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A new family of route formulations for split delivery vehicle routing problems

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Abstract

We propose a new family of formulations with route-based variables for the split delivery vehicle routing problem with and without time windows. Each formulation in this family is characterized by the maximum number of different quantities of demand that can be delivered to a customer during a vehicle visit. As opposed to previous formulations in the literature, the exact delivery quantities are not always explicitly known in this new family. The validity of these formulations is ensured by an exponential set of non-robust constraints. We also explore a property of optimal solutions that allows us to specify a minimum delivery quantity based on customer demand and vehicle capacity, and this number is often greater than one. We use this property to reduce the number of possible delivery quantities in our formulations, improving the solution times of the computationally strongest formulation in the family. In addition, we propose new variants of non-robust cutting planes that strengthen the formulations, which are limited-memory subset-row covering inequalities and limited-memory strong k -path inequalities. Finally, we develop a branch-cut-and-price algorithm to solve our formulations enriched with the proposed valid inequalities, which resorts to state-of-the-art algorithmic enhancements. We show how to effectively manage the non-robust cuts when solving the pricing problem that dynamically generates route variables. Numerical results indicate that our formulations and BCP algorithm establish new state-of-the-art results for the variant with time windows since all benchmark instances with 50 customers and many instances with 100 customers are solved to optimality for the first time. Several instances of the variant without time windows are solved to proven optimality for the first time.

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1 Introduction

The applications of the vehicle routing problem (VRP) are ubiquitous and play an important role in promoting effective logistics operations that contribute to economic, environmental and sustainable goals (Shapiro, 2007; Bektaş and Laporte, 2011). The success of these applications rests on the results of intensive research developments obtained by the VRP community over more than 60 years. Such efforts have involved the design of a number of mathematical formulations and solution approaches for theoretical and applied variants, continuously pushing the boundaries of the size and type of problems that one can expect to solve in practice (Toth and Vigo, 2014; Braekers et al., 2016).

In the traditional variants of the capacitated VRP, one must design a set of least-cost routes in a way to visit each customer exactly once using a homogeneous fleet of vehicles available in a single depot such that the total demand delivered in each route does not exceed the vehicle capacity. In certain variants, such as the VRP with time windows (VRPTW), customers might only be available for service during a certain period of time throughout the day, known as a time window, and service times might vary among customers. In this paper, we are mainly interested in an extension of this latter variant, known in the literature as the split delivery VRPTW (SDVRPTW), in which customers may be visited more than once, if beneficial, so that their demands are split between two or more vehicles.

The SDVRPTW adds a degree of operational flexibility by relaxing the VRPTW and encompassing the decision of how much to deliver to each customer. Multiple visits allow us to include customers with demands that are larger than the vehicle capacity. Additionally, multiple visits can be beneficial even if a customer’s demand fits in a single vehicle since these split deliveries can promote significant savings by increasing the utilization of the vehicles’ capacity. As pointed out originally by Dror and Trudeau (1989, 1990) and Archetti et al. (2006) for the variant without time windows (SDVRP), the savings can reach up to 50% of routing costs. However, the benefits of multiple visits come in exchange for increased difficulty in modeling and solving these variants with respect to their nonsplit counterparts, especially regarding exact approaches (Bianchessi and Irnich, 2019; Munari and Savelsbergh, 2022; Gouveia et al., 2022).

1.1 Related literature

Despite its practical benefits and theoretical relevance, the SDVRPTW has received relatively little attention in the literature. Mullaseril and Dror (1996) presented the first attempt to model the SDVRPTW using a column generation scheme. Their formulation relies on the replication of customers, followed by the definition of split configurations in advance so that the problem becomes an instance of the VRPTW. A first standard branch-and-price scheme tailored for the SDVRPTW and with no initial assumptions on the number of splits was presented by Feillet et al. (2006), in which the decision on how much to deliver is addressed at the master problem level, whereas feasible routes are determined in the pricing subproblem, which consists of an elementary shortest path problem with resource constraints (ESPPRC). Desaulniers (2010) proposed an innovative branch-cut-and-price (BCP) algorithm based on extreme delivery patterns, which are determined in the pricing subproblem together with their corresponding routes, and the actual delivery quantities are determined through convex combinations of these extreme patterns at the master problem level. Archetti et al. (2011a) enhanced this approach by implementing acceleration techniques on the subproblem by means of a tabu search heuristic, as well as presenting novel valid inequalities. Luo et al. (2017) presented a BCP algorithm that extends the extreme delivery pattern concept to the SDVRPTW with linear weight-related costs and takes advantage of acceleration techniques in their label-setting pricing algorithm.

More recently, tailored branch-and-cut (BC) algorithms have shown superior performance for the SDVRPTW variant and have become the state-of-the-art exact approaches. Bianchessi and Irnich (2019) proposed a tailored BC method for the SDVRPTW based on a relaxed commodity flow formulation. The authors presented new types of valid inequalities, which, together with

other well-known cuts from the literature, are used in their BC to strengthen the linear relaxation of the relaxed model, as well as cut off integer solutions that are infeasible for the SDVRPTW. Munari and Savelsbergh (2022) introduced three novel compact formulations for the SDVRP and SDVRPTW and proposed a BC algorithm based on a relaxation of their best-performing formulation. Different from previous approaches, their BC algorithm locally extends the relaxed model by inserting new variables and the so-called regularity property constraints every time an infeasible integer solution is found in the BC tree. The same BC algorithm was used by Munari and Savelsbergh (2020) to develop a column generation-based heuristic that consists of adding to the relaxed formulation a set of time-feasible routes that are generated in advance. Other heuristics have also been proposed specifically for the SDVRPTW and related variants, but using different strategies. Frizzell and Giffin (1995) developed a construction heuristic given by a look-ahead approach, in addition to improvement heuristics based on moving and exchange customer between routes. Mullaseril et al. (1997) adapted the construction and improvement heuristics of Dror and Trudeau (1990) to solve a variant applied to livestock feed distribution that was modeled as a split-delivery capacitated rural postman problem with time windows on arcs. Belfiore and Yoshizaki (2009) also addressed a real-life variant, modeled as a heterogeneous fleet SDVRPTW, by proposing constructive heuristics and a scatter search algorithm. Finally, Ho and Haugland (2004) developed a tabu search heuristic for the SDVRPTW, based on traditional move operators (relocate, exchange and 2-opt*) and a new move operator called relocate split. The authors considered modified Solomon’s instances in their computational experiments. Except from Munari and Savelsbergh (2020), none of the mentioned heuristics have been tested on the exact same benchmark instances considered in the experiments performed with the recent exact approaches (Archetti et al., 2011a; Bianchessi and Irnich, 2019; Munari and Savelsbergh, 2022).

The aforementioned state-of-the-art exact approaches can effectively solve most benchmark instances with up to 50 customers, but they often become ineffective if the number of customers increases. For example, Bianchessi and Irnich (2019) reported proven optimal solutions for 104 of 168 instances with 50 customers, while this number decreased to 5 of 168 for instances with 100 customers. For the same instances, Munari and Savelsbergh (2020, 2022) presented proven optimal solutions for 123 50-customer instances, whereas they reported relatively large integrality gaps for most 100-customer instances. This behavior has not been observed in the results reported in the literature for 100 customer instances of traditional VRP variants, such as the VRPTW, when they are solved using a BCP method (see, e.g., Sadykov et al. (2021)), suggesting that there could be room for improvement in the computational solution of the SDVRPTW.

All of the aforementioned algorithms can also be applied for the SDVRP, i.e., the variant without time windows. However, only Munari and Savelsbergh (2022) presented results for both the SDVRPTW and the SDVRP. Exact approaches proposed in the literature specifically for the SDVRP include those by Jin et al. (2008); Moreno et al. (2010); Archetti et al. (2011b, 2014); Ozbaygin et al. (2018); Gouveia et al. (2022). These approaches were not extended to include time windows or verified on SDVRPTW instances. Again, tailored branch-and-cut algorithms dominate the state of the art for the SDVRP. The current best algorithms by Gouveia et al. (2022) and by Munari and Savelsbergh (2022) are able to solve to optimality most of the literature instances with 50 customers and a small proportion of instances having between 64 and 100 customers.

1.2 Contributions

The main contributions of this paper are summarized as follows.

- We explore a property that is satisfied by at least one optimal solution of the problem, which is valid for both the SDVRP and the SDVRPTW. This property allows us to specify a minimum delivery quantity when visiting a customer on a route. Efficiency of our solution approach depends on the ratio between the minimum delivery quantity and average customer demand. The larger is this ratio, the faster is our algorithm.
- We present a new family of route-based formulations for the problem, as well as a BCP

algorithm for solving them. Compared to previous column generation-based approaches in the literature, the exact delivery quantities are not always explicitly known, neither in the pricing problem nor in the master problem. Covering of customer demand is guaranteed by an exponential family of constraints separated dynamically using a maximum flow-based algorithm. These constraints are supported by a flow graph representation of a solution, which offers theoretical and practical advantages, as we show in this paper.

- We propose variants of non-robust valid inequalities designed to improve the strength of our formulations, which are limited-memory subset-row covering inequalities and limited-memory strong k -path inequalities. Additionally, we show how to effectively manage novel non-robust valid inequalities when solving the pricing problem of the column generation procedure.
- We numerically compare the strength and the solution time of the formulations in the newly proposed family using instances of the SDVRPTW with different characteristics.
- Finally, we show that our BCP algorithm outperforms the state-of-the-art exact approaches for the SDVRPTW. Numerous literature instances are solved to optimality for the first time, including all instances with 50 customers and many instances with 100 customers. Some SDVRP instances are also solved to optimality for the first time.

1.3 Organization of the paper

The remainder of this paper is organized as follows. In Section 2, we define the problem and state its known properties of optimal solutions. A new property is introduced in Section 3. In Section 4, a new family of formulations is presented, as well as known and novel valid inequalities for this family. Our BCP algorithm is described in Section 5. The results of computational experiments are shown in Section 6, in which we numerically compare the formulations from the proposed family and test our BCP algorithm on benchmark instances for both the SDVRPTW and the SDVRP. Section 7 outlines the major contributions and future research directions.

2 Problem definition and known properties

We define the SDVRPTW over a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, where $\mathcal{V} = \{0, n+1\} \cup \mathcal{C}$ and $\mathcal{A} = \{(i, j) : i, j \in \mathcal{V}, i \neq n+1, j \neq 0, i \neq j\}$. Nodes 0 and $n+1$ are the source and sink representations of the depot, respectively. Set $\mathcal{C} = \{1, 2, \dots, n\}$ represents customer nodes. For each arc $a \in \mathcal{A}$, a cost $c_a \geq 0$ and a travel time $t_a \geq 0$ is defined. For the sake of simplicity, when $a = (i, j)$, we may drop the parenthesis and the comma, and replace $c_{(i,j)}$ and $t_{(i,j)}$ with c_{ij} and t_{ij} , respectively. We suppose that the triangle inequality holds for both costs and travel times. In practice, triangle inequality always holds for travel times. Thus, in the case in which arc costs are equal to arc travel times, this assumption comes without loss of generality. Each customer $i \in \mathcal{C}$ has a rational positive demand $d_i > 0$ to be fulfilled by one or more vehicles, and this customer is available for service within a nonempty interval $[e_i, l_i]$, where e_i and l_i stand for the earliest and latest times for starting service, respectively. The time window of the depot $[e_0, l_0] = [e_{n+1}, l_{n+1}]$ defines the planning horizon. In the case of an early arrival at a customer, waiting before starting the service is allowed. The service time for each customer $i \in \mathcal{C}$ is assumed to be constant and is embedded into the travel times of all arcs (i, j) leaving i . A homogeneous fleet of H vehicles with rational, positive capacity $Q > 0$ is available. A route $r = (v_0^r = 0, v_1^r, \dots, v_{n_r}^r = n+1)$ visiting n_r nodes in graph \mathcal{G} , starting at time $t_0^r = e_0$ and delivering amount $d_k^r \geq 0$ of demand to every customer v_k^r , $1 \leq k < n_r$, is feasible if the total delivered demand $\sum_{k=1}^{n_r-1} d_k^r$ does not exceed Q , and the start of service time t_k^r at every node is within its time window. We define as q_i^r the total demand delivered via route r to customer i (in possibly several visits). Times of the service start t_k^r , $1 \leq k \leq n_r$, are determined recursively as $t_k^r = \max\{t_{k-1}^r + t_{a_k^r}, e_{v_k^r}\}$, where a_k^r is the arc (v_{k-1}^r, v_k^r) . The cost c^r of route r is the total of the costs of the arcs that it traverses: $c^r = \sum_{k=1}^{n_r} c_{a_k^r}$. A feasible solution of the problem consists

of a set of at most H feasible routes in which the total delivered quantity to every customer $i \in \mathcal{C}$ is at least equal to d_i . A split customer in a solution is a customer who is visited by two or more vehicles. The aim of the SDVRPTW is to find a feasible solution that minimizes the total route cost. A related variant, the (capacitated) split delivery vehicle routing problem (SDVRP), is the most basic one, in which time windows, service times, and route timings are not defined.

There always exists an optimal solution to the SDVRPTW that satisfies the following properties:

Property 1. (Dror and Trudeau, 1990; Feillet et al., 2006) *Two routes in the solution share at most one single split customer;*

Property 2. (Feillet et al., 2006) *Each route is elementary: it visits each customer at most once;*

Property 3. (Feillet et al., 2006) *Each arc between customer nodes is traversed at most once;*

Property 4. (Desaulniers, 2010) *For each pair of arcs between two customers, at most one is traversed; and*

Property 5. (Archetti et al., 2011a) *All delivery quantities are integers if demand and vehicle capacity are integers.*

In the next section, we present a new property that generalizes Property 5.

3 Flow graph solution representation and a useful property

The flow graph solution representation introduced in this section can be used to verify the feasibility of a set of routes with respect to the customer demand. This graph has theoretical and practical importance since we rely on it to prove the validity of our formulations and to separate some of our valid inequalities.

Let $\tilde{\mathcal{R}} = \{r_1, r_2, \dots, r_{|\tilde{\mathcal{R}}|}\}$ be a set of time-feasible routes (i.e., satisfying time window constraints), in which delivery quantities are not defined. For ease of presentation, we consider that routes r_i and r_j are different whenever $i \neq j$, even if they follow the same sequence of arcs. We now construct the following valued graph $\mathcal{F}(\tilde{\mathcal{R}})$ to check whether set $\tilde{\mathcal{R}}$ defines a feasible SDVRPTW solution. In the case of positive answer, delivery quantities are determined. The values of arcs in $\mathcal{F}(\tilde{\mathcal{R}})$ correspond to their capacities.

The set of nodes in $\mathcal{F}(\tilde{\mathcal{R}})$ is $\{0\} \cup \tilde{\mathcal{R}} \cup \mathcal{C} \cup \{n+1\}$. Nodes 0 and $n+1$ are the source and the sink, respectively. The first set of arcs $\mathcal{A}_1(\tilde{\mathcal{R}})$ connects the source with each of the route nodes in $\tilde{\mathcal{R}}$. The capacity of these arcs is Q . The second set of arcs $\mathcal{A}_2(\tilde{\mathcal{R}})$ connects the route nodes in $\tilde{\mathcal{R}}$ with customer nodes in \mathcal{C} : arc (r, i) belongs to $\mathcal{A}_2(\tilde{\mathcal{R}})$ if and only if route $r \in \tilde{\mathcal{R}}$ visits customer $i \in \mathcal{C}$. The capacity of these arcs is ∞ . Finally, the third set of arcs $\mathcal{A}_3(\tilde{\mathcal{R}})$ connects each customer node to the sink. The capacity of an arc $(i, n+1) \in \mathcal{A}_3(\tilde{\mathcal{R}})$ is d_i . Figure 1 provides an illustration of graph $\mathcal{F}(\tilde{\mathcal{R}})$, where $\tilde{\mathcal{R}} = \{r_1 = \{0, 1, 2, 3, 6\}, r_2 = \{0, 2, 3, 6\}, r_3 = \{0, 4, 5, 6\}\}$ is a set of routes serving customers $\mathcal{C} = \{1, 2, 3, 4, 5\}$ with demands $d = \{10, 20, 30, 40, 10\}$, respectively. The capacity of the vehicle is $Q = 30$. From the construction of the graph, we deduce the following observation.

Observation 1. *A set $\tilde{\mathcal{R}}$ of time-feasible routes forms a feasible SDVRPTW solution if and only if the maximum flow f in graph $\mathcal{F}(\tilde{\mathcal{R}})$ has a value of $\sum_{i \in \mathcal{C}} d_i$. In such a case, values f_a , $a \in \mathcal{A}_2$, correspond to the delivery quantities for every route in $\tilde{\mathcal{R}}$ to each customer in \mathcal{C} .*

The maximum flow f in the example flow graph in Figure 1 has value 90, which is smaller than $\sum_{i \in \mathcal{C}} d_i = 110$. Thus, this set of routes does not form a feasible solution.

Let now $\bar{q} = \gcd(Q, d_1, d_2, \dots, d_n)$ be the greatest common divisor of the capacity and all customer demands. Since all these values are rational and positive, value \bar{q} exists (Weil, 1983).

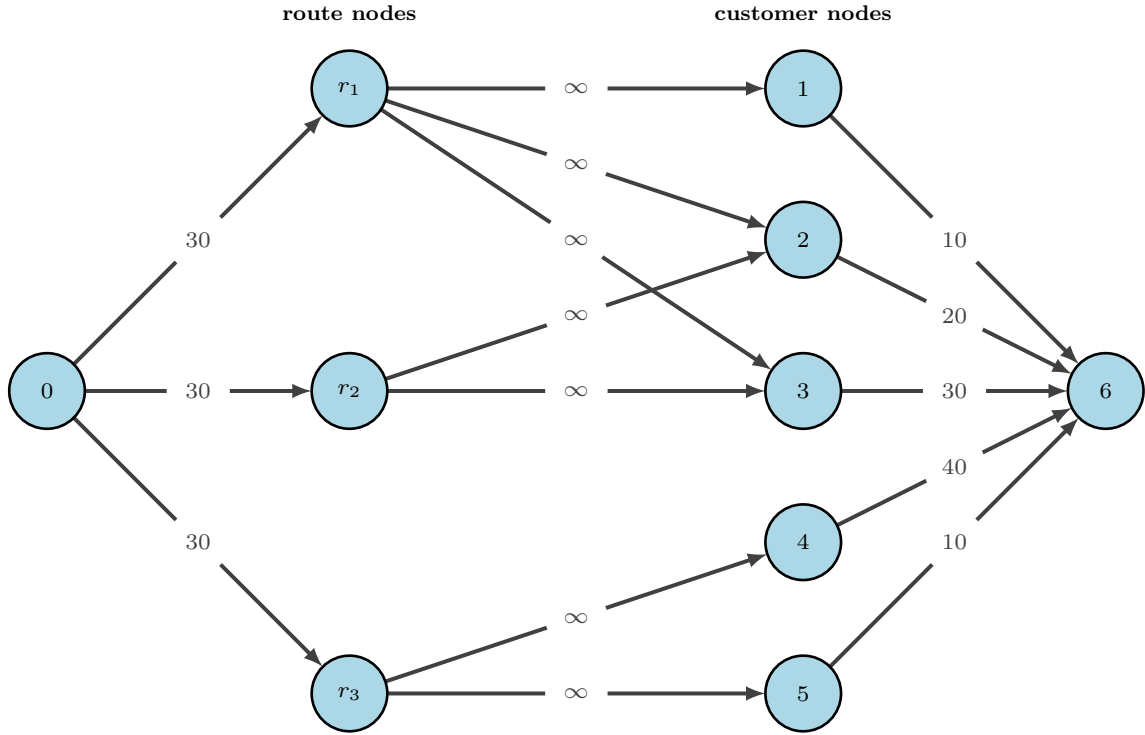


Figure 1: An example of the flow graph $\mathcal{F}(\tilde{\mathcal{R}})$.

Property 6. *There exists an optimal solution in which all delivery quantities in all routes are multiples of \bar{q} .*

Proof. Let I be a problem instance. Define a new instance I' derived from I by dividing all demands and capacity by \bar{q} . In this case, any solution s' of I' is feasible if, and only if, there is an equivalent feasible solution s of I with the same cost, and the same delivery quantities multiplied by \bar{q} . Since, by Property 5, there exists an optimal solution s' of I' in which all the delivery quantities are integer, the corresponding solution s of I is also optimal and satisfies Property 6. \square

4 Mathematical formulations

In this section, we introduce a new family of formulations based on route variables. Let \mathcal{R} be the set of all feasible SDVRPTW routes satisfying Property 6. We define $D_i = \{\bar{q}, 2\bar{q}, \dots, d_i\}$ as the set of all possible delivery quantities to customer $i \in \mathcal{C}$ via any route in \mathcal{R} . Zero delivery is not included in D_i due to the triangle inequalities.

Imposing elementarity for routes may make the route generation subproblem difficult. Thus, Property 2 is not imposed for routes in \mathcal{R} . Given a route $r \in \mathcal{R}$, customer $i \in \mathcal{C}$ and delivery quantity $q \in D_i$, we define as b_{iq}^r the number of times r makes a visit to i in which the delivery quantity equals q .

4.1 Base formulation

Let h_{rS} be a binary value that is equal to 1 if and only if route r enters subset $S \subseteq \mathcal{C}$. For every route $r \in \mathcal{R}$, we define a nonnegative integer variable θ_r , which represents the number of vehicles

that follow route r . We now state our first formulation, which we denote as (F0).

$$(F0): \quad \text{Min} \quad \sum_{r \in \mathcal{R}} c^r \theta_r, \quad (1)$$

$$\text{s.t.} \quad \sum_{r \in \mathcal{R}} h_{r,S} \theta_r \geq \left\lceil \sum_{i \in S} d_i / Q \right\rceil, \quad \forall S \subseteq \mathcal{C}, \quad (2)$$

$$\theta_r \in \mathbb{Z}^+, \quad \forall r \in \mathcal{R}. \quad (3)$$

Objective Function (1) minimizes the sum of routing costs. Constraints (2) correspond to the strong k -path inequalities introduced by Baldacci et al. (2008) and used for the SDVRPTW in Archetti et al. (2011a). They are a strengthened version of well-known rounded capacity cuts, which have the same right-hand-side. The difference is that the coefficient $h_{r,S}$ is equal to one even if route r quits and enters subset S several times. Finally, the integrality of the θ_r variables is ensured by (3).

We now show that these inequalities suffice to define the set of feasible solutions to our problem. We first denote by $\tilde{\mathcal{R}}(\bar{\theta}) = \{r_1, r_2, \dots, r_{|\tilde{\mathcal{R}}(\bar{\theta})|}\}$ the set of routes corresponding to an integer solution $\bar{\theta}$: every route $r \in \mathcal{R}$ is added $\bar{\theta}_r$ times to $\tilde{\mathcal{R}}(\bar{\theta})$, and the information about the delivered quantities is removed from r . Again, we consider that $r_i \neq r_j$ are different whenever $i \neq j$, although there may be multiple routes in $\tilde{\mathcal{R}}(\bar{\theta})$ representing the same route $r \in \mathcal{R}$ in case $\bar{\theta}_r > 1$.

Theorem 1. *If an integer solution $\bar{\theta}$ satisfies Constraints (2), then set $\tilde{\mathcal{R}}(\bar{\theta})$ of routes forms a feasible SDVRPTW solution.*

Proof. Suppose that set $\tilde{\mathcal{R}}(\bar{\theta})$ does not form a feasible SDVRPTW solution. By Observation 1, it follows that, in the flow graph $\mathcal{F}(\tilde{\mathcal{R}}(\bar{\theta}))$, the maximum flow value is strictly less than $\sum_{i \in \mathcal{C}} d_i$. By the min-cut-max-flow theorem, the value of a minimum cut C in this graph is strictly smaller than $\sum_{i \in \mathcal{C}} d_i$. We denote by $\mathcal{R}(C)$ the set of route nodes $r \in \tilde{\mathcal{R}}(\bar{\theta})$ in graph $\mathcal{F}(\tilde{\mathcal{R}}(\bar{\theta}))$ such that arc $(0, r)$ is included in cut C . Set $\mathcal{R}(C)$ is not empty since the cut cannot contain all arcs in $\mathcal{A}_3(\tilde{\mathcal{R}}(\bar{\theta}))$ and cannot contain any arc in $\mathcal{A}_2(\tilde{\mathcal{R}}(\bar{\theta}))$. We also denote by $\mathcal{C}(C)$ the set of customer nodes $i \in \mathcal{C}$ in graph $\mathcal{F}(\tilde{\mathcal{R}}(\bar{\theta}))$ such that arc $(i, n+1)$ is not included in cut C . No arc in $\mathcal{A}_2(\tilde{\mathcal{R}}(\bar{\theta}))$ crosses cut C in the direction of the sink since otherwise the value of the cut would be ∞ . Therefore, no route in $\tilde{\mathcal{R}}(\bar{\theta}) \setminus \mathcal{R}(C)$ enters subset $\mathcal{C}(C)$ of customers. Conversely, all routes in $\mathcal{R}(C)$ enter $\mathcal{C}(C)$; otherwise, C would not be a minimum cut. We therefore have

$$\sum_{r \in \mathcal{R}} Q h_{r, \mathcal{C}(C)} \bar{\theta}_r = |\mathcal{R}(C)| \cdot Q < \sum_{i \in \mathcal{C}(C)} d_i. \quad (4)$$

The last inequality in (4) is true because the value of cut C is $\sum_{i \in \mathcal{C} \setminus \mathcal{C}(C)} d_i + |\mathcal{R}(C)| \cdot Q$, which should be smaller than $\sum_{i \in \mathcal{C}} d_i = \sum_{i \in \mathcal{C} \setminus \mathcal{C}(C)} d_i + \sum_{i \in \mathcal{C}(C)} d_i$. Canceling the term $\sum_{i \in \mathcal{C} \setminus \mathcal{C}(C)} d_i$ in both expressions, we obtain the result. By dividing Inequality (4) by Q , we obtain $\sum_{r \in \mathcal{R}} h_{r, \mathcal{C}(C)} \bar{\theta}_r < \sum_{i \in \mathcal{C}(C)} d_i / Q$, and Constraint (2) for set $\mathcal{C}(C)$ of customers is violated by solution $\bar{\theta}$. Thus, our assumption is wrong, and set $\tilde{\mathcal{R}}(\bar{\theta})$ forms a feasible SDVRPTW solution. \square

Corollary 1. *Constraints (2) can be exactly separated for integer solutions $\bar{\theta}$ of Formulation (F0) in polynomial time by defining graph $\mathcal{F}(\tilde{\mathcal{R}}(\bar{\theta}))$ and finding a minimum cut in it.*

The minimum cut in the example flow graph in Figure 1 has customer nodes 4 and 5 at the sink side of the cut. The subset formed by these two customers induces a violated inequality (2). Only one route in $\tilde{\mathcal{R}}$ enters this subset, whereas at least two routes are required.

Observation 2. *Formulation (F0) is correct even without any information about delivery quantities in routes $r \in \mathcal{R}$.*

Although this observation denotes a nice theoretical property, it is not always possible to use Formulation (F0) in practice. We denote as \mathcal{R}^0 the set of routes without specification of delivery quantities. Obviously, \mathcal{R}^0 is smaller than \mathcal{R} . Set \mathcal{R}^0 is not finite: i) for the standard SDVRP; and ii) for the SDVRPTW if there exists a cycle with zero total travel time. This happens because routes in \mathcal{R}^0 are not necessarily elementary and are not constrained by the vehicle capacity. Thus, the length of routes is not restricted in the SDVRP case. This is also the case for the the SDVRPTW with zero travel times, as a route may indefinitely circulate along the cycle with zero travel time. Even for the SDVRPTW with only positive travel times, Formulation (F0) is weak, as shown in the computational experiments.

4.2 A family of partially discretized formulations

Formulation (F0) can be strengthened in the following way. As the triangle inequalities are satisfied, there exists an optimal solution with no zero deliveries. Thus, we can reduce the number of variables in Formulation (F0) by retaining only variables θ_r that correspond to routes delivering at least \bar{q} at every visit. The minimum delivery of one unit was already imposed by Archetti et al. (2011b). We denote this strengthened formulation as (F1). Moreover, there exists an optimal solution $\bar{\theta}$ to the linear relaxation of (F1) in which every route r such that $\bar{\theta}_r > 0$ delivers exactly \bar{q} at every visit; i.e., r has length of at most Q/\bar{q} . We denote the set of all such routes as $\mathcal{R}^1 \subseteq \mathcal{R}$.

There are further ways to strengthen (F0) if we consider the information about full and partial deliveries along routes. Recall that b_{iq}^r is the number of times that a route $r \in \mathcal{R}$ includes a visit to i with delivery quantity equal to q . We denote as $b_{iF}^r = b_{id_i}^r$ the number of times that route $r \in \mathcal{R}$ delivers full demand to customer $i \in \mathcal{C}$. Additionally, let $b_{iP}^r = \sum_{q \in D_i \setminus \{0, d_i\}} b_{iq}^r$ be the number of times that route $r \in \mathcal{R}$ delivers partial demand to customer $i \in \mathcal{C}$. The following constraints are valid for the SDVRPTW:

$$\sum_{r \in \mathcal{R}} (2b_{iF}^r + b_{iP}^r) \theta_r \geq 2, \quad \forall i \in \mathcal{C}. \quad (5)$$

These constraints are a special case of the strong minimum number of vehicles (SVM) inequalities used by Archetti et al. (2011a).

We denote as (F2) the formulation with Objective Function (1) and Constraints (2), (3) and (5). Let $\mathcal{R}^2 = \{r \in \mathcal{R} : b_{iq}^r = 0, \forall i \in \mathcal{C}, \forall q \in D_i \setminus \{\bar{q}, d_i\}\}$ be the set of routes in which the delivery quantity in every visit to customer $i \in \mathcal{C}$ is equal either to \bar{q} or to d_i .

Observation 3. *There exists an optimal solution $\bar{\theta}$ to the linear relaxation of Formulation (F2) such that $\bar{\theta}_r = 0$ for all $r \notin \mathcal{R}^2$.*

This observation follows from the coefficients of variables in Formulation (F2) depending only on the information whether a delivery is full or partial and not depending on exact delivery quantities. Then, any route $r' \notin \mathcal{R}^2$ is dominated, i.e., can be replaced by a route $r \in \mathcal{R}^2$ in any optimal solution of the relaxation without losing feasibility and optimality of this solution. Thus, we can restrict the set of routes in Formulation (F2) to \mathcal{R}^2 without compromising the validity of the dual bound provided by its linear relaxation.

Restricting the number of different delivery quantities to a customer is useful to speed up the dynamic generation of route variables, as will be shown in Section 5. We now extend the case with at most two different delivery quantities per customer to any integer $K \geq 2$. As will be shown by computational experiments, increasing the value of K renders the formulation stronger, possibly at the expense of slower generation of route variables. We start with the observation that the following constraints are valid for the SDVRPTW:

$$\sum_{r \in \mathcal{R}} \sum_{q \in D_i} qb_{iq}^r \theta_r \geq d_i, \quad \forall i \in \mathcal{C}. \quad (6)$$

For some customers $i \in \mathcal{C}$, we derive new inequalities as follows. Let us define $\mathcal{C}(K) = \{i \in \mathcal{C} : K\bar{q} < d_i\}$ as the set containing each customer $i \in \mathcal{C}$ to which K deliveries of size \bar{q} are

not enough to satisfy demand d_i . For a given customer $i \in \mathcal{C}(K)$, we multiply Inequality (6) by $(K-1)/(d_i - \epsilon)$, where $\epsilon > 0$ is a constant significantly smaller than \bar{q} . Next, we apply Chvátal-Gomory rounding on both sides of the resulting inequality and obtain

$$\sum_{r \in \mathcal{R}} \sum_{q \in D_i} \left\lceil \frac{(K-1)qb_{iq}^r}{d_i - \epsilon} \right\rceil \theta_r \geq \left\lceil \frac{(K-1)d_i}{d_i - \epsilon} \right\rceil. \quad (7)$$

Since $d_i/(d_i - \epsilon)$ is slightly greater than 1, the right-hand side of (7) is equal to K . Additionally, given that b_{iq}^r is integer, the following holds for the coefficient in the left-hand side of these inequalities:

$$b_{iq}^r \left\lceil \frac{(K-1)q}{d_i - \epsilon} \right\rceil \geq \left\lceil \frac{(K-1)qb_{iq}^r}{d_i - \epsilon} \right\rceil. \quad (8)$$

Hence, we can rewrite (7) as

$$\sum_{r \in \mathcal{R}} \sum_{q \in D_i} b_{iq}^r \left\lceil \frac{(K-1)q}{d_i - \epsilon} \right\rceil \theta_r \geq K. \quad (9)$$

For a given K and delivery quantity $q \in D_i$ of a route r , the rounded-up coefficient in the left-hand side of these inequalities is a step function that assumes integer values k from 1 to K , depending on the value of q . For example, for $K = 2$, the possible values of $\lceil q/(d_i - \epsilon) \rceil$ with $q \in D_i$ are 1, if $q < d_i$; and 2, if $q = d_i$. Note that in this case, Inequalities (9) are the same as (5). For $K = 3$, $\lceil 2q/(d_i - \epsilon) \rceil$ results in 1, if $q < d_i/2$; in 2, if $d_i/2 \leq q < d_i$; and in 3, if $q = d_i$. The first two cases correspond to partial deliveries, while the last is a full delivery. Extending this analysis to an arbitrary K , we have that for given a delivery quantity $q \in D_i$, $\left\lceil \frac{(K-1)q}{d_i - \epsilon} \right\rceil$ is equal to k , if $\frac{(k-1)d_i}{K-1} \leq q < \frac{kd_i}{K-1}$, for $k = 1, \dots, K$. Using this observation, we define the binary value g_{iq}^k that assumes the value of 1 if and only if $\frac{(k-1)d_i}{K-1} \leq q < \frac{kd_i}{K-1}$, and use it to rewrite Inequality (9) as follows:

$$\sum_{r \in \mathcal{R}} \sum_{q \in D_i} \sum_{k=1}^K b_{iq}^r g_{iq}^k k \theta_r \geq K. \quad (10)$$

We denote as (FK) the formulation with Objective Function (1), Constraints (2) and (3), Constraints (10) for $i \in \mathcal{C}(K)$, and Constraints (6) for $i \in \mathcal{C} \setminus \mathcal{C}(K)$. We define set \mathcal{R}^K of routes in which the delivery quantity of every visit to a customer $i \in \mathcal{C}(K)$ is the minimum nonzero value in $\left[\frac{(k-1)d_i}{K-1}, \frac{kd_i}{K-1} \right)$ that is a multiple of \bar{q} , for some $k \in \{1, \dots, K\}$; and the delivery quantity of every visit to a customer $i \in \mathcal{C} \setminus \mathcal{C}(K)$ is a nonzero multiple of \bar{q} . Let $D_i(K)$ be the set of such delivery quantities, defined as follows:

$$D_i(K) = \bigcup_{k=1}^K \left\{ \min\{l\bar{q}\} : l\bar{q} \in \left[\frac{(k-1)d_i}{K-1}, \frac{kd_i}{K-1} \right), l \in \mathbb{N} \right\} \quad \text{if } i \in \mathcal{C}(K), \quad (11)$$

$$D_i(K) = \{l\bar{q} : \forall l = 1, \dots, d_i/\bar{q}\} \quad \text{if } i \in \mathcal{C} \setminus \mathcal{C}(K). \quad (12)$$

Then, $\mathcal{R}^K = \{r \in \mathcal{R} : b_{iq}^r = 0, \forall i \in \mathcal{C}, \forall q \notin D_i(K)\}$. Note that for customers in $\mathcal{C} \setminus \mathcal{C}(K)$, set $D_i(K)$ includes all nonzero multiples of \bar{q} , from \bar{q} to d_i . For this reason, we call it a full discretization of d_i . Conversely, for customers in $\mathcal{C}(K)$, at least one multiple of \bar{q} is not included in $D_i(K)$, and hence we have a partial discretization.

Figure 2 illustrates the set of delivery quantities $D_i(K)$ for different values of K , considering $\bar{q} = 5$ and a customer i with demand $d_i = 40$. Since $d_i/\bar{q} = 8$, we observe partial discretizations for $K = 2$ to 7, and a full discretization for $K = 8$. For instance, in the partial discretization with $K = 4$, the possible delivery quantities are 5, 15, 30 and 40, which are the minimum multiples of \bar{q} inside the ranges $[0, 13.33)$, $[13.33, 26.67)$, $[26.67, 40)$ and $[40, 53.33)$, obtained from the expression $\left[\frac{(k-1)d_i}{K-1}, \frac{kd_i}{K-1} \right)$ for each $k = 1, 2, 3$ and 4. As indicated in the figure, each of these intervals have size $13.33 (= d_i/(K-1))$.

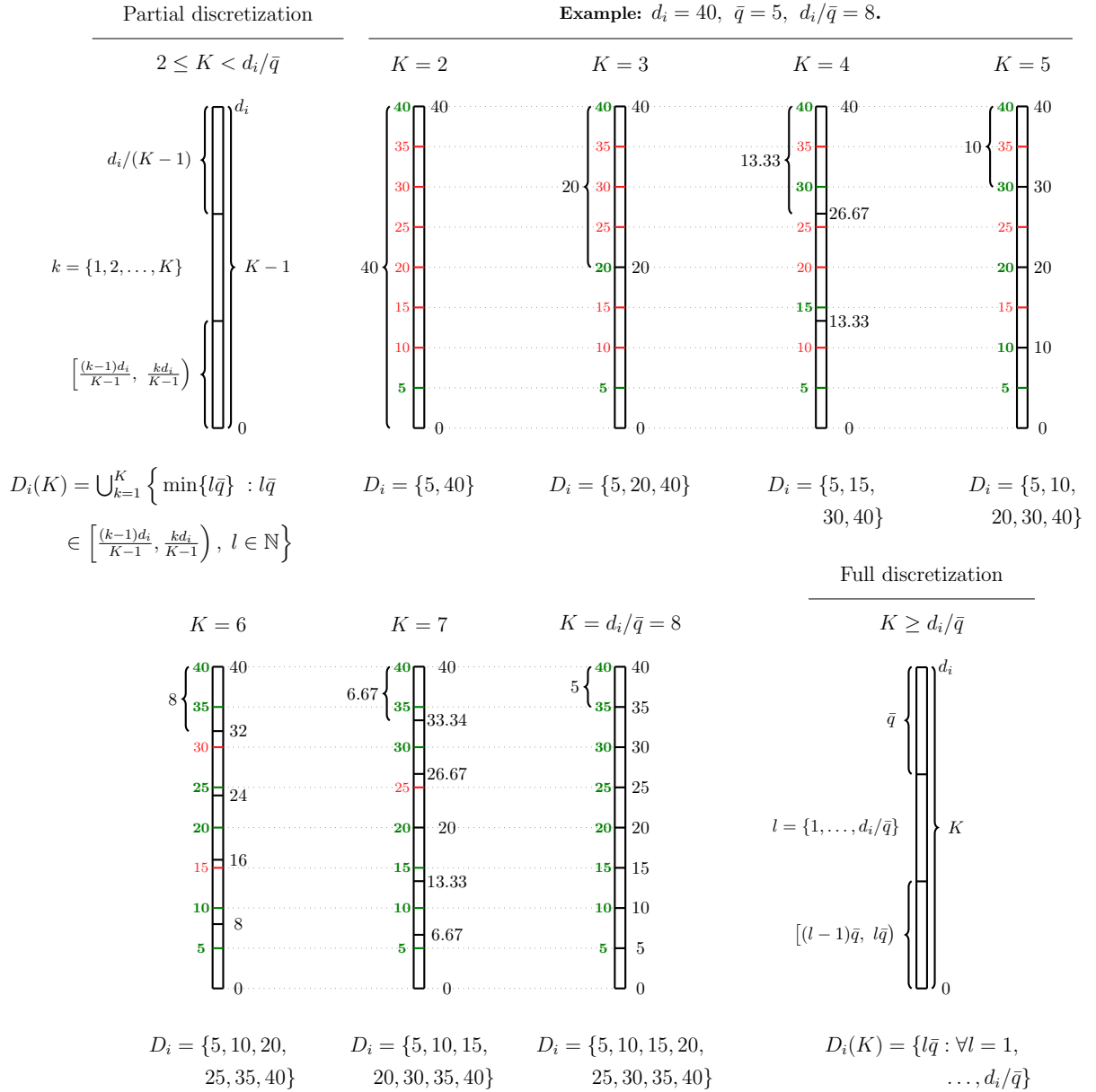


Figure 2: From partial to full discretization – a numerical example.

Observation 4. *There exists an optimal solution $\bar{\theta}$ to the linear relaxation of Formulation (FK) such that $\bar{\theta}_r = 0$ for all $r \notin \mathcal{R}^K$.*

Let $K_{\max} = \max_{i \in \mathcal{C}} d_i/\bar{q}$. We call Formulation (FK $_{\max}$) the fully discretized formulation since all possible delivery quantities according to Property 6 are considered and $\mathcal{R}^K = \mathcal{R}$. Strong k -path inequalities (2) are redundant for this formulation but still useful as cutting planes. Formulations (FK) with $K > K_{\max}$ are equivalent to (FK $_{\max}$). Formulations (FK) with $K < K_{\max}$ are partially discretized. For such formulations, the exact separation of strong k -path inequalities (2) is necessary to ensure the feasibility of integer solutions.

4.3 Valid inequalities

To further strengthen the proposed formulations, we present well-known valid inequalities that are adapted to the split delivery variants and propose novel families of limited-memory subset-row inequalities and limited-memory strong k -path inequalities.

4.3.1 Rounded capacity inequalities.

Let x_{ij}^r be the number of times route $r \in \mathcal{R}$ traverses arc $(i, j) \in \mathcal{A}$. Constraints

$$\sum_{r \in \mathcal{R}} \sum_{\substack{(i,j) \in \mathcal{A}: \\ |\{i,j\} \cap S|=1}} x_{ij}^r \theta_r \geq 2 \left\lceil \sum_{i \in S} d_i / Q \right\rceil, \quad \forall S \subseteq \mathcal{C}, \quad (13)$$

are known in the literature as rounded capacity inequalities (RCIs) or weak k -path inequalities (Desaulniers, 2010; Archetti et al., 2011a). They were introduced by Laporte and Nobert (1983) for the CVRP. RCIs have been separated in virtually all branch-and-cut and branch-cut-and-price algorithms in the literature for the SDVRP and the SDVRPTW due to their importance for obtaining strong lower bounds.

4.3.2 Limited-memory subset-row packing inequalities.

Cuts of the next family are adapted from the subset-row inequalities introduced by Jepsen et al. (2008) for the CVRP. They can be obtained by Chvátal-Gomory rounding of the set packing constraints, stating that every customer can be visited at most once. However, since the split delivery variants do not include this requirement, we modify these constraints to consider only the visits in which the delivery quantity is strictly greater than half of its demand. Hence, for each customer i , the number of such visits must be at most one, i.e.:

$$\sum_{r \in \mathcal{R}} \sum_{\substack{q \in D_i: \\ q > d_i/2}} b_{iq}^r \theta_r \leq 1, \quad \forall i \in \mathcal{C}. \quad (14)$$

By considering subsets $S \subseteq \mathcal{C}$ of size three and applying Chvátal-Gomory rounding of Constraints (14) for customers in S with multiplier $1/2$, we obtain the following subset-row packing inequalities (SRPIs) that are valid for Formulation (FK) with $K \geq 1$:

$$\sum_{r \in \mathcal{R}} \left\lceil \sum_{i \in S} \sum_{\substack{q \in D_i: \\ q > d_i/2}} \frac{1}{2} b_{iq}^r \right\rceil \theta_r \leq 1, \quad \forall S \subseteq \mathcal{C}, |S| = 3. \quad (15)$$

A weaker version of the SRPIs was used for the SDVRPTW by Archetti et al. (2011a). In their cuts, condition $q > d_i/2$ is restricted to $q = d_i$. Note that any formulation (FK) with $K \geq 1$ still involves only variables θ_r , $r \in \mathcal{R}^K$. Of course, inequalities (15) are just an example of SRPIs that can be derived from (14). In general, one can define a rational multiplier $\mu_i/\eta \in [0, 1]$, $\eta > 0$, for each customer $i \in \mathcal{C}$ and obtain inequality (16) by means of Chvátal-Gomory rounding. The reader is referred to Appendix A for further details on the definition of general SRPIs.

$$\sum_{r \in \mathcal{R}} \left\lceil \sum_{i \in \mathcal{C}} \sum_{\substack{q \in D_i: \\ q > d_i/2}} \frac{\mu_i}{\eta} b_{iq}^r \right\rceil \theta_r \leq \left\lceil \sum_{i \in \mathcal{C}} \frac{\mu_i}{\eta} \right\rceil \quad (16)$$

We also use elementarity cuts to impose Property 2 in any integer solution. This cuts state that every customer can be visited at most once in any route:

$$\sum_{r \in \mathcal{R}} \left[\sum_{q \in D_i} \frac{1}{2} b_{iq}^r \right] \theta_r \leq 0, \quad \forall i \in \mathcal{C}. \quad (17)$$

Constraints (17) can be obtained from inequalities (16) by considering a larger set of values q , and by setting $\eta = 2$, $\mu_i = 1$, $\mu_{i'} = 0$ for all $i' \in \mathcal{C} \setminus \{i\}$. Thus, these constraints are also included in the family of subset-row packing inequalities.

It is known in the literature that a large number of active subset-row cuts can render the dynamic generation of route variables very expensive (Jepsen et al., 2008; Pecin et al., 2017a). Thus, we adopt the following limited memory technique introduced by Pecin et al. (2017a). Each limited-memory SRPI (lm-SRPI) $\sum_{r \in \mathcal{R}} \alpha(r, \boldsymbol{\mu}, \eta, M) \theta_r \leq \left\lfloor \sum_{i \in \mathcal{C}} \frac{\mu_i}{\eta} \right\rfloor$ is associated with an arc memory $M \subseteq \mathcal{A}$. Algorithm 1 shows the computation of function $\alpha(r, \boldsymbol{\mu}, \eta, M)$. From its definition, it follows that

$$\alpha(r, \boldsymbol{\mu}, \eta, M) \leq \left\lfloor \sum_{i \in \mathcal{C}} \sum_{\substack{q \in D_i: \\ q > d_i/2}} \frac{\mu_i}{\eta} b_{iq}^r \right\rfloor \quad (18)$$

for any $M \subseteq \mathcal{A}$. Therefore, lm-SRPIs are valid for the problem.

Algorithm 1: Function $\alpha(r, \boldsymbol{\mu}, \eta, M)$

```

 $\alpha \leftarrow 0, \phi \leftarrow 0$ 
for  $k = 1$  to  $n_r - 1$  do
  if  $a_k^r = (i, j) \notin M$  then  $\phi \leftarrow 0$ 
  if  $d_k^r > d_j/2$  then
     $\phi \leftarrow \phi + \mu_j$ 
    if  $\phi \geq \eta$  then  $\alpha \leftarrow \alpha + 1, \phi \leftarrow \phi - \eta$ 
return  $\alpha$ 

```

4.3.3 Limited-memory subset-row covering inequalities.

Novel subset-row covering inequalities (SRCIs) can be obtained by Chvátal-Gomory rounding of constraints stating that, for every customer, there should be at least one nonzero delivery. Again, let $\mu_i/\eta \in [0, 1), \eta > 0$, be a rational multiplier defined for customer $i \in \mathcal{C}$. Then, the cut is as follows:

$$\sum_{r \in \mathcal{R}} \left[\sum_{\substack{i \in \mathcal{C} \\ q > 0}} \sum_{q \in D_i:} \frac{\mu_i}{\eta} b_{iq}^r \right] \theta_r \geq \left\lceil \sum_{i \in \mathcal{C}} \frac{\mu_i}{\eta} \right\rceil \quad (19)$$

We developed a computational approach in the spirit of the work by Pecin et al. (2017b) to determine which multipliers to consider (see Appendix A) and, again, we rely on the limited-memory technique to reduce the impact of SRCIs on the solution time. Each limited-memory SRCI (lm-SRCI) $\sum_{r \in \mathcal{R}} \beta(r, \boldsymbol{\mu}, \eta, M) \theta_r \geq \left\lceil \sum_{i \in \mathcal{C}} \frac{\mu_i}{\eta} \right\rceil$ is associated with an arc memory $M \subseteq \mathcal{A}$. Algorithm 2 shows the computation of function $\beta(r, \boldsymbol{\mu}, \eta, M)$. From its definition, it follows that

$$\beta(r, \boldsymbol{\mu}, \eta, M) \geq \left\lceil \sum_{i \in \mathcal{C}} \sum_{\substack{q \in D_i: \\ q > 0}} \frac{\mu_i}{\eta} b_{iq}^r \right\rceil \quad (20)$$

for any $M \subseteq \mathcal{A}$. Therefore, lm-SRCIs are valid for the problem.

Algorithm 2: Function $\beta(r, \mu, \eta, M)$

```

 $\beta \leftarrow 0, \psi \leftarrow 0$ 
for  $k = 1$  to  $n_r - 1$  do
  if  $a_k^r = (i, j) \notin M$  then  $\psi \leftarrow 0$ 
  if  $d_k^r > 0$  then
     $\psi \leftarrow \psi - \mu_j$ 
    if  $\psi < 0$  then  $\beta \leftarrow \beta + 1, \psi \leftarrow \psi + \eta$ 
return  $\beta$ 

```

4.3.4 Limited-memory strong k -path inequalities.

Finally, we adapt the limited-memory technique to inequalities (2). Each limited-memory strong k -path inequality (lm-SKPI) $\sum_{r \in \mathcal{R}} \gamma(r, S, M) \theta_r \geq \lceil \sum_{i \in S} d_i / Q \rceil$ is associated with a set S of customers and an arc memory $M \subseteq \mathcal{A} \setminus \{(i, j)\}_{\{i, j\} \subseteq S}$. The memory can only include arcs with at least one node outside set S . Algorithm 3 shows the pseudocode to compute $\gamma(r, S, M)$. From its definition, it follows that $\gamma(r, S, M) \geq h_{rS}$ for any $M \subseteq \mathcal{A}$. Thus, lm-SKPIs are valid for the problem. In Algorithm 3, value σ is used to “remember” visits to set S . If $\sigma = 0$, set S has not yet been visited or a visit has been already “forgotten” as the memory has been left. If $\sigma = 1/2$, set S has been visited already as one still “remembers” it. Particular values 0 and 1/2 have been chosen, as they are used later in (24).

Algorithm 3: Function $\gamma(r, S, M)$

```

 $\gamma \leftarrow 0, \sigma \leftarrow 0$ 
for  $k = 1$  to  $n_r - 1$  do
  if  $v_{k-1}^r \in S$  and  $v_k^r \notin S$  then  $\sigma \leftarrow 1/2$ 
  if  $a_k^r \notin M$  then  $\sigma \leftarrow 0$ 
  if  $v_{k-1}^r \notin S$  and  $v_k^r \in S$  then
     $\sigma \leftarrow \sigma - 1/2$ 
    if  $\sigma = -1/2$  then  $\gamma \leftarrow \gamma + 1, \sigma \leftarrow 0$ 
return  $\gamma$ 

```

5 Branch-cut-and-price algorithm

In this section, we describe the branch-cut-and-price (BCP) algorithm to solve Formulation (FK) for a fixed $K \in \{0, 1, 2, \dots, K_{\max}\}$, together with the valid inequalities presented in Section 4.3. The linear relaxation of (FK), or the master problem, is solved by the column generation procedure. On every iteration of this procedure, the restricted master problem (RMP) is solved considering a restricted subset of variables θ , others being fixed to zero. Let $(\bar{\pi}, \bar{\rho}, \bar{\zeta}, \bar{\xi}, \bar{\tau})$ be an optimal dual solution of the RMP. The dual value $\bar{\pi}_i \geq 0$ corresponds to Constraint (9) if $i \in \mathcal{C}(K)$ and to Constraint (6) if $i \in \mathcal{C} \setminus \mathcal{C}(K)$. If $K \leq 1$ then Constraints (6) and (9) are not defined, and we assume $\bar{\pi}_i = 0$ for all $i \in \mathcal{C}$. Let O be the set of active RCIs, and let $S^o \subseteq \mathcal{C}$ define rounded capacity inequality $o \in O$ with dual value $\bar{\rho}_o > 0$. Let $\delta(S^o)$ also be the set of arcs in \mathcal{A} which have exactly one node in S^o . Let P be the set of active lm-SRPIs, and tuple (μ^p, η^p, M^p) defines limited-memory subset-row packing inequality $p \in P$ with dual value $\bar{\zeta}_p < 0$. Let U be the set of active lm-SRCIs, and tuple (μ^u, η^u, M^u) defines limited-memory subset-row covering inequality $u \in U$ with dual value $\bar{\xi}_u > 0$. Let W be the set of active lm-SKPIs, and pair (S^w, M^w) defines limited-memory strong k -path inequality $w \in W$ with dual value $\bar{\tau}_w > 0$.

5.1 Pricing problem

To determine whether the current solution to the RMP is optimal for the master problem, we must find the minimum reduced cost among all variables θ_r , $r \in \mathcal{R}^K$. We represent set \mathcal{R}^K of routes as resource-constrained paths in multi-graph $\mathcal{G}'(K) = (\mathcal{V}, \mathcal{A}'(K))$. Every arc (i, j) in the original graph \mathcal{G} is replaced by multiple arcs (i, j, q) , $q \in D_j(K)$, between nodes i and j in $\mathcal{G}'(K)$. Sets $D_j(K)$ are defined in (11)-(12) for $K \geq 2$. We set $D_j(0) = \{0\}$ and $D_j(1) = \{\bar{q}\}$ for all $j \in \mathcal{C}$, and $D_{n+1}(K) = \{0\}$ for all $K \geq 0$. We define disposable time and capacity resources with accumulated resource consumption bounds $[e_i, l_i]$ and $[0, Q]$, respectively, for every node $i \in \mathcal{V}$. The time resource consumption of every arc $(i, j, q) \in \mathcal{A}'$ equals t_{ij} , and the capacity resource consumption of this arc equals q . Thus, the set of resource-feasible paths in multigraph $\mathcal{G}'(K)$ corresponds to the set of routes in \mathcal{R}^K . The capacity resource is redundant for $K = 0$ and may be skipped.

The reduced cost $\bar{c}_{(i,j,q)}$ of every arc $(i, j, q) \in \mathcal{A}'(K)$ equals

$$\bar{c}_{(i,j,q)} = c_{ij} - \sum_{\substack{o \in \mathcal{O}: \\ (i,j) \in \delta(S^o)}} \bar{\rho}_o - \begin{cases} 0, & \text{if } K < 2, \\ \bar{k}(j, q, K) \cdot \bar{\pi}_j, & \text{if } K \geq 2 \text{ and } j \in \mathcal{C}(K), \\ q\bar{\pi}_j, & \text{if } K \geq 2 \text{ and } j \in \mathcal{C} \setminus \mathcal{C}(K), \end{cases} \quad (21)$$

where $\bar{k}(j, q, K)$ is equal to the value k that satisfies $\frac{(k-1)d_j}{K-1} \leq q < \frac{kd_j}{K-1}$.

Let $\mathcal{A}'_r(K)$ be the set of arcs traversed by the resource-constrained path in $\mathcal{G}'(K)$, corresponding to a route $r \in \mathcal{R}^K$. Then, the reduced cost \bar{c}^r of the resource-constrained path r and its related variable θ_r is equal to

$$\bar{c}^r = \sum_{(i,j,q) \in \mathcal{A}'_r(K)} \bar{c}_{(i,j,q)} - \sum_{p \in P} \alpha(r, \boldsymbol{\mu}^p, \eta^p, M^p) \bar{\zeta}_p - \sum_{u \in U} \beta(r, \boldsymbol{\mu}^u, \eta^u, M^u) \bar{\xi}_u - \sum_{w \in W} \gamma(r, S^w, M^w) \bar{\tau}_w. \quad (22)$$

To find the resource-constrained path r in $\mathcal{G}'(K)$ corresponding to the best reduced cost, we use the bucket-graph based bidirectional labeling algorithm proposed by Sadykov et al. (2021). Every label L represents a partial path $\mathcal{G}'(K)$, which is either forward (starting from node 0) or backward (starting from node $n+1$). Every label L is characterized by a vector $(\bar{c}^L, v^L, t^L, q^L, \boldsymbol{\phi}^L, \boldsymbol{\psi}^L, \boldsymbol{\sigma}^L)$, where \bar{c}^L is the reduced cost of the partial path, v^L is the terminating node, and t^L and q^L are the accumulated time and capacity resource consumption, respectively. Finally, $\boldsymbol{\phi}^L$, $\boldsymbol{\psi}^L$, and $\boldsymbol{\sigma}^L$ are vectors of states corresponding to active limited-memory cuts. The lengths of these vectors are $|P|$, $|U|$, and $|W|$. These states are computed in the same way as in Functions α , β , and γ presented in Section 4.3. To adapt the bucket-graph labeling algorithm to our problem, we must define label initialization, extension, domination, and concatenation functions.

The initial forward label is defined as $(0, 0, e_0, 0, \mathbf{0}, \mathbf{0}, \mathbf{0})$, and the initial backward label is defined as $(0, 0, l_{n+1}, Q, \mathbf{0}, \mathbf{0}, \mathbf{0})$. In the backward labeling algorithm, the direction of arcs is reversed. The function that extends a label L' in the forward or the backward direction along an arc $a = (i, j, q)$, such that $i = v^{L'}$ to obtain label L , is presented in Algorithm 4. It returns true if extension is feasible. For ease of presentation, we define $\mu_0^p = \mu_{n+1}^p = 0$ for every $p \in P$ and $\mu_0^u = \mu_{n+1}^u = 0$ for every $u \in U$.

A label L dominates label L' if $v^L = v^{L'}$, $q^L \leq q^{L'}$, $t^L \leq t^{L'}$ ($q^L \geq q^{L'}$ and $t^L \geq t^{L'}$ for backward labels), and

$$\bar{c}^L - \sum_{\substack{p \in P: \\ \phi_p^L > \phi_p^{L'}}} \bar{\zeta}_p + \sum_{\substack{u \in U: \\ \psi_u^L > \psi_u^{L'}}} \bar{\xi}_u + \sum_{\substack{w \in W: \\ \sigma_w^L > \sigma_w^{L'}}} \bar{\tau}_w \leq \bar{c}^{L'}. \quad (23)$$

For a set of labels \mathcal{L} and a label L' , the bucket-graph labeling algorithm uses inequality $\bar{c}^{L'} < \min_{L \in \mathcal{L}} \bar{c}^L$ as a sufficient condition for nondomination of label L' by any label in \mathcal{L} . This sufficient

Algorithm 4: Extension of label L' along arc $a = (i, j, q) \in \mathcal{A}'$ to obtain label L

$\bar{c}^L \leftarrow \bar{c}^{L'} + \bar{c}_a, v^L \leftarrow j, \phi^L \leftarrow \phi^{L'}, \psi^L \leftarrow \psi^{L'}, \sigma^L \leftarrow \sigma^{L'}$

if forward direction **then**

$q^L \leftarrow q^{L'} + q, t^L \leftarrow \max\{t^{L'} + t_{ij}, e_j\}$
if $q^L > Q$ **or** $t^L > l_j$ **then return** false

if L is a backward label **then**

$q^L \leftarrow q^{L'} - q, t^L \leftarrow \min\{t^{L'} - t_{ij}, l_j\}$
if $q^L < 0$ **or** $t^L < e_j$ **then return** false

for $p \in P$ **do**

if $(i, j) \notin M^p$ **then** $\phi_p^L \leftarrow 0$
if $q > d_j/2$ **then**
 $\phi_p^L \leftarrow \phi_p^L + \mu_j^p$
if $\phi_p^L \geq \eta^p$ **then** $\phi_p^L \leftarrow \phi_p^L - \eta^p, \bar{c}^L \leftarrow \bar{c}^L - \bar{\zeta}_p$

for $u \in U$ **do**

if $(i, j) \notin M^u$ **then** $\psi_u^L \leftarrow 0$
if $q > 0$ **then**
 $\psi_u^L \leftarrow \psi_u^L - \mu_j^u$
if $\psi_u^L < 0$ **then** $\psi_u^L \leftarrow \psi_u^L + \eta^u, \bar{c}^L \leftarrow \bar{c}^L - \bar{\xi}_u$

for $w \in W$ **do**

if $i \in S^w$ **and** $j \notin S^w$ **then** $\sigma_w^L \leftarrow 1/2$
if $(i, j) \notin M^w$ **then** $\sigma_w^L \leftarrow 0$
if $i \notin S^w$ **and** $j \in S^w$ **then**
 $\sigma_w^L \leftarrow \sigma_w^L - 1/2$
if $\sigma_w^L = -1/2$ **then** $\sigma_w^L \leftarrow 0, \bar{c}^L \leftarrow \bar{c}^L - \bar{\tau}_w$

return true

condition remains valid in our case since $\bar{\zeta}_p \leq 0$ for all $p \in P$, $\bar{\xi}_u \geq 0$ for all $u \in U$, and $\bar{\tau}_w \geq 0$ for all $w \in W$.

The partial paths represented by a forward label \vec{L} and a backward label \bar{L} can be concatenated along an arc $(i, j, q) \in \mathcal{A}'$ if $i = v^{\vec{L}}, j = v^{\bar{L}}, q^{\vec{L}} + q \leq q^{\bar{L}}, t^{\vec{L}} + t_{ij} \leq t^{\bar{L}}$. The reduced cost $\bar{c}(\vec{L}, \bar{L}, i, j, q)$ of the path obtained by such concatenation can be computed as

$$\bar{c}(\vec{L}, \bar{L}, i, j, q) = \bar{c}^{\vec{L}} + \bar{c}_{(i,j,q)} + \bar{c}^{\bar{L}} - \sum_{\substack{p \in P: \\ (i,j) \in M^p, \\ \phi_p^{\vec{L}} + \phi_p^{\bar{L}} \geq \eta^p}} \bar{\zeta}_p + \sum_{\substack{u \in U: \\ (i,j) \in M^u, \\ \psi_u^{\vec{L}} + \psi_u^{\bar{L}} \geq \eta^u}} \bar{\xi}_u + \sum_{\substack{w \in W: \\ (i,j) \in M^w, \\ \sigma_w^{\vec{L}} + \sigma_w^{\bar{L}} = 1}} \bar{\tau}_w + \sum_{\substack{w \in W: \\ \{i,j\} \subseteq S^w}} \bar{\tau}_w. \quad (24)$$

The bucket-graph labeling algorithm uses a lower bound on the reduced cost of any path obtained by concatenation along an arc $a = (i, j, q)$ of a forward label \vec{L} and any backward label in a given set $\vec{\mathcal{L}}$. We use the following lower bound in our case:

$$\min_{\vec{L} \in \vec{\mathcal{L}}} \bar{c}(\vec{L}, \bar{L}, i, j, q) = \bar{c}^{\vec{L}} + \bar{c}_{(i,j,q)} + \min_{\vec{L} \in \vec{\mathcal{L}}} \bar{c}^{\vec{L}} + \sum_{\substack{w \in W: \\ \{i,j\} \subseteq S^w}} \bar{\tau}_w. \quad (25)$$

Bound (25) is valid since the part related to limited-memory cuts in (24) is nonnegative.

5.2 Cut separation

To separate the RCIs, we use four algorithms presented in Lysgaard et al. (2004): i) the connected component heuristic; ii) the max-flow-based algorithm, which separates fractional capacity inequalities; iii) the greedy construction heuristics; and iv) the heuristic which inspects the pool of previously generated inequalities and performs a fast local search for each of them.

Our approach for separating the strong k -path inequalities is as follows. First, we use a greedy construction heuristic similar to that for the separation of RCIs. Then, the separation of fractional strong k -path inequalities is performed by finding the minimum cut in the flow graph, similar to that described in Section 3. This graph is based on the current fractional solution $\bar{\theta}$ and the set of routes $\bar{\mathcal{R}} = \{r \in \mathcal{R}^K : \bar{\theta}_r > 0\}$. The capacity of each arc $(0, r)$ connecting the source to route node r is set to $Q\bar{\theta}_r$. After finding the minimum cut in the flow graph, the candidate set $S \subseteq \mathcal{C}$ of customers is constructed according to the proof of Theorem 1. As shown by the theorem, this separation algorithm is exact for integer solutions $\bar{\theta}$. Finally, we use a variant of the connected component heuristic called the route-based algorithm proposed by Archetti et al. (2011a).

We separate only 3-row subset-row packing and covering cuts. Preliminary experiments showed that cuts with more rows do not significantly improve the quality of the linear relaxation on average but take a noticeable time. The 3-row cuts are separated by the enumeration of all triples of customers. Elementarity cuts (17) are also separated by enumeration.

The limited memory for strong k -path inequalities and subset-row cuts is obtained in the same way as proposed by Pecin et al. (2017b). For each violated cut, a minimal memory is generated such that the coefficients of the route variables $\bar{\theta}_r > 0$ in the limited-memory cut are equal to coefficients of these variables in the full-memory cut.

5.3 Other algorithmic components

Our BCP algorithm resorts to the following algorithmic enhancements.

- In the labelling algorithm, the labels are stored in buckets according to their consumption of the capacity resource (SDVRP) or the time resource (SDVRPTW). Bucket-graph based labeling algorithm (Sadykov et al., 2021) is used to reduce the number of dominance checks. Before using the exact labeling algorithm, the labeling heuristic is used, in which only the best label (according to the reduced cost) is kept in every bucket.
- As stated before, elementarity constrains are not imposed in the labeling algorithm. Instead, we impose partial elementarity by using the ng -path relaxation introduced by Baldacci et al. (2011). For a customer $i \in \mathcal{C}$, its ng -neighborhood includes eight closest customers including i itself.
- To improve the convergence of column generation, we use automatic dual price smoothing stabilization technique. The dual solution passed to the pricing problem is a convex combination of the optimal dual solution of the current RMP and the dual solution that gave the best Lagrangian bound so far. The convex combination multiplier is adjusted dynamically according to the approach proposed by Pessoa et al. (2018).
- After each convergence of column generation, the bucket graph used in the labeling algorithm is filtered using a reduced cost argument. A bucket arc defined by a bucket and an outgoing arc is eliminated if it is proved that there does not exist an improving solution containing any route passing by this bucket arc.
- After bucket arc elimination, we run the elementary route enumeration procedure. This procedure was initially developed by Baldacci et al. (2008) for classic VRPs. It consists in enumeration of all elementary routes with reduced cost smaller than the current primal-dual gap, i.e. all routes which may participate in an improving solution. After successful enumeration, the pricing problem is solved by inspection instead of the labeling algorithm. If the number of enumerated routes is smaller than 3'000, the node is finished by solving the master problem as a MIP.

The procedure is a dynamic-programming-based algorithm that keeps the best partial route with each customer visit pattern (and each endpoint) as a state. Since the number of possible patterns may grow exponentially on n , it uses a reduced cost argument to prune a

large number of states. For SDVRPs, the maintained states do not only depend on which customers are visited but also on the corresponding delivery quantities, since partial routes visiting the same set of customers but with different delivery quantities cannot dominate each other. While at most 2^n visit patterns are possible in classic VRPs, this number increases to $(K + 1)^n$ in SDVRPs, when using the proposed formulations. Therefore, the route enumeration procedure becomes impractical for $K > 2$, and we use it only for formulations (F0), (F1), and (F2).

Details of these enhancements are presented in (Sadykov et al., 2021).

In addition, we use the ILS-based matheuristic proposed by Alvarez and Munari (2022) to generate initial upper bounds of the optimal values of the instances. Before launching our BCP algorithm, we run the matheuristic with the time limit equal to $t = 8 \lceil 4^{\log_2(n/16)} \rceil$ seconds. For example, the time limit is 24, 80, and 320 seconds for instances with 25, 50, and 100 customers, respectively. The value of the best solution (plus a small epsilon) is then used as the initial upper bound in the BCP algorithm. A good upper bound is extremely useful for the bucket arc elimination and for the elementary route enumeration.

5.4 Branching

If no violated cutting planes are found for the current fractional solution, or the tailing off condition is attained, we branch. The tailing off condition is satisfied 1) either when the primal-dual gap decreases by less than 1.5% in three rounds of cut separation, which are not necessarily consecutive, 2) or when the average exact pricing time exceeds 10 seconds.

We use two branching strategies, described as follows. Let $\bar{x}_{ij} = \sum_{r \in \mathcal{R}^K} x_{ij}^r \bar{\theta}_r$. First, we branch on expressions $\bar{x}_{ij} + \bar{x}_{ji}$ for all $\{i, j\} \subset \mathcal{C}$ (i.e., edges between customers), $\bar{x}_{0,i} + \bar{x}_{i,n+1}$ for all $i \in \mathcal{C}$ (i.e., edges between the depot and customers), and $\sum_{i \in \mathcal{C}} \frac{1}{2}(\bar{x}_{0,i} + \bar{x}_{i,n+1})$ (i.e., the number of used vehicles). The best branching expression is determined using the multiphase strong branching scheme described by Sadykov et al. (2021).

If none of the above expressions is fractional for the current solution $\bar{\theta}$, we perform the following Ryan&Foster (Ryan and Foster, 1981; Desrochers and Soumis, 1989) branching. We find a pair $\{i, j\} \subset \mathcal{C}$ such that $\sum_{r \in \mathcal{R}^K: h_{ri} + h_{rj} = 2} \bar{\theta}_r$ is fractional and impose the constraint that customers i and j should be on the same route in one branch and on different routes in another branch. These constraints are imposed in the pricing problem by introducing additional binary resources. Desaulniers (2010) showed that the combination of branching on edges and Ryan&Foster branching is sufficient to fulfill the integrality requirements for route variables θ . Again, the best Ryan&Foster branching pair is determined using the multiphase strong branching.

To improve the quality of branching candidates, we use the multiphase strong branching procedure. During phase zero, we select up to 100 candidates from most fractional pairs $\{i, j\}$ and from branching history. Then in phase one, for each candidate and each branch, we only resolve the restricted master without any column generation. Up to five best candidates (according to the product rule) are chosen for the phase two. In it, only heuristic column generation is performed. Finally, the best candidate is chosen also according to the product rule. Further details of the multiphase strong branching are given by Pecin et al. (2017b); Sadykov et al. (2021).

6 Computational results

In this section, we numerically investigate the strength of the proposed family of formulations and verify the performance of our BPC algorithm using benchmark instances with and without time windows. The algorithm is coded in the C++ programming language on top of the generic BCP library BaPCod (Sadykov and Vanderbeck, 2021) with its VRPSolver extension (Pessoa et al.,

2020), containing the implementation of the labeling algorithm for the SPPRC. We use IBM CPLEX version 20.1, as the general-purpose LP and MIP solver. We performed the numerical evaluation of our BCP algorithm on a server with nodes having the 2.6GHz Cascade Lake Intel Xeon Skylake Gold 6240 processor with 36 cores and 196 GB of RAM. Up to 36 instances were run in parallel on each node, each run using a single core.

6.1 Instances

We benchmark our algorithm on standard literature test instances of the SDVRPTW and SDVRP. For the SDVRPTW, we use the same instances tested in (Desaulniers, 2010; Archetti et al., 2011a; Bianchessi et al., 2019; Munari and Savelsbergh, 2022). These instances are derived from the classic Solomon’s VRPTW instances with 50 and 100 customers (56 instances of each size). The SDVRPTW instances are obtained from original instances by allowing for split deliveries and limiting the vehicle capacity to 30, 50, and 100. Therefore, there are 168 instances for each size.

We also use modified SDVRPTW instances. First, we generate instances with 75 customers by removing the last 25 customers from the instances with 100 customers. There are 168 such instances. Secondly, we create new instances by modification of instances with $\bar{q} = 10$ (these are all instances in class C, and instances in class RC with 50 customers). For this, we add a random integer value in $[-3, 3]$ to the demand of every customer $i \in \mathcal{C}$ such that $d_i \in [1.2d_{\min}, 0.8d_{\max}]$. Then we verify that $\bar{q} = 1$ and that the value $\lceil \sum_{i \in \mathcal{C}} d_i / 30 \rceil$ stays the same as in the original instance. We regenerated each instance until these two condition are verified. We denote these new instances as 50P, 75P, and 100P, depending on the instance size. There are 201 such instances. Thus, the total number of instances is 705.

As in Bianchessi and Irnich (2019), we preprocess instances in the following way. First, we round arc travel times to one decimal precision. Then, the travel time for each arc is replaced by the shortest path between its tail and head nodes. Finally, the travel time of each arc is completed with the service time of its tail node.

For the SDVRP, we use the same four classes of instances tested in (Archetti et al., 2014; Ozbaygin et al., 2018; Gouveia et al., 2022; Munari and Savelsbergh, 2022). First class SD contains 21 instances, having from 8 to 288 customers. Second class S contains 14 instances, having from 50 to 100 customers. The third class p contains 42 instances, having from 50 to 199 customers. Finally, fourth class eil contains 11 instances having from 21 to 100 customers. Each instance has four variations depending on whether distances (and travel costs) are rounded and whether the number of vehicles is fixed to the minimum possible number. Instances with unlimited and limited fleet are referred to as UF and LF, respectively, if distances are considered to be Euclidean distances. Similarly, we use UF-r and LF-r to refer to instances with rounded Euclidean distances.

6.2 Strength of formulations (FK) for different values K

First, we numerically investigate the dependency of the quality of the lower bound and the solution time of the master problem on value K . As a reminder, the master problem is the linear relaxation of Formulation (FK) enriched by cutting planes. The master problem is solved by applying the column and cut generation procedure. It terminates either when no violated cutting planes are found or when the tailing-off criterion is satisfied. In this experiment, we set the initial upper bound equal to the optimal value of the instance. We do so to exclude the random impact of improving the primal solution during the column and cut generation procedure.

This investigation is performed using the SDVRPTW test instances from the literature with 50 customers. All of the optimal solutions for these instances were obtained by us during preliminary experiments. The set of instances is divided into two groups depending on the value K_{\max} . The first group contains instances in classes C and RC, for which $K_{\max} = 4$. We tested Formulations (F0), (F1), (F2), and (F4) for these instances. The second group contains

instances in class R for which $K_{\max} = 36$. We tested Formulations (F0), (F1), (F2), (F4), (F8), (F16), and (F36) for these instances.

In Figure 3, we show the plots for different instance groups and different values of the vehicle capacity. In these plots, Gap is the average difference between the optimal value and the master problem lower bound as a percentage of the former (right scale), and $Time$ is the average solution time in seconds of the column and cut generation procedure (left scale).

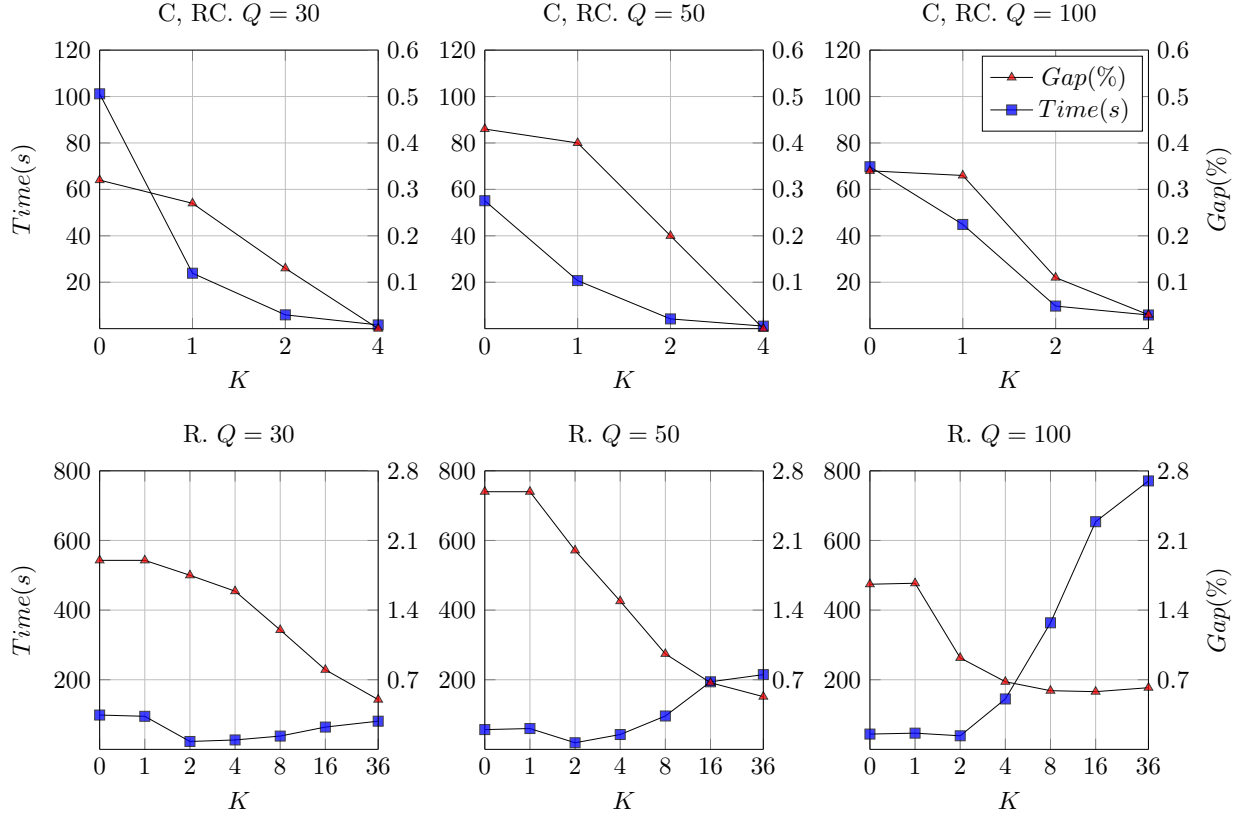


Figure 3: Dependency of the optimality gap and the solution time of master problem on value K

As expected, the strength of the master problem increases (the optimality gap decreases) with increasing value K . However, this decrease in the gap is steeper for instances with a smaller vehicle capacity ($Q = 30$ and $Q = 50$) and less steep for instances with longer routes ($Q = 100$). The optimality gap is almost nullified for $K = K_{\max}$ for instances with small K_{\max} . For instances in class R, the gap remains significant even for $K = K_{\max}$.

The decrease of the solution time with the increase of value K for instances in the first group is less intuitive. This outcome occurs because the linear relaxation of (FK_{\max}) becomes stronger with the increase in value K and thus a smaller number of cut generation rounds is necessary to attain convergence. In addition, the size of graph $\mathcal{G}'(K)$ remains reasonable since K_{\max} is still small. Thus, Formulation (FK_{\max}) is a clear choice for instances in the first group.

The choice of the best formulation for instances in the second group is not as obvious. It is only clear that Formulations (F0) and (F1) are dominated by (F2), both in terms of the optimality gap and the solution time. However, when K increases beyond the value of 2, the solution time increases due to a larger size of graph $\mathcal{G}'(K)$, and the optimality gap decreases. A slight increase of the gap between (F16) and (F36) for $Q = 100$ is caused by the root termination criterion which depends on the route generation subproblem time.

It is clear that there is a trade-off between the root primal-dual gap and the root solution time. The slope gradient is, however, different for different vehicle capacities, i.e., for instances with

different average route lengths. When routes are short on average, the optimality gap decrease is steep when value K is increasing, whereas the increase in the solution time does not have this result. The opposite outcome occurs for instances with relatively long routes ($Q = 100$): the gain in terms of the optimality gap is small for large values of K in comparison with $K = 2$, whereas the loss in terms of the solution time is very large and reaches one order of magnitude for large values of K . This observation may indicate that Formulation (FK_{\max}) should be chosen for instances with smaller vehicle capacity and Formulation (F2) for instances with larger vehicle capacity. This intuition is confirmed by the next round of experiments, in which we solve the benchmark and novel SDVRPTW instances using our BCP algorithm on Formulations (F2) and (FK_{\max}).

6.3 BCP performance for the SDVRPTW instances

We first test the impact of new families of cutting planes on the BCP performance for the SDVRPTW instances. In all tested BCP variants, we use rounded capacity cuts and limited-memory subset-row cuts with three rows, as these cuts are shown to be useful in the literature. Thus, we compare the following variants of the BCP algorithm.

- $BCP_{\max_{\text{all}}}$ — BCP over formulation (FK_{\max}) with all cutting planes.
- $BCP_{\max_{\text{all}}-SKPI}$ — BCP over formulation (FK_{\max}) with all cutting planes except limited-memory strong k -path inequalities.
- $BCP_{\max_{\text{all}}-SRCI}$ — BCP over formulation (FK_{\max}) with all cutting planes except limited-memory subset-row covering inequalities.
- $BCP_{\max_{\text{all}}-SKPI-SRCI}$ — BCP over formulation (FK_{\max}) with all cutting planes except limited-memory subset-row covering inequalities and strong k -path inequalities.
- $BCP2_{\text{all}}$ — BCP over formulation (F2) with all cutting planes.
- $BCP2_{\text{all}}-SRCI$ — BCP over formulation (F2) with all cutting planes except limited-memory subset-row covering inequalities.

We do not consider variants of BCP2 without strong k -path inequalities as the latter are necessary for the validity of formulation (F2).

We would like to note that variant $BCP_{\max_{\text{all}}-SKPI-SRCI}$ can be equivalently obtained by defining an appropriate VRPSolver model (Pessoa et al., 2020). In this model, two vertices are defined for each customer, one node for more-than-a-half delivery, and another one for smaller deliveries. For each customer, its packing set includes only the first vertex, but its elementarity set includes the both vertices. Multiple arcs are defined between pairs of vertices, one arc for every possible delivery quantity, corresponding to the head vertex of the arc. Ryan&Foster branching is defined on elementarity sets, and not on packing sets as usual. Definition of resources is the same as for the CVRP and VRPTW. Enumeration should be turned off as the sufficient condition defined in (Pessoa et al., 2020) is not satisfied.

In Table 1, we present the results for different BCP variants aggregated for all SDVRPTW instances. We use the standard time limit of one hour for the combined matheuristic and BCP time. In the columns, we show the BCP variant, the total number of instances solved to optimality, the geometric mean BCP time (which excludes the matheuristic time), the average primal-dual gap in the root node and the average final primal-dual gap.

We can see from the table that the both new families of cuts contribute to reducing the average root and final gaps. Strong k -path inequalities also contribute to the increased number of optimally solved instances. However, subset-row covering inequalities does not allow us to solve more instances to optimality. Thus, we do not use these cuts in the subsequent experiments.

Concerning formulations (F2) and (FK_{\max}), it is clear that the latter is more efficient. However, it does not dominate the former, as there are instances and even instance classes which

Table 1: Aggregated results for different BCP variants for all 705 SDVRPTW instances (one-hour time limit)

BCP variant	Opt	Time(s)	Root gap(%)	Final gap(%)
BCPmax _{all} -SKPI-SRCI	361	360.40	0.87	0.72
BCPmax _{all} -SKPI	358	362.56	0.86	0.72
BCPmax _{all} -SRCI	368	354.53	0.84	0.70
BCPmax _{all}	363	349.39	0.82	0.69
BCP2 _{all} -SRCI	273	734.92	2.42	2.00
BCP2 _{all}	273	730.12	2.39	1.93

are better solved by (F2). Thus, in the Table 2, we show detailed comparison between variants BCPmax_{all}-SRCI and BCP2_{all}-SRCI.

The results in Table 2 are aggregated by instance class, instance size and vehicle capacity. The columns in the table show the instance class, the number of customers (n), the vehicle capacity (Q), and the number of instances in the group ($\#$). The instances in groups “P” are the modified ones. Then, for the both BCP variants, the table shows the average number of nodes in the BCP tree, the average final optimality gap, the geometric mean pure BCP time in seconds, and the number of instances solved to optimality. The last two columns show (for the literature instances) the number of instances for which the best literature lower bounds are improved, and the number of instances solved to optimality for the first time. The last two lines in Table 2 aggregate statistics for all instances and for a subset of instances which we denote as “Challenging”. These instances have $Q = 100$ and $K_{\max} > 10$, i.e., instances 50P, 75P and 100P in class C, instances 50P, 75 and 100 in class RC, and all instances in class R. These instances are particularly hard for the fully discretized formulation (FK_{\max}), as the size of multi-graph $\mathcal{G}'(K_{\max})$ is large and routes are long.

Although variant BCPmax_{all}-SRCI is significantly more efficient than variant BCP2_{all}-SRCI, the latter is more performant for instances in the “Challenging” subset. Variant BCP2_{all}-SRCI solves to optimality 16 more instances in this subset, and on average two times faster for it.

If we compare our results with the literature, our best configuration solves 214 instances within one hour against 127 instances solved by Munari and Savelsbergh (2022), 109 instances solved by Bianchessi et al. (2019), and 94 instances solved by Archetti et al. (2011a). In total, 84 instances are solved to optimality for the first time by at least one of the configurations.

We emphasize the exceptional results for instances in class C with 100 customers by configuration $K = K_{\max}$. All 51 instances are solved to optimality in a relatively short time, whereas only seven of them were previously solved in the literature within a one-hour time limit and all of them for $Q = 100$. These results allow us to suggest that instances with small K_{\max} are easier to solve than instances with large K_{\max} , such as instances in class R with 50 and 100 customers and in class RC with 100 customers. Note that the difficulty of instances in class RC depends on their size: $K_{\max} = 40$ for instances with 100 customers and $K_{\max} = 4$ for instances with 50 customers. The detailed results are presented in the supplementary materials.

If one wishes to know how the size of parameter K impacts the number of split customers in the solutions and, how many visits, on average, each split customers receives, Appendix C presents some aggregated results in a similar format to Table 2. In synthesis, the percentage of split customers is below 30%, even with small capacities, for all the instances tested. The number of splits per split customer is rarely greater than one, i.e. split customers generally receive at most 2 visits. In addition, information about how much of each inequalities are added to the master problem is also presented.

We also run our BCP algorithm with a twelve-hour time limit passing the best-known solution value as initial upper bound for configurations BCP2_{all}-SRCI, BCPmax_{all}-SRCI and BCP10_{all}-SRCI. The latter variant is an “intermediate” one between the first two, it uses formulation (FK) in which $K = \min\{10, K_{\max}\}$. The most important outcome from these experiments

Table 2: Aggregated results for two BCP variants for SDVRPTW instances (one-hour time limit)

Class	n	Q	#	BCP _{2all} -SRCI				BCP _{maxall} -SRCI				Improvement		
				Nodes	Gap(%)	Time(s)	Opt	Nodes	Gap(%)	Time(s)	Opt	LB*	Opt*	
C	50	30	17	7.9	0.03	69.37	16	1.0	0.00	1.58	17	0	0	
		50	17	18.8	0.00	23.23	17	1.0	0.00	1.26	17	0	0	
		100	17	3.5	0.00	12.95	17	1.9	0.00	5.28	17	0	0	
	50P	30	17	138.8	0.15	1749.04	10	121.0	0.10	990.05	12	-	-	
		50	17	145.6	0.59	849.51	9	21.1	0.00	416.22	17	-	-	
		100	17	70.5	0.04	448.51	16	32.9	0.21	1096.61	13	-	-	
	75	30	17	235.8	0.60	3391.29	3	5.1	0.00	13.32	17	-	-	
		50	17	159.6	0.44	608.84	9	3.7	0.00	11.82	17	-	-	
		100	17	104.4	0.32	413.02	11	8.5	0.00	46.52	17	-	-	
	75P	30	17	52.3	1.29	3425.55	0	100.2	0.53	3424.89	0	-	-	
		50	17	58.1	1.89	3424.65	0	37.0	0.98	3328.79	1	-	-	
		100	17	48.5	1.52	3359.07	2	7.0	1.57	3428.91	1	-	-	
	100	30	17	131.5	0.62	3281.85	0	3.6	0.00	22.20	17	15	15	
		50	17	168.9	0.69	3280.58	0	10.1	0.00	46.75	17	17	17	
		100	17	32.9	0.02	196.78	16	2.5	0.00	31.08	17	14	14	
	100P	30	17	36.5	5.18	3282.42	0	29.1	1.32	3284.37	0	-	-	
		50	17	31.5	2.68	3282.01	0	8.3	1.50	3290.48	0	-	-	
		100	17	24.1	1.47	2718.41	5	2.9	1.90	3361.35	0	-	-	
	R	50	30	23	311.8	2.25	3520.69	0	157.5	0.23	1722.90	18	11	7
			50	23	350.5	1.47	1626.10	8	51.5	0.00	491.01	23	12	12
			100	23	51.1	0.00	69.48	23	27.5	0.10	380.11	20	17	17
		75	30	23	61.4	5.50	3425.53	0	43.8	2.05	3427.79	0	-	-
			50	23	94.7	4.20	3425.25	0	30.5	1.63	3345.96	1	-	-
			100	23	96.0	1.39	1033.95	9	11.3	1.04	1611.50	6	-	-
100		30	23	45.1	8.34	3282.81	0	22.5	2.39	3285.66	0	23	0	
		50	23	44.0	6.04	3281.88	0	8.2	2.40	3310.83	0	23	0	
		100	23	40.0	2.58	1856.39	3	3.5	2.08	2439.90	3	22	2	
RC		50	30	16	1.3	0.00	246.52	16	1.0	0.00	2.00	16	0	0
			50	16	1.0	0.00	3.16	16	1.0	0.00	1.04	16	0	0
			100	16	1.0	0.00	1.31	16	1.0	0.00	1.24	16	0	0
	50P	30	16	2.5	0.00	60.28	16	247.8	0.01	210.41	13	-	-	
		50	16	6.5	0.00	52.59	16	8.9	0.00	206.12	16	-	-	
		100	16	4.8	0.00	22.30	16	5.3	0.03	450.84	15	-	-	
	75	30	16	64.4	1.31	3426.25	0	78.5	0.33	2999.38	3	-	-	
		50	16	119.4	1.33	3424.76	0	41.5	0.46	3236.50	2	-	-	
		100	16	166.6	1.09	2560.25	3	6.9	1.51	2916.94	3	-	-	
	100	30	16	29.6	7.29	3283.99	0	17.1	0.52	3285.49	0	16	0	
		50	16	27.9	6.35	3283.94	0	7.1	0.77	3293.42	0	16	0	
		100	16	59.0	6.35	3311.52	0	2.5	1.81	3385.98	0	14	0	
Total			705		734.92	273			354.53	368	200	84		
Challenging			168		746.92	77			1574.53	61	92	19		

Table 3: Results for three BCP variants for the SDVRP instances (two-hour time limit)

Configuration	Class	#	BCP _{2all-SRCI}				BCP _{10all-SRCI}				BCP _{maxall-SRCI}				CG limit
			Nodes	Gap(%)	Time(s)	Opt	Nodes	Gap(%)	Time(s)	Opt	Nodes	Gap(%)	Time(s)	Opt	
LF-r	eil	11	49.9	0.90	370.52	6	31.7	0.29	630.42	8	9.5	0.15	1308.12	4	5
	p	42	33.1	51.27	5758.62	1	39.6	13.71	5864.04	3	11.0	2.98	5686.13	3	2
	S	14	64.6	3.96	3661.46	3	90.6	1.97	4233.85	3	35.1	1.94	4157.07	3	0
	SD	21	28.1	4.28	403.94	9	111.6	0.66	108.60	12	111.8	0.66	102.42	12	0
LF	eil	11	53.5	0.99	367.21	5	37.7	0.34	630.46	7	4.5	0.73	1388.51	3	4
	p	42	35.5	45.80	5745.34	1	33.0	12.34	5755.23	2	11.6	3.09	6145.36	3	0
	S	14	78.7	4.18	3417.39	3	98.7	2.03	4102.42	4	49.9	2.05	4332.25	5	0
	SD	21	34.2	6.17	423.61	9	27.0	0.68	63.18	13	28.8	0.68	63.28	13	0
UF-r	eil	11	62.3	1.10	402.46	5	42.6	0.26	656.58	9	25.5	0.62	1438.83	5	3
	p	42	33.4	53.48	5781.69	1	37.1	12.97	5813.81	2	12.5	2.71	5814.34	3	1
	S	14	67.4	3.54	3804.23	2	102.3	1.79	4037.46	3	23.0	1.19	4134.06	3	1
	SD	21	30.2	4.85	562.00	8	41.3	0.51	109.20	12	45.4	0.51	120.57	12	0
UF	eil	11	70.5	1.12	366.88	5	42.3	0.38	618.08	7	29.5	0.73	1236.54	4	4
	p	42	37.2	45.70	5730.41	1	38.8	12.16	5916.83	2	10.3	2.78	5981.23	3	0
	S	14	81.1	3.71	3446.15	3	113.6	1.73	4050.14	4	36.3	1.36	4174.28	4	2
	SD	21	31.5	6.04	515.68	8	127.6	0.62	153.23	11	144.2	0.62	155.19	11	0
Total		352	43.0	25.49	2089.02	70	57.0	6.59	1597.51	102	33.0	1.92	1776.10	91	22

is that formulation $\text{BCP}_{\text{maxall-SRCI}}$ manages to prove optimality for all remaining instances with 50 customers. In addition, another 13 optimal solutions for instances with 100 customers are found. All additional optimal solutions found in these experiments are listed in the supplementary materials.

6.4 BCP performance for the SDVRP instances

We test three variants of our BCP algorithm on the the full test set of SDVRP instances. These variants are $\text{BCP}_{\text{maxall-SRCI}}$, $\text{BCP}_{2\text{all-SRCI}}$, and $\text{BCP}_{10\text{all-SRCI}}$. As is common in the literature, the time limit of two hours is imposed for these instances, and it includes the initial matheuristic time.

Table 3 summarizes the results obtained by the three tested BCP variants. Recall that we consider four versions of the original test set of 88 SDVRP instances, obtained by the combination of unlimited- and limited-fleet and Euclidean and rounded Euclidean distances (as described in Section 6.1). Results are aggregated for each combination, instance class and BCP variant. In the table, we give, the average number of nodes processed in the branch-and-bound tree, the average final primal-dual gap, the geometric mean pure BCP time and the number of instances solved to optimality (Opt). For variant $\text{BCP}_{\text{maxall-SRCI}}$, one additional column indicate the number of instances for which the first convergence of column generation could not terminate within the time limit. For other variants, column generation could always terminate.

Results show that the best performing BCP variant for the SDVRP instances is clearly $\text{BCP}_{10\text{all-SRCI}}$. The detailed results obtained with this variant are presented in Appendix ???. Variant $\text{BCP}_{\text{maxall-SRCI}}$ is on the second place overall. However it is not robust, as even a lower bound cannot be obtained for some instances with a very large value of K_{max} . Variant $\text{BCP}_{10\text{all-SRCI}}$ solved to optimality 102 instances, and fourteen of them for the first time. We compare these numbers with the state-of-the-art branch-and-cut algorithms of Munari and Savelsbergh (2022) and Gouveia et al. (2022). The latter solves 106 instances to optimality from the total 352 instances. Our algorithm improved best known lower bounds for 116 instances or almost 50% of open instances. If we consider only the smaller test set of 224 instances, considered by Munari and Savelsbergh (2022), we solve 96 instances to optimality, which is 11 instances more than in the latter paper. Our algorithm improved best known lower bounds for 51 instances in the reduced set among 108 previously open ones. The detailed results are also presented in the supplementary materials.

In table 4 we present aggregated results for variant $\text{BCP}_{10\text{all-SRCI}}$. For each class of instances, we show the number of instances solved to optimality, the number of improved lower bounds in comparison to the literature, and the number of instances solved to optimality for the first time.

Table 4: Aggregated results of variant BCP10_{all-SRCI} for the SDVRP instances (two-hour time limit)

Class	#	LF-r			LF			UF-r			UF		
		Opt	LB*	Opt*	Opt	LB*	Opt*	Opt	LB*	Opt*	Opt	LB*	Opt*
eil	11	8	5	2	7	6	2	9	5	3	7	6	2
p	42	3	16	1	2	9	0	2	16	0	2	7	0
S	14	3	2	0	4	1	0	3	5	0	4	2	0
SD	21	12	9	1	13	9	1	12	9	1	11	9	1
Total	88	26	32	4	26	25	3	26	35	4	24	24	3

We also performed an additional test with the time limit of twelve hours and starting with the best upper bound between the matheuristic, best known solution value in the literature and upper bounds obtained in our benchmark runs. The additional optimal solutions obtained in this test are presented in the supplementary materials. Aggregated statistics on the solutions (percentage of split customers etc) and on the number of cuts added to the master problem, can be found in Appendix B.

7 Conclusions

We have proposed a new family of partially discretized route-based formulations (FK) for split delivery vehicle routing problems, where K is the maximum number of different delivery quantities allowed when visiting a customer. We have shown experimentally that, with the increase in K , the formulation becomes stronger but might become more challenging to solve. In the strongest fully discretized formulation (FK_{\max}), all possible delivery quantities are considered.

The proposed formulations are based on a new property that holds for at least one optimal solution of the problem. This property provides a minimum delivery quantity based on customer demand and vehicle capacity, and it allows us to reduce K_{\max} for some instances, in turn improving the solution times observed for the strongest formulation. This property is likely to benefit other formulations, as well as other exact and heuristic approaches in the literature.

To effectively solve the formulations, we have designed a BCP algorithm that resorts to new and state-of-the-art algorithmic improvements. We have developed the limited-memory variant of subset-row covering inequalities and strong k -path inequalities, reducing the impact of non-robust valid inequalities. Moreover, we have shown how to consider them while solving the pricing problem that dynamically generates the route variables.

Experimental results on benchmark instances of the SDVRPTW indicate the excellent performance of our BCP algorithm. In total, 101 instances are solved to optimality for the first time, including all instances with 50 customers and all instances with 100 customers and small value K_{\max} . Formulation (FK_{\max}) has the best average performance. However, formulation (F2) is more efficient for instances with long routes and large value K_{\max} . Formulation (FK) with $K = \max\{10, K_{\max}\}$ is the best one for the SDVRP instances (i.e., instances without time windows). Several SDVRP instances were solved to optimality for the first time, and the best known lower bounds were improved for many or them. In comparison with the literature, our BCP algorithm is especially efficient for instances with longer routes. Based on these results, the proposed formulations and BCP algorithm establish a new state-of-the-art for the SDVRPTW, and are highly competitive with the best approach in the literature for the SDVRP. This indicates that column generation-based approaches can have better-or-equal performance in comparison to pure branch-and-cut approaches.

We believe there are interesting future research topics related to improving and extending our solution approach. For example, the master problem of our strongest formulation (FK_{\max})

may take a long time to be solved, especially for instances with long routes. One possible way to make it faster is to avoid the discretization of delivery quantities since it reduces the size of the graph in the pricing problem. This can be done, e.g., by inserting load flow variables into the master problem, as proposed by Munari and Savelsbergh (2020), or by defining extreme delivery patterns together with the generation of routes, as introduced by Desaulniers (2010). Another possible improvement to our BCP algorithm concerns the insufficient strength of Formulation (FK_{\max}) for instances with a relatively large value of K_{\max} . Root optimality gaps may still be high for such instances, even after adding non-robust cuts. One should search for other families of valid inequalities. Separation of cuts based on Chvátal-Gomory rounding of demand covering constraints with different multipliers might be useful to reduce optimality gaps. Thus, developing efficient separation algorithms for such cuts is a promising research direction.

The proposed solution approach could be extended to other variants with additional attributes, such as multiple depots (Gouveia et al., 2022), heterogeneous fleet (Belfiore and Yoshizaki, 2009), pickup and delivery (Casazza et al., 2021), and others. Such extensions could be made following the generic modeling approach by Pessoa et al. (2020). Of course, the numerical efficiency of these extensions remains to be seen for each variant separately. Another important extension concerns the case in which the service time for a customer depends on delivery quantity. This dependence might come from non-negligible loading or unloading times (Li et al., 2020).

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A Subset-row inequalities

In this appendix we provide further details on the subset-row cuts defined in Section 4.

A.1 Subset-row packing inequalities

Inequalities (15) can be referred to as 3SRPIs since they were derived from exactly three packing inequalities in (14), i.e., from a set of multipliers in which there are exactly three nonzero numerators μ_i . To find strong SRPIs, Pecin et al. (2017b) performed a computational study of the complete set partitioning polytope $CSPP_{=}(p)$, defined below together with the related

complete set packing polytope $CSPP_{\leq}(p)$ and complete set covering polytope $CSPP_{\geq}(p)$.

$$CSPP_{=}(p) = \text{Conv} \left\{ B_p x = \mathbb{1}, x \in \{0, 1\}^{2^p-1} \right\}. \quad (26)$$

$$CSPP_{\leq}(p) = \text{Conv} \left\{ B_p x \leq \mathbb{1}, x \in \{0, 1\}^{2^p-1} \right\}. \quad (27)$$

$$CSPP_{\geq}(p) = \text{Conv} \left\{ B_p x \geq \mathbb{1}, x \in \{0, 1\}^{2^p-1} \right\}. \quad (28)$$

In the definition of these polytopes, B_p is a binary matrix with p rows and all distinct $2^p - 1$ nonzero columns, and $\mathbb{1}$ represents the p -dimensional all-ones vector. Due to the combinatorial explosion of their approach, the authors managed to study the $CSPP_{=}(p)$ only for $p \leq 5$, and they concluded that the SRPIs associated with the following set of multipliers are facet inducing:

- 3 rows: $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$;
- 4 rows: $\left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$;
- 5 rows: $\left(\frac{2}{4}, \frac{2}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{3}{4}, \frac{3}{4}, \frac{2}{4}, \frac{2}{4}, \frac{1}{4}\right),$
 $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$

Later, Bulhões et al. (2018) determined new families of multipliers that induce facets of $CSPP_{=}(p)$ for arbitrarily large values of p , and they showed that, with very few exceptions, every facet-inducing inequality for $CSPP_{\leq}(p)$ is also facet inducing for $CSPP_{=}(p)$, and vice versa.

Observation 5. *The existence of routes $r \in \mathcal{R}$ with nonbinary coefficients in (14) has no impact on the strength of the multipliers found by Pecin et al. (2017b) and Bulhões et al. (2018).*

This follows from such routes not being part of an integer solution.

A.2 Subset-row covering inequalities

We developed a computational approach to analyze the complete set covering polytope $CSPP_{\geq}(p)$ (see (28)) in the spirit of the work by Pecin et al. (2017b). Despite the inferior scalability of this approach for $CSPP_{\geq}(p)$ due to a larger number of extreme points, we managed to find the following multipliers, the associated SRCIs of which are facet-inducing:

- 3 rows: $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$;
- 4 rows: $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$;
- 5 rows: $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{2}{4}, \frac{2}{4}, \frac{3}{4}\right),$
 $\left(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right), \left(\frac{2}{4}, \frac{2}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right), \left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}\right), \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5}\right), \left(\frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{4}{5}\right).$

Observation 6. *In contrast to the packing case, one might find stronger SRCIs by analyzing a generalization of $CSPP_{\geq}(p)$ in which the extreme points are not necessarily binary.*

Preliminary experiments showed that the larger number of extreme points of this generalized polytope renders our computational approach impractical.

B Aggregated statistics for benchmark SDVRP tests

Table 5 aggregates statistics on the benchmark tests with formulation $\text{BCP10}_{\text{all-SRCI}}$. For each test configuration and instance class, we list the percentage of partially discretized customers in column “Part. Disc.(%)”, the percentage of customers with splits in the column “Split cust.(%)”, the average number of splits per split customer in the column “Splits”, the total of cuts added to the master in the column “# Cuts”, as well as for each inequality, the specific quantity of cuts added in columns “# R1C”, “# CAP” and “# SKP”. At last, column “Opt” counts the number of optimal solutions found.

Table 5: Aggregated statistics on the number of partially discretized customers, splits per split customer and cuts for the SDVRP (two-hours time limit)

Configuration	Class	#	$\text{BCP10}_{\text{all-SRCI}}$							
			Part. Disc.(%)	Split cust.(%)	# Splits	# Cuts	# R1C	# CAP	# SKP	Opt
LF-r	eil	11	73.4	3.3	1.2	9664	4369	1945	3349	8
	p	42	91.9	28.2	1.2	7969	850	4238	2880	3
	S	14	85.7	18.3	1.2	15419	1564	7971	5884	3
	SD	21	0.0	53.6	1.1	6062	309	3425	2329	12
LF	eil	11	73.4	3.1	1.0	10085	4799	1775	3512	7
	p	42	91.9	27.4	1.1	7623	576	4467	2580	2
	S	14	85.7	17.0	1.2	17206	1557	10057	5592	4
	SD	21	0.0	53.8	1.0	5037	116	3159	1761	13
UF-r	eil	11	73.4	2.5	1.1	9941	5039	1955	2947	9
	p	42	91.9	24.8	1.3	7814	748	4227	2839	2
	S	14	85.7	15.0	1.3	15362	1573	8831	4958	3
	SD	21	0.0	51.4	1.1	7618	262	4516	2840	12
UF	eil	11	73.4	2.2	1.0	8160	4161	1636	2363	7
	p	42	91.9	23.9	1.2	7677	607	4527	2543	2
	S	14	85.7	16.1	1.2	17399	1666	10677	5056	4
	SD	21	0.0	50.3	1.1	26057	298	18958	6801	11

C Aggregated statistics for benchmark SDVRPTW tests

Table 6 aggregates statistics on the benchmark tests and the results obtained with formulations $\text{BCP2}_{\text{all-SRCI}}$ and $\text{BCPmax}_{\text{all-SRCI}}$. For each class of instances, we list \bar{q} , the percentage of partially discretized customers in column “Part. Disc.(%)” (only for variant $\text{BCP2}_{\text{all-SRCI}}$), the percentage of customers with splits in the column “Split cust.(%)”, the average number of splits per split customer in the column “Splits”, the total of cuts added to the master in the column “# Cuts”, as well as for each inequality, the specific quantity of cuts added in columns “# R1C”, “# CAP” and “# SKP”. At last, column “Opt” counts the number of optimal solutions found.

Table 6: Aggregated statistics on the number of partially discretized customers, splits per split customer and cuts for the SDVRPTW (one-hour time limit)

Class	n	Q	q	#	BCP2 _{all} -SRC1										BCPmax _{all} -SRC1																																		
					Part. Disc.(%)					Split cust.(%)					# Splits					# Cuts					# R1C					# CAP					# SKP					Opt									
					Part. Disc.(%)	Split cust.(%)	# Splits	# Cuts	# R1C	# CAP	# SKP	Opt	Part. Disc.(%)	Split cust.(%)	# Splits	# Cuts	# R1C	# CAP	# SKP	Opt	Part. Disc.(%)	Split cust.(%)	# Splits	# Cuts	# R1C	# CAP	# SKP	Opt	Part. Disc.(%)	Split cust.(%)	# Splits	# Cuts	# R1C	# CAP	# SKP	Opt													
C	50	30	10	17	18.0	9.9	1.0	3922	6	3277	640	16	16	545	0	545	0	17	18.0	9.9	1.0	545	0	545	0	17	18.0	9.9	1.0	3922	6	3277	640	16	16	545	0	17											
		50	10	17	18.0	5.8	1.0	3003	76	2348	579	17	17	267	0	267	0	17	18.0	5.8	1.0	267	0	267	0	17	18.0	5.8	1.0	3003	76	2348	579	17	17	267	0	17											
		100	10	17	18.0	0.5	1.0	693	147	384	162	17	17	170	0	170	0	17	18.0	0.5	1.0	170	0	170	0	17	18.0	0.5	1.0	693	147	384	162	17	17	170	0	17											
	50P	30	1	17	100.0	26.5	1.1	56938	507	33080	23351	10	10	17071	165	11317	5588	12	100.0	26.5	1.1	17071	165	11317	5588	12	100.0	26.5	1.1	56938	507	33080	23351	10	10	17071	165	11317	5588	12									
		50	1	17	100.0	11.5	1.1	27087	541	18179	8367	9	9	2282	223	1191	868	17	100.0	11.5	1.1	2282	223	1191	868	17	100.0	11.5	1.1	27087	541	18179	8367	9	9	2282	223	1191	868	17									
		100	1	17	100.0	4.0	1.0	8336	2806	2300	3230	16	16	4025	2112	752	1161	13	100.0	4.0	1.0	4025	2112	752	1161	13	100.0	4.0	1.0	8336	2806	2300	3230	16	16	4025	2112	752	1161	13									
	R	75	30	10	17	21.3	11.8	1.0	90396	624	53322	36450	3	3	1159	5	1733	112	17	21.3	11.8	1.0	1159	5	1733	112	17	21.3	11.8	1.0	90396	624	53322	36450	3	3	1159	5	1733	112	17								
			50	10	17	21.3	2.3	1.0	44788	1405	34793	8590	14	14	1044	44	887	113	17	21.3	2.3	1.0	1044	44	887	113	17	21.3	2.3	1.0	44788	1405	34793	8590	14	14	1044	44	887	113	17								
			100	10	17	21.3	0.7	1.0	14696	4228	5704	4764	11	11	1806	705	638	463	17	21.3	0.7	1.0	1806	705	638	463	17	21.3	0.7	1.0	14696	4228	5704	4764	11	11	1806	705	638	463	17								
		75P	30	1	17	100.0	24.2	1.1	51929	650	29152	22128	0	0	25.1	464	15920	6946	0	100.0	24.2	1.1	25.1	464	15920	6946	0	100.0	24.2	1.1	51929	650	29152	22128	0	0	25.1	464	15920	6946	0								
50			1	17	100.0	8.8	1.1	34985	867	22267	11851	0	0	9.6	8183	4926	2440	1	100.0	8.8	1.1	9.6	8183	4926	2440	1	100.0	8.8	1.1	34985	867	22267	11851	0	0	9.6	8183	4926	2440	1									
100			1	17	100.0	4.2	1.1	16472	3197	6031	7243	2	2	3.1	725	1	1	100.0	4.2	1.1	3.1	725	1	1	1	100.0	4.2	1.1	16472	3197	6031	7243	2	2	3.1	725	1	1											
RC		100	30	10	17	22.0	10.0	1.0	83362	785	46822	35755	0	0	10.4	2	1681	108	17	22.0	10.0	1.0	10.4	2	1681	108	17	22.0	10.0	1.0	83362	785	46822	35755	0	0	10.4	2	1681	108	17								
			50	10	17	22.0	2.8	1.0	55313	1635	42492	11186	0	0	4.1	74	1991	461	17	22.0	2.8	1.0	4.1	74	1991	461	17	22.0	2.8	1.0	55313	1635	42492	11186	0	0	4.1	74	1991	461	17								
			100	10	17	22.0	1.1	1.0	7155	1379	4355	1420	16	16	1.1	888	582	145	17	22.0	1.1	1.0	1.1	888	582	145	17	22.0	1.1	1.0	7155	1379	4355	1420	16	16	1.1	888	582	145	17								
		100P	30	1	17	100.0	25.9	1.2	38957	1300	20136	17521	0	0	25.9	223	7510	2962	0	100.0	25.9	1.2	25.9	223	7510	2962	0	100.0	25.9	1.2	38957	1300	20136	17521	0	0	25.9	223	7510	2962	0								
	50		1	17	100.0	12.6	1.1	27848	965	15731	11152	0	0	12.6	2986	1830	853	0	100.0	12.6	1.1	12.6	2986	1830	853	0	100.0	12.6	1.1	27848	965	15731	11152	0	0	12.6	2986	1830	853	0									
	100		1	17	100.0	6.8	1.0	11267	1520	6543	3204	5	5	5.4	483	297	0	100.0	6.8	1.0	5.4	483	297	0	100.0	6.8	1.0	11267	1520	6543	3204	5	5	5.4	483	297	0	100.0	6.8	1.0	11267	1520	6543	3204	5	5	5.4	483	297
	R	50	30	1	23	96.0	23.3	1.0	96912	1230	64388	31294	0	0	25.0	706	15408	7001	18	96.0	23.3	1.0	25.0	706	15408	7001	18	96.0	23.3	1.0	96912	1230	64388	31294	0	0	25.0	706	15408	7001	18								
			50	1	23	96.0	9.2	1.2	47886	2412	27137	18337	8	8	10.4	620	2654	2325	23	96.0	9.2	1.2	10.4	620	2654	2325	23	96.0	9.2	1.2	47886	2412	27137	18337	8	8	10.4	620	2654	2325	23								
			100	1	23	96.0	4.1	1.0	5872	2306	1908	1659	23	23	3.7	2059	498	1028	20	96.0	4.1	1.0	3.7	2059	498	1028	20	96.0	4.1	1.0	5872	2306	1908	1659	23	23	3.7	2059	498	1028	20								
		75	30	1	23	96.0	24.6	1.2	47306	942	25686	20678	0	0	24.6	387	6359	3818	0	96.0	24.6	1.2	24.6	387	6359	3818	0	96.0	24.6	1.2	47306	942	25686	20678	0	0	24.6	387	6359	3818	0								
50			1	23	96.0	9.2	1.1	34100	1254	21683	11164	0	0	9.4	735	2132	1798	1	96.0	9.2	1.1	9.4	735	2132	1798	1	96.0	9.2	1.1	34100	1254	21683	11164	0	0	9.4	735	2132	1798	1									
100			1	23	96.0	3.3	1.1	16155	5636	5912	4606	9	9	3.2	1465	509	646	6	96.0	3.3	1.1	3.2	1465	509	646	6	96.0	3.3	1.1	16155	5636	5912	4606	9	9	3.2	1465	509	646	6									
100		30	1	23	95.0	26.2	1.2	33304	1229	16904	15170	0	0	26.2	220	2833	1595	0	95.0	26.2	1.2	26.2	220	2833	1595	0	95.0	26.2	1.2	33304	1229	16904	15170	0	0	26.2	220	2833	1595	0									
		50	1	23	95.0	10.6	1.1	24960	1018	14836	9106	0	0	10.6	1671	257	600	0	95.0	10.6	1.1	10.6	1671	257	600	0	95.0	10.6	1.1	24960	1018	14836	9106	0	0	10.6	1671	257	600	0									
		100	1	23	95.0	2.1	1.0	11353	3729	3909	3715	3	3	2.1	787	181	183	3	95.0	2.1	1.0	2.1	787	181	183	3	95.0	2.1	1.0	11353	3729	3909	3715	3	3	2.1	787	181	183	3									
RC		50	30	10	16	26.0	13.1	1.0	1386	2	1288	97	16	16	682	0	682	0	16	26.0	13.1	1.0	682	0	682	0	16	26.0	13.1	1.0	1386	2	1288	97	16	16	682	0	16										
	50		10	16	26.0	5.5	1.0	381	0	381	0	16	16	5.0	0	197	0	16	26.0	5.5	1.0	5.0	0	197	0	16	26.0	5.5	1.0	381	0	381	0	16	16	5.0	0	197	0	16									
	100		10	16	26.0	1.3	1.0	86	1	85	0	16	16	1.3	20	59	2	16	26.0	1.3	1.0	1.3	20	59	2	16	26.0	1.3	1.0	86	1	85	0	16	16	1.3	20	59	2	16									
	50P	30	1	16	100.0	27.4	1.1	4644	36	3404	1204	16	16	24.33	26	1787	620	13	100.0	27.4	1.1	24.33	26	1787	620	13	100.0	27.4	1.1	4644	36	3404	1204	16	16	24.33	26	1787	620	13									
		50	1	16	100.0	12.8	1.0	2969	26	2612	331	16	16	15.72	72	1096	404	16	100.0	12.8	1.0	15.72	72	1096	404	16	100.0	12.8	1.0	2969	26	2612	331	16	16	15.72	72	1096	404	16									
		100	1	16	100.0	6.5	1.2	785	127	453																																							

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