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Roots of outer automorphisms of free groups and centralizers of abelian subgroups of $\text{Out}(F_N)$

Yassine Guerch

December 15, 2022

Abstract

Let $N \geq 2$ and let $\text{Out}(F_N)$ be the outer automorphism group of a nonabelian free group of rank N . Let $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ be the finite index subgroup of $\text{Out}(F_N)$ which is the kernel of the natural action of $\text{Out}(F_N)$ on $H_1(F_N, \mathbb{Z}/3\mathbb{Z})$. We show that $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is an *R-group*, that is, for every $\phi, \psi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$, if there exists $k \in \mathbb{N}^*$ such that $\phi^k = \psi^k$, then $\phi = \psi$. This answers a question of Handel and Mosher. We then use the fact that $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is an *R-group* in order to prove that the normalizer in $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ of every abelian subgroup of $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is equal to its centralizer. We finally give an alternative proof of a result, due to Feighn and Handel, that the centralizer of an element of $\text{Out}(F_N)$ which has only finitely many periodic orbits of conjugacy classes of maximal cyclic subgroups of F_N is virtually abelian. ¹

1 Introduction

Let $N \geq 2$ and let $\text{Out}(F_N)$ be the outer automorphism group of a nonabelian free group F_N . When studying dynamical or algebraic properties of $\text{Out}(F_N)$, one often needs to pass to a finite index subgroup in order to avoid periodic behaviours. A finite index subgroup of particular importance for such considerations is the subgroup $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$, which is the kernel of the natural action of $\text{Out}(F_N)$ on the first homology group of F_N with coefficients in $\mathbb{Z}/3\mathbb{Z}$. The group $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is an analogue of the congruence subgroups of arithmetic lattices, which are finite index subgroups of the considered lattices having the advantage of being torsion free. The group $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ similarly satisfies numerous aperiodic properties. For instance, it is torsion free [BFH1, Corollary 5.7.6], every virtually abelian subgroup of $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is abelian [HM4], and any conjugacy class of a free factor of F_N which has a periodic orbit by a subgroup of $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is in fact fixed [HM3, Theorem II.3.1]. We notice that we could have replaced $\mathbb{Z}/3\mathbb{Z}$ in the definition of $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ by any other $\mathbb{Z}/m\mathbb{Z}$ with $m \geq 3$ without changing the aperiodic properties of the group. On the contrary, the group $\text{IA}_N(\mathbb{Z}/2\mathbb{Z})$ is not torsion free as it contains the involution sending every element of some basis of F_N to its inverse.

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This is why the group $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is predominant when studying rigidity properties of $\mathrm{Out}(F_N)$. It is for instance of central importance in the proof of the Tits alternative due to Bestvina, Feighn and Handel [BFH1, BFH3]. The fact that periodic conjugacy classes of free factors and elements of F_N are fixed by subgroups of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is a key step in the proof of decomposition theorems due to Handel and Mosher [HM3] and in their alternative regarding the second bounded cohomology of subgroups of $\mathrm{Out}(F_N)$ [HM2]. It also intervenes in the proof of the rigidity of the abstract commensurator of $\mathrm{Out}(F_N)$ due to Farb and Handel [FaH] for $N \geq 4$ and Horbez and Wade [HW] for $N \geq 3$. Finally, it also appears in the recent proof of the measure equivalence rigidity of $\mathrm{Out}(F_N)$ due to Guirardel and Horbez [GH2].

In this article, we prove another aperiodic property of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$, which is the content of the following theorem.

Theorem 1.1. *Let $N \geq 2$ and let $\phi, \psi \in \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$. Suppose that there exists $k \in \mathbb{N}^*$ such that $\phi^k = \psi^k$. Then $\phi = \psi$.*

This answers a question of Handel and Mosher [HM4, Question 1.4]. Following Kontorovič [Kon] (see also [Bau]), Theorem 1.1 implies that $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is an *R-group*. This implies in particular (see Corollary 4.2) that the normalizer in $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ of any cyclic subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is equal to its centralizer.

Another consequence of Theorem 1.1 is that one can enlarge the recognition theorem due to Feighn and Handel [FH2, Theorem 5.3] to a larger class of elements of $\mathrm{Out}(F_N)$. Indeed, the recognition theorem only applies to *forward rotationless* elements. By [FH2, Lemma 4.42], there exists $K_N \in \mathbb{N}^*$ such that, for every $\phi \in \mathrm{Out}(F_N)$, the element ϕ^{K_N} is forward rotationless. Thus, we have the following corollary.

Corollary 1.2. *Let $N \geq 2$ and let $K_N \in \mathbb{N}^*$ be the constant given by [FH2, Lemma 4.42]. Let $\phi, \psi \in \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$. Then $\phi = \psi$ if and only if ϕ^{K_N} and ψ^{K_N} satisfy the hypotheses of the recognition theorem [FH2, Theorem 5.3].*

Recall that, in an *R-group*, the normalizer of every cyclic subgroup is equal to its centralizer. We may ask whether this property extends to every *abelian* subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$. This is in fact the case as shown by the following theorem.

Theorem 1.3 (see Theorem 5.1). *Let $N \geq 2$. For every abelian subgroup $H \subseteq \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$, the normalizer of H in $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is equal to its centralizer.*

Note that Theorem 1.3 is no longer true if we remove the assumption of being abelian. We thus deduce this new algebraic characterization of abelian subgroups of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$.

Corollary 1.4. *Let H be a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$. Then H is abelian if and only if its normalizer in $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ equals its centralizer.*

Theorem 1.3 extends a similar result due to Handel and Mosher for abelian *UPG* subgroups of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ [HM4, Proposition 1.3]. It also enlarges the known informations regarding both abelian subgroups of $\mathrm{Out}(F_N)$ (see for instance the work of Feighn

and Handel [FH1]) and centralizers of subgroups of $\text{Out}(F_N)$, see for instance the work of Rodenhausen and Wade [RW], Algom-Kfir and Pfaff [AKP] and Andrew and Martino [AM].

As we will see in Corollary 5.2, Theorem 1.3 gives obstructions to the existence of subgroups of $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ isomorphic to some semidirect products of groups. Indeed, if H is a subgroup of $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ isomorphic to $A \rtimes B$ where A is abelian, then Theorem 1.3 implies that H is in fact isomorphic to the direct product of A and B .

Finally, our techniques also enable us to understand centralizers of some elements of $\text{Out}(F_N)$. The following theorem also follows from the work of Feighn and Handel [FH1] regarding abelian subgroups of $\text{Out}(F_N)$ (see also the work of Mutanguha [Mut] regarding centralizers of *atoroidal* elements of $\text{Out}(F_N)$).

Theorem 1.5 (Theorem 5.3). *[FH1] Let $N \geq 2$ and let $\phi \in \text{Out}(F_N)$ be an outer automorphism which has only finitely many periodic orbits of conjugacy classes of maximal cyclic subgroups of F_N . The centralizer of ϕ in $\text{Out}(F_N)$ is virtually abelian.*

We note that the work of Feighn and Handel uses the technology of train tracks, while the proof of Mutanguha uses some actions on limit trees. Our proof relies on isometric actions of $\text{Out}(F_N)$ on some Gromov-hyperbolic spaces, so that the techniques used in this paper significantly differs from the other two proofs.

We now sketch the proof of Theorem 1.1, the proofs of Theorems 1.3 and 1.5 following essentially the same lines. Let $\phi, \psi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ and let $k \in \mathbb{N}^*$ be such that $\phi^k = \psi^k$. The idea of the proof is to show that the group $H = \langle \phi, \psi \rangle$ is abelian. Since $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is torsion free and since $\phi^k = \psi^k$, this will imply that H is cyclic and this will conclude the proof.

The proof is by induction on N , the case $N = 1$ being immediate. Consider a maximal proper H -invariant *free factor system* \mathcal{F} of F_N (see Section 2.1). If \mathcal{F} is *sporadic*, then one can canonically associate to \mathcal{F} a Bass-Serre tree S of F_N whose conjugacy classes of vertex stabilizers are contained in \mathcal{F} . The action of F_N on S naturally extends to an action of the preimage \tilde{H} of H in $\text{Aut}(F_N)$. A thorough investigation of the action of \tilde{H} then shows that H is necessarily abelian. This investigation relies on the induction hypothesis in order to deal with the action of \tilde{H} on vertex stabilizers of S in F_N .

Suppose now that \mathcal{F} is *nonsporadic*. Then H acts by isometries on a Gromov hyperbolic space called the *free factor graph of F_N relative to \mathcal{F}* and denoted by $\text{FF}(F_N, \mathcal{F})$. Maximality of \mathcal{F} implies that H contains a loxodromic element by results of Handel and Mosher [HM3, Theorem A] and Guirardel and Horbez [GH1, Theorem 7.1]. Moreover, since H is abelian, it acts with a finite orbit on the Gromov boundary $\partial_\infty \text{FF}(F_N, \mathcal{F})$ of $\text{FF}(F_N, \mathcal{F})$. A thorough analysis of stabilizers of points in $\partial_\infty \text{FF}(F_N, \mathcal{F})$ based on their description by Guirardel and Horbez [GH1] and Horbez and Wade [HW] then concludes the proof. The analysis is essentially done in Proposition 4.1.

We now present the structure of the article. In Section 2, we recall basic definitions associated with $\text{Out}(F_N)$ such as free factor systems, splittings and properties of the action of $\text{Out}(F_N)$ on the relative free factor graph. Sections 3 and 4 are devoted to the proof of Theorem 1.1. Section 3 focuses on the sporadic case and we finish the proof

of Theorem 1.1 in Section 4. Finally, in Section 5, we prove Theorem 1.3 regarding the normalizer of abelian subgroups of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ and Theorem 1.5 regarding the centralizer of some elements of $\mathrm{Out}(F_N)$.

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2 Preliminaries

2.1 Free factor systems of F_N

Let $N \geq 2$ and let F_N be a nonabelian free group of rank N . In this section, we present the definition of free factor systems of F_N , which are specific finite sets of conjugacy classes of subgroups of F_N . First, we give the definition of some larger collections of conjugacy classes of subgroups of F_N called *subgroup systems*.

Definition 2.1 (Subgroup system). A *subgroup system* is a finite (possibly trivial) set \mathcal{A} of conjugacy classes of finitely generated subgroups of F_N .

There exists a natural partial order on the set of subgroup systems of F_N , where $\mathcal{A}_1 \leq \mathcal{A}_2$ if for every subgroup A of F_N such that $[A] \in \mathcal{A}_1$, there exists $[B] \in \mathcal{A}_2$ such that $A \subseteq B$. The subgroup system \mathcal{A}_2 is then an *extension* of \mathcal{A}_1 .

The group $\mathrm{Out}(F_N)$ has a natural action on the set of subgroup systems, and this action preserves the partial order defined above. Given a subgroup system \mathcal{A} , we denote by $\mathrm{Out}(F_N, \mathcal{A})$ the subgroup of $\mathrm{Out}(F_N)$ preserving \mathcal{A} .

Let \mathcal{A} be a subgroup system of F_N . An element g of F_N is *\mathcal{A} -peripheral* if there exists $[A] \in \mathcal{A}$ such that $g \in A$. Otherwise, we say that g is *\mathcal{A} -nonperipheral*. A subgroup of F_N is *\mathcal{A} -nonperipheral* if it contains an *\mathcal{A} -nonperipheral* element, and is *\mathcal{A} -peripheral* otherwise.

We now present an important family of subgroup systems, the *free factor systems*. A *free factor system* is a subgroup system $\mathcal{F} = \{[A_1], \dots, [A_k]\}$ of F_N such that there exists a subgroup B of F_N with $F_N = A_1 * \dots * A_k * B$.

Definition 2.2 (Sporadic extension). Let $\mathcal{F}_1 \leq \mathcal{F}_2$ be two free factor systems of F_N . The extension $\mathcal{F}_1 \leq \mathcal{F}_2$ is *sporadic* if one of the following holds:

- (1) there exist $[A], [B] \in \mathcal{F}_1$ such that $\mathcal{F}_2 = (\mathcal{F}_1 - \{[A], [B]\}) \cup \{[A * B]\}$;
- (2) there exist $[A] \in \mathcal{F}_1$ and $g \in F_N$ such that $\mathcal{F}_2 = (\mathcal{F}_1 - \{[A]\}) \cup \{[A * \langle g \rangle]\}$;
- (3) there exists $g \in F_N$ such that $\mathcal{F}_2 = \mathcal{F}_1 \cup \{[\langle g \rangle]\}$.

Otherwise, we say that the extension $\mathcal{F}_1 \leq \mathcal{F}_2$ is *nonsporadic*.

A free factor system \mathcal{F} of F_N is *sporadic* if the extension $\mathcal{F} \leq \{[F_N]\}$ is sporadic. Note that, for a sporadic free factor system \mathcal{F} , either $\mathcal{F} = \{[A]\}$ or $\mathcal{F} = \{[A], [B]\}$ for some subgroups $A, B \subseteq F_N$.

Let $\phi \in \text{Out}(F_N)$ and let \mathcal{F} be a free factor system of F_N . Suppose that ϕ fixes every element of \mathcal{F} . Then, for every $[A] \in \mathcal{F}$, by malnormality of A , the element ϕ induces an element $\phi|_A \in \text{Out}(A)$.

2.2 Splittings of F_N and twists automorphisms

A *splitting* of F_N is an F_N -equivariant homeomorphism class \mathcal{S} of a minimal, simplicial action of F_N on a simplicial tree S . Here, minimal means that S does not contain a proper F_N -invariant subtree. If \mathcal{F} is a free factor system of F_N , an (F_N, \mathcal{F}) -*splitting* \mathcal{S} is a splitting of F_N such that, for every $[A] \in \mathcal{F}$, the group A is elliptic in \mathcal{S} . If v is a vertex of S , we denote by G_v its stabilizer. Let V be a set of representatives of the F_N -orbits of vertices in S .

A splitting \mathcal{S} is *free* if edge stabilizers of S are trivial. Note that, in this case, the set $\{[G_v]\}_{v \in V}$ is a free factor system of F_N . Given a free factor system \mathcal{F} of F_N , a *Grushko (F_N, \mathcal{F}) -free splitting* is an (F_N, \mathcal{F}) -free splitting such that, for every $v \in V$ with nontrivial stabilizer, we have $[G_v] \in \mathcal{F}$.

The group $\text{Aut}(F_N)$ acts on the right on the set of splittings of F_N by precomposition of the action. This action passes to the quotient to give an action of $\text{Out}(F_N)$. Note that this action preserves the set of free splittings.

Let \mathcal{F} be a free factor system of F_N . Given two (F_N, \mathcal{F}) -splittings \mathcal{S} and \mathcal{S}' , we say that \mathcal{S} is a *refinement* of \mathcal{S}' if there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S' is obtained from S by collapsing some orbits of edges. Two (F_N, \mathcal{F}) -splittings are *compatible* if they have a common refinement which is an (F_N, \mathcal{F}) -splitting.

Let \mathcal{S} be a splitting of F_N and let S be a representative of \mathcal{S} . The subgroup $\text{Stab}(\mathcal{S})$ of elements of $\text{Out}(F_N)$ which fix \mathcal{S} has a natural description by a result of Levitt [Lev, Proposition 4.2]. Indeed, there exists a natural homomorphism from $\text{Stab}(\mathcal{S})$ to the group of graph automorphisms $\text{Aut}_{gr}(\overline{F_N \setminus S})$ of the underlying graph of $F_N \setminus S$. We denote the kernel of this homomorphism by $K(\mathcal{S})$. Moreover, there exists a natural homomorphism $K(\mathcal{S}) \rightarrow \prod_{v \in V} \text{Out}(G_v)$ given by the action on vertex stabilizers. The kernel of this homomorphism, denoted by $T(\mathcal{S})$, is the *group of twists* of \mathcal{S} .

Lemma 2.3. [CLus, Lemma 5.3] *Let \mathcal{S} be a splitting of F_N whose edge stabilizers are all nontrivial. The group $T(\mathcal{S})$ is central in $K(\mathcal{S})$.*

Let \mathcal{F} be a sporadic free factor system of F_N . Then one can naturally associate to \mathcal{F} a Grushko (F_N, \mathcal{S}) -free splitting \mathcal{S} , which is the *Bass-Serre tree of F_N associated with \mathcal{F}* . A representative $S \in \mathcal{S}$ has exactly one orbit of edges. The splitting \mathcal{S} is fixed by $\text{Out}(F_N, \mathcal{F})$. Moreover, by [Lev, Proposition 3.1], the group of twists of \mathcal{S} is isomorphic to a direct product of two free (maybe cyclic) groups.

2.3 Properties of the subgroup $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$

Let $N \geq 2$ and let

$$\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z}) = \ker(\mathrm{Out}(F_N) \rightarrow \mathrm{Aut}(H_1(F_N, \mathbb{Z}/3\mathbb{Z}))).$$

In this section, we recall some properties of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ that will be used in the proof of Theorem 1.1. Most of the following results state the aperiodicity of some orbits associated with the action of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ on some natural sets.

Proposition 2.4. [BFH1, Corollary 5.7.6] *The group $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is torsion free.*

Theorem 2.5. [HM3, Theorem II.3.1] *Let H be a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ and let \mathcal{F} be an H -periodic free factor system. Then \mathcal{F} is fixed by H and every element $[A] \in \mathcal{F}$ is fixed by H .*

Theorem 2.6. [HM3, Theorem II.4.1] *Let H be a subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$. Then every H -periodic conjugacy class of some element of F_N is fixed by H .*

Theorem 2.7. [HM4, Theorem 1.1] *Let H be a virtually abelian subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$. Then H is abelian and finitely generated.*

The fact, stated in Theorem 2.7, that any abelian subgroup H of $\mathrm{Out}(F_N)$ is finitely generated also follows from the work of Bass and Lubotzky [BL].

Lemma 2.8. [HW, Lemma 2.6] *Let $H \subseteq \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ and let \mathcal{S} be an H -periodic free splitting. Then \mathcal{S} is fixed by H and $H \subseteq K(\mathcal{S})$.*

Lemma 2.9. *A subgroup H of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is abelian if and only if it does not contain a nonabelian free group.*

Proof. By the Tits alternative due to Bestvina, Feighn and Handel [BFH1, BFH3], either H is virtually solvable, or it contains a nonabelian free group. By [BFH2, Theorem 1.1] (see also [Ali, Corollary 1.3]), either H is virtually abelian or H contains a nonabelian free group. By Theorem 2.7, either H is abelian or it contains a nonabelian free group. \square

2.4 Relative free factor graph and relative arational trees

Let \mathcal{F} be a free factor system of F_N . The *free factor graph* of F_N relative to \mathcal{F} , denoted by $\mathrm{FF}(F_N, \mathcal{F})$, is the graph whose vertices are the (F_N, \mathcal{F}) -free splittings of F_N , two such splittings being adjacent if they have a common refinement which is free or if they share a common elliptic \mathcal{F} -nonperipheral element. By a result of Guirardel and Horbez [GH1, Proposition 2.11] (see also the work of Handel and Mosher [HM1] for the case $\mathcal{F} = \emptyset$), the graph $\mathrm{FF}(F_N, \mathcal{F})$ is Gromov-hyperbolic. In the rest of the section, we describe the group of isometries of $\mathrm{FF}(F_N, \mathcal{F})$ as well as its Gromov boundary.

The group $\mathrm{Out}(F_N, \mathcal{F})$ acts naturally on $\mathrm{FF}(F_N, \mathcal{F})$ by isometries. An outer automorphism $\phi \in \mathrm{Out}(F_N, \mathcal{F})$ is *fully irreducible relative to \mathcal{F}* if there does not exist a proper free factor system $\mathcal{F} < \mathcal{F}'$ fixed by a power of ϕ . These elements are in fact the loxodromic elements of $\mathrm{FF}(F_N, \mathcal{F})$.

Theorem 2.10. [Gup, Theorem A] *Let \mathcal{F} be a nonsporadic free factor system of F_N . An element $\phi \in \text{Out}(F_N, \mathcal{F})$ is a loxodromic element of $\text{FF}(F_N, \mathcal{F})$ if and only if ϕ is fully irreducible relative to \mathcal{F} .*

Theorem 2.10 was proved by Gupta [Gup] and later extended by Guirardel and Horbez to outer automorphisms of free products of groups [GH1, Theorem 4.1].

The following theorem was proved by Handel and Mosher [HM3] in the finitely generated case and by Guirardel and Horbez [GH1] in the general case.

Theorem 2.11. [GH1, Theorem 7.1][HM3, Theorem A] *Let H be a subgroup of $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ and let \mathcal{F} be a maximal proper H -invariant free factor system. Suppose that \mathcal{F} is nonsporadic. Then H contains a fully irreducible outer automorphism relative to \mathcal{F} .*

We now describe the Gromov boundary of $\text{FF}(F_N, \mathcal{F})$. An (F_N, \mathcal{F}) -free factor is a subgroup of F_N which arises as a point stabilizer of some (F_N, \mathcal{F}) -free splitting. An (F_N, \mathcal{F}) -free factor is *proper* if it is \mathcal{F} -nonperipheral and not equal to F_N . Note that, if A is an (F_N, \mathcal{F}) -free factor, then \mathcal{F} induces a free factor system of A , denoted by $\mathcal{F}|_A$.

An (F_N, \mathcal{F}) -arational tree is an \mathbb{R} -tree T equipped with an F_N -action by isometries such that T is not a Grushko (F_N, \mathcal{F}) -free splitting and such that, for every proper (F_N, \mathcal{F}) -free factor A , the action of A on its minimal tree is a Grushko $(A, \mathcal{F}|_A)$ -free splitting.

We record the following fact, which is a consequence of the description of the Gromov boundary of $\text{FF}(F_N, \mathcal{F})$. It is due to Hamenstädt [Ham] for the case $\mathcal{F} = \emptyset$, and Guirardel and Horbez [GH1] for the general case.

Proposition 2.12. [GH1, Theorem 3.4] *Let \mathcal{F} be a nonsporadic free factor system of F_N and let H be a subgroup of $\text{Out}(F_N, \mathcal{F})$. If H has a finite orbit in $\partial_\infty \text{FF}(F_N, \mathcal{F})$, then H has a finite index subgroup which fixes the homothety class of an (F_N, \mathcal{F}) -arational tree.*

We now describe the stabilizer in $\text{Out}(F_N, \mathcal{F})$ of the homothety class $[T]$ of an arational (F_N, \mathcal{F}) -tree T . We have a natural homomorphism

$$\text{SF}: \text{Stab}([T]) \rightarrow \mathbb{R}_+^*$$

given by the stretching factor, whose kernel is denoted by $\text{Stab}_{\text{isom}}(T)$. The homomorphism SF has the following properties.

Lemma 2.13. [GH1, Lemma 6.2, Proposition 6.3, Corollary 6.12] (1) *The image of SF is cyclic.*

(2) *For every $\phi \in \text{Stab}([T])$, we have $\text{SF}(\phi) \neq 1$ if and only if ϕ is fully irreducible relative to \mathcal{F} .*

The following proposition is extracted from [HW], where it is attributed to Guirardel and Levitt. Given an (F_N, \mathcal{F}) -arational tree T , it describes an (F_N, \mathcal{F}) -splitting canonically associated with a subgroup of $\text{Stab}_{\text{isom}}(T)$.

Proposition 2.14. [HW, Lemmas 5.3, 5.6, Theorem 5.4] Let \mathcal{F} be a nonsporadic free factor system and let T be an (F_N, \mathcal{F}) -arational tree. Let $H \subseteq \text{Stab}_{\text{isom}}(T) \cap \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ be a subgroup. There exists an (F_N, \mathcal{F}) -splitting $\mathcal{S}_{T,H}$ fixed by the normalizer of H in $\text{Out}(F_N, \mathcal{F})$ with the following properties.

- (1) Every edge stabilizer is nontrivial.
- (2) There exists a partition $V = V_0 \sqcup V_1$ of the vertices of $\mathcal{S}_{T,H}$ such that
 - (a) for every $v \in V_1$, the conjugacy class $[G_v]$ is contained in \mathcal{F} ;
 - (b) the set V_0 consists in a unique F_N -orbit of a vertex $v_0 \in V_0$. The group G_{v_0} is a nonabelian free group. Moreover, the homomorphism

$$H \cap K(\mathcal{S}_{T,H}) \rightarrow \text{Out}(G_{v_0})$$

has trivial image.

We will also use the following corollary, which is an easy consequence of Proposition 2.14 (2)(b) (see [GH1, Proposition 6.10] for a proof).

Corollary 2.15. [GH1, Proposition 6.10] Let \mathcal{F} be a nonsporadic free factor system and let T be an (F_N, \mathcal{F}) -arational tree. For every $\phi \in \text{Stab}_{\text{isom}}(T) \cap \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$, there exist a nonabelian free group F of F_N and a representative $\Phi \in \phi$ such that $\Phi(F) = F$ and $\Phi|_F = \text{id}_F$.

3 The sporadic case

In this section, we prove Theorem 1.1 when the elements $\phi, \psi \in \text{IA}_N(\mathbb{Z}, 3\mathbb{Z})$ fix a sporadic free factor system. We first need some preliminary lemmas.

Lemma 3.1. Let $\Phi, \Psi \in \text{Aut}(F_N)$. Suppose that there exists $k \in \mathbb{N}^*$ such that $\Phi^k = \Psi^k$ and that the image ϕ of Φ in $\text{Out}(F_N)$ is equal to the image ψ of Ψ and is contained in $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$. Then $\Phi = \Psi$.

Proof. We only need to show that Φ and Ψ commute. Indeed, in this case, the group $\langle \phi, \psi \rangle$ is abelian. By Proposition 2.4, the group $\langle \phi, \psi \rangle$ is also torsion free. Thus, since the kernel of $\text{Aut}(F_N) \rightarrow \text{Out}(F_N)$ is a nonabelian free group, the group $\langle \Phi, \Psi \rangle$ is free abelian. Since $\Phi^k = \Psi^k$, we have in fact $\Phi = \Psi$.

So let us prove that Φ commutes with Ψ . Let K be the normal subgroup of $\langle \Phi, \Psi \rangle$ consisting in all its inner automorphisms and let K_0 be the subgroup of F_N consisting in all elements $g \in F_N$ with $\text{ad}_g \in K$.

We first treat the case where K_0 is cyclic, generated by $g \in F_N$. Since K is a normal subgroup of $\langle \Phi, \Psi \rangle$, the group K_0 is preserved by $\langle \Phi, \Psi \rangle$. By Theorem 2.6, since $\phi, \psi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$, the element g is fixed by $\langle \Phi, \Psi \rangle$. Note that, since $\phi = \psi$, we have $\Phi\Psi^{-1} \in K$. Since K_0 is cyclic, we have $\Phi = \Psi \circ \text{ad}_{g^m}$ for some $m \in \mathbb{Z}$. Since g is fixed

by Ψ , we see that $\Phi^k = \Psi^k \circ \text{ad}_{g^{km}}$. Since $\Phi^k = \Psi^k$, we have $\text{ad}_{g^{km}} = \text{id}$, so that $g = e$ and $\Phi = \Psi$. This concludes the proof when K_0 is cyclic.

Suppose now that K_0 is a nonabelian free group. Since Φ^k is central in $\langle \Phi, \Psi \rangle$, it fixes every element $g \in K_0$. Since $\phi, \psi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$, by Theorem 2.6, both ϕ and ψ fix the conjugacy class of every element in K_0 . Thus, there exist $g_\phi, g_\psi \in F_N$ such that, for every $g \in K_0$, we have $\Phi(g) = g_\phi g g_\phi^{-1}$ and $\Psi(g) = g_\psi g g_\psi^{-1}$.

We claim that g_ϕ and g_ψ are fixed by Φ^k . We prove the result for g_ϕ , the arguments for g_ψ being similar. Since the group K is normal in $\langle \Phi, \Psi \rangle$, the automorphism Φ preserves K_0 . In particular, for every $g \in K_0$, we have $g_\phi g g_\phi^{-1} \in K_0$. Thus, for every $g \in K_0$, we have

$$\Phi^k(g_\phi g g_\phi^{-1}) = \Phi^k(g_\phi) g \Phi^k(g_\phi^{-1}) = g_\phi g g_\phi^{-1},$$

where the first equality follows from the fact that g is fixed by Φ^k . Thus, for every $g \in K_0$, the element $\Phi^k(g_\phi)^{-1} g_\phi$ commutes with g . Since this is true for every $g \in K_0$ and since the group K_0 is nonabelian, we have in fact $\Phi^k(g_\phi) = g_\phi$.

Thus, the group $K'_0 = \langle K_0, g_\phi, g_\psi \rangle$ is fixed elementwise by Φ^k . Since K_0 is a nonabelian free group and since $\phi, \psi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$, by Theorem 2.6, the automorphisms Φ and Ψ act on K'_0 by a global conjugation by respectively g_ϕ and g_ψ .

Let $g \in K_0$ be such that $\Phi = \text{ad}_g \circ \Psi$, which exists since $\phi = \psi$. We claim that $g \in \langle g_\psi \rangle$. Indeed, otherwise $\langle g, g_\psi \rangle$ is a nonabelian free group of rank 2. But $g_\phi = g g_\psi$, so that, for every $i \in \mathbb{N}^*$, the element $\Phi^i(g_\psi)$ can be written as a nontrivial product of elements in $\langle g_\psi \rangle$ and elements in $\langle g \rangle$ (recall that Φ acts on K'_0 by a global conjugation by g_ϕ). This contradicts the fact that $\Phi^k(g_\psi) = g_\psi$.

Hence we have $g \in \langle g_\psi \rangle$. Thus, the element g is fixed by Ψ . Therefore, we see that Ψ commutes with ad_g . Since $\Psi^{-1} \circ \Phi = \text{ad}_g$, we see that Ψ commutes with Φ . This concludes the proof. \square

Lemma 3.2. *Let $\Phi \in \text{Aut}(F_N)$ with $\phi = [\Phi] \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$. Let $a, b \in F_N$ and $k \in \mathbb{N}$ be such that*

$$\prod_{i=0}^k \Phi^i(a) = \prod_{i=0}^k \Phi^i(b).$$

Then $a = b$.

Proof. Let $a, b \in F_N$ be such that $\prod_{i=0}^k \Phi^i(a) = \prod_{i=0}^k \Phi^i(b)$. Let $A = \prod_{i=0}^{k-1} \Phi^i(a)$ and let $B = \prod_{i=0}^{k-1} \Phi^i(b)$. Note that, for every $c \in \{a, b\}$, we have

$$\prod_{i=0}^k \Phi^i(c) = c \Phi(C),$$

so that

$$a^{-1}b = \Phi(AB^{-1}).$$

Moreover, we have $A\Phi^k(a) = B\Phi^k(b)$. Thus,

$$\Phi^k(a^{-1}b) = (\Phi^k(a))^{-1}\Phi^k(b) = (\Phi^k(a))^{-1}B^{-1}A\Phi^k(a).$$

Thus, the outer automorphism ϕ^{k+1} sends the conjugacy class of AB^{-1} to the conjugacy class of $B^{-1}A$. Since $AB^{-1} = BB^{-1}AB^{-1}$, we see that ϕ^{k+1} sends the conjugacy class of AB^{-1} to itself. Thus the orbit of $[AB^{-1}]$ under iteration of ϕ is periodic. By Theorem 2.6, we see that the conjugacy class of AB^{-1} is fixed by ϕ . Thus, we have $[AB^{-1}] = [a^{-1}b]$, that is

$$[a\Phi(a)\dots\Phi^k(a)\Phi^k(b^{-1})\dots\Phi(b^{-1})b^{-1}] = [a^{-1}b].$$

Since

$$[a\Phi(a)\dots\Phi^k(a)\Phi^k(b^{-1})\dots\Phi(b^{-1})b^{-1}] = [\Phi(a)\dots\Phi^k(a)\Phi^k(b^{-1})\dots\Phi(b^{-1})b^{-1}a],$$

there exists $h \in F_N$ with

$$[\Phi(a)\dots\Phi^k(a)\Phi^k(b^{-1})\dots\Phi(b^{-1})] = [ha^{-1}bh^{-1}a^{-1}b].$$

Using this argument inductively, we see that there exist $g, h_0, \dots, h_k \in F_N$ with

$$ga^{-1}bg^{-1} \prod_{i=0}^k h_i \Phi^i(a^{-1}b) h_i^{-1} = e.$$

Since Φ preserves the conjugacy class of $a^{-1}b$, there exist $g, g_0, \dots, g_k \in F_N$ with

$$ga^{-1}bg^{-1} \prod_{i=0}^k g_i a^{-1} b g_i^{-1} = e.$$

Thus some product of conjugates of $a^{-1}b$ is trivial. By for instance [CLyn, Corollary 4.5], this implies that $a^{-1}b = e$, which concludes the proof. \square

Lemma 3.3. *Let $\phi, \psi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ and let $k \in \mathbb{N}^*$ be such that $\phi^k = \psi^k$. Suppose that there exists a $\langle \phi, \psi \rangle$ -invariant sporadic free factor system \mathcal{F} of F_N such that, for every $[A] \in \mathcal{F}$, we have $\phi|_A = \psi|_A$. Then $\phi = \psi$.*

Proof. Since \mathcal{F} is sporadic, the Bass-Serre tree \mathcal{S} of F_N associated with \mathcal{F} is a Grushko (F_N, \mathcal{F}) -free splitting fixed by $\text{Out}(F_N, \mathcal{F})$, and in particular by $\langle \phi, \psi \rangle$. Since we have $\langle \phi, \psi \rangle \subseteq \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$, by Lemma 2.8, we also have $\langle \phi, \psi \rangle \subseteq K(\mathcal{S})$. We distinguish between two cases, according to the distinct cases in the definition of a sporadic free factor system.

Suppose first that $\mathcal{F} = \{[A], [B]\}$, so that $F_N = A * B$. For every $\theta \in K(\mathcal{S})$, let Θ be the unique representative of θ such that $\Theta(A) = A$ and $\Theta(B) = B$. By [Lev, Proposition 4.2], the map $\theta \mapsto (\Theta|_A, \Theta|_B)$ gives an isomorphism between $K(\mathcal{S})$ and $\text{Aut}(A) \times \text{Aut}(B)$. Note that, by assumption on ϕ and ψ , the outer automorphism class of $\Phi|_A$ (resp. $\Phi|_B$) is the same as the one of $\Psi|_A$ (resp. $\Psi|_B$). Since $\phi|_A, \psi|_A \in \text{IA}(A, \mathbb{Z}/3\mathbb{Z})$ and $\phi|_B, \psi|_B \in \text{IA}(B, \mathbb{Z}/3\mathbb{Z})$, by Lemma 3.1, we have $\Phi|_A = \Psi|_A$ and $\Phi|_B = \Psi|_B$. Thus, we have $\phi = \psi$.

Suppose now that $\mathcal{F} = \{[A]\}$ and let $g \in F_N$ be such that $F_N = A * \langle g \rangle$. For every $\theta \in K(\mathcal{S})$, let Θ be the unique representative of θ such that $\Theta(A) = A$ and $\Theta(g) = gg\theta$ with $g\theta \in A$. By [Lev, Proposition 4.2], the map $\theta \mapsto (g\theta, \Theta|_A)$ gives an isomorphism between $K(\mathcal{S})$ and $A \rtimes \text{Aut}(A)$. As in the first case, by Lemma 3.1, we have $\Phi|_A = \Psi|_A$.

It now suffices to prove that $g_\phi = g_\psi$. But $g_{\phi^k} = g_{\psi^k}$. Thus, we have

$$g_{\phi^k} = \prod_{i=0}^{k-1} \Phi^i(g_\phi) = g_{\psi^k} = \prod_{i=0}^{k-1} \Psi^i(g_\psi) = \prod_{i=0}^{k-1} \Phi^i(g_\psi),$$

where the last equality follows from the fact that $\Phi|_A = \Psi|_A$ and from the fact that $g_\psi \in A$. By Lemma 3.2, we have $g_\phi = g_\psi$. This concludes the proof. \square

4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 in the general case. First, we need a proposition regarding stabilizers of (F_N, \mathcal{F}) -arational trees which is of independent interest.

Proposition 4.1. *Let H be a subgroup of $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ and let \mathcal{F} be a nonsporadic free factor system fixed by H . Suppose that H fixes the homothety class of an (F_N, \mathcal{F}) -arational tree T and, for every $[A] \in \mathcal{F}$, that the image of the homomorphism $H \rightarrow \text{Out}(A)$ is abelian. Then H is abelian.*

Proof. By Lemma 2.9, it suffices to show that H does not contain a nonabelian free group.

Note that the stretching factor homomorphism SF associated with $\text{Stab}([T])$ has abelian image. Thus, the group H contains a nonabelian free group if and only if so does $K = \text{Stab}_{\text{isom}}(T) \cap H$.

Let $\mathcal{S}_{T,K}$ be the canonical (F_N, \mathcal{F}) -splitting associated with T and K given by Proposition 2.14. Let $V = V_0 \sqcup V_1$ be the associated partition of the vertices, let $v \in V_0$ and let v_1, \dots, v_k be representatives of the F_N -orbits in V_1 . Note that, by Proposition 2.14 (2)(b) the set V_0 consists in a unique F_N -orbit. Up to taking a finite index subgroup of K , we have a homomorphism

$$\Lambda: K \rightarrow \text{Out}(G_v) \times \prod_{i=1}^k \text{Out}(G_{v_i}).$$

By Proposition 2.14 (2)(a), for every $i \in \{1, \dots, k\}$, there exists $[A] \in \mathcal{F}$ such that $[G_{v_i}] = [A]$. Thus, by assumption, for every $i \in \{1, \dots, k\}$, the image of K in $\text{Out}(G_{v_i})$ is abelian. Moreover, by Proposition 2.14 (2)(b), the image of K in $\text{Out}(G_v)$ is trivial. Hence the image of Λ is abelian.

Thus H contains a nonabelian free subgroup if and only if so does $\ker(\Lambda)$. But $\ker(\Lambda)$ is contained in the group of twists of \mathcal{S}_T . Recall that, by Proposition 2.14 (1), every edge stabilizer of \mathcal{S}_T is nontrivial. Hence, by Lemma 2.3, the group of twists of \mathcal{S}_T is central in $K(\mathcal{S})$. Thus, the group $\ker(\Lambda)$ is abelian. This concludes the proof. \square

Proof of Theorem 1.1. Let $\phi, \psi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ and suppose that there exists $k \in \mathbb{N}^*$ such that $\phi^k = \psi^k$. We prove that $\phi = \psi$ by induction on the rank N of F_N . When $N = 1$, we have $\text{IA}_N(\mathbb{Z}/3\mathbb{Z}) = \{\text{id}\}$ and there is nothing to prove.

Suppose that $N \geq 2$ and let $H = \langle \phi, \psi \rangle$. Let $\mathcal{F} = \{[A_1], \dots, [A_\ell]\}$ be a maximal proper H -invariant free factor system. Since $\phi, \psi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$, by Theorem 2.5, every element of \mathcal{F} is fixed by H . Therefore, for every $i \in \{1, \dots, \ell\}$, we have a natural homomorphism $H \rightarrow \text{Out}(A_i)$ whose image is contained in $\text{IA}_N(A_i, \mathbb{Z}/3\mathbb{Z})$. By induction, for every $i \in \{1, \dots, \ell\}$, we have $\phi|_{A_i} = \psi|_{A_i}$. Hence the image of the homomorphism

$$\Lambda: H \rightarrow \prod_{i=1}^{\ell} \text{Out}(A_i)$$

is cyclic. Thus, if \mathcal{F} is sporadic, by Lemma 3.3, we see that $\phi = \psi$.

Therefore, we may suppose that \mathcal{F} is a nonsporadic free factor system. We prove that the group H is abelian. Since $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is torsion free by Proposition 2.4 and since $\phi^k = \psi^k$, this will conclude the proof. By Lemma 2.9, it suffices to prove that H does not contain a nonabelian free group.

We claim that ϕ and ψ are fully irreducible relative to \mathcal{F} . Indeed, otherwise, we may suppose that some power of ϕ fixes a free factor system \mathcal{F}' with $\mathcal{F} < \mathcal{F}' < \{[F_N]\}$. By Theorem 2.5, the element ϕ fixes \mathcal{F}' . Thus, $\phi^k = \psi^k$ fixes \mathcal{F}' . By Theorem 2.5, we see that ψ and H fix \mathcal{F}' . This contradicts the maximality of \mathcal{F} .

Hence ϕ and ψ are fully irreducible relative to \mathcal{F} . Thus, by Theorem 2.10, they act loxodromically on the Gromov hyperbolic free factor graph $\text{FF}(F_N, \mathcal{F})$ relative to \mathcal{F} . Note that both ϕ and ψ fix exactly two points in $\partial_\infty \text{FF}(F_N, \mathcal{F})$ by hyperbolicity of $\text{FF}(F_N, \mathcal{F})$. Since $\phi^k = \psi^k$, it follows that ϕ and ψ fix the same two points in the Gromov boundary $\partial_\infty \text{FF}(F_N, \mathcal{F})$. Thus, the group H has a finite orbit in $\partial_\infty \text{FF}(F_N, \mathcal{F})$.

By Proposition 2.12, H has a finite index subgroup H' which fixes the homothety class of an arational (F_N, \mathcal{F}) -tree. Note that H contains a nonabelian free group if and only if so does H' .

But the group

$$\Lambda(H') \subseteq \prod_{i=1}^k \text{Out}(A_i)$$

is cyclic by hypothesis. Thus, we can apply Proposition 4.1 to show that H' is abelian. Hence H does not contain a nonabelian free group. This concludes the proof. \square

Following Kontorovič [Kon] (see also [Bau]), Theorem 1.1 implies that the group $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is an R -group. The following properties then follow for instance from [FW, Proposition 2.2].

Corollary 4.2. *The group $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ satisfies the following properties.*

- (1) *For every $\phi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$, the normalizer of $\langle \phi \rangle$ in $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ coincides with its centralizer.*
- (2) *For every $\phi, \psi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$, if there exist $m, n \in \mathbb{Z}^*$ such that ϕ^m and ψ^n commute, then ϕ and ψ commute.* \square

5 Abelian subgroups of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$

In this section, we study some properties of abelian subgroups of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$. These properties are mainly consequences of Proposition 4.1 and Corollary 4.2.

Theorem 5.1. *Let $N \geq 2$ and let H be an abelian subgroup of $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$. The normalizer of H in $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is equal to its centralizer.*

Proof. Let $\phi \in N_{\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})}(H)$. In order to prove Theorem 5.1, it suffices to prove that the group $H' = \langle \phi, H \rangle$ is abelian.

We prove the result by induction on N , the case $N = 1$ being immediate. Suppose that $N \geq 2$ and let \mathcal{F} be a maximal proper H' -invariant free factor system of F_N . We distinguish between two cases, according to the nature of \mathcal{F} .

Case A. Suppose that \mathcal{F} is nonsporadic.

Since \mathcal{F} is maximal, by Theorem 2.11, the group H' contains a fully irreducible outer automorphism relative to \mathcal{F} . By Theorem 2.10, the group H' contains a loxodromic element of the relative free factor graph $\mathrm{FF}(F_N, \mathcal{F})$, hence has unbounded orbits in $\mathrm{FF}(F_N, \mathcal{F})$.

Suppose first that H contains a loxodromic element ψ of $\mathrm{FF}(F_N, \mathcal{F})$. Then ψ has exactly two periodic points in $\partial_\infty \mathrm{FF}(F_N, \mathcal{F})$, which are its attracting and repelling fixed points. Since H is abelian, the group H has also at most two finite orbits in $\partial_\infty \mathrm{FF}(F_N, \mathcal{F})$, and such an H -finite orbit exists. This implies that H' has a finite orbit in $\partial_\infty \mathrm{FF}(F_N, \mathcal{F})$. By Proposition 2.12, the group H' has a finite index normal subgroup H'_0 which fixes the homothety class of an (F_N, \mathcal{F}) -arational tree T .

Let H_0 be a characteristic subgroup of H of finite index and contained in H'_0 (it exists since H is finitely generated by Theorem 2.7). Let $k \in \mathbb{N}^*$ be such that $\phi^k \in H'_0$. Then the element ϕ^k normalizes H_0 .

Note that, since $\langle H_0, \phi^k \rangle \subseteq \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$, by Theorem 2.5, for every $[A] \in \mathcal{F}$, we have a homomorphism $\langle H_0, \phi^k \rangle \rightarrow \mathrm{Out}(A)$ whose image is abelian by induction hypothesis. Hence, by Proposition 4.1, the group $\langle H_0, \phi^k \rangle$ is abelian. By Corollary 4.2 (2), we see that ϕ is in the centralizer of H .

Suppose now that H does not contain any loxodromic element of $\mathrm{FF}(F_N, \mathcal{F})$. Since H is a normal subgroup of H' and since H' has unbounded orbits, by for instance [HW, Proposition 4.2], the group H has a (not necessarily unique) finite orbit in $\partial_\infty \mathrm{FF}(F_N, \mathcal{F})$.

By Proposition 2.12, the group H contains a finite index subgroup H_0 which fixes the homothety class of an (F_N, \mathcal{F}) -arational tree T . Since H is abelian, it is finitely generated by Theorem 2.7. Hence we may suppose that H_0 is also a characteristic subgroup of H , so that $H' \subseteq N(H_0)$.

By Lemma 2.13, since H_0 does not contain any fully irreducible element of F_N relative to \mathcal{F} , we have in fact $H_0 \subseteq \mathrm{Stab}_{\mathrm{isom}}(T)$. Let \mathcal{S}_{T, H_0} be the (F_N, \mathcal{F}) -splitting of F_N given by Proposition 2.14. The splitting \mathcal{S}_{T, H_0} is fixed by the normalizer of H_0 in $\mathrm{Out}(F_N, \mathcal{F})$. In particular, it is fixed by H' . Let $V = V_0 \amalg V_1$ be the associated partition of the vertices, let $v \in V_0$ and let v_1, \dots, v_k be representatives of the F_N -orbits in V_1 . Note that, by Proposition 2.14 (2)(b) the set V_0 consists in a unique F_N -orbit.

Recall that, by Proposition 2.14 (2)(a), for every $i \in \{1, \dots, k\}$, we have $[G_{v_i}] \in \mathcal{F}$. Since $H' \subseteq \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$, Theorem 2.5 implies that, for every $i \in \{1, \dots, k\}$, the group H' fixes the conjugacy class of G_{v_i} . Since V_0 consists in a unique F_N -orbit, we see that $H' \subseteq K(\mathcal{S}_{T, H_0})$. Therefore, we have a homomorphism

$$\Lambda_0: H' \rightarrow \text{Out}(G_v) \times \prod_{i=1}^k \text{Out}(G_{v_i}).$$

By induction and Proposition 2.14 (2)(a), for every $i \in \{1, \dots, k\}$, the homomorphism $H' \rightarrow \text{Out}(G_{v_i})$ has abelian image.

By Proposition 2.14 (2)(b), the image of the homomorphism $H_0 \rightarrow \text{Out}(G_v)$ is trivial. By [HW, Lemma 5.5], this implies that the image of the homomorphism $H \rightarrow \text{Out}(G_v)$ is also trivial. Since $H' = \langle H, \phi \rangle$, the homomorphism $H' \rightarrow \text{Out}(G_v)$ has cyclic image. Thus H' contains a nonabelian free group if and only if $\ker(\Lambda_0)$ contains a nonabelian free group.

By Proposition 2.14 (1), every edge stabilizer of \mathcal{S}_{T, H_0} is nontrivial. By Lemma 2.3, the kernel $\ker(\Lambda_0)$ is abelian. This shows that H' does not contain a nonabelian free group and, by Lemma 2.9, that H' is abelian. This proves Theorem 5.1 when \mathcal{F} is nonsporadic.

Case B. Suppose that \mathcal{F} is sporadic.

The group H' then fixes the splitting \mathcal{S} associated with the Bass-Serre tree of F_N relative to \mathcal{F} . Let V be a set of representatives of the orbits of vertices of $\mathcal{S} \in \mathcal{S}$.

Since $H' \subseteq \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$, by Lemma 2.8, we have $H' \subseteq K(\mathcal{S})$. Thus, we have a homomorphism

$$\Lambda: H' \rightarrow \prod_{v \in V} \text{Out}(G_v)$$

whose kernel is $T(\mathcal{S}) \cap H'$. Since \mathcal{S} is the Bass-Serre tree of F_N associated with \mathcal{F} , for every $v \in V$, we have $[G_v] \in \mathcal{F}$. By induction hypothesis, the image of Λ is abelian.

Recall that, by Lemma 2.9, in order to prove that H' is abelian, it suffices to prove that H' does not contain a nonabelian free group. Note that, since the image of Λ is abelian, the group H' contains a nonabelian free group if and only if $T(\mathcal{S}) \cap H'$ contains a nonabelian free group. Note also that, since H is normal in $H' = \langle \phi, H \rangle$, for every $\psi \in H'$, there exist $k_\psi \in \mathbb{Z}$ and $h_\psi \in H$ such that $\psi = \phi^{k_\psi} h_\psi$. We distinguish between several cases, according to the intersection $H' \cap T(\mathcal{S})$.

Subcase 1. Suppose that, for every $\psi \in T(\mathcal{S}) \cap H'$, we have $k_\psi = 0$.

In that case, we see that $T(\mathcal{S}) \cap H' \subseteq H$. Since H is abelian, this implies that $T(\mathcal{S}) \cap H'$ is abelian. Thus, H' is abelian.

Subcase 2. Suppose that there exists $\psi \in T(\mathcal{S}) \cap H'$ with $k_\psi \neq 0$ and that $H \cap T(\mathcal{S}) = \{1\}$.

In that case, H and $H' \cap T(\mathcal{S})$ are normal subgroups of H' with trivial intersection. Thus, the groups H and $H' \cap T(\mathcal{S})$ commute. Therefore, ψ commutes with every element

of H . Since H is abelian and since $h_\psi \in H$, this implies that ϕ^{k_ψ} commutes with every element of H . By Corollary 4.2 (2), we see that ψ commutes with every element of H .

Subcase 3. Suppose that there exists $\psi \in T(\mathcal{S}) \cap H'$ with $k_\psi \neq 0$ and that $H \cap T(\mathcal{S}) \neq \{1\}$.

Recall that $T(\mathcal{S})$ is isomorphic to $A \times B$, where A and B are free (maybe cyclic) groups. Since H is abelian, there exist $T_A \in A$ and $T_B \in B$ such that $H \cap T(\mathcal{S}) \subseteq \langle T_A, T_B \rangle$.

Note that ψ normalizes the groups $H \cap T(\mathcal{S})$, A and B . Since A and B are free groups and since $\psi \in T(\mathcal{S})$, the element ψ centralizes T_A and T_B . Since $\psi \in T(\mathcal{S})$ and since A and B are free groups, one of the following holds:

- (i) we have $\psi \in \langle T_A, T_B \rangle$;
- (ii) the element T_A is trivial and $\psi \in \langle A, T_B \rangle$;
- (iii) the element T_B is trivial and $\psi \in \langle B, T_A \rangle$.

Suppose that T_A and T_B are not trivial. In that case, Assertion (i) holds for every $\psi \in H' \cap T(\mathcal{S})$. Thus, the group $H' \cap T(\mathcal{S})$ is contained in the abelian group $\langle T_A, T_B \rangle$. Hence the group $H' \cap T(\mathcal{S})$ does not contain a nonabelian free group. By Lemma 2.9, the group H' is abelian.

Up to exchanging the roles of A and B , suppose that Assertion (ii) holds. Then every element $\psi \in H' \cap T(\mathcal{S})$ with $k_\psi \neq 0$ is contained in $\langle A, T_B \rangle$. Since Assertion (ii) holds, we have $H \cap T(\mathcal{S}) \subseteq \langle T_B \rangle \subseteq B$. Since the centralizer of T_B in $A \times B$ is $\langle A, T_B \rangle$, the group $H' \cap T(\mathcal{S})$ contains a nonabelian free group if and only if $H' \cap A$ contains a nonabelian free group.

Since $H \cap T(\mathcal{S}) \subseteq B$, the groups $H' \cap A$ and H are normal subgroups of H' with trivial intersection. Thus, they commute with each other. The conclusion is now similar to the one in Case 2.

As we have ruled out every case, this concludes the proof of Theorem 5.1. \square

From Theorem 5.1, we immediately deduce the following corollary, which gives obstructions to the existence of subgroups of $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ isomorphic to some semidirect products of groups.

Corollary 5.2. *Let H be a subgroup of $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ isomorphic to a semidirect product $A \rtimes B$, where A is abelian. Then H is isomorphic to the direct product $A \times B$.*

In particular, if H is a subgroup of $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ which fits in a short exact sequence

$$1 \rightarrow A \rightarrow H \rightarrow B \rightarrow 1$$

where A is abelian and B is a free group, then H is isomorphic to $A \times B$. \square

Given an element $\phi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$, we denote by $\text{Fix}(\phi)$ the set of conjugacy classes of maximal cyclic subgroups of F_N fixed by ϕ . The following result follows from the work of Feighn and Handel [FH1] (see also the work of Mutanguha [Mut] regarding centralizers of *atoroidal* elements of $\text{Out}(F_n)$). Corollary 4.2 (1) allows us to extend this result to normalizers.

Theorem 5.3. [FH1] *Let $\phi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ be such that $\text{Fix}(\phi)$ is finite. The normalizer of $\langle \phi \rangle$ in $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$ is abelian.*

Proof. Let H be the centralizer of ϕ in $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$. By Corollary 4.2 (1), it suffices to prove that H is abelian. The proof is by induction on N , the result for $N = 1$ being immediate. Let \mathcal{F} be a maximal proper H -invariant free factor system of F_N . As in the proof of Theorem 5.1, we distinguish between two cases, according to the nature of \mathcal{F} .

Suppose that \mathcal{F} is sporadic and let \mathcal{S} be the splitting of F_N associated with the Bass-Serre tree of F_N relative to \mathcal{F} . Let V be a set of representatives of orbits of vertices in \mathcal{S} . As in the proof of Theorem 5.1, we have a homomorphism $\Lambda: H \rightarrow \prod_{v \in V} \text{Out}(G_v)$ whose image is abelian by induction hypothesis.

We claim that the kernel of Λ is abelian. Indeed, by Lemma 2.9, it suffices to prove that the kernel of Λ does not contain a nonabelian free group. Recall that the kernel of Λ is $H \cap T(\mathcal{S})$ and that the group $T(\mathcal{S})$ is isomorphic to a direct product $G_1 \times G_2$ of two free (maybe cyclic) groups. Recall also that both G_1 and G_2 are normal subgroups of $K(\mathcal{S})$. In particular, let $a \in G_1$ and $b \in G_2$ be such that $ab \in H \cap T(\mathcal{S})$. Then both a and b centralizes ϕ . Thus, $H \cap T(\mathcal{S})$ contains a nonabelian free group if and only if $H \cap G_1$ or $H \cap G_2$ contains a nonabelian free group.

For every $i \in \{1, 2\}$, the map $H \cap G_i \rightarrow F_N$ which sends $\psi \in H \cap G_i$ to its twistor is injective (see for instance [Lev, Proposition 3.1]). By for instance [HW, Lemma 2.9], for every $i \in \{1, 2\}$ and every $\psi \in H \cap G_i$, the element ϕ fixes the conjugacy class of the twistor associated with ψ . Thus, if there exists $i \in \{1, 2\}$ such that $H \cap G_i$ contains a nonabelian free group, then $\text{Fix}(\phi)$ is infinite. Therefore, since $\text{Fix}(\phi)$ is finite, for every $i \in \{1, 2\}$, the group $H \cap G_i$ does not contain a nonabelian free group. Therefore, the kernel of Λ does not contain a nonabelian free group. By Lemma 2.9, this shows that H is abelian. This concludes the proof when \mathcal{F} is sporadic.

Suppose now that \mathcal{F} is nonsporadic. We claim that ϕ is fully irreducible relative to \mathcal{F} . Indeed, by maximality of \mathcal{F} and Theorem 2.11, the group H contains a fully irreducible outer automorphism ψ relative to \mathcal{F} . If $\phi = \psi$, we are done.

Otherwise, by Theorem 2.10, the element ψ is a loxodromic element of the relative free factor graph $\text{FF}(F_N, \mathcal{F})$. Thus ψ has exactly two fixed points in the Gromov boundary $\partial_\infty \text{FF}(F_N, \mathcal{F})$.

Since ϕ commutes with ψ , the group $\langle \phi, \psi \rangle$ has a finite orbit in $\partial_\infty \text{FF}(F_N, \mathcal{F})$. By Proposition 2.12, the group $\langle \phi, \psi \rangle$ has a finite index subgroup K which fixes the homothety class of an (F_N, \mathcal{F}) -arational tree T . Let $k \in \mathbb{N}^*$ be such that $\phi^k \in K$.

We claim that the element ϕ^k is not contained in the kernel of the stretching factor homomorphism $\text{SF}: K \rightarrow \mathbb{R}_+^*$. Indeed, since $\text{Fix}(\phi)$ is finite, by Theorem 2.6, so is $\text{Fix}(\phi^k)$. Therefore, there do not exist a nonabelian free group F of F_N and a representative Φ^k of ϕ^k such that $\Phi^k(F) = F$ and $\Phi^k|_F = \text{id}_F$. Thus, by Corollary 2.15, we see that $\phi^k \notin \text{Stab}_{\text{isom}}(T)$. This proves the claim.

Thus, we have $\text{SF}(\phi^k) \neq 1$. By Lemma 2.13, the element ϕ^k is fully irreducible relative to \mathcal{F} . Thus, ϕ is fully irreducible relative to \mathcal{F} . This proves the claim.

By Theorem 2.10, the element ϕ fixes exactly two points in $\partial_\infty \text{FF}(F_N, \mathcal{F})$. Hence H has a finite orbit in $\partial_\infty \text{FF}(F_N, \mathcal{F})$. By Proposition 2.12, the group H virtually fixes the

homothety class of an (F_N, \mathcal{F}) -arational tree. As in the proof of Theorem 5.1, we can apply Proposition 4.1 to show that H is virtually abelian. By Theorem 2.7, the group H is abelian. This concludes the proof. \square

References

- [AKP] Y. Algom-Kfir and C. Pfaff. *Normalizers and centralizers of cyclic subgroups generated by lone axis fully irreducible outer automorphisms*. New York J. Math. **23** (2017) 365–381.
- [Ali] E. Alibegović. *Translation lengths in $\text{Out}(F_n)$* . Geom. Dedicata **92** (2002) 87–93.
- [AM] N. Andrew and A. Martino. *Centralisers of linear growth automorphisms of free groups*. Preprint [arXiv:2205.12865].
- [Bau] G. Baumslag. *Some aspects of groups with unique roots*. Acta Math. **104** (1960) 217–303.
- [BFH1] M. Bestvina, M. Feighn, and M. Handel. *The Tits alternative for $\text{Out}(F_n)$ I: Dynamics of exponentially growing automorphisms*. Ann. of Math. **151** (2000) 517–623.
- [BFH2] M. Bestvina, M. Feighn, and M. Handel. *Solvable subgroups of $\text{Out}(F_n)$ are virtually abelian*. Geom. Dedicata **104** (2004) 71–96.
- [BFH3] M. Bestvina, M. Feighn, and M. Handel. *The Tits alternative for $\text{Out}(F_n)$ II: A Kolchin type theorem*. Ann. of Math. **161** (2005) 1–59.
- [BL] H. Bass and A. Lubotzky. *Linear-central filtrations on groups, the mathematical legacy of Wilhelm Magnus: groups, geometry and special functions*. Contemp. Math. **169**, Amer. Math. Soc., 1994.
- [CLus] M.M. Cohen and M. Lustig. *The conjugacy problem for Dehn twist automorphisms of free groups*. Comment. Math. Helv. (2) **74** (1999) 179–200.
- [CLyn] D.E. Cohen and R.C. Lyndon. *Free bases for normal subgroups of free groups*. Trans. Amer. Math. Soc. **108** (1963) 526–537.
- [FaH] B. Farb and M. Handel. *Commensurations of $\text{Out}(F_n)$* . Pub. Math. IHÉS **105** (2007) 1–48.
- [FH1] M. Feighn and M. Handel. *Abelian subgroups of $\text{Out}(F_n)$* . Geom. Topol. **13** (2009) 1657–1727.
- [FH2] M. Feighn and M. Handel. *The recognition theorem for $\text{Out}(F_n)$* . Groups Geom. Dyn. (1) **5** (2011) 39–106.
- [FW] T.H. Fay and G.L. Walls. *\overline{R} -groups*. J. Algebra **212** (1999) 375–393.

- [GH1] V. Guirardel and C. Horbez. *Boundaries of relative factor graphs and subgroup classification for automorphisms of free products*. *Geom. Topol.* **26** (2022) 71–126.
- [GH2] V. Guirardel and C. Horbez. *Measure equivalence rigidity of $\text{Out}(F_N)$* . Preprint [arXiv:2103.03696].
- [Gup] R. Gupta. *Loxodromic elements for the relative free factor complex*. *Geom. Dedicata* **196** (2018) 91–121.
- [Ham] U. Hamenstädt. *The boundary of the free splitting graph and of the free factor graph*. Preprint [arXiv:1211.1630].
- [HM1] M. Handel and L. Mosher. *Relative free splitting and free factor complexes*. Preprint [arXiv:1407.3508].
- [HM2] M. Handel and L. Mosher. *Hyperbolic actions and 2nd bounded cohomology of subgroups of $\text{Out}(F_n)$. Part II: Finite lamination subgroups*. Preprint [arXiv:1702.08050], to appear in *Mem. Amer. Math. Soc.*
- [HM3] M. Handel and L. Mosher. *Subgroup Decomposition in $\text{Out}(F_n)$* . *Mem. Amer. Math. Soc.* **264** (2020).
- [HM4] M. Handel and L. Mosher. *Virtually abelian subgroups of $\text{IA}_n(\mathbb{Z}/3)$ are abelian*. *Michigan Math. J.* **69** (2020) 465–485.
- [HW] C. Horbez and R.D. Wade. *Commensurations of subgroups of $\text{Out}(F_N)$* . *Trans. Amer. Math. Soc.* (4) **373** (2020) 2699–2742.
- [Kon] P.G. Kontorovič. *Groups with a separation basis III*. *Mat. Sb.* **19** (1946) 287–308.
- [Lev] G. Levitt. *Automorphisms of hyperbolic groups and graphs of groups*. *Geom. Dedicata.* (1) **114** (2005) 49–70.
- [Mut] J.P. Mutanguha. *Limit trees for free group automorphisms: universality*. Preprint [arXiv:2210.01275].
- [RW] M. Rodenhausen and R.D. Wade. *Centralisers of Dehn twist automorphisms of free groups*. *Math. Proc. Cambridge Philos. Soc.* **159** (2015) 89–114.

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