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***Abstract*—This paper investigates the robust semi-global output regulation of linear two-time-scale systems with input saturation.**

***Index Terms*—Two-time-scale, output regulation, input saturation.**

I. Introduction

THE systems evolving on both fast and slow time scales (also known as two-time-scale systems) appears in many practical applications such as robot [1], biology [2] and electric power management [3], [4]. Since two-time-scale systems (TTSSs) often subject to high dimensionality and numerical issues, the traditional control design techniques are not applicable. Appropriate methodological control tools are therefore needed, see e.g., [5], [6]. As far as we know, the control research of TTSSs mainly focus on the stabilization problem [7]–[9], few results handle the output regulation problem of TTSSs [10], [11], although the output regulation problem has been a fundamental control problem since 1970s and arises in a variety of practical control problem such as the controlling a spacecraft with disturbances, controlling a missile to track the moving target and so on.

Notation. $\mathbb{Z}^+$ is the set of strictly positive integers, $\mathbb{R}^{m\times n}$ denotes the set of $m\times n$ real matrices. The notation I_n stands for the n-dimensional unit matrix with $n\in\mathbb{Z}^+$. For a given real symmetric matrix P, $P>0$ means that P is a positive definite matrix. The notation $\|\cdot\|$ denotes the Euclidean norm for vectors or the induced 2-norm for matrices depending on the context. For a piecewise continuous bounded function $v:[0,\infty)\to\mathbb{R}^m$, and $T\geq 0$, $\|v(t)\|_{\infty,T}\triangleq \sup_{t\geq T}\|v(t)\|_\infty$. The function $f:[0,\infty)^2\to\mathbb{R}^{m\times n}$ is said to be $O(\varepsilon^n)$ if there exist positive constants k and ε^* strictly positive such that $\|f(t,\varepsilon)\|\leq k\varepsilon^n$, for all $t\in[0,\infty)$ and $\varepsilon\in[0,\varepsilon^*]$.

II. Problem Statement

Consider the following two-time-scale system

$$\begin{cases}\dot{x}=A_{11}x+A_{12}z+B_1\sigma(u)+F_1v,\\ \varepsilon\dot{z}=A_{21}x+A_{22}z+B_2\sigma(u)+F_2v,\\ e=C_1x+C_2z+Qv\end{cases}\tag{1}$$

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where $x\in\mathbb{R}^{n_x}$ and $z\in\mathbb{R}^{n_z}$ are the slow and the fast states, respectively, $\varepsilon>0$ is a small positive parameter inducing the time-scale separation between the slow and the fast dynamics, $u\in\mathbb{R}^p$ is the control input, $e\in\mathbb{R}^q$ is the output regulation error, $v\in\mathbb{R}^{n_v}$ is the state of the exosystem, representing both external disturbances and time-varying references input and it is generated by an autonomous exosystem of the form

$$\dot{v}=Sv.\tag{2}$$

Matrices S, Q, A_{ij}, B_i, F_i, C_i, $i,j=1,2$, are known constant matrices of appropriate dimensions. $\sigma(\cdot)$ is a vector-valued saturation function with

$$\sigma(u)=(\bar{\sigma}(u_1),\bar{\sigma}(u_2),\ldots,\bar{\sigma}(u_p)),\tag{3}$$

where

$$\bar{\sigma}(u_i)=\begin{cases}u_i, & \text{if } |u_i|\leq\Upsilon\\ -\Upsilon, & \text{if } u_i<-\Upsilon\\ \Upsilon, & \text{if } u_i>\Upsilon.\end{cases}$$

The first goal is to design a state-feedback controller

$$\begin{aligned}\dot{\eta}&=\Phi\eta+\Gamma e,\\ u&=K_1x+K_2z+G(x,z,\eta),\end{aligned}\tag{4}$$

which handles the semig-lobal output regulation problem for TTSS (1), as formalized next.

Definition 1. *The Semi-global output regulation problem of TTSS* (1) *is solved under the designed controller, if for any priori given compact subsets* $\mathbb{X}\subset\mathbb{R}^{n_x}$, $\mathbb{Z}\subset\mathbb{R}^{n_z}$, $\mathbb{V}\subset\mathbb{R}^{n_v}$, *and* $\bar{\mathbb{V}}\subset\mathbb{R}^{n_vq}$ *all containing the origin, there exists* $\bar{\varepsilon}>0$ *such that for any* $\varepsilon\in(0,\bar{\varepsilon}]$,

1) (Internal Stability) *When* $v=0$, *the equilibrium point* $(x,z,\eta)=(0,0,0)$ *of the corresponding closed-loop system composed of* (1) *and* (4) *is asymptotically stable with* $\mathbb{X}\times\mathbb{Z}\times\bar{\mathbb{V}}$ *being contained in its basin of attraction.*
2) (Output Regulation) *For all* $(x(0),z(0),v(0),\eta(0))\in\mathbb{X}\times\mathbb{Z}\times\mathbb{V}\times\bar{\mathbb{V}}$, *the trajectories of the closed-loop system composed of* (1), (2) *and* (4) *are bounded for all* $t\geq 0$, *and satisfies* $\lim_{t\to\infty}\|e(t)\|=0$.

Compared to stabilization problems, for handling the problem in definition 1, not only the internal stability should be ensured, but disturbance rejection and practical tracking problems also need to be simultaneously handled, while the TTSS subjects to input saturation. On the one hand, the input saturation would bring the nonlinearity of the system, on the other hand, the two-time-scale feature would bring the numerical issues, which would make the standard control

design techniques no longer applicable. All of these lead to the difficulty of controller design and stability analysis. To solve these issues, the next four assumptions and one Lemma are presented.

Assumption 1. *The eigenvalues of matrix S are semi-simple with zero real parts.*

Assumption 1 is common and standard for ensuring the neutrally stability of the exosystem.

Assumption 2. *The matrix A_{22} is invertible.*

Assumption 2 is essential to separate the slow and fast dynamics, which is standard in the singularly perturbed literature.

Assumption 3. *The pairs (A_0, B_0) and (A_{22}, B_2) are asymptotically null controllable with bounded controls (ANCBC), i.e.*

1) *The pairs (A_0, B_0) and (A_{22}, B_2) are stabilizable*
2) *All eigenvalues of A_0, A_{22} are in the closed left half s-plane,*

where $A_0 := A_{11} - A_{12}A_{22}^{-1}A_{21}$, $B_0 := B_1 - A_{12}A_{22}^{-1}B_2$.

Assumption 3 (also being used in [12], [13]) is common and instrumental for the design of semi-global asymptotically stabilizing feedback gains for the reduced order and boundary layer subsystems.

Lemma 1. *[14] Under Assumption 3, for any $\epsilon \in (0, 1]$, there exist unique real symmetric matrices $P_1 > 0$, $P_2 > 0$, which solve the following algebraic Riccati equations:*

$$A_0^\top P_1(\epsilon) + P_1(\epsilon)A_0 - 2P_1(\epsilon)B_0B_0^\top P_1(\epsilon) + \epsilon I_{n_x} = 0, \quad (5)$$

$$A_{22}^\top P_2(\epsilon) + P_2(\epsilon)A_{22} - 2P_2(\epsilon)B_2B_2^\top P_2(\epsilon) + \epsilon I_{n_z} = 0. \quad (6)$$

Moreover, $\lim_{\epsilon\to 0} P_1(\epsilon) = 0_{n_x\times n_x}$, $\lim_{\epsilon\to 0} P_2(\epsilon) = 0_{n_z\times n_z}$.

Assumption 4. *The matrix* $\begin{pmatrix} A_\varepsilon - \lambda I & B_\varepsilon \\ C & 0 \end{pmatrix}$ *has independent rows for each λ being an eigenvalues of S, where* $A_\varepsilon = \begin{pmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\varepsilon} & \frac{A_{22}}{\varepsilon} \end{pmatrix}$ $B_\varepsilon = \begin{pmatrix} B_1 \\ \frac{B_2}{\varepsilon} \end{pmatrix}$, $C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$.

Assumption 4 is standard and instrumental for the output regulation of TTSSs (1).

III. Main result

In this section, the semi-global stabilization and output regulation problem of TTSSs are investigated.

A. Semi-global stabilization of TTSSs

In this subsection, the semi-global stabilization problem of TTSSs is studied. The goal is to design a controller

$$u = K_1x + K_2z + G(x, z, \eta), \quad (7)$$

such that the semi-global stabilization problem defined next can be solved for the following TTSSs

$$\begin{cases} \dot\eta = \Phi\eta + \Gamma(C_1x + C_2z), \\ \dot x = A_{11}x + A_{12}z + B_1\sigma(u), \\ \varepsilon\dot z = A_{21}x + A_{22}z + B_2\sigma(u), \end{cases} \quad (8)$$

where the internal model unit of $\eta \in \mathbb{R}^{n_vq}$ is designed with

$$\Phi = \begin{pmatrix} 0_q & I_q & 0_q & \dots & 0_q \\ 0_q & 0_q & I_q & \dots & 0_q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -s_0I_q & -s_1I_q & -s_2I_q & \dots & -s_{r-1}I_q \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} I_q & 0_q & \dots & 0_q \end{pmatrix}^\top, \quad (9)$$

and the real numbers $s_0, \dots, s_{r-1}$ denote the coefficients of the minimal polynomial of the matrix S.

Definition 2. *The semi-global stabilization problem of system (8) is solved under the designed controller, if for any priori given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{Z} \subset \mathbb{R}^{n_z}$, and $\bar{\mathbb{V}} \subset \mathbb{R}^{n_vq}$ all containing the origin, there exists $\bar\varepsilon > 0$ such that for any $\varepsilon \in (0, \bar\varepsilon]$ and all $(x(0), z(0), \eta(0)) \in \mathbb{X}\times\mathbb{Z}\times\bar{\mathbb{V}}$, $\lim_{t\to\infty}\|x(t)\| = 0$, $\lim_{t\to\infty}\|z(t)\| = 0$, $\lim_{t\to\infty}\|\eta(t)\| = 0$.*

Before the controller design, the next lemma is introduced.

Lemma 2. *Suppose Assumption 2 holds. There exists $\bar\varepsilon > 0$ such that for any $\varepsilon \in (0, \bar\varepsilon]$, the origin of the system (10) is exponentially stable,*

$$\begin{pmatrix} \dot x \\ \varepsilon\dot z \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} \quad (10)$$

where $\Lambda_{ij} := A_{ij} + B_iK_j$, for $i, j = 1, 2$, $K_1 := (1 - K_2A_{22}^{-1}B_2)K_0 + K_2A_{22}^{-1}A_{21}$, $K_0 = B_0^\top P_1(\epsilon)$, $K_2 := B_2^\top P_2(\epsilon)$, $P_1(\epsilon)$, $P_2(\epsilon)$ are the solutions of (5) and (6).

Proof. For the stability analysis, Chang transformation is introduced for the TTSS (10) to separate the slow dynamics from the fast ones, see Chapter 3 in [5], which is presented as follows

$$\begin{pmatrix} x_s \\ z_f \end{pmatrix} := T_c^{-1}\begin{pmatrix} x \\ z \end{pmatrix}, \quad (11)$$

where ${T_c}^{-1} := \begin{pmatrix} I_{n_x} - \varepsilon HL & -\varepsilon H \\ L & I_{n_z} \end{pmatrix}$, and the matrices L and H are the solution of the following equations

$$\begin{aligned} &\Lambda_{21} - \Lambda_{22}L + \varepsilon L\Lambda_{11} - \varepsilon L\Lambda_{12}L = 0, \\ &\Lambda_{12} - H\Lambda_{22} + \varepsilon\Lambda_{11}H - \varepsilon\Lambda_{12}LH - \varepsilon HL\Lambda_{12} = 0. \end{aligned} \quad (12)$$

As a result, the system (10) in the (x_s, z_f) coordinates is

$$\begin{pmatrix} \dot x_s \\ \dot z_f \end{pmatrix} = \begin{pmatrix} A_s + B_sK_s & 0 \\ 0 & \frac{A_f + B_fK_2}{\varepsilon} \end{pmatrix}\begin{pmatrix} x_s \\ z_f \end{pmatrix}, \quad (13)$$

where

$$\begin{aligned} &A_s := A_0 - \varepsilon A_{12}A_{22}^{-1}L(A_{11} - A_{12}L), \\ &B_s := B_0 - \varepsilon A_{12}A_{22}^{-1}LB_1, K_s := K_1 - K_2L, \\ &A_f := A_{22} + \varepsilon LA_{12}, B_f := B_2 + \varepsilon LB_1, \\ &B_d := B_1 - HB_2 - \varepsilon HLB_1. \end{aligned} \quad (14)$$

From the definition of L, H, we have

$$\begin{aligned} &A_s + B_sK_s = (1 + O(\varepsilon))(A_0 + B_0K_0), \\ &A_f + B_fK_2 = (1 + O(\varepsilon))(A_{22} + B_2K_2). \end{aligned}$$

From (5) and (6), $A_0 + B_0K_0$ and $A_{22} + B_2K_2$ are both Hurwitz. Thus, there exist There exists $\bar\varepsilon > 0$ such that for

any $\varepsilon \in (0, \bar{\varepsilon}]$, $A_s + B_s K_s$ and $A_f + B_f K_2$ are Hurwitz, i.e., the origin of the system (10) is exponentially stable. □

Based on Lemma 2, it can be obtained that for $\varepsilon \in (0, \bar{\varepsilon}]$, the intersection of the spectrum of $\Lambda_\varepsilon := \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \frac{\Lambda_{21}}{\varepsilon} & \frac{\Lambda_{22}}{\varepsilon} \end{pmatrix}$ (which is Hurwitz) and Φ (which is neutrally stable) is empty. Thus, the matrix M is uniquely defined as the solution of

$$M\Lambda_\varepsilon = \Phi M + \Gamma C, \tag{15}$$

where $C = (C_1, C_2)$. Besides, Since Φ is neutrally stable, there exists a real symmetric matrix $P_3 > 0$, which solves the following equation:

$$\Phi^\top P_3 + P_3 \Phi^\top = 0. \tag{16}$$

Then the controller (8) can be designed with K_1, K_2 defined in Lemma 2, and

$$G(x,z,\eta) = -B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon(\epsilon) T_c^{-1} \xi + \epsilon B_\varepsilon^\top M^\top P_3 (\eta - M\xi),$$

where $P_\varepsilon(\epsilon) := \text{diag}\{P_1(\epsilon), \varepsilon P_2(\epsilon)\}$ and $\xi := (x, z)$.

The main point of applying the low gain feedback technique is to ensure that $\|u(t)\|_\infty \le \Upsilon$, so $\sigma(u) = u$, for all $t \ge 0$. In this way, TTSS (8) can be rewritten as follows

$$\begin{cases} \dot{\xi} = \Lambda_\varepsilon \xi + B_\varepsilon G(x,z,\eta), \\ \dot{\eta} = \Phi\eta + \Gamma(C_1 x + C_2 z). \end{cases} \tag{17}$$

Consider the Lyapunov function

$$V := \xi^\top (T_c^{-1})^\top P_\varepsilon(\epsilon) T_c^{-1} \xi + \epsilon (\eta - M\xi)^T P_3 (\eta - M\xi). \tag{18}$$

Due to the fact that $(x(0), z(0), \eta(0)) \in \mathbb{X} \times \mathbb{Z} \times \bar{\mathbb{V}}$ and $\mathbb{X}$, $\mathbb{Z}$, $\bar{\mathbb{V}}$ are compact subsets, there always exists a constant $c > 0$, such that

$$\sup_{\epsilon \in (0,1], (x(0),z(0),\eta(0)) \in \mathbb{X}\times\mathbb{Z}\times\bar{\mathbb{V}}} V(0) \le c. \tag{19}$$

Let $L_V(c) = \{\{(x,z,\eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_v \times q} : V \le c\}$. Since $\lim_{\epsilon \to 0} \|P_1(\epsilon)\|_\infty = 0$ and $\lim_{\epsilon \to 0} \|P_2(\epsilon)\|_\infty = 0$, it can be obtained that $\lim_{\epsilon \to 0} \|K_1(\epsilon)\|_\infty = 0$, $\lim_{\epsilon \to 0} \|K_2(\epsilon)\|_\infty = 0$, $\lim_{\epsilon \to 0} \|B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon(\epsilon) T_c^{-1}\|_\infty = 0$, so that $\lim_{\epsilon \to 0} \|u(\epsilon)\|_\infty = 0$. Thus, there exists an $\epsilon^* \in (0, 1]$, such that for all $\epsilon \in (0, \epsilon^*]$ and $(x,z,\eta) \in L_v(c)$, $\|u\|_\infty \le \Upsilon$. Let $\epsilon \in (0, \epsilon^*]$. In this case, for $(x,z,\eta) \in L_v(c)$, TTSS (8) can be rewritten as (17). Thus, the derivative of V along with (8) yields, for $(x,z,\eta) \in L_v(c)$,

$$\begin{aligned} \dot{V} =& x_s^\top (A_s^\top P_1(\epsilon) + P_1(\epsilon) A_s - 2P_1(\epsilon) B_s K_s) x_s \\ &+ z_f^\top (A_f^\top P_2(\epsilon) + P_2(\epsilon) A_f - 2P_2(\epsilon) B_f K_2) z_f \\ &+ 2\xi^\top (T_c^{-1})^\top P_\varepsilon(\epsilon) T_c^{-1} B_\epsilon G(x,z,\eta) \\ &+ 2\epsilon(\eta - M\xi)^T P_3 (\Phi\eta + \Gamma C\xi - M(\Lambda_\varepsilon \xi + B_\varepsilon G(x,z,\eta))) \\ \le& -(1 - O(\varepsilon))(\epsilon x_s^\top x_s + \epsilon z_f^\top z_f) \\ &+ 2\xi^\top (T_c^{-1})^\top P_\varepsilon(\epsilon) T_c^{-1} B_\epsilon G(x,z,\eta) \\ &+ 2\epsilon(\eta - M\xi)^T P_3 (\Phi(\eta - M\xi) - M B_\varepsilon G(x,z,\eta)). \end{aligned}$$

Thus, there exists $\bar{\varepsilon} > 0$, such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, $\frac{1}{2} - O(\varepsilon) > 0$. Then, for $\varepsilon \in (0, \bar{\varepsilon}]$ and $(x,z,\eta) \in L_v(c)$,

$$\dot{V} \le -\frac{\epsilon}{2} x_s^\top x_s - \frac{\epsilon}{2} z_f^\top z_f - G^\top(x,z,\eta) G(x,z,\eta). \tag{20}$$

Thus, if $(x(0), z(0), \eta(0)) \in L_V(c)$, then $(x(t), z(t), \eta(t)) \in L_V(c), \forall t \ge 0$. Indeed, (20) is satisfied for all $t \ge 0$. By using La Salle's arguments, we can prove that the state of the closed-loop system (7)-(8) converges to the set $\{(x,z,\eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_v \times q} : x = 0, z = 0, G(x,z,\eta) = 0\} = \{0\} \times \{0\} \times \{B_\varepsilon^\top M^\top P_3 \eta = 0\}$. Thus, $\lim_{t \to \infty} \|x(t)\| = 0$, $\lim_{t \to \infty} \|z(t)\| = 0$.

Since ξ converges to zero, the dynamics of η reduces to

$$\dot{\eta} = \Phi\eta.$$

Meanwhile, based on the Lemma 4.4 in [15], under Assumption 4, the pair $(B_\varepsilon^\top M^\top P_3, \Phi)$ is observable. Then, $\lim_{t \to \infty} \|\eta(t)\| = 0$. Thus, the equilibrium point $(x,z,\eta) = (0,0,0)$ of the corresponding closed-loop system (7)-(8) is asymptotically stable with $\mathbb{X} \times \mathbb{Z} \times \bar{\mathbb{V}}$ being contained in its basin of attraction. In this way, it can also be obtained that the matrix $J := \begin{pmatrix} \Gamma C & \Phi \\ \tilde{\Lambda}_\varepsilon & \epsilon B_\varepsilon B_\varepsilon^\top M^\top P_3 \end{pmatrix}$ is Hurwitz, where $\tilde{\Lambda}_\varepsilon = \Lambda_\varepsilon - B_\varepsilon B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon T_c^{-1} - \epsilon B_\varepsilon B_\varepsilon^\top M^\top P_3 M$.

It is noted that the above result can only be obtained when the exact solutions of (12) and (15) can be obtained. However, since ε is very small, it might be hard to get exact solutions L, H for (12) and M for (15). Thus, the approximate solution for (15) is provided in the next lemma.

Lemma 3. *Suppose Assumption 1-3 hold. The equation* (21) *has an unique solution* $\bar{M}$*,*

$$\bar{M}\Lambda = \Phi\bar{M}\bar{E} + \Gamma C, \tag{21}$$

where $\bar{E} = diag\{I_{n_x}, 0\}$, $\Lambda := \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}$ *has the same definition as in* (10)*. Besides, it satisfies* $M = \bar{M}E + O(\varepsilon)$*, when* Λ_ε *is Hurwitz.*

Proof. From Lemma 2, there exists $\bar{\varepsilon}_1 > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_1]$, Λ_ε is Hurwitz. Define $E = \text{diag}\{I_{n_x}, \varepsilon I_{n_z}\}$. Thus, $\Lambda = E\Lambda_\varepsilon$ and $\Lambda^{-1} = \Lambda_\varepsilon^{-1} E^{-1}$. In this way, solving the equation (21) is equal to solving

$$\bar{M} = \Phi\bar{M}\bar{E}\Lambda^{-1} + \Gamma C\Lambda^{-1}, \tag{22}$$

which is also equal to solving

$$(I_{n_v q \times (n_x + n_z)} - \Phi \otimes \bar{E}\Lambda^{-1}) \text{Vec}(\bar{M}) = \text{Vec}(\Gamma C \Lambda^{-1}). \tag{23}$$

From (13), it can also be obtained that $\Lambda_\varepsilon^{-1} = T_c A_D^{-1} T_c^{-1}$, where $A_D^{-1} = \begin{pmatrix} \Lambda_s^{-1} & 0 \\ 0 & \varepsilon\Lambda_f^{-1} \end{pmatrix}$, $\Lambda_s = A_s + B_s K_s$ and $\Lambda_f = A_f + B_f K_2$. Thus,

$$\begin{aligned} \bar{E}\Lambda^{-1} =& \bar{E} T_c A_D^{-1} T_c^{-1} E^{-1} \\ =& \begin{pmatrix} I_{n_x} & \varepsilon H \\ 0 & 0 \end{pmatrix} A_D T_c^{-1} E^{-1} \\ =& \begin{pmatrix} \Lambda_s^{-1} & \varepsilon^2 H \Lambda_f^{-1} \\ 0 & 0 \end{pmatrix} T_c^{-1} E^{-1} \\ =& \begin{pmatrix} \Lambda_s^{-1}(I_{n_x} - \varepsilon HL) + \varepsilon^2 H\Lambda_f^{-1} L & -\Lambda_s^{-1} H + \varepsilon^2 H \Lambda_f^{-1} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

It is obviously that, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, $\Lambda_s^{-1}(I_{n_x} - \varepsilon HL) + \varepsilon^2 H\Lambda_f^{-1} L$ is always Hurwitz. Thus, matrix $I_{n_v q \times (n_x + n_z)} - \Phi \otimes \bar{E}\Lambda^{-1}$ has no zero eigenvalues,

which means that equation (21) has an unique solution $\bar{M}$.

From (15) and (21), we have

$$(M-\bar{M}E)\Lambda_\varepsilon=\Phi(M-\bar{M}E)+\Phi\bar{M}\bar{E}. \tag{24}$$

Since Λ_ε is Hurwitz, it can be obtained that,

$$(M-\bar{M}E)=\Phi(M-\bar{M}E)\Lambda_\varepsilon^{-1}+\varepsilon\Phi\bar{M}(I_{n_x+n_z}-\bar{E})\Lambda_\varepsilon^{-1}.$$

Thus, we have

$$\begin{aligned}&(I_{n_vq\times(n_x+n_z)}-\Phi\otimes\Lambda_\varepsilon^{-1})\text{Vec}(M-\bar{M}E)\\=&\varepsilon\text{Vec}(\Phi\bar{M}(I_{n_x+n_z}-\bar{E})\Lambda_\varepsilon^{-1}).\end{aligned}$$

Since the intersection of the spectrum of Λ_ε and Φ is empty, $I_{n_vq\times(n_x+n_z)}-\Phi\otimes\Lambda_\varepsilon^{-1}$ is invertible. Thus

$$\begin{aligned}&\|\text{Vec}(M-\bar{M}E)\|\\=&\varepsilon\|(I_{n_vq\times(n_x+n_z)}-\Phi\otimes\Lambda_\varepsilon^{-1})^{-1}\text{Vec}(\Phi\bar{M}(I_{n_x+n_z}-\bar{E})\Lambda_\varepsilon^{-1})\|\\=&O(\varepsilon),\end{aligned}$$

which also means that $M=\bar{M}E+O(\varepsilon)$. The proof is complete. □

Then, based on the approximate solution, the controller can be designed as follows,

$$u=K_1x+K_2z-G_c\xi+\epsilon B^\top\bar{M}^\top P_3(\eta-\bar{M}\bar{E}\xi), \tag{25}$$

where $B=(B_1^\top,B_2^\top)^\top$, K_1, K_2, $\bar{M}$, P_i, $i=1,2,3$, have the same definition as in above, $G_c=(B_1^\top P_1(\epsilon)-B_2^\top(\Lambda_{12}\Lambda_{22}^{-1})^\top P_1(\epsilon)+B_2^\top P_2(\epsilon)\Lambda_{22}^{-1}\Lambda_{21},B_2^\top P_2(\epsilon))$.

The next theorem is obtained.

Theorem 1. *Suppose Assumptions 1-4 hold. For any priori given compact subsets $\mathbb{X}\subset\mathbb{R}^{n_x}$, $\mathbb{Z}\subset\mathbb{R}^{n_z}$, and $\bar{\mathbb{V}}\subset\mathbb{R}^{n_vq}$ all containing the origin, there exists $\bar{\varepsilon}>0$ such that for any $\varepsilon\in(0,\bar{\varepsilon}]$, the stabilization problem is solvable for system* (8) *with the controller* (25)*.*

Proof. From the definition of T_c^{-1} in (11), we have

$$\begin{aligned}&B_\varepsilon^\top(T_c^{-1})^\top P_\varepsilon(\epsilon)T_c^{-1}\\=&B_\varepsilon^\top(T_c^{-1})^\top\begin{pmatrix}P_1(\epsilon)&0\\0&\varepsilon P_2(\epsilon)\end{pmatrix}\begin{pmatrix}I_{n_x}-\varepsilon HL&-\varepsilon H\\L&I_{n_z}\end{pmatrix}\\=&B_\varepsilon^\top\begin{pmatrix}I_{n_x}-\varepsilon L^\top H^\top&L^\top\\-\varepsilon H^\top&I_{n_z}\end{pmatrix}\begin{pmatrix}P_1(\epsilon)+O(\varepsilon)&O(\varepsilon)\\\varepsilon P_2(\epsilon)L&\varepsilon P_2(\epsilon)\end{pmatrix}\\=&B_\varepsilon^\top\begin{pmatrix}P_1(\epsilon)+O(\varepsilon)&O(\varepsilon)\\\varepsilon(H^\top P_1(\epsilon)+P_2(\epsilon)L+O(\varepsilon))&\varepsilon(P_2(\epsilon)+O(\varepsilon))\end{pmatrix}\\=&\begin{pmatrix}B_1^\top P_1(\epsilon)+B_2^\top(H^\top P_1(\epsilon)+P_2(\epsilon)L)&B_2^\top P_2(\epsilon)\end{pmatrix}+O(\varepsilon).\end{aligned}$$

From (12), it can be obtained that

$$L=\Lambda_{22}^{-1}\Lambda_{21}+O(\varepsilon),\ H=\Lambda_{12}\Lambda_{22}^{-1}+O(\varepsilon).$$

Thus

$$B_\varepsilon^\top(T_c^{-1})^\top P_\varepsilon(\epsilon)T_c^{-1}=G_c+O(\varepsilon). \tag{26}$$

From Lemma 3, we have

$$M=\bar{M}E+O(\varepsilon)=\bar{M}\bar{E}+O(\varepsilon). \tag{27}$$

Define $\bar{J}:=\begin{pmatrix}\Gamma C&\Phi\\\bar{\Lambda}_\varepsilon&\epsilon B_\varepsilon B^\top\bar{M}^\top P_3\end{pmatrix}$, where $\bar{\Lambda}_\varepsilon=\Lambda_\varepsilon-B_\varepsilon(G_c+\epsilon B^\top\bar{M}^\top P_3\bar{M}\bar{E})$. From (26) and (27), we have $\bar{J}=J+O(\varepsilon)$. Thus, there exist small enough $\bar{\varepsilon}>0$ and $\epsilon^*\in(0,1]$, such that for all $\varepsilon\in(0,\bar{\varepsilon}]$ and $\epsilon\in(0,\epsilon^*]$, J and $\bar{J}$ are both Hurwitz. Moreover, by using the approximate solution $\bar{M}$, we can find a set $\mathcal{B}(r):=\{\{(x,z,\eta)\in\mathbb{R}^{n_x}\times\mathbb{R}^{n_z}\times\mathbb{R}^{n_vq}:\|(x,z,\eta)\|\le r\}\supset L_V(c)$. Similarly, we have $\lim_{\epsilon\to0}\|u(\epsilon)\|_\infty=0$. Then, we can find an $\epsilon^*\in(0,1]$, such that for all $\epsilon\in(0,\epsilon^*]$ and $(x,z,\eta)\in\mathcal{B}(r)$, $\|u\|_\infty\le\Upsilon$, which also ensure that for all $\epsilon\in(0,\epsilon^*]$ and $(x,z,\eta)\in L_V(c)$, $\|u\|_\infty\le\Upsilon$. Let $\epsilon\in(0,\epsilon^*]$. Then, we can also ensure that $\|u\|_\infty\le\Upsilon$, $\forall t\ge0$, i.e., $\sigma(u(t))=u(t)$, $\forall t\ge0$, by properly choosing a large enough r, due to the continuity of the solution to the controller parameters. Thus, the semi-global stabilization problem of system (8) can be solved under the controller (25). □

B. Semi-global Output Regulation of TTSSs

In this subsection, the semi-global output regulation problem of TTSSs is studied.

Similar to the stabilization controller (25), an output regulation controller is designed as

$$\begin{aligned}&\dot{\eta}=\Phi\eta+\Gamma e,\\&u=K_1x+K_2z-G_c\xi+\epsilon B_\varepsilon^\top E\bar{M}^\top P_3(\eta-E\bar{M}\xi),\end{aligned} \tag{28}$$

where the matrices G_c, K_1, K_2, $\bar{M}$ and P_i, $i=1,2,3$ have same definition as in Theorem 1. We have the next theorem.

Theorem 2. *Suppose Assumptions 1-4 hold. For any priori given compact subsets $\mathbb{X}\subset\mathbb{R}^{n_x}$, $\mathbb{Z}\subset\mathbb{R}^{n_z}$, $\mathbb{V}\subset\mathbb{R}^{n_v}$, and $\bar{\mathbb{V}}\subset\mathbb{R}^{n_vq}$ all containing the origin, there exists $\bar{\varepsilon}>0$ such that for any $\varepsilon\in(0,\bar{\varepsilon}]$, the output regulation problem is solvable for system* (1)-(2) *with the controller* (28)*.*

Proof. Based on Theorem 1, the matrix $\bar{J}$ is Hurwitz. Thus, there exist matrices Π and Σ such that

$$\begin{pmatrix}\Sigma\\\Pi\end{pmatrix}S=\bar{J}\begin{pmatrix}\Sigma\\\Pi\end{pmatrix}+\begin{pmatrix}\Gamma Q\\F_\varepsilon\end{pmatrix},$$

where $F_\varepsilon=(F_1^\top,\frac{F_2^\top}{\varepsilon})^\top$. It noted that the first equation above leads to $C\Pi+Q=0$, since S and Φ have the same eigenvalues [16, Theorem 1.7, pages 24-26].

Define the following coordinate transformation

$$\bar{\xi}:=\xi-\Pi v,\bar{\eta}:=\eta-\Sigma v. \tag{29}$$

When $\|u(t)\|_\infty\le\Upsilon$, the closed-loop system (1), (28) in the coordinates is

$$\begin{pmatrix}\dot{\bar{\xi}}\\\dot{\bar{\eta}}\end{pmatrix}=\bar{J}\begin{pmatrix}\bar{\xi}\\\bar{\eta}\end{pmatrix},\ e=C\bar{\xi}. \tag{30}$$

Consider the Lyapunov function

$$\bar{V}:=\bar{\xi}^\top(T_c^{-1})^\top P_\varepsilon(\epsilon)T_c^{-1}\bar{\xi}+\epsilon(\bar{\eta}-M\bar{\xi})^TP_3(\bar{\eta}-M\bar{\xi}). \tag{31}$$

Due to the fact that $(x(0),z(0),v(0),\eta(0))\in\mathbb{X}\times\mathbb{Z}\times\mathbb{V}\times\bar{\mathbb{V}}$ and $\mathbb{X}$, $\mathbb{Z}$, $\bar{\mathbb{V}}$, $\bar{\mathbb{V}}$ are compact subsets, there always exists a constant $\bar{c}>0$, such that

$$\sup_{\epsilon\in(0,1],(x(0),z(0),v(0),\eta(0))\in\mathbb{X}\times\mathbb{Z}\times\mathbb{V}\times\bar{\mathbb{V}}}\bar{V}(0)\le\bar{c}. \tag{32}$$

Let $L_{\bar{V}}(\bar{c}) = \{\{(x,z,v,\eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_v q} : \bar{V} \leq c\}$. With a similar proof of Lemma (3), the following equation has an unique solution $(\bar{\Sigma}^\top, \bar{\Pi}^\top)^\top$,

$$\tilde{E}\begin{pmatrix} \bar{\Sigma} \\ \bar{\Pi} \end{pmatrix} S = \tilde{J}\begin{pmatrix} \bar{\Sigma} \\ \bar{\Pi} \end{pmatrix} + \begin{pmatrix} \Gamma Q \\ F \end{pmatrix},$$

where $\tilde{E} = \text{diag}\{I_{n_v q}, \bar{E}\}$, $\tilde{J} = \begin{pmatrix} \Gamma C & \Phi \\ E\bar{\Lambda}_\varepsilon & \epsilon B B^\top \bar{M}^\top P_3 \end{pmatrix} = \text{diag}\{I_{n_v q}, E\}\bar{J}$. Besides, since $\bar{J}$ is Hurwitz, $\begin{pmatrix} \Sigma \\ \Pi \end{pmatrix} = \begin{pmatrix} \bar{\Sigma} \\ \bar{\Pi} \end{pmatrix} + O(\varepsilon)$. By using approximate solutions $\bar{M}$, $\bar{\Sigma}$ and $\bar{\Pi}$, we can find a set $\bar{\mathcal{B}}(r) := \{\{(x,z,v,\eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_v q} : \|(x,z,v,\eta)\| \leq r\} \supset L_{\bar{V}}(\bar{c})$. Similarly, we have $\lim_{\epsilon \to 0} \|u(\epsilon)\|_\infty = 0$. Then, we can find an $\epsilon^* \in (0,1]$, such that for all $\epsilon \in (0,\epsilon^*]$ and $(x,z,\eta) \in \bar{\mathcal{B}}(r)$, $\|u\|_\infty \leq \Upsilon$, which also ensure that for all $\epsilon \in (0,\epsilon^*]$ and $(x,z,\eta) \in L_{\bar{V}}(\bar{c})$, $\|u\|_\infty \leq \Upsilon$. Let $\epsilon \in (0,\epsilon^*]$. In this case, for $(x,z,\eta) \in L_v(c)$, TTSS (1), (28) can always be rewritten as (30). Since $\bar{J}$ is Hurwitz, the origin of the system is stable, which means that $\lim_{t \to \infty} \|e(t)\| = 0$. The proof is complete. □

The above result is also suitable for single time scale linear systems. Thus, the next corollary is proposed.

Corollary 1. *Suppose Assumption 3 holds. Consider system*

$$\begin{cases} \dot{x} = A_0 x + B_0 \sigma(u) + F v, \\ e = Cx + Qv, \end{cases} \tag{33}$$

where v is generated by (2)*. Then, for any priori given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{V} \subset \mathbb{R}^{n_v}$ and $\bar{\mathbb{V}} \subset \mathbb{R}^{n_v \times q}$ all containing the origin, there always exists an $\epsilon^* \in (0,1]$, such that for all $\epsilon \in (0,\epsilon^*]$, the regulator*

$$\begin{aligned} \dot{\eta} &= \Phi\eta + \Gamma e, \\ u &= K_0 x - B^\top P_1(\epsilon) x + \epsilon B^\top M^\top P_3(\eta - Mx), \end{aligned} \tag{34}$$

with P_1, P_3 being the solutions of (5)*,* (16) *and M being the solution of*

$$M(A_0 + B_0 K_0) = \Phi M + \Gamma C, \tag{35}$$

solves the robust output regulation problem, i.e.,

1) *When $v = 0$, the equilibrium point $(x,\eta) = (0,0)$ of the corresponding closed-loop system* (33)-(34) *is asymptotically stable with $\mathbb{X} \times \bar{\mathbb{V}}$ being contained in its basin of attraction.*
2) *For all $(x(0), v(0), \eta(0)) \in \mathbb{X} \times \mathbb{V} \times \bar{\mathbb{V}}$, the trajectories of the closed-loop system* (33)-(34) *are bounded for all $t \geq 0$, and satisfies $\lim_{t \to \infty} \|e(t)\| = 0$.*

The proof is similar with the one in above, thus it is omitted here.

IV. Illustrative Example

In this section, an example is present to illustrate the obtained results.

V. Conclusion

The robust semi-global output regulation problem was investigated for linear TTSSs with input saturation.

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