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Robust Semi-global output regulation of uncertain two-time-scale systems with input saturation [★]

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Abstract

This paper investigates the robust semi-global output regulation of uncertain linear two time scales systems with input saturation. In this setup, one cannot apply off-the-shelf techniques to reject structural uncertainties due to numerical issues caused by the two time scales but also due to technical issues generated by the input saturation. To solve this problem we combine internal model principle with low gain technique and singular perturbation theory. Explicitly, the output regulation controller design is based on the computation of the solutions of a Sylvester equation which are hard to get in the two time scales settings. To overcome this, we propose an easily solvable equation and the existence and uniqueness of its solution are guaranteed under some standard assumptions. Accordingly, an internal model based controller is designed such that the semi-global robust output regulation problem can be solved. In order to cope with the input saturation we impose that the ultimate upper bound of the infinity-norm of the exosystem state is limited and satisfies certain constraints. Finally, a numerical example is provided to illustrate the effectiveness of the results.

Key words: Two-time-scale, robust output regulation, input saturation, internal model.

1 Introduction

Systems involving processes that evolve on both fast and slow time scales appear in many practical applications such as robotics [1], biology [2] and electric power management [3,4]. A small positive parameter commonly is introduced to mathematically describe the time-scale separation between the fast and slow dynamics. Thus, numerical issues often appear and hamper the applica-

tion of the traditional control design techniques. Tailored methodological tools for two-time-scale systems (TTSSs) are therefore needed, see e.g., [5–8].

In this context, we focus on the robust regulation problem for uncertain linear time-invariant TTSSs. This problem has been a fundamental control problem since 1970s [9–11] and is relevant in many applications such as the control of unmanned aerial vehicles [12], spacecrafts [13] or power systems [14]. By achieving robust output regulation, we mean that the system's output asymptotically tracks some reference input regardless the external disturbance and some structural uncertainty, while the internal stability is ensured. It is noteworthy that the reference input and external disturbance are generated by an exosystem. Over the last decade, the problem received an increasing attention and the internal model principle has been widely utilized for the robust output regulation of linear uncertain systems [15,16] and nonlinear uncertain systems [17–20]. However, to the best of our knowledge, only few results focus

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on systems exhibiting two time scales. Ignoring the structural uncertainty, the output regulation problem for linear Takagi-Sugeno fuzzy TTSSs has been handled in [21]. For specific classes of nonlinear TTSSs in which only the slow subsystem is subject to the disturbance generated by the exosystem and the output regulation error is independent of the fast subsystem the problem has been addressed in [22]. It is important to note that the control designs in [21,22] are not robust against the structural uncertainty since they require the exosystem's states. In other words, the output tracking error of the system cannot asymptotically converge to the origin even if the structural uncertainty is very small.

Moreover, most works on robust output regulation problem ignore the input saturation, which often appears in practice due to the limitations of physical devices. If the input saturation is not carefully handled it may either damage the installation or lead to poor control performances. The small gain design technique is an effective method for input saturation problem of asymptotically null-controllable linear system [23–25], and has been utilized for the output regulation of linear systems [26] and singular linear systems [27] with input saturation. Furthermore, for linear systems [28,29] and discrete-time singular linear systems [30] with anti-stable eigenvalues and subject to input saturation, the output regulation problem is achieved within the specific initial region, and the nonlinear design technique is used as in [31,32] to enlarge corresponding initial region. However the proposed controllers are not robust against structured uncertainty which is an important property in the output regulation setting as pointed out in [15,16,33]. Moreover, the aforementioned results are for single time scale systems. As far as we know, no results are available for robust output regulation of TTSSs with input saturation.

In this context, we consider the plant modeled by uncertain linear TTSSs subject to external disturbance and input saturation. Our objective is to design a controller, which handle the robust semi-global output regulation problem. To avoid the continuous monitoring of exosystem, an internal model is introduced as in [9,10,34]. We first investigate the semi-global stabilization of the augmented system consisting of the internal model system and the original uncertain TTSS without the external disturbance. Due to the numerical issues caused by the two time-scale evolution and the input saturation non-linearity, the traditional control design techniques for single time scale linear systems in [10,15] cannot be directly applied here. Thus, the low gain feedback technique is combined with singular perturbation theory to design the stabilizing controller. It is also noted that, different from [22,21], a more general case is considered, i.e., both the slow and fast subsystems are subject to external disturbance, and the output tracking error relies on both the fast and slow states. In this case, solving the corresponding Sylvester equation as constructed in [10,15] would suffer from numerical issues. Thus, an eas-

ily solvable Sylvester equation is proposed and the existence and uniqueness of its solution are guaranteed under some standard assumptions. Accordingly, the stabilizing controller is redesigned to ensure that the origin of the augmented system is asymptotically stable regardless some small structural uncertainty. Then, an internal model based tracking controller is further designed such that the robust semi-global output regulation problem can be handled. The main contributions of this paper is threefold.

- 1) The robust semi-global output regulation problem is handled for uncertain linear systems exhibiting two time scales and input saturation.
- 2) An internal model based controller is designed by combining the low gain technique and singular perturbation theory. As a result, the monitoring of the exosystem's states can be avoided and the output regulation error can converge to the origin asymptotically regardless some small structural uncertainty.
- 3) To avoid the numerical issues, an easily solvable equation is proposed to provide the solution of corresponding Sylvester function needed for control design. Besides, the existence and uniqueness of its solution are guaranteed under some standard assumptions.

The rest of the paper is organized as follows. The problem formulation is stated in Section II. The robust semi-global output regulation of uncertain linear TTSS with input saturation is investigated in Section III. An illustrative example is presented in Section IV. Conclusions are drawn in Section V.

Notation. $\mathbb{R}^{m \times n}$ denotes the set of $m \times n$ real matrices. The notation I_n stands for the n -dimensional unit matrix. For a given real symmetric matrix P , $P > 0$ means that P is a positive definite matrix. The notation $\|\cdot\|$ denotes the Euclidean norm for vectors or the induced 2-norm for matrices depending on the context. For a matrix A , $\text{Vec}(A)$ denotes the vectorization of A . For a piecewise continuous bounded function $v : [0, \infty) \rightarrow \mathbb{R}^m$, and $T \geq 0$, $\|v(t)\|_{\infty, T} \triangleq \sup_{t \geq T} \|v(t)\|_{\infty}$. The function $f : [0, \infty)^2 \rightarrow \mathbb{R}^{m \times n}$ is said to be $O(\varepsilon)$ if there exist positive constants k and ε^* strictly positive such that $\|f(t, \varepsilon)\| \leq k\varepsilon$, for all $t \in [0, \infty)$ and $\varepsilon \in [0, \varepsilon^*]$.

2 Problem Statement

In this work, we address the problem of robust output regulation for the class of TTSSs described below. Notice that, for convenience, we often neglect the time argument of the variables.

$$\begin{cases} \dot{x} = A_{11}(w)x + A_{12}(w)z + B_1(w)\sigma(u) + F_1(w)v, \\ \varepsilon \dot{z} = A_{21}(w)x + A_{22}(w)z + B_2(w)\sigma(u) + F_2(w)v, \\ e = C_1(w)x + C_2(w)z + Q(w)v, \end{cases} \quad (1)$$

where $x \in \mathbb{R}^{n_x}$ and $z \in \mathbb{R}^{n_z}$ are the slow and the fast states, respectively, ε is a small positive parameter describing the time-scale separation between the slow and the fast dynamics, $u \in \mathbb{R}^p$ is the control input, $e \in \mathbb{R}^q$ is the output regulation error, $v \in \mathbb{R}^{n_v}$ is the state of the exosystem, representing both external disturbances and time-varying references input and it is generated by an autonomous exosystem of the form

$$\dot{v} = Sv, \quad (2)$$

where S is a constant matrix with appropriate dimensions. The matrices $Q(w)$, $A_{ij}(w)$, $B_i(w)$, $F_i(w)$, $C_i(w)$, $i, j = 1, 2$, are all continuous matrix functions of $w \in \mathbb{W}$ with appropriate dimensions, where \mathbb{W} is an open neighborhood of the origin. For the sake of convenience, $(Q(0), A_{ij}(0), B_i(0), F_i(0), C_i(0))$ is denoted by $(Q, A_{ij}, B_i, F_i, C_i)$, $i, j = 1, 2$, which are all known constant matrices. $\sigma(\cdot)$ is a vector-valued saturation function with

$$\sigma(u) = (\bar{\sigma}(u_1), \bar{\sigma}(u_2), \dots, \bar{\sigma}(u_p)), \quad (3)$$

where

$$\bar{\sigma}(u_i) = \begin{cases} u_i, & \text{if } |u_i| \leq \Upsilon \\ -\Upsilon, & \text{if } u_i < -\Upsilon \\ \Upsilon, & \text{if } u_i > \Upsilon, \end{cases}$$

where Υ is the saturation level. The goal of this paper is to design a state-feedback controller

$$\begin{aligned} \dot{\eta} &= \Phi\eta + \Gamma e, \\ u &= K_1x + K_2z + G(x, z, \eta), \end{aligned} \quad (4)$$

where $\eta \in \mathbb{R}^{n_\eta}$, such that the semi-global output regulation problem of TTSS (1) can be handled, as formalized next. It is noted that the design of controller (4) does not require the information of the state of the exosystem and is independent of parameter ε .

Definition 1 *The robust semi-global output regulation problem of TTSS (1) is solved under the controller in the form of (4), if for any priori given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{Z} \subset \mathbb{R}^{n_z}$, and $\mathbb{V} \subset \mathbb{R}^{n_v}$ all containing the origin, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$,*

- (1) (Internal Stability) *When $v = 0$ and $w = 0$, the equilibrium point $(x, z, \eta) = (0, 0, 0)$ of the corresponding closed-loop system composed of (1) and (4) is asymptotically stable with $\mathbb{X} \times \mathbb{Z} \times \mathbb{V}$ being contained in its basin of attraction.*
- (2) (Robust Output Regulation) *There exists a compact subset $\mathbb{V} \subset \mathbb{R}^{n_v}$ containing the origin and some open neighborhood \mathbb{W} of the origin, such that, for any $w \in \mathbb{W}$, any $(x(0), z(0), v(0), \eta(0)) \in \mathbb{X} \times \mathbb{Z} \times \mathbb{V} \times \mathbb{V}$, the trajectories of the closed-loop system composed of (1), (2) and (4) are bounded for all $t \geq 0$, and satisfies $\lim_{t \rightarrow \infty} \|e(t)\| = 0$.*

We remark that the compact set \mathbb{V} , i.e. the size of the admissible exosignals, cannot be arbitrarily large due to the presence of the input saturation. On the other hand, the size of the compact set of initial conditions $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{Z} \subset \mathbb{R}^{n_z}$, and $\mathbb{V} \subset \mathbb{R}^{n_v}$ can be arbitrarily chosen, i.e. we are in a semi-global context. Furthermore, note that to handle the robust semi-global output regulation problem in above, controller (4) should be applicable for any $\varepsilon \in (0, \bar{\varepsilon}]$ and $w \in \mathbb{W}$ to ensure both disturbance rejection and asymptotic tracking, i.e., the proposed controller should be robust to parameters ε and w . Due to the numerical issues caused by the small positive parameter ε , the eigenvalues of the control matrix and the exact solution of the corresponding Sylvester equation would be hard to obtained. The control design techniques for the single time scale system subject to input saturation in [26,29] are no longer applicable. The nonlinearity caused by input saturation would be harder to handle and the stability analysis would be more complicated.

To achieve the goal, the next four assumptions and one Lemma are presented.

Assumption 1 *The eigenvalues of matrix S are semi-simple with zero real parts.*

Assumption 1 is common and standard for ensuring the neutrally stability of the exosystem.

Assumption 2 *The matrix A_{22} is invertible.*

Assumption 2 is essential to decouple the slow and fast dynamics, which is standard in the singularly perturbed literature, see, e.g., [5].

Assumption 3 *The pairs (A_0, B_0) and (A_{22}, B_2) are asymptotically null controllable with bounded controls (ANCBC), i.e.*

- (1) *The pairs (A_0, B_0) and (A_{22}, B_2) are stabilizable*
- (2) *All eigenvalues of A_0 , A_{22} are in the closed left half s -plane,*

where $A_0 := A_{11} - A_{12}A_{22}^{-1}A_{21}$, $B_0 := B_1 - A_{12}A_{22}^{-1}B_2$.

Assumption 3 (also being used in [23–25]) is common and instrumental for the design of semi-global asymptotically stabilizing feedback gains for the reduced order and boundary layer subsystems. We recall now the following Lemma from [25], which is a direct consequence of precious assumption.

Lemma 1 ([25]) *Under Assumption 3, for any $\gamma \in (0, 1]$, there exist unique real symmetric matrices $P_1 > 0$, $P_2 > 0$, which solve the following algebraic Riccati equations:*

$$A_0^\top P_1 + P_1 A_0 - 2P_1 B_0 B_0^\top P_1 + \gamma I_{n_x} = 0, \quad (5)$$

$$A_{22}^\top P_2 + P_2 A_{22} - 2P_2 B_2 B_2^\top P_2 + \gamma I_{n_z} = 0. \quad (6)$$

Moreover, $\lim_{\gamma \rightarrow 0} P_1(\gamma) = 0_{n_x \times n_x}$, $\lim_{\gamma \rightarrow 0} P_2(\gamma) = 0_{n_z \times n_z}$.

Finally, we introduce the following non-resonance condition, necessary in the context output regulation (see, e.g. [9]).

Assumption 4 The matrix $\begin{pmatrix} A_\varepsilon - \lambda I & B_\varepsilon \\ C & 0 \end{pmatrix}$ has independent rows for each λ being an eigenvalues of S , where $A_\varepsilon = \begin{pmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\varepsilon} & \frac{A_{22}}{\varepsilon} \end{pmatrix}$, $B_\varepsilon = \begin{pmatrix} B_1 \\ \frac{B_2}{\varepsilon} \end{pmatrix}$, $C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$.

Assumption 4 is standard and instrumental for the output regulation of TTSSs (1), which has also been used in [9].

3 Main result

In this section, the semi-global stabilization and output regulation problem of TTSSs are investigated.

3.1 Semi-global stabilization of TTSSs

In this subsection, the semi-global stabilization problem of TTSSs is studied. The goal is to design a controller

$$u = K_1 x + K_2 z + G(x, z, \eta), \quad (7)$$

such that the semi-global stabilization problem defined next can be solved for the following augmented TTSSs

$$\begin{cases} \dot{\eta} = \Phi \eta + \Gamma(C_1 x + C_2 z), \\ \dot{x} = A_{11} x + A_{12} z + B_1 \sigma(u), \\ \varepsilon \dot{z} = A_{21} x + A_{22} z + B_2 \sigma(u), \end{cases} \quad (8)$$

where the internal model unit of $\eta \in \mathbb{R}^{n_v q}$ is designed as in [9,10] with

$$\Phi = \begin{pmatrix} 0 & I_q & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -s_0 I_q & -s_1 I_q & -s_2 I_q & \dots & -s_{n_v-1} I_q \end{pmatrix}, \quad \Gamma = \begin{pmatrix} I_q & 0_q & \dots & 0_q \end{pmatrix}^\top, \quad (9)$$

and the real numbers s_0, \dots, s_{n_v-1} denote the coefficients of the minimal polynomial of the matrix S .

Definition 2 The semi-global stabilization problem of system (8) is solved under the controller in the form of (7), if for any priori given compact subsets $\mathbb{X} \subset$

\mathbb{R}^{n_x} , $\mathbb{Z} \subset \mathbb{R}^{n_z}$, and $\bar{\mathbb{V}} \subset \mathbb{R}^{n_v q}$ all containing the origin, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$ and all $(x(0), z(0), \eta(0)) \in \mathbb{X} \times \mathbb{Z} \times \bar{\mathbb{V}}$, $\lim_{t \rightarrow \infty} \|x(t)\| = 0$, $\lim_{t \rightarrow \infty} \|z(t)\| = 0$, $\lim_{t \rightarrow \infty} \|\eta(t)\| = 0$.

Before the controller design, next lemma is introduced.

Lemma 2 ([5]) Suppose Assumption 2 holds. There exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, the origin of the system (10) is exponentially stable,

$$\begin{pmatrix} \dot{x} \\ \varepsilon \dot{z} \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \quad (10)$$

where $\Lambda_{ij} := A_{ij} + B_i K_j(\gamma)$, for $i, j = 1, 2$, $K_1(\gamma) := (1 - K_2(\gamma) A_{22}^{-1} B_2) K_0(\gamma) + K_2(\gamma) A_{22}^{-1} A_{21}$, $K_0(\gamma) := B_0^\top P_1(\gamma)$, $K_2(\gamma) := B_2^\top P_2(\gamma)$, $P_1(\gamma)$, $P_2(\gamma)$ are the solutions of (5) and (6) with $\gamma > 0$.

The design of the controller (7) is based on a two-step procedure. Firstly, the control gains K_1 and K_2 are design based on Lemma 2 for the stabilization of two-time-scale system of (x, z) . Then the so-called forwarding approach inspired by [20] is applied as follows to design $G(x, z, \eta)$ for the stabilization of the system of η .

Based on Lemma 2, there exists $\bar{\varepsilon}_1 > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_1]$, the intersection of the spectrum of $\Lambda_\varepsilon := \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \frac{\Lambda_{21}}{\varepsilon} & \frac{\Lambda_{22}}{\varepsilon} \end{pmatrix}$ (which is Hurwitz) and Φ (which is neutrally stable) is empty. Thus, the matrix M is uniquely defined as the solution of the following Sylvester function

$$M \Lambda_\varepsilon = \Phi M + \Gamma C. \quad (11)$$

Since Φ is neutrally stable, there exists a real symmetric matrix $P_3 > 0$, which solves the following inequality:

$$\Phi^\top P_3 + P_3 \Phi \leq 0. \quad (12)$$

With assuming that the exact solution of (11) is obtained, the next Theorem is firstly proposed, whose proof is given in the appendix.

Theorem 1 Suppose Assumptions 1-4 hold. For any priori given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{Z} \subset \mathbb{R}^{n_z}$, and $\bar{\mathbb{V}} \subset \mathbb{R}^{n_v q}$ all containing the origin, there exists $\bar{\varepsilon} > 0$ such that $\forall \varepsilon \in (0, \bar{\varepsilon}]$, the semi-global stabilization problem is solvable for system (8) with controller (7) when

- K_1, K_2 are defined as in Lemma 2,
- $G(x, z, \eta) = -B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1} \xi + \gamma B_\varepsilon^\top M^\top P_3(\eta - M\xi)$ with $\xi := (x, z)$, $P_\varepsilon(\gamma) := \text{diag}\{P_1(\gamma), \varepsilon P_2(\gamma)\}$,

$P_1(\gamma)$ and $P_2(\gamma)$ as defined in Lemma 2, M and Φ as defined (11) and (12), respectively.

Remark 1 Due to the two-time-scale feature, the traditional control design techniques for input saturation are not applicable. Thus, the singular perturbed theory is combined with low gain feedback technique to design the stabilizing controller (7). It is noted that $\|\gamma B_\varepsilon^\top M^\top P_3\|_\infty$ and $\|B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1}\|_\infty$ will not go to infinity when ε tends to be zero from (35) and (36) as shown in the proof of Theorem 1 in Appendix.

However, since ε is very small, it is hard to get exact solutions M for (11) due to the numerical issues. The controller designed in Theorem 1 would be hard to apply, which is also one of the challenges in this context. To handle it, an easily solved equation is proposed to replace (11) in the next lemma.

Lemma 3 Suppose Assumption 1-3 hold. There exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, the equation (13) has an unique solution \bar{M} ,

$$\bar{M}\Lambda = \Phi\bar{M}\bar{E} + \Gamma C, \quad (13)$$

where $\bar{E} = \text{diag}\{I_{n_x}, 0\}$, $\Lambda := \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}$ has the same definition as in (10). Besides, it satisfies $ME^{-1} = \bar{M} + O(\varepsilon)$ with $E = \text{diag}\{I_{n_x}, \varepsilon I_{n_z}\}$.

Proof 1 From Lemma 2, there exists $\bar{\varepsilon}_1 > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_1]$, Λ_ε is Hurwitz. Then, $\Lambda = E\Lambda_\varepsilon$ and $\Lambda^{-1} = \Lambda_\varepsilon^{-1}E^{-1}$. In this way, solving the equation (13) is equal to solving

$$\bar{M} = \Phi\bar{M}\bar{E}\Lambda^{-1} + \Gamma C\Lambda^{-1}, \quad (14)$$

which is also equal to solving

$$(I_{n_v q \times (n_x + n_z)} - \Phi \otimes \bar{E}\Lambda^{-1}) \text{Vec}(\bar{M}) = \text{Vec}(\Gamma C\Lambda^{-1}). \quad (15)$$

From (27), we have $\Lambda_\varepsilon^{-1} = T_c A_D^{-1} T_c^{-1}$, where $A_D^{-1} = \begin{pmatrix} \Lambda_s^{-1} & 0 \\ 0 & \varepsilon \Lambda_f^{-1} \end{pmatrix}$, $\Lambda_s = A_s + B_s K_s$ and $\Lambda_f = A_f + B_f K_f$. Thus,

$$\begin{aligned} \bar{E}\Lambda^{-1} &= \bar{E}T_c A_D^{-1} T_c^{-1} E^{-1} \\ &= \begin{pmatrix} I_{n_x} & \varepsilon H \\ 0 & 0 \end{pmatrix} A_D T_c^{-1} E^{-1} \\ &= \begin{pmatrix} \Lambda_s^{-1} & \varepsilon^2 H \Lambda_f^{-1} \\ 0 & 0 \end{pmatrix} T_c^{-1} E^{-1} \end{aligned}$$

$$= \begin{pmatrix} \Lambda_s^{-1}(I_{n_x} - \varepsilon H L) + \varepsilon^2 H \Lambda_f^{-1} L & -\Lambda_s^{-1} H + \varepsilon^2 H \Lambda_f^{-1} \\ 0 & 0 \end{pmatrix}.$$

Obviously, there exists $0 < \bar{\varepsilon}_3 \leq \bar{\varepsilon}_1$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_3]$, $\Lambda_s^{-1}(I_{n_x} - \varepsilon H L) + \varepsilon^2 H \Lambda_f^{-1} L$ is always Hurwitz. Thus, matrix $I_{n_v q \times (n_x + n_z)} - \Phi \otimes \bar{E}\Lambda^{-1}$ has no zero eigenvalues, which means that equation (13) has an unique solution \bar{M} .

From (11) and (13), we have

$$(M - \bar{M}E)\Lambda_\varepsilon = \Phi(M - \bar{M}E) + \Phi\bar{M}(E - \bar{E}). \quad (16)$$

Since Λ_ε is Hurwitz, it can be obtained that,

$$(ME^{-1} - \bar{M}) = \Phi(ME^{-1} - \bar{M})\Lambda^{-1} + \varepsilon\Phi\bar{M}(I_{n_x + n_z} - \bar{E})\Lambda^{-1}.$$

Thus, we have

$$\begin{aligned} &(I_{n_v q \times (n_x + n_z)} - \Phi \otimes \Lambda^{-1}) \text{Vec}(ME^{-1} - \bar{M}) \\ &= \varepsilon \text{Vec}(\Phi\bar{M}(I_{n_x + n_z} - \bar{E})\Lambda^{-1}). \end{aligned}$$

Since the intersection of the spectrum of Λ_ε and Φ is empty, $I_{n_v q \times (n_x + n_z)} - \Phi \otimes \Lambda^{-1}$ is invertible. Thus

$$\begin{aligned} &\|\text{Vec}(ME^{-1} - \bar{M})\| \\ &= \varepsilon \|(I_{n_v q \times (n_x + n_z)} - \Phi \otimes \Lambda^{-1})^{-1} \text{Vec}(\Phi\bar{M}(I_{n_x + n_z} - \bar{E})\Lambda^{-1})\| \\ &= O(\varepsilon), \end{aligned}$$

which also means that $ME^{-1} = \bar{M} + O(\varepsilon)$. The proof is complete.

From Lemma 3, the solution \bar{M} of (13) can always be exactly obtained. Besides, when ε is very small, \bar{M} can be regarded as an approximated solution of (11).

Accordingly, the controller can be designed as follows,

$$u = K_1 x + K_2 z - G_c \xi + \gamma B^\top \bar{M}^\top P_3(\eta - \bar{M}\bar{E}\xi), \quad (17)$$

where $B = (B_1^\top, B_2^\top)^\top$, K_1 , K_2 , \bar{M} , P_i , $i = 1, 2, 3$, have the same definition as in above, $G_c = (B_1^\top P_1(\gamma) - B_2^\top (\Lambda_{12} \Lambda_{22}^{-1})^\top P_1(\gamma) + B_2^\top P_2(\gamma) \Lambda_{22}^{-1} \Lambda_{21}, B_2^\top P_2(\gamma))$.

Then, the next theorem is obtained, whose proof is given in the appendix.

Theorem 2 Suppose Assumptions 1-4 hold. For any priori given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{Z} \subset \mathbb{R}^{n_z}$, and $\mathbb{V} \subset \mathbb{R}^{n_v q}$ all containing the origin, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, the semi-global stabilization problem is solvable for system (8) with controller (17).

Remark 2 It is noted that, controller (17) is independent of ε and is the approximation of the de-

signed controller in Theorem 1 with $G(x, z, \eta) = -G_c \xi + \gamma B^\top \bar{M}^\top P_3(\eta - \bar{M} \bar{E} \xi) + O(\varepsilon)\eta + O(\varepsilon)\eta$. Besides, due to the robustness of the closed-loops systems and the continuity of its solution to the controller parameters, with small enough ε , the controller (17) can still remain unsaturated, and the stabilization of systems (8) can be ensured under controller (17).

3.2 Semi-global Output Regulation of TTSSs

In this subsection, the semi-global output regulation problem of TTSSs is studied. To handle the robust output regulation problem of linear two-time-scale systems with input saturation, the internal model principle is combined with low gain feedback technique and singular perturbed theory to design the output regulation controller.

Based on the stabilizing controller (17), the output regulation controller is designed as

$$\begin{aligned} \dot{\eta} &= \Phi \eta + \Gamma e, \\ u &= K_1 x + K_2 z - G_c \xi + \gamma B^\top \bar{E} \bar{M}^\top P_3(\eta - \bar{E} \bar{M} \xi), \end{aligned} \quad (18)$$

where the matrices G_c , K_1 , K_2 , \bar{M} and P_i , $i = 1, 2, 3$ have same definition as in (17). Then, the next theorem is proposed, whose proof is given in the appendix.

Theorem 3 *Suppose Assumptions 1-4 hold. Given $\Upsilon > 0$, there exists a compact set $\mathbb{V} \subset \mathbb{R}^{n_v}$ containing the origin, and, for any priori given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{Z} \subset \mathbb{R}^{n_z}$, and $\bar{\mathbb{V}} \subset \mathbb{R}^{n_v \times q}$ all containing the origin, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, the robust semi-global output regulation problem stated in Definition 1 is solvable for system (1)-(2) with the controller (18).*

Remark 3 *Based on the internal model design, the output regulation problem can be transformed into a stabilization problem with the corresponding coordinate transformation, so the output regulation controller (18) has a similar form with stabilizing controller (17). Benefiting from the internal model based design, the proposed controller is structurally robust which guarantee the rejection of the small structural disturbance.*

3.3 The application to single time scale linear systems

The above result is also suitable for single time scale linear systems. Consider system

$$\begin{cases} \dot{x} = A_0(w)x + B_0(w)\sigma(u) + F_0(w)v, \\ e = C(w)x + Q(w)v, \end{cases} \quad (19)$$

where $(A_0(w), B_0(w), F_0(w), C_0(w), Q_0(w))$ are all continuous matrix functions of $w \in \mathbb{W}$ with appropriate dimensions and v is generated by exosystem (2). For convenience, $(A_0(0), B_0(0), F_0(0), C_0(0), Q_0(0))$ is denoted by $(A_0, B_0, F_0, C_0, Q_0)$. Suppose the matrix

$\begin{pmatrix} A_0 - \lambda I & B_0 \\ C_0 & 0 \end{pmatrix}$ has independent rows for each λ being an eigenvalues of S .

Then, the next corollary is proposed. The proof is similar with that of Theorem 3, thus it is omitted here.

Corollary 1 *Suppose Assumptions 1, 3 hold. Given $\Upsilon > 0$, there exists a compact set $\mathbb{V} \subset \mathbb{R}^{n_v}$ containing the origin, and, for any priori given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, and $\bar{\mathbb{V}} \subset \mathbb{R}^{n_v \times q}$ all containing the origin, and all $(x(0), v(0), \eta(0)) \in \mathbb{X} \times \mathbb{V} \times \bar{\mathbb{V}}$, there always exists an $\gamma^* \in (0, 1]$, such that for all $\gamma \in (0, \gamma^*]$, the regulator*

$$\begin{aligned} \dot{\eta} &= \Phi \eta + \Gamma e, \\ u &= K_0(\gamma)x - B_0^\top P_1(\gamma)x + \gamma B_0^\top M^\top P_3(\eta - M_0 x), \end{aligned} \quad (20)$$

with $K_0(\gamma)$ defined as in Lemma 2, P_1, P_3 being the solutions of (5), (12) and M_0 being the solution of

$$M(A_0 + B_0 K_0) = \Phi M_0 + \Gamma C, \quad (21)$$

solves the robust output regulation problem stated in Definition 1.

4 Illustrative Example

To illustrate the effectiveness of the obtained results, consider two time scales system (1) with $\varepsilon = 0.01$,

$$\begin{aligned} A_{11}(w) &= \begin{pmatrix} 10w_1 - 5 & 2w_1 - 1 \\ 1 & 4 \end{pmatrix}, \quad A_{12}(w) = \begin{pmatrix} -2 & -3 \\ w_1 & 2 \end{pmatrix}, \\ A_{21}(w) &= \begin{pmatrix} 1+w_2 & 2+2w_2 \\ -1-w_2 & 1 \end{pmatrix}, \quad A_{22}(w) = \begin{pmatrix} 0 & 1 \\ -1-w_2 & w_2 \end{pmatrix}, \\ B_1(w) &= \begin{pmatrix} -1 \\ w_1 - 2 \end{pmatrix}, \quad B_2(w) = \begin{pmatrix} w_2 - 1 \\ 2 + w_2 \end{pmatrix}, \\ F(w) &= \begin{pmatrix} w_1 - 0.2 & 0 & 0.1(w_2 - 1) & -w_2 \\ 0.3 & 0.1 & 0 & 0.1 \end{pmatrix}^\top, \\ C(w) &= \begin{pmatrix} 0 & w_1 & 1 & 1 + w_2 \end{pmatrix}, \quad Q(w) = \begin{pmatrix} -0.5 - w_1 & 0 \end{pmatrix}, \end{aligned}$$

and exosystem (2) with $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus, Assumptions 1 and 2 are satisfied.

Then, it can be obtained that $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21} = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}$, $B_0 = B_1 - A_{12}A_{22}^{-1}B_2 = \begin{pmatrix} -8 & 0 \end{pmatrix}^\top$. Assumption 3 is satisfied.

The simulation is presented with $(w_1, w_2) = (0.01, 0.01)$, $(x_1(0), x_2(0), z_1(0), z_2(0)) = (-2, -5, 5, 15)$, $v(0) = (-22, 0)$ and $\Upsilon = 1$. Let $\gamma = 0.005$, then from (5) and (6), it can be obtained that

$$P_1 = \begin{pmatrix} 0.0072 & -0.0008 \\ -0.0008 & 0.0224 \end{pmatrix}, P_2 = \begin{pmatrix} 0.0302 & 0.0018 \\ 0.0018 & 0.0337 \end{pmatrix}.$$

$$\text{Then, } K_0 := B_0^\top P_1 = \begin{pmatrix} 0.0576 & -0.0066 \end{pmatrix}, K_2 = B_2^\top P_2 = \begin{pmatrix} 0.0266 & -0.0655 \end{pmatrix}, K_1 = (1 - K_2 A_{22}^{-1} B_2) K_0 + K_2 A_{22}^{-1} A_{21} = \begin{pmatrix} 0.0194 & -0.1643 \end{pmatrix}.$$

Denote $\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus, from (12), we

can choose $P_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. From (13), it has

$$\bar{M} = \begin{pmatrix} -0.2826 & 2.9112 & -5.7181 & -0.4541 \\ -0.9537 & 0.1569 & -2.6812 & 2.1078 \end{pmatrix},$$

$$G_c = \begin{pmatrix} -0.0187 & 0.1642 & -0.0266 & 0.0655 \end{pmatrix}.$$

Then, controller (17) is designed based on Theorem 3.

Simulation results are presented in Fig. 1-3, which show that the output error converges to origin asymptotically without input being saturated, which confirms the effectiveness of Theorem 3.

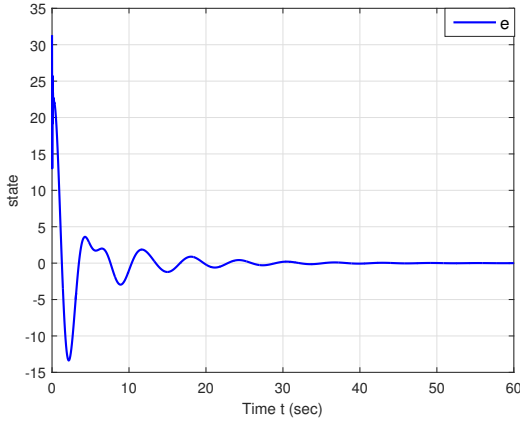


Fig. 1. Output regulation error evolution of system (1).

To test the robustness of the designed controller to structural disturbance w , let $\mathbb{W} = \{w | w_1 \in (-0.01, 0.01), w_2 \in (-0.01, 0.01)\}$. It can be verified that $\tilde{J}(w)$ is Hurwitz for all $w \in \mathbb{W}$, which ensure the internal stability of system (1). Simulation results are presented in Table 1. Table 1 shows that

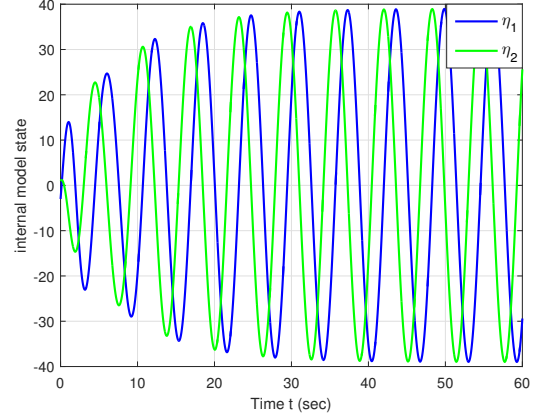


Fig. 2. State evolution of the internal model.

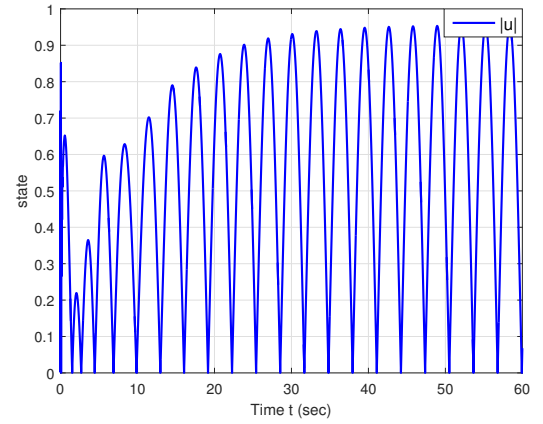


Fig. 3. Evolution of $\|u\|_\infty$ in the form of (18).

the corresponding upper bound λ_{\min} on the real part of all eigenvalues of $\tilde{J}(w)$ and $\|u(t)\|_{\infty,0}$ for a few w when $(x_1(0), x_2(0), z_1(0), z_2(0)) = (-2, -5, 5, 15)$, $v(0) = (-22, 0)$ and input saturation is ignored.

Table 1 Influence of w

w	λ_{\min}	$\ u(t)\ _{\infty,0}$
$w_1 = -0.005, w_2 = 0.006$	-0.1245	0.7853
$w_1 = 0.01, w_2 = 0.01$	-0.1165	0.9327
$w_1 = -0.01, w_2 = 10.01$	-0.1489	0.7628
$w_1 = -0.03, w_2 = 0.02$	-0.0011	0.7657
$w_1 = -0.05, w_2 = 0.06$	-0.0008	1.0983
$w_1 = 0.05, w_2 = -0.02$	-0.1298	1.2277
$w_1 = 0.1, w_2 = -0.2$	0.6897	∞

For the sake of comparison, the simulation for the controller designed by directed solving the output regulator equation is run with same initial state condition as follows. The controller can be designed in a form similar to

the one in as follows,

$$u = g(x, z, v) = K_1 x + K_2 z + Gv, \quad (22)$$

where K_1, K_2 have same definition as in above, $G = \Gamma_c - K\Pi_c$, and Γ_c, Π_c are the solution of the following output regulator equation,

$$\begin{aligned} A\Pi_c + B\Gamma_c + F &= E\Pi_c S, \\ C\Pi_c + Q &= 0, \end{aligned} \quad (23)$$

where $A = EA_\varepsilon, B = EB_\varepsilon$. Thus,

$$\begin{aligned} \Gamma_c &= \begin{pmatrix} -0.0477 & -0.1315 \end{pmatrix} + O(\varepsilon) \\ \Pi_c &= \begin{pmatrix} -0.2762 & 0.0908 \\ 0.0092 & -0.4762 \\ 0.19 & -0.73 \\ 0.31 & 0.73 \end{pmatrix} + O(\varepsilon). \end{aligned}$$

Here, the error part $O(\varepsilon)$ of the solution (Γ_c, Π_c) is ignored in the simulation, as ε is small.

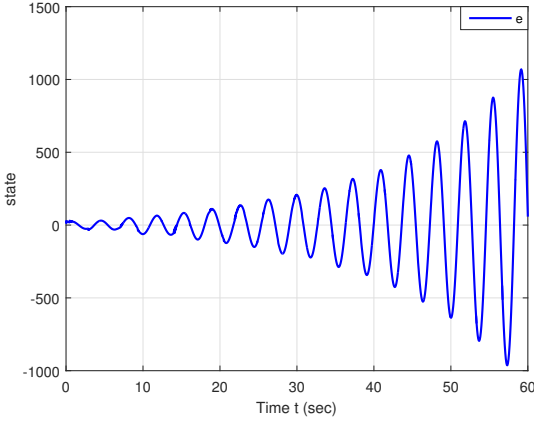


Fig. 4. Output regulation error evolution of system (1) with controller (22).

The obtained evolution of the regulation error and input are given in Fig. 4-5. Obviously, under the proposed controller (18), the system (1) has better performance. It is noted that the input is saturated as shown in Fig. 5 and the controller (22) is not robust to the structure uncertainty, so that the output regulation is not achieved under the controller (22) as shown in Fig. 4.

5 Conclusion

The robust semi-global output regulation problem was investigated for linear TTSSs with input saturation and structural uncertainty. An internal model based state feedback control law has been proposed with combining the low gain feedback technique and singular perturbed

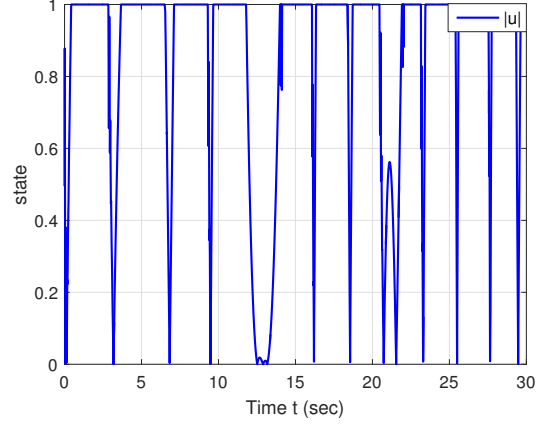


Fig. 5. Evolution of $\|u\|$ in the form of (22).

theory. As a result, the output regulation error can converge to the origin asymptotically regardless small structural uncertain parameters. It would be interesting in the future to consider the robust output regulation problem for nonlinear TTSSs.

6 Appendix

6.1 Proof of Theorem 1

For proving Theorem 1, next Lemma is presented firstly.

Lemma 4 Suppose Assumptions 1-4 hold. Consider system

$$\begin{cases} \dot{\eta} = \Phi\eta + \Gamma(C_1 x + C_2 z), \\ \dot{x} = A_{11}x + A_{12}z + B_1 u, \\ \varepsilon \dot{z} = A_{21}x + A_{22}z + B_2 u, \\ u = K_1(\gamma)x + K_2(\gamma)z + G(x, z, \eta), \end{cases} \quad (24)$$

where $K_1(\gamma), K_2(\gamma)$ and $G(x, z, \eta)$ have the same definition as in Theorem 1. Then, for any priori given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}, \mathbb{Z} \subset \mathbb{R}^{n_z}$, and $\bar{\mathbb{V}} \subset \mathbb{R}^{n_v}$ all containing the origin, and for any $\gamma \in (0, 1]$, for all $(x(0), z(0), \eta(0)) \in \mathbb{X} \times \mathbb{Z} \times \bar{\mathbb{V}}$, there exist $\bar{\varepsilon} > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}]$, $\lim_{t \rightarrow \infty} \|x(t)\| = 0, \lim_{t \rightarrow \infty} \|z(t)\| = 0, \lim_{t \rightarrow \infty} \|\eta(t)\| = 0$. Moreover, there exists $r > 0$, such that for any $t \geq 0, (x(t), z(t), \eta(t)) \in \mathcal{B}(r) := \{(x, z, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_v} : \|(x, z, \eta)\| \leq r\}$.

Proof 2 For the stability analysis, Chang transformation is introduced for the TTSS to separate the slow dynamics from the fast ones, see Chapter 3 in [5], which is presented as follows

$$\begin{pmatrix} x_s \\ z_f \end{pmatrix} := T_c^{-1} \begin{pmatrix} x \\ z \end{pmatrix}, \quad (25)$$

where $T_c^{-1} := \begin{pmatrix} I_{n_x} - \varepsilon HL & -\varepsilon H \\ L & I_{n_z} \end{pmatrix}$, and the matrices L and H are the solution of the following equations

$$\begin{aligned} \Lambda_{21} - \Lambda_{22}L + \varepsilon L\Lambda_{11} - \varepsilon L\Lambda_{12}L &= 0, \\ \Lambda_{12} - H\Lambda_{22} + \varepsilon \Lambda_{11}H - \varepsilon \Lambda_{12}LH - \varepsilon H\Lambda_{12} &= 0. \end{aligned} \quad (26)$$

As a result, the system in the (x_s, z_f) coordinates is

$$\begin{pmatrix} \dot{x}_s \\ \dot{z}_f \end{pmatrix} = \begin{pmatrix} \Lambda_s & 0 \\ 0 & \frac{\Lambda_f}{\varepsilon} \end{pmatrix} \begin{pmatrix} x_s \\ z_f \end{pmatrix} + T_c^{-1} B_\varepsilon G(x, z, \eta), \quad (27)$$

where

$$\begin{aligned} \Lambda_s &:= A_s + B_s K_s, \Lambda_f := A_f + B_f K_2, \\ A_s &:= A_0 - \varepsilon A_{12} A_{22}^{-1} L (A_{11} - A_{12} L), \\ B_s &:= B_0 - \varepsilon A_{12} A_{22}^{-1} L B_1, K_s := K_1 - K_2 L, \\ A_f &:= A_{22} + \varepsilon L A_{12}, B_f := B_2 + \varepsilon L B_1, \end{aligned} \quad (28)$$

From the definition of L , H , we have

$$\begin{aligned} A_s + B_s K_s &= (1 + O(\varepsilon))(A_0 + B_0 K_0), \\ A_f + B_f K_2 &= (1 + O(\varepsilon))(A_{22} + B_2 K_2). \end{aligned}$$

From (5) and (6), $A_0 + B_0 K_0$ and $A_{22} + B_2 K_2$ are both Hurwitz. Thus, there exist $\bar{\varepsilon}_1 > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_1]$, $A_s + B_s K_s$ and $A_f + B_f K_2$ are Hurwitz.

Recall that $P_\varepsilon(\gamma) := \text{diag}\{P_1(\gamma), \varepsilon P_2(\gamma)\}$ and $\xi := (x, z)$. Consider the Lyapunov function candidate

$$\begin{aligned} V &:= \xi^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1} \xi + \gamma(\eta - M\xi)^\top P_3(\eta - M\xi) \\ &= x_s^\top P_1 x_s + \varepsilon z_f^\top P_2 z_f + \gamma(\eta - M\xi)^\top P_3(\eta - M\xi). \end{aligned} \quad (29)$$

Thus, the derivative of V along with (24) yields,

$$\begin{aligned} \dot{V} &= x_s^\top (A_s^\top P_1(\gamma) + P_1(\gamma) A_s - 2P_1(\gamma) B_s K_s) x_s \\ &\quad + z_f^\top (A_f^\top P_2(\gamma) + P_2(\gamma) A_f - 2P_2(\gamma) B_f K_2) z_f \\ &\quad + 2\xi^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1} B_\varepsilon G(x, z, \eta) \\ &\quad + 2\gamma(\eta - M\xi)^\top P_3(\Phi\eta + \Gamma C\xi - M(\Lambda_\varepsilon \xi + B_\varepsilon G(x, z, \eta))) \\ &\leq - (1 - O(\varepsilon))(\gamma x_s^\top x_s + \gamma z_f^\top z_f) \\ &\quad + 2\xi^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1} B_\varepsilon G(x, z, \eta) \\ &\quad + 2\gamma(\eta - M\xi)^\top P_3(\Phi(\eta - M\xi) - M B_\varepsilon G(x, z, \eta)). \end{aligned}$$

Thus, there exists $0 < \bar{\varepsilon}_2 \leq \bar{\varepsilon}_1$, such that for any $\varepsilon \in (0, \bar{\varepsilon}_2]$, $\frac{1}{2} - O(\varepsilon) > 0$. Then, for $\varepsilon \in (0, \bar{\varepsilon}_2]$,

$$\dot{V} \leq -\frac{\gamma}{2} x_s^\top x_s - \frac{\gamma}{2} z_f^\top z_f - 2G^\top(x, z, \eta)G(x, z, \eta). \quad (30)$$

By using La Salle's arguments, we can prove that the

state of the closed-loop system (24) converges to the set $\{(x, z, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_v \times q} : x = 0, z = 0, G(x, z, \eta) = 0\} = \{0\} \times \{0\} \times \{B_\varepsilon^\top M^\top P_3 \eta = 0\}$. Thus, $\lim_{t \rightarrow \infty} \|x(t)\| = 0$, $\lim_{t \rightarrow \infty} \|z(t)\| = 0$. Since ξ converges to zero, the dynamics of η reduces to

$$\dot{\eta} = \Phi \eta.$$

Based on the Proposition 2 in [20], under Assumption 4, the pair $(B_\varepsilon^\top M^\top P_3, \Phi)$ is observable. Then,

$$\lim_{t \rightarrow \infty} \|\eta(t)\| = 0. \text{ Denote } J := \begin{pmatrix} \Phi & \Gamma C \\ \gamma B_\varepsilon B_\varepsilon^\top M^\top P_3 & \tilde{\Lambda}_\varepsilon \end{pmatrix},$$

where $\tilde{\Lambda}_\varepsilon = \Lambda_\varepsilon - B_\varepsilon B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon T_c^{-1} - \gamma B_\varepsilon B_\varepsilon^\top M^\top P_3 M$. Then TTSSs (24) can be rewritten as

$$\begin{pmatrix} \dot{\eta} \\ \dot{\xi} \end{pmatrix} = J \begin{pmatrix} \eta \\ \xi \end{pmatrix}. \quad (31)$$

Thus, for $\varepsilon \in (0, \bar{\varepsilon}_2]$, J is Hurwitz.

Meanwhile, from (30), there exists a class \mathcal{KL} function β_s , such that $\forall t \geq 0$,

$$\begin{aligned} &\|(x_s(t), \sqrt{\varepsilon} z_f(t), \eta(t) - M\xi(t))\| \\ &\leq \beta_s(\|(x_s(0), z_f(0), \eta(0))\|, t). \end{aligned}$$

Due to the fact that $(x(0), z(0), \eta(0)) \in \mathbb{X} \times \mathbb{Z} \times \bar{\mathbb{V}}$, and $\mathbb{X}, \mathbb{Z}, \bar{\mathbb{V}}$ are compact subsets, there exists $r_1 > 0$ such that

$$\|(x_s(t), \eta(t) - M\xi(t))\| \leq r_1, \forall t \geq 0. \quad (32)$$

It is noted that a upper bound of $\|z_f(t)\|$ independent of ε can not be obtained from (30). Thus, the next Lyapunov function candidate is further introduced,

$$V_f := \varepsilon z_f^\top P_2 z_f. \quad (33)$$

With a similar proof in above, it can be obtained that, for $\varepsilon \in (0, \bar{\varepsilon}_2]$,

$$\begin{aligned} \dot{V}_f &\leq -\frac{\gamma}{2} z_f^\top z_f - 2G^\top(x, z, \eta)G(x, z, \eta) \\ &\quad + 2\gamma(\eta - M\xi)^\top P_3 M B_\varepsilon G(x, z, \eta) \\ &\quad - 2x_s^\top P_1 (B_1 - H B_2 - \varepsilon H L B_1) G(x, z, \eta) \\ &\leq -\frac{\gamma}{2} z_f^\top z_f + \|\gamma B_\varepsilon^\top M^\top P_3(\eta - M\xi)\|^2 \\ &\quad + \|(B_1 - H B_2 - \varepsilon H L B_1)^\top P_1 x_s\|^2 \end{aligned}$$

From Lemma 3, it can be obtained that $B_\varepsilon^\top M^\top = B^\top \bar{M}^\top + O(\varepsilon)$. Then, from (32), it can be obtained that,

$$\dot{V}_f \leq -\frac{\gamma}{2} z_f^\top z_f + r_2. \quad (34)$$

where $r_2 > (\|\gamma B^\top \bar{M}^\top P_3\|^2 + \|(B_1 - HB_2)^\top P_1\|^2 + O(\varepsilon))r_1^2$. Thus, there exists a class \mathcal{KL} function β_f , such that $\forall t \geq 0$,

$$\|z_f(t)\| \leq \beta_f(\|(x(0), z(0))\|, t) + \sqrt{\frac{2r_2}{\gamma}}.$$

Thus, there exist $\varepsilon_2 > 0$ and $r > 0$ such that for any $\varepsilon \in (0, \varepsilon_2]$, and $t \geq 0$, $(x(t), z(t), \eta(t)) \in \mathcal{B}(r) := \{(x, z, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_{vq}} : \|(x, z, \eta)\| \leq r\}$.

Then, the proof of Theorem 1 is presented.

From Lemma 1, it can be obtained that

$$\lim_{\gamma \rightarrow 0} \|P_1(\gamma)\|_\infty = 0, \quad \lim_{\gamma \rightarrow 0} \|P_2(\gamma)\|_\infty = 0.$$

Then, from the definition of K_1 and K_2 in Lemma 2, it is easily obtained that $\lim_{\gamma \rightarrow 0} \|K_1(\gamma)\|_\infty = 0$ and $\lim_{\gamma \rightarrow 0} \|K_2(\gamma)\|_\infty = 0$. Besides, from Lemma 3,

$$B_\varepsilon^\top M^\top = B^\top \bar{M}^\top + O(\varepsilon), \quad (35)$$

thus $\lim_{\gamma \rightarrow 0} \|\gamma B_\varepsilon^\top M^\top P_3\|_\infty = 0$.

From the definition of T_c^{-1} in (25), it can be obtained that

$$\begin{aligned} & B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon(\gamma) \\ &= \left((B_1^\top - B_2^\top H^\top + O(\varepsilon))P_1(\gamma) (B_2^\top + O(\varepsilon))P_2(\gamma) \right). \end{aligned} \quad (36)$$

Thus, $\lim_{\gamma \rightarrow 0} \|B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1}\|_\infty = 0$.

Since $(x(0), z(0), \eta(0)) \in \mathbb{X} \times \mathbb{Z} \times \bar{\mathbb{V}}$, and $\mathbb{X} \times \mathbb{Z} \times \bar{\mathbb{V}}$ is the compact set, it can be obtained that $\lim_{\gamma \rightarrow 0} \|u(\gamma)\|_{\infty,0} = 0$.

From Lemma 4, for any $\gamma \in (0, 1]$ there exist $\bar{\varepsilon}_1 > 0$ and $r > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_1]$, and $t \geq 0$, $(x(t), z(t), \eta(t)) \in \mathcal{B}(r) := \{(x, z, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_{vq}} : \|(x, z, \eta)\| \leq r\}$. Thus, there exists an $\gamma^* \in (0, 1]$, such that for all $\gamma \in (0, \gamma^*]$, $\|u(t)\|_{\infty,0} \leq \Upsilon$. Let $\gamma \in (0, \gamma^*]$. In this case, for all $(x(0), z(0), \eta(0)) \in \mathbb{X} \times \mathbb{Z} \times \bar{\mathbb{V}}$, TTSS (8) can always be rewritten as (24). Thus, with similar proof of Lemma 4, there exist $0 < \bar{\varepsilon} \leq \bar{\varepsilon}_1$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}]$, $\lim_{t \rightarrow \infty} \|x(t)\| = 0$, $\lim_{t \rightarrow \infty} \|z(t)\| = 0$, $\lim_{t \rightarrow \infty} \|\eta(t)\| = 0$. Thus, the equilibrium point $(x, z, \eta) = (0, 0, 0)$ of the corresponding closed-loop system (7)-(8) is asymptotically stable with $\mathbb{X} \times \mathbb{Z} \times \bar{\mathbb{V}}$ being contained in its basin of attraction. Thus, the semi-global stabilization problem of system (8) can be solved. The proof is complete.

6.2 Proof of Theorem 2

From (36), it can be obtained that

$$\begin{aligned} & B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1} \\ &= \left((B_1^\top - B_2^\top H^\top + O(\varepsilon))P_1(\gamma) (B_2^\top + O(\varepsilon))P_2(\gamma) \right) T_c^{-1} \\ &= \left(B_1^\top P_1(\gamma) + B_2^\top (H^\top P_1(\gamma) + P_2(\gamma)L) B_2^\top P_2(\gamma) \right) + O(\varepsilon). \end{aligned}$$

Meanwhile, from (26), it can be obtained that

$$L = \Lambda_{22}^{-1} \Lambda_{21} + O(\varepsilon), \quad H = \Lambda_{12} \Lambda_{22}^{-1} + O(\varepsilon).$$

Thus

$$B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1} = G_c + O(\varepsilon). \quad (37)$$

From Lemma 2, 3 and the proof in above, there exists $\bar{\varepsilon}_1 > 0$, such that for any $\varepsilon \in (0, \bar{\varepsilon}_1]$, we have that the matrix J is Hurwitz, and

$$M = \bar{M}E + O(\varepsilon) = \bar{M}\bar{E} + O(\varepsilon). \quad (38)$$

Define $\bar{J} := \begin{pmatrix} \Phi & \Gamma C \\ \gamma B_\varepsilon B^\top \bar{M}^\top P_3 & \bar{\Lambda}_\varepsilon \end{pmatrix}$, where $\bar{\Lambda}_\varepsilon = \Lambda_\varepsilon - B_\varepsilon(G_c + \gamma B^\top \bar{M}^\top P_3 \bar{M} \bar{E})$. From (37) and (38), we have

$$\bar{J} = J + O(\varepsilon).$$

Thus, with a similar proof of theorem 4, there exists small enough $0 < \bar{\varepsilon}_2 \leq \bar{\varepsilon}_1$, such that for all $\varepsilon \in (0, \bar{\varepsilon}]$, J and \bar{J} are both Hurwitz.

Then, from (38) and the continuity of the solution to the controller parameters, for any $\gamma \in (0, 1]$, with a similar proof of Lemma 4, there similarly exist $0 < \bar{\varepsilon} < \varepsilon_2$ and $r > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, and $t \geq 0$, $(x(t), z(t), \eta(t)) \in \mathcal{B}(r) := \{(x, z, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_{vq}} : \|(x, z, \eta)\| \leq r\}$. Similarly, it can be obtained that there exists $\gamma^* \in (0, 1]$, such that for all $\gamma \in (0, \gamma^*]$ and for all $(x(0), z(0), \eta(0)) \in \mathbb{X} \times \mathbb{Z} \times \bar{\mathbb{V}}$, $\|u(t)\|_{\infty,0} \leq \Upsilon$. Let $\gamma \in (0, \gamma^*]$. Then, we can ensure that $\sigma(u(t)) = u(t)$, $\forall t \geq 0$, and TTSS (8) with the controller (17) can always be rewritten as

$$\begin{pmatrix} \dot{\eta} \\ \dot{\xi} \end{pmatrix} = \bar{J} \begin{pmatrix} \eta \\ \xi \end{pmatrix}. \quad (39)$$

Since \bar{J} is Hurwitz, the origin of the corresponding closed-loop system (8)-(17) is asymptotically stable with $\mathbb{X} \times \mathbb{Z} \times \bar{\mathbb{V}}$ being contained in its basin of attraction. The semi-global stabilization problem of system

(8) can be solved under the controller (17).

6.3 Proof of Theorem 3

Denote

$$\tilde{J}(w) := \begin{pmatrix} \Phi & \Gamma C(w) \\ \gamma B_\varepsilon(w) B^\top(0) \bar{M}^\top P_3 & \tilde{\Lambda}_\varepsilon \end{pmatrix},$$

where $C(w) = \begin{pmatrix} C_1(w) & C_2(w) \end{pmatrix}$, $\tilde{\Lambda}_\varepsilon = A_\varepsilon(w) + B_\varepsilon(W)K - B_\varepsilon(W)(G_c + \gamma B^\top(0) \bar{M}^\top P_3 \bar{M} \bar{E})$, $A_\varepsilon(w) = \begin{pmatrix} A_{11}(w) & A_{12}(w) \\ \frac{A_{21}(w)}{\varepsilon} & \frac{A_{22}(w)}{\varepsilon} \end{pmatrix}$, $B_\varepsilon(w) = \begin{pmatrix} B_1(w) \\ \frac{B_2(w)}{\varepsilon} \end{pmatrix}$.

Based on Theorem 2, there exists small enough $\bar{\varepsilon}_1 > 0$, such that for all $\varepsilon \in (0, \bar{\varepsilon}_1]$, the matrix \tilde{J} is Hurwitz. Thus, there exists an open neighborhood \mathbb{W}_1 of the origin, such that for any $w \in \mathbb{W}_1$, $\tilde{J}(w)$ is Hurwitz. Then, the intersection of the spectrum of S and \tilde{J} is empty. There exist matrices Π and Σ uniquely defined such that

$$\begin{pmatrix} \Sigma \\ \Pi \end{pmatrix} S = \tilde{J}(w) \begin{pmatrix} \Sigma \\ \Pi \end{pmatrix} + \begin{pmatrix} \Gamma Q(w) \\ F_\varepsilon \end{pmatrix}, \quad (40)$$

where $F_\varepsilon = (F_1^\top, \frac{F_2^\top}{\varepsilon})^\top$. It is noted that the first equation above leads to $C(w)\Pi + Q(w) = 0$, since S and Φ have the same eigenvalues [9, Theorem 1.7, pages 24-26].

Define the following coordinate transformation

$$\bar{\xi} := \xi - \Pi v, \bar{\eta} := \eta - \Sigma v. \quad (41)$$

When $\|u(t)\|_{\infty,0} \leq \Upsilon$, the closed-loop system (1), (18) in the coordinates $(\bar{\eta}, \bar{\xi})$ is

$$\begin{pmatrix} \dot{\bar{\eta}} \\ \dot{\bar{\xi}} \end{pmatrix} = \tilde{J}(w) \begin{pmatrix} \bar{\eta} \\ \bar{\xi} \end{pmatrix}, \quad e = C\bar{\xi}. \quad (42)$$

As Assumption 4 holds, there exists an open neighborhood $\mathbb{W}_2 \subset \mathbb{W}_1$ of the origin, such that for any $w \in \mathbb{W}_2$, the output regulation solution of system (1) exists and can be defined as $\xi := \Pi_c v$, $u := \Gamma_c v$, where $\Pi_c := (\Pi_x, \Pi_z)$ and Γ_c satisfy

$$\begin{aligned} \Pi_x S &= A_{11}(w)\Pi_x + A_{12}(w)\Pi_z + B_1(w)\Gamma_c + F_1(w), \\ \varepsilon \Pi_z S &= A_{21}(w)\Pi_x + A_{22}(w)\Pi_z + B_2(w)\Gamma_c + F_2(w), \\ 0 &= C_1(w)\Pi_x + C_2(w)\Pi_z + Q(w). \end{aligned} \quad (43)$$

It can be easily obtained that the output regulation of system (1) can be achieved under controller (18), when there is no input saturation. Thus, we also have $\xi = \Pi v$ and $u = K_1 \Pi_x v + K_2 \Pi_z v - G_c \Pi v + \gamma B_\varepsilon^\top E \bar{M}^\top P_3 (\Sigma v -$

$E \bar{M} \Pi v)$. Thus, $\Pi = \Pi_c$ and $\lim_{\gamma \rightarrow 0} K_1 \Pi_x + K_2 \Pi_z - G_c \Pi + \gamma B_\varepsilon^\top E \bar{M}^\top P_3 (\Sigma - E \bar{M} \Pi) = \lim_{\gamma \rightarrow 0} \gamma B_\varepsilon^\top E \bar{M}^\top P_3 \Sigma = \Gamma_c$.

It is noted that the exact solution (Π_x, Π_z, Γ_c) of (43) is hard to be obtained. The approximate solution $(\bar{\Pi}_x, \bar{\Pi}_z, \bar{\Gamma}_c)$ is introduced in place of (Π_x, Π_z, Γ_c) , which satisfying

$$\begin{aligned} \bar{\Pi}_x S &= A_{11} \bar{\Pi}_x + A_{12} \bar{\Pi}_z + B_1 \bar{\Gamma}_c + F_1, \\ 0 &= A_{21} \bar{\Pi}_x + A_{22} \bar{\Pi}_z + B_2 \bar{\Gamma}_c + F_2, \\ 0 &= C_1 \bar{\Pi}_x + C_2 \bar{\Pi}_z + Q. \end{aligned} \quad (44)$$

With a similar proof of Lemma 3, there exist small enough $0 < \bar{\varepsilon}_2 \leq \varepsilon_1$, such that for all $\varepsilon \leq \bar{\varepsilon}_2$, the equation (44) has an solution, and $(\Pi_x, \Pi_z, \Gamma_c) = (\bar{\Pi}_x, \bar{\Pi}_z, \bar{\Gamma}_c) + O(\varepsilon) + O(w)$.

It is noted that the exact solution (Π, Σ) of (40) is hard to be obtained. The approximate solution $(\bar{\Pi}, \bar{\Sigma})$ is introduced in place of (Π, Σ) , which satisfying

$$\hat{E} \begin{pmatrix} \bar{\Sigma} \\ \bar{\Pi} \end{pmatrix} S = \hat{J} \begin{pmatrix} \bar{\Sigma} \\ \bar{\Pi} \end{pmatrix} + \begin{pmatrix} \Gamma Q \\ F \end{pmatrix}, \quad (45)$$

where $\hat{E} = \text{diag}\{I_{n_v q}, \bar{E}\}$, $\hat{J} = \begin{pmatrix} \Phi & \Gamma C \\ \gamma B B^\top \bar{M}^\top P_3 & E \bar{\Lambda}_\varepsilon \end{pmatrix} = \text{diag}\{I_{n_v q}, E\} \bar{J}$. With a similar proof of Lemma 3, the equation (45) has an unique solution, and $(\Pi, \Sigma) = (\bar{\Pi}, \bar{\Sigma}) + O(\varepsilon) + O(w)$. Thus, with the approximate solution $(\bar{\Pi}, \bar{\Sigma})$, for any given compact sets $\mathbb{X}, \mathbb{Z}, \mathbb{V}, \bar{\mathbb{V}}$ all containing the origin, there always exist compact sets $\mathbb{X}_r \subset \mathbb{R}^{n_x}$, $\mathbb{Z}_r \subset \mathbb{R}^{n_z}$, $\bar{\mathbb{V}}_r \subset \mathbb{R}^{n_v q}$ containing the origin such that for all $(x(0), z(0), v(0), \eta(0)) \in \mathbb{X} \times \mathbb{Z} \times \mathbb{V} \times \bar{\mathbb{V}}$, $(\bar{\xi}(0), \bar{\eta}(0)) \in \mathbb{X}_r \times \mathbb{Z}_r \times \bar{\mathbb{V}}_r$.

Consider the Lyapunov function

$$\bar{V} := \bar{\xi}^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1} \bar{\xi} + \gamma (\bar{\eta} - M \bar{\xi})^\top P_3 (\bar{\eta} - M \bar{\xi}).$$

With a similar proof of theorem 2, there exist $0 < \bar{\varepsilon}_3 \leq \varepsilon_2$ and $r > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_3]$, and $t \geq 0$, $(\bar{\xi}(t), \bar{\eta}(t)) \in \bar{\mathcal{B}}(r) := \{(\bar{\xi}, \bar{\eta}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_v q} : \|(\bar{\xi}, \bar{\eta})\| \leq r\}$. From (18), we have $u = (K - G_c)(\bar{\xi} + \Pi v) + \gamma B_\varepsilon^\top E \bar{M}^\top P_3 (\bar{\eta} - E \bar{M}(\bar{\xi} + \Pi v)) + \gamma B_\varepsilon^\top E \bar{M}^\top P_3 \Sigma v$, where $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$. Thus, $\lim_{\gamma \rightarrow 0} u = B_\varepsilon^\top E \bar{M}^\top P_3 \Sigma v = \Gamma_c v = \bar{\Gamma}_c v + O(\varepsilon)v + O(w)v$. Denote $\mathbb{V} = \{v \in \mathbb{R}^{n_v} : \|v\| \leq \bar{v}\}$ with $\bar{v} > 0$ and

$$\sup_{|v| \leq \bar{v}, t \geq t_0} |\bar{\Gamma}_c e^{S(t-t_0)} v|_\infty < \Upsilon.$$

In this way, there exist $0 < \bar{\varepsilon} \leq \varepsilon_3$ and an open neighborhood $\mathbb{W} \subset \mathbb{W}_2$ of the origin, such that there exists

a small enough $\gamma^* \in (0, 1]$, such that for all $\varepsilon \in (0, \bar{\varepsilon}]$, $w \in \mathbb{W}$ and $\gamma \in (0, \gamma^*]$, $\|u(t)\|_{\infty, 0} \leq \Upsilon$. Let $\gamma \in (0, \gamma^*]$, then TTSS (1), (18) can always be rewritten as (42). Since $\tilde{J}(w)$ is Hurwitz, the origin of the system is stable, which means that $\lim_{t \rightarrow \infty} \|e(t)\| = \lim_{t \rightarrow \infty} \|C(w)\xi\| = 0$. The proof is complete.

References

- [1] B. Siciliano, W. J. Book, A singular perturbation approach to control of lightweight flexible manipulators, *The International Journal of Robotics Research* 7 (4) (1988) 79–90.
- [2] R. Wang, T. Zhou, Z. Jing, L. Chen, Modelling periodic oscillation of biological systems with multiple timescale networks, *Systems Biology* 1 (1) (2004) 71–84.
- [3] G. Peponides, P. Kokotovic, J. Chow, Singular perturbations and time scales in nonlinear models of power systems, *IEEE Transactions on Circuits and systems* 29 (11) (1982) 758–767.
- [4] N. Jiang, H.-D. Chiang, A two-time scale dynamic correction method for fifth-order generator model undergoing large disturbances, *IEEE Transactions on Power Systems* 31 (5) (2015) 3616–3623.
- [5] P. Kokotović, H. K. Khalil, J. O’reilly, *Singular perturbation methods in control: analysis and design*, SIAM, 1999.
- [6] H. K. Khalil, *Nonlinear systems* (3rd Edition), Upper Saddle River, 2002.
- [7] J. B. Rejeb, I.-C. Morărescu, A. Girard, J. Daafouz, Stability analysis of a general class of singularly perturbed linear hybrid systems, *Automatica* 90 (2018) 98–108.
- [8] W. Yang, Y.-W. Wang, C. Wen, J. Daafouz, Exponential stability of singularly perturbed switched systems with all modes being unstable, *Automatica* 113 (2020) 108800.
- [9] C. I. Byrnes, F. D. Priscoli, A. Isidori, *Output regulation of uncertain nonlinear systems*, Springer, 1997.
- [10] J. Huang, *Nonlinear output regulation: theory and applications*, SIAM, 2004.
- [11] L. Wang, L. Marconi, C. Wen, H. Su, Pre-processing nonlinear output regulation with non-vanishing measurements, *Automatica* 111 (2020) 108616.
- [12] A. T. Tran, N. Sakamoto, M. Sato, K. Muraoka, Control augmentation system design for quad-tilt-wing unmanned aerial vehicle via robust output regulation method, *IEEE Transactions on Aerospace and Electronic Systems* 53 (1) (2017) 357–369.
- [13] Z. Chen, J. Huang, Attitude tracking and disturbance rejection of rigid spacecraft by adaptive control, *IEEE Transactions on Automatic Control* 54 (3) (2009) 600–605.
- [14] A. Silani, M. Cucuzzella, J. M. Scherpen, M. J. Yazdanpanah, Robust output regulation for voltage control in dc networks with time-varying loads, *Automatica* 135 (2022) 109997.
- [15] L. Paunonen, Controller design for robust output regulation of regular linear systems, *IEEE Transactions on Automatic Control* 61 (10) (2015) 2974–2986.
- [16] D. Liang, J. Huang, Robust output regulation of linear systems by event-triggered dynamic output feedback control, *IEEE Transactions on Automatic Control* 66 (5) (2020) 2415–2422.
- [17] A. Serrani, A. Isidori, Global robust output regulation for a class of nonlinear systems, *Systems & Control Letters* 39 (2) (2000) 133–139.
- [18] A. Serrani, A. Isidori, L. Marconi, Semi-global nonlinear output regulation with adaptive internal model, *IEEE Transactions on Automatic Control* 46 (8) (2001) 1178–1194.
- [19] T. Liu, J. Huang, Cooperative robust output regulation for a class of nonlinear multi-agent systems subject to a nonlinear leader system, *Automatica* 108 (2019) 108501.
- [20] D. Astolfi, L. Praly, L. Marconi, Harmonic internal models for structurally robust periodic output regulation, *Systems & Control Letters* 161 (2022) 105154.
- [21] S. Chen, Output regulation of nonlinear singularly perturbed systems based on t-s fuzzy model, *Journal of Control Theory & Applications* 3 (4) (2005) 399–403.
- [22] J. Yu, C. Mu, X. Mu, S. Chen, Output regulation of nonlinear singularly perturbed systems, *Computers & Mathematics with Applications* 46 (10) (2003) 1571–1579.
- [23] W. Schmitendorf, B. Barmish, Null controllability of linear systems with constrained controls, *SIAM Journal on control and optimization* 18 (4) (1980) 327–345.
- [24] Z. Lin, A. Saberi, Semi-global exponential stabilization of linear systems subject to input saturation via linear feedbacks, *Systems & Control Letters* 21 (3) (1993) 225–239.
- [25] Z. Lin, *Low gain feedback*, Berlin, Germany: Springer, 1999.
- [26] Z. Lin, A. A. Stoorvogel, A. Saberi, Output regulation for linear systems subject to input saturation, *Automatica* 32 (1) (1996) 29–47.
- [27] W. Lan, J. Huang, Semiglobal stabilization and output regulation of singular linear systems with input saturation, *IEEE Transactions on Automatic Control* 48 (7) (2003) 1274–1280.
- [28] R. D. Santis, Output regulation for linear systems with anti-stable eigenvalues in the presence of input saturation, *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal* 10 (5) (2000) 423–438.
- [29] R. De Santis, A. Isidori, On the output regulation for linear systems in the presence of input saturation, *IEEE Transactions on Automatic Control* 46 (1) (2001) 156–160.
- [30] E. Jafari, T. Binazadeh, Robust output regulation in discrete-time singular systems with actuator saturation and uncertainties, *IEEE Transactions on Circuits and Systems II: Express Briefs* 67 (2) (2019) 340–344.
- [31] B. M. Chen, T. H. Lee, K. Peng, V. Venkataramanan, Composite nonlinear feedback control for linear systems with input saturation: theory and an application, *IEEE Transactions on automatic control* 48 (3) (2003) 427–439.
- [32] Y. He, B. M. Chen, C. Wu, Composite nonlinear control with state and measurement feedback for general multivariable systems with input saturation, *Systems & Control Letters* 54 (5) (2005) 455–469.
- [33] M. Bin, D. Astolfi, L. Marconi, About robustness of control systems embedding an internal model, *IEEE Transactions on Automatic Control* doi:10.1109/TAC.2022.3151574.
- [34] M. Bin, J. Huang, A. Isidori, L. Marconi, M. Mischiati, E. Sontag, Internal models in control, bioengineering, and neuroscience, *Annual Review of Control, Robotics, and Autonomous Systems* 5 (2022) 55–79.