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A DESCRIPTION OF THE SARKISOV PROGRAM OF HOROSPHERICAL VARIETIES VIA MOMENT POLYTOPES

ENRICA FLORIS¹ AND BORIS PASQUIER¹

ABSTRACT. Let X and Y be horospherical Mori fibre spaces which are birational equivariantly with respect to the group action. Then, there is a horospherical Sarkisov program from X/S to Y/T .

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1. INTRODUCTION

Horospherical varieties are examples of varieties endowed with an action of a linear algebraic group, with finitely many orbits. In particular, they have an open orbit and are therefore rational. An important class of horospherical varieties is given by toric varieties, on which acts the group \mathbb{G}_m^n . Similarly to toric varieties, horospherical varieties admit a combinatorial description. Indeed, to a horospherical variety Z and an ample divisor D one can attach a polytope Q_D , called *the moment polytope*, describing the geometry of the variety and of the action. For instance, some facets of Q_D (those not contained in a wall of the dominant chamber) are in bijection with the divisors of the variety which are invariant by the group.

In [Pas15] the second named author described a minimal model program for horospherical varieties completely in terms of moment polytopes by considering a one-parameter family of polytopes of the form $\{Q_{D+\epsilon K_Z}\}_{\epsilon \in \mathbb{Q}}$. For small values of ϵ the polytope $Q_{D+\epsilon K_Z}$ still defines the same variety Z , but, since K_Z is not pseudoeffective, as ϵ grows, facets of the polytope start collapsing. Eventually,

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the dimension of the polytope drops, defining a fibration to a variety of smaller dimension.

This process, for a suitable choice of D , is a minimal model program and ends with a Mori fibre space $X \rightarrow S$.

The natural question arises then of the relation between two different horospherical Mori fibre spaces which are outcome of two MMP for two different ample divisors.

Two birational Mori fibre spaces, without any further structure, are connected by a *Sarkisov program* by the cornerstone results of [Cor95] and [HM13]. A Sarkisov program is a sequence of diagrams, called *links*, of one of the following forms

$$\begin{array}{cccc}
 I & II & III & IV \\
 \begin{array}{ccc} X' & \dashrightarrow & Y \\ \downarrow & & \downarrow \psi \\ X & & T \\ \downarrow \phi & \swarrow & \\ S & & \end{array} &
 \begin{array}{ccc} X' & \dashrightarrow & Y' \\ \downarrow & & \downarrow \\ X & & Y \\ \downarrow \phi & & \downarrow \psi \\ S & \equiv & T \end{array} &
 \begin{array}{ccc} X & \dashrightarrow & Y' \\ \downarrow \phi & & \downarrow \\ S & & Y \\ & \searrow & \downarrow \psi \\ & & T \end{array} &
 \begin{array}{ccc} X & \dashrightarrow & Y \\ \downarrow \phi & & \downarrow \psi \\ S & & T \\ & \searrow & \swarrow \\ & & R \end{array}
 \end{array}$$

where $W' \rightarrow W$ are extremal divisorial contractions, horizontal dashed arrows are isomorphisms in codimension 1 and all the other arrows are extremal contractions. In type IV, we have two cases depending on whether $S \rightarrow R$ and $T \rightarrow R$ are fibrations (type IV_m) or small birational maps (type IV_s).

If X/S and Y/T are two Mori fibre spaces carrying the action of a connected group and which are birational equivariantly with respect to the group, then by [Flo20] there is an equivariant Sarkisov program, that is one in which all the arrows in the links are equivariant with respect to the group.

In this work, we aim to produce a Sarkisov program in the spirit of [Pas15] and prove the following.

Theorem 1. *Let G be a connected reductive algebraic group. Let X and Y be horospherical G -varieties which are G -equivariantly birational. Assume moreover that there are Mori fibre space structures X/S and Y/T . Then, there is a horospherical Sarkisov program from X/S to Y/T .*

What a horospherical Sarkisov program is will be made more precise in the last section where we state and prove Theorem 40, which directly implies Theorem 1, but the rough idea is that if X/S and Y/T are outcomes of two MMP on the same Z , we consider the two-parameter family

$$\{Q_{(1-\delta)D+\delta D'+\epsilon K_Z}\}_{(\delta,\epsilon)\in\mathbb{Q}^2}.$$

We consider the polyhedron Ω of dimension two of the (δ, ϵ) such that $Q_{(1-\delta)D+\delta D'+\epsilon K_Z}$ is non-empty. There is a polyhedral decomposition of Ω given by the points (δ, ϵ) such that the corresponding moment polytope defines the same variety. We show then that we can "read" the Sarkisov links on a part of the boundary of Ω .

In the appendix are some examples of this horospherical Sarkisov program, showing how to effectively implement our methods.

2. HOROSPHERICAL VARIETIES

We begin by recalling very briefly the Luna-Vust theory of horospherical embeddings and by setting the notation used in the rest of the paper. For more details on horospherical varieties, we refer the reader to [Pas08], and for basic results on Luna-Vust theory of spherical embeddings, we refer to [Kno91].

Let G be connected reductive algebraic group. A closed subgroup H of G is said to be *horospherical* if it contains the unipotent radical U of a Borel subgroup B of G . This is equivalent to say that ([Pas08, Prop. and Rem. 2.2]), there exists a parabolic subgroup P containing H such that the map $G/H \rightarrow G/P$ is a torus fibration; or to say that there exists a parabolic subgroup P containing H such that H is the kernel of finitely many characters of P .

Note that $B \subset P = N_G(H)$. We also fix a maximal torus T of B . Then we denote by S the set of simple roots of (G, B, T) . Also denote by R the subset of S of simple roots of P . Let $X(T)$ (resp. $X(T)^+$) be the lattice of characters of T (resp. the set of dominant characters). Similarly, we define $X(P)$ and $X(P)^+ = X(P) \cap X(T)^+$. Note that the lattice $X(P)$ and the dominant chamber $X(P)^+$ are generated by the fundamental weights ϖ_α with $\alpha \in S \setminus R$ and the weights of the center of G .

We denote by M the sublattice of $X(P)$ consisting of characters of P vanishing on H . The rank of M is called the *rank of G/H* and denoted by n . Let $N := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.

For any free lattice \mathbb{L} , we denote by $\mathbb{L}_{\mathbb{Q}}$ the \mathbb{Q} -vector space $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$.

For any simple root $\alpha \in S \setminus R$, the restriction of the coroot α^\vee to M is a point of N , which we denote by α_M^\vee .

Definition 1. A G/H -embedding is a couple (X, x) , where X is a normal algebraic G -variety and x a point of X such that $G \cdot x$ is open in X and isomorphic to G/H . The variety X is called a *horospherical variety*.

By abuse of notation, we often forget the point x , so that we call X a G/H -embedding. But there are several non-isomorphic G/H -embeddings (X, x) for the same horospherical variety X . Two points x_1 and x_2 differ by an element of the torus P/H , which acts on the right on G/H . Similarly to toric varieties (which are $(\mathbb{C}^*)^n$ -embeddings with the above definition), G/H -embeddings are classified by colored fans in $N_{\mathbb{Q}}$. For example, for the toric variety \mathbb{P}^2 we have different non-isomorphic $(\mathbb{C}^*)^2$ -embeddings, whose fans are the same up to the action of $\text{SL}_2(\mathbb{Z})$. In this paper, since we define horospherical varieties by their colored fans, or equivalently by some of their moment polytopes, we are implicitly fixing a G/H -embedding up to isomorphism.

Definition 2. (1) A colored cone of $N_{\mathbb{Q}}$ is an couple $(\mathcal{C}, \mathcal{F})$ where \mathcal{C} is a convex cone of $N_{\mathbb{Q}}$ and \mathcal{F} is a set of colors (called the set of colors of the colored cone), such that

- (i) \mathcal{C} is generated by finitely many elements of N and contains $\{\alpha_M^\vee \mid \alpha \in \mathcal{F}\}$,
- (ii) \mathcal{C} does not contain any line and \mathcal{F} does not contain any α such that α_M^\vee is zero.

(2) A colored face of a colored cone $(\mathcal{C}, \mathcal{F})$ is a couple $(\mathcal{C}', \mathcal{F}')$ such that \mathcal{C}' is a face of \mathcal{C} and \mathcal{F}' is the set of $\alpha \in \mathcal{F}$ satisfying $\alpha_M^\vee \in \mathcal{C}'$. A colored fan is a finite set \mathbb{F} of colored cones such that

- (i) any colored face of a colored cone of \mathbb{F} is in \mathbb{F} ,
- (ii) and any element of $N_{\mathbb{Q}}$ is in the interior of at most one colored cone of \mathbb{F} .

The main result of Luna-Vust Theory of spherical embeddings is the one-to-one correspondence between colored fans and isomorphic classes of G/H -embeddings (see for example [Kno91]). It generalizes the classification of toric varieties in terms of fans, case where $G = (\mathbb{C}^*)^n$ and $H = \{1\}$. We will rewrite this result in Section 2.1 for projective horospherical varieties in terms of polytopes and describe explicitly the correspondence.

If X is a G/H -embedding, we denote by \mathbb{F}_X the *colored fan* of X in $N_{\mathbb{Q}}$ and we denote by \mathcal{F}_X the subset $\cup_{(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X} \mathcal{F}$ of $S \setminus R$, which we call the *set of colors* of X .

We now recall the description of divisors of horospherical varieties.

We denote by X_1, \dots, X_r the G -stable irreducible divisors of a G/H -embedding X . For any $i \in \{1, \dots, r\}$, we denote by x_i the primitive element in N of the colored edge associated to X_i . The B -stable and not G -stable irreducible divisors of a G/H -embedding X are the closures in X of B -stable irreducible divisors of G/H , which are the inverse images by the torus fibration $G/H \rightarrow G/P$ of the Schubert divisors of the flag variety G/P . The B -stable irreducible divisors of G/H are indexed by simple roots of $S \setminus R$, we write them D_{α} with $\alpha \in S \setminus R$.

We can now recall the characterization of Cartier, \mathbb{Q} -Cartier and ample divisors of horospherical varieties due to M. Brion in the more general case of spherical varieties ([Bri89]). This will permit to define a polytope associated to a divisor of a horospherical variety.

Theorem 2. (Section 3.3, [Bri89]) *Let G/H be a horospherical homogeneous space. Let X be a G/H -embedding. Then every divisor of X is equivalent to a linear combination of X_1, \dots, X_r and D_{α} with $\alpha \in S \setminus R$. Now, let $D = \sum_{i=1}^r a_i X_i + \sum_{\alpha \in S \setminus R} a_{\alpha} D_{\alpha}$ be a \mathbb{Q} -divisor of X .*

(1) *D is \mathbb{Q} -Cartier if and only if there exists a piecewise linear function h_D , linear on each colored cone of \mathbb{F}_X , such that for any $i \in \{1, \dots, r\}$, $h_D(x_i) = a_i$ and for any $\alpha \in \mathcal{F}_X$, $h_D(\alpha_M^{\vee}) = a_{\alpha}$.*

(2) *Suppose that D is a divisor (i.e. a_1, \dots, a_r and the a_{α} with $\alpha \in S \setminus R$ are in \mathbb{Z}). Then D is Cartier if moreover, for any colored cone $(\mathcal{C}, \mathcal{F})$ of \mathbb{F}_X , the linear function $(h_D)|_{\mathcal{C}}$, can be define as an element of M (instead of $M_{\mathbb{Q}}$ for \mathbb{Q} -Cartier divisors).*

(3) *Suppose that D is \mathbb{Q} -Cartier. Then D is ample, resp. nef if and only if the piecewise linear function h_D is strictly convex, resp. convex, and for any $\alpha \in (S \setminus R) \setminus \mathcal{F}_X$, we have $h_D(\alpha_M^{\vee}) < a_{\alpha}$, resp. $\leq a_{\alpha}$.*

(4) *Suppose that D is Cartier. Let \tilde{Q}_D be the polytope in $M_{\mathbb{Q}}$ defined by the following inequalities, where $\chi \in M_{\mathbb{Q}}$: for any colored cone $(\mathcal{C}, \mathcal{F})$ of \mathbb{F}_X , $(h_D) + \chi \geq 0$ on \mathcal{C} , and for any $\alpha \in (S \setminus R) \setminus \mathcal{F}_X$, $\chi(\alpha_M^{\vee}) + a_{\alpha} \geq 0$. Note that here the weight of the canonical section of D is $v^0 := \sum_{\alpha \in S \setminus R} a_{\alpha} \varpi_{\alpha}$. Then the G -module $H^0(X, D)$ is the direct sum, with multiplicity one, of the irreducible G -modules of highest weights $\chi + v^0$ with χ in $\tilde{Q}_D \cap M$.*

In all the paper, a divisor of a horospherical variety is always supposed to be B -stable, i.e. of the form $\sum_{i=1}^r a_i X_i + \sum_{\alpha \in S \setminus R} a_{\alpha} D_{\alpha}$.

Corollary 3. *A G/H -embedding X is \mathbb{Q} -factorial if and only if all the cones in \mathbb{F}_X are simplicial and for any $\alpha \in \mathcal{F}_X$, α_M^{\vee} generates a ray of \mathbb{F}_X . Moreover, in that case, the Picard number of X is the number of their B -stable prime divisors minus the rank n (Equality (4.1.1) [Pas08]).*

In what follows we denote by $WDiv(X)_{\mathbb{Q}}$ (resp. $WDiv_0(X)_{\mathbb{Q}}$) the vector space of B -invariant (linearly equivalent to zero) \mathbb{Q} -Weil divisors on a horospherical variety X and by $Amp(X)$ (resp. $Nef(X)$) the cones in $WDiv(X)_{\mathbb{Q}}$ of ample (resp. nef) divisors. Both cones are polyhedral and, if X is \mathbb{Q} -factorial, full-dimensional.

2.1. Projective horospherical varieties and polytopes. In this section we recall how many properties of G/H -embeddings can be formulated in terms of moment polytopes. Theorem 2 gives the following

Proposition 4. (Corollary 2.8, [Pas15]) *Let X be a projective G/H -embedding and $D = \sum_{i=1}^r a_i X_i + \sum_{\alpha \in S \setminus R} a_{\alpha} D_{\alpha}$ be a \mathbb{Q} -divisor of X . Suppose that D is \mathbb{Q} -Cartier and ample.*

(1) *The polytope \tilde{Q}_D defined in Theorem 2 is of maximal dimension in $M_{\mathbb{Q}}$ and we have*

$$\tilde{Q}_D = \{m \in M_{\mathbb{Q}} \mid \langle m, x_i \rangle \geq -a_i, \forall i \in \{1, \dots, r\} \text{ and } \langle m, \alpha_M^{\vee} \rangle \geq -a_{\alpha}, \forall \alpha \in \mathcal{F}_X\}.$$

(2) *Let $v^0 := \sum_{\alpha \in S \setminus R} a_{\alpha} \varpi_{\alpha}$. The polytope $Q_D := v^0 + \tilde{Q}_D$ is contained in the dominant chamber $X(P)^+$ of $X(P)$ and it is not contained in any wall of the dominant chamber.*

(3) *Let $(\mathcal{C}, \mathcal{F})$ be a maximal colored cone of \mathbb{F}_X , then the element $v^0 - (h_D)|_{\mathcal{C}}$ of $M_{\mathbb{Q}}$ is a vertex of Q_D . In particular, if D is Cartier, then Q_D is a lattice polytope (i.e. has its vertices in $v^0 + M$).*

(4) *Conversely, let v be a vertex of Q_D . We define \mathcal{C}_v to be the cone of $N_{\mathbb{Q}}$ generated by inward-pointing normal vectors of the facets of Q_D containing v . We set $\mathcal{F}_v = \{\alpha \in S \setminus R \mid v \text{ the corresp. wall of the dominant chamber}\}$. Then $(\mathcal{C}_v, \mathcal{F}_v)$ is a maximal colored cone of \mathbb{F}_X .*

The polytope Q_D is called the *moment polytope* of (X, D) (or of D), and the polytope \tilde{Q}_D the *pseudo-moment polytope* of (X, D) (or of D).

The projective G/H -embeddings are classified in terms of G/H -polytopes (defined below in Definition 3), and we can give an explicit construction of a G/H -embedding from a G/H -polytope.

Definition 3. Let Q be a polytope in $X(P)_{\mathbb{Q}}^+$ (not necessarily a lattice polytope). We say that Q is a G/H -polytope, if its direction is $M_{\mathbb{Q}}$ and if it is contained in no wall $W_{\alpha, P}$ with $\alpha \in S \setminus R$.

Let Q and Q' be two G/H -polytopes in $X(P)_{\mathbb{Q}}^+$. Consider any polytopes \tilde{Q} and \tilde{Q}' in $M_{\mathbb{Q}}$ obtained by translations from Q and Q' respectively. We say that Q and Q' are equivalent G/H -polytopes if the following conditions are satisfied.

- (1) There exist an integer j and $2j$ affine half-spaces $\mathcal{H}_1^+, \dots, \mathcal{H}_j^+$ and $\mathcal{H}'_1^+, \dots, \mathcal{H}'_j^+$ of $M_{\mathbb{Q}}$ (respectively delimited by the affine hyperplanes $\mathcal{H}_1, \dots, \mathcal{H}_j$ and $\mathcal{H}'_1, \dots, \mathcal{H}'_j$) such that \tilde{Q} is the intersection of the \mathcal{H}_i^+ , \tilde{Q}' is the intersection of the \mathcal{H}'_i^+ , and for all $i \in \{1, \dots, j\}$, \mathcal{H}_i^+ is the image of \mathcal{H}'_i^+ by a translation.
- (2) With the notation of (1), for all subset J of $\{1, \dots, j\}$, the intersections $\cap_{i \in J} \mathcal{H}_i \cap Q$ and $\cap_{i \in J} \mathcal{H}'_i \cap Q'$ have the same dimension.
- (3) Q and Q' intersect exactly the same walls of the dominant chamber.

Remark that this definition does not depend on the choice of \tilde{Q} and \tilde{Q}' . We now give a classification of projective horospherical varieties in terms of polytopes.

Proposition 5. (Proposition 2.10, [Pas15]) *The correspondence between moment polytopes and colored fans gives a bijection between the set of equivalence classes of G/H -polytopes and (isomorphism classes of) projective G/H -embeddings.*

Proposition 6. (Proposition 2.11 and Remark 2.12, [Pas15]) *Let Q be a G/H -polytope. Up to multiplying Q by an integer, there exists a very ample Cartier divisor D of the corresponding G/H -embedding X such that $Q = Q_D$. More precisely, X is isomorphic to the closure of the G -orbit $G \cdot [\sum_{\chi \in (v^0+M) \cap Q} v_\chi]$ in $\mathbb{P}(\oplus_{\chi \in (v^0+M) \cap Q} V(\chi))$.*

Lemma 7. *Let X be a G/H -embedding and let D be a nef divisor on X . Then there is a horospherical subgroup $H \subsetneq H' \subseteq G$ such that Q_D is the polytope of a G/H' -embedding X' and a suitable multiple of D defines a morphism $X \rightarrow X'$.*

Proof. If D is nef, the polytope Q_D can be defined as in Proposition 4. The colored fan that we can then construct is not necessarily \mathbb{F}_X , but the colored fan of a horospherical G -variety Y (not necessarily a G/H -embedding), where the G -equivariant map ϕ_D associated to the nef divisor goes from X to Y .

Indeed, up to multiplying D or Q by an integer, the proof of [Pas15, Proposition 2.11] and [Pas15, Remark 2.12] gives the projective G -equivariant map

$$\begin{aligned} \phi_D : X &\longrightarrow \mathbb{P}(H^0(X, D)^\vee) \\ x &\longmapsto [s \mapsto s(x)] \end{aligned}$$

whose image is the closure of the G -orbit $G \cdot [\sum_{\chi \in (v^0+M) \cap Q} v_\chi]$ in $\mathbb{P}(\oplus_{\chi \in (v^0+M) \cap Q} V(\chi))$. This closure is the variety Y by applying Proposition 6 to $Q = Q_D$. Thus Q is a G/H' -polytope with $H \subseteq H'$, and Y is the G/H' -embedding corresponding to Q . \square

By the duality between colored fans and moment polytopes we easily get that if X is a \mathbb{Q} -factorial G/H -embedding then for every ample B -invariant divisor D the polytope Q_D is simple. We can go further and give the following result, which is a translation of Corollary 3 in terms of polytopes, by using Proposition 4 (3) and (4). The matrix inequality $Ax \geq B$ comes from Proposition 4 (1).

Lemma 8. *Let X be a G/H -embedding and let D be an ample B -invariant divisor. The polytope \tilde{Q}_D can be defined by a matrix inequality $Ax \geq B$, where the lines of A are given by the x_i 's and the α_M^\vee 's, and the column matrix B is given by minus the coefficients of D . For any vertex v of \tilde{Q}_D , denote I_v the set of lines of A such that $A_{I_v}x = B_{I_v}$.*

Then X is \mathbb{Q} -factorial if and only if A_{I_v} is surjective for any vertex v of \tilde{Q}_D .

Similarly, we have the following result.

Lemma 9. *Let X be a G/H -embedding and let D be an ample B -invariant divisor. Let D' be a B -invariant divisor of X and denote by B' the column matrix is given by minus the coefficients of D' .*

Then D' is \mathbb{Q} -Cartier if and only if for any vertex v of \tilde{Q}_D , B' is in the image of A_{I_v} .

Remark 1. The existence of G -equivariant morphisms between horospherical varieties can be characterized in terms of colored fans [Kno91] or equivalently in terms of moment polytopes [Pas15, section 2.4]. In this text, we will rather use Lemma 7.

3. MINIMAL MODEL PROGRAM

In this section, we recall the results in [Pas15] where the second-named author describes a minimal model program from a horospherical variety in terms of a one-parameter family of polytopes.

We start by recalling some standard terminology for the minimal model program for projective varieties. We refer to [KM98] for the basic notions on the minimal model program.

Let X be a projective variety with terminal \mathbb{Q} -factorial singularities such that K_X is not nef. Then by the cone and contraction theorem there is a morphism $\varphi: X \rightarrow Y$ such that $\rho(X) = \rho(Y) + 1$ and for every curve C in X which is contracted to a point by φ we have $K_X \cdot C < 0$. Moreover

- if $\dim Y < \dim X$ then φ is called a Mori fibre space;
- if $\dim Y = \dim X$ and $\text{Exc}(\varphi)$ has codimension 1 in X then φ is said to be divisorial;
- if $\dim Y = \dim X$ and $\text{Exc}(\varphi)$ has codimension at least 2 in X then φ is said to be small.

In particular, the total space of a Mori fiber space is always assumed to be terminal \mathbb{Q} -factorial.

In the last case, by [HM07] and [HM10], there is $\varphi^+: X^+ \rightarrow Y$ such that $\text{Exc}(\varphi^+)$ has codimension at least 2 in X^+ and for every curve C in X^+ contracted by φ^+ we have $K_{X^+} \cdot C > 0$. The data of φ and φ^+ is called a *flip*.

In the second and third case Y is birational to X . We set $X_1 = Y$ and $\varphi_1 = \varphi$ in the second case and $X_1 = X^+$ and $\varphi_1 = (\varphi^+)^{-1} \circ \varphi$ in the third case. If K_{X_1} is not nef, then by the cone and contraction theorem there is again a morphism $X_1 \rightarrow Y_1$. An *MMP*, or minimal model program, is a sequence of birational maps $\varphi_i: X_{i-1} \dashrightarrow X_i$ obtained as above. We say that it *terminates* if there is an integer k such that K_{X_k} is nef or there is a Mori fibre space $X_k \rightarrow T$. A proper morphism with connected fibres is called a *contraction*. A birational morphism $\varphi: X \rightarrow Y$ such that $\rho(X) = \rho(Y) + 1$ is called an *extremal contraction*. We say that a morphism φ is *K-negative* (resp. *K-positive*) if for every curve C in X which is contracted to a point by φ we have $K_X \cdot C < 0$ (resp. $K_X \cdot C > 0$).

3.1. HMMP scaled by an ample divisor. Let (X, D) be a polarized horospherical variety: X is a G/H -embedding and D is a B -stable ample \mathbb{Q} -divisor of X . Write $D = \sum_{i=1}^r b_i X_i + \sum_{\alpha \in S \setminus R} b_\alpha D_\alpha$. An anticanonical divisor of X is $-K_X = \sum_{i=1}^r X_i + \sum_{\alpha \in S \setminus R} a_\alpha D_\alpha$, where a_α are integers greater or equal to 2, and given by an explicit formula [Bri97, Th. 4.2].

We consider the one-parameter family of polytopes $(\tilde{Q}^\epsilon)_{\epsilon \geq 0}$ defined by $\langle x, x_i \rangle \geq -b_i + \epsilon$ for all $i \in \{1, \dots, r\}$ and $\langle x, \alpha_M^\vee \rangle \geq -b_\alpha + \epsilon a_\alpha$ for any $\alpha \in S \setminus R$. Equivalently, let A be the matrix associated to the linear map $\varphi(m) = (\langle m, x_i \rangle_{i=1 \dots r}, \langle m, \alpha_M^\vee \rangle_{\alpha \in \mathcal{F}})$. Let B be the column matrix whose coordinates are minus the coefficients of D ; and C the column matrix whose coordinates are the coefficients of $-K_X$.

We define $(Q^\epsilon)_{\epsilon \geq 0}$ by $Q^\epsilon = \tilde{Q}^\epsilon + \sum_{\alpha \in S \setminus R} b_\alpha - \epsilon a_\alpha \varpi_\alpha$.

Remark 2. For small $\epsilon \in \mathbb{Q}$, $\tilde{Q}^\epsilon = \tilde{Q}_{D+\epsilon K_X}$ and $Q^\epsilon = Q_{D+\epsilon K_X}$ is the moment polytope associated to the ample divisor $D + \epsilon K_X$.

Theorem 10. (Corollary 3.16 and Section 4, [Pas15]) *For every ample divisor D over a \mathbb{Q} -Gorenstein horospherical variety X , there exists $\epsilon_{max} > 0$, and, there*

exist non-negative integers k, j_0, \dots, j_k , rational numbers $\alpha_{i,j}$ for $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, j_i\}$ and $\alpha_{k, j_k+1} \in \mathbb{Q}_{>0} \cup \{+\infty\}$ ordered as follows with the convention that $\alpha_{i, j_i+1} = \alpha_{i+1, 0}$ for any $i \in \{0, \dots, k-1\}$:

- (1) $\alpha_{0,0} = 0$;
- (2) for any $i \in \{0, \dots, k\}$, and for any $j < j'$ in $\{0, \dots, j_i + 1\}$ we have $\alpha_{i,j} < \alpha_{i,j'}$;

and such that the different G/H -embedding associated to the polytopes in the family $(Q^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ are given by the following intervals:

- (1) $X_{i,0}$ when $\epsilon \in [\alpha_{i,0}, \alpha_{i,1}[$, with $i \in \{0, \dots, k\}$;
- (2) $X_{i,j}$ when $\epsilon \in]\alpha_{i,j}, \alpha_{i,j+1}[$, with $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$;
- (3) $Y_{i,j}$ when $\epsilon = \alpha_{i,j}$ with $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$;
- (4) T such that $\dim T < \dim X$ when $\epsilon = \alpha_{k, j_k+1} = \epsilon_{max}$.

Moreover we get dominant G -equivariant morphisms:

- (1) $\phi_{i,j} : X_{i,j-1} \rightarrow Y_{i,j}$ for any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$;
- (2) $\phi_{i,j}^+ : X_{i,j} \rightarrow Y_{i,j}$ for any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$;
- (3) $\phi_i : X_{i, j_i} \rightarrow X_{i+1, 0}$ for any $i \in \{0, \dots, k-1\}$;
- (4) and $\phi : X_{k, j_k} \rightarrow T$.

For every i, j the morphism $\phi_{i,j}$ is K -negative, the morphism $\phi_{i,j}^+$ is K -positive and their exceptional loci have codimension at least 2. For every i the morphism ϕ_i is a K -negative divisorial contraction and ϕ is a fibration.

We will refer to the series of maps of Theorem 10 as *HMMP*, for horospherical minimal model program.

Remark 3. We notice that the algorithm described in Theorem 10 is not necessarily an MMP, as the morphisms involved are not extremal contractions. Nevertheless, if X is terminal \mathbb{Q} -factorial, it is an MMP for a general choice of D , in particular ϕ is a Mori fiber space.

Remark 4. In Theorem 10, we can also use the family $(\tilde{Q}^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$. By replacing the walls of the dominant chamber by the inequalities in $Ax \geq B + \epsilon C$ coming from an $\alpha \in S \setminus R$, the study of the family and associated horospherical variety is simpler.

4. RESOLUTIONS AND HOROSPHERICAL MMP

The main goal of this section is to prove that if X/T is a Mori fiber space and $Z \rightarrow X$ a resolution of the singularities of X , then there is a euclidian open set in $WDiv(Z)_{\mathbb{Q}}$ of ample divisors A such that the HMMP from Z with scaling of A ends with X/T .

Lemma 11. *Let X and Y be G/H -embeddings. Then there is a smooth G/H -embedding Z and $Z \rightarrow X \times Y$ a resolution of the indeterminacy of $X \dashrightarrow Y$.*

Proof. The existence of a smooth resolution of the indeterminacy of $X \dashrightarrow Y$ is a consequence of the same result for toric varieties. Indeed, we can first unpick colors both for X and Y to obtain toroidal varieties. Then, as for toric varieties we choose a common smooth subdivision of the two fans. \square

Lemma 12. *Let X and Z be terminal and \mathbb{Q} -factorial G/H -embeddings such that Z is a resolution of the singularities of X .*

Let E_1, \dots, E_k be the exceptional divisors of $\phi: Z \rightarrow X$. Then, for any non-negative rational numbers d_1, \dots, d_k , we have $Q_D = Q_{\phi^*(D) + \sum_{i=1}^k d_i E_i}$.

Proof. Since X has terminal singularities, we have $K_Z = \phi^* K_X + \sum_{i=1}^k a_i E_i$ with $a_i > 0$ for every i .

We denote by X_1, \dots, X_r the G -stable irreducible divisors of X . For any $i \in \{1, \dots, r\}$, we denote by x_i the primitive element in N of the colored edge associated to X_i . Then the G -stable irreducible divisors of Z are $X_1, \dots, X_r, E_1, \dots, E_k$. We denote by e_i the primitive element in N of the colored edge associated to E_i . Let $D = \sum_{i=1}^r b_i X_i + \sum_{\alpha \in S \setminus R} b_\alpha D_\alpha$ be an ample B -stable divisor of X . Denote by c_1, \dots, c_k the rational numbers such that $\phi^*(D) = \sum_{i=1}^r b_i X_i + \sum_{j=1}^k c_j E_j + \sum_{\alpha \in S \setminus R} b_\alpha D_\alpha$. Then, for any non-negative rational numbers d_1, \dots, d_k , the polytope Q_D coincides with

$$Q_{\phi^*(D) + \sum_{i=1}^k d_i E_i} := \{m \in M_{\mathbb{Q}} \mid \langle m, x_i \rangle \geq -b_i, \langle m, e_j \rangle \geq -c_j - d_j, \langle m, \alpha_M^\vee \rangle \geq -a_\alpha \text{ for all } i, j, \alpha\}.$$

By [Bri89], the divisors D and $\phi^*(D)$ are the divisors associated to the same piecewise linear function h_D defined in Theorem 2. Therefore, the coefficients d_j are the values $h_D(e_j)$. In terms of polytopes, since D is ample, this means that $\mathcal{H}_j := \{m \in M_{\mathbb{Q}} \mid \langle m, e_j \rangle = -c_j\}$ intersects Q_D along a face of Q_D . In particular, intersecting Q_D with the halfspace $\{m \in M_{\mathbb{Q}} \mid \langle m, e_j \rangle \geq -c_j - d_j\}$ does not change the polytope. \square

Lemma 13. *Let X/T be a Mori fibre space such that X is a G/H -embedding. The set of ample B -stable \mathbb{Q} -divisors D of X such that the HMMP from X scaled by D gives $X \rightarrow T$ is an open cone C_1 in $WDiv(X)_{\mathbb{Q}}$.*

Proof. Let R be the extremal ray of $NE(X)$ corresponding to the Mori fibration $X \rightarrow T$. Let F be the dual facet of R in $Nef(X)_{\mathbb{Q}}$. By abuse of notation we denote by F the facet $F + WDiv_0(X)_{\mathbb{Q}}$ of the inverse image of $Nef(X)_{\mathbb{Q}}$ in $WDiv(X)_{\mathbb{Q}}$.

Then the HMMP from X scaled by D gives $X \rightarrow T$ if and only if the half line $D + \mathbb{Q}_+ K_X$ intersects the boundary of $Nef(X)_{\mathbb{Q}} + WDiv_0(X)_{\mathbb{Q}}$ in the relative interior \mathring{F} of F . Equivalently, D is in $C'_1 := \mathring{F} - \mathbb{Q}_+ K_X = \mathbb{Q}_+(\mathring{F} - K_X)$.

Note that $-K_X \cdot C > 0$ for any $[C] \in R$, so that $-K_X$ is not in F and then C'_1 is of maximal dimension. This implies that $D' - \epsilon K_X$ is ample for any $D' \in \mathring{F}$ and $\epsilon > 0$ small enough and then C'_1 intersects $Amp(X)_{\mathbb{Q}} + WDiv_0(X)_{\mathbb{Q}}$ non trivially.

Finally, the set $C_1 = C'_1 \cap Amp(X)_{\mathbb{Q}} + WDiv_0(X)_{\mathbb{Q}}$ is the sought open cone. \square

Proposition 14. *Let X/T be a Mori fibre space such that X is a G/H -embedding. Let $Z \rightarrow X$ be a resolution of singularities in the category of G/H -embeddings. Then there is an euclidian open neighborhood U_X of $WDiv(Z)_{\mathbb{Q}}$ such that every divisor in U_X is ample and for every $A \in U_X$ the Mori fiber space X/T is the outcome of the HMMP from Z with scaling of A .*

Proof. Let C_1 be the open cone in $WDiv(X)_{\mathbb{Q}}$ of Lemma 13. Let $D \in C_1$. Let η be small enough such that $D - \eta K_X$ ample.

We have $WDiv(Z)_{\mathbb{Q}} = \phi^*(WDiv(X)_{\mathbb{Q}}) \oplus Vect(E_1, \dots, E_k)$, and $\phi^*(Nef(X))$ is a face of $Nef(Z) = Amp(Z)$. Set

$$A := \phi^*(D - \eta K_X) + \sum_{i=1}^k b_i E_i.$$

There exists an open polyhedron Pol_D of \mathbb{Q}^k containing 0 in its boundary, such that for any $(b_1, \dots, b_k) \in Pol_D$ the divisor A is ample and $d_j := b_j + \eta a_j > 0$ for all $j \in \{1, \dots, k\}$.

Since $A + \eta K_Z = \phi^*(D) + \sum_{i=1}^k d_i E_i$, we have $Q_{A+\eta K_Z} = Q_D$ by Lemma 12. This, together with Lemma 13, proves that the HMMP scaled by A ends with the Mori fiber space X/T . Indeed, the inequalities coming from the exceptional divisors E_i are necessary to define the polytope Q_A but not the polytope $Q_{A+\eta K_Z}$ and the polytopes $Q_{A+\epsilon K_Z}$ with $\epsilon \geq \eta$.

Choose U_X to be the set of divisors $A = \phi^*(D - \eta K_X) + \sum_{i=1}^k b_i E_i$, with $D \in C_1$, $\eta > 0$ such that $D - \eta K_X$ is ample and $(b_1, \dots, b_k) \in Pol_{D, \eta}$. Remark that by construction, since $Amp(X)$ and $Amp(Z)$ are polyhedral, the condition $D - \eta K_X$ ample is given by an inequality depending linearly on the coefficients of D , and the polyhedron Pol_D is given by inequalities depending linearly on the coefficients of D and δ . In particular, U_X is also an open polyhedron. \square

5. TWO-PARAMETERS FAMILIES OF POLYTOPES

In this section, we study some two-parameters families of polytopes. This section can be read independently from the rest of the paper, except for subsection 5.4.

5.1. A two-parameters family of polytopes: definitions and first properties. Let $\mathbb{K} = \mathbb{Q}$ or \mathbb{R} . Let n and p be two positive integers. Denote by I_0 the set $\{1, \dots, p\}$. Then two matrices $A \in M_{p \times n}(\mathbb{Q})$ and $D \in M_{p \times 1}(\mathbb{Q})$ define a polyhedron

$$P := \{x \in \mathbb{K}^n \mid Ax \geq D\}.$$

Remark 5. Suppose that P is non-empty. Then P is a polytope, that is a bounded polyhedron, if and only if the following condition is satisfied.

Condition 1. There is no non-zero $x \in \mathbb{Q}^n$ satisfying $Ax \geq 0$.

The first implication is not difficult. Conversely, if P is not bounded, P contains at least an affine half-line, and x can be taken to be a generator of the direction of this half-line.

Note that Condition 1 implies that A is injective.

To define a two-parameters family of polytopes, we fix A satisfying Condition 1 and we define divisors depending on two rational parameters δ and ϵ .

Let B, B' and C in $M_{p \times 1}(\mathbb{Q})$. Set $I_0 := \{1, \dots, p\}$ and define

$$\begin{aligned} D: \quad \mathbb{Q}^2 &\rightarrow \mathbb{Q}^p \\ (\delta, \epsilon) &\mapsto (1 - \delta)B + \delta B' + \epsilon C. \end{aligned}$$

Note that D is an affine map.

Definition 4. Given A, B, B' and C as above, we define for any $(\delta, \epsilon) \in \mathbb{Q}^2$:

$$P^{\delta, \epsilon} := \{x \in \mathbb{K}^n \mid Ax \geq D(\delta, \epsilon)\} \cap \mathbb{Q}^n.$$

We do not exclude the case where some lines of A are zero. Notice that $P^{\delta, \epsilon}$ can be empty, even for all $(\delta, \epsilon) \in \mathbb{Q}^2$.

Condition 1 implies that for any $(\delta, \epsilon) \in \mathbb{Q}^2$, the set $P^{\delta, \epsilon}$ is a polytope (possibly empty).

Now, we want to describe some equivalent classes of polytopes in this family, looking at their faces that correspond to lines of A . A face $F^{\delta, \epsilon}$ of $P^{\delta, \epsilon}$ is given by some equalities in $Ax \geq D(\delta, \epsilon)$ and then is associated to some $I \subseteq I_0$. We formalize this below.

For any matrix \mathcal{M} and any $i \in I_0$, we denote by \mathcal{M}_i the matrix consisting of the line i of \mathcal{M} . More generally, for any subset I of I_0 we denote by \mathcal{M}_I the matrix consisting of the lines $i \in I$ of \mathcal{M} . For any subset I of I_0 , we can identify D_I with the affine map

$$\begin{aligned} D_I: \quad \mathbb{Q}^2 &\rightarrow \mathbb{Q}^{|I|} \\ (\delta, \epsilon) &\mapsto (1 - \delta)B_I + \delta B'_I + \epsilon C_I. \end{aligned}$$

Let $(\delta, \epsilon) \in \mathbb{Q}^2$. Denote by $\mathcal{H}_i^{\delta, \epsilon}$ the hyperplane $\{x \in \mathbb{Q}^n \mid A_i x = D_i(\delta, \epsilon)\}$. For any $I \subseteq I_0$, denote by $F_I^{\delta, \epsilon}$ the face of $P^{\delta, \epsilon}$ defined by

$$F_I^{\delta, \epsilon} := \left(\bigcap_{i \in I} \mathcal{H}_i^{\delta, \epsilon} \right) \cap P^{\delta, \epsilon}.$$

Note that for any face $F^{\delta, \epsilon}$ of $P^{\delta, \epsilon}$ there exists a unique maximal $I \subseteq I_0$ such that $F^{\delta, \epsilon} = F_I^{\delta, \epsilon}$ (we include the empty face and $P^{\delta, \epsilon}$ itself).

Definition 5. Let $I \subseteq I_0$. Define Ω_I to be the set of $(\delta, \epsilon) \in \mathbb{Q}^2$ such that $F_I^{\delta, \epsilon}$ is not empty. Define ω_I to be the subset of Ω_I such that, if $I' \subseteq I_0$ satisfies $F_I^{\delta, \epsilon} = F_{I'}^{\delta, \epsilon}$, then $I' \subseteq I$.

In other words,

$$\Omega_I = \{(\delta, \epsilon) \mid \text{there is } x \in \mathbb{Q}^n \text{ such that } A_I x = D_I(\delta, \epsilon) \text{ and } Ax \geq D(\delta, \epsilon)\}$$

and

$$\omega_I = \{(\delta, \epsilon) \mid \text{there is } x \in \mathbb{Q}^n \text{ such that } A_I x = D_I(\delta, \epsilon) \text{ and } A_{\bar{I}} x > D_{\bar{I}}(\delta, \epsilon)\}.$$

To simplify the notation, we often write i instead of $\{i\}$, for any $i \in I_0$.

Remark 6. If $(\delta, \epsilon) \in \omega_\emptyset$, then the polytope $P^{\delta, \epsilon}$ is of dimension n (i.e. has a non-empty interior). And, for any $i \in I_0$, if $(\delta, \epsilon) \in \omega_i$ and $A_i \neq 0$, then $F_i^{\delta, \epsilon}$ is a facet of $P^{\delta, \epsilon}$.

5.2. A polyhedral partition of \mathbb{Q}^2 . The following lemma describes the first properties of the sets Ω_I and ω_I .

Lemma 15. *Let $I \subseteq I_0$.*

- (1) *The sets Ω_I and ω_I are convex subsets of \mathbb{Q}^2 .*
- (2) *The set ω_I is open, for the euclidean topology, inside $\{(\delta, \epsilon) \mid D_I(\delta, \epsilon) \in \text{Im } A_I\} = D_I^{-1} \text{Im } A_I$.*

(3) *There are four cases: either ω_I is empty, or it is a point, or it is a convex part of an affine line (a segment, a half-line or a line), or it is a non-empty open set in \mathbb{Q}^2 .*

(4) *If ω_I is not empty, we have $\omega_I \subset \Omega_I \subset \overline{\omega_I}$.*

Proof. (1) If Ω_I is either empty or has cardinality 1, the statement is obvious. We assume then that Ω_I has at least two points, (δ_1, ϵ_1) and (δ_2, ϵ_2) . Then there exist x_1 and x_2 in \mathbb{Q}^n such that $A_I x_i = D_I(\delta_i, \epsilon_i)$ and $A_{\bar{I}} x_i \geq D_{\bar{I}}(\delta_i, \epsilon_i)$ for $i = 1, 2$. For any rational number $t \in [0, 1]$, we get $A_I(tx_1 + (1-t)x_2) = D_I(t\epsilon_1 + (1-t)\epsilon_2, t\delta_1 + (1-t)\delta_2)$ and $A_{\bar{I}}(tx_1 + (1-t)x_2) \geq D_{\bar{I}}(t\epsilon_1 + (1-t)\epsilon_2, t\delta_1 + (1-t)\delta_2)$, so that $t(\delta_1, \epsilon_1) + (1-t)(\delta_2, \epsilon_2)$ is in Ω_I .

Replacing \geq by $>$, we prove the convexity of ω_I .

(2) The inclusion $\omega_I \subseteq D_I^{-1} \text{Im } A_I$ follows from the definition of ω_I .

If $D_I^{-1} \text{Im } A_I$ is a point or if ω_I is empty, there is nothing to prove.

Let $(\delta_0, \epsilon_0) \in \omega_I$. Then there is x_0 such that $A_I x_0 = D_I(\delta_0, \epsilon_0)$ and $A_{\bar{I}} x_0 > D_{\bar{I}}(\delta_0, \epsilon_0)$.

Assume that $D_I^{-1} \text{Im } A_I$ is one-dimensional. Let $(\delta_1, \epsilon_1) \neq (\delta_0, \epsilon_0)$ in $D_I^{-1} \text{Im } A_I$ and let x_1 be such that $A_I x_1 = D_I(\delta_1, \epsilon_1)$. Then $A_I((1-t)x_0 + tx_1) = D_I((1-t)\epsilon_0 + t\epsilon_1, (1-t)\delta_0 + t\delta_1)$ and for $t \in \mathbb{Q}$ small enough $A_{\bar{I}}((1-t)x_0 + tx_1) > D_{\bar{I}}((1-t)\epsilon_0 + t\epsilon_1, (1-t)\delta_0 + t\delta_1)$.

Assume that $D_I^{-1} \text{Im } A_I$ is two-dimensional. Let (δ_1, ϵ_1) and (δ_2, ϵ_2) in $D_I^{-1} \text{Im } A_I$ such that the three (δ_i, ϵ_i) 's are not in the same line. Let x_1 and x_2 be such that $A_I x_1 = D_I(\delta_1, \epsilon_1)$ and $A_I x_2 = D_I(\delta_2, \epsilon_2)$. Then

$$A_I((1-t_1-t_2)x_0 + t_1x_1 + t_2x_2) = D_I((1-t_1-t_2)\epsilon_0 + t_1\epsilon_1 + t_2\epsilon_2, (1-t_1-t_2)\delta_0 + t_1\delta_1 + t_2\delta_2)$$

and for (t_1, t_2) in a neighborhood of 0 in \mathbb{Q}^2 , we have

$$A_{\bar{I}}((1-t_1-t_2)x_0 + t_1x_1 + t_2x_2) > D_{\bar{I}}((1-t_1-t_2)\epsilon_0 + t_1\epsilon_1 + t_2\epsilon_2, (1-t_1-t_2)\delta_0 + t_1\delta_1 + t_2\delta_2).$$

This concludes the proof of the statement.

(3) It follows from the two previous statements.

(4) The first inclusion is obvious.

To prove the second one, remark that for all $(\delta_1, \epsilon_1) \in \omega_I$ and $(\delta_2, \epsilon_2) \in \Omega_I$, the segment $\{(1-t)(\delta_1, \epsilon_1) + t(\delta_2, \epsilon_2) \mid t \in [0, 1]\}$ is contained in ω_I . Indeed, let x_1 and x_2 be such that $A_I x_i = D_I(\delta_i, \epsilon_i)$, $A_{\bar{I}} x_i > D_{\bar{I}}(\delta_i, \epsilon_i)$. Then, for any $t \in (0, 1]$, we set $x_t := tx_1 + (1-t)x_2$ and $(\delta_t, \epsilon_t) := (t\epsilon_1 + (1-t)\epsilon_2, t\delta_1 + (1-t)\delta_2)$. Thus $A_I x_t = D_I(\delta_t, \epsilon_t)$ and $A_{\bar{I}} x_t > D_{\bar{I}}(\delta_t, \epsilon_t)$.

This remark implies directly that if $(\delta_2, \epsilon_2) \in \Omega_I$ then $(\delta_2, \epsilon_2) \in \overline{\omega_I}$, as soon as ω_I is not empty. \square

Remark 7. Lemma 15 is still true if we consider polytopes in an \mathbb{R} -vector space. The proof is the same after replacing \mathbb{Q} by \mathbb{R} everywhere.

Lemma 16. *Let $I \subseteq I_0$ be such that ω_I is not empty. Then $\Omega_I = \overline{\omega_I}$ and Ω_I is polyhedral in \mathbb{Q}^2 .*

Proof. If ω_I is reduced to a point, by Lemma 15 we have nothing to prove, so we suppose that ω_I contains at least two points.

For $(\delta, \epsilon) \in \mathbb{R}^2$ we also set

$$P^{\delta, \epsilon} := \{x \in \mathbb{R}^n \mid Ax \geq (1-\delta)B + \delta B' + \epsilon C\}.$$

and we define as before $F_I^{\delta, \epsilon}$.

For $I \subseteq I_0$ we set

$$\Omega_I(\mathbb{R}) = \{(\delta, \epsilon) \in \mathbb{R}^2 \mid F_I^{\delta, \epsilon} \neq \emptyset\}$$

and

$$\omega_I(\mathbb{R}) = \{(\delta, \epsilon) \in \mathbb{R}^2 \mid \text{if } I' \subseteq I_0 \text{ satisfies } F_{I'}^{\delta, \epsilon} = F_I^{\delta, \epsilon}, \text{ then } I' \subseteq I\}.$$

Since the matrices A , B , B' and C have rational coefficients, if $(\delta, \epsilon) \in \mathbb{Q}^2$ and if there exists $x \in \mathbb{R}^n$, such that $A_I x = D_I(\delta, \epsilon)$, then there is $x' \in \mathbb{Q}^n$, such that $A_I x' = D_I(\delta, \epsilon)$; and x' can be chosen arbitrarily close to x . In particular, if $A_{\bar{I}} x > D_{\bar{I}}(\delta, \epsilon)$, then x' can be chosen such that $A_{\bar{I}} x' > D_{\bar{I}}(\delta, \epsilon)$. Hence, $\omega_I = \omega_I(\mathbb{R}) \cap \mathbb{Q}^2$.

Moreover, $\Omega_I = \bigcup_{I' \subseteq I' \subseteq I_0} \omega_{I'}$ (over \mathbb{R} and \mathbb{Q}), then we also have $\Omega_I(\mathbb{Q}) = \Omega_I(\mathbb{R}) \cap \mathbb{Q}^2$.

We first prove that $\Omega_I(\mathbb{R})$ is closed. This, together with Lemma 15 will imply that $\Omega_I(\mathbb{R}) = \overline{\omega_I(\mathbb{R})}$.

Let $(\delta_k, \epsilon_k)_{k \in \mathbb{N}}$ be a sequence of elements in $\Omega_I(\mathbb{R})$ converging to $(\bar{\delta}, \bar{\epsilon})$ in \mathbb{R}^2 . The elements (δ_k, ϵ_k) are contained in a compact set K of \mathbb{R}^2 . Then, for every $k \in \mathbb{N}$, the polytope P^{δ_k, ϵ_k} is contained in the polytope

$$P^K := \{x \in \mathbb{R}^n \mid Ax \geq \text{Min}_{(\delta, \epsilon) \in K} D(\delta, \epsilon)\}$$

where $\text{Min}_{(\delta, \epsilon) \in K} D(\delta, \epsilon)$ is the vector whose i -th coordinate is $\text{Min}_{(\delta, \epsilon) \in K} D(\delta, \epsilon)_i$. The set P^K is compact by Remark 5.

By definition, for any $k \in \mathbb{N}$, there exists $x_k \in \mathbb{R}^n$ such that $A_I x_k = D_I(\delta_k, \epsilon_k)$ and $A_{\bar{I}} x_k \geq D_{\bar{I}}(\delta_k, \epsilon_k)$. Since the $x_k \in P^K$ that is compact, there is $\{k_m\}_{m \in \mathbb{N}}$ such that $k_m \rightarrow \infty$ as m tends to infinity, such that $(x_{k_m})_{m \in \mathbb{N}}$ converges to $\bar{x} \in \mathbb{R}$. Then \bar{x} satisfies $A_I \bar{x} = D_I(\bar{\delta}, \bar{\epsilon})$ and $A_{\bar{I}} \bar{x} \geq D_{\bar{I}}(\bar{\delta}, \bar{\epsilon})$. This implies that $(\bar{\delta}, \bar{\epsilon})$ is also in Ω_I .

We now prove that $\Omega_I(\mathbb{R})$ is polyhedral. If $\omega_I(\mathbb{R})$ is either empty, or a point, or a convex part of an affine line, then there is nothing to prove. Suppose that $\omega_I(\mathbb{R})$ is open in \mathbb{R}^2 . Then the boundary of $\Omega_I(\mathbb{R})$ is

$$\Omega_I(\mathbb{R}) \setminus \omega_I(\mathbb{R}) = \bigcup_{I' \subsetneq I' \subseteq I_0} \omega_{I'}(\mathbb{R}) = \bigcup_{I' \subsetneq I' \subseteq I_0} \Omega_{I'}(\mathbb{R}).$$

But for every I' such that $I' \subsetneq I' \subseteq I_0$, the set $\Omega_{I'}(\mathbb{R})$ is either empty, or a point, or a closed convex part of an affine line. We proved that the boundary of $\Omega_I(\mathbb{R})$ is a finite union of closed convex parts of affine lines. Hence, $\Omega_I(\mathbb{R})$ is polyhedral in \mathbb{R}^2 .

To conclude, we have to prove that the vertices of $\Omega_I(\mathbb{R})$ are rational and that the maximal half lines in the boundary of $\Omega_I(\mathbb{R})$ are rational half-lines, that is, they have a rational extremity and rational direction.

A vertex of $\Omega_I(\mathbb{R})$ is (δ, ϵ) such that there is $I' \neq I$ with $\{(\delta, \epsilon)\} = \omega_{I'}(\mathbb{R})$. By Lemma 15 we have $\{(\delta, \epsilon)\} = D_{\bar{I}'}^{-1} \text{Im } A_{I'} \in \mathbb{Q}^2$.

In the same way, a component of the boundary is of the form $\omega_{I'}(\mathbb{R})$ with $I' \neq I$. By Lemma 15, we have $\omega_{I'}(\mathbb{R}) \subseteq D_{\bar{I}'}^{-1} \text{Im } A_{I'}$ open. The latter is an affine subspace defined over \mathbb{Q} , thus its direction is rational. The extremities are vertices of $\Omega_I(\mathbb{R})$ and are then rational. \square

We now study inclusions between the ω 's, generalizing the easy following fact: for any $J \subseteq I \subseteq I_0$, we have $\Omega_I \subseteq \Omega_J$.

Lemma 17. *Let $J \subseteq I \subseteq I_0$, such that $\omega_I \neq \emptyset$.*

- (1) *If ω_J is not empty and has the same dimension of ω_I , then $\omega_I \subseteq \omega_J$.*
- (2) *If the images of A_I and A_J have the same codimension k , then ω_J is not empty.*
- (3) *There exists $J \subseteq I$ as above and such that $\omega_I \subseteq \omega_J$ and for any $j \in J$ the codimension of the image of $A_{J \setminus \{j\}}$ is less than k .*

Proof. (1) If ω_J is not empty, Lemma 16 implies that ω_J is the interior part Ω_J . Since ω_I is not empty, it is as well the interior part of Ω_I . As $\Omega_I \subseteq \Omega_J$, we conclude that $\omega_I \subseteq \omega_J$.

(2) We prove the claim by induction on $h = |I \setminus J|$. If $h = 0$ the claim is true. We assume now that the claim is true for h and let J be such that $|I \setminus J| = h + 1$. Let $i \in I \setminus J$ and $J' = J \cup \{i\}$. By inductive hypothesis $\omega_{J'}$ is not empty. Let $(\delta, \epsilon) \in \omega_{J'}$ and let $x \in \mathbb{Q}^n$ such that $A_{J'}x = D_{J'}(\delta, \epsilon)$ and $A_{\bar{J}'}x > D_{\bar{J}'}(\delta, \epsilon)$. By the hypothesis on the images of $A_{J'}$ and A_J , there is $y \in \text{Ker}(A_J) \setminus \text{Ker}(A_{J'})$. We can choose y such that $A_i y > 0$. Then for $t > 0$ small enough, we have $A_J(x + ty) = D_J(\delta, \epsilon)$, $A_i(x + ty) > D_i(\delta, \epsilon)$ and $A_{\bar{J}'}(x + ty) > D_{\bar{J}'}(\delta, \epsilon)$. This proves that $(\delta, \epsilon) \in \omega_J$.

(3) It is enough to take J minimal such that the images of A_I and A_J have the same codimension, and apply the previous statement. □

Lemma 18. *Let $I \subseteq I_0$ be such that $\omega_I = \{(\delta_0, \epsilon_0)\}$. Suppose that the image of A_I is of codimension 2. There is $i \in I$ such that the image of $A_{I \setminus \{i\}}$ is of codimension 1. For any such i , the point (δ_0, ϵ_0) belongs to $\Omega_{I \setminus \{i\}} \setminus \omega_{I \setminus \{i\}}$.*

Proof. Let $i \in I$ be such that there exists rational numbers λ_j for $j \in I \setminus \{i\}$ such that $A_i = \sum_{j \in I \setminus \{i\}} \lambda_j A_j$. Then the image of $A_{I \setminus \{i\}}$ is of codimension one. Let $(\delta_1, \epsilon_1) \neq (\delta_0, \epsilon_0)$ such that $D_{I \setminus \{i\}}(\delta_1, \epsilon_1)$ is in $\text{Im}(A_{I \setminus \{i\}})$. Let x and y in \mathbb{Q}^n such that $A_I x = D_I(\delta_0, \epsilon_0)$, $A_{\bar{I}}x > D_{\bar{I}}(\delta_0, \epsilon_0)$ and $A_{I \setminus \{i\}}y = D_{I \setminus \{i\}}(\delta_1, \epsilon_1)$. For any $t \in \mathbb{Q}$, define $z_t := (1 - t)x + ty$ and $(\delta_t, \epsilon_t) := (1 - t)(\delta_0, \epsilon_0) + t(\delta_1, \epsilon_1)$. Then for any t we have $A_{I \setminus \{i\}}z_t = D_{I \setminus \{i\}}(\delta_t, \epsilon_t)$ and for any t small enough we have $A_{\bar{I}}z_t > D_{\bar{I}}(\delta_t, \epsilon_t)$. Now if $A_i y > 0$ we have $A_i z_t > D_i(\delta_t, \epsilon_t)$ for any $t > 0$; and if $A_i y < 0$ we have $A_i z_t > D_i(\delta_t, \epsilon_t)$ for any $t < 0$. Hence, for any t small enough, either positive or negative, we have $A_{I \setminus \{i\}}z_t > D_{I \setminus \{i\}}(\delta_t, \epsilon_t)$ and (δ_t, ϵ_t) is in $\omega_{I \setminus \{i\}}$. We have proved that ω_I is in the closure of $\omega_{I \setminus \{i\}}$.

But $\omega_I = \{(\delta_0, \epsilon_0)\}$ cannot be in $\omega_{I \setminus \{i\}}$, because the relation $A_i = \sum_{j \in I \setminus \{i\}} \lambda_j A_j$ implies that if $A_{I \setminus \{i\}}x = D_{I \setminus \{i\}}(\delta_0, \epsilon_0)$ then $A_I x = D_I(\delta_0, \epsilon_0)$. Hence ω_I is in $\Omega_{I \setminus \{i\}} \setminus \omega_{I \setminus \{i\}}$. □

Remark 8. A similar result could be proved from a one-dimensional ω_I . Let $I \subset I_0$ such that ω_I is an open convex part of an affine line. Suppose that the image of A_I is of codimension 1. Then there exists $i \in I$ such that ω_I is a subset of $\Omega_{I \setminus \{i\}} \setminus \omega_{I \setminus \{i\}}$ and $A_{I \setminus \{i\}}$ is surjective.

Lemma 19. *Let I and J such that ω_I and ω_J are not empty. Then $\omega_{I \cap J}$ is not empty and contains the strict convex hull of any element of ω_I with any element of ω_J .*

Proof. Let x and y in \mathbb{Q}^n be such that $A_I x = D_I(\delta_1, \epsilon_1)$, $A_{\bar{I}}x > D_{\bar{I}}(\delta_1, \epsilon_1)$, $A_J x = D_J(\delta_2, \epsilon_2)$ and $A_{\bar{J}}x > D_{\bar{J}}(\delta_2, \epsilon_2)$.

For any $t \in]0, 1[$, $z_t := (1-t)x + ty$ and $(\delta_t, \epsilon_t) := (1-t)(\delta_1, \epsilon_1) + t(\delta_2, \epsilon_2)$ satisfy $A_{I \cap J} z_t = D_{I \cap J}(\delta_t, \epsilon_t)$, $A_{\overline{I \cap J}} z_t > D_{\overline{I \cap J}}(\delta_t, \epsilon_t)$. Hence, $\omega_{I \cap J}$ contains the strict convex hull of (δ_1, ϵ_1) and (δ_2, ϵ_2) . In particular, $\omega_{I \cap J}$ is not empty. \square

Remark 9. If $I = \{i\}$ and $A_i \neq 0$, or more generally if A_I is surjective, Lemma 15 implies that ω_I is open (possibly empty).

If $I = \{i\}$ and $A_i \neq 0$, for any $(\delta, \epsilon) \in \Omega_I \setminus \omega_I$, either the polytope $P^{\delta, \epsilon}$ can be defined without the line i , in other words

$$P^{\delta, \epsilon} = \{x \in \mathbb{Q}^n \mid A_{\overline{\{i\}}} x \geq D_{\overline{\{i\}}}(\delta, \epsilon)\},$$

or the polytope $P^{\delta, \epsilon}$ is not of dimension n . Indeed, if $(\delta, \epsilon) \in \Omega_I \setminus \omega_I$, and $P^{\delta, \epsilon}$ has maximal dimension, then $F_i^{\delta, \epsilon}$ is not a facet of $P^{\delta, \epsilon}$ or equals another $F_j^{\delta, \epsilon}$ and in both cases the inequality $A_i x \geq D_i(\delta, \epsilon)$ is superfluous in the definition of $P^{\delta, \epsilon}$.

If I is such that A_I is invertible, then $F_I^{\delta, \epsilon}$ is either empty or a vertex of $P^{\delta, \epsilon}$. If $(\delta, \epsilon) \in \Omega_I$, then $P^{\delta, \epsilon}$ is simple at the corresponding vertex if $(\delta, \epsilon) \in \omega_I$. But the converse is false: it can happen that $P^{\delta, \epsilon}$ is simple but $(\delta, \epsilon) \notin \omega_I$ if an inequality $A_i x \geq D_i(\delta, \epsilon)$ with $i \notin I$ is superfluous in the definition of $P^{\delta, \epsilon}$ and $F_I^{\delta, \epsilon} = F_{I \cup \{i\}}^{\delta, \epsilon}$.

5.3. Generality of the polarization. In this section we prove some preparatory results. Throughout the section by *general* we mean *in a Zariski open set*.

Lemma 20. *We can choose B and B' general such that for any $I \subseteq I_0$, we have the following.*

- If the image of A_I has codimension 1 in \mathbb{Q}^I , then ω_I is an open convex part of an affine line (possibly empty).
- If the image of A_I has codimension 2 in \mathbb{Q}^I , then ω_I is either empty or a point.
- If the image of A_I has codimension 3 in \mathbb{Q}^I , then ω_I is empty.

Proof. The set $\{D_I(\delta, \epsilon) \mid (\delta, \epsilon) \in \mathbb{Q}^2\}$ is the affine subspace passing through B_I and directed by $B'_I - B_I$ and C_I . It is a plane for B and B' general (that is, if $B' - B$ is not colinear to C). Now, to have the three conditions above, it is enough to choose B and B' such that:

- if the image of A_I has codimension 1 in \mathbb{Q}^I , then B_I and B'_I are not in the image of A_I ;
- if the image of A_I has codimension 2 in \mathbb{Q}^I , then $B'_I - B_I$ is not in the vector subspace $\text{Im}(A_I) + \mathbb{Q}C_I$ (of codimension at least one in \mathbb{Q}^I), and if C_I is in $\text{Im}(A_I)$ then B_I is not in $\text{Im}(A_I)$;
- if the image of A_I has codimension 3 in \mathbb{Q}^I , $B'_I - B_I$ is not in $\text{Im}(A_I)$ and B_I is not in the vector subspace $\text{Im}(A_I) + \mathbb{Q}C_I + \mathbb{Q}(B'_I - B_I)$ (of codimension at least one in \mathbb{Q}^I).

This, together with Lemma 15, means that it is enough to choose B and B' outside finitely many proper linear subspaces of \mathbb{Q}^p , thus B and B' in an open set of \mathbb{Q}^p . \square

From now, we assume that B and B' general in the sense of Lemma 20.

Corollary 21. *There is a finite union of convex parts of affine lines*

$$L = \partial\Omega_\emptyset \cup \bigcup_{I \subseteq I_0, \dim \Omega_I \leq 1} \Omega_I \subseteq \mathbb{Q}^2$$

such that if $(\delta, \epsilon) \notin L$ then $P^{\delta, \epsilon}$ is an n -dimensional simple polytope or it is empty.

Proof. Assume that $P^{\delta, \epsilon}$ is not empty. If $P^{\delta, \epsilon}$ has of dimension n and is not simple, then there exists a vertex v of $P^{\delta, \epsilon}$ that is contained in at least $n + 1$ facets. Then there is $I \subseteq I_0$ of cardinality $p \geq n + 1$ such that the vertex is $F_I^{\delta, \epsilon} = \{v\}$. In particular, if we choose I maximal among the subsets I of I_0 such that $F_I^{\delta, \epsilon} = \{v\}$, we have $(\delta, \epsilon) \in \omega_I$. But A_I is a $p \times n$ matrix with $p \geq n + 1$ so it cannot be surjective. By Lemma 20, ω_I is either a segment or a point.

If $P^{\delta, \epsilon}$ is of dimension at most n , then $(\delta, \epsilon) \notin \omega_\emptyset$, which is open and non-empty by hypothesis. Then (δ, ϵ) is in the boundary of Ω_\emptyset , which is a finite union of convex parts of affine lines.

We set then

$$L = \partial\Omega_\emptyset \cup \bigcup_{I \subseteq I_0, \dim \Omega_I \leq 1} \Omega_I.$$

□

Remark 10. Notice that the condition we impose on $\Omega_\emptyset \setminus L$ is stronger than the polytope $P^{\delta, \epsilon}$ being simple. Indeed, the vertices of $P^{\delta, \epsilon}$ for $(\delta, \epsilon) \in \Omega_\emptyset \setminus L$ are contained in exactly n affine hyperplanes $\mathcal{H}_i := \{x \in \mathbb{Q}^n \mid A_i x = D_i(\delta, \epsilon)\}$, while a vertex of an affine polytope is contained in exactly n facets.

Now we prove that if B and B' are general, two sets ω_I and ω_J of dimension 0 or 1 intersect or are aligned only in specific cases.

Notation/Construction 1. Let $I \subseteq I_0$ be such that the image of A_I is contained in a hyperplane, and such that for any $i \in I$, $A_{I \setminus \{i\}}$ is surjective. Then there are rational numbers λ_i^I with $i \in I$ such that $\sum_{i \in I} \lambda_i^I A_i = 0$. We notice that $\lambda_i^I \neq 0$ for every $i \in I$. Indeed, if there is j such that $\lambda_j^I = 0$, then $\sum_{i \in I \setminus \{j\}} \lambda_i^I X_i$ is a nontrivial equation for the lines of $A_{I \setminus \{j\}}$. We can assume that $\sum_{i \in I} \lambda_i^I C_i$ is zero or one. We fix such numbers.

Let $I \subseteq I_0$ such that ω_I is not empty, the image of A_I has codimension at least two and such that for any $i \in I$, the image of $A_{I \setminus \{i\}}$ has codimension one. In particular, the image of A_I has codimension exactly two. Then there are rational numbers λ_i^I and λ_i^{II} with $i \in I$ such that $\sum_{i \in I} \lambda_i^I A_i = 0$ and $\sum_{i \in I} \lambda_i^{II} A_i = 0$ are two independent relations. For B and B' general as in Lemma 20, the point in ω_I is the only solution of the linear system

$$\begin{cases} (\sum_{i \in I} \lambda_i^I C_i)\epsilon + (\sum_{i \in I} \lambda_i^I (B'_i - B_i))\delta + \sum_{i \in I} \lambda_i^I B_i & = 0 \\ (\sum_{i \in I} \lambda_i^{II} C_i)\epsilon + (\sum_{i \in I} \lambda_i^{II} (B'_i - B_i))\delta + \sum_{i \in I} \lambda_i^{II} B_i & = 0 \end{cases}$$

therefore $\sum_{i \in I} \lambda_i^I C_i$ and $\sum_{i \in I} \lambda_i^{II} C_i$ cannot be simultaneously zero. After perhaps replacing λ_i^{II} with $\lambda_i^{II} - \lambda_i^I$, we can assume that $\sum_{i \in I} \lambda_i^I C_i = 1$ and $\sum_{i \in I} \lambda_i^{II} C_i = 0$.

Moreover, for any $i \in I$, either λ_i^I or λ_i^{II} is not zero. Indeed, if there is j such that $\lambda_j^I = 0$ and $\lambda_j^{II} = 0$, then $\sum_{i \in I \setminus \{j\}} \lambda_i^I X_i$ and $\sum_{i \in I \setminus \{j\}} \lambda_i^{II} X_i$ are nontrivial linearly independent equations for the lines of $A_{I \setminus \{j\}}$.

We fix such numbers.

Remark 11. Let $I \subseteq I_1$ be such that $\text{Im } A_I$ and $\text{Im } A_{I_1}$ have codimension 1 and are defined by the same equation $\sum_{i \in I} \lambda_i X_i = 0$. Assume moreover that $I = \{i \mid \lambda_i \neq 0\}$. Then for every $j \in I$ the morphism induced by $A_{I_1 \setminus \{j\}}$ is surjective. Indeed, consider the projection $\pi: \mathbb{Q}^{I_1} \rightarrow \mathbb{Q}^{I_1 \setminus \{j\}}$. The image of $A_{I_1 \setminus \{j\}}$ is the projection of the hyperplane defined by $\sum_{i \in I} \lambda_i X_i = 0$. Therefore $A_{I_1 \setminus \{j\}}$ is surjective.

Remark 12. Let $I \neq J$ be such that the images of A_I and A_J have codimension d and for any $i \in I$ and $j \in J$, the images of $A_{I \setminus \{i\}}$ and $A_{J \setminus \{j\}}$ have codimension $d - 1$. Then $I \not\subseteq J$ and $J \not\subseteq I$. Indeed if we had $I \subsetneq J$, there would be $j \in J \setminus I$ and $\text{Im}(A_I)$ would have codimension at most the codimension of $\text{Im}(A_{J \setminus \{j\}})$.

Lemma 22. Let $I \neq J$ be such that ω_I and ω_J are non-empty open convex parts of the same affine line, and such that for any $i \in I$ and $j \in J$, $A_{I \setminus \{i\}}$ and $A_{J \setminus \{j\}}$ are surjective. Then B and B' satisfy a quadratic or linear condition.

Proof. Set $B'' := B' - B$. By remark 12 we have $I \not\subseteq J$ and $J \not\subseteq I$. Since ω_I and ω_J are non-empty open convex parts of affine lines, the images of A_I and A_J have codimension at least one. Since $A_{I \setminus \{i\}}$ and $A_{J \setminus \{j\}}$ are surjective, they have codimension exactly one.

The affine line containing ω_I (resp. ω_J) has as equation $\sum_{i \in I} \lambda_i^I D_i(\delta, \epsilon) = 0$ (resp. $\sum_{j \in J} \lambda_j^J D_j(\delta, \epsilon) = 0$), i.e.

$$\left(\sum_{i \in I} \lambda_i^I C_i \right) \epsilon + \left(\sum_{i \in I} \lambda_i^I B_i'' \right) \delta + \sum_{i \in I} \lambda_i^I B_i = 0, \text{ resp. } \left(\sum_{j \in J} \lambda_j^J C_j \right) \epsilon + \left(\sum_{j \in J} \lambda_j^J B_j'' \right) \delta + \sum_{j \in J} \lambda_j^J B_j = 0.$$

Those two equations by hypothesis define the same line. It follows from Construction 1 that the coefficients of ϵ are either both zero or both one.

If $\sum_{i \in I} \lambda_i^I C_i = \sum_{j \in J} \lambda_j^J C_j = 0$, since the two equations define the same line, we have

$$\left(\sum_{i \in I} \lambda_i^I B_i'' \right) \left(\sum_{j \in J} \lambda_j^J B_j \right) = \left(\sum_{j \in J} \lambda_j^J B_j'' \right) \left(\sum_{i \in I} \lambda_i^I B_i \right).$$

This condition is nontrivial: let $i \in I$ and $j \in J$ such that $i \notin J$ and $j \notin I$, then the coefficient in the quadratic condition in $B_i'' B_j$ is $\lambda_i^I \lambda_j^J$ and this is non zero by Construction 1.

If $\sum_{i \in I} \lambda_i^I C_i = \sum_{j \in J} \lambda_j^J C_j = 1$, then $\sum_{i \in I} \lambda_i^I B_i'' = \sum_{j \in J} \lambda_j^J B_j''$ and $\sum_{i \in I} \lambda_i^I B_i = \sum_{j \in J} \lambda_j^J B_j$. \square

Corollary 23. Assume that B and B' are general. Then for any I and J subsets of I_0 such that ω_I and ω_J are nonempty open convex parts of affine lines, ω_I and ω_J are contained in the same affine line if and only if $\omega_{I \cap J}$ is a non-empty open convex part of an affine line. Moreover in this case, we have $\omega_{I \cap J} \supseteq \omega_I \cup \omega_J$.

Proof. Let I and J be subsets of I_0 such that ω_I and ω_J are non-empty open convex parts of the same affine line L . By Lemma 17, there exist $I' \subseteq I$ and $J' \subseteq J$ such that $\omega_{I'}$ and $\omega_{J'}$ are non-empty open segments contained in L , and such that for every $i \in I'$ and $j \in J'$, $A_{I' \setminus \{i\}}$ and $A_{J' \setminus \{j\}}$ are surjective. By Lemma 22, for B and B' general, we must have $I' = J'$. In particular $I \cap J \supset I'$ and $\omega_I \subset \omega_{I \cap J} \subseteq \omega_{I'}$ so that $\omega_{I \cap J}$ is a non-empty open convex part of an affine line.

Conversely, if ω_I , ω_J and $\omega_{I \cap J}$ are non-empty open convex part of an affine line, then they are contained in the same affine line because $\omega_I \subset \omega_{I \cap J} \supset \omega_J$. \square

Lemma 24. Let $I \neq J$ be such that $\omega_I = \omega_J$ is reduced to a point, and such that for any $i \in I$ and $j \in J$, the images of $A_{I \setminus \{i\}}$ and $A_{J \setminus \{j\}}$ have codimension one. Then B and B' satisfy quadratic conditions.

Proof. Set $B'' := B' - B$. By remark 12 we have $I \not\subseteq J$ and $J \not\subseteq I$. Since ω_I and ω_J are points, the images of A_I and A_J have codimension at least two. Since

$A_{I \setminus \{i\}}$ and $A_{J \setminus \{j\}}$ have codimension exactly one, the images of A_I and A_J have codimension exactly two.

Let (δ_0, ϵ_0) be such that $\omega_I = \omega_J = \{(\delta_0, \epsilon_0)\}$. Then (δ_0, ϵ_0) is the unique solution of the two following systems:

$$(\mathcal{S}_I) \quad \begin{cases} \sum_{i \in I} \lambda_i^I D_i(\delta, \epsilon) = 0 \\ \sum_{i \in I} \lambda_i^{II} D_i(\delta, \epsilon) = 0 \end{cases} \quad \text{and} \quad (\mathcal{S}_J) \quad \begin{cases} \sum_{j \in J} \lambda_j^J D_j(\delta, \epsilon) = 0 \\ \sum_{j \in J} \lambda_j^{JJ} D_j(\delta, \epsilon) = 0 \end{cases}.$$

By Construction 1, we can write these two systems as follows:

$$\begin{cases} \epsilon + (\sum_{i \in I} \lambda_i^I B_i'') \delta + \sum_{i \in I} \lambda_i^I B_i = 0 \\ (\sum_{i \in I} \lambda_i^{II} B_i'') \delta + \sum_{i \in I} \lambda_i^{II} B_i = 0 \end{cases} \quad \text{and} \quad \begin{cases} \epsilon + (\sum_{j \in J} \lambda_j^J B_j'') \delta + \sum_{j \in J} \lambda_j^J B_j = 0 \\ (\sum_{j \in J} \lambda_j^{JJ} B_j'') \delta + \sum_{j \in J} \lambda_j^{JJ} B_j = 0 \end{cases}.$$

The equations in the second line are multiple one of the other, thus

$$(\mathcal{C}_1) : \left(\sum_{i \in I} \lambda_i^{II} B_i'' \right) \left(\sum_{j \in J} \lambda_j^J B_j \right) = \left(\sum_{j \in J} \lambda_j^{JJ} B_j'' \right) \left(\sum_{i \in I} \lambda_i^I B_i \right).$$

The equations in the first line imply $\left((\sum_{i \in I} \lambda_i^I B_i'') - (\sum_{j \in J} \lambda_j^J B_j'') \right) \delta + \sum_{i \in I} \lambda_i^I B_i - \sum_{j \in J} \lambda_j^J B_j = 0$. This equation is a multiple of $(\sum_{i \in I} \lambda_i^{II} B_i'') \delta + \sum_{i \in I} \lambda_i^{II} B_i = 0$, therefore

$$(\mathcal{C}_2) : \left(\left(\sum_{i \in I} \lambda_i^I B_i'' \right) - \left(\sum_{j \in J} \lambda_j^J B_j'' \right) \right) \sum_{i \in I} \lambda_i^I B_i - \left(\sum_{i \in I} \lambda_i^I B_i - \sum_{j \in J} \lambda_j^J B_j \right) \sum_{i \in I} \lambda_i^{II} B_i'' = 0.$$

To conclude, it is enough to prove that the conditions (\mathcal{C}_1) and (\mathcal{C}_2) cannot be both trivial. Suppose that (\mathcal{C}_1) is trivial. For any $j \in J$ such that $j \notin I$ and for any $i \in I$, $\lambda_j^J \lambda_i^{II} = 0$, but $(\lambda_i^{II})_{i \in I}$ is not zero, so that $\lambda_j^J = 0$. The condition $\lambda_j^J = 0$ implies that $\lambda_j^J \neq 0$, and then for any $i \in I$ the coefficient of $B_j B_i''$ in (\mathcal{C}_2) is $\lambda_j^J \lambda_i^{II}$, but $(\lambda_i^{II})_{i \in I}$ is not zero, so that there exists $i \in I$ such that $\lambda_j^J \lambda_i^{II} \neq 0$, and (\mathcal{C}_2) is not trivial. \square

Corollary 25. *For B and B' general we have the following: for any $I, J \subseteq I_0$ such that ω_I and ω_J are singletons, we have $\omega_I = \omega_J$ if and only if $\omega_{I \cap J}$ is a singleton, equal to $\omega_I = \omega_J$.*

Proof. Let I and J be subsets of I_0 such that $\omega_I = \omega_J = \{(\delta_0, \epsilon_0)\}$. By Lemma 17, there exist $I' \subseteq I$ and $J' \subseteq J$ such that $\omega_{I'} = \omega_{J'}$ and for any $i \in I'$ and $j \in J'$, the images of $A_{I' \setminus \{i\}}$ and $A_{J' \setminus \{j\}}$ have codimension one. By Lemma 24, for general B and B' , we must have $I' = J'$. In particular $I \cap J \supseteq I'$ and $\omega_{I \cap J} \subseteq \omega_{I'} = \{(\delta_0, \epsilon_0)\}$. By Lemma 19 the set $\omega_{I \cap J}$ is not empty, then it must be $\omega_{I \cap J} = \{(\delta_0, \epsilon_0)\}$.

Conversely, if ω_I , ω_J and $\omega_{I \cap J}$ are singletons then $\omega_I = \omega_{I \cap J} = \omega_J$ because $\omega_I \subset \omega_{I \cap J} \supset \omega_J$. \square

Lemma 26. *Let I and J be such that $\omega_I = \{(\delta_0, \epsilon_0)\}$, ω_J is an open convex part of an affine line, and ω_I is contained in this affine line. Suppose that for every $i \in I$, the image of $A_{I \setminus \{i\}}$ has codimension one and for every $j \in J$, $A_{J \setminus \{j\}}$ is surjective.*

Then either $J \subseteq I$ or B and B' satisfy a quadratic condition.

Proof. Set $B'' = B' - B$.

The affine line containing both ω_I and ω_J has equation

$$\left(\sum_{j \in J} \lambda_j^J C_j\right)\epsilon + \left(\sum_{j \in J} \lambda_j^J B_j''\right)\delta + \sum_{j \in J} \lambda_j^J B_j = 0,$$

where $\sum_{j \in J} \lambda_j^J C_j$ is zero or one and (δ_0, ϵ_0) is the unique solution of the system

$$(\mathcal{S}_I) : \begin{cases} \epsilon + \left(\sum_{i \in I} \lambda_i^I B_i''\right)\delta + \sum_{i \in I} \lambda_i^I B_i & = 0 \\ \left(\sum_{i \in I} \lambda_i^I B_i''\right)\delta + \sum_{i \in I} \lambda_i^I B_i & = 0 \end{cases}.$$

If $\sum_{j \in J} \lambda_j^J C_j = 0$ we have

$$\left(\sum_{i \in I} \lambda_i^I B_i''\right)\left(\sum_{j \in J} \lambda_j^J B_j\right) = \left(\sum_{i \in I} \lambda_i^I B_i\right)\left(\sum_{j \in J} \lambda_j^J B_j''\right).$$

And if $\sum_{j \in J} \lambda_j^J C_j = 1$, we have

$$\left(\sum_{i \in I} \lambda_i^I B_i''\right)\left(\sum_{j \in J} \lambda_j^J B_j\right) = \left(\sum_{i \in I} \lambda_i^I B_i\right)\left(\sum_{j \in J} \lambda_j^J B_j''\right).$$

In both cases, if there exists $j \in J$ that is not in I , then there exists $i \in I$ such that the coefficient of $B_j B_i''$ is non zero (it is $\lambda_i^I \lambda_j^J$ in the first case and $\lambda_i^I \lambda_j^J$ is the second case) so that the condition is not trivial. \square

Corollary 27. *For B and B' general we have the following: for any I and J subsets of I_0 such that ω_I is a singleton and ω_J is an open convex part of an affine line ℓ , the affine line ℓ contains ω_I if and only if $\omega_{I \cap J}$ is a non-empty open convex part of ℓ .*

Proof. By Lemma 17, there exist $I' \subseteq I$ and $J' \subseteq J$ such that $\omega_{I'} = \omega_I$, $\omega_J \subset \omega_{J'}$ and for any $i \in I'$ and $j \in J'$, the image of $A_{I' \setminus \{i\}}$ has codimension one and $A_{J' \setminus \{j\}}$ is surjective. Then, by Lemma 26, , for general B and B' , we must have $J' \subseteq I'$.

In particular, $J' \subseteq I \cap J$ and by Lemma 17 we have $\omega_{I \cap J} \subset \omega_{J'}$, so that $\omega_{I \cap J}$ is either empty or an open convex part of an affine line (the same affine line as for ω_J).

By with Lemma 19, the set $\omega_{I \cap J}$ is not empty. \square

We end this subsection with the following lemma in order to avoid the case where three affine lines intersect into a point except in only one situation.

Lemma 28. *Let I , J and K be distinct subsets of I_0 such that the images of A_I , A_J and A_K have codimension one, and such that for any $i \in I$, $j \in J$ and $k \in K$, $A_{I \setminus \{i\}}$, $A_{J \setminus \{j\}}$ and $A_{K \setminus \{k\}}$ are surjective. If the three affine lines generated by ω_I , ω_J and ω_K intersect, then either B and B' satisfy some linear or quadratic condition or $I \cup J = I \cup K = J \cup K = I \cup J \cup K$.*

Proof. Set $B'' := B' - B$.

By Lemma 22, if the three lines are not distinct, we are done. Suppose then that the three lines are not distinct and meet at the point (δ_0, ϵ_0) .

Let $L = I, J, K$. The affine line containing ω_L has equation $\sum_{\ell \in L} \lambda_\ell^L D_\ell(\delta, \epsilon) = 0$. If we set

$$\begin{cases} a_L = \sum_{\ell \in L} \lambda_\ell^L C_\ell \\ b_L = \sum_{\ell \in L} \lambda_\ell^L B_\ell'' \\ c_L = \sum_{\ell \in L} \lambda_\ell^L B_\ell \end{cases}$$

then the equation becomes $a_L\epsilon + b_L\delta + c_L = 0$. By hypothesis the point (a_L, b_L, c_L) belongs to the plane $a\epsilon_0 + b\delta_0 + c = 0$ for $L = I, J, K$.

By Construction 1 we have $a_L \in \{0, 1\}$.

Assume that there is L such that $a_L = 0$. Without loss of generality we can assume that $L = I$. Notice that then only I is such that $a_I = 0$, because two affine lines distinct and parallel do not meet.

Then $b_I(c_J - c_K) = c_I(b_J - b_K)$, that is,

$$\left(\sum_{i \in I} \lambda_i^I B_i''\right) \left(\sum_{j \in J} \lambda_j^J B_j - \sum_{k \in K} \lambda_k^K B_k\right) = \left(\sum_{i \in I} \lambda_i^I B_i\right) \left(\sum_{j \in J} \lambda_j^J B_j'' - \sum_{k \in K} \lambda_k^K B_k''\right).$$

Assuming that we do not have $I \cup J = I \cup K = J \cup K = I \cup J \cup K$ we prove that this condition is not trivial. There are three cases, either $J \cup K \neq I \cup J \cup K$, or $I \cup J \neq I \cup J \cup K$, or $I \cup K \neq I \cup J \cup K$. The last two cases are proved in the same way.

If $J \cup K \neq I \cup J \cup K$ let $i \in I$ be such that $i \notin J \cup K$. As $J \neq K$ we can assume without loss of generality that there is $j \in J, j \notin K$. Then the coefficient of $B_i'' B_j$ in the quadratic condition is $\lambda_i^I \lambda_j^J$ which is non zero by Construction 1.

If $I \cup J \neq I \cup J \cup K$ let $k \in K$ such that $k \notin I \cup J$. Then for every $i \in I$ the coefficient of $B_i'' B_k$ in the quadratic condition is $\lambda_i^I \lambda_k^K$ which is non zero by Construction 1.

Assume that $a_L = 1$ for every L . Without loss of generality we can assume that $I \cup J \neq I \cup J \cup K$. We prove that the condition

$$(b_I - b_J)(c_I - c_K) = (b_I - b_K)(c_I - c_J)$$

verified by the coefficients of the two linear equations is not trivial. By replacing b_L and c_L we get

$$\begin{aligned} & \left(\sum_{i \in I} \lambda_i^I B_i'' - \sum_{j \in J} \lambda_j^J B_j''\right) \left(\sum_{i \in I} \lambda_i^I B_i - \sum_{k \in K} \lambda_k^K B_k\right) \\ &= \left(\sum_{i \in I} \lambda_i^I B_i'' - \sum_{k \in K} \lambda_k^K B_k''\right) \left(\sum_{i \in I} \lambda_i^I B_i - \sum_{j \in J} \lambda_j^J B_j\right). \end{aligned}$$

Let $k \in K$ be such that $k \notin I \cup J$. As $I \neq J$ we can assume without loss of generality that there is $i \in I, i \notin J$. Then the coefficient in the quadratic condition of $B_i'' B_k$ is $\lambda_i^I \lambda_k^K$ which is non zero by Construction 1. □

Remark 13. Let L be such that $\dim D_L^{-1}(Im A_L) = 0$, $\omega_L = \{(\delta_0, \epsilon_0)\}$ and such that for every $\ell \in L$ the image of $A_{L \setminus \{\ell\}}$ has codimension 1 in $\mathbb{Q}^{|L|}$.

Then there are two linearly independent relations $\sum_{\ell \in L} \lambda_\ell X_\ell = 0$ and $\sum_{\ell \in L} \mu_\ell X_\ell = 0$ on the lines of A_L . We set $I = \{\ell \in L \mid \lambda_\ell \neq 0\}$ and $J = \{\ell \in L \mid \mu_\ell \neq 0\}$. We can suppose that I and J are proper in L . Then, as the image of $A_{L \setminus \{\ell\}}$ has codimension 1 in $\mathbb{Q}^{|L|}$ for every $\ell \in L$, the images of A_I and A_J have codimension 1 and we have $I \cup J = L$.

Viceversa, given proper subsets I and J in L such that $I \cup J = L$ and the images of A_I and A_J have codimension 1, we have two such independent relations on the lines of A_L .

5.4. Polyhedral decomposition and the geography of models. Let G be a reductive group and $H \subseteq G$ be a horospherical subgroup. We choose a basis of $M_{\mathbb{Q}}$ so that $M_{\mathbb{Q}} \cong \mathbb{Q}^n$. We fix a horospherical embedding Z of G/H and set $p = r + |S \setminus R|$. Recall that, by the notation given in Section 2, r is the number of G -stable prime divisor of Z and $|S \setminus R|$ is the number of B -stable prime divisor of G/H . In particular, p is the number of B -stable prime divisor of Z . Let A be the $p \times n$ matrix associated to the linear map $\varphi(m) = (\langle m, x_i \rangle_{i=1 \dots r}, \langle m, \alpha_M^\vee \rangle_{\alpha \in S \setminus R})$. Denote by $J_0 \subseteq I_0$ the set of indices $S \setminus R$.

Let $B = (-d_1, \dots, -d_r, (-d_\alpha)_{\alpha \in S \setminus R})$ and $B' = (-d'_1, \dots, -d'_r, (-d'_\alpha)_{\alpha \in S \setminus R})$ be such that $D = \sum_{i=1}^d d_i Z_i + \sum_{\alpha \in S \setminus R} d_\alpha D_\alpha$ and $D' = \sum_{i=1}^d d'_i Z_i + \sum_{\alpha \in S \setminus R} d'_\alpha D_\alpha$ are ample divisors on two varieties X and Y respectively, in the sense that \tilde{Q}_D and $\tilde{Q}_{D'}$ are pseudo-moment polytopes for X and Y respectively. Notice that it may occur that some D_i 's are not divisors of X or Y as we observe in Lemma 13. Let $C = (c_1, \dots, c_r, (c_\alpha)_{\alpha \in S \setminus R})$ be such that $c_i > 0$ for every i and $c_\alpha > 0$ for every α . Since X is projective, $\tilde{Q}_D = \{x \in M_{\mathbb{Q}} \mid Ax \geq B\}$ is a polytope and then Condition 1 is verified by A .

From now on, following remark 4, if a polytope Q is defined by $\{x \in M_{\mathbb{Q}} \mid Ax \geq D\}$, with $D = (-d_1, \dots, -d_r, (-d_\alpha)_{\alpha \in S \setminus R})$, we will say that it is the pseudo-moment polytope associated to the moment polytope $v^0 + Q$ with $v^0 = \sum_{\alpha \in S \setminus R} d_\alpha \varpi_\alpha$, or similarly the pseudo-moment polytope of the associated G/H -embedding as soon as $v^0 + Q$ is a G/H -polytope (Definition 3).

We suppose that B and B' are general in the sense of the previous subsection, that is, that they belong to the Zariski open subset of the ample cone such that the conditions of Corollaries 21, 23, 25, 27 and Lemma 28 are satisfied.

Then we define the following locally closed subsets of Ω_\emptyset . We denote by $\text{Aff}(S)$ the affine space generated by a set S .

$$\begin{aligned} U_2 &= \{(\delta, \epsilon) \in \Omega_\emptyset \mid (\delta, \epsilon) \in \omega_I \Rightarrow \dim \omega_I = 2\} \\ U_1 &= \{(\delta, \epsilon) \in \Omega_\emptyset \mid (\delta, \epsilon) \notin U_2, (\delta, \epsilon) \in \omega_I \cap \omega_J \Rightarrow \dim \text{Aff } \omega_I \cap \text{Aff } \omega_J = 1\} \\ U_0 &= \{(\delta, \epsilon) \in \Omega_\emptyset \mid \exists I, \{(\delta, \epsilon)\} = \omega_I\} \\ U'_0 &= \Omega_\emptyset \setminus (U_2 \cup U_1 \cup U_0). \end{aligned}$$

Note that U_0, U'_0 are finite sets by the generality assumptions on B and B' .

Notation 1. Let (δ, ϵ) be such that $P^{\delta, \epsilon}$ is not empty.

(1) If $(\delta, \epsilon) \in U_2$, by Definition 3 and Proposition 5, the polytope $P^{\delta, \epsilon}$ is the pseudo-moment polytope of a G/H -embedding which we denote by $X^{\delta, \epsilon}$.

(2) If $(\delta, \epsilon) \in U_1$, by Lemma 7 there is $H' \supseteq H$ such that $P^{\delta, \epsilon}$ is the pseudo-moment polytope of a G/H' -embedding which we denote by $Y^{\delta, \epsilon}$.

(3) If $(\delta, \epsilon) \in U_0$, by Lemma 7 there is $H' \supseteq H$ such that $P^{\delta, \epsilon}$ is the pseudo-moment polytope of a G/H' -embedding which we denote by $Z^{\delta, \epsilon}$.

Remark 14. If $(\delta, \epsilon) \in \omega_\emptyset$, by Definition 3 and Proposition 5 the polytope $P^{\delta, \epsilon}$ is the pseudo-moment polytope associated to a G/H -polytope. On the other hand, if $(\delta, \epsilon) \in \Omega_\emptyset \setminus \omega_\emptyset$, then there is $H \subsetneq H'$ such that $P^{\delta, \epsilon}$ is the pseudo-moment polytope associated to a G/H' -polytope. In this case, H' is defined as follows: let M'_\emptyset be the linear subspace generated by $P^{\delta, \epsilon}$ and M' the sublattice $M'_\emptyset \cap M$ of M , let R' be the union of R with the set of simple roots $\alpha \in S \setminus R$ such that $P^{\delta, \epsilon} \subset \mathcal{H}_\alpha^{\delta, \epsilon}$, and define P' to be the parabolic subgroup containing B whose simple roots are R' ; then H' is the kernel of characters of $M' \subset \mathfrak{X}(P')$.

Remark 15. By Corollary 21, the generality assumption on B and B' and Lemma 8 for every $(\delta, \epsilon) \in U_2$ the variety $X^{\delta, \epsilon}$ is \mathbb{Q} -factorial. In particular, by Corollary 3, its Picard number is

$$(1) \quad \rho(X^{\delta, \epsilon}) = |I_1^{\delta, \epsilon} \sqcup J_0| - n$$

where $I_1^{\delta, \epsilon} := \{i \in I_0 \setminus J_0 \mid (\delta, \epsilon) \in \omega_i\}$. Since $(\delta, \epsilon) \in U_2$, we can replace $I_1^{\delta, \epsilon}$ by $I_2^{\delta, \epsilon} = \{i \in I_0 \setminus J_0 \mid (\delta, \epsilon) \in \Omega_i\}$ in the formula.

Let $(\delta_0, \epsilon_0) \in U_2$. Let $\ell: \mathbb{Q} \rightarrow \mathbb{Q}^2$ be the parametrisation of a rational affine line such that $\ell(0) = (\delta_0, \epsilon_0)$. Let $\bar{t} > 0$ be the minimum such that $\ell(\bar{t}) \in \Omega_\emptyset \setminus U_2$. There is a G -equivariant morphism from X^{δ_0, ϵ_0} to the variety corresponding to $P^{\ell(\bar{t})}$. Indeed, for every $t \in [0, \bar{t})$, the polytope $P^{\ell(t)}$ is the moment polytope of a polarized variety $(X^{\ell(t)}, D^{\ell(t)})$, and all the $P^{\ell(t)}$ for $t \in [0, \bar{t})$ are equivalent. Thus the corresponding varieties are isomorphic to X^{δ_0, ϵ_0} . The divisor $D(\ell(\bar{t}))$ is nef, but not ample, on X^{δ_0, ϵ_0} . Hence, by Lemma 7 there is a G equivariant morphism from X^{δ_0, ϵ_0} to the variety corresponding to $P^{\ell(\bar{t})}$.

5.4.1. *If $\ell(\bar{t}) \in U_1$.* We assume in this paragraph that $\ell(\bar{t}) = (\delta_1, \epsilon_1) \in U_1$. The main result of this paragraph is Proposition 33, which describes the different sorts of G -equivariant morphisms we can get from X^{δ_0, ϵ_0} to Y^{δ_1, ϵ_1} . We start with some preparatory lemmas and remarks.

Lemma 29. *Let $(\delta, \epsilon) \in U_1$. Let I be such that $(\delta, \epsilon) \in \omega_I$ and $\dim \omega_I = 1$. Assume moreover that I is minimal with the property. Denote by $I_+ := \{i \in I \mid \lambda_i^I > 0\}$ and $I_- := \{i \in I \mid \lambda_i^I < 0\}$. Note that $I = I_+ \sqcup I_-$.*

Then $(\delta, \epsilon) \in \omega_\emptyset$ if and only if both I_+ and I_- are not empty.

Proof. Let $\sum_{i \in I} \lambda_i^I X_i = 0$ be a relation satisfied by the lines of A_I as in Construction 1. Notice that $\lambda_i^I \neq 0$ for any $i \in I$ by minimality of I .

Fix $x \in \mathbb{Q}^n$ such that $A_I x = D_I(\delta, \epsilon)$ and $A_{I^-} x \geq D_{I^-}(\delta, \epsilon)$.

Assume first that I_+ and I_- are not empty. Let $i \in I_+$, then

$$A_i = - \sum_{j \in I_+, j \neq i} \frac{\lambda_j^I}{\lambda_i^I} A_j + \sum_{j \in I_-} \frac{-\lambda_j^I}{\lambda_i^I} A_j.$$

By the minimality of I , the matrix $A_{I \setminus \{i\}}$ is surjective. Therefore, there exists $y \in \mathbb{Q}^n$ such that $A_j y > 0$ for any $j \in I \setminus \{i\}$ and $A_j y$ big enough for $j \in I_-$ so that $A_i y$ is also positive. Then for any $t > 0$ small enough, we have $A(x + ty) > D(\delta, \epsilon)$. This means that $(\delta, \epsilon) \in \omega_\emptyset$.

Conversely, suppose that $I^+ = I$ and $I_- = \emptyset$. First, we prove that $\sum_{i \in I} \lambda_i^I D_i(\delta, \epsilon) = 0$. Since $(\delta, \epsilon) \in \omega_I$, there is $x \in \mathbb{Q}^n$ such that $Ax \geq D(\delta, \epsilon)$ and $A_I x = D_I(\delta, \epsilon)$. Thus $0 = \sum_{i \in I} \lambda_i^I A_i x = \sum_{i \in I} \lambda_i^I D_i(\delta, \epsilon)$.

Now, let $y \in \mathbb{Q}^n$ such that $Ay \geq D(\delta, \epsilon)$. Then $0 = \sum_{i \in I} \lambda_i^I A_i y \geq \sum_{i \in I} \lambda_i^I D_i(\delta, \epsilon) = 0$, so that $A_I y = D_I(\delta, \epsilon)$. In particular, $(\delta, \epsilon) \notin \omega_\emptyset$.

The same occurs if $I^- = I$. \square

Remark 16. The proof implies in particular that, if both I_+ and I_- are not empty, then there is $y \in \mathbb{Q}^n$ such that $A_I y > 0$.

Lemma 30. *Let $(\delta, \epsilon) \in U_1 \cap \omega_\emptyset$. Let I be such that $(\delta, \epsilon) \in \omega_I$ and $\dim \omega_I = 1$. Assume moreover that I is minimal with the property. Let $i \in I_0 \setminus J_0$. Then $(\delta, \epsilon) \in \Omega_i \setminus \omega_i$ if and only if I_+ or I_- equals $\{i\}$.*

In particular, there is at most one $i \in I_0 \setminus J_0$, such that (δ, ϵ) is in $\Omega_i \setminus \omega_i$.

Proof. Let $\sum_{i \in I} \lambda_i^I X_i = 0$ be a relation satisfied by the lines of A_I as in Construction 1. Notice that $\lambda_i^I \neq 0$ for any $i \in I$ by minimality of I .

Suppose that there exists $i \in I_0 \setminus J_0$ such that $(\delta, \epsilon) \in \Omega_i \setminus \omega_i$. Then there exists $J \subseteq I_0$ containing i such that $(\delta, \epsilon) \in \omega_J$. Pick such J minimal. By Lemma 17(1), since $(\delta, \epsilon) \notin \omega_i$, the image of A_J has codimension either one or two. By assumption on (δ, ϵ) , ω_J is not a point, so that ω_J is one-dimensional and then by Corollary 23 and the generality assumption, $I \subset J$. By minimality of J , we have $J = I \cup \{i\}$.

Assume by contradiction that $i \notin I$. With Notation 1, we have $\lambda_i^J = 0$, that is, J and I give the same relation.

Since $(\delta, \epsilon) \in \omega_\emptyset$, by Lemma 29 and Remark 16 and 11, there exists $y \in \mathbb{Q}^n$ such that $A_I y > 0$ and such that $A_i y = 0$. Let $x \in \mathbb{Q}^n$ such that $A_J x = D_J(\delta, \epsilon)$ and $A_{\bar{J}} x > D_{\bar{J}}(\delta, \epsilon)$. Then for any $t > 0$ small enough $A_i(x + ty) = D_i(\delta, \epsilon)$ and $A_{\bar{i}}(x + ty) > D_{\bar{i}}(\delta, \epsilon)$. This is a contradiction with $(\delta, \epsilon) \notin \omega_i$. Then $i \in I$ and $I = J$.

By assumption the matrix $A_{I \setminus \{i\}}$ is surjective. If $I \setminus \{i\}$ is not contained in either I_+ or I_- , we can find $y \in \mathbb{Q}^n$ such that $A_j y > 0$ for any $j \in I \setminus \{i\}$ and $\sum_{j \in I \setminus \{i\}} \lambda_j^I A_j y = 0$, which implies that $A_i y = 0$. Let $x \in \mathbb{Q}^n$ be such that $A_I x = D_I(\delta, \epsilon)$ and $A_{\bar{I}} x > D_{\bar{I}}(\delta, \epsilon)$. Then, for any $t > 0$ small enough we have $A_i(x + ty) = D_i(\delta, \epsilon)$ and $A_{\bar{i}}(x + ty) > D_{\bar{i}}(\delta, \epsilon)$. This is a contradiction with $(\delta, \epsilon) \notin \omega_i$. We conclude with Lemma 29 and the hypothesis $(\delta, \epsilon) \in \omega_\emptyset$, that $\{i\} = I_+$ or I_- .

Conversely, suppose that $I_+ = \{i\}$. Note that $(\delta, \epsilon) \in \Omega_i$ is obvious. Since there exists $x \in \mathbb{Q}^n$ such that $A_I x = D_I(\delta, \epsilon)$, we have $D_i(\delta, \epsilon) = \sum_{j \in I \setminus \{i\}} \frac{-\lambda_j^I}{\lambda_i^I} D_j(\delta, \epsilon)$.

For any $y \in \mathbb{Q}^n$, such that $A_{\bar{i}} y \geq D_{\bar{i}}(\delta, \epsilon)$ we have

$$A_i y = \sum_{j \in I \setminus \{i\}} \frac{-\lambda_j^I}{\lambda_i^I} A_j y \geq \sum_{j \in I \setminus \{i\}} \frac{-\lambda_j^I}{\lambda_i^I} D_j(\delta, \epsilon) = D_i(\delta, \epsilon).$$

And we have equality if and only if $A_{I \setminus \{i\}} y = D_{I \setminus \{i\}}(\delta, \epsilon)$. Then $(\delta, \epsilon) \notin \omega_i$.

For the last statment, note that if $|I| = 1$ then A_I is a zero line and $i \in J_0$; and if $|I| = 2$ with $|I_+| = 1$ then the two lines of A_I generate the same ray of $N_{\mathbb{Q}}$ and at most one could be the primitive element of the ray, so that the other line has index in J_0 . \square

Lemma 31. *Let $(\delta, \epsilon) \in U_1 \cap \Omega_\emptyset \setminus \omega_\emptyset$. Let I be such that $(\delta, \epsilon) \in \omega_I$ and $\dim \omega_I = 1$. Assume moreover that I is minimal with the property. Let $i \in I_0 \setminus I$. Then either $(\delta, \epsilon) \notin \Omega_{I \cup \{i\}}$ or $(\delta, \epsilon) \in \omega_{I \cup \{i\}}$.*

Proof. Let $i \in I_0 \setminus I$ such that $(\delta, \epsilon) \in \Omega_{I \cup \{i\}}$. There exists $J \subseteq I_0$ containing i and I such that $(\delta, \epsilon) \in \omega_J$. By hypothesis on (δ, ϵ) the set ω_J is one-dimensional and Lemma 17 implies that $\omega_J \subseteq \omega_{I \cup \{i\}}$. In particular, $(\delta, \epsilon) \in \omega_{I \cup \{i\}}$. \square

Corollary 32. *Let $(\delta, \epsilon) \in U_1$.*

(1) *Assume that $(\delta, \epsilon) \in \omega_\emptyset$. If there is $i \in I_0 \setminus J_0$ such that I_+ or I_- equals $\{i\}$, then the G -stable prime divisors of $Y^{\delta, \epsilon}$ are in bijection with $I_1^{\delta, \epsilon} = I_2^{\delta, \epsilon} \setminus \{i\}$. Otherwise the G -stable prime divisors of $Y^{\delta, \epsilon}$ are in bijection with $I_1^{\delta, \epsilon} = I_2^{\delta, \epsilon}$.*

(2) Assume that $(\delta, \epsilon) \in \Omega_\emptyset \setminus \omega_\emptyset$. Then the G -stable prime divisors of $Y^{\delta, \epsilon}$ are in bijection with

$$\{i \in I_0 \setminus I \mid (\delta, \epsilon) \in \omega_{I \cup \{i\}}\} = \{i \in I_0 \setminus I \mid (\delta, \epsilon) \in \Omega_{I \cup \{i\}}\} = I_2^{\delta, \epsilon} \setminus (I_2^{\delta, \epsilon} \cap I).$$

Proof. The set of B -stable prime divisors of $Y^{\delta, \epsilon}$ is the union of G -stable divisors and colors of the horospherical homogeneous space. The G -stable prime divisors correspond bijectively to the facets $F_i^{\delta, \epsilon}$ with $i \in I_0 \setminus J_0$ that are not equal to some $F_j^{\delta, \epsilon}$ with $j \in J_0$.

Assume that $(\delta, \epsilon) \in \omega_\emptyset$. Thus $Y^{\delta, \epsilon}$ is a G/H -embedding, colors are indexed by $S \setminus I$ or equivalently by J_0 . If there are i and $j \in I_0$ with $i \neq j$ and such that $F_i^{\delta_1, \epsilon_1} = F_j^{\delta_1, \epsilon_1}$ is a facet, then A_i and A_j are colinear, one of i or j is in J_0 , and $I = i, j$. Lemma 30 implies the claim.

Assume that $(\delta, \epsilon) \in \Omega_\emptyset \setminus \omega_\emptyset$. Then $Y^{\delta, \epsilon}$ is a G/H' -embedding with $H' \subsetneq H$ as explained in Remark 14. In the proof of Lemma 29, we prove that $P^{\delta, \epsilon}$ generates the affine subspace $\{x \in M_\mathbb{Q} \mid A_I x = D_I(\delta, \epsilon)\}$, so that $M' = \ker(A_I) \cap M$. Moreover, the colors of G/H' are indexed by $J_0 \setminus (J_0 \cap I)$. If for some $i, j \in I_0$, $F_i^{\delta, \epsilon} = F_j^{\delta, \epsilon}$ is a facet, then $i, j \notin I$, A_i and A_j are colinear modulo the vector space $N''_\mathbb{Q}$ generated by the lines of A_I . If $i \neq j$ it would give another relation to the one given by I , then $i = j$. Note also that for any $i \in I_0 \setminus I$, $F_i^{\delta, \epsilon} = F_{I \cup \{i\}}^{\delta, \epsilon}$, and if it is not empty then $A_i \notin N''_\mathbb{Q}$. Indeed, if $A_i \in N''_\mathbb{Q}$, we have another relation to the one given by I . Hence Lemma 31 implies the claim. \square

Proposition 33. *Let $(\delta_0, \epsilon_0) \in U_2$. Let $\ell: \mathbb{Q} \rightarrow \mathbb{Q}^2$ be the parametrisation of a rational affine line such that $\ell(0) = (\delta_0, \epsilon_0)$. Let $\bar{t} > 0$ be the minimum such that $\ell(\bar{t}) \in \Omega_\emptyset \setminus U_2$. Assume that $\ell(\bar{t}) = (\delta_1, \epsilon_1) \in U_1$.*

Set I minimal such that $(\delta_1, \epsilon_1) \in \omega_I$.

The morphism from X^{δ_0, ϵ_0} to Y^{δ_1, ϵ_1} is an extremal contraction and one of the following occurs.

- (1) *If $I = I_+$ or I_- then $\dim Y^{\delta_1, \epsilon_1} < \dim X^{\delta_0, \epsilon_0}$, Y^{δ_1, ϵ_1} is \mathbb{Q} -factorial and $\rho(X) = \rho(Y) + 1$.*
- (2) *If $|I| \geq 2$ and I_+ or I_- is $\{i\}$ with $i \in I_0 \setminus J_0$ then Y^{δ_1, ϵ_1} is \mathbb{Q} -factorial and $X^{\delta_0, \epsilon_0} \rightarrow Y^{\delta_1, \epsilon_1}$ is an extremal divisorial contraction or an isomorphism.*
- (3) *In the other cases $X^{\delta_0, \epsilon_0} \rightarrow Y^{\delta_1, \epsilon_1}$ is a small extremal contraction.*

Proof. Assume that $I = I_+$ or I_- . By Corollary 32 we have $(\delta_1, \epsilon_1) \in \Omega_\emptyset \setminus \omega_\emptyset$ and by Remark 14 we have $\dim Y^{\delta_1, \epsilon_1} < \dim X^{\delta_0, \epsilon_0}$.

By Lemma 31, the B -stable prime divisors of Y^{δ_1, ϵ_1} are in bijection with $(I_2^{\delta_1, \epsilon_1} \sqcup J_0) \setminus I$. Let $F_J^{\delta_1, \epsilon_1}$ be a face of P^{δ_1, ϵ_1} , choose J such that $(\delta_1, \epsilon_1) \in \omega_J$. Then $I \subseteq J$ and ω_J is one-dimensional, so that the restriction of $A_{J \setminus I}$ to $\ker(A_I)$ is surjective. This implies, with Lemma 8 applied to the matrix of lines of $A_{I_0 \setminus I}$ restricted to $\ker(A_I)$, that Y^{δ_1, ϵ_1} is \mathbb{Q} -factorial. Then its Picard number is $|(I_2^{\delta_1, \epsilon_1} \sqcup J_0) \setminus I| - (n - |I| + 1) = |I_2^{\delta_1, \epsilon_1} \sqcup J_0| - n - 1$. But $I_2^{\delta_1, \epsilon_1} = I_2^{\delta_0, \epsilon_0}$ so that the relative Picard number of $X^{\delta_0, \epsilon_0} \rightarrow Y^{\delta_1, \epsilon_1}$ is 1.

Assume that I_+ or I_- is $\{i\}$ with $i \in I_0 \setminus J_0$ and $|I| \geq 2$. By Lemma 29 we have $(\delta_1, \epsilon_1) \in \omega_\emptyset$. By Remark 14 we have $\dim Y^{\delta_1, \epsilon_1} = \dim X^{\delta_0, \epsilon_0}$.

By Corollary 32, the B -stable prime divisors of Y^{δ_1, ϵ_1} are in bijection with $(I_2^{\delta_1, \epsilon_1} \setminus \{i\}) \sqcup J_0 = (I_2^{\delta_0, \epsilon_0} \setminus \{i\}) \sqcup J_0$. For the sake of shortness we denote by $K \setminus \{i\}$ the set $K \cap (I_0 \setminus \{i\})$.

Let $F_J^{\delta_1, \epsilon_1}$ be a face of P^{δ_1, ϵ_1} , choose J such that $(\delta_1, \epsilon_1) \in \omega_J$.

If A_J is not surjective then $I \subseteq J$ and $A_{J \setminus \{i\}}$ is surjective. This implies, with Lemma 8, that Y^{δ_1, ϵ_1} is \mathbb{Q} -factorial. Then its Picard number is $|(I_2^{\delta_0, \epsilon_0} \setminus \{i\}) \sqcup J_0| - n$. This is the Picard number of X^{δ_0, ϵ_0} minus one if $i \in I_2^{\delta_0, \epsilon_0}$ and the Picard number of X^{δ_0, ϵ_0} if not. In that latter case, since P^{δ_1, ϵ_1} can be defined as well without the line i , it is an equivalent G/H -polytope to P^{δ_0, ϵ_0} , and then $X^{\delta_0, \epsilon_0} = Y^{\delta_1, \epsilon_1}$.

In the other cases, again y Lemma 29 and Remark 14 we have $\dim Y^{\delta_1, \epsilon_1} = \dim X^{\delta_0, \epsilon_0}$.

Corollary 32 implies that the B -stable prime divisors of Y^{δ_1, ϵ_1} are the same as the ones of X^{δ_0, ϵ_0} , they are therefore in bijection with $I_2^{\delta_0, \epsilon_0} \sqcup J_0$.

Let $F_J^{\delta_1, \epsilon_1}$ be a face of P^{δ_1, ϵ_1} , choose J such that $(\delta_1, \epsilon_1) \in \omega_J$. If A_J is not surjective, then ω_J is one-dimensional and $I \subseteq J$. Then a B -stable \mathbb{Q} -divisor of Y^{δ_1, ϵ_1} is \mathbb{Q} -Cartier if and only if its coefficients satisfy the equation $\sum_{i \in I} \lambda_i^I X_i$. Also note that for $I = J$, A_J is not surjective.

Then the relative Picard number of $X^{\delta_0, \epsilon_0} \rightarrow Y^{\delta_1, \epsilon_1}$ is 1. \square

Remark 17. If $I^+ = I$ or $I^- = I$, for every $i \in I$ the polytope $P^{\delta, \epsilon}$ is contained in $\mathcal{H}_i^{\delta, \epsilon} = \{x \in \mathbb{Q}^n \mid A_i x = D_i^{\delta, \epsilon}\}$. There are two possible cases for I . Either $I = \{i\}$ and $A_i = 0$, so that $P^{\delta, \epsilon}$ has dimension n , the associated horospherical variety X has the same rank as G/H , the same lattice M but its open homogeneous space has one color less (α_i); or $A_i \neq 0$ for any $i \in I$ (and $|I| \geq 2$), so that $P^{\delta, \epsilon}$ is of maximal dimension an affine subspace directed by $\ker(A_I)$, then it has dimension $n - |I| + 1$, and the associated horospherical variety X has rank $n - |I| + 1$, its lattice is $M \cap \ker A_I$ and the colors of its open homogeneous space are the colors of G/H whose index of line is not in I .

If $I^+ = \{i\}$ or $I^- = \{i\}$, then the condition $A_i x \geq D_i(\delta, \epsilon)$ is superfluous in the definition of $P^{\delta, \epsilon}$.

5.4.2. If $\ell(\bar{t}) \in U_0$. We assume in this paragraph that $\ell(\bar{t}) = (\delta_2, \epsilon_2) \in U_0$. We want to study the morphisms from X^{δ_0, ϵ_0} to Z^{δ_2, ϵ_2} .

Let L be such that $\{(\delta_2, \epsilon_2)\} = \omega_L$. By Lemma 18, there is I such that ω_I has dimension one and $(\delta_2, \epsilon_2) \in \Omega_I$. Then all but a finite set of points of ω_I are in $\omega_I \cap U_1$. Choose $(\delta_1, \epsilon_1) \in \omega_I \cap U_1$ close to (δ_2, ϵ_2) .

By Lemma 7 there are G -equivariant morphisms from X^{δ_0, ϵ_0} to Y^{δ_1, ϵ_1} and Z^{δ_2, ϵ_2} , and from Y^{δ_1, ϵ_1} to Z^{δ_2, ϵ_2} . Moreover the morphism from X^{δ_0, ϵ_0} to Z^{δ_2, ϵ_2} factorizes through Y^{δ_1, ϵ_1} .

We suppose from now on that $\{(\delta_2, \epsilon_2)\} = \omega_L \subset \Omega_\emptyset \setminus \omega_\emptyset$.

Notation/Construction 2. Let L be such that $\omega_L = \{(\bar{\delta}, \bar{\epsilon})\} \subset \Omega_\emptyset \setminus \omega_\emptyset$. Then, for general B and B' , the image of A_L has codimension two. By Lemma 17, we can suppose up to taking a subset of L that for every $\ell \in L$ the image of $A_{L \setminus \{\ell\}}$ has codimension 1 in $\mathbb{Q}^{|L|}$.

We have two possible cases.

(1) If ω_L is a vertex of $\Omega_\emptyset \setminus \omega_\emptyset$, by remark 13 there exist I and J subsets of L such that, for any $i \in I$ and any $j \in J$, $A_{I \setminus \{i\}}$ and $A_{J \setminus \{j\}}$ are surjective, and ω_I, ω_J are segments of the Mori's polygonal chain. By Lemma 29 we can fix two linearly

independent equations for the lines of A_L

$$\begin{aligned} (\mathcal{R}_I) \quad & \sum_{i \in I} \lambda_i^I X_i = 0, \quad \lambda_i^I > 0 \quad \forall i \in I \\ (\mathcal{R}_J) \quad & \sum_{j \in J} \lambda_j^J X_j = 0, \quad \lambda_j^J > 0 \quad \forall j \in J \end{aligned}$$

We suppose that $\omega_I \subseteq \{\delta < \bar{\delta}\}$ and that $\omega_J \subseteq \{\delta > \bar{\delta}\}$.

We can moreover assume that $\sum_{i \in I} \lambda_i^I C_i = 1$ and $\sum_{j \in J} \lambda_j^J C_j = 1$.

(2) If ω_L is not a vertex of $\Omega_\emptyset \setminus \omega_\emptyset$, then there is $I \subseteq L$ such that, for any $i \in I$ the matrix $A_{I \setminus \{i\}}$ is surjective, and ω_I is a segment of $\Omega_\emptyset \setminus \omega_\emptyset$ containing ω_L . By Lemma 29, we can fix two linearly independent equations for the lines of A_L

$$\begin{aligned} (\mathcal{R}_I) \quad & \sum_{i \in I} \lambda_i^I X_i = 0, \quad \lambda_i^I > 0 \quad \forall i \in I \\ (\mathcal{R}_L) \quad & \sum_{j \in L} \lambda_j^L X_j = 0, \quad \lambda_j^L \neq 0 \quad \forall j \in L \setminus I \end{aligned}$$

We verify that $\lambda_j^L \neq 0 \quad \forall j \in L \setminus I$. Indeed if the second equation has one zero coefficient λ_j^L for some $j \in L \setminus I$, then $A_{L \setminus \{j\}}$ would also have an image of codimension 2, contradicting the minimality of L . Moreover, the coefficients with indexes in $L \setminus I$ in the relation (\mathcal{R}_L) do not have all the same sign.

Indeed, if not, we can suppose that they are positive, and by adding to (\mathcal{R}_L) a positive multiple of (\mathcal{R}_I) , we would obtain a second equation, linearly independent with (\mathcal{R}_I) , with positive coefficients, associated to some $J \subsetneq I$. By Lemma 29 the set ω_J would be contained in $\Omega_\emptyset \setminus \omega_\emptyset$, contradicting the hypothesis on ω_L , as it would be the vertex between the segments ω_I and ω_J .

With the same argument we can also suppose that $\lambda_j^L \geq 0$ for any $j \in I$ but some are zero.

We set $(L \setminus I)_+ = \{j \in L \setminus I \mid \lambda_j^L > 0\}$ and $(L \setminus I)_- = \{j \in L \setminus I \mid \lambda_j^L < 0\}$.

We can moreover assume that $\sum_{i \in I} \lambda_i^I C_i = 1$ and either $\sum_{j \in L} \lambda_j^L C_j = 1$, or $\sum_{j \in L} \lambda_j^L C_j = 0$ and $\sum_{j \in L} \lambda_j^L (B'_j - B_j) > 0$. Indeed if $\sum_{j \in L} \lambda_j^L C_j < 0$ we take the opposite and add a positive multiple of the first equation and we obtain the wanted equation with $\sum_{j \in L} \lambda_j^L C_j > 0$. And if $\sum_{j \in L} \lambda_j^L C_j = 0$ and $\sum_{j \in L} \lambda_j^L (B'_j - b_j) > 0$, we take the opposite and add a positive multiple of the first equation and we obtain $\sum_{i \in I} \lambda_i^I C_i > 0$.

Lemma 34. *Let $(\delta, \epsilon) \in U_0 \cap \Omega_\emptyset \setminus \omega_\emptyset$ and assume that (δ, ϵ) is a vertex. Let $L \subseteq I_0$ be a minimal subset such that $\omega_L = \{(\delta, \epsilon)\}$. Let $i \in I_0 \setminus L$. Then either $(\delta, \epsilon) \notin \Omega_{L \cup \{i\}}$ or $(\delta, \epsilon) \in \omega_{L \cup \{i\}}$.*

Proof. If $(\delta, \epsilon) \in \Omega_{L \cup \{i\}} \setminus \omega_{L \cup \{i\}}$, there exists $J \subseteq I_0$ containing i and L such that $(\delta, \epsilon) \in \omega_J$. By Lemma 17, we have $\omega_{L \cup \{i\}} = \omega_L = \omega_J$. In particular $(\delta, \epsilon) \in \omega_{L \cup \{i\}}$. \square

Lemma 35. *Let $(\delta, \epsilon) \in U_0 \cap \Omega_\emptyset \setminus \omega_\emptyset$ and assume that (δ, ϵ) is not a vertex of $\Omega_\emptyset \setminus \omega_\emptyset$. Let $L \subseteq I_0$ be a minimal subset such that $\omega_L = \{(\delta, \epsilon)\}$. Let I be a minimal subset such that $\omega_L \subseteq \omega_I$ and $\dim \omega_I = 1$. Let $i \in I_0 \setminus I$. If $(\delta, \epsilon) \in \Omega_{I \cup \{i\}} \setminus \omega_{I \cup \{i\}}$ then $i \in L \setminus I$ and either $L \setminus I_+ = \{i\}$ or $L \setminus I_- = \{i\}$. In particular, there are at most 2 indices i such that $(\delta, \epsilon) \in \Omega_{I \cup \{i\}} \setminus \omega_{I \cup \{i\}}$. If there are 2 such indices, then $|L| = |I| + 2$ and the restriction of $A_{L \setminus I}$ to $\ker(A_I)$ consists of twice the same line if $L \setminus I \subseteq I_0 \setminus J_0$.*

Proof. The proof is similar to the proof of Lemma 30.

If $(\delta, \epsilon) \in \Omega_{I \cup \{i\}} \setminus \omega_{I \cup \{i\}}$, then there exists $J \subseteq I_0$ containing i and I such that $(\delta, \epsilon) \in \omega_J$ and ω_J is one or zero-dimensional. The set ω_J cannot have dimension

one, otherwise Lemma 17 would imply that $(\delta, \epsilon) \in \omega_J \subseteq \omega_{I \cup \{i\}}$. Hence, $\omega_J = \omega_L$ and $J = L \cup \{i\}$.

We prove that $i \in L$. Assume that it is not the case. Since the coefficients λ_j^L do not have the same sign for $j \in L \setminus I$, there exist positive μ_j^L such that $\sum_{j \in L \setminus I} \lambda_j^L \mu_j^L = 0$. By Remark 11 there exists $y \in \ker(A_I)$ such that $A_j y = \mu_j^L > 0$ for any $j \in L \setminus I$ and so that $A_i y = 0$. Since $(\delta, \epsilon) \in \omega_{L \cup \{i\}}$, there is $x \in M_{\mathbb{Q}}$ is such that $A_{L \cup \{i\}} x = D_{L \cup \{i\}}(\delta, \epsilon)$ and $A_{\overline{L \cup \{i\}}} x > D_{\overline{L \cup \{i\}}}(\delta, \epsilon)$. Thus for t small, we have $A_{I \cup \{i\}}(x + ty) = D_{I \cup \{i\}}(\delta, \epsilon)$ and $A_{L \setminus I}(x + ty) > D_{L \setminus I}(\delta, \epsilon)$ proving that $(\delta, \epsilon) \in \omega_{I \cup \{i\}}$, which is a contradiction. Thus $i \in L$.

We prove now that all the λ_j^L for $j \in L \setminus (I \cup \{i\})$ have the same sign. Again, by contradiction, if the λ_j^L for $j \in L \setminus (I \cup \{i\})$ do not have the same sign, by Remark 11 we can find $y \in \ker(A_I)$ such that $A_j y > 0$ for any $j \in L \setminus (I \cup \{i\})$ and $A_i y = 0$. Let $x \in \mathbb{Q}^n$ such that $A_L x = D_L(\delta, \epsilon)$ and $A_{\overline{L}} x > D_{\overline{L}}(\delta, \epsilon)$. Then for any $t > 0$ small enough $A_{I \cup \{i\}}(x + ty) = D_{I \cup \{i\}}(\delta, \epsilon)$ and $A_{\overline{I \cup \{i\}}}(x + ty) > D_{\overline{I \cup \{i\}}}(\delta, \epsilon)$. This is a contradiction, as $(\delta, \epsilon) \notin \omega_{I \cup \{i\}}$.

Finally, as the coefficients of (\mathcal{R}_L) do not have all the same sign, λ_i^L has opposite sign than λ_j^L for $j \in L \setminus (I \cup \{i\})$.

The last statement is not difficult. □

Corollary 36. *Let $(\delta, \epsilon) \in U_0 \cap \partial\Omega_0$. Let L be minimal such that $\omega_L = \{(\delta, \epsilon)\}$.*

- (1) *If ω_L is a vertex, the set of G -stable prime divisors of $Z^{\delta, \epsilon}$ is in bijection with $I_2^{\delta, \epsilon} \setminus (I_2^{\delta, \epsilon} \cap L)$;*
- (2) *if ω_L is not a vertex and there is $i \in I_0 \setminus J_0$ such that either $L \setminus I_+$ or $L \setminus I_-$ is $\{i\}$, the set of G -stable prime divisors of $Z^{\delta, \epsilon}$ is in bijection with $I_2^{\delta, \epsilon} \setminus \{i\}$;*
- (3) *otherwise, the set of G -stable prime divisors of $Z^{\delta, \epsilon}$ is in bijection with $I_2^{\delta, \epsilon}$.*

Proof. By Remark 14, the variety $Z^{\delta, \epsilon}$ is a G/H' -embedding with $H' \subsetneq H$. We let M' be the lattice of characters vanishing on H and N' its dual. Note that $N'_{\mathbb{Q}}$ is the quotient $N_{\mathbb{Q}}/N''_{\mathbb{Q}}$ where $N''_{\mathbb{Q}}$ is the subspace generated by the lines of A_L .

The G -stable prime divisors of $Z^{\delta, \epsilon}$ correspond bijectively to the facets $F_i^{\delta, \epsilon}$ with $i \in I_0 \setminus J_0$ that are not equal to some $F_j^{\delta, \epsilon}$ with $j \in J_0$.

Assume that ω_L is a vertex. The polytope $P^{\delta, \epsilon}$ generates the affine subspace $\{x \in M_{\mathbb{Q}} \mid A_L x = D_L(\delta, \epsilon)\}$, so that $M' = \ker(A_L) \cap M$. Moreover, the colors of G/H' are indexed by $J_0 \setminus (J_0 \cap L)$. If for some $i, j \in I_0$, $F_i^{\delta, \epsilon} = F_j^{\delta, \epsilon}$ is a facet, then, $i, j \notin L$, A_i and A_j are colinear modulo $N''_{\mathbb{Q}}$. If $i \neq j$ we would have another relation other than the two given by L . Thus we have $i = j$.

Note also that for any $i \in I_0 \setminus L$, we have $F_i^{\delta, \epsilon} = F_{L \cup \{i\}}^{\delta, \epsilon}$. If the latter is non empty and if $A_i \in N''_{\mathbb{Q}}$ we would have another relation other than the two given by L so that $A_i \notin N''_{\mathbb{Q}}$. Hence by Lemma 34 the G -stable prime divisors of the variety $Z^{\delta, \epsilon}$ are in bijection with

$$\{i \in I_0 \setminus L \mid (\delta, \epsilon) \in \omega_{L \cup \{i\}}\} = \{i \in I_0 \setminus L \mid (\delta, \epsilon) \in \Omega_{L \cup \{i\}}\} = I_2^{\delta, \epsilon} \setminus (I_2^{\delta, \epsilon} \cap L).$$

Assume that ω_L is not a vertex. The polytope $P^{\delta, \epsilon}$ generates the affine subspace $\{x \in M_{\mathbb{Q}} \mid A_I x = D_I(\delta, \epsilon)\}$, so that $M' = \ker(A_I) \cap M$. Moreover, the colors of G/H' are indexed by $J_0 \setminus (J_0 \cap I)$. If for some $i \neq j \in I_0$, $F_i^{\delta, \epsilon} = F_j^{\delta, \epsilon}$ is a facet, then A_i and A_j are colinear modulo the vector space $N'_{\mathbb{Q}}$ generated by the lines of

A_I , and then $L = I \cup \{i, j\}$. Hence, by Lemma 35, the G -stable prime divisors of the variety $Z^{\delta, \epsilon}$ are in bijection with

$$\{i \in I_0 \setminus I \mid (\delta, \epsilon) \in \omega_{I \cup \{i\}}\} = I_2^{\delta, \epsilon} \setminus \{i\}$$

if $L \setminus I_+$ or $L \setminus I_-$ is $\{i\}$ with $i \in I_0 \setminus J_0$, and else with

$$\{i \in I_0 \setminus I \mid (\delta, \epsilon) \in \omega_{I \cup \{i\}}\} = I_2^{\delta, \epsilon}.$$

□

Proposition 37. *Let $(\delta_0, \epsilon_0) \in U_2$. Let $\ell: \mathbb{Q} \rightarrow \mathbb{Q}^2$ be the parametrisation of a rational affine line such that $\ell(0) = (\delta_0, \epsilon_0)$. Let $\bar{t} > 0$ be the minimum such that $\ell(\bar{t}) \in \Omega_\emptyset \setminus U_2$. Assume that $\ell(\bar{t}) = (\delta_2, \epsilon_2) \in U_0 \cap \Omega_\emptyset \setminus \omega_\emptyset$.*

Let L be a minimal set such that $\{(\delta_2, \epsilon_2)\} = \omega_L$. The morphism from X^{δ_0, ϵ_0} to Z^{δ_2, ϵ_2} has relative Picard number at most 2.

Proof. Suppose first that ω_L is a vertex.

By Corollary 36, the B -stable prime divisors of Z^{δ_2, ϵ_2} are in bijection with $(I_2^{\delta_2, \epsilon_2} \sqcup J_0) \setminus L$. Let $F_J^{\delta_2, \epsilon_2}$ be a face of P^{δ_2, ϵ_2} , choose J such that $(\delta_2, \epsilon_2) \in \omega_J$. Then $L \subseteq J$ and $\omega_J = \omega_L$, so that the restriction of $A_{J \setminus L}$ to $\ker(A_L)$ is surjective. This implies, by Lemma 8 applied to the matrix of lines of $A_{I_0 \setminus L}$ restricted to $\ker(A_L)$, that Z^{δ_2, ϵ_2} is \mathbb{Q} -factorial. Then its Picard number is

$$|(I_2^{\delta_2, \epsilon_2} \sqcup J_0) \setminus L| - (n - |L| + 2) = |I_2^{\delta_2, \epsilon_2} \sqcup J_0| - n - 2.$$

Since $I_2^{\delta_2, \epsilon_2} \supseteq I_2^{\delta_0, \epsilon_0}$, the relative Picard number of $X^{\delta_0, \epsilon_0} \rightarrow Z^{\delta_2, \epsilon_2}$ is at most 2.

Suppose now that ω_L is not a vertex. By Corollary 36, the B -stable prime divisors of Z^{δ_2, ϵ_2} are in bijection with a subset of $(I_2^{\delta_2, \epsilon_2} \sqcup J_0) \setminus I$ of cocardinality at most 1. Let $F_J^{\delta_2, \epsilon_2}$ be a face of P^{δ_2, ϵ_2} , choose J such that $(\delta_2, \epsilon_2) \in \omega_J$. Then $I \subseteq J$. Assume that ω_J is one-dimensional. Then the restriction of $A_{J \setminus I}$ to $\ker(A_L)$ is surjective.

Assume that ω_J has dimension zero. Then $\omega_J = \omega_L$, so that the restriction of $A_{J \setminus I}$ to $\ker(A_I)$ implies the same equation for every such J .

If the cocardinality of a subset of $(I_2^{\delta_2, \epsilon_2} \sqcup J_0) \setminus I$ is one, then this equation does not occur for B -stable divisors and then Z^{δ_2, ϵ_2} is \mathbb{Q} -factorial. In any case, the Picard number of Z^{δ_2, ϵ_2} is $|(I_2^{\delta_2, \epsilon_2} \sqcup J_0) \setminus I| - (n - |I| + 1) - 1 = |I_2^{\delta_2, \epsilon_2} \sqcup J_0| - n - 2$. But $I_2^{\delta_2, \epsilon_2}$ is either $I_2^{\delta_0, \epsilon_0}$ or the union of $I_2^{\delta_0, \epsilon_0}$ with i as in Lemma 35, so that the relative Picard number of $X^{\delta_0, \epsilon_0} \rightarrow Z^{\delta_2, \epsilon_2}$ is 1 or 2.

□

Remark 18. If $\omega_L = \{(\delta_2, \epsilon_2)\}$ is a vertex and the relative Picard number of $X^{\delta_0, \epsilon_0} \rightarrow Z^{\delta_2, \epsilon_2}$ is not 2, then there exists $i \in I_2^{\delta_2, \epsilon_2} \setminus I_2^{\delta_0, \epsilon_0}$. In particular, ω_L is a vertex of Ω_i that does not contain (δ_0, ϵ_0) . Hence, there exists $K \subset I_0$, such that $\omega_K \subset \omega_\emptyset$ is one dimensional with one extremity equals to ω_L . If ω_L is not a vertex, such ω_K exists by definition.

5.5. Mori polygonal chain. We start with the following definition.

Definition 6. The Mori polygonal chain of the family $P^{\delta, \epsilon}$ is

$$MPC := (\Omega_\emptyset \setminus \omega_\emptyset) \cap \{(\delta, \epsilon) \in \mathbb{Q}^2, | 0 \leq \delta \leq 1\}.$$

Throughout this section we suppose that $(0, 0)$ and $(1, 0)$ are in ω_\emptyset , that for every $i \in I_0 \setminus J_0$ there exist negative ϵ_0 and ϵ_1 such that $(0, \epsilon_0)$ and $(1, \epsilon_1)$ are in ω_i and that $C \geq 0$. Then $\{(\delta, \epsilon) \in \mathbb{Q}^2 \mid \epsilon \leq 0 \text{ and } 0 \leq \delta \leq 1\}$ is contained in ω_\emptyset .

Note that, with the above hypothesis, the Mori polygonal chain is contained in the half plane defined by $\epsilon > 0$. It is polygonal because $\Omega_\emptyset \setminus \omega_\emptyset$ is a union of one-dimensional Ω_I .

We say that ω_I is a *segment* of the Mori polygonal chain if $\dim \omega_I = 1$ and the intersection of ω_I with MPC is not empty. If $(\delta, \epsilon) \in MPC$, then either $P^{\delta, \epsilon}$ is not of maximal dimension or there exists i such that $A_i = 0$ and $D_i(\delta, \epsilon) = 0$.

By Lemma 29, if ω_I is a segment of MPC such that $A_{I \setminus \{j\}}$ is surjective for every $j \in I$, then there is an equation $\sum_{i \in I} \lambda_i^I X_i = 0$ for the lines of A_I such that $\lambda_i^I > 0$ for every $i \in I$.

Definition 7. Let $K \subseteq I_0$ be such that $\dim \omega_K = 1$ and the codimension of the image A_K is 1. Let $\sum_{i \in K} \lambda_i^K X_i = 0$ be an equation for the lines of A_K . The slope of ω_K is

$$sl_K = \begin{cases} \frac{\sum_{i \in K} \lambda_i^K (B'_i - B_i)}{\sum_{i \in K} \lambda_i^K C_i} & \text{if } \sum_{i \in K} \lambda_i^K C_i \neq 0, \\ \infty & \text{if } \sum_{i \in K} \lambda_i^K C_i = 0. \end{cases}$$

Remark 19. The segment ω_K is included in the line defined by

$$\epsilon \left(\sum_{i \in K} \lambda_i^K C_i \right) + \delta \left(\sum_{i \in K} \lambda_i^K (B'_i - B_i) \right) + \sum_{i \in K} \lambda_i^K B_i = 0$$

and sl_K is the slope of this line.

Proposition 38. Let $(\delta_2, \epsilon_2) \in U_0 \cap MPC$ and let L be minimal such that $\omega_L = \{(\delta_2, \epsilon_2)\}$. We refer to the notation of Construction 2. Then there is a partition $L = K^0 \sqcup \dots \sqcup K^{r+1}$, with $r \geq 0$, such that

(1) for every $s \in \{0, \dots, r+1\}$, for every $h, k \in K^s$ we have $-\frac{\lambda_h^I}{\lambda_h^J} = -\frac{\lambda_k^I}{\lambda_k^J} =: \nu_s \in [-\infty, 0]$; for every $s < s' \in \{0, \dots, r+1\}$, we have $\nu_s > \nu_{s'}$.

(2) The set $K_s = L \setminus K^s$ is such that the codimension of the image of A_{K_s} is 1 and for every $k \in K_s$ the map $A_{K_s \setminus \{k\}}$ is surjective. If $K \subseteq L$ is such that the codimension of the image of A_K is 1 and for every $k \in K$ the map $A_{K \setminus \{k\}}$ is surjective, then $K = K_s$ for some $s \in \{0, \dots, r+1\}$. Moreover, if (δ_2, ϵ_2) is a vertex then $K_0 = I$ and $K_{r+1} = J$ and if it is not then $K_0 = I$ and $K^0 = L \setminus I$.

Note that, by Lemma 18, ω_{K_s} is not empty and with an extremity equals to ω_L .

(3) The slope of ω_{K_s} is

$$sl_{K_s} = \frac{\sum_{i \in I} \lambda_i^I (B'_i - B_i) + \nu_s \sum_{j \in J} \lambda_j^J (B'_j - B_j)}{1 + d\nu_s}$$

with $d = 1$ if (δ_2, ϵ_2) is a vertex and $d = \sum_{j \in L} \lambda_j^L C_j$ otherwise.

(4) Up to a rotation, the slopes decrease when s increases (see picture below).

Proof. Let $K \subseteq L$ be such that the codimension of the image of A_K is 1 and for every $k \in K$ the map $A_{K \setminus \{k\}}$ is surjective. Then there is an equation

$$(\mathcal{R}_K) \quad \sum \lambda_k^K X_k = 0$$

satisfied by the lines of A_K .

Assume that (δ_2, ϵ_2) is a vertex and let I, J be as in Construction 2. The relation (\mathcal{R}_K) is a linear combination of (\mathcal{R}_I) and (\mathcal{R}_J) . Thus there are μ_I, μ_J such that $(\mathcal{R}_K) = \mu_I(\mathcal{R}_I) + \mu_J(\mathcal{R}_J)$. We notice that if $\mu_I = 0$, then by the minimality of K we have $K = J$ and if $\mu_J = 0$ then by the minimality of K we have $K = I$.

Assume that (δ_2, ϵ_2) is not a vertex and let I be as in Construction 2. The relation (\mathcal{R}_K) is a linear combination of (\mathcal{R}_I) and (\mathcal{R}_L) . Thus there are μ_I, μ_L such that $(\mathcal{R}_K) = \mu_I(\mathcal{R}_I) + \mu_L(\mathcal{R}_L)$. We notice that if $\mu_L = 0$ then by minimality of K we have $K = I$ and if $\mu_L \neq 0$ then $L \setminus I \subseteq K$.

If (δ_2, ϵ_2) is not a vertex, we set $J = \{j \in L \mid \lambda_j^L \neq 0\}$. Notice that J gives the relation (\mathcal{R}_L) .

If $\mu_I \lambda_k^I + \mu_J \lambda_k^J \neq 0$ for every k , then the image of $A_{L \setminus \{k\}}$ has the same codimension as the image of A_L , and this is a contradiction. Therefore the set $\{k \mid \mu_I \lambda_k^I + \mu_J \lambda_k^J = 0\}$ is non-empty and K is the complement of this set in L .

We set $N = \{-\lambda_k^I / \lambda_k^J\} = \{\nu_0 > \dots > \nu_{r+1}\}$ with the convention that $\nu_{r+1} = -\infty$.

We set $K_s = \{k \mid -\lambda_k^I / \lambda_k^J = \nu_s\}$ and $K_s = L \setminus K^s$.

This proves (1) and (2).

As for (3), the slope sl_{K_s} is

$$\frac{\sum_{k \in K_s} \lambda_k^{K_s} (B'_k - B_k)}{\sum_{k \in K_s} \lambda_k^{K_s} C_k}.$$

Since up to a multiple $\lambda_k^{K_s} = \lambda_k^I + \nu_s \lambda_k^J$, we get the third part of the statement, with the assumption that, if (δ_2, ϵ_2) is a vertex, we have $\sum_{i \in I} \lambda_i^I C_i = 1$ and $\sum_{j \in J} \lambda_j^J C_j = 1$, and otherwise $\sum_{i \in I} \lambda_i^I C_i = 1$ and $\sum_{j \in L} \lambda_j^L C_j = 1$ or 0.

Set

$$a = \sum_{i \in I} \lambda_i^I (B'_i - B_i) = sl_I$$

$$b = \sum_{j \in J} \lambda_j^J (B'_j - B_j) = sl_J \text{ and}$$

$$d = \begin{cases} 1 & \text{if } (\delta_2, \epsilon_2) \text{ is a vertex and} \\ \sum_{j \in L} \lambda_j^L C_j & \text{(which is 1 or 0) otherwise.} \end{cases}$$

To prove (4), set $f(x) = \frac{a + bx}{1 + dx}$. Note that $sl_{K_s} = f(\nu_s)$. And apply a rotation

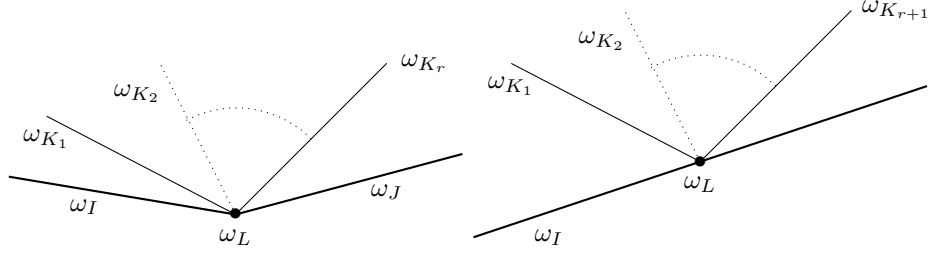
$\frac{1}{1+a^2} \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}$ in order to replace $sl_I = a$ by $+\infty$ (vertical direction). This way,

the slopes sl_K are replaced by relative slopes $relsl_K := \frac{a sl_K + 1}{a - sl_K}$. Set $g(x) = \frac{a^2 + 1 + x(ab + d)}{x(ad - b)}$ so that $relsl_{K_s} = g(\nu_s)$.

We now prove that $ad - b < 0$. By Construction 2, if (δ_2, ϵ_2) is a vertex we have $\omega_I \subseteq \{\delta < \delta_2\}$ and $\omega_J \subseteq \{\delta > \delta_2\}$. By the convexity of Ω_θ the slope sl_I is smaller than sl_J , so that $ad - b < 0$. If (δ_2, ϵ_2) is not a vertex and $d = 1$, similarly we have $sl_I < sl_J = \frac{b}{d}$, so that $ad - b < 0$. And if $d = 0$, we have $b > 0$ (still by Construction 2).

In all cases, since g is increasing, $relsl_{K_s}$ is decreasing from $+\infty$ ($s = 0$ and $\nu_s = 0$) to $\frac{ab+d}{ad-b}$ ($\nu_s = -\infty$).

□



6. PROOF OF THE MAIN THEOREM

Let G be a reductive group and $H \subseteq G$ be a horospherical subgroup. We choose a basis of $M_{\mathbb{Q}}$ so that $M_{\mathbb{Q}} \cong \mathbb{Q}^n$. We fix a horospherical embedding Z of G/H and set $p = r + |S \setminus R|$. Let A be the matrix associated to the linear map $\varphi(m) = (\langle m, x_i \rangle_{i=1 \dots r}, \langle m, \alpha_M^\vee \rangle_{\alpha \in S \setminus R})$. Denote by $J_0 \subseteq I_0$ the set of indices $S \setminus R$.

Let $B = (-d_1, \dots, -d_r, -d_\alpha)$ and $B' = (-d'_1, \dots, -d'_r, -d'_\alpha)$ be such that $\sum d_i Z_i + \sum d_\alpha D_\alpha$ and $\sum d'_i Z_i + \sum d'_\alpha D_\alpha$ are ample divisors. Let $-K_Z = \sum c_i Z_i + \sum c_\alpha D_\alpha$. Let $C = (c_1, \dots, c_r, c_\alpha)$.

Suppose now that for any $i \in I_0 \setminus J_0$ there exist negative ϵ_0 and ϵ_1 such that $(0, \epsilon_0)$ and $(1, \epsilon_1)$ are in ω_i . Then for any δ , the intersection $\omega_i \cap \{(\delta, \epsilon) \mid \epsilon \geq 0\}$ is an open segment (possibly empty) with one extremity at $(\delta, 0)$. Then the family $(P^{\delta, \epsilon})_{\epsilon \in \mathbb{Q}_{\geq 0}}$ describes a HMMP.

Proposition 39. *For any $\delta \in [0, 1]$ in the complement of a finite set, the HMMP described by the family $(P^{\delta, \epsilon})_{\epsilon \in \mathbb{Q}_{\geq 0}}$ is an MMP.*

Proof. By Theorem 10, the family $(P^{\delta, \epsilon})_{\epsilon \in \mathbb{Q}_{\geq 0}}$ describes an HMMP. Let U'' be the set of δ such that there is a 1-dimensional set ω_I included in $\{\delta\} \times \mathbb{Q}$. Let $p_1: \mathbb{Q}^2 \rightarrow \mathbb{Q}$ be the projection onto the first factor and let $\delta \in [0, 1] \setminus p_1(U_0 \cup U'_0) \cup U''$. By Proposition 33, the HMMP described by the family $(Q^{\delta, \epsilon})_{\epsilon \in \mathbb{Q}_{\geq 0}}$ consists of extremal contractions. □

Theorem 40. *Let G be a connected reductive algebraic group. Let X and Y be horospherical G -varieties which are G -equivariantly birational. Assume moreover that there are Mori fibre space structures X/S and Y/T . Then, there is a horospherical Sarkisov program from X/S to Y/T .*

More precisely, there are a resolution of the indeterminacy Z of $X \dashrightarrow Y$, divisors A_X and A_Y , a 2-dimensional polyhedron Ω , points (δ_X, ϵ_X) , (δ_Y, ϵ_Y) , (δ_T, ϵ_T) , $(\delta_S, \epsilon_S) \in \Omega$ and singletons $\omega_{L_1}, \dots, \omega_{L_\ell} \subseteq \partial\Omega$ such that

- (1) every point $(\delta, \epsilon) \in \Omega$ defines the horospherical variety of pseudo-moment polytope $Q_{(1-\delta)A_X + \delta A_Y - \epsilon K_Z}$;
- (2) (δ_X, ϵ_X) , (δ_Y, ϵ_Y) , (δ_T, ϵ_T) , (δ_S, ϵ_S) correspond, respectively, to X, Y, S, T ;
- (3) $(\delta_S, \epsilon_S), (\delta_T, \epsilon_T) \in \partial\Omega$;
- (4) if we denote $\omega_{L_i} = \{(\delta_i, \epsilon_i)\}$, we have $0 < \delta_1 < \dots < \delta_\ell < 1$;
- (5) every ω_{L_i} defines a Sarkisov link $W_i/V_i \dashrightarrow W_{i+1}/V_{i+1}$ where $W_1/V_1 = X/S$ and $W_{\ell+1}/V_{\ell+1} = Y/T$.

Proof. Since the birational map $X \dashrightarrow Y$ is G -equivariant, X and Y are both G/H -embedding with the same horospherical homogeneous space G/H .

By Lemma 11 and Proposition 14 there is a smooth resolution Z of the indeterminacies of $X \dashrightarrow Y$ and there are euclidean open sets U_X and U_Y of $WDiv(Z)_{\mathbb{Q}}$ such that every divisor in U_X (resp. U_Y) is ample and for every $A \in U_X$ (resp. in U_Y) the Mori fiber space X/T (resp. Y/S) is the outcome of the HMMP from Z with scaling of A .

We pick $(A_X, A_Y) \in U_X \times U_Y$ satisfying the generality conditions of (5.3). Since the open set determined in (5.3) is Zariski open, we can find such (A_X, A_Y) .

Let $B = (-d_1, \dots, -d_r, (-d_\alpha)_{\alpha \in S \setminus R})$ and $B' = (-d'_1, \dots, -d'_r, (-d'_\alpha)_{\alpha \in S \setminus R})$ and $C = (c_1, \dots, c_r, (c_\alpha)_{\alpha \in S \setminus R})$ be such that

$$\begin{aligned} A_X &= \sum_{i=1}^d d_i Z_i + \sum_{\alpha \in S \setminus R} d_\alpha D_\alpha \\ A_Y &= \sum_{i=1}^d d'_i Z_i + \sum_{\alpha \in S \setminus R} d'_\alpha D_\alpha \\ -K_Z &= \sum_{i=1}^d c_i Z_i + \sum_{\alpha \in S \setminus R} c_\alpha D_\alpha. \end{aligned}$$

Let Ω_I and ω_I be the polytopes of Definition 5.

Thus there are two segments ω_I and ω_J of the Mori polygonal chain such that every (δ, ϵ) in a euclidean neighborhood of ω_I satisfies $X^{\delta, \epsilon} \cong X$ and if $(\delta_1, \epsilon_1) \in \omega_I$ then $Y^{\delta_1, \epsilon_1} \cong T$. The same holds for J .

The segments ω_I and ω_J disconnect the chain. Let $\mathcal{C} \subseteq MPC$ be such that $\omega_I \cup \mathcal{C} \cup \omega_J$ is connected. Let $\omega_{L_1}, \dots, \omega_{L_h}$ be the points in $U_0 \cap \mathcal{C}$. We notice that $U'_0 \cap \partial\Omega_\emptyset = \emptyset$. Indeed, if $(\delta, \epsilon) \in \omega_I \cap \omega_J$ and ω_I, ω_J are not aligned, then the convex hull of ω_I and ω_J is non-empty and open (and thus of dimension 2), proving that $(\delta, \epsilon) \in \omega_\emptyset$.

We prove that every ω_{L_i} describes a link. We write ω_L for simplicity.

Let $L = K^0 \sqcup \dots \sqcup K^{r+1}$ be the partition existing by Proposition 38. We fix a euclidean neighborhood Δ of ω_L and for any $i \in \{0, \dots, r\}$, let X_i be the horospherical variety corresponding to a point in the open set delimited by ω_{K_i} and $\omega_{K_{i+1}}$. If ω_L is not a vertex, let X_{r+1} be the horospherical variety corresponding to a point in the open set delimited by $\omega_{K_{r+1}}$ and ω_I . Let t be r if ω_L is a vertex and $r+1$ if not. We denote by T_0 , respectively T_{t+1} , the horospherical variety corresponding to a point in MPC on the left, respectively on the right, of ω_L .

We prove first that

Claim 41. *the only possible divisorial contractions between two varieties X_i and $X_{i\pm 1}$ are $X_1 \rightarrow X_0$ and $X_{t-1} \rightarrow X_t$.*

Recall that we have Mori fibrations $X_0 \rightarrow T_0$ and $X_t \rightarrow T_{t+1}$. Let $s \in \{1, \dots, r\}$. First assume that ω_L is a vertex. Then K_s is such that $K_s^- = K^0 \sqcup \dots \sqcup K^{s-1}$ and $K_s^+ = K^{s+1} \sqcup \dots \sqcup K^{r+1}$. By Proposition 33, around ω_{K_s} we have flips except if $s=1$ and $K^0 = \{i\}$ with $i \in I_0 \setminus J_0$ or $s=r$ and $K^{r+1} = \{i\}$ with $i \in I_0 \setminus J_0$.

Assume now that ω_L is not a vertex. Let $s \in \{1, \dots, t\}$, then K_s is such that $K_s^- = K^0 \sqcup K^1 \sqcup \dots \sqcup K^{s-1}$ and $K_s^+ = K^0 \sqcup K^{s+1} \sqcup \dots \sqcup K^{r+1}$. Since K_+^0 and K_-^0 are non-empty (by Notation 2), by Proposition 33 around ω_{K_s} we have flips except if $s=1$ and $K_-^0 = \{i\}$ with $i \in I_0 \setminus J_0$ or $s=r+1$ and $K_+^0 = \{i\}$ with $i \in I_0 \setminus J_0$. This finishes the proof of the claim.

Let R be the variety corresponding to ω_L . Notice that we have fibrations from $T_0 \rightarrow R$ and $T_{t+1} \rightarrow R$. There are three cases:

- (1) $R \cong T_0 \cong T_{t+1}$;
- (2) $R \cong T_0$ or $R \cong T_{t+1}$, and we are not in case 1;
- (3) $R \not\cong T_0$ and $R \not\cong T_{t+1}$.

In case 1, by Remark 18, we have $t \geq 1$. In particular, $sl_{K_1} < sl_{K_0}$, or $sl_{K_t} > sl_{K_{t+1}}$ (with the convention that $K_{r+2} = I$ if ω_L is not a vertex). A priori, sl_{K_1} and sl_{K_t} could be ∞ , but the next paragraph proves that it cannot happen.

If $sl_{K_1} < sl_{K_0}$, then the Mori fibration $X_0 \dashrightarrow R$ factors through $X_0 \dashrightarrow Y \rightarrow R$ where Y is the horospherical variety corresponding to a point of ω_{K_1} . Then the map $X_0 \dashrightarrow Y$ is either an isomorphism or a K -negative contraction. It cannot be a K -negative contraction because $sl_{K_1} < sl_{K_0}$. Then $X_0 = Y$, which implies that $X_1 \rightarrow X_0$ is a divisorial contraction. Similarly, if $sl_{K_t} > sl_{K_{t+1}}$, $X_{t-1} \rightarrow X_t$ has to be a divisorial contraction.

Thus there is $i = 1$ or $i = t - 1$ such that $\rho(X_i) = \rho(R) + 2$. Assume it is $i = 1$. By Proposition 37 and Claim 41, for every $i \in \{1, \dots, t - 1\}$ we have $\rho(X_i) = \rho(X_1)$ so that $\rho(X_{t-1}) = \rho(X_t) + 1$ and thus $X_{t-1} \rightarrow X_t$ is also a divisorial contraction. We have a type II link.

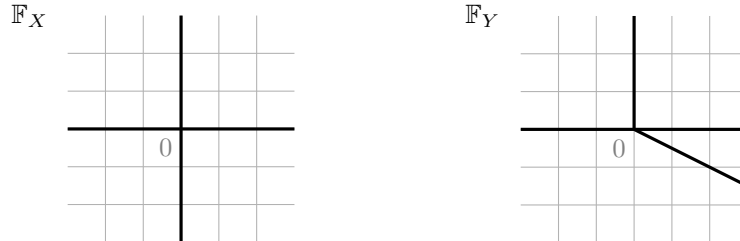
In case 2, assume that $R \cong T_0$. Again by Proposition 33 the variety R is \mathbb{Q} -factorial. Thus $R \not\cong T_{t+1}$ implies $\rho(T_{t+1}) = \rho(R) + 1$ and $\rho(X_t) = \rho(R) + 2$. By Proposition 37 and Claim 41, for every $i \in \{1, \dots, t\}$ we have $\rho(X_i) = \rho(X_t) = \rho(R) + 2 = \rho(T_0) + 2$. Since $\rho(X_0) = \rho(T_0) + 1$, the map $X_1 \rightarrow X_0$ is a divisorial contraction and we have a type I link. Similarly if $R \cong T_{r+1}$ we get a type III link.

In case 3 we have $\rho(T_0) = \rho(R) + 1$ and $\rho(T_{t+1}) = \rho(R) + 1$. Moreover $\rho(X_0) = \rho(T_0) + 1$ and $\rho(X_t) = \rho(T_{t+1}) + 1$. By Proposition 37 and Claim 41, for every $i \in \{0, \dots, t - 1\}$ the map $X_i \dashrightarrow X_{i+1}$ is an isomorphism in codimension 1 and we get a type IV link. \square

Remark 20. If ω_L is not a vertex then we cannot have a link of type IV with fibrations $T_0 \rightarrow R$ and $T_{t+1} \rightarrow R$. Indeed from ω_I (both side) to ω_L we get two birational maps, one of the two can be divisorial (if condition of Lemma 35 occurs), but not in case 3.

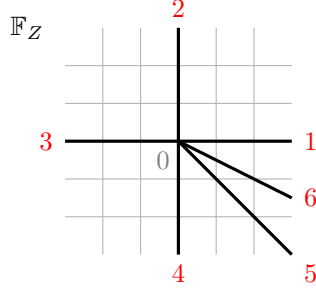
APPENDIX A. TORIC EXAMPLE

Set $X := \mathbb{P}^1 \times \mathbb{P}^1$, $S = \mathbb{P}^1$, Y the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ and $T = \mathbb{P}^1$. Fans of X and Y are the following:



Here $G = (\mathbb{C}^*)^2$ and coincides with the Borel subgroup.

A resolution of X and Y is the toric variety Z given by the following fan, where we index the edges as in the picture:



We let $p: Z \rightarrow X$ and $q: Z \rightarrow Y$ be the two morphisms resolving the indeterminacy of $X \dashrightarrow Y$.

We will consider the following matrix $A := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \\ 2 & -1 \end{pmatrix}$.

A G -stable divisor of Z is of the form $D = \sum_{i=1}^6 d_i D_i$. Note that D_1, D_2, D_3 and D_4 are also prime G -stable divisor of X and D_1, D_2, D_3 and D_6 are also prime G -stable divisor of Y .

Let $D = \sum_{i=1}^6 d_i D_i$ be a divisor on Z and B be the column matrix associated to $(-d_1, \dots, -d_6)$. The divisor D is ample if and only if for any $I \subseteq \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{1, 6\}\}$ we have $A_{\bar{I}}(A_{\bar{I}}^{-1}B_I) > B_I$.

This inequality system reduces to the following system, thus defining the ample cone of Z :

$$(2) \quad d_1 + d_5 > d_6, \quad d_2 + d_6 > 2d_1, \quad d_3 + d_5 > d_4, \quad d_4 + d_6 > 2d_5.$$

The polytope Q_D is the pseudo-moment polytope of (X, p_*D) if and only if for every $I \subseteq \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$ we have $A_{\bar{I}}(A_{\bar{I}}^{-1}B_I) > B_I$. Equivalently, if and only if the following inequalities are satisfied

$$(3) \quad d_1 + d_3 > 0, \quad d_2 + d_4 > 0, \quad d_5 > d_1 + d_4, \quad d_6 > 2d_1 + d_4.$$

Note that the first two inequalities correspond to the condition for p_*D to be ample and the last two other correspond to the fact that the lines 5 and 6 are not necessary to define Q_D .

Similarly, Q_D is the pseudo-moment polytope of (Y, q_*D) if and only if for every $I \subseteq \{\{1, 2\}, \{2, 3\}, \{3, 6\}, \{1, 6\}\}$ we have $A_{\bar{I}}(A_{\bar{I}}^{-1}B_I) > B_I$. Equivalently, if and only if the following inequalities are satisfied

$$(4) \quad d_1 + d_3 > 0, \quad d_2 + d_6 > 2d_1, \quad d_4 > 2d_3 + d_6, \quad d_5 > d_3 + d_6.$$

Note that the first two inequalities correspond to the condition for q_*D to be ample and the last two other correspond to the fact that the lines 4 and 5 are not necessary to define Q_D .

To run the horospherical Sarkisov program, we have to choose D, D' such that

(1) Q_D and $Q_{D'}$ are pseudo-moment polytopes of (X, p_*D) and (Y, q_*D') respectively;

(2) there exist $\epsilon < 0$ and $\epsilon' < 0$ such that $D + \epsilon K_Z$ and $D' + \epsilon' K_Z$ are ample over Z .

(3) the HMMP with scaling of D (resp. D') ends with X/S (resp. Y/T).

Condition (1) is given by (3) and (4).

For condition (2), note that $-K_Z = \sum_{i=1}^6 D_i$ and thus C is the column matrix associated to $(1, \dots, 1)$. Then $D + \epsilon K_Z$ is ample over Z if and only if

$$d_1 + d_5 > d_6 + \epsilon, \quad d_2 + d_6 > 2d_1, \quad d_3 + d_5 > d_4 + \epsilon \text{ and } d_4 + d_6 > 2d_5.$$

Hence, there exists $\epsilon < 0$ such that $D + \epsilon K_Z$ is ample over Z if and only if $d_2 + d_6 > 2d_1$ and $d_4 + d_6 > 2d_5$. Similarly, there exists $\epsilon' < 0$ such that $D' + \epsilon' K_Z$ is ample over Z if and only if $d'_2 + d'_6 > 2d'_1$ and $d'_4 + d'_6 > 2d'_5$.

As for condition (3), note first that the HMMP from any ample divisor of Y ends with $Y \rightarrow S$, because only one extremal ray of $NE(Y)$ is K -negative. If $d_1 + d_3 < d_2 + d_4$ (resp. $d_1 + d_3 > d_2 + d_4$) the HMMP from D gives the first (resp. the second) projection $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$.

Since D and D' are given up to linearly equivalence, we can choose them such that $d_1 = d_2 = d'_1 = d'_2 = 0$. In particular, Q_D and $Q_{D'}$ have a "south-west" vertex at 0 (but the same is not true for $Q^{\epsilon, \lambda}$ for $\epsilon \neq 0$). The conditions on D and D' are then

$$d_3 > 0, \quad d_6 > d_4 > 0 \text{ and } d_4 < d_5 < \frac{1}{2}(d_4 + d_6), \text{ with either } d_3 < d_4 \text{ or } d_3 > d_4;$$

$$d'_3 > 0, \quad d'_6 > 0, \quad d'_4 > 2d'_3 + d'_6 \text{ and } d'_3 + d'_6 < d'_5 < \frac{1}{2}(d'_4 + d'_6).$$

For example, we can choose B and B' to be the column matrices associated to $(0, 0, -1, -2, -5/2, -4)$ and $(0, 0, -1, -6, -7/2, -2)$ respectively.

In Figure 1, we draw Ω_\emptyset for this choice of B, B' . We denote by \tilde{X} is the blow-up of X at a point and \tilde{Y} is the blow-up of \mathbb{F}_1 at a point in the (-1) -section. Note that $L_1 = \{1, 3, 4, 5\}$ and $L_2 = \{1, 3, 5, 6\}$. Also note that $(1/2, 0)$ is in U'_6 .

In the above example, we have $d_3 < d_4$. If we choose B to be the column matrices associated to $(0, 0, -6, -1, -3/2, -3)$ (and same B' as before), then $d_3 > d_4$. Thus we obtain the horospherical Sarkisov program illustrated in Figure 2. Note that the first Sarkisov link is the type IV link from the first to the second projection $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$. The fifth line of A does not correspond to a divisor of X or Y , but is added in order to get Z to be smooth. The strategy given in the paper also works by choosing a (not necessarily smooth) resolution of indeterminacies Z' , but non-terminal varieties can appear. In this example, if we forget the fifth line of A , we can obtain a unique Sarkisov link of type II with $X' = Y' = Z'$.

APPENDIX B. A RANK ONE HOROSPHERICAL EXAMPLE

Choose a connected algebraic group G such that

$$S \backslash R = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$$

with the notation of section 2. Let $\chi = \varpi_1 + \varpi_2 - \varpi_3 - \varpi_4$ and $H \subset P$ be the kernel in P of the character χ . In particular, $M = \mathbb{Z}\chi$. We identify χ with 1 in $M \simeq \mathbb{Z}$. Then $N \simeq \mathbb{Z}$, $\alpha_{1M}^\vee = \alpha_{2M}^\vee = 1$, $\alpha_{3M}^\vee = \alpha_{4M}^\vee = -1$ and $\alpha_{5M}^\vee = 0$.

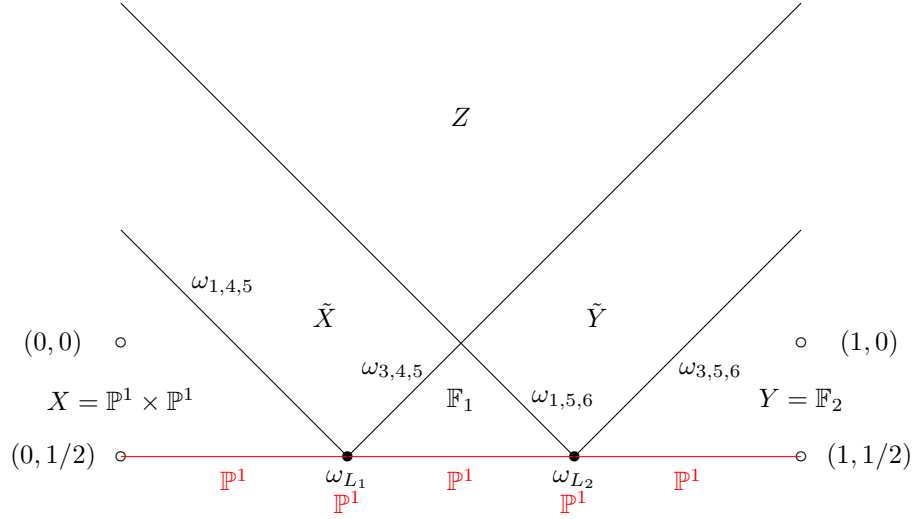
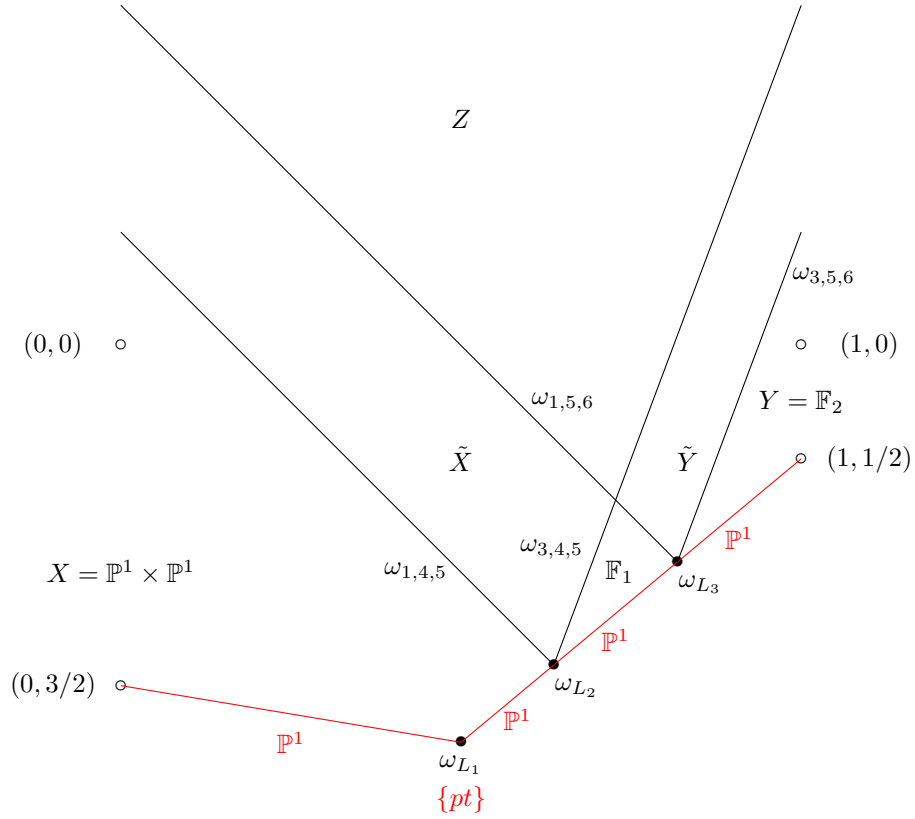
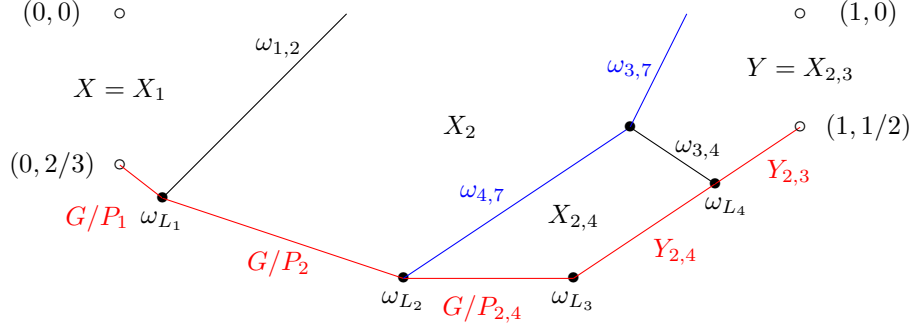
FIGURE 1. Scheme of Ω_θ for the first toric example.FIGURE 2. Scheme of Ω_θ for the second toric example.

FIGURE 3. Scheme of Ω_θ for the rank one horospherical example.



Since G/H is of rank one, projective G/H -embeddings are uniquely determined by their colors, that is, by subsets of $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Denote by X_K the projective G/H -embedding such that $\mathcal{F}_{X_K} = \{\alpha_k \mid k \in K\}$. In particular, $Z := X_\emptyset$ is a common resolution of all X_K .

Denote by P_K the parabolic subgroup of G containing B whose the set of simple roots is $\{\alpha_k \mid k \in K\} \cup R$.

Let $H' \subset P_5$ be the kernel in P_5 of the character χ . It is a horospherical subgroup associated to the same lattice M . Denote by Y_K the projective G/H' -embedding Y such that $\mathcal{F}_{Y_K} = \{\alpha_k \mid k \in K\}$.

Suppose that the coefficients a_{α_i} 's are respectively 2, 3, 2, 3 and 2 for any $i = 1, \dots, 5$. We consider A to be the column matrix associated to

$$(\alpha_{1M}^\vee, \alpha_{2M}^\vee, \alpha_{3M}^\vee, \alpha_{4M}^\vee, \alpha_{5M}^\vee, 1, -1) = (1, 1, -1, -1, 0, 1, -1).$$

Then C is the column matrix associated to $(2, 3, 2, 3, 2, 1, 1)$.

Let B and B' be the column matrices associated to $-(0, 1, 7, 6, 5, 2, 2)$ and $-(2, 0, 6, 7, 1, 3, 7)$ respectively. Then the associated horospherical Sarkisov program gives a link of type IV_m , followed by a link of type III, a link of type IV_m and a last one of type IV_s .

We then obtain the horospherical Sarkisov program illustrated in Figure 3 (we only draw Ω_θ for $\epsilon \geq 0$ and $0 \leq \delta \leq 1$). In red we draw the one dimensional ω_I 's giving fibrations, in blue the ones giving divisorial contractions and in black the ones giving flips.

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