizes $p \otimes p|_{C\alpha_3}$.

Then $(f_{\beta'_1}^{-1} \circ f_{\beta_1})(\alpha_2) = \alpha'_2$, considering the definable maps, and not their germs. We want to show that $k = [f_{\beta'_1}^{-1} \circ f_{\beta_1}]$ has the required properties. It then remains to prove that $\beta_1\beta'_1 \vDash p^{\otimes 2}|_{C\alpha_2} \cup p^{\otimes 2}|_{C\alpha'_2}$ and that $\alpha_2 \vDash t|_{C\beta_1\beta'_1}$. We know by Corollary 1.30 that $p(x) \otimes p(y) \otimes t(z) = t(z) \otimes p(x) \otimes p(y)$. Thus, it suffices to prove that $\beta_1\beta'_1 \vDash p^{\otimes 2}|_{C\alpha_2} \cup p^{\otimes 2}|_{C\alpha'_2}$.

By construction of β_1, β'_1 , we know that $\beta_1 \vDash p|_{C\alpha_3}$ and that $\beta'_1 \vDash p|_{C\alpha_3}$. Then, by the third point of Proposition 2.21, we deduce that $\beta_1 \vDash p|_{C\alpha_2}$ and that $\beta'_1 \vDash p|_{C\alpha'_2}$. Since p commutes with itself, it remains to show that $\beta'_1 \vDash p|_{C\alpha_2\beta_1}$ and $\beta_1 \vDash p|_{C\alpha'_2\beta'_1}$. By symmetry of the construction, it suffices to prove the second point. Recall $(*)$: $\beta_1 \vDash p|_{C\beta'_1\alpha_3}$. Since $\alpha'_2 \in dcl(C\beta'_1\alpha_3)$, we have indeed that $\beta_1 \vDash p|_{C\alpha'_2\beta'_1}$.

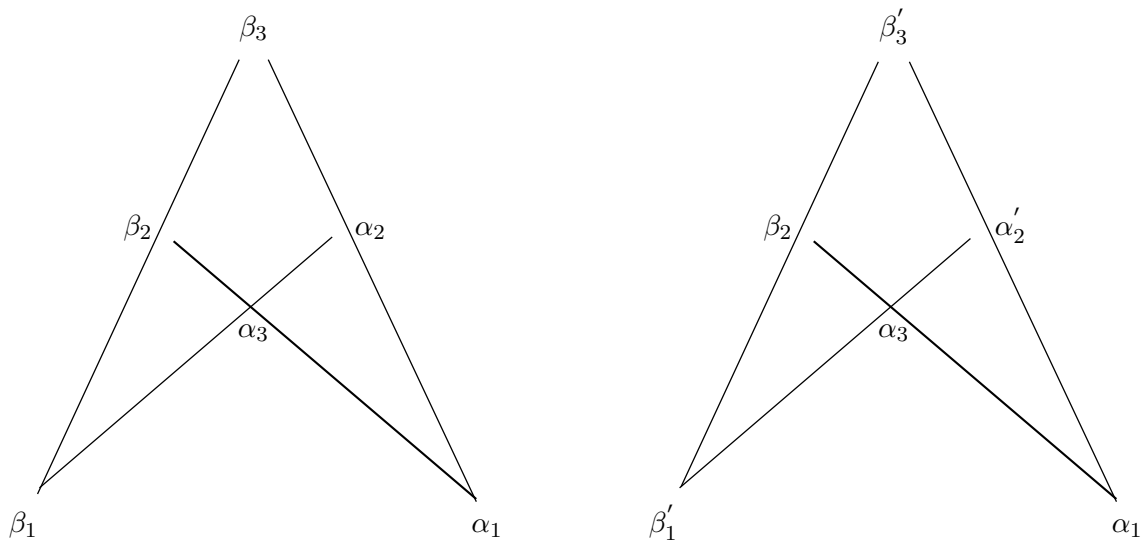
Now, let us assume the stronger hypotheses of Theorem 2.16, and prove finiteness. Let k in K be such that $k \vDash K|_{C\alpha_2}$ and $k(\alpha_2) = \alpha'_2$. Let us show that $k \in acl(C\alpha_2\alpha'_2)$. As K is the image of $p \otimes p$ (see the second point of Remark 2.27), we can apply Remark 1.28. We then find β_1, β'_1 realizations of $p = tp(b_1/C)$ such that $\beta_1\beta'_1 \vDash p \otimes p|_{C\alpha_2}$ and $k = [f_{\beta'_1}^{-1} \circ f_{\beta_1}]$. Then, by symmetry, α_2 realizes $t|_{C\beta_1\beta'_1}$. Thus, we have $f_{\beta'_1}^{-1} \circ f_{\beta_1}(\alpha_2) = k(\alpha_2) = \alpha'_2$.

In order to symmetrize the information available on α_2 and α'_2 , let us show that $\beta_1\beta'_1 \vDash p \otimes p|_{C\alpha'_2}$. Let $\alpha_3 = f_{\beta_1}(\alpha_2) = f_{\beta'_1}(\alpha'_2)$. Then, as $\beta_1\beta'_1 \vDash p \otimes p|_{C\alpha_2}$, we have, by symmetry, $\beta'_1 \vDash p|_{C\beta_1\alpha_2}$, so $\beta'_1 \vDash p|_{C\beta_1\alpha_2\alpha_3}$. Moreover, $\beta_1 \vDash p|_{\alpha_2}$, so, by Proposition 2.21, we have $\beta_1 \vDash p|_{\alpha_3}$. So $\beta'_1\beta_1 \vDash p \otimes p|_{C\alpha_3}$. So, by symmetry, $\beta_1\beta'_1 \vDash p \otimes p|_{C\alpha_3}$. So $\beta_1 \vDash p|_{C\alpha_3\beta'_1}$, so $\beta_1 \vDash p|_{C\beta'_1\alpha'_2}$. Also, applying Proposition 2.21 to the hypothesis " $\beta'_1 \vDash p|_{C\alpha_3}$ ", we have $\beta'_1 \vDash p|_{C\alpha'_2}$. Then, by definition of a tensor product, we have $\beta_1\beta'_1 \vDash p \otimes p|_{C\alpha'_2}$, as stated.

Let β_2 realize $tp(b_2/M)|_{C\alpha_3\beta_1\beta'_1\alpha_2\alpha'_2}$. So $\beta_1\beta_2 \equiv_C \beta'_1\beta_2 \equiv_C b_1b_2$. Then, let $\beta_3\beta'_3$ be a couple such that $\beta_1\beta_2\beta_3 \equiv_C \beta'_1\beta_2\beta'_3 \equiv_C b_1b_2b_3$. Then, since $\beta_2 \perp_C \beta_1\alpha_2$, and $\beta_1 \perp_C \alpha_2$,

we have, by Lemma 1.19, $\beta_2\beta_1 \downarrow_C \alpha_2$, so $\alpha_2 \downarrow_C \beta_2\beta_1$, so $\alpha_2 \downarrow_C \beta_3\beta_2\beta_1$. Then, by stationarity, we have $\alpha_2\beta_1\beta_2\beta_3 \equiv_C a_2b_1b_2b_3$. So $\alpha_2\alpha_3\beta_1\beta_2\beta_3 \equiv_C a_2a_3b_1b_2b_3$, because $\alpha_3 = f_{\beta_1}(\alpha_2)$. Let $\alpha_1 := g_{\beta_2}(\alpha_3)$. Then $\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 \equiv_C a_1a_2a_3b_1b_2b_3$. By symmetric arguments, we also have $\alpha_1\alpha'_2\alpha_3\beta'_1\beta_2\beta'_3 \equiv_C a_1a_2a_3b_1b_2b_3$.

Thus, we get the following configurations, which have the thick line in common:



By Lemma 2.25, the germ $k = [f_{\beta'_1}^{-1} \circ f_{\beta_1}] = [f_{\beta'_1}^{-1} \circ g_{\beta_2}^{-1} \circ g_{\beta_2} \circ f_{\beta_1}]$ is algebraic over $C\beta_3\beta'_3$. Besides, using the stronger hypotheses of Theorem 2.16, we know that $\beta_1\beta'_1 \in \text{acl}(C\alpha_3\alpha_2\alpha'_2)$, so $k \in \text{acl}(C\alpha_3\alpha_2\alpha'_2)$. If we show that $\alpha_3 \downarrow_{C\alpha_2\alpha'_2} \beta_3\beta'_3$, we can then apply Proposition 1.10, to deduce that $k \in \text{acl}(C\alpha_2\alpha'_2)$. To that end, it suffices to show that $\alpha_3 \downarrow_C \alpha_2\beta_3\beta'_3$.

We have seen above that $\beta'_1 \vDash p|_{C\beta_1\alpha_2}$. So $\beta'_1 \downarrow_C \beta_1\alpha_2$. Moreover, by choice of β_2 , we have $\beta_2 \downarrow_C \beta'_1\beta_1\alpha_2$. The type $tp(\beta_1\alpha_2/C)$ being generically stable, because $\beta_1\alpha_2 \equiv_C b_1a_2$, we may apply Lemma 1.19, which yields $\beta_2\beta_1\alpha_2 \downarrow_C \beta'_1$. By symmetry, we deduce that $\beta'_1 \downarrow_C \beta_2\beta_1\alpha_2$, so $\beta'_1 \downarrow_{C\beta_2} \beta_1\beta_2\beta_3\alpha_1\alpha_2\alpha_3$. So $\beta'_3 \downarrow_{C\beta_2} \beta_1\beta_2\beta_3\alpha_1\alpha_2\alpha_3$. As $\beta'_3 \downarrow_C \beta_2$, we have, by transitivity, $\beta'_3 \downarrow_C \beta_1\beta_2\beta_3\alpha_1\alpha_2\alpha_3$. So $\beta'_3 \downarrow_C \beta_3\alpha_2\alpha_3$. Since $\alpha_3 \downarrow_C \beta_3\alpha_2$, and $tp(\beta_3\alpha_2/C)$ is generically stable, we can apply Lemma 1.19 again, to get $\beta'_3\beta_3\alpha_2 \downarrow_C \alpha_3$. Now, the type $tp(\beta'_3\beta_3\alpha_2/C)$ is extendible, for it is the tensor product $tp(\beta'_3/C) \otimes tp(\beta_3\alpha_2/C)$. Thus, we may apply Proposition 1.18, to deduce that $\alpha_3 \downarrow_C \alpha_2\beta_3\beta'_3$, as desired.

Therefore, applying Proposition 1.10, we finally prove that $k \in \text{acl}(C\alpha_2\alpha'_2)$. This holds for all realizations of the partial type defined by “ $k \vDash K|_{C\alpha_2}$ and $k(\alpha_2) = \alpha_2$ ” so, by compactness, there are only finitely many $k \in K$ satisfying $k \vDash K|_{C\alpha_2}$ and $k(\alpha_2) = \alpha_2$. \square

The following lemma shows that the collection of germs is, in some sense, homogeneous. It will be used for several key results.

Lemma 2.31. 1. Let g, f_1, f_2 be such that gf_1f_2 realizes $G \otimes F \otimes F|_C$. Then, there exists b_2^2 realizing $tp(b_2/C)$ such that $g \circ f_1 = [g_{b_2^2}] \circ f_2$, $b_2^2 \downarrow_C gf_1$ and $b_2^2 \downarrow_C gf_2$.

2. Let g, g', f_1 be such that $gg'f_1$ realizes $G \otimes G \otimes F|_C$. Then, there exists b_1^2 realizing $tp(b_1/C)$ such that $g \circ f_1 = g' \circ [f_{b_1^2}]$, $b_1^2 \downarrow_C gf_1$ and $b_1^2 \downarrow_C g'f_1$.

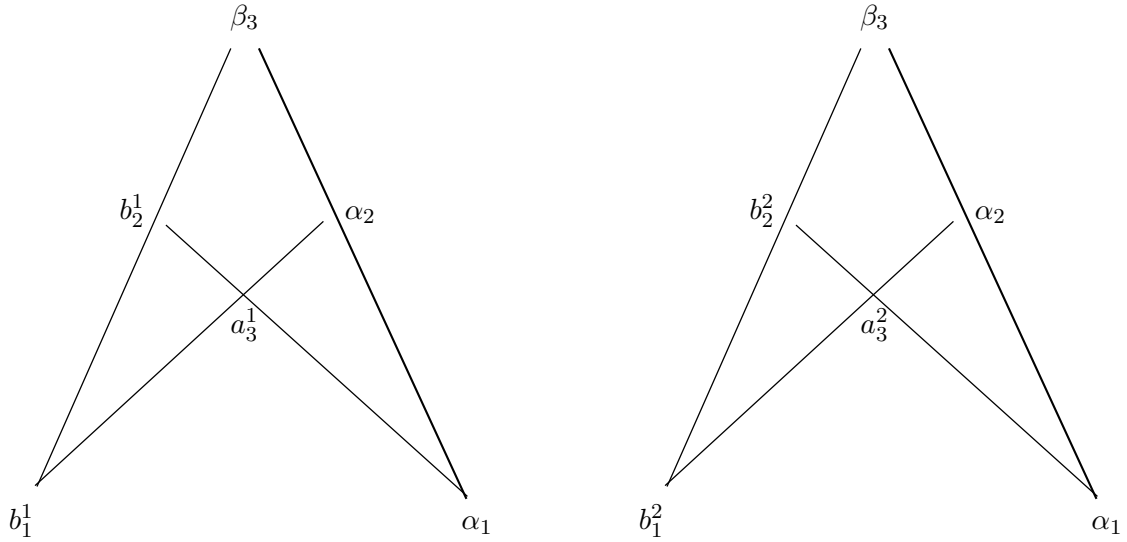
Proof. Let us prove the first result. Let $r(x, y, z)$ be the tensor product $(tp(b_2/M)(x) \otimes tp(b_1/M)(y) \otimes tp(b_1/M)(z))|_C$.

By the first point of Remark 2.27, there exist b_2^1, b_1^1, b_1^2 such that $[g_{b_2^1}] = g$, $[f_{b_1^1}] = f_1$, $[f_{b_1^2}] = f_2$ and $b_2^1 b_1^1 b_1^2 \vDash r(x, y, z)$. We look for a suitable element b_2^2 . Let $\beta_3 \in acl(Cb_2^1 b_1^1)$ be such that $\beta_3 b_2^1 b_1^1 \equiv_C b_3 b_2 b_1$. Let α_2 be a realization of $tp(a_2/M)|_{Cb_1^1 b_1^2 b_2^1 \beta_3}$. Then, by stationarity, we have $\alpha_2 b_1^1 b_2^1 \beta_3 \equiv_C a_2 b_1 b_2 b_3$. Let $a_3^1 = f_{b_1^1}(\alpha_2)$ and $\alpha_1 = g_{b_2^1}(a_3^1)$.

By choice of α_2 , we have $\alpha_2 \downarrow_C b_1^1 b_2^1 b_1^2$. By hypothesis, we also have $b_2^2 \downarrow_C b_1^1 b_1^2$. Thus, we may apply Lemma 1.19, to deduce that $\alpha_2 b_1^1 b_2^1 \downarrow_C b_1^2$. So, by Proposition 1.10, we have $\alpha_1 \alpha_2 a_3^1 b_1^1 b_2^1 \beta_3 \downarrow_C b_1^2$ (*).

Then, by symmetry and stationarity, $\alpha_1 \alpha_2 \beta_3 b_1^2 \equiv_C a_1 a_2 b_3 b_1$. Let b_2^2, a_3^2 be such that $b_2^2 a_3^2 \alpha_1 \alpha_2 \beta_3 b_1^2 \equiv_C b_2 a_3 a_1 a_2 b_3 b_1$. We will show that b_2^2 has the required properties.

We end up with the following quadrangles, which have the thick line in common:



We know that $b_2^2 \downarrow_C b_1^2$, because $b_2 \downarrow_C b_1$. Let us show that $b_2^2 \downarrow_C b_2^1 b_1^1$. By (*), we know that $\alpha_1 \alpha_2 a_3^1 b_1^1 b_2^1 \beta_3 \downarrow_C b_1^2$. Then, as $tp(\alpha_1 \alpha_2 a_3^1 b_1^1 b_2^1 \beta_3/C)$ is generically stable, we can apply symmetry, to get $b_1^2 \downarrow_C \alpha_1 \alpha_2 a_3^1 b_1^1 b_2^1 \beta_3$. So, by Proposition 1.10, we have $b_2^2 \downarrow_{C\beta_3} \alpha_1 \alpha_2 a_3^1 b_1^1 b_2^1$. Since $b_2^2 \downarrow_C \beta_3$, by transitivity, we have $b_2^2 \downarrow_C \beta_3 \alpha_1 \alpha_2 a_3^1 b_1^1 b_2^1$ (**). By monotonicity, we deduce $b_2^2 \downarrow_C b_2^1 b_1^1$. Hence, we have indeed $b_2^2 \downarrow_C gf_1$.

In order to prove that $b_2^2 \downarrow_C gf_2$, we will show that $b_2^2 \downarrow_C b_2^1 b_1^1$. By construction, we know that $b_1^2 \downarrow_C b_1^1 b_2^1$, so $b_1^2 \downarrow_C \beta_3 b_2^1$. Since $\beta_3 \downarrow_C b_2^1$, by Lemma 1.19, we have $\beta_3 b_1^2 \downarrow_C b_2^1$.

so $b_2^2 b_1^2 \downarrow_C b_2^1$. Applying Lemma 1.19 again, we have $b_2^2 \downarrow_C b_2^1 b_1^2$. Therefore, we have $b_2^2 \downarrow_C g f_2$.

Besides, we know that $g_{b_2^1} \circ f_{b_1^1}(\alpha_2) = \alpha_1 = g_{b_2^2} \circ f_{b_1^2}(\alpha_2)$. It then remains to show that $\alpha_2 \downarrow_C b_2^1 b_1^1 b_2^2 b_1^2$, so that we can conclude equality of the germs of $g_{b_2^1} \circ f_{b_1^1}$ and $g_{b_2^2} \circ f_{b_1^2}$. It suffices to prove that $\alpha_2 \downarrow_C b_1^2 b_2^1 \beta_3$, which is true by choice of α_2 .

In order to prove the second statement, it suffices to swap the roles of b_2^2 and b_1^2 . The nine points that are obtained have the same properties in both cases, only the order in which they are defined is different. \square

We can now prove that K behaves like the generic of a group:

Corollary 2.32. *Let (f_1, f_2, f_3, f_4) be a family of elements of F which realizes the tensor product $F^{\otimes 4}|_C$. Let $E \supseteq C$. Assume that $f_1 f_2$ realizes $F \otimes F|_{E f_3 f_4}$. Then, there exist $f, f' \in F$ such that*

1. $(f_1^{-1} \circ f_2) \circ (f_3^{-1} \circ f_4) = f^{-1} \circ f'$
2. $f f'$ realizes $F \otimes F|_{C \cup f_1^{-1} \circ f_2}$ and $f f'$ realizes $F \otimes F|_{E \cup f_3 f_4}$.
3. Therefore, $(f_1^{-1} \circ f_2) \circ (f_3^{-1} \circ f_4) \vDash K|_{C \cup f_1^{-1} \circ f_2}$ and $(f_1^{-1} \circ f_2) \circ (f_3^{-1} \circ f_4) \vDash K|_{E \cup f_3 f_4}$.

Note that neither the hypotheses nor the conclusion are symmetric : f_1, f_2 and f, f' are possibly more generic than f_3, f_4 .

Proof. Let g_2 realize $G|_{E f_1 f_2 f_3 f_4}$. Then, by Lemma 2.31, there exists $g_3 \in G$ such that $g_2 \circ f_3 = g_3 \circ f_2$ and $g_3 \downarrow_C g_2 f_3$. We know that $g_2 \vDash G|_{C f_1 f_2 f_3 f_4}$, and $f_2 f_3 f_4 \vDash F^{\otimes 3}|_{C f_1}$. By choice of g_2 , this implies $g_2 f_2 f_3 f_4 \vDash G \otimes F^{\otimes 3}|_{C f_1}$. Then, by symmetry, we have $f_2 f_3 \vDash F \otimes F|_{C g_2 f_1 f_4}$.

Then, by Lemma 2.28 applied to $D = C g_2 f_1 f_4$, we have $f_3 \circ f_2^{-1} \vDash L|_{C g_2 f_1 f_3 f_4}$. So $g_2 \circ f_3 \circ f_2^{-1} \downarrow_{C g_2} f_1 f_3 f_4$. In other words, $g_3 \downarrow_{C g_2} f_1 f_3 f_4$. Since $g_3 \downarrow_C g_2$, we then have $g_3 \downarrow_C g_2 f_1 f_3 f_4$. Thus, by stationarity, $g_3 \vDash G|_{C g_2 f_1 f_3 f_4}$. Moreover, we have chosen g_2 so that $g_2 \vDash G|_{E f_1 f_2 f_3 f_4}$, so $g_3 g_2 f_4 \vDash G \otimes G \otimes F|_{C f_1 f_3}$. Then, we can again apply Lemma 2.31, for the germs g_2, g_3 and f_4 . We thus obtain a germ $f_5 \in F$ such that $g_3 \circ f_5 = g_2 \circ f_4$ and $f_5 \downarrow_C g_2 f_4$. We will show that $f = f_1$ and $f' = f_5$ have the required properties.

Compute : $f_1^{-1} \circ f_2 \circ f_3^{-1} \circ f_4 = f_1^{-1} \circ g_3^{-1} \circ g_2 \circ f_4 = f_1^{-1} \circ f_5$, these being equalities of germs. In other words, $f_5 = f_2 \circ f_3^{-1} \circ f_4$. In particular, $f_5 \downarrow_C f_1$, so, by stationarity, $f_1 f_5 \vDash F \otimes F|_C$.

Then, let us prove that $f_1^{-1} \circ f_5 \vDash K|_{C f_1^{-1} \circ f_2}$. By the fourth point of Remark 2.27, it is the same as showing that $f_5^{-1} \circ f_1 \vDash K|_{C f_1^{-1} \circ f_2}$. By definition of K , it then suffices to show that $f_5 f_1 \vDash F \otimes F|_{C \cup f_1^{-1} \circ f_2}$.

By Lemma 2.28, we know that $f_1^{-1} \circ f_2 \vDash K|_{E f_1}$, so $(f_1^{-1} \circ f_2, f_1) \vDash K \otimes F|_E$. So, by symmetry, f_1 realizes $F|_{E \cup f_1^{-1} \circ f_2}$, so a fortiori f_1 realizes $F|_{C \cup f_1^{-1} \circ f_2}$. It then remains to show that $f_5 \vDash F|_{C f_1 \cup (f_1^{-1} \circ f_2)}$, i.e. $f_5 \vDash F|_{C f_1 f_2}$. By stationarity, it is enough to prove that $f_5 \downarrow_C f_1 f_2$. Using the hypotheses on (f_1, f_2, f_3, f_4) , we have, by symmetry, $f_2 f_3 \vDash F \otimes F|_{C f_1 f_4}$ and $f_2 \vDash F|_{E f_1 f_3 f_4}$. So, by Lemma 2.28 applied to $D = C f_1 f_2$, we have

$f_3^{-1} \circ f_4 \vDash K|_{Cf_1f_2f_4}$. So $f_2 \circ f_3^{-1} \circ f_4 \downarrow_{Cf_2} f_1f_4$. As $f_2 \circ f_3^{-1} \circ f_4 = f_5 \downarrow_C g_2f_4$, we have, by transitivity, $f_5 \downarrow_C f_1f_2f_4$. So $f_5 \downarrow_C f_1f_2$, as desired.

To conclude, let us show that the germ $f_1^{-1} \circ f_5$ realizes $K|_{Ef_3f_4}$. By symmetry and by definition of K , it suffices to show that f_5f_1 realizes $F \otimes F|_{Ef_3f_4}$. We know by hypothesis that f_1 realizes $F|_{Ef_3f_4}$. By stationarity, it remains to show that $f_5 \downarrow_C Ef_1f_3f_4$.

On the one hand, by hypothesis (and symmetry), we have $f_2 \downarrow_C Ef_1f_3f_4$. So $f_2 \circ f_3^{-1} \circ f_4 \downarrow_{Cf_3f_4} Ef_1$, i.e. $f_5 \downarrow_{Cf_3f_4} Ef_1$ (*). On the other hand, Lemma 2.28 implies that $f_2 \circ f_3^{-1}$ realizes $L|_{Cf_1f_3f_4}$, so $f_2 \circ f_3^{-1} \downarrow_C f_1f_3f_4$, so $f_5 \downarrow_{Cf_4} f_1f_3$. We also know that $f_5 \downarrow_C g_2f_4$. So, by transitivity, $f_5 \downarrow_C f_1f_3f_4$. So, by transitivity in (*), we have $f_5 \downarrow_C Ef_1f_3f_4$, as desired. Finally, we can indeed pick $f = f_1$ and $f' = f_5$. \square

Corollary 2.33. *Let k_1, k_2 be realizations of $K|_C$, and $D \supseteq C$, such that $k_1 \vDash K|_{Dk_2}$. Then $k_1 \circ k_2 \vDash K|_{Dk_2}$ and $k_2 \circ k_1 \vDash K|_{Dk_2}$.*

Proof. Let us show that $k_2 \circ k_1 \vDash K|_{Dk_2}$, the other result being more straightforward. By definition of K , there are f_3, f_4 such that $(f_3, f_4) \vDash F \otimes F|_C$ and $f_3^{-1} \circ f_4 = k_2$. Let f_1, f_2 in F be such that $(f_1, f_2) \vDash F \otimes F|_{Df_3f_4}$. Then, if $k' = f_1^{-1} \circ f_2$, we know that $k' \vDash K|_{Df_3f_4}$, so in particular $k' \vDash K|_{Dk_2}$, so $k' \vDash_{Dk_2} k_1$. So, it suffices to prove that $k_2 \circ k' \vDash K|_{Dk_2}$.

Since F commutes with itself, we can apply Corollary 2.32 to the family (f_2, f_1, f_4, f_3) . It yields that $f_2^{-1} \circ f_1 \circ f_4^{-1} \circ f_3$ realizes $K|_{Dk_2}$. Then, the inverse $f_3^{-1} \circ f_4 \circ f_1^{-1} \circ f_2$ still realizes $K|_{Dk_2}$. In other words, $k_2 \circ k' \vDash K|_{Dk_2}$, as stated.

To show that $k_1 \circ k_2 \vDash K|_{Dk_2}$, we also apply Corollary 2.32, without permuting functions, nor considering inverses. \square

Proposition 2.34. *The set Γ is closed under composition of germs.*

Proof. Let $k_1, k_2, k_3, k_4 \in K$. Let $D = Ck_1k_2k_3k_4$. Let $k \vDash K|_D$. Then, using Corollary 2.33 four times, we can easily show that $k_1 \circ k_2 \circ k_3 \circ k_4 \circ k \vDash K|_D$.

Finally, we notice that $k_1 \circ k_2 \circ k_3 \circ k_4 = (k_1 \circ k_2 \circ k_3 \circ k_4 \circ k) \circ (k^{-1})$. Since k^{-1} and $(k_1 \circ k_2 \circ k_3 \circ k_4 \circ k)$ are in K , the germ $k_1 \circ k_2 \circ k_3 \circ k_4$ is indeed in $K \circ K = \Gamma$. \square

Corollary 2.35. *Composition of germs induces a definable group structure on the type-definable set Γ .*

Proof. Proposition 2.21 and Corollary 2.22 imply that the composition of germs is associative, for it is induced by composition of functions.

Moreover, by the fourth point of Remark 2.27, K is closed under taking inverses, and so is Γ . Besides, we have proved that Γ is closed under composition.

Finally, the germ of the identity is indeed in Γ , for, if $k \in K$, then $id = k \circ k^{-1} \in \Gamma$. \square

2.5 Properties of the group

Proposition 2.36. *The type-definable group Γ is connected, with generic K .*

Proof. First, recall that K is a C -definable type. We will prove the following :

$$Stab_{\Gamma}(K) = \Gamma \tag{1}$$

Let $g \in \Gamma(M)$. Let k be a realization of $K|_M$ (in an elementary extension of M). By definition of Γ , and since M is sufficiently saturated, there exist $k_1, k_2 \in K(M)$ such that $g = k_1 \circ k_2$. Then, applying Corollary 2.33 twice, we show that $g \circ k = k_1 \circ k_2 \circ k$ still realizes $K|_M$. Thus (1) holds. We then apply Lemma 2.5, to conclude that K is the unique generic of Γ . \square

Definition 2.37. Let Y be the set of couples (γ, α) where $\gamma \in \Gamma$ and $\alpha \models tp(a_2/C)$. Let E be the equivalence relation on Y defined by $(\gamma, \alpha) E (\gamma', \alpha')$ if and only if there exists $\sigma \models K|_{C\gamma\gamma'\alpha\alpha'}$ such that $(\sigma \cdot \gamma)(\alpha) = (\sigma \cdot \gamma')(\alpha')$.

Then, the set Y is type-definable over C , and E is relatively C -definable. Let X be the type-definable set Y/E .

Proposition 2.38. *For each $\sigma \in \Gamma$, the map $(\gamma, a) \mapsto (\sigma \cdot \gamma, a)$ factorizes through the equivalence relation E , and this induces a definable action of Γ on X .*

Proof. Let $\sigma \in \Gamma$. Pick (γ, a) and (γ', a') that are in the same E -class. Let us show that $(\sigma \cdot \gamma, a)$ and $(\sigma \cdot \gamma', a')$ are in the same E -class. By assumption, there exists $\tau \models K|_{C\gamma\gamma'aa'}$ such that $(\tau \cdot \gamma) \cdot a = (\tau \cdot \gamma') \cdot a'$. In fact, by completeness of the type $K|_{C\gamma\gamma'aa'}$, the equality holds for all such τ . Let τ_1 realize $K|_{C\sigma\gamma\gamma'aa'}$. Then, by genericity of K and Lemma 2.5, the element $\tau_2 = \tau_1 \cdot \sigma$ also realizes $K|_{C\sigma\gamma\gamma'aa'}$. Thus, we have $(\tau_2 \cdot \gamma) \cdot a = (\tau_2 \cdot \gamma') \cdot a'$, which implies $(\tau_1 \cdot \sigma \cdot \gamma) \cdot a = (\tau_1 \cdot \sigma \cdot \gamma') \cdot a'$, which proves the result. \square

Remark 2.39. Let P_2 denote the type-definable set of realizations of $tp(a_2/C)$. Then P_2 embeds definably into X , via the map $a \mapsto (1, a)/E$. Moreover, the action of Γ on X extends the ‘‘generic action’’ of K on P_2 .

Proposition 2.40. 1. *The action of Γ on X is transitive.*

2. *The (image of the) type $tp(a_2/C)$ is generic in the space X , which is connected.*

3. *The action of Γ on X is faithful.*

4. *Under the stronger hypotheses of Theorem 2.16, the action is almost free: the stabilizers are finite.*

Proof. 1. To prove transitivity, it suffices to show that, given a, a' realizing $tp(a_2/C)$, there exists $\sigma \in \Gamma$ such that $(\sigma, a) E (1, a')$. Given such a, a' , let a'' realize $tp(a_2/C)|_{Ca\alpha'}$. Then, by Lemma 2.30, there exist τ_1, τ_2 such that $\tau_1 \models K|_{Ca} \cup K|_{Ca''}$, $\tau_2 \models K|_{Ca'} \cup K|_{Ca''}$, $\tau_1(a) = a''$ and $\tau_2(a'') = a'$. Let σ be $\tau_2 \cdot \tau_1 \in \Gamma$. Since we already know that Γ acts on X , it suffices to note that τ_1 sends the class $(1, a)/E$ to $(1, a'')/E$, which is then sent by τ_2 to $(1, a')/E$, as desired.

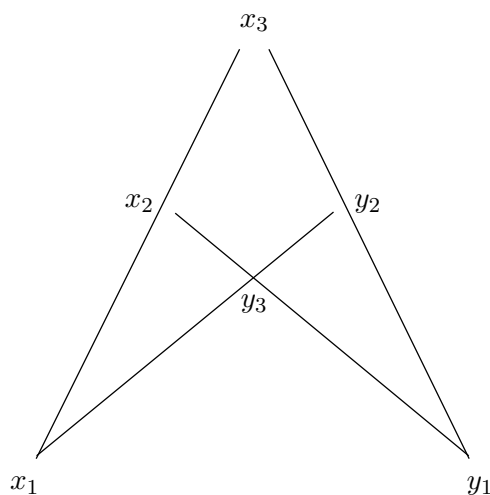
2. Let us show that the class of $(1, a_2)$ is generic in X . First, by Lemma 2.30, the stabilizer of the type $q = tp([1, a_2]_E/C)$ contains $K|_C$. Moreover, this stabilizer is a C -type-definable subgroup of Γ . Since K generates Γ , the stabilizer of q has to be Γ itself, which proves genericity of q and connectedness of X .

3. Let $g \in \Gamma$ be an element that acts trivially on X . Let us show that $g = 1$. We know that there exist $k_1, k_2 \in K$ such that $g = k_1^{-1} \circ k_2$. Then, by the hypothesis on g , we deduce that, for all a realizing $tp(a_2/C)|_{Ck_1k_2}$, we have $k_1(a) = k_2(a)$. Thus, by definition of a germ, we get $k_1 = k_2$, which implies $g = 1$.

4. Now, let us work under the stronger hypotheses of Theorem 2.16. By transitivity of the action, it suffices to show that the stabilizer of the E -class of $(1, a_2)$ is finite. Let $\sigma \in \Gamma$ such that $\sigma \cdot (1, a_2) \in E(1, a_2)$. By definition, there exists τ realizing $K|_{C\sigma a_2}$ such that $\tau \cdot \sigma(a_2) = \tau(a_2)$. Now, by Lemma 2.30, the element $\tau(a_2)$ realizes $tp(a_2/C)|_{Ca_2}$. Moreover, by genericity of K , we know that $\tau \cdot \sigma$ realizes $K|_{C\sigma a_2}$. Thus, by the second point of said lemma, we deduce that $\tau \cdot \sigma \in acl(Ca_2\tau(a_2))$. Then, $\sigma \in acl(Ca_2\tau)$. However, the element τ satisfies $\tau \downarrow_C a_2\sigma$, so we have $\tau \downarrow_{Ca_2} \sigma$. Then, $\sigma \in acl(Ca_2)$. Finally, by compactness, the stabilizer of the E -class of $(1, a_2)$ is finite, as desired. \square

2.6 Building a quadrangle

We shall now construct a partial algebraic quadrangle equivalent over M to the initial one. Let $b'_1, b'_2, b'_3 \in M$ be such that $b'_1 b'_2 b'_3 a_3 \equiv_C b_1 b_2 b_3 a_3$. Consider the following quadrangle:



with the following definitions :

- $x_1 = [f_{b'_1}^{-1}] \circ [f_{b_1}]$
- $x_2 = [f_{b'_1}^{-1}] \circ [g_{b'_2}^{-1}] \circ [g_{b_2}] \circ [f_{b'_1}]$
- $x_3 = x_2 \circ x_1 = [f_{b'_1}^{-1}] \circ [g_{b'_2}^{-1}] \circ [g_{b_2}] \circ [f_{b_1}]$
- $y_1 = x_3(y_2)$
- $y_2 = a_2$

- $y_3 = x_1(y_2)$.

There are several facts to check, in order to make sure this is well-defined and equivalent to the original quadrangle. Note that, by Remark 2.13, proving the equivalence with the original quadrangle yields that the quadrangle is a partial algebraic quadrangle generically stable over M .

Claim 2.41. *For $i = 1, 2, 3$, we have $\text{acl}(Mx_i) = \text{acl}(Mb_i)$.*

Proof. For $i = 1, 2$, this is a consequence of the definition of partial algebraic quadrangles, and of Proposition 1.21. For instance, the element x_1 is interalgebraic over M with the germ $[f_{b_1}]$, which is, by Proposition 1.21, interdefinable over $C \subseteq M$ with the canonical base $Cb(a_2a_3/Cb_1)$, which is, by definition, interalgebraic over C with b_1 . For $i = 3$, we use Lemma 2.25 and Proposition 1.21. \square

Claim 2.42. *The elements x_1, x_2, x_3 realize $K|_M$, and we have $x_1 \downarrow_M x_2$.*

Proof. Since K is generically stable, it suffices to check that these elements realize $K|_C$, and that each is independent from M over C . In fact, since K is the unique generic of Γ , it suffices to check it for x_1 and x_2 , and to prove the independence $x_1 \downarrow_M x_2$.

For $x_1 = [f_{b_1}^{-1}] \circ [f_{b_1}]$, we know that $b_1 \downarrow_C M$ and $b_1' \in M$, so that $x_1 \downarrow_{C[f_{b_1}']} M$. Then, applying Lemma 2.28, we have $x_1 \downarrow_C [f_{b_1}']$. Since $tp(x_1/C) = K|_C$ is generically stable, we can apply transitivity, to get $x_1 \downarrow_C M$, as desired.

For $x_2 = [f_{b_1}^{-1}] \circ [g_{b_2}^{-1}] \circ [g_{b_2}] \circ [f_{b_1}']$, the ideas are similar: one easily proves that $x_2 \downarrow_{C[f_{b_1}'] [g_{b_2}']} M$. Then, by Lemma 2.31, let $f \in F$ be such that $[g_{b_2}] \circ [f_{b_1}'] = [g_{b_2}'] \circ f$ and $f \downarrow_C [f_{b_1}'] [g_{b_2}']$. Then, by Lemma 2.28, we have $[f_{b_1}^{-1}] \circ f \downarrow_C [f_{b_1}'] [g_{b_2}']$, i.e. $x_2 \downarrow_C [f_{b_1}'] [g_{b_2}']$. We can then apply transitivity, just as before. \square

Claim 2.43. *The elements $y_1 = x_3(y_2)$ and $y_3 = x_1(y_2)$ are well-defined, and satisfy the following : $y_1 = f_{b_1}^{-1} \circ g_{b_2}^{-1}(a_1)$ and $y_3 = f_{b_1}^{-1}(a_3)$.*

Proof. Let us first show that the elements are well-defined. Since K is generically stable, and $tp(a_2/C)$ commutes with it, it suffices to check that $x_3 \downarrow_C y_2$ and $x_1 \downarrow_C y_2$. These verifications are left to the reader.

For the equalities, we can easily compute that $b_1b_2 \downarrow_C a_2$ and $g_{b_2} \circ f_{b_1}(y_2) = a_1 \downarrow_C b_1'b_2'$, so that $y_1 = x_3(y_2) = [f_{b_1}^{-1}] \circ [g_{b_2}^{-1}] \circ [g_{b_2}] \circ [f_{b_1}](a_2) = [f_{b_1}^{-1}] \circ [g_{b_2}^{-1}](a_1) = f_{b_1}^{-1} \circ g_{b_2}^{-1}(a_1)$.

Similarly, we have $b_1 \downarrow_C a_2$, and $f_{b_1}(a_2) = a_3 \downarrow_C b_1'$, which implies the following: $y_3 = x_1(y_2) = [f_{b_1}^{-1}] \circ [f_{b_1}](a_2) = [f_{b_1}^{-1}](a_3) = f_{b_1}^{-1}(a_3)$. \square

Claim 2.44. *For $i = 1, 2, 3$, we have $\text{acl}(My_i) = \text{acl}(Ma_i)$.*

Proof. This is easily deduced from the identities proved above, and the fact that $y_2 = a_2$. \square

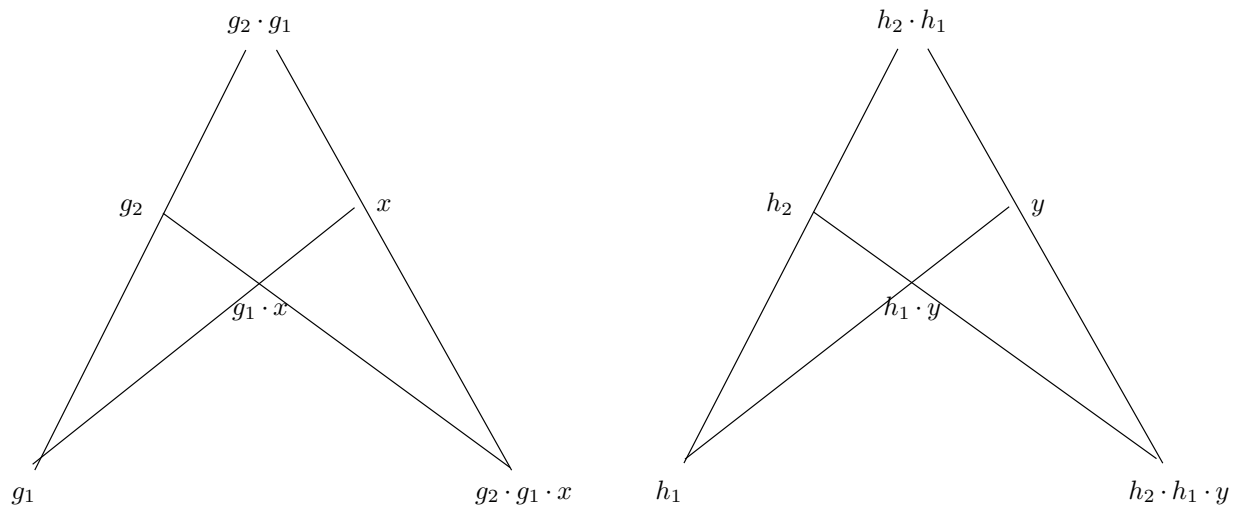
2.7 Uniqueness

To conclude, let us prove that a configuration captures the structure of the group and its action, up to some notion of correspondence.

Proposition 2.45. *Let (G, X) and (H, Y) be type-definable transitive faithful actions, where G, H and X, Y are connected, type-definable with generically stable generics. Let M be a sufficiently saturated model over which everything is defined. Let $(g_1, g_2, g_2 \cdot g_1, g_2 \cdot g_1 \cdot x, x, g_1 \cdot x)$ and $(h_1, h_2, h_2 \cdot h_1, h_2 \cdot h_1 \cdot y, y, h_1 \cdot y)$ be configurations built as in Proposition 2.10, for (G, X) and (H, Y) respectively. Assume that these configurations are equivalent over M . Then, there exist type-definable sets $S \leq G \times H$ and $T \subseteq X \times Y$, and finite normal subgroups $N_1 \triangleleft G$, $N_2 \triangleleft H$ such that:*

1. *The subgroup S induces the graph of a group isomorphism $G/N_1 \simeq H/N_2$.*
2. *The set T is an S -invariant finite correspondence between X and Y .*

Proof. Let $C \subset M$ be a small algebraically closed set of parameters over which everything is defined, and which captures the interalgebraicities. Thus, we have the following configurations, which are equivalent over C :



For $1 \leq i \leq 2$, let $c_i = (g_i, h_i) \in G \times H$. Also, let $c_3 = c_2 \cdot c_1 = (g_2 \cdot g_1, h_2 \cdot h_1) \in G \times H$. Let $p_i = tp(c_i/M)$, for $1 \leq i \leq 3$. Note that p_i is generically stable over $acl(C) = C$. By assumption and interalgebraicity, we have $c_2 \downarrow_M c_1$ and $c_3 \downarrow_M c_1$. Moreover, by definition, we have $c_2 \cdot c_1 = c_3$. Thus, we have $c_1 = (g_1, h_1) \in Stab^r(p_2, p_3)$. Let us also define $S = Stab^r(p_2)$ and $Z = Stab^r(p_2, p_3)$. Let $c'_1 \in M$ be such that $c'_1 \equiv_C c_1$. So, there exist $g \in G(M)$ and $h \in H(M)$ such that $c'_1 = (g, h) \in G \times H$.

Claim 2.46. *The following equality of type-definable sets holds : $S \cdot c'_1 = Z$.*

Proof. Let $s \in S$. Let γ realize $p_2|_{Cc'_1s}$. Then, by definition of S as a stabilizer, we have $\gamma \cdot s \models p_2|_{Cc'_1s}$. Moreover, we know that $c_1 \in \text{Stab}^r(p_2, p_3)$, so c'_1 is in $\text{Stab}^r(p_2, p_3)$ as well. Thus, as $\gamma \cdot s$ realizes $p_2|_{Cc'_1s}$, we deduce that $\gamma \cdot s \cdot c'_1$ realizes $p_3|_{Cc'_1s}$. Since we chose γ realizing $p_2|_{Cc'_1s}$, we can conclude that $s \cdot c'_1 \in \text{Stab}^r(p_2, p_3) = Z$. As $s \in S$ was arbitrary, we have just proved that $S \cdot c'_1 \subseteq Z$. The other inclusion is proved similarly. \square

Let $\pi : G \times H \rightarrow G$ be the canonical projection. Since π is a group morphism, the claim implies that $\pi(S) \cdot \pi(c'_1) = \pi(Z)$. In particular, as $c_1 = (g_1, h_1) \in Z$, we get $g^{-1} \cdot g_1 \in \pi(S)$. However, by construction, g_1 is, in the group G , generic over M . Since g is in M , this implies that $g^{-1} \cdot g_1$ is also generic over M . Therefore, the subgroup $\pi(S) \leq G$ contains an element generic over M , so it is equal to the whole group G . Symmetrically, the projection of S to the second coordinate is equal to H .

Let $N_1 = \{n_1 \in G \mid (n_1, 1) \in S\}$ and $N_2 = \{n_2 \in H \mid (1, n_2) \in S\}$. It is easy to check that $N_1 \triangleleft G$ and $N_2 \triangleleft H$. Let us show that N_1 is finite; the proof for N_2 will be symmetric. Let $n_1 \in N_1(M)$. Then, we know that $c_2 = (g_2, h_2)$ realizes $p_2|_{Cn_1}$. Then by definition of S , the element $(g_2 \cdot n_1, h_2)$ realizes $p_2|_{Cn_1}$. In particular, we have $g_2 \cdot n_1 \in \text{acl}(Ch_2)$. Thus, since we are in a group, we deduce $n_1 \in \text{acl}(Cc_2)$. However, we know that $c_2 \downarrow_C n_1$. So, by Proposition 1.10, we have $n_1 \in \text{acl}(C)$. As $n_1 \in N_1$ was arbitrary, we deduce by compactness that N_1 is finite. Thus, the subgroup $S \leq G \times H$ induces a subgroup $\Sigma \leq G/N_1 \times H/N_2$, which defines the graph of a definable group isomorphism $G/N_1 \rightarrow H/N_2$.

Now, by saturation of M , let $(x_0, y_0) \in X(M) \times Y(M)$ be such that $(x_0, y_0) \equiv_C (g_1 \cdot x, h_1 \cdot y)$. Let us consider the following type-definable set : $T = S \cdot (x_0, y_0) \subseteq X \times Y$. Let $q_3 = \text{tp}(g_1 \cdot x, h_1 \cdot y/M)$ and $q_1 = \text{tp}(g_2 \cdot g_1 \cdot x, h_2 \cdot h_1 \cdot y/M)$. Then, q_1 and q_3 are generically stable over $C = \text{acl}(C)$ and, by hypothesis, we have $(g_2, h_2) \in \text{Stab}(q_3, q_1)$.

Claim 2.47. *The set T is closed under the action of $S \leq G \times H$. Moreover, for all $x_1 \in X, y_1 \in Y$, the sets $T \cap (\{x_1\} \times Y)$ and $T \cap (X \times \{y_1\})$ are finite and nonempty.*

Proof. Since T is the S -orbit of a point in the space $X \times Y$, it is closed under the action of S . Moreover, we have proved that S projects onto G and onto H . Therefore, by transitivity of the actions of G on X and Y , the projections $T \rightarrow X$ and $T \rightarrow Y$ are onto. Thus, it remains to prove finiteness of the fibers, so to speak. By symmetry, it suffices to prove that, for any $y_1 \in Y$, the type-definable set $T_{y_1} := T \cap (X \times \{y_1\})$ is finite. Note that, if $y_1, y_2 \in Y$, if $\gamma \in S$ is an element such that $\gamma \cdot y_1 = y_2$, then $\gamma \cdot T_{y_1} \subseteq T_{y_2}$. Then, since we also have $\gamma^{-1} \cdot y_2 = y_1$, the equality $\gamma \cdot T_{y_1} = T_{y_2}$ holds. So, by transitivity of the action of G on Y , it suffices to prove that T_{y_1} is finite, for *some* $y_1 \in Y$. We shall prove that T_{y_0} is finite.

By compactness and saturation of M , it suffices to prove that, if $(a, y_0) \in T_{y_0}(M)$, then $a \in \text{acl}(Cy_0)$. So, let $a \in X(M)$ be such that $(a, y_0) \in T_{y_0}$. Let us prove that $a \in \text{acl}(Cy_0)$. By saturation of M , and definition of T , there exists $(g, h) \in S(M)$ such that $(a, y_0) = (g \cdot x_0, h \cdot y_0)$. In particular, we have $h \cdot y_0 = y_0$. Now, recall that $\text{Stab}^r(p_2) = S$, and $(g_2, h_2) \models p_2|_M$. Therefore, we have $(g_2 \cdot g, h_2 \cdot h) \models p_2|_M$. In particular, as $(g_2, h_2) \in \text{Stab}(q_3, q_1)$, the element $c := (g_2 \cdot g, h_2 \cdot h)$ is also in $\text{Stab}(q_3, q_1)$.

Also, by symmetry for the generically stable types $q_3|_C$ and $p_2|_C$, we have $(x_0, y_0) \models q_3|_{Cc}$. Thus, $c \cdot (x_0, y_0) \models q_1|_{Cc}$. In particular, we have $g_2 \cdot g \cdot x_0 \in \text{acl}(Ch_2 \cdot h \cdot y_0)$. So $g \cdot x_0 \in \text{acl}(Cg_2, h_2 \cdot h \cdot y_0) \subseteq \text{acl}(Cg_2, h_2, h \cdot y_0)$. Now, recall that $\text{acl}(Cg_2) = \text{acl}(Ch_2)$, that $h \cdot y_0 = y_0$, and $a = g \cdot x_0$. Thus, we have $a \in \text{acl}(Cg_2, y_0)$. Since $a, y_0 \in M$ and g_2 is generic over M , we have $g_2 \downarrow_C ay_0$, so $g_2 \downarrow_{Cy_0} a$. Therefore, by Proposition 1.10, we have $a \in \text{acl}(Cy_0)$, as desired. \square

Thus, we can use the set T to define an S -equivariant finite correspondence between X and Y . \square

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