# Adding Direction Constraints to the 1-2-3 Conjecture

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**Abstract**

In connection with the so-called 1-2-3 Conjecture, we introduce and study a new variant of proper labellings, obtained when aiming at designing, for an oriented graph, an oriented colouring through the sums of labels incident to its vertices. Formally, for an oriented graph $\vec{G}$ and a $k$-labelling $\ell : A(\vec{G}) \to \{1, \ldots, k\}$ of its arcs, for every vertex $v \in V(\vec{G})$, one can compute the sum $\sigma(v)$ of labels assigned by $\ell$ to its incident arcs. We call $\ell$ an oriented labelling if the sum function $\sigma$ indeed forms an oriented colouring of $\vec{G}$. That is, for any two arcs $\overrightarrow{ab}$ and $\overrightarrow{cd}$ of $\vec{G}$, if $\sigma(a) = \sigma(d)$, then we must have $\sigma(b) \neq \sigma(c)$. We denote by $\chi_{\vec{\Sigma}}(\vec{G})$ the smallest $k$ such that oriented $k$-labellings of $\vec{G}$ exist (if any).

We study this new parameter in general and in particular contexts. In particular, we observe that there is no constant bound on $\chi_{\vec{\Sigma}}(\vec{G})$ in general, contrarily to the undirected case. Still, we establish connections between this parameter and others, such as the oriented chromatic number, from which we deduce other types of bounds, some of which we improve upon for some classes of oriented graphs. We also investigate other aspects of this parameter, such as the complexity of determining $\chi_{\vec{\Sigma}}(\vec{G})$ for a given oriented graph $\vec{G}$, or the possible relationships between $\chi_{\vec{\Sigma}}(\vec{G})$ and the underlying graph $G$ of $\vec{G}$.

*Keywords:* 1-2-3 Conjecture; proper labelling; oriented colouring; oriented graph.

## 1. Introduction

This work deals mainly with a generalisation of the so-called **1-2-3 Conjecture** to **oriented graphs**; we thus start by introducing all material related to that conjecture.

Let $G$ be a graph[1]. For any $k \geq 1$, a *$k$-labelling* $\ell$ of $G$ is an assignment $\ell : E(G) \to \{1, \ldots, k\}$ of *labels* (from $\{1, \ldots, k\}$) to the edges of $G$. For every vertex $v$ of $G$, one can now compute its incident *sum* of labels by $\ell$, denoted by $\sigma_\ell(v)$ (or simply by $\sigma(v)$ in case no confusion is possible), which is nothing but $\sum_{vu \in E(G)} \ell(vu)$, the sum of labels assigned to the edges incident to $v$. If we have $\sigma(u) \neq \sigma(v)$ for all $uv \in E(G)$, that is, if no two adjacent vertices of $G$ have the same sum, or, equivalently, if the resulting sums by $\ell$ form a proper colouring[1] of $G$, then $\ell$ is said to be *proper*. Finally, we denote by $\chi_\Sigma(G)$ the smallest $k \geq 1$, if any, such that proper $k$-labellings of $G$ exist.

It is not too complicated to observe that $\chi_\Sigma(G)$ is well defined provided $G$ does not contain $K_2$, the complete graph on two vertices, as a connected component. Thus, throughout this work, unless specified otherwise, we always implicitly consider graphs excluding $K_2$ as a connected component, whenever dealing with these notions.

The 1-2-3 Conjecture (introduced in 2004 by Karoński, Łuczak, and Thomason [14]), now, is a presumption on the maximum value that $\chi_\Sigma(G)$ can reach for a graph $G$:

[1] Throughout this work, unless specified otherwise, the generic term *graph* always refers to an undirected simple graph, while the generic term *colouring* always refer to a vertex-colouring.

**1-2-3 Conjecture.** *If $G$ is a graph, then $\chi_\Sigma(G) \leq 3$.*

The 1-2-3 Conjecture has received quite some attention over the past decades, covering several of its aspects (see [20] for a survey). Among other notable aspects, the conjecture was verified for 3-colourable graphs [14], determining $\chi_\Sigma(G)$ for a given graph $G$ is NP-complete in general [9] but polynomial-time doable for bipartite graphs [24], and $\chi_\Sigma(G) \leq 5$ is known to hold for all graphs [13]. Quite some attention has also been dedicated to generalising the 1-2-3 Conjecture to proper labellings fulfilling additional properties, and/or to structures that are more general than graphs such as hypergraphs and directed graphs.

The current work is precisely about a new generalisation of the 1-2-3 Conjecture to oriented graphs. It has to be mentioned that a previous series of works, [1, 3, 6, 7, 12], has already initiated this thread of research, based on the following notions and observations. Given a digraph $D$ and a labelling $\ell$ of $D$, note that, for every vertex $v$ of $D$, one can, this time, compute two possible sums for $v$: its *incoming sum* $\sigma^-(v)$ and its *outgoing sum* $\sigma^+(v)$, being the sums of labels assigned to the arcs incoming to $v$ and outgoing from $v$, respectively. Given that there are now two sum parameters to play with, there are multiple ways to consider that $\ell$ is proper, that is, that every two of its adjacent vertices are distinguished by $\ell$. In [7], the authors consider that two vertices are distinguished if their relative sums (differences of their outgoing sums and incoming sums) differ. In [1, 3, 6, 12], the authors consider that two vertices are distinguished if one particular of the two sums of one vertex is different from a particular of the two sums of the second vertex (leading to four possible distinction conditions). Long story short, for all the resulting variations of proper labellings, the best possible upper bounds on the parameters corresponding to $\chi_\Sigma$ were proved, through more or less complicated proofs, which is somewhat surprising as one could expect such results to be harder to establish than the original 1-2-3 Conjecture.

The notions we investigate in the current work stand as another way to generalise the 1-2-3 Conjecture to oriented graphs which is, in some sense, closer to the original problem. For the sake of keeping the current introduction short, we postpone our formal definitions to Section 2. For now, let us just mention that we introduce and investigate labellings of oriented graphs for which the resulting sums form an **oriented colouring**, which is, roughly put, a proper colouring where the arcs joining any two colour classes are oriented the same way. Again, these notions will be properly surveyed throughout Section 2.

This paper is organised as follows. We start with some preliminaries in Section 2, covering some notation, terminology, and early remarks to be useful throughout. In particular, we recall some notions related to oriented colouring, we get to defining formally our notion of oriented labelling and the associated parameter $\chi_{\vec{\Sigma}}$, and we raise first observations on it (in particular, we show that $\chi_{\vec{\Sigma}}(\vec{G})$ can be arbitrarily large for an oriented graph $\vec{G}$). In Section 3, we raise a few connections between $\chi_{\vec{\Sigma}}(\vec{G})$ and other parameters of $\vec{G}$, such as $\chi_\Sigma(G)$, $\chi_{\rm o}(\vec{G})$, and $\Delta(\vec{G})$. In Sections 4 and 5, we investigate the cases where $\vec{G}$ has maximum degree 2 and where $\vec{G}$ is an oriented tree, for which we establish tight upper bounds on $\chi_{\vec{\Sigma}}(\vec{G})$. A few complexity aspects are considered in Section 6. We finish off with concluding words in Section 7.

## 2. Notation, terminology, and early remarks

### *2.1. Oriented graphs and oriented colourings*

Let us start by recalling notions revolving around oriented graphs, oriented colourings, and the oriented chromatic number. Given a graph $G$, by *orienting* every edge $uv$ to either the arc $\overrightarrow{uv}$ (from $u$ to $v$) or the arc $\overrightarrow{vu}$ (from $v$ to $u$), one obtains an *oriented graph* $\vec{G}$,

being an *orientation* of $G$. Note that this notation is convenient in that, given an oriented graph $\vec{G}$, we can simply refer to its underlying graph as $G$. Also, up to considering simple graphs $G$ only (which we do throughout), note that any oriented graph $\vec{G}$ cannot contain a directed cycle of length 2 (*i.e.*, two opposite arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$), which is the main difference between oriented graphs and digraphs (in which such directed cycles are allowed).

Throughout this work, the notation and terminology we employ to deal with oriented graphs are rather standard. In case anything is unclear, we thus refer the reader to any monograph on the topic. Let us just recall, for now, that if $\vec{G}$ is an oriented graph, then we denote by $V(\vec{G})$ and $A(\vec{G})$ its vertex and arc sets. Unless specified otherwise, any notion or terminology usually defined for undirected graphs, when used for $\vec{G}$, is with respect to $G$. For instance, when referring to the degree of a vertex of $\vec{G}$, or to the fact that $\vec{G}$ is bipartite or has other properties of interest, we actually mean that it is $G$ that has these properties. A path or cycle of $G$, when considered in $\vec{G}$, is said to be an *oriented path* or *cycle*, respectively, in $\vec{G}$. In particular, the orientation of that path or cycle does not have to fulfil particular properties. On the other hand, when saying that a path or cycle of $\vec{G}$ is *directed*, we mean all its arcs go to the same direction. A *$k$-path* refers to a path of length $k$, while a *$k$-cycle* refers to a cycle of length $k$.

Let now $\vec{G}$ be an oriented graph. A *$k$-colouring* $\phi$ of $\vec{G}$ is an assignment of colours $V(\vec{G}) \to \{1, \ldots, k\}$ from $\{1, \ldots, k\}$ to the vertices of $\vec{G}$. We say that $\phi$ is *oriented* if, for every two distinct colours $\alpha$ and $\beta$ assigned by $\phi$, all arcs of $\vec{G}$ joining a vertex with colour $\alpha$ and a vertex with colour $\beta$ go from the former to the latter, or, *vice versa*, from the latter to the former. Another classical way for this property to be expressed is through the lens of homomorphisms, by saying that $\phi$ is oriented if it stands as a homomorphism from $\vec{G}$ to some oriented graph $\vec{H}$ with order $k$. In such cases, we sometimes also call $\phi$ an *$\vec{H}$-colouring* of $\vec{G}$. Now, the *oriented chromatic number* $\chi_o(\vec{G})$ of $\vec{G}$ is defined as the smallest $k$ such that oriented $k$-colourings of $\vec{G}$ exist. This parameter has also been considered in the context of classes of graphs, by defining, for a class $\mathcal{F}$ of graphs, the parameter $\chi_o(\mathcal{F})$ as the maximum value of $\chi_o(\vec{G})$ for some $G \in \mathcal{F}$.

These notions and parameters gave birth to a tremendous amount of research works, dedicated mainly to determining the oriented chromatic number of several classical families of graphs. Given the amount of works involved, it would be a bit daring to give here a fully detailed and complete survey on this topic. Instead, we will, throughout this work, disseminate several known results and facts on oriented colourings here and there, as they are needed to get the full extent of our contribution. Yet, the interested reader can refer to [23] for a recent survey on the oriented chromatic number and related notions.

Before going on, it is worth discussing a very peculiar aspect behind oriented colourings, being that having $\chi_o(\vec{G}) \leq k$ and $\chi_o(\vec{H}) \leq k$ for two oriented graphs $\vec{G}$ and $\vec{H}$ does not guarantee that $\chi_o(\vec{G} + \vec{H}) \leq k$, where $\vec{G} + \vec{H}$ denotes the disjoint union of $\vec{G}$ and $\vec{H}$. This is because the set of oriented graphs on $k$ vertices to which $\vec{G}$ admits a homomorphism might be disjoint from the set of those to which $\vec{H}$ admits a homomorphism. For this reason, the oriented chromatic number of disconnected oriented graphs has been receiving a special focus in several works of the literature. In the current, initiating work, we voluntarily focus on connected oriented graphs only, for the sake of converging to a first understanding of our new labelling notion. Thus, throughout this work, unless specified otherwise, every oriented graph we consider is assumed connected.

### *2.2. Oriented labellings*

We now get to defining the main notions to be investigated throughout this work, which, as mentioned earlier, can be seen as a combination of proper labellings and oriented

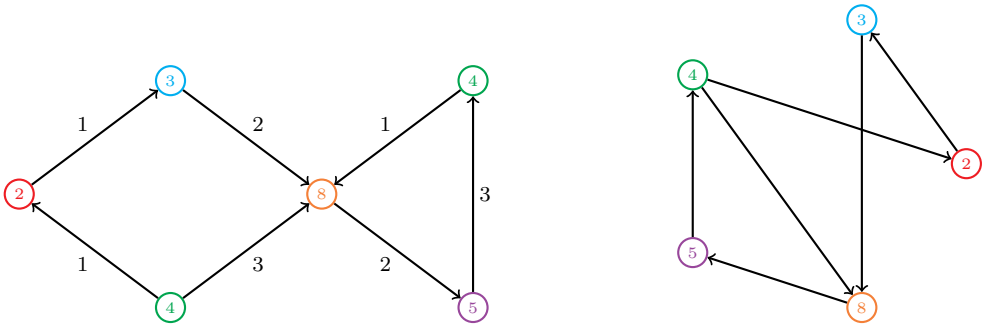



Figure 1: An oriented graph $\vec{G}$ together with an oriented 3-labelling $\ell$ (a), for which $\sigma_\ell$ is actually a homomorphism to $\vec{H}$ (b), and thus $\ell$ is an $\vec{H}$-oriented 3-labelling of $\vec{G}$. In (a), the numbers in the vertices indicate the sums obtained by $\ell$.

colourings. Let $\vec{G}$ be an oriented graph, and let $\ell$ be a $k$-labelling of $\vec{G}$. Note that, $\vec{G}$ being oriented, $\ell$ transposes naturally to $G$, and, thus, the sum $\sigma(v)$ by $\ell$ of any vertex $v$ of $\vec{G}$ can therefore be computed similarly as in $G$. This is exactly how we compute sums in the new variant we introduce. In other words, the sums by $\ell$ in $\vec{G}$ we consider are those computed by taking all arcs incident to the vertices into account, regardless of their directions. Now, we say that $\ell$ is *oriented* if the sums obtained for the vertices of $\vec{G}$ form an oriented colouring of $\vec{G}$. That is to say, if $\overrightarrow{ab}$ and $\overrightarrow{cd}$ are two arcs of $\vec{G}$ with $\sigma(a) = \sigma(d)$, then $\sigma(b) \neq \sigma(c)$. To be even more precise, assuming $\ell$ is an oriented labelling of $\vec{G}$ such that the colouring function $\sigma_\ell$ forms a homomorphism from $\vec{G}$ to some oriented graph $\vec{H}$ (where $V(\vec{H}) \subset \mathbb{N}$), we will sometimes call $\ell$ an *$\vec{H}$-oriented labelling* (see Figure 1 for an illustration). Last, we define $\chi_{\vec{\Sigma}}(\vec{G})$ as the smallest $k$ (if any) such that oriented $k$-labellings of $\vec{G}$ exist. By observations to be raised in what follows, it is not too complicated to establish that $\chi_{\vec{\Sigma}}(\vec{G})$ is always well defined, unless $G$ contains $K_2$ as a connected component. Thus, throughout this work, unless specified otherwise, we always consider $\chi_{\vec{\Sigma}}(\vec{G})$ in contexts where $G$ excludes $K_2$ as a connected component (an assumption actually implied by another one we mentioned earlier, being that we focus on connected oriented graphs only).

### *2.3. Remarks on the possible magnitude of $\chi_{\vec{\Sigma}}$*

The very first question one could naturally ask, is whether $\chi_{\vec{\Sigma}}(\vec{G})$ can be arbitrarily large for some oriented graph $\vec{G}$, or whether, just as hypothesised by the 1-2-3 Conjecture, there is an absolute constant upper bound on the parameter $\chi_{\vec{\Sigma}}$. Note that this is a legitimate wonder, as, due to the definitions involved, proper labellings and oriented labellings are objects that are not so distant. Namely:

**Observation 2.1.** *Every oriented labelling of an oriented graph $\vec{G}$ also stands as a proper labelling of $G$. Consequently, $\chi_\Sigma(G) \leq \chi_{\vec{\Sigma}}(\vec{G})$.*

*Proof.* This is just because, by any labelling $\ell$ of $\vec{G}$, by definition for every vertex $v$ the sum $\sigma(v)$ is computed the same way in both $\vec{G}$ and $G$. Also, if $\ell$ is oriented, then, by definition of an oriented colouring, no two adjacent vertices of $\vec{G}$ get the same sum. This means $\ell$ is proper in $G$, and the inequality in the statement thus follows. □

Since the 1-2-3 Conjecture, if true, would be tight (this follows for instance from the fact that it is NP-hard to decide whether $\chi_\Sigma(G) \leq 2$ for a graph $G$, see [9]), Observation 2.1

implies that there exist arbitrarily many oriented graphs $\vec{G}$ with $\chi_{\vec{\Sigma}}(\vec{G}) \geq 3$: just consider any orientation $\vec{G}$ of a graph $G$ with $\chi_{\Sigma}(G) = 3$. Thus, if an absolute constant upper bound on $\chi_{\vec{\Sigma}}$ exists, then it must be at least 3.

It turns out that such an upper bound does not exist. We can actually construct several families of oriented graphs $\vec{G}$ with $\chi_{\vec{\Sigma}}(\vec{G})$ being arbitrarily large, some of which will be introduced later in this work. For now, we introduce an easy construction that involves particular classical notions of cliques for oriented colouring.

For an oriented graph $\vec{G}$, a *relative clique* $S \subseteq V(\vec{G})$ is a set of vertices of $\vec{G}$ that must receive pairwise distinct colours by any oriented colouring of $\vec{G}$. The point is that if $S$ is a relative clique of $\vec{G}$, then $\chi_o(\vec{G}) \geq |S|$. For these reasons, relative cliques have received quite some attention in the literature (see e.g. [21] and the pointers therein). Fortunately, relative cliques are easy to describe (see e.g. [19]):

**Lemma 2.2.** *A subset $S \subseteq V(\vec{G})$ of vertices of an oriented graph $\vec{G}$ is a relative clique if and only if every two vertices of $S$ are either adjacent or joined by a directed 2-path.*

In particular, an oriented graph $\vec{G}$ is an *oriented clique* if $\chi_o(\vec{G}) = |V(\vec{G})|$, or, in other words, if $V(\vec{G})$ is a relative clique of $\vec{G}$. Oriented cliques also received quite some attention, since they hold as the oriented counterpart of the notion of cliques for proper colouring. Planar oriented cliques were notably studied in [21], while oriented cliques with bounded maximum degree were studied e.g. in [11]. In particular, in connection with the so-called Postage Stamp Problem, in the latter work the authors were able to show the following:

**Lemma 2.3** ([11])**.** *For every $\Delta$, there exists an oriented clique with maximum degree $\Delta$ and order $\frac{\Delta^2}{7} + \mathcal{O}(\Delta)$.*

We now prove that $\chi_{\vec{\Sigma}}(\vec{G})$ can be arbitrarily large, for some oriented graphs $\vec{G}$. This relies mainly on a straight observation.

**Lemma 2.4.** *If $\vec{G}$ is an oriented graph with a relative clique $S$ containing only vertices of degree $\Delta$, then $\chi_{\vec{\Sigma}}(\vec{G}) \geq \frac{|S|+\Delta-1}{\Delta}$.*

*Proof.* By any $k$-labelling $\ell$ of $\vec{G}$, the vertices of $S$ must all have their sum lying in $\{\Delta, \dots, k\Delta\}$, which is a set of cardinality $(k-1)\Delta + 1$. Now, since all vertices of $S$ must have distinct colours in any oriented colouring of $\vec{G}$, for $\ell$ to be oriented we must have $(k-1)\Delta + 1 \geq |S|$, and thus $k \geq \frac{|S|+\Delta-1}{\Delta}$. □

**Theorem 2.5.** *There is no $k \geq 1$ such that $\chi_{\vec{\Sigma}}(\vec{G}) \leq k$ for every oriented graph $\vec{G}$.*

*Proof.* Assume this is wrong, and that such a $k$ exists. Choose a $\Delta$ such that $\Delta^2 > 7k\Delta - 7\Delta + 7$, and consider the following construction. Start from an oriented clique $\vec{G}$ with maximum degree $\Delta$ and order $\frac{\Delta^2}{7} + \mathcal{O}(\Delta)$; such a $\vec{G}$ exists by Theorem 2.5. Denote by $S$ the current vertices in $\vec{G}$. We now further modify $\vec{G}$, if needed, in the following way. Note that the maximum degree of a vertex in $S$ is $\Delta$. However, at this point, nothing guarantees that all vertices in $S$ have degree $\Delta$. So, we consider every vertex $v$ of $S$ with degree less than $\Delta$ in turn, and attach to $v$ new pendant arcs (oriented arbitrarily) going to new degree-1 vertices, so that the degree of $v$ becomes exactly $\Delta$. Once this is achieved for all vertices of $S$, all its vertices have degree $\Delta$ in $\vec{G}$, the resulting oriented graph.

Due to how $\vec{G}$ was obtained, we still have the property that the vertices of $S$ form a relative clique of $\vec{G}$ (recall Lemma 2.2). Now, by Lemma 2.4, we must have $k \geq \chi_{\vec{\Sigma}}(\vec{G}) \geq \frac{|S|+\Delta-1}{\Delta}$. However, the fact that $\Delta^2 > 7k\Delta - 7\Delta + 7$ implies that $(k-1)\Delta + 1 < \frac{\Delta^2}{7}$. From this we deduce $|S| > (k-1)\Delta + 1$, thus $k < \frac{|S|+\Delta-1}{\Delta}$, a contradiction. So, $\vec{G}$ admits no oriented $k$-labellings at all, which contradicts the hypothesis on $k$. □

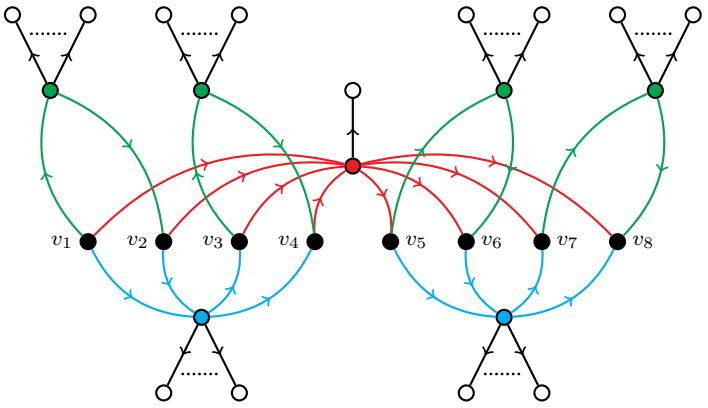



Figure 2: Illustration of the construction in the proof of Theorem 3.1, for $\Delta = 3$. Red vertices and arcs are added during the first step, blue vertices and arcs are added during the second one, and green vertices and arcs are added during the third one. White vertices are degree-1 vertices added at the end of the construction to ensure the whole oriented graph is locally irregular.

## 3. Connections between $\chi_{\vec{\Sigma}}$ and other parameters

As seen earlier through Theorem 2.5, the parameter $\chi_{\vec{\Sigma}}(\vec{G})$ can be arbitrarily large for an oriented graph $\vec{G}$. It thus makes sense to wonder, now, about ways to bound this parameter in terms of other parameters of $\vec{G}$. In what follows, we consider a few natural candidate parameters, such as $\chi_\Sigma(G)$ and $\chi_o(\vec{G})$. Through connections with the latter parameter, we get to establishing upper bounds on $\chi_{\vec{\Sigma}}(\vec{G})$ in terms of $\Delta(G)$.

### *3.1.* $\chi_{\vec{\Sigma}}(\vec{G})$ *versus* $\chi_\Sigma(G)$

Through another construction, in the next result we establish that there exist oriented graphs $\vec{G}$ that are *locally irregular* (*i.e.*, that do not have two adjacent vertices with the same degree) and that require arbitrarily many labels in their oriented labellings. In other words, for such oriented graphs $\vec{G}$, even though $\chi_\Sigma(G) = 1$, the parameter $\chi_{\vec{\Sigma}}(\vec{G})$, on the other hand, is arbitrarily large. Note that we already had a somewhat similar result in hand, due to Theorem 2.5 and the fact that $\chi_\Sigma(G) \leq 5$ holds for every graph $G$ (as proved in [13]). The fact, however, that we have such a construction resulting in a locally irregular graph is, in our opinion, even more significant.

**Theorem 3.1.** *There exist oriented graphs $\vec{G}$ with $\chi_\Sigma(G) = 1$ and $\chi_{\vec{\Sigma}}(\vec{G})$ arbitrarily large.*

*Proof.* Let $n = 2^\Delta$ for some $\Delta \geq 2$ be any power of 2, and consider the following construction of an oriented graph $\vec{G}$ (illustrated in Figure 2). Start from $n$ isolated vertices $v_1, \ldots, v_n$. Split the set $S = \{v_1, \ldots, v_n\}$ into the two sets $S_1 = \{v_1, \ldots, v_{\frac{n}{2}}\}$ and $S_2 = \{v_{\frac{n}{2}+1}, \ldots, v_n\}$ of equal cardinality, $n/2$. Now, add a new vertex $x$ to $\vec{G}$, and add arcs so that $x$ is an out-neighbour of all vertices in $S_1$ and an in-neighbour of all vertices in $S_2$. Note that this makes any vertex of $S_1$ and any vertex of $S_2$ being joined by a directed 2-path. Now repeat this process, that is split $S_1$ into two parts of equal size, add a new vertex being an in-neighbour of half its vertices and an out-neighbour of the other half, do the same for $S_2$, and iterate until singletons are obtained when splitting subsets.

Once the process achieves, in the resulting $\vec{G}$ all pairs of $v_i$'s are joined by a directed 2-path. Also, since the process finishes in $\log_2 n = \Delta$ rounds, all $v_i$'s have degree precisely $\Delta$. Now, to make sure $\vec{G}$ is locally irregular, we can consider every vertex $x$ of $\vec{G}$ that is neither a degree-1 vertex nor one of the $v_i$'s, and attach new degree-1 vertices to it (through arcs

in any direction) until its degree becomes e.g. $n+1$ (what is desired being actually that this degree is different from $\Delta$). At the end, $\overrightarrow{G}$ is actually bipartite, one of its partition classes containing the $v_i$'s (of degree $\Delta$) and all degree-1 vertices, while the other contains all other vertices, of degree $n+1$ (adjacent to some of the $v_i$'s and degree-1 vertices). So, $\overrightarrow{G}$ is locally irregular and $\chi_\Sigma(G) = 1$. On the other hand, note that $S$ contains $n$ vertices of degree $\Delta$, every two of which are joined by a directed 2-path. In other words, $S$ is a relative clique (by Lemma 2.2) of $\overrightarrow{G}$ containing $n$ vertices of degree $\Delta$. Thus, by Lemma 2.4, we have $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G}) \geq \frac{|S|+\Delta-1}{\Delta} = \frac{2^\Delta+\Delta-1}{\Delta}$. The claim can now be established by noticing that all these arguments apply whatever value of $\Delta \geq 2$ we plug into the construction. □

A consequence of Theorem 3.1 and of an observation to be stated later on (Observation 6.1), is that there exist graphs $G$ for which the possible orientations $\overrightarrow{G}$ are very diverse with respect to $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$. In particular, there exist graphs $G$ that have both orientations $\overrightarrow{G}$ with $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G}) = 1$, and orientations $\overrightarrow{G'}$ with $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G'})$ being arbitrarily large. In other words, looking at the underlying graph $G$ is, in general, far from being a good method to determine how large will $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$ be, for an oriented graph $\overrightarrow{G}$.

Another consequence of the construction given to prove Theorem 3.1 (or, rather, of slight modifications of that construction), is that there exist oriented graphs $\overrightarrow{G}$ for which $\chi_o(\overrightarrow{G}) - \chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$ is arbitrarily large. This shows that an upper bound to be given in later Corollary 3.5 can be arbitrarily bad in general. Note that this can be proved through more straight examples; we come back to this after the observation.

**Observation 3.2.** *For an oriented graph $\overrightarrow{G}$, the difference $\chi_o(\overrightarrow{G}) - \chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$ can be arbitrarily large.*

*Proof.* We get the result by modifying further the construction described in the proof of Theorem 3.1. Namely, let $\overrightarrow{G}$ be obtained from the same construction, with the exception that, when adding pendant vertices to make $\overrightarrow{G}$ locally irregular, we add more degree-1 vertices to achieve certain degrees for the vertices. Precisely, we do it in the following way. Recall that the $v_i$'s have degree $\Delta$, and set $\alpha = \lceil \frac{2^\Delta+\Delta-1}{\Delta} \rceil$. Denote by $X = \{x_1, \ldots, x_p\}$ the set of vertices of $\overrightarrow{G}$ that have degree more than 1 and are not some of the $v_i$'s. That is, $X$ contains the vertices we added to make sure every two of the $v_i$'s are joined by a directed 2-path. Now, start by adding degree-1 vertices adjacent to $x_1$ so that its degree becomes $\alpha\Delta + 1$. Now, for every $i = 2, \ldots, p$ in turn such that we have treated $x_{i-1}$, add degree-1 vertices adjacent to $x_i$ so that its degree becomes $\alpha d(x_{i-1}) + 1$. These modifications imply that, by any $\alpha'$-labelling of $\overrightarrow{G}$ with $\alpha' \geq \alpha$, any two of the $x_i$'s cannot have the same sum, and, similarly, the $v_i$'s and $x_i$'s cannot have the same sum.

Since any two of the $v_i$'s are joined by a directed 2-path, and there are $n$ such vertices, we have $\chi_o(\overrightarrow{G}) \geq n$. Now, for the same reasons, and because the $v_i$'s have degree $\Delta = \log_2 n$, for any oriented $k$-labelling of $\overrightarrow{G}$, the value of $k$ must guarantee that $|\{\Delta, \ldots, k\Delta\}| = (k-1)\Delta + 1 \geq n = 2^\Delta$, and thus we must have $k \geq \alpha$. It is easy to see, now, that due to the structure of $\overrightarrow{G}$, and, in particular, because the degrees of some of its vertices are so different, there exist oriented labellings of $\overrightarrow{G}$ assigning labels from $\{1, \ldots, \alpha\}$. For instance, to obtain such a labelling, one can first assign labels arbitrarily to the arcs incident to the $v_i$'s so that their sums are pairwise different (e.g. so that their sums form the set $\{\Delta, \ldots, \Delta + n - 1\}$), and then assign label 1 to all other arcs. Thus, we have $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G}) \leq \alpha$, and the result follows since, as $n$ grows, the difference $n - \frac{n+\log_2 n-1}{\log_2 n}$ gets larger and larger. □

As mentioned earlier, through other peculiar, simpler oriented graphs, we can reach the same conclusion as in Observation 3.2. Namely, consider any tournament $\overrightarrow{T}$ on $n$ vertices. Clearly, $\chi_o(\overrightarrow{T}) = \chi(T) = n$. Also, we have $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{T}) \leq \chi_\Sigma(T)$, while $\chi_\Sigma(T) = 3$ (see e.g. [8]).

*3.2.* $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$ *versus* $\chi_o(\overrightarrow{G})$ *(and* $\Delta(G)$*)*

Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be two oriented graphs, where $V(\overrightarrow{H}) \subset \mathbb{N}$. Let also $\ell$ be a labelling of $\overrightarrow{G}$. For an integer $k \geq 1$ and any vertex $v$ of $\overrightarrow{G}$, we define $\sigma_\ell^{[k]}(v)$ (or simply $\sigma^{[k]}(v)$) as the *sum* $\sigma(v)$ of $v$ *modulo* $k$ by $\ell$, that is, $\sigma_\ell^{[k]}(v) = \sigma_\ell(v) \bmod k$. We now say that $\ell$ is an $\overrightarrow{H}$*-oriented labelling modulo* $k$ if the resulting colouring function $\sigma_\ell^{[k]}$ forms a homomorphism of $\overrightarrow{G}$ to $\overrightarrow{H}$. Under that assumption, note that, for any two vertices $u$ and $v$ of $\overrightarrow{G}$, we have $\sigma(u) \neq \sigma(v)$ whenever $\sigma^{[k]}(u) \neq \sigma^{[k]}(v)$. The main point behind these notions is that, assuming some oriented graph $\overrightarrow{G}$ is known to admit a homomorphism $\phi$ to some oriented graph $\overrightarrow{H}$ on $k$ vertices, it is possible to build upon $\phi$ to design labellings that yield the same colours/sums as $\phi$ modulo $k$, provided we consider labellings assigning sufficiently large labels. Such techniques actually appeared in previous works, such as [2, 5, 8, 14, 16, 17, 24]; we recall them in what follows, as they will be useful in the next sections.

Let $\overrightarrow{G}$ be an oriented graph, given together with a $k$-labelling $\ell$. Consider $\overrightarrow{P} = v_1 \dots v_p$, an oriented walk[2] of $\overrightarrow{G}$. Let us insist on the fact that $\overrightarrow{P}$ is not assumed to be directed, and that its $p-1$ arcs can be in any direction. By *+-switching* $\overrightarrow{P}$ *modulo* $k$, we mean adding 1 to the label assigned by $\ell$ to the first arc of $\overrightarrow{P}$ (*i.e.*, joining $v_1$ and $v_2$), subtracting 1 to the second arc of $\overrightarrow{P}$, adding 1 to the third arc, subtracting 1 to the fourth one, and so on, where all operations over the labels are modulo $k$ (where, for simplicity, we assume that adding 1 to label $k$ cycles to label 1, and the other way round, subtracting 1 to label 1 cycles to label $k$). Analogously, we define *−-switching* $\overrightarrow{P}$ *modulo* $k$ as the similar procedure, but starting with subtracting 1 to the label of the first arc, before alternating between adding and subtracting 1 modulo $k$. In most contexts, for convenience, after +-switching and −-switching oriented walks in $\overrightarrow{G}$, we still denote by $\ell$ the resulting $k$-labelling.

As observed in previous works, switching walks has convenient properties when it comes to building labellings resulting in desired sums/colours modulo some $k$.

**Observation 3.3.** *Let* $\overrightarrow{G}$ *be an oriented graph, and* $\ell$ *be a* $k$*-labelling of* $\overrightarrow{G}$*. By +-switching or −-switching any oriented walk* $v_1 \dots v_p$ *of* $\overrightarrow{G}$*, the sums of* $v_2, \dots, v_{p-1}$ *by* $\ell$ *are not altered modulo* $k$*. Meanwhile, the sum of* $v_1$ *is altered by* $+1$ *(when +-switching) or* $-1$ *(otherwise) modulo* $k$*, while, depending on the parity of* $p$*, the sum of* $v_p$ *can be altered by* $+1$ *or* $-1$*.*

From Observation 3.3, it can be proved that, in some contexts, a given (not necessarily proper) $k$-colouring $\phi$ with colour classes $V_0, \dots, V_{k-1}$ of some graph can serve as a "layout" to design a labelling where the resulting sums/colours match $\phi$ modulo $k$. Again, this is a classical approach in the field; for more details, we would advise the reader to refer e.g. to [4] for comprehensive proofs.

**Theorem 3.4** (see e.g. Theorems 3.7 to 3.9 of [4])**.** *If* $\overrightarrow{G}$ *is an oriented graph, then:*

- $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G}) \leq \chi_o(\overrightarrow{G})$ *if* $\overrightarrow{G}$ *is not bipartite and* $\chi_o(\overrightarrow{G}) \not\equiv 2 \bmod 4$*; and*
- $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G}) \leq \chi_o(\overrightarrow{G}) + 1$ *otherwise.*

It is worth mentioning that some of the bounds in Theorem 3.4 cannot be improved upon in general, particularly those about oriented bipartite graphs. As an illustration, recall that some bipartite graphs $G$ verify $\chi_\Sigma(G) = 3 = \chi(G) + 1$. It is worth recalling also that, as we proved previously in Observation 3.2, there are oriented graphs $\overrightarrow{G}$ with $\chi_o(\overrightarrow{G}) - \chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$ arbitrarily large; thus, the bounds in Theorem 3.4 are not always good.

[2]Recall that a *walk* in a graph is a path in which vertices and edges can repeat.

Although Theorem 3.4 provides a general upper bound on $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$ for every oriented graph $\overrightarrow{G}$, this might not feel quite satisfactory, as the oriented chromatic number is a parameter that is not that easy to get a good intuition on. Fortunately, known upper bounds on the oriented chromatic number exist, some of which involve more classical graph parameters, such as the maximum degree. In the following, we summarise the such bounds that seem the most interesting to us (but others could also be derived; see e.g. [23]).

**Corollary 3.5.** *For an oriented graph $\overrightarrow{G}$:*

- *if $\Delta(\overrightarrow{G}) \leq 2$, then $\chi_o(\overrightarrow{G}) \leq 5$ (see e.g. [23]); thus, $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G}) \leq 6$;*
- *if $\Delta(\overrightarrow{G}) \leq 3$, then $\chi_o(\overrightarrow{G}) \leq 8$ (see [10]); thus, $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G}) \leq 9$;*
- *if $\Delta(\overrightarrow{G}) = \Delta$, then $\chi_o(\overrightarrow{G}) \leq 2\Delta^2 2^\Delta$ (see [15]); thus, $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G}) \leq 2\Delta^2 2^\Delta + 1$.*

## 4. Oriented graphs with maximum degree 2

In this section, we deal with oriented graphs with maximum degree 2, *i.e.*, oriented paths and oriented cycles. We start off by considering oriented paths $\overrightarrow{P}$, for which we always have $\chi_o(\overrightarrow{P}) \leq 3$ (this upper bound holding more generally for oriented trees, see [23]). Thus, by Theorem 3.4, we deduce that we have $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{P}) \leq 4$. We reduce this upper bound to 3, which is best possible (consider any directed path of length at least 5).

**Theorem 4.1.** *If $\overrightarrow{P}$ is an oriented path, then $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{P}) \leq 3$.*

*Proof.* Let us denote by $\overrightarrow{H_1}$ and $\overrightarrow{H_2}$ the 3-vertex tournaments with vertex set $V(\overrightarrow{H_1}) = V(\overrightarrow{H_2}) = \{0,1,2\}$, and arc sets $A(\overrightarrow{H_1}) = \{\overrightarrow{01}, \overrightarrow{12}, \overrightarrow{20}\}$ and $A(\overrightarrow{H_2}) = \{\overrightarrow{02}, \overrightarrow{21}, \overrightarrow{10}\}$. It can be noted that the following, which actually holds more generally for all oriented graphs with maximum degree 2, is true:

**Claim 4.2.** *Let $\ell_1$ be any 3-labelling of $\overrightarrow{P}$, and let $\ell_2$ be the 3-labelling obtained from $\ell_1$ by turning all assigned 1's into 2's, and vice versa. Then, for any vertex $v$ of $\overrightarrow{P}$:*

- *if $\sigma^{[3]}_{\ell_1}(v) = 1$, then $\sigma^{[3]}_{\ell_2}(v) = 2$;*
- *if $\sigma^{[3]}_{\ell_1}(v) = 2$, then $\sigma^{[3]}_{\ell_2}(v) = 1$;*
- *if $\sigma^{[3]}_{\ell_1}(v) = 0$, then $\sigma^{[3]}_{\ell_2}(v) = 0$.*

*Particularly, if $\ell_1$ is an $\overrightarrow{H_1}$-oriented labelling of $\overrightarrow{P}$ modulo 3, then $\ell_2$ is an $\overrightarrow{H_2}$-oriented labelling of $\overrightarrow{P}$ modulo 3, and vice versa.*

*Proof of the claim.* This can be established by considering the possible values that can be assigned to the at most two arcs incident to $v$, and comparing $\sigma^{[3]}_{\ell_1}(v)$ and $\sigma^{[3]}_{\ell_2}(v)$. ⋄

Back to the proof of Theorem 4.1, let us denote by $v_1, \ldots, v_p$ the consecutive vertices of $\overrightarrow{P}$ from any one end-vertex to the other, and by $\overrightarrow{a_1}, \ldots, \overrightarrow{a_{p-1}}$ its arcs, where $\overrightarrow{a_i}$ joins $v_i$ and $v_{i+1}$ for every $i \in \{1, \ldots, p-1\}$. Let also $\phi$ be an $\overrightarrow{H_1}$-colouring of $\overrightarrow{P}$ (which clearly exists). Free to permute colours, we may suppose that $\phi(v_1) = 2$. We start by assigning label 2 to $\overrightarrow{a_1}$, and then a label in $\{1,2,3\}$ to the arcs $\overrightarrow{a_2}, \ldots, \overrightarrow{a_{p-1}}$ one by one in turn, starting with $\overrightarrow{a_2}$, so that, for every $i \in \{1, \ldots, p-1\}$, we have $\sigma^{[3]}(v_i) = \phi(v_i)$. Note that this is possible, since we are assigning labels in $\{1,2,3\}$. In what follows, we also require that $\ell(\overrightarrow{a_{p-1}}) \neq 2$. To guarantee this, note that we can, in the process above, consider $\overrightarrow{H_2}$ instead of $\overrightarrow{H_1}$;

when leading the process with still starting from $\overrightarrow{a_1}$ being assigned label 2 (and requiring $\phi(v_1) = 2$), we get that $\overrightarrow{a_{p-1}}$ cannot be assigned label 2 as well this time (note that this is indeed sort of similar to switching the oriented path $v_2 \dots v_p$ modulo 3). Below, we denote by $\overrightarrow{H}$ the one of $\overrightarrow{H_1}$ and $\overrightarrow{H_2}$ we eventually led the process with, to guarantee this property.

The only reason why the resulting labelling, $\ell$, might be not oriented, is because of $v_p$, since we have no control over its sum, which is exactly $\ell(\overrightarrow{a_{p-1}})$. Recall that $\ell(\overrightarrow{a_{p-1}}) \neq 2$.

- If $\ell(\overrightarrow{a_{p-1}}) = 1$, then, regardless of $\phi(v_p)$, note that $\ell$ is oriented, since $v_p$ is the only vertex of $\overrightarrow{P}$ with sum 1.
- If $\ell(\overrightarrow{a_{p-1}}) = 3$ and $\ell(\overrightarrow{a_{p-2}}) = 3$, then we have $\sigma(v_{p-1}) = 6 > 3 = \sigma(v_p)$, while $v_{p-1}$ and $v_p$ are the only adjacent vertices with the same sum modulo 3. So, $\ell$ is oriented.
- If $\ell(\overrightarrow{a_{p-1}}) = 3$ but $\ell(\overrightarrow{a_{p-2}}) \in \{1, 2\}$, then $\phi(v_{p-1}) \in \{1, 2\}$. If the direction of the arc joining $\sigma^{[3]}(v_{p-1})$ and $\sigma^{[3]}(v_p) = 0$ of $\overrightarrow{H}$ matches that of $\overrightarrow{a_{p-1}}$, then $\ell$ is actually an $\overrightarrow{H}$-oriented 3-labelling of $\overrightarrow{P}$ modulo 3. Otherwise, we swap 1's and 2's by $\ell$, resulting in another 3-labelling $\ell'$ of $\overrightarrow{P}$. As mentioned in Claim 4.2, $\ell'$ is mostly an $\overrightarrow{H'}$-oriented 3-labelling of $\overrightarrow{P}$ modulo 3, where $\overrightarrow{H'}$ is the tournament in $\{\overrightarrow{H_1}, \overrightarrow{H_2}\} \setminus \{\overrightarrow{H}\}$. Note that the sum of $v_p$ did not change modulo 3, and, now, the direction of the arc joining $\sigma^{[3]}_{\ell'}(v_{p-1})$ and $\sigma^{[3]}_{\ell'}(v_p) = 0$ of $\overrightarrow{H'}$ matches that of $\overrightarrow{a_{p-1}}$. From this, $\ell'$ is oriented. □

We now turn our attention to oriented cycles, for which the oriented chromatic number is well understood. Indeed, it is known, see [23], that if $\overrightarrow{C}$ is an oriented cycle, then $\chi_o(\overrightarrow{C}) \in \{2, 3, 4, 5\}$. More precisely, $\chi_o(\overrightarrow{C}) = 2$ if and only if $\overrightarrow{C}$ contains sources and sinks only (what we call an *alternating oriented cycle* below), while $\chi_o(\overrightarrow{C}) = 5$ if and only if $\overrightarrow{C}$ is the directed 5-cycle. Meanwhile, there is a characterisation, involving, with respect to a virtual orientation, the number of forward arcs and backward arcs, of when $\chi_o(\overrightarrow{C}) = 3$ and when $\chi_o(\overrightarrow{C}) = 4$. For both values, there exist infinitely many oriented cycles with that oriented chromatic number.

As a starting point, let us wonder about the maximum value of $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{C})$ for an oriented cycle $\overrightarrow{C}$. Since we always have $\chi_o(\overrightarrow{C}) \leq 5$, by Theorem 3.4 we always have $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{C}) \leq 6$ (and even $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{C}) \leq 5$ since the directed 5-cycle is not bipartite). Regarding the directed 5-cycle $\overrightarrow{C}$, it is actually easy to see that $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{C}) \leq 3$.

**Observation 4.3.** *If $\overrightarrow{C}$ is the directed 5-cycle, then $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{C}) \leq 3$.*

*Proof.* Consider the labelling of $\overrightarrow{C}$ assigning labels $1, 1, 2, 3, 3$ to the consecutive arcs. □

Regarding alternating oriented cycles $\overrightarrow{C}$, we have:

**Theorem 4.4.** *If $\overrightarrow{C}$ is an alternating oriented cycle, then $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{C}) \leq 3$.*

*Proof.* Since $\overrightarrow{C}$ has adjacent vertices with the same degree, $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{C}) > 1$. So our goal, now, is to design an oriented 3-labelling $\ell$ of $\overrightarrow{C}$. Because $\overrightarrow{C}$ has sources and sinks only, $\overrightarrow{C}$ is bipartite with bipartition $U \cup V$, and all arcs being oriented, say, from $U$ to $V$ (thus all sources lie in $U$ while all sinks lie in $V$). In what follows, we actually design a 2-labelling that is almost oriented, and we eventually introduce a few 3's to get rid of conflicts, if any.

So, again, for now, we are building a 2-labelling of $\overrightarrow{C}$. We start by assigning label 2 to all arcs of $\overrightarrow{C}$. This way, note that, currently, $\sigma^{[2]}(v) = 0$ for every vertex $v$. Now:

- If, say, $|U|$ is even, then by repeatedly considering an oriented path $\overrightarrow{P} = u \dots v$ joining two vertices $u$ and $v$ of $U$ with $\sigma^{[2]}(u) = \sigma^{[2]}(v) = 0$, and +-switching $\overrightarrow{P}$ modulo 2, by Observation 3.3 we get to a situation where $\sigma^{[2]}(v) = 0$ for all $v \in V$, while $\sigma^{[2]}(v) = 1$ for all $v \in U$. Since all arcs of $\overrightarrow{C}$ go from $U$ to $V$, the resulting $\ell$ is oriented.

- Otherwise, both $|U|$ and $|V|$ are odd. Let $u^*$ be any vertex of $U$. By the same arguments as in the previous case, by switching oriented paths modulo 2 we can get to a situation where $\sigma^{[2]}(v) = 1$ for all $v \in U \setminus \{u^*\}$ and $\sigma^{[2]}(v) = 0$ for all $v \in V \cup \{u^*\}$. Since $\sigma^{[2]}(u^*) = 0$, both arcs incident to $u^*$ are assigned the same label (1 or 2) by $\ell$. If this label is 1, then we are done when changing to 3 the label assigned to the two arcs incident to $u^*$, since the two neighbours of $u^*$ remain of even sum, while $u^*$ becomes the only vertex of $\overrightarrow{C}$ with sum 6. Now, if this label is 2, then we swap all 1's assigned by $\ell$ into 2's, and *vice versa*. Note that this preserves the parity of all sums. However, we now fall back into the previous case where both arcs incident to $u^*$ are assigned label 1, and we can deal with it the exact same way. In all cases, here we thus end up with an oriented 3-labelling of $\overrightarrow{C}$. □

In the peculiar case of general directed cycles, we have the following:

**Theorem 4.5.** *If $\overrightarrow{C}$ is a directed cycle, then $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{C}) \leq 3$.*

*Proof.* We obtain an oriented 3-labelling $\ell$ of $\overrightarrow{C}$ as follows:

- If $k \equiv 0 \bmod 3$, then we assign labels $1, 2, 3, 1, 2, 3, \dots, 1, 2, 3$ to the consecutive arcs of $\overrightarrow{C}$. As a result, note that $\ell$ is an $\overrightarrow{H}$-oriented 3-labelling of $\overrightarrow{C}$, for $\overrightarrow{H}$ being the 3-vertex tournament with vertex set $\{3, 4, 5\}$ and arc set $\{\overrightarrow{35}, \overrightarrow{54}, \overrightarrow{43}\}$.

- If $k \equiv 1 \bmod 3$, the we assign labels $3, 1, 2, 3, 1, 2, 3, \dots, 1, 2, 3$ to the consecutive arcs of $\overrightarrow{C}$. Here, note that $\ell$ is an $\overrightarrow{H}$-oriented 3-labelling of $\overrightarrow{C}$, for $\overrightarrow{H}$ being the 4-vertex oriented graph with vertex set $\{3, 4, 5, 6\}$ and arc set $\{\overrightarrow{35}, \overrightarrow{54}, \overrightarrow{43}, \overrightarrow{56}, \overrightarrow{64}\}$.

- If $k \equiv 2 \bmod 3$, the we assign labels $3, 1, 1, 2, 3, 1, 2, 3, \dots, 1, 2, 3$ to the consecutive arcs of $\overrightarrow{C}$. Here, $\ell$ is an $\overrightarrow{H}$-oriented 3-labelling of $\overrightarrow{C}$, for $\overrightarrow{H}$ being the 5-vertex oriented graph with vertex set $\{2, 3, 4, 5, 6\}$ and arc set $\{\overrightarrow{35}, \overrightarrow{54}, \overrightarrow{43}, \overrightarrow{56}, \overrightarrow{64}, \overrightarrow{23}, \overrightarrow{42}\}$. □

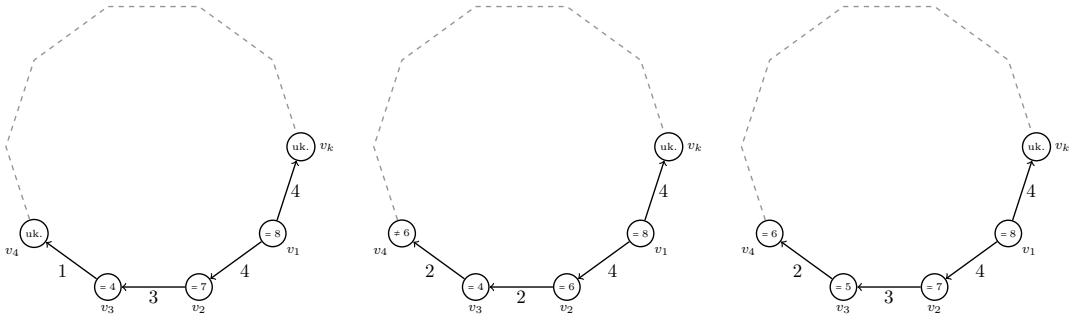
Before going on, let us go beyond the application of Theorem 3.4 mentioned earlier, to establish the following general bound.

**Theorem 4.6.** *If $\overrightarrow{C}$ is an oriented cycle, then $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{C}) \leq 4$.*

*Proof.* We denote by $v_1, \dots, v_k$ the consecutive vertices of $\overrightarrow{C}$, and by $\overrightarrow{a_1}, \dots, \overrightarrow{a_k}$ its arcs (where $\overrightarrow{a_i}$ joins $v_i$ and $v_{i+1}$ for every $i \in \{1, \dots, k-1\}$, and $\overrightarrow{a_k}$ joins $v_k$ and $v_1$).

By Theorem 3.4 and Observation 4.3, we can focus on cases where $\chi_o(\overrightarrow{C}) = 4$. This means $\overrightarrow{C}$ is neither an alternating oriented cycle, nor the directed 5-cycle. Also, by Theorem 4.5, $\overrightarrow{C}$ is not a directed cycle. By the characterisation of oriented cycles with oriented chromatic number 4 (see [23]), we also know that there must be three consecutive arcs oriented the same direction in $\overrightarrow{C}$. Due to all these arguments, w.l.o.g. we may suppose $\overrightarrow{a_k} = \overrightarrow{v_1 v_k}$, $\overrightarrow{a_1} = \overrightarrow{v_1 v_2}$, $\overrightarrow{a_2} = \overrightarrow{v_2 v_3}$, and $\overrightarrow{a_3} = \overrightarrow{v_3 v_4}$.

Let us consider the following 4-labelling of $\overrightarrow{C}$. We first set $\ell(\overrightarrow{a_k}) = 4$. From here, we then assign a label in $\{1, 2, 3, 4\}$ to the arcs $\overrightarrow{a_{k-1}}, \dots, \overrightarrow{a_3}$ one by one, following this order, so that the resulting colours for $v_k, \dots, v_4$, modulo 4, form a homomorphism to $\overrightarrow{H}$, the 3-vertex tournament with vertex set $\{1, 2, 3\}$ and arc set $\{\overrightarrow{12}, \overrightarrow{23}, \overrightarrow{31}\}$. This is possible,




(a) $\ell(\overrightarrow{a_4}) = 1$ (b) $\ell(\overrightarrow{a_4}) = 2$, $\sigma(v_4) \neq 6$ (c) $\ell(\overrightarrow{a_4}) = 2$, $\sigma(v_4) = 6$

Figure 3: Cases considered in the proof of Theorem 4.6. Numbers in vertices indicate the sums obtained by the given labelling. "uk." means that a sum can basically be anything, and is not relevant here.

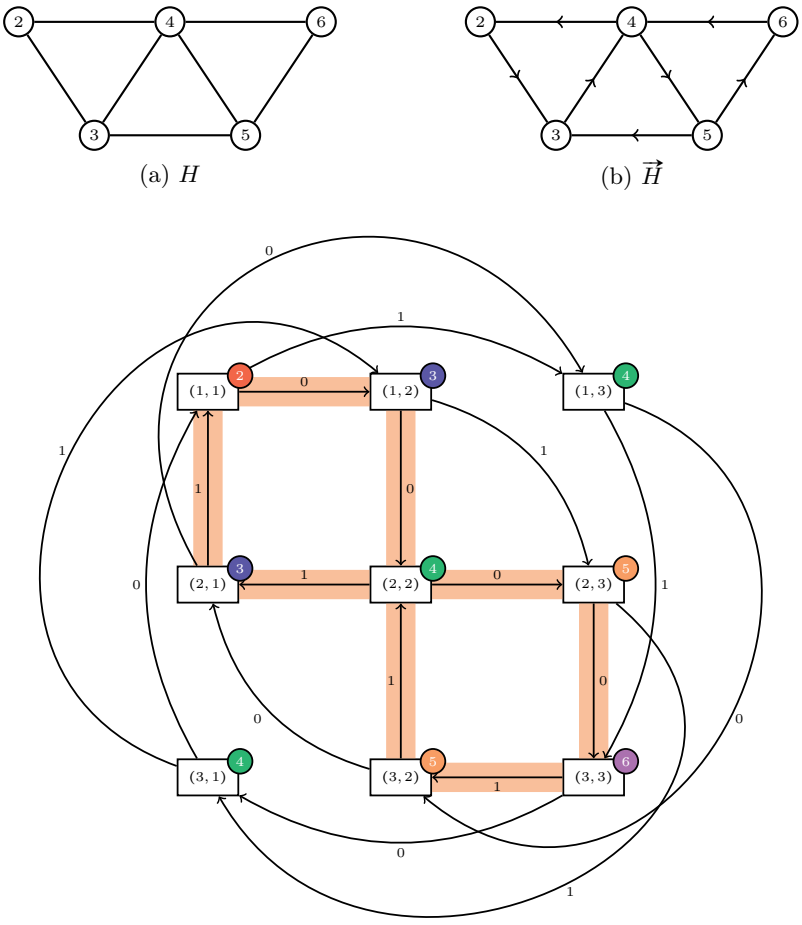
since we are using labels in $\{1, \ldots, 4\}$. Actually, not only this is possible, but also there are actually three possible ways to do it, depending only on the label we assign to $\overrightarrow{a_{k-1}}$ (in other words, this is because $\sigma^{[4]}(v_k)$ can take any of the three values in $\{1, 2, 3\}$). Among these three possible ways to label $\overrightarrow{a_{k-1}}, \ldots, \overrightarrow{a_3}$, at least two of these must assign a label different from 4 to $\overrightarrow{a_3}$, thus a label in $\{1, 2, 3\}$, and thus at least one of them must assign a label in $\{1, 2\}$ to $\overrightarrow{a_3}$. For our $\ell$, we fix one such way to label $\overrightarrow{a_{k-1}}, \ldots, \overrightarrow{a_3}$.

We now achieve the labelling (assigning a label to $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$) as follows (see Figure 3).

- If $\ell(\overrightarrow{a_3}) = 1$, then $\sigma(v_4) \leq 5$. We here set $\ell(\overrightarrow{a_1}) = 4$ and $\ell(\overrightarrow{a_2}) = 3$. This way, we get $\sigma(v_1) = 8$, $\sigma(v_2) = 7$, and $\sigma(v_3) = 4$. Note that only $v_3$ has sum 4 while its neighbours have different sums. Similarly, $v_1$ is the only vertex with sum 8, while its two incident arcs are outgoing arcs. From these arguments, the resulting $\ell$ is oriented.
- If $\ell(\overrightarrow{a_3}) = 2$, then we consider two cases.
  - If $\sigma(v_4) \neq 6$, then we set $\ell(\overrightarrow{a_1}) = 4$ and $\ell(\overrightarrow{a_2}) = 2$. As a result, $\sigma(v_3) = 4$ (and only $v_3$ has this sum), $\sigma(v_1) = 8$ (and only $v_1$ has sum 8), and $\sigma(v_2) = 6$. For similar reasons as earlier, and because $\sigma(v_4) \neq 6$, the resulting labelling $\ell$ is oriented.
  - Otherwise, $\sigma(v_4) = 6$. We set $\ell(\overrightarrow{a_1}) = 4$ and $\ell(\overrightarrow{a_2}) = 3$. This way, $\sigma(v_3) = 5$, $\sigma(v_1) = 8$, and $\sigma(v_2) = 7$. Again, $v_1$ is the only vertex with sum 8 while its two incident arcs are outgoing. Note also that $\sigma^{[4]}(v_2) = 3$, $\sigma^{[4]}(v_3) = 1$, and $\sigma^{[4]}(v_4) = 2$. Thus, the resulting sums of $v_2$, $v_3$, and $v_4$ actually match the homomorphism to $\overrightarrow{H}$ modulo 4. So, again, $\ell$ is oriented. □

At this point, the main question is whether the bound in Theorem 4.6 can be decreased to 3 for all oriented cycles or not. Note that this might sound plausible, as, by a 3-labelling, we can generate five sums for a degree-2 vertex, namely those in $\{2, 3, 4, 5, 6\}$, while oriented cycles, in general, have oriented chromatic number at most 4. In what follows, we answer that question. Before that, we first need to introduce a few more concepts, which we think are of independent interest, as they notably borrow concepts from automata theory.

To start with, throughout what follows, we need to describe oriented cycles through binary words, which we do formally as follows. Let $\overrightarrow{C}$ be an oriented cycle with vertices $v_0, \ldots, v_{k-1}$ and arcs $\overrightarrow{a_0}, \ldots, \overrightarrow{a_{k-1}}$, where $\overrightarrow{a_i}$ joins $v_i$ and $v_{i+1}$ for every $i \in \{0, \ldots, k-1\}$ (again, indices are modulo $k$). We define $\mathrm{bin}(\overrightarrow{C}) = \alpha_0, \ldots, \alpha_{k-1}$ as the word on alphabet $\{0, 1\}$, where, for every $i \in \{0, \ldots, k-1\}$, we have $\alpha_i = 0$ if $\overrightarrow{a_i} = \overrightarrow{v_i v_{i+1}}$, and $\alpha_i = 1$ otherwise, if $\overrightarrow{a_i} = \overrightarrow{v_{i+1} v_i}$. Generally speaking, $\mathrm{bin}(\overrightarrow{C})$ is actually not unique, as $\overrightarrow{C}$ can be described by




(a) $H$ (b) $\overrightarrow{H}$

(c) $T(\overrightarrow{H}, 3)$

Figure 4: Illustration of the concepts introduced in Section 4. (a) shows a graph $H$, while (b) shows one of its orientations $\overrightarrow{H}$. (c) depicts the oriented transition graph $T(\overrightarrow{H}, 3)$. In (c), coloured numbers indicate the sum $x + y$ for all vertices $(x, y)$ of $T(\overrightarrow{H}, 3)$. Highlighted in orange, is given a directed closed walk showing that any oriented cycle with binary word 00001111 admits an $\overrightarrow{H}$-oriented 3-labelling.

several binary words, depending on which vertices we define as $v_1$ and $v_2$. As an example, note that the oriented cycle with vertex set $\{v_0, v_1, v_2\}$ and arc set $\{\overrightarrow{v_0v_1}, \overrightarrow{v_1v_2}, \overrightarrow{v_0v_2}\}$ could be described by any of the six binary words 001, 010, 100, 110, 101, and 011. Thus, $\text{bin}(\overrightarrow{C})$ should rather be perceived as one possible representation of $\overrightarrow{C}$, equivalent to others through swapping its 0's and 1's, and translating its letters.

Now let $\overrightarrow{H}$ be an oriented graph, such that $\overrightarrow{C}$ admits an $\overrightarrow{H}$-oriented 3-labelling $\ell$. Note that $\overrightarrow{H}$ must be an orientation of (a subgraph of) $H$, the graph depicted in Figure 4(a). Indeed, for a degree-2 vertex to have sum 2, its two incident arcs must be assigned label 1, labels 1 and 2 to have sum 3, labels 1 and 3 or label 2 to have sum 4, labels 2 and 3 to have sum 5, and label 3 to have sum 6. In particular, no two adjacent vertices of $\overrightarrow{C}$ can have sum 2 and 5, 2 and 6, or 3 and 6.

Let now $v_i$ and $v_{i+1}$ be two adjacent vertices of $\overrightarrow{C}$, being consecutive w.r.t. the ordering of the vertices of $\overrightarrow{C}$. Assume $\ell(\overrightarrow{a_{i-1}}) = \alpha$, $\ell(\overrightarrow{a_i}) = \beta$, and $\ell(\overrightarrow{a_{i+1}}) = \gamma$. Note that, from a more local point of view, taking the ordering into account, this configuration can be described as having a pair $(l_1, l_2)$ of two labels around $v_i$ and a pair $(l_3, l_4)$ of two labels around $v_{i+1}$, verifying a number of properties, such as having $l_2 = l_3 = \beta$, and, for $\ell$ to be $\overrightarrow{H}$-

oriented, that, in $\overrightarrow{H}$, the direction of the arc joining $l_1 + l_2$ (which is $\sigma(v_i)$) and $l_3 + l_4$ (being $\sigma(v_{i+1})$) should match that of $\overrightarrow{a_i}$ in $\overrightarrow{C}$. Since this is supposed to hold for every two consecutive vertices of $\overrightarrow{C}$, from these ideas we come up with a notion of possible pairs of labels assigned by $\ell$ that can be consecutive along $\overrightarrow{C}$ or not.

We capture this under the following terminology (see Figure 4 for an illustration). Set $k \geq 1$. W.r.t. $\overrightarrow{H}$, we define the *oriented transition graph* $T(\overrightarrow{H}, k)$ as follows. For every pair $(l, l')$ of label, where $l, l' \in \{1, \ldots, k\}$, we have a vertex $(l, l')$ in $T(\overrightarrow{H}, k)$. For any two vertices $(x, x')$ and $(y, y')$ in $T(\overrightarrow{H}, k)$, we add an arc from $(x, x')$ to $(y, y')$ if $x' = y$ and $\overrightarrow{H}$ has an arc joining $x + x'$ and $y + y'$ (in any direction). Now, assuming $T(\overrightarrow{H}, k)$ has an arc joining $(x, y)$ and $(y, z)$, we label that arc with either 0 or 1 in the following way: if $\overrightarrow{H}$ contains an arc from $x+y$ to $y+z$, then we label the arc of $T(\overrightarrow{H}, k)$ with 0, while, otherwise, $\overrightarrow{H}$ contains an arc from $y + z$ to $x + y$, then we label the arc of $T(\overrightarrow{H}, k)$ with 1. In other words, the label assigned to the arc joining $(x, y)$ and $(y, z)$ indicates the direction of the corresponding arc (joining $x + y$ and $y + z$) in $\overrightarrow{H}$, being either forward (0) or backward (1).

Now, for any oriented cycle $\overrightarrow{C}$, it is not too complicated to see that finding an $\overrightarrow{H}$-oriented 3-labelling of $\overrightarrow{C}$ is equivalent to finding a directed closed walk of $T(\overrightarrow{H}, 3)$ with the same binary word (when reading the successive labels of the arcs traversed as going along) as a binary word $\mathrm{bin}(\overrightarrow{C})$ describing $\overrightarrow{C}$. To make it more clear, consider the example of an oriented cycle $\overrightarrow{C}$ with $\mathrm{bin}(\overrightarrow{C}) = 00001111$. As depicted in Figure 4(c), for $\overrightarrow{H}$ being the oriented graph in Figure 4(b), $T(\overrightarrow{H}, 3)$ contains a directed closed walk whose associated binary word is exactly 00001111. Looking at the consecutive labels along this directed closed walk, we deduce that there is a way to assign consecutive labels $1, 2, 2, 3, 3, 2, 2, 1$ to the arcs of $\overrightarrow{C}$ so that an $\overrightarrow{H}$-oriented 3-labelling results. Particularly, this will generate consecutive sums $2, 3, 4, 5, 6, 5, 4, 3$ to the vertices, with the arcs joining these vertices being directed so that, indeed, a homomophism from $\overrightarrow{C}$ to $\overrightarrow{H}$ is obtained.

Another nice aspect of this method, is that directed closed walks of oriented transition graphs can be combined to yield larger ones, and thus oriented labellings of more oriented cycles. For instance, regarding the previous example, the fact that $T(\overrightarrow{H}, 3)$ has a directed closed walk with binary word 00001111 actually means that any oriented cycle with binary word $(00001111)^*$ (*i.e.*, any binary word resulting from arbitrarily many concatenations of 00001111) also admits an $\overrightarrow{H}$-oriented 3-labelling.

The main interest behind this method, is that it yields another way to determine whether a given oriented cycle $\overrightarrow{C}$ admits an oriented 3-labelling: one can just consider all orientations $\overrightarrow{H}$ of the graph $H$ in Figure 4(a), and determine whether there exists, in $T(\overrightarrow{H}, 3)$, an oriented closed walk whose associated binary word also describes $\overrightarrow{C}$. Given that the number of non-isomorphic orientations of $H$ is very limited (64), this all yields an approach that is very convenient to implement through computer programs to try to come up with examples of (small) oriented cycles $\overrightarrow{C}$ with $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{C}) > 3$.

Particularly, through this method, we were able to observe that there exist many oriented cycles on at least 14 vertices that do not admit any oriented 3-labelling at all. It is worth mentioning that this is true both for oriented cycles with oriented chromatic number 3 (an example can be constructed from the binary word 01111000110100) and oriented chromatic number 4 (an example of binary word is 11011100011101011000111). This implies that Theorem 4.6 is actually best possible in general.

**Theorem 4.7.** *There exist oriented cycles $\overrightarrow{C}$ with $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{C}) = 4$.*

## 5. Oriented trees

It is known that, for every oriented tree $\overrightarrow{T}$, we have $\chi_o(\overrightarrow{T}) \leq 3$ (see [23]). Thus, from Theorem 3.4, we deduce that $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{T}) \leq 4$ holds for every oriented tree $\overrightarrow{T}$. In the next result, we improve this general upper bound down to 3 for these oriented graphs. Again, this is best possible (consider any oriented tree containing a directed 5-path having its inner vertices being of degree 2).

**Theorem 5.1.** *If $\overrightarrow{T}$ is an oriented tree, then $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{T}) \leq 3$.*

*Proof.* Let us choose any vertex $r$ of $\overrightarrow{T}$ as its root, which, in the usual way, defines a virtual root-to-leaves orientation of $T$, through which every non-root vertex is adjacent to a unique parent (a vertex closer to the root) and every non-leaf vertex is adjacent to sons (vertices farther from the root). The *depth* of a vertex $v$ of $\overrightarrow{T}$ is the distance between $v$ and $r$ in $T$.

Defining $\overrightarrow{H}$ as the 3-vertex tournament with vertex set $\{0,1,2\}$ and arc set $\{\overrightarrow{01}, \overrightarrow{12}, \overrightarrow{20}\}$, we prove the claim by starting designing a 3-labelling $\ell$ of $\overrightarrow{T}$ which is almost an $\overrightarrow{H}$-oriented 3-labelling modulo 3, and, then, performing slight changes to this labelling so that an oriented 3-labelling results. Before proving this, let us raise some remarks.

Let $\phi$ be any $\overrightarrow{H}$-colouring of $\overrightarrow{T}$, which is easily verified to exist (see e.g. [23]). We claim that we can design a 3-labelling $\ell$ of $\overrightarrow{T}$ where, for every vertex $v \neq r$, we have $\sigma^{[3]}(v) = \phi(v)$. Indeed, consider the following process. Let us consider the vertices of $\overrightarrow{T}$ one by one, in decreasing order of their depths (considering vertices with the same depth in arbitrary order). Whenever considering a vertex $v$ this way, we aim at assigning a label to the incident arc going to its parent. Note that this guarantees, when considering $v$, that only the incident arc going to its parent remains to be labelled (trivially, note that if $v$ is a leaf of $\overrightarrow{T}$, then indeed only the incident arc going to its parent is unlabelled). So, whenever considering a new vertex $v$ in this process, we simply compute its current sum by the labelling (being the sum of the labels assigned earlier to the arcs incident to $v$ going to its sons), and just assign to the remaining arc incident to $v$, going to its parent, a label in $\{1,2,3\}$ so that $\sigma^{[3]}(v)$ gets equal to $\phi(v)$. This is always possible, since we are considering sums modulo 3.

Note that, by the process above, we have no control over $\sigma^{[3]}(r)$. In case, by the resulting 3-labelling $\ell$ of $\overrightarrow{T}$, we do have $\sigma^{[3]}(r) = \phi(r)$, then $\ell$ is an $\overrightarrow{H}$-oriented 3-labelling and we are done. So we may suppose this does not happen.

In what follows, we take the structure of $\overrightarrow{T}$ into account to tweak the process above a bit, and get our desired conclusion in all cases.

- **Case 1:** $\overrightarrow{T}$ contains a vertex adjacent to at least two leaves.

  Assume $r$ is a vertex of $\overrightarrow{T}$ adjacent to two leaves $x$ and $y$ (through arcs in any direction). We start by running the same labelling process as above with $r$ as root, without treating $x$ and $y$ in the process (that is, when considering vertices of depth 1, we just omit these two vertices). We also run this process with $\phi$ being an $\overrightarrow{H}$-colouring of $\overrightarrow{T}$ where $\phi(r) = 0$ (which can be guaranteed, since $\overrightarrow{H}$ is vertex-transitive). Note then that we have $\phi(x), \phi(y) \in \{1,2\}$. Once the process achieves, we obtain $\ell$, a partial 3-labelling of $\overrightarrow{T}$ where only the arcs joining $r$, and $x$ and $y$ remain to be labelled, and we have $\sigma^{[3]}(v) = \phi(v)$ for every vertex $v \notin \{r,x,y\}$.

  We consider the possible values that $\sigma^{[3]}(r)$ can currently have. Recall that $\phi(r) = 0$.

  - **Case 1.1:** $\sigma^{[3]}(r) = 0$.

If both $\overrightarrow{rx}$ and $\overrightarrow{ry}$ arcs, or both $\overrightarrow{xr}$ and $\overrightarrow{yr}$ are arcs, then we assign label 3 to both arcs by $\ell$. Otherwise, if, say, $\overrightarrow{xr}$ and $\overrightarrow{ry}$ are arcs, then we assign label 2 to $\overrightarrow{xr}$ and label 1 to $\overrightarrow{ry}$. In both cases, we claim that the resulting $\ell$ is oriented. If $\sigma^{[3]}(v) = \phi(v)$ for every vertex $v$ of $\overrightarrow{T}$, then $\ell$ is actually an $\overrightarrow{H}$-oriented labelling. The only situation where this might be not achieved, is because we have $\sigma^{[3]}(r) = \sigma^{[3]}(x) = \sigma^{[3]}(y) = 0$, but, in such a case, we actually have $\sigma(x) = \sigma(y) = 3$ and $\sigma(r) > 3$ with the two arcs joining $r$, and $x$ and $y$ going the same direction, while $\sigma^{[3]}(v) = \phi(v)$ for every vertex $v \notin \{x, y\}$. From these arguments, it can be deduced that $\ell$ is oriented.

- **Case 1.2:** $\sigma^{[3]}(r) = 1$.

  If, say, $\overrightarrow{xr}$ is an arc, then we assign label 2 to $\overrightarrow{xr}$ and label 3 to the arc joining $r$ and $y$. Otherwise, both $\overrightarrow{rx}$ and $\overrightarrow{ry}$ are arcs, in which case we assign label 1 to both $\overrightarrow{rx}$ and $\overrightarrow{ry}$. In all cases, note that $\sigma^{[3]}(v) = \phi(v)$ for every vertex $v$ of $\overrightarrow{T}$ but maybe one of $x$ and $y$, in which case that vertex has sum 3, while $r$ has sum strictly more than 3. Similarly as in the previous case, $\ell$ is thus oriented.

- **Case 1.3:** $\sigma^{[3]}(r) = 2$.

  This case can be treated similarly as **Case 1.2**. If, say, $\overrightarrow{rx}$ is an arc, then we assign label 1 to $\overrightarrow{rx}$ and label 3 to the arc joining $r$ and $y$. Otherwise, both $\overrightarrow{xr}$ and $\overrightarrow{yr}$ are arcs, in which case we assign label 2 to $\overrightarrow{xr}$ and $\overrightarrow{yr}$. We then deduce that $\ell$ is oriented for similar reasons as earlier.

- **Case 2:** no vertex of $\overrightarrow{T}$ is adjacent to at least two leaves.

  In this case, we can suppose that $\overrightarrow{T}$ contains at least three leaves, as otherwise $\overrightarrow{T}$ would be an oriented path and the result would follow from Theorem 4.1.

  Let $r$ be any vertex adjacent to a leaf, $x$. Just as in **Case 1**, we start by running the labelling process with root $r$ so that $x$ is omitted, and $\phi(r) = 0$. As a result, we get $\ell$, a partial 3-labelling where only the arc joining $r$ and $x$ remains to be labelled. For similar reasons as above, note that if $\sigma^{[3]}(r) = 0$, then we get an oriented 3-labelling of $\overrightarrow{T}$ upon assigning label 3 to the arc joining $r$ and $x$ (we get $\sigma^{[3]}(v) = \phi(v)$ for every $v \neq x$, and $r$ and $x$ are the only two adjacent vertices with the same sum modulo 3, that of $r$ being strictly larger than that of $x$).

  In what follows, we assume that $\overrightarrow{xr}$ is an arc, but the arguments can be symmetrised when $\overrightarrow{rx}$ is an arc. If $\sigma^{[3]}(r) = 1$, then note that, upon assigning label 2 to $\overrightarrow{xr}$, the resulting labelling $\ell$ is actually an $\overrightarrow{H}$-oriented labelling (as $\sigma^{[3]}(v) = \phi(v)$ for every vertex $v$). So the remaining case is when, currently, $\sigma^{[3]}(r) = 2$.

  Since $\overrightarrow{T}$ was assumed to have at least three leaves, and no vertex of $\overrightarrow{T}$ is adjacent to at least two leaves, there exist two non-leaf vertices $r', r''$ and two leaves $x', x''$ such that $r \notin \{r', r''\}$, $x \notin \{x', x''\}$, $r'$ is the parent of $x'$, and $r''$ is the parent of $x''$. Two main situations can now occur.

  - **Case 2.1:** one of the two arcs incident to $x'$ and $x''$ is not directed from the leaf towards its parent, with the parent being assigned colour 0 by $\phi$.

    W.l.o.g., assume this is the case for $x'$. To be clear, the conditions mean that either $\overrightarrow{r'x'}$ is the arc joining $r'$ and $x'$, or $\overrightarrow{x'r'}$ is the arc but $\phi(r') \neq 0$. Since $\phi(x') \neq \phi(r')$, note that the label assigned by $\ell$ to the arc joining $x'$ and $r'$ is exactly $\phi(x')$ (modulo 3), and is thus different from $\phi(r')$ (modulo 3).

We modify $\ell$ further as follows. Let $\overrightarrow{P'} = x' \dots r$ be the unique oriented path from $x'$ to $r$ in $\overrightarrow{T}$. Now, either +-switch or −-switch $\overrightarrow{P'}$ modulo 3, choosing the option making the label assigned to the arc joining $x'$ and $r'$ be $\phi(r')$ modulo 3. By Lemma 3.3, recall that only the sums of $r$ and $x'$ changed modulo 3. Regarding $x'$, note that, by the resulting labelling, we now have $\sigma^{[3]}(x') = \sigma^{[3]}(r') = \phi(r')$, but $\sigma(x') < \sigma(r')$ (while, currently, omitting $r$ and $x$, only $x'$ and $r'$ verify this). Regarding $r$, note that we now have $\sigma^{[3]}(r) \in \{0, 1\}$. If $\sigma^{[3]}(r) = 1$, then, as previously, we are done when assigning label 2 to $\overrightarrow{xr}$. Otherwise, if $\sigma^{[3]}(r) = 0$, then we assign label 3 to $\overrightarrow{xr}$. Note that this yields $\sigma^{[3]}(x) = \sigma^{[3]}(r) = \phi(r) = 0$ (while we have $\sigma(r) > \sigma(x)$), while the only other adjacent vertices that might verify this property are $x'$ and $r'$. However, note that if $\sigma^{[3]}(x') = \sigma^{[3]}(r') = \phi(r') = 0$, then the resulting labelling $\ell$ is oriented, since, by assumption, both $\overrightarrow{xr}$ and $\overrightarrow{x'r'}$ are arcs. Of course, if $\sigma^{[3]}(x') = \sigma^{[3]}(r') = \phi(r') \neq 0$, then also by the same arguments $\ell$ is oriented.

– **Case 2.2:** $\overrightarrow{r'x'}$ and $\overrightarrow{r'', x''}$ are arcs, and $\phi(r') = \phi(r'') = 0$.

These assumptions mean that $\phi(x') = \phi(x'') = 1$, and, thus, $\ell(\overrightarrow{r'x'}) = \ell(\overrightarrow{r''x''}) = 1$. Here, let $\overrightarrow{P'} = x' \dots r$ and $\overrightarrow{P''} = x'' \dots r$ be the unique oriented paths of $\overrightarrow{T}$ from $x'$ and $x''$, respectively, to $r$.

∗ If, say, $\overrightarrow{P'}$ has odd length, then we modify $\ell$ by −-switching $\overrightarrow{P'}$ modulo 3. As a result, as in **Case 2.1**, we get $\sigma^{[3]}(x') = \sigma^{[3]}(r') = \phi(r') = 0$ (and, again, only $x'$ and $r'$ verify this at the moment), and thus $\sigma(x') < \sigma(r')$. Now, since $\overrightarrow{P'}$ has odd length, note that we now have $\sigma^{[3]}(r) = 1$. By then assigning label 2 to $\overrightarrow{xr}$, the resulting labelling of $\overrightarrow{T}$ is oriented.

∗ Otherwise, both $\overrightarrow{P'}$ and $\overrightarrow{P''}$ have even length. We here −-switch both $\overrightarrow{P'}$ and $\overrightarrow{P''}$ modulo 3. As a result, we get $\sigma^{[3]}(x') = \sigma^{[3]}(x'') = \sigma^{[3]}(r') = \sigma^{[3]}(r'') = \phi(r') = \phi(r'') = 0$ (and only $x'$ and $r'$, and $x''$ and $r''$ verify this), and thus $\sigma(x') < \sigma(r')$ and $\sigma(x'') < \sigma(r'')$. Also, since $\overrightarrow{P'}$ and $\overrightarrow{P''}$ have even length, we, again, get $\sigma^{[3]}(r) = 1$. So, once more, we are done when assigning label 2 to $\overrightarrow{xr}$.

In all cases, we thus end up with an oriented 3-labelling of $\overrightarrow{T}$, which concludes. □

## 6. Complexity results

In this section, we investigate the computational complexity of some decision problems related to our new labelling notion. Particularly, we investigate the complexity of determining whether a graph $G$ admits "good" orientations $\overrightarrow{G}$ (w.r.t. $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$), and of determining $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$ for a given oriented graph $\overrightarrow{G}$.

### *6.1. Finding "good" orientations $\overrightarrow{G}$ of $G$ with respect to $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$*

We here investigate the complexity of determining whether a given graph $G$ admits an orientation $\overrightarrow{G}$ with $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$ small. Wondering about this problem makes sense, since $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$ can be arbitrarily large for some $\overrightarrow{G}$, even if $\chi_{\Sigma}(G)$ is small (recall Theorem 3.1).

We start off by, in some sense, reversing Observation 2.1.

**Observation 6.1.** *If $\ell$ is a proper labelling of some graph $G$, then $G$ admits an orientation $\overrightarrow{G}$ in which $\ell$ is an oriented labelling. Thus, there is an orientation $\overrightarrow{G}$ with $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G}) = \chi_{\Sigma}(G)$.*

*Proof.* Just consider, as $\overrightarrow{G}$, the orientation of $G$ obtained by considering every edge $uv$ of $G$, and orienting $uv$ towards the one of $u$ and $v$ with the largest sum by $\sigma_\ell$. □

From Observation 6.1, we determine the following:

**Theorem 6.2.** *For any $k \geq 1$, there is a polynomial-time reduction which, from a graph $G$, yields a graph $G'$ such that $\chi_\Sigma(G) \leq k$ if and only if there is an orientation $\overrightarrow{G'}$ of $G'$ with $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G'}) \leq k$.*

*Proof.* We pretend that it suffices to take $G' = G$. Indeed, if $G$ admits a proper $k$-labelling $\ell$, then, by Observation 6.1, there is an orientation $\overrightarrow{G'}$ of $G'$ in which $\ell$ is an oriented labelling. Now, by Observation 2.1, if $G'$ has an orientation $\overrightarrow{G'}$ admitting an oriented $k$-labelling $\ell$, then $\ell$ is a proper labelling of $G' = G$. Thus, the statement holds. □

Our current knowledge on the 1-2-3 Conjecture now results in the following:

**Corollary 6.3.** *Let $k \geq 1$. The problem of deciding whether a given graph $G$ admits an orientation $\overrightarrow{G}$ with $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G}) \leq k$ is:*

- *in $\mathsf{P}$ for $k = 1$;*
- *$\mathsf{NP}$-complete for $k = 2$;*
- *solvable in constant time for every $k \geq 5$.*

*Furthermore, if the 1-2-3 Conjecture was shown to hold, then the problem would be constant-time solvable for every $k \geq 3$.*

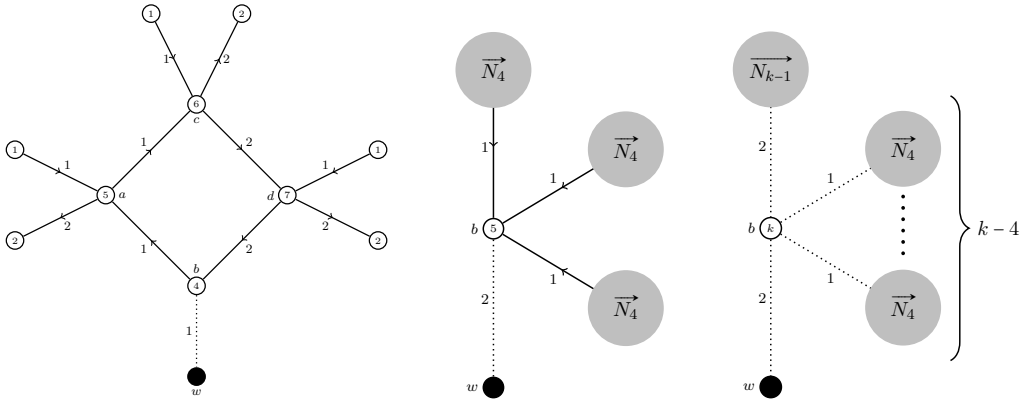
*Proof.* Obviously, the problem lies in $\mathsf{NP}$ for every $k \geq 1$. Now, the items follow from Theorem 6.2. Particularly, for a given graph $G$, determining if $\chi_\Sigma(G) \leq 1$ is equivalent to checking whether $G$ has adjacent vertices with the same degree, which can be done in polynomial time. Determining if $\chi_\Sigma(G) \leq 2$ was proved to be $\mathsf{NP}$-hard in [9]. Also, it was proved in [13] that all graphs admit proper 5-labellings, so every instance of the problem of deciding whether $\chi_\Sigma(G) \leq k$, is positive for every $k \geq 5$. Note that if the 1-2-3 Conjecture was proved, then the same would also hold when $k \in \{3, 4\}$. □

*6.2. Determining $\chi_{\overrightarrow{\Sigma}}(\overrightarrow{G})$ for a given oriented graph $\overrightarrow{G}$*

Before proceeding with the proof of our main result in this section (Theorem 6.6), we need to introduce some gadgets first. It is worth emphasising now that, for technical reasons, these gadgets are oriented graphs for which we do not specify the direction of some of their arcs. The directions of these arcs, which are represented by dotted edges in Figure 5, will be clarified whenever needed later on. What is important is that the properties of the gadgets we exhibit are independent of the directions of these arcs.

Before we introduce the gadgets we need, we first have to introduce a special oriented graph $\mathcal{T}$. Regarding the vertices of $\mathcal{T}$, we have $V(\mathcal{T}) = \mathbb{N}^+$. Regarding the arcs of $\mathcal{T}$ (where, to avoid any ambiguity, any arc $\overrightarrow{uv}$ is here denoted as $(u, v)$):

- $(1, 5)$, $(1, 6)$, and $(1, 7)$ are arcs;
- $(5, 2)$, $(6, 2)$, and $(7, 2)$ are arcs;
- $(i, i + 1)$ is an arc for every $i \geq 4$, except for $i \in \{13, 21\}$, for which $(i, i - 1)$ is an arc;
- $(i, 4)$ is an arc for every $i \geq 6$;




(a) $\overrightarrow{N_4}$ (b) $\overrightarrow{N_5}$ (c) $\overrightarrow{N_k}$

Figure 5: Gadgets used in Section 6. Black vertices are roots. Numbers in white vertices indicate the sums obtained by the depicted labelling. "$\overrightarrow{N_k}$" indicates that a copy of $\overrightarrow{N_k}$ is attached at some vertex through its root. Dotted edges represent arcs for which the direction depend on some parameters.

- $(10, 12)$, $(13, 11)$, $(21, 23)$, $(10, 21)$, $(13, 21)$ are arcs.

The first gadget, $\overrightarrow{N_4}$, is depicted in Figure 5(a). Throughout what follows we will sometimes deal with the vertices and arcs of $\overrightarrow{N_4}$ following the terminology from that figure. We call vertex $b$ the *center* of $\overrightarrow{N_4}$, vertex $w$ the *root* of $\overrightarrow{N_4}$, and the arc joining $b$ and $w$ the *root arc* of $\overrightarrow{N_4}$. Note that, in Figure 5(a), the direction of the root arc is actually not determined. This is because, as explained earlier, depending on the context we will need to consider copies of $\overrightarrow{N_4}$ in which the root arc is oriented in one direction or the other. Regardless of the direction of that arc, $\overrightarrow{N_4}$ has peculiar labelling properties.

**Lemma 6.4.** *Regardless of the direction of the root arc, in any oriented 2-labelling of $\overrightarrow{N_4}$, we must have $\sigma(b) = 4$, and the root arc must be assigned label 1. Furthermore, such oriented 2-labellings of $\overrightarrow{N_4}$ exist, and, omitting $w$, some of them are $\mathcal{T}$-oriented 2-labellings.*

*Proof.* Assume an oriented 2-labelling $\ell$ of $\overrightarrow{N_4}$ exists. Note that each of $a$, $c$, and $d$ is both an in-neighbour of a degree-1 vertex and an out-neighbour of a degree-1 vertex. Then, for $\ell$ to be oriented, for each of $a$, $c$, and $d$ the two incident arcs being incident to a degree-1 vertex must be assigned distinct labels, 1 and 2. Regardless of how these two labels are actually assigned by $\ell$, the labels assigned to these pendant arcs thus contribute for 3 to $\sigma(a)$, $\sigma(c)$, and $\sigma(d)$. Now, so that $\sigma(a) \neq \sigma(c)$, note that we must have $\ell(\overrightarrow{ba}) \neq \ell(\overrightarrow{cd})$, and, similarly, so that $\sigma(c) \neq \sigma(d)$, note that we must have $\ell(\overrightarrow{ac}) \neq \ell(\overrightarrow{db})$. So there are four possible cases: either $(\ell(\overrightarrow{ba}), \ell(\overrightarrow{db})) = (1, 1)$ (and thus $(\ell(\overrightarrow{cd}), \ell(\overrightarrow{ac})) = (2, 2)$), $(\ell(\overrightarrow{ba}), \ell(\overrightarrow{db})) = (2, 2)$ (and thus $(\ell(\overrightarrow{cd}), \ell(\overrightarrow{ac})) = (1, 1)$), $(\ell(\overrightarrow{ba}), \ell(\overrightarrow{db})) = (1, 2)$ (and thus $(\ell(\overrightarrow{cd}), \ell(\overrightarrow{ac})) = (2, 1)$), and $(\ell(\overrightarrow{ba}), \ell(\overrightarrow{db})) = (2, 1)$ (and thus $(\ell(\overrightarrow{cd}), \ell(\overrightarrow{ac})) = (1, 2)$). In the first two of these cases, note that we would get $\sigma(a) = \sigma(d) = 6$ while $\overrightarrow{ac}$ and $\overrightarrow{cd}$ are arcs, contradicting that $\ell$ is oriented. For the last two cases, note that we get $\sigma(c) = 6$, and $\{\sigma(a), \sigma(d)\} = \{5, 7\}$. Meanwhile, we currently have $\ell(\overrightarrow{ba}) + \ell(\overrightarrow{db}) = 3$, which means that, so that $\sigma(b) \neq 5$ (which would contradict that $\ell$ is oriented), the root arc must be assigned label 1, so that $\sigma(b) = 4$. We thus reach the conclusions of the statement in both possible situations (one of which is depicted in Figure 5(a), to convince the reader that the last part of the statement holds). □

We now generalise gadget $\overrightarrow{N_4}$ through the following attachment operation. Assume $\overrightarrow{G}$ is an oriented graph with a vertex $v$. Let $\overrightarrow{H}$ be another oriented graph with a degree-1

vertex $u$. In $\vec{G}$, by *attaching (a copy of)* $\vec{H}$ *at* $v$ *through* $u$, we mean adding a new copy of $\vec{H}$ to $\vec{G}$, and identifying $u$ and $v$.

We define $\overrightarrow{N_5}$ as the gadget depicted in Figure 5(b), obtained from an arc (in any direction) joining two vertices $b$ and $w$, by attaching three copies of $\overrightarrow{N_4}$ at $b$ through their roots, the root arcs of these copies of $\overrightarrow{N_4}$ being directed as shown in Figure 5(b). We define $w$ as the *root* of $\overrightarrow{N_5}$, $b$ as its *center*, and the arc joining $w$ and $b$ as its *root arc*.

Now, assuming gadget $\overrightarrow{N_{k-1}}$ was defined for some $k \geq 6$, we define $\overrightarrow{N_k}$ as follows (see Figure 5(c)). Start from two vertices $b$ and $w$, joined by an arc in any direction. Attach, at $b$, one copy of $\overrightarrow{N_{k-1}}$ and $k-4$ copies of $\overrightarrow{N_4}$ through their roots, and orient their root arcs as indicated by $\mathcal{T}$. That is, orient the root arcs so that their directions meet the respective ones joining vertices $k$ and $k-1$ of $\mathcal{T}$ (for $\overrightarrow{N_k}$ and $\overrightarrow{N_{k-1}}$), and vertices $k$ and 4 (for $\overrightarrow{N_k}$ and $\overrightarrow{N_4}$). We call $w$ the *root* of $\overrightarrow{N_k}$, the arc joining $b$ and $w$ its *root arc*, and $b$ its *center*. Note that the direction of the root arc of $\overrightarrow{N_k}$ is not fixed.

**Lemma 6.5.** *For any $k \geq 5$, regardless of the direction of the root arc, in any oriented 2-labelling of $\overrightarrow{N_k}$, we must have $\sigma(b) = k$, and the root arc must be assigned label 2. Furthermore, such oriented 2-labellings of $\overrightarrow{N_k}$ exist, and, omitting $w$, some of them are $\mathcal{T}$-oriented 2-labellings.*

*Proof.* We prove this by induction on $k$. For $k = 5$, note that three copies of $\overrightarrow{N_4}$ were attached to $b$. By Lemma 6.4, by any oriented 2-labelling $\ell$ of $\overrightarrow{N_5}$, the root arcs of these three copies must be assigned label 1, and the center of any of the three copies must have sum 4. Thus, so that $\sigma(b) \neq 4$, the root arc must be assigned label 2, and $\sigma(b) = 5$. It can be observed that, due to how the root arcs have been oriented, $\ell$ indeed exists, and that $\ell$ can even be designed so that it is indeed a $\mathcal{T}$-oriented 2-labelling.

Now consider any general value of $k \geq 6$. Let $\ell$ be any oriented 2-labelling $\ell$ of $\overrightarrow{N_k}$. By construction, in $\overrightarrow{N_k}$, to $b$ were attached $k-4$ copies of $\overrightarrow{N_4}$, in which the root arcs are assigned label 1 and the center has sum 4, and one copy of $\overrightarrow{N_{k-1}}$, in which the root arc is assigned label 2 and the center has sum $k-1$. Thus, so that $\sigma(b) \neq k-1$, the root arc must be assigned label 2, which yields $\sigma(b) = k$. Again, how we directed the root arcs guarantees such a 2-labelling is indeed oriented. Now, due to Lemma 6.4 and the induction hypothesis, by combining $\mathcal{T}$-oriented 2-labellings of copies of $\overrightarrow{N_4}$ and $\overrightarrow{N_{k-1}}$, and assigning label 2 to the root arc, we can, actually, even design $\mathcal{T}$-oriented 2-labellings of $\overrightarrow{N_k}$. □

We are now ready to prove our main result in this section. We define Oriented 2-Labelling as the decision problem where an oriented graph $\vec{G}$ is given as input, and the question is to determine whether oriented 2-labellings of $\vec{G}$ exist, or, in other words, whether $\chi_{\vec{\Sigma}}(\vec{G}) \leq 2$. We prove this problem is NP-complete. An interesting fact is that we prove this to hold even when $\vec{G}$ is bipartite (and planar), which contrasts with the undirected context, in which determining whether $\chi_{\Sigma}(G) \leq 2$ for a given graph $G$ is NP-complete in general [9], but polynomial-time solvable when $G$ is bipartite [24].

**Theorem 6.6.** Oriented 2-Labelling *is NP-complete, even for instances where $\vec{G}$ is bipartite and planar.*

*Proof.* Since Oriented 2-Labelling is obviously in NP, we focus on proving it is NP-hard. This is done by reduction from Cubic Monotone 1-in-3 SAT, in which a 3CNF formula $F$ is given, where $F$ contains clauses $C_1, \ldots, C_m$ each of which contains exactly three distinct variables (in positive form only) from a set $\{x_1, \ldots, x_n\}$, and the task is to *1-in-3 satisfy* $F$, *i.e.*, to find a truth assignment to the variables such that each clause contains exactly one true variable. In this cubic version of the problem (which remains

NP-hard, see [18]), each variable is further assumed to appear once in exactly three distinct clauses. In other words, the bipartite graph modelling the structure of $F$ can be assumed to be cubic (and even planar, additionally). From an instance $F$ of CUBIC MONOTONE 1-IN-3 SAT, we construct, in polynomial time, an oriented graph $\vec{G}$, such that $F$ is 1-in-3 satisfiable if and only if $\vec{G}$ admits oriented 2-labellings.

We proceed as follows. We start from $\vec{G}$ being the oriented bipartite graph modelling the structure of $F$. That is, for every clause $C$ of $F$ we have a *clause vertex* $v_C$ in $\vec{G}$, for every variable $x$ of $F$ we have a *variable vertex* $v_x$, and whenever a variable $x$ appears in a clause $C$ of $F$, we have a *formula arc* $\overrightarrow{v_x v_C}$ in $\vec{G}$. Note that all arcs are oriented from the variable vertices to the clause vertices. We now modify $\vec{G}$ further as follows[3]:

- To every clause vertex $v_C$, we attach eleven copies of $\overrightarrow{N_4}$, one copy of $\overrightarrow{N_{20}}$, one copy of $\overrightarrow{N_{22}}$, and one copy of $\overrightarrow{N_{23}}$ through their roots.
- To every variable vertex $v_x$, we attach three copies of $\overrightarrow{N_4}$, one copy of $\overrightarrow{N_{11}}$, and one copy of $\overrightarrow{N_{12}}$ through their roots.

Remember that whenever attaching a copy of some $\overrightarrow{N_k}$ in an oriented graph, we also need to specify the direction of its root arc. In the present case, for every $\overrightarrow{N_k}$ we have attached at some clause vertex or variable vertex, we orient its root arc towards the clause or variable vertex if $k \in \{4, 12, 20\}$, while we orient it the other way in all other cases, *i.e.*, if $k \in \{11, 22, 23\}$. This achieves the construction of $\vec{G}$.

Note that $\vec{G}$ is clearly obtained in polynomial time from $F$. Also, since the gadgets $\overrightarrow{N_k}$ are mostly oriented trees, and are thus bipartite and planar, and the bipartite graph modelling the structure of $F$ also has these properties by assumption, it can also be noted that $\vec{G}$ is bipartite and planar.

We claim that we have the desired equivalence between $F$ and $\vec{G}$.

- Assume first that there exists an oriented 2-labelling $\ell$ of $\vec{G}$. Recall that, by Lemmas 6.4 and 6.5, for every $\overrightarrow{N_k}$ we have attached to some clause or variable vertex $v$, the root arc brings 1 (if $k = 4$) or 2 (otherwise) to $\sigma(v)$, while, in $\overrightarrow{N_k}$, the center has sum $k$ and is adjacent to $v$. For these reasons, due to the gadgets we have attached at every clause vertex $v_C$, the sum of $v_C$ by $\ell$ must lie in $\{20, 21, 22, 23\}$, taking account that the three formula arcs incident to $v_C$ can be assigned any value in $\{1, 2\}$ by $\ell$. Actually, $v_C$ must have sum 21, since copies of $\overrightarrow{N_{20}}$, $\overrightarrow{N_{22}}$, and $\overrightarrow{N_{23}}$ were attached at $v_C$. This implies one formula arc incident to $v_C$ must be assigned label 2 by $\ell$, while the other two must be assigned label 1. By the same reasoning, it can be deduced that every variable vertex $v_x$ must have sum 10 or 13, and thus its three incident formula arcs must be assigned the same label.

  Now imagine that having a formula arc $\overrightarrow{v_x v_C}$ assigned label 1 by $\ell$ models the fact that variable $x$ brings truth value *false* to clause $C$ by a truth assignment, while having it assigned label 2 models that $x$ brings *true* to $C$. The fact that a clause vertex $v_C$ must be incident to exactly one formula arc assigned label 2 thus models that $C$ is considered satisfied only if it contains exactly one true variable. The fact that a variable vertex $v_x$ must have its three incident arcs assigned the same label

[3]Some numbers in the reduction, such as the values of $k$ for which we attach copies of $\overrightarrow{N_k}$, might seem a bit large and arbitrary. Some of these numbers could indeed be optimised, but this would result in a less clear proof, for, in our opinion, a very limited benefit (having e.g. Theorem 6.6 holding for oriented graphs with "small" maximum degree).

models that $x$ bring the same truth value to all clauses that contain it. From this, we can naturally derive a truth assignment to the variables of $F$ 1-in-3 satisfying it, by setting to *true* all variables $x$ such that the clause vertex $v_x$ has its three incident formula arcs assigned label 2 by $\ell$, and setting to *false* all others.

- Assume now $F$ admits a truth assignment $\phi$ that 1-in-3 satisfies it. To obtain an oriented 2-labelling of $\vec{G}$, we first consider every variable vertex $v_x$, and assign label 1 to its three incident formula arcs if $x$ is set to *false* by $\phi$, while we assign label 2 to these three arcs otherwise. Now, for all copies of $\overrightarrow{N_{11}}$, $\overrightarrow{N_{12}}$, $\overrightarrow{N_{20}}$, $\overrightarrow{N_{21}}$, and $\overrightarrow{N_{22}}$ attached to the variable vertices and clause vertices, we extend the labelling so that the root arcs are assigned label 2, in a $\mathcal{T}$-oriented way, which is possible by Lemma 6.5. Similarly, for all copies of $\overrightarrow{N_4}$ attached to the variable vertices and clause vertices, we extend the labelling so that their root arcs are assigned label 1, in a $\mathcal{T}$-oriented way, which can be done, according to Lemma 6.4. Note that all arcs of $\vec{G}$ are now labelled, and, due to the resulting sums (in particular, all clause vertices have sum 21 while all variable vertices have sum 10 or 13), and due to how we oriented the arcs of $\vec{G}$ w.r.t. $\mathcal{T}$, it is not too hard to check that the resulting labelling is $\mathcal{T}$-oriented. □

## 7. Conclusion and perspectives

In connection with proper labellings and the 1-2-3 Conjecture, we have introduced a new generalisation of these notions to oriented graphs, resulting in the definition of oriented labellings and of the parameter $\chi_{\vec{\Sigma}}$. Many aspects behind these notions seemed interesting to us, which is why we did our best, in this exploratory work, to establish several results of different natures. In particular, we proved that $\chi_{\vec{\Sigma}}(\vec{G})$ can be arbitrarily large for an oriented graph $\vec{G}$, which led us to investigate efficient and accurate ways to bound this parameter. As a result, we established that $\chi_{\vec{\Sigma}}(\vec{G}) \leq \chi_{\rm o}(\vec{G})$, which is of interest given how investigated the oriented chromatic number has been. From here, we then investigated easy classes of oriented graphs, for which we managed to prove tight results. We also considered side aspects, such as the complexity of determining $\chi_{\vec{\Sigma}}(\vec{G})$ for an oriented graph $\vec{G}$.

Of course, many directions for further work on this topic could be considered, including both ways to improve some of our results and new questions. We end up this work with a list of questions and problems that sound interesting to us.

- Note that we did not raise a general conjecture on the exact value of $\chi_{\vec{\Sigma}}(\vec{G})$ for any oriented graph $\vec{G}$, that would stand as an analogue of the 1-2-3 Conjecture for our problem. The main reasons why we did not dare, are that $\chi_{\vec{\Sigma}}(\vec{G})$ can be arbitrarily large, and also that the upper bounds we have established, for example the bounds involving $\chi_{\rm o}(\vec{G})$ and $\Delta(\vec{G})$ (recall Theorem 3.4 and Corollary 3.5), can be far from good in general, recall Observation 3.2. So, we are not quite sure how $\chi_{\vec{\Sigma}}(\vec{G})$ should be expressed in general, and we believe this is an appealing question.

- When dealing with classes of oriented graphs in Sections 4 and 5, one great tool we used was the modulo method that lies behind the proof of Theorem 3.4. Recall that this led to tight bounds for some classes of oriented graphs, such as oriented trees $\vec{T}$ (infinitely many of which verify $\chi_{\vec{\Sigma}}(\vec{T}) = \chi_{\rm o}(\vec{T})$, recall Theorem 5.1) and oriented cycles $\vec{C}$ (some of which even verify $\chi_{\vec{\Sigma}}(\vec{C}) = \chi_{\rm o}(\vec{C}) + 1$, as mentioned in Section 4). However, as mentioned earlier (recall Observation 3.2), it seems that relying on oriented colourings is not a good way to proceed in general, if one aims at designing oriented labellings using labels from $\{1, \ldots, \chi_{\vec{\Sigma}}(\vec{G})\}$ for an oriented graph

$\vec{G}$. The main reason is that, through $\vec{H}$-oriented labellings modulo $k$, we actually design a homomorphism to an oriented graph $\vec{H}$ that is not a tournament (consider pairs of sums with the same value modulo $k$). To overcome this, in the proofs of Theorems 4.6 and 5.1, we had to go beyond this method by allowing a few adjacent vertices to have the same sum modulo $k$, in safe contexts. So, although this sounds rather difficult to us, we think a crucial point would be to wonder about other, more permissive ways to design oriented labellings from scratch.

- One class of oriented graphs to consider next, could be that of oriented graphs with maximum degree 3. One problem, however, is that their oriented chromatic number is not fully understood. At the moment, we know that the oriented chromatic number of oriented subcubic graphs is at most 8 (see [10]), while it is conjectured it should be at most 7 (see [22]). Our best tool to prove bounds on $\chi_{\vec{\Sigma}}$ being the modulo method, which requires to have good knowledge on the oriented chromatic number of the class in consideration, we are not sure it would be worth considering oriented subcubic graphs at this point. The same goes, of course, for any class of oriented graphs for which the maximum value of the oriented chromatic is not fully identified.

- Regarding complexity questions and our results from Section 6, one could wonder about the complexity of the problem of determining whether a given oriented graph $\vec{G}$ verifies $\chi_{\vec{\Sigma}}(\vec{G}) \leq k$, for any fixed $k \geq 2$. Recall that we proved that this problem is NP-complete for $k = 2$ (Theorem 6.6). One could also wonder about the converse of Corollary 6.3, that is, about the problem of determining whether a graph $G$ admits orientations $\vec{G}$ that are "bad" w.r.t $\chi_{\vec{\Sigma}}(\vec{G})$.

- As mentioned in Section 2, a peculiar aspect of oriented colouring is that considering disconnected oriented graphs stands as a challenge by itself. Notable differences in our case (recall that we focused on connected oriented graphs) can already be pointed out, even for oriented graphs with maximum degree 2. Indeed, as pointed out in Section 4, the smallest oriented graphs $\vec{G}$ with maximum degree 2 and $\chi_{\vec{\Sigma}}(\vec{G}) = 4$ we were able to find have order 14. In the case of disconnected oriented graphs with maximum degree 2, smaller examples exist: consider e.g. the disjoint union of the two non-isomorphic tournaments on three vertices. More generally speaking, we wonder whether disconnected oriented graphs can show particular behaviours w.r.t. our problem. Be aware, for instance, that the upper bounds on the oriented chromatic number in terms of the maximum degree are not the same for disconnected oriented graphs (see [23]). Thus, our Corollary 3.5 would be different in that setting.

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