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New route formulations for the Split-Delivery VRP

Ruslan Sadykov

Inria Centre at the University of Bordeaux



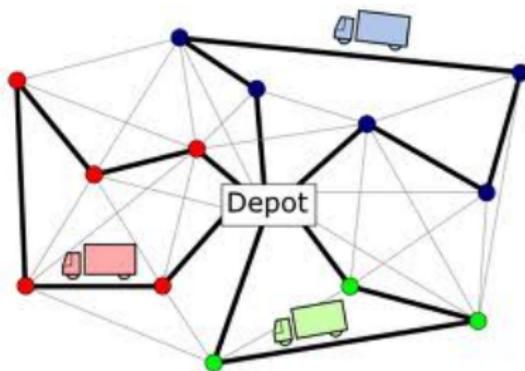
Joint work with **Isaac Balster** (Inria),
Teobaldo Bulhões (UFPB, Brazil), and **Pedro Munari** (UFSC, Brazil)

Workshop on the Inventory Routing Problem
Ecole des Ponts Paristech, May 20, 2022

From standard CVRP to SDVRP's

Classic Capacitated Vehicle Routing Problem - CVRP

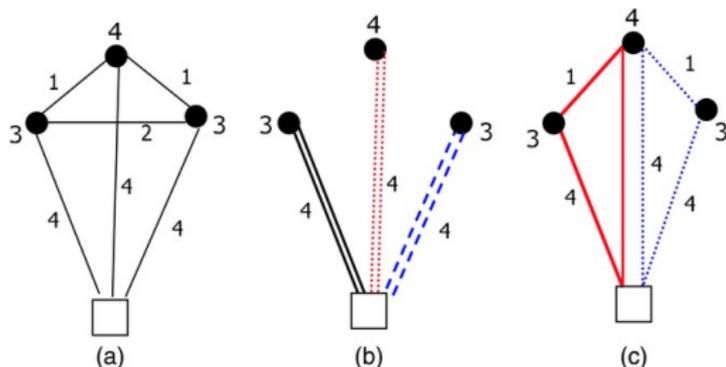
- ▶ Objective: minimise routing costs.



Split delivery variants – SDVRP's

- ▶ The single visit requirement for customers is relaxed.
- ▶ Each client can now be visited by one or more vehicles.

Practical motivation



Instance (a) with $Q = 5$. The cost is 24 (with 3 vehicles) for the CVRP (b) and 18 (with 2 vehicles) for the SDVRP. Source: [Archetti and Speranza, 2012].

- ▶ Routing savings can reach up to 50% [Archetti et al., 2006].

Research motivation

CG-based approaches and BCP algorithms:

Feillet, Dejax, Gendreau and Gueguen (2006)

Jin, Liu and Eksioğlu (2008)

Moreno, De Aragão and Uchoa (2010)

Desaulniers (2010)

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Munari and Savelsbergh (2020)

BC algorithms (current state-of-the-art):

Archetti, Bianchessi and Speranza (2014)

Ozbaygin, Karasan and Yaman (2018)

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The SDVRP has a similar structure to the IRP!

Current BCP algorithms for the SDVRP's

- ▶ Based on extreme delivery patterns [Desaulniers, 2010]
- ▶ Pricing problem is harder than the standard RCSP
- ▶ To use the standard RCSP solver [Sadykov et al., 2021], we need to discretize delivery quantities
- ▶ Are there route formulations which allow us to use the standard RCSP solver without full discretization?

Base formulation for SDVRP's

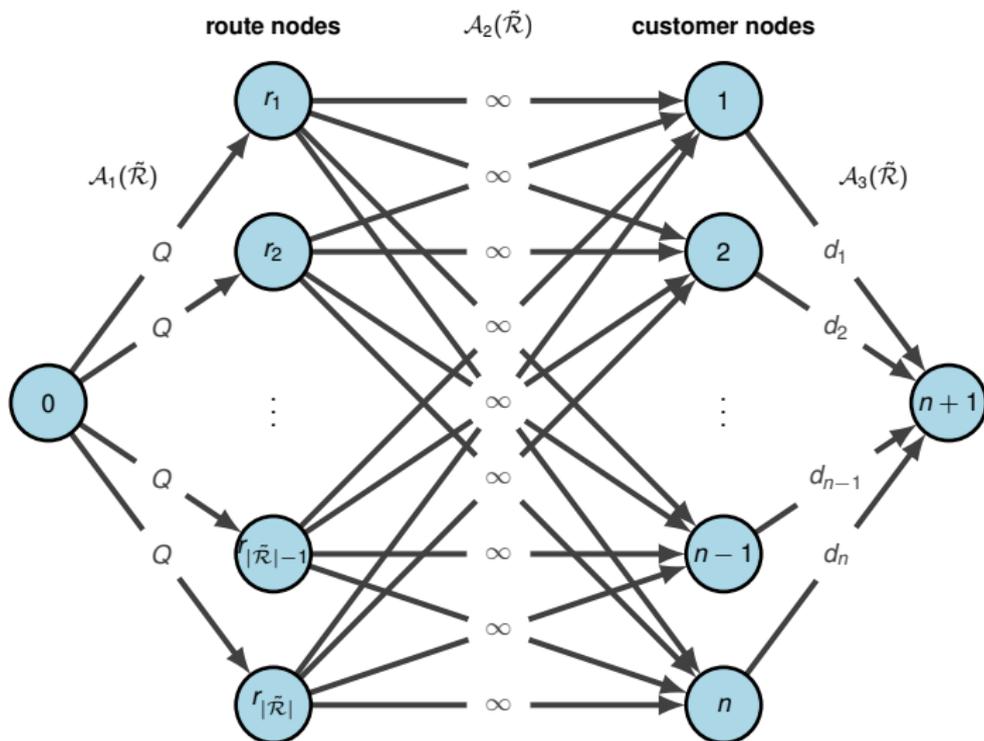
- ▶ \mathcal{C} — set of customers
- ▶ \mathcal{R} — set of elementary [and time-feasible] routes.
- ▶ c^r — cost of route $r \in \mathcal{R}$
- ▶ $h_{rS} = 1$ iff route $r \in \mathcal{R}$ enters subset $S \subseteq \mathcal{C}$ of customers
- ▶ θ_r — number of vehicles which follow route $r \in \mathcal{R}$ (variable)

$$\begin{aligned} \text{(F0): Min} \quad & \sum_{r \in \mathcal{R}} c^r \theta_r, \\ \text{s.t.} \quad & \sum_{r \in \mathcal{R}} h_{rS} \theta_r \geq \left\lceil \sum_{i \in S} d_i / Q \right\rceil, & \forall S \subseteq \mathcal{C}, \\ & \theta_r \in \mathbb{Z}_+, & \forall r \in \mathcal{R}. \end{aligned}$$

- ▶ Constraints are strong k -path inequalities
[Baldacci et al., 2008, Archetti et al., 2011].
- ▶ No information about delivery quantities in route variables!

Flow graph $\mathcal{F}(\tilde{\mathcal{R}})$ to show correctness of (F0)

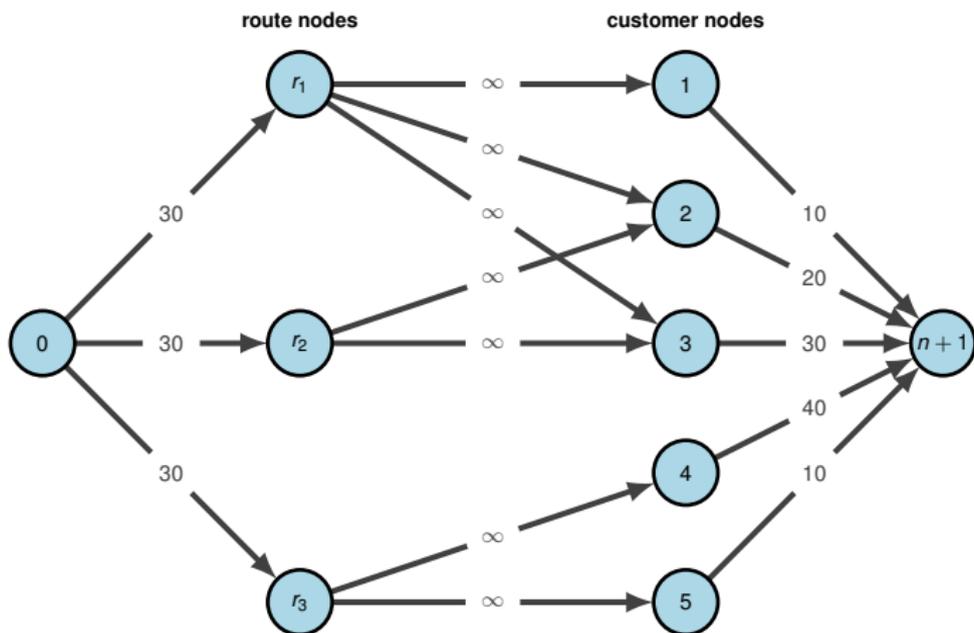
$\tilde{\mathcal{R}}$ is the set of routes in the solution of (F0)



An example of flow graph $\mathcal{F}(\tilde{\mathcal{R}})$

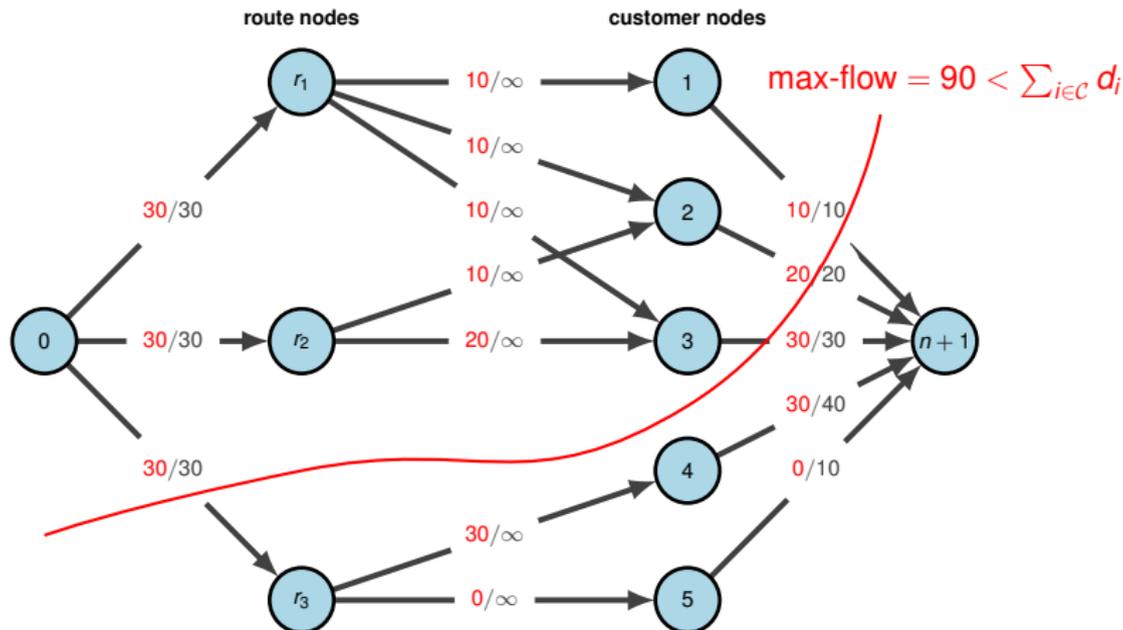
Customers $\mathcal{C} = \{1, 2, 3, 4, 5\}$ with demands
 $d = \{10, 20, 30, 40, 10\}$, and vehicle capacity $Q = 30$.

$$\tilde{\mathcal{R}} = \left\{ r_1 = \{0, 1, 2, 3, 6\}, r_2 = \{0, 2, 3, 6\}, r_3 = \{0, 4, 5, 6\} \right\}$$



Checking feasibility with $\mathcal{F}(\tilde{\mathcal{R}})$

The max-flow value in $\mathcal{F}(\tilde{\mathcal{R}})$ tells us if $\tilde{\mathcal{R}}$ is a feasible solution.

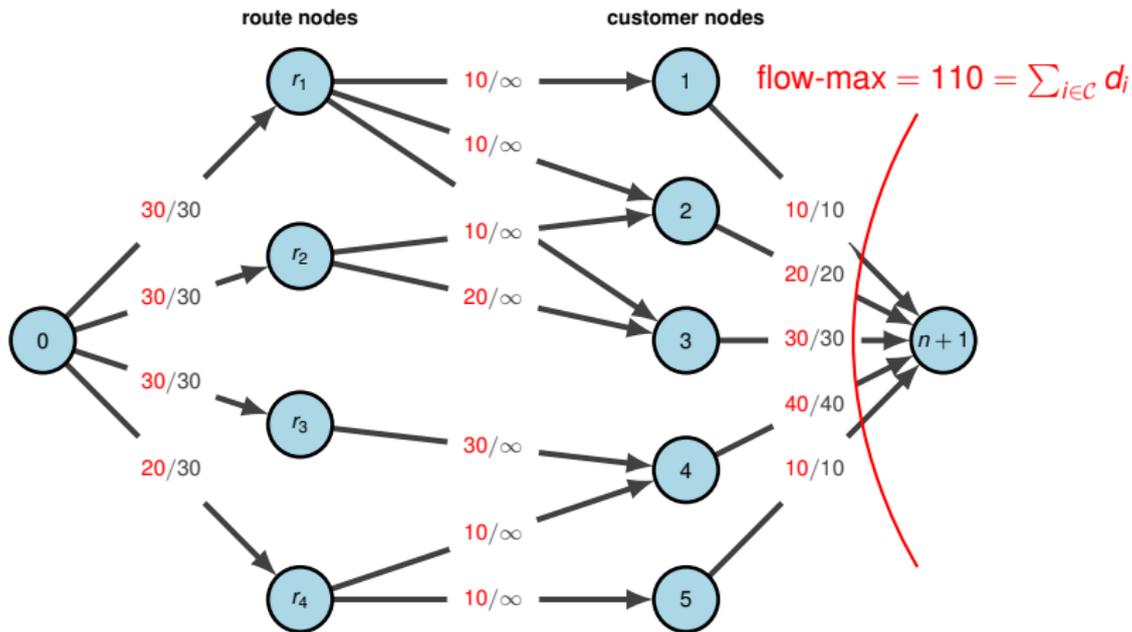


$$\sum_{r \in \tilde{\mathcal{R}}} h_{r, \{4,5\}} \theta_r(1) < \lceil \sum_{i \in \{4,5\}} d_i / Q \rceil \quad (\lceil 50/30 \rceil = 2)$$

\Rightarrow strong k -path inequality for $S = \{4, 5\}$ is violated

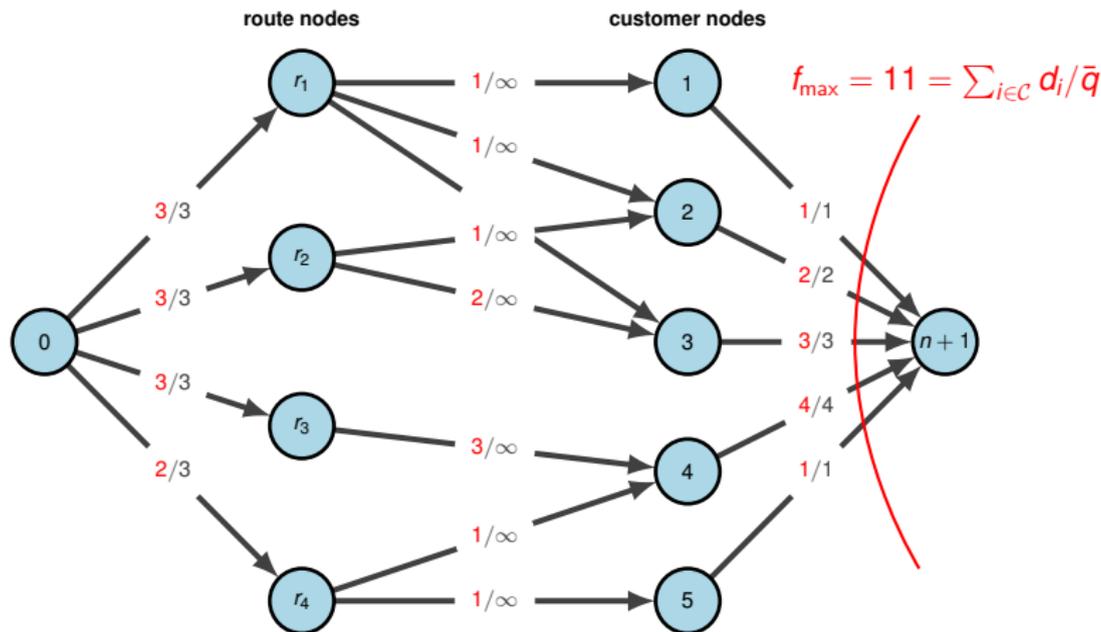
Checking feasibility with $\mathcal{F}(\tilde{\mathcal{R}})$ (II)

$$\tilde{\mathcal{R}}^* = \left\{ r_1 = \{0, 1, 2, 3, 6\}, r_2 = \{0, 2, 3, 6\}, r_3 = \{0, 4, 6\}, r_4 = \{0, 4, 5, 6\} \right\}$$



A dominance rule for optimal solutions

Divide arc capacities in $\mathcal{F}(\tilde{\mathcal{R}})$ by $\bar{q} = \gcd(Q, d_1, d_2, \dots, d_n)$.



Dominance rule: *There exists an optimal solution in which all delivery quantities in all routes are multiples of \bar{q}*

Strengthened formulation (F2)

- ▶ \mathcal{R}' — set of all resource-feasible routes (but not necessarily elementary)
- ▶ $D_i = \{\bar{q}, 2\bar{q}, \dots, d_i\}$ — possible delivery quantities to $i \in \mathcal{C}$.
- ▶ $b_{iF}^r = b_{i,d_i}^r$ — # of times $r \in \mathcal{R}'$ delivers full demand to $i \in \mathcal{C}$.
- ▶ $b_{iP}^r = \sum_{q \in D_i \setminus \{d_i\}} b_{iq}^r$ — # of times $r \in \mathcal{R}'$ delivers partial demand to i .

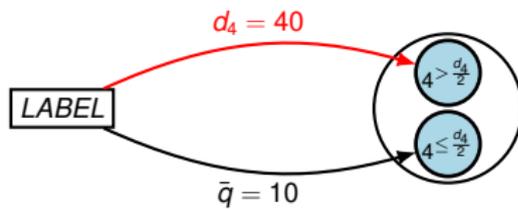
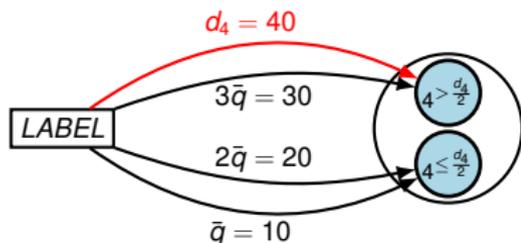
(F2) : Objective and all constraints in (F0)

$$\sum_{r \in \mathcal{R}'} (2b_{iF}^r + b_{iP}^r)\theta_r \geq 2, \quad \forall i \in \mathcal{C}. \quad (*)$$

(*) is a special case of strong minimum number of vehicles (SVM) constraints from [\[Archetti et al., 2011\]](#).

Pricing problem for formulation (F2)

Example: $i = 4$, $d_4 = 40$, $\bar{q} = 10$.



- ▶ Arcs incoming to nodes i with delivery $q \notin \{\bar{q}, d_i\}$ can be removed without compromising correctness
- ▶ Their removal does not weaken formulation (F2)

A family of formulations (FK)

A valid inequality for a customer $i \in \mathcal{C}$

$$\sum_{r \in \mathcal{R}'} \sum_{q \in D_i} (qb_{iq}^r) \theta_r \geq d_i, \quad (*)$$

where b_{iq}^r is the # of times $r \in \mathcal{R}'$ visits $i \in \mathcal{C}$ delivering $q \in D_i$.

Given $K < d_i/\bar{q}$, after Chvátal-Gomory rounding with multiplier $\frac{K-1}{d_i-\epsilon}$:

$$\sum_{r \in \mathcal{R}'} \sum_{q \in D_i} \sum_{k=1}^K (b_{iq}^r g_{iq}^k k) \geq K, \quad (**)$$

where $g_{iq}^k = 1$ iff $\frac{(k-1)d_i}{K-1} \leq q < \frac{kd_i}{K-1}$.

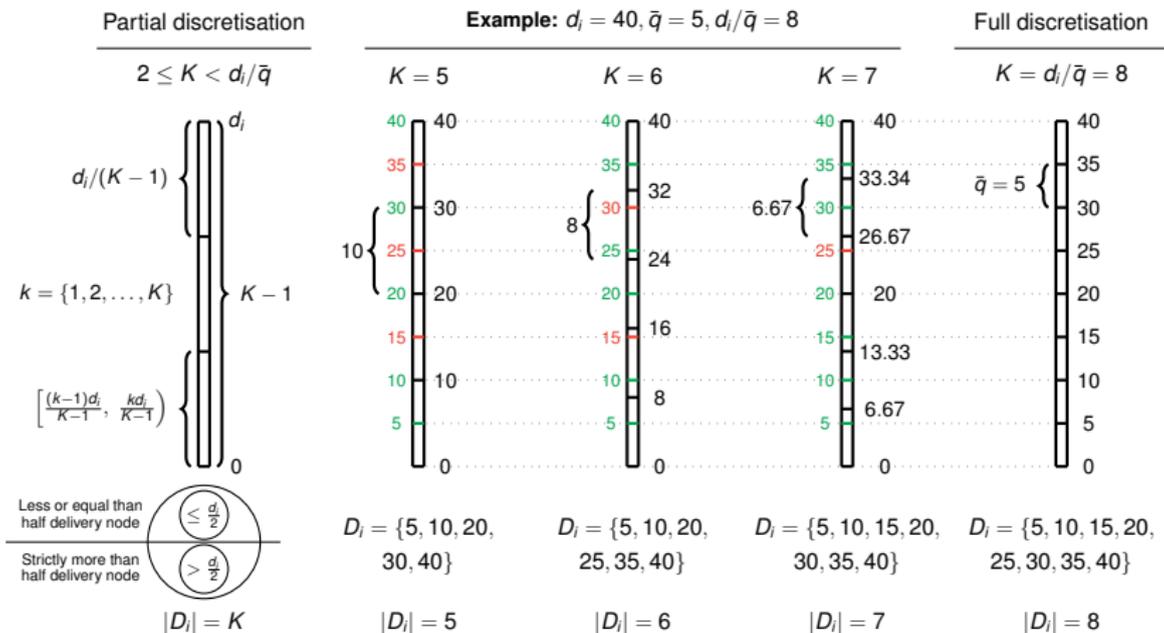
(FK) : Objective and all constraints in (F0)

Inequalities (*) $\forall i \in \mathcal{C} : K \geq d_i/\bar{q}$

Inequalities (**) $\forall i \in \mathcal{C} : K < d_i/\bar{q}$

From partial to full discretisation: illustration

- ▶ Number of incoming arcs for vertices $i \in \mathcal{C}$ in the pricing for (FK) is at most K .
- ▶ Full discretisation formulation (FK_{max}), $K_{\max} = \max_{i \in \mathcal{C}} \left\{ \frac{d_i}{\bar{q}} \right\}$.



Valid inequalities

x_{ij}^r — # of times $r \in \mathcal{R}'$ follows arc $(i, j) \in \mathcal{A}$, $i, j \in \mathcal{C} \cap \{0\}$.

- ▶ Rounded capacity inequalities:

$$\sum_{r \in \mathcal{R}} \sum_{\substack{(i,j) \in \mathcal{A}: \\ |\{i,j\} \cap \mathcal{S}|=1}} x_{ij}^r \theta_r \geq 2 \left\lceil \sum_{i \in \mathcal{S}} d_i / Q \right\rceil, \quad \forall \mathcal{S} \subseteq \mathcal{C}.$$

- ▶ 3-row subset-row packing inequalities:

$$\sum_{r \in \mathcal{R}} \left[\sum_{i \in \mathcal{S}} \sum_{\substack{q \in D_i: \\ q > d_i/2}} \frac{1}{2} b_{iq}^r \right] \theta_r \leq 1, \quad \forall \mathcal{S} \subseteq \mathcal{C}, |\mathcal{S}| = 3.$$

Valid inequalities (II)

- ▶ 3-row subset-row covering inequalities:

$$\sum_{r \in \mathcal{R}} \left[\sum_{i \in S} \sum_{\substack{q \in D_i: \\ q > 0}} \frac{1}{2} b_{iq}^r \right] \theta_r \geq 2, \quad \forall S \subseteq \mathcal{C}, |S| = 3.$$

- ▶ Limited memory technique ([Pecin et al., 2017]) is used for all non-robust cuts.

Implementation

- ▶ C++ libraries BaPCod [Sadykov and Vanderbeck, 2021] and VRPSolver extension [Pessoa et al., 2020] are used to leverage all the latest advances on exact solution of the classic CVRP
- ▶ VRPSolver is extended with
 - ▶ separation procedures for strong k -path inequalities
 - ▶ covering sets (to support limited-memory Chvátal-Gomory rank-1 covering cuts and strong k -path inequalities in the pricing)
- ▶ Branching on arcs and Ryan-and-Foster branching

Computational evaluation

Instance sets

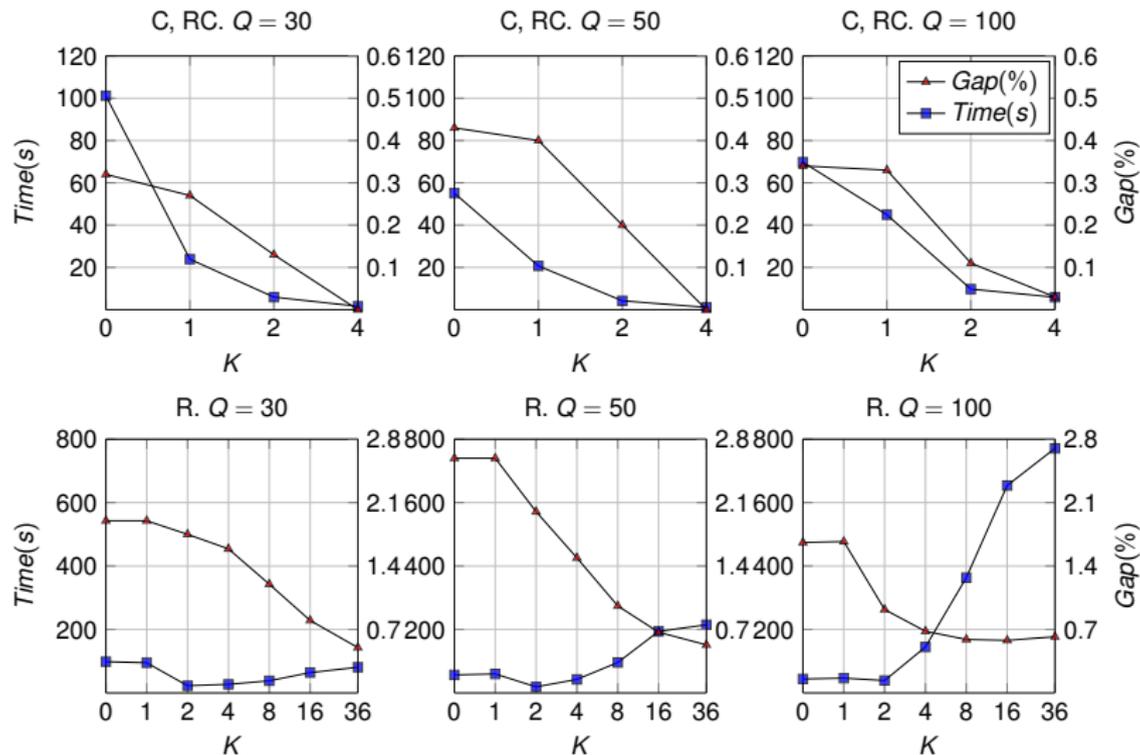
- ▶ **SDVRPTW** – 504 test instances, derived from 56 classic Solomon's VRPTW instances, having $n = \{25, 50, 100\}$ and $Q = \{30, 50, 100\}$.
- ▶ **SDVRP** – 352 test instances, derived from 88 instances (S, SD, eil, p), limiting, or not, the size of the fleet (LF/UF) and rounding, or not, distances (LF-r/UF-r).

Initial upper bounds

- ▶ We use an ILS-based matheuristic proposed by [\[Alvarez and Munari, 2022\]](#) to generate initial upper bounds.

Comparison of formulations (*FK*)

Root node results for all SDVRPTW instances with $n = 50$.



Comparison with the state-of-the-art on the SDVRPTW

n	Benchmark run – 3600s				Long run – 18000s		
	(FK_{\max})	MS22	BI19	A11	$(F2)$	(FK_{\max})	Best($F2, FK_{\max}$)
25	168	168	168	168	168	168	168 (0)
50	152 (27)	123	104	86	136	168	168 (40)
100	54 (48)	4	5	8	24	55	56 (50)
{50, 100}	206	127	109	94	160	223	224 (90)
{25, 50, 100}	374 (75)	295	277	262	328	391	392 (90)
Ol average Gap (%)	1.66	-	-	-	3.02	1.56	1.57

MS22: Munari and Savelsbergh (2022)

BI19: Bianchessi and Irnich (2019)

A11: Archetti et al. (2011)

- ▶ Formulation (FK_{\max}) finds 374 optimal solutions, 75 for the first time, within one hour benchmark tests.
- ▶ Formulations $(F2)$ and (FK_{\max}) all together find 392 optimal solutions, 90 for the first time, within five hours.

Comparison with the state-of-the-art on the SDVRP

Formulation (FK), $K = \min(K_{\max}, 10)$

Tests	Model or reference – test set size	Opt	Opt*	LB*
Benchmark run – 7200s	FK (MH) – 352	94 (88 [†])	10 (6 [†])	121 (53 [†])
	Munari and Savelsbergh (2022) – 224 [†]	85	-	-
	Gouveia et al. (2021) – 352	106	-	-
Long run – 18000s	FK (MH) – 352	112 (106 [†])	14 (10 [†])	130 (53 [†])
	FK (BKS) – 352	121 (115 [†])	19 (15 [†])	134 (53 [†])
	Best of long runs – 352	123 (117 [†])	20 (16 [†])	136 (54 [†])

[†] number of corresponding instances in the reduced test set considered in Munari and Savelsbergh (2022).

- ▶ Formulation (FK) finds 94 (88) optimal solutions, 10 (6) for the first time, within two hours benchmark tests.
- ▶ Our best results overall account for 123 (117) optimal solutions, 20 (16) for the first time, within five hours.

Conclusions

- ▶ A new family of partially discretised route formulations (FK) for SDVRP's.
- ▶ A new dominance rule (\bar{q}) for optimal SDVRP's solutions.
- ▶ Experimentally (FK) becomes stronger with $K \uparrow$
- ▶ BCP algorithm is the new state-of-the-art for the SDVRPTW

Perspectives

- ▶ Our BCP algorithm can be easily extended to other variants such as multiple depots [Gouveia et al., 2021], heterogeneous fleet [Belfiore and Yoshizaki, 2009], using the generic VRPSolver model.
- ▶ Further strengthening of formulation (FK_{\max}) requires a generalized RCSP solver for the pricing
- ▶ We are bad for at finding good primal solutions!
- ▶ Extension to inventory and/or production routing problems?

Inventory Routing Problem: the structure

We consider the same variant as [Desaulniers et al., 2016]:

- ▶ Production quantities p_0^t of a single commodity for every period $t \in \mathcal{T} = 1, \dots, T$.
- ▶ Homogeneous fleet of H vehicles with capacity Q
- ▶ Demands d_i^t of each clients $i \in \mathcal{C}$ in each period $t \in \mathcal{T}$
- ▶ Inventory capacities C_i^t for every $(i, t) \in \mathcal{C} \cup \{0\} \times \mathcal{T}$
- ▶ Maximum-Level (ML) replenishment policy
- ▶ Initial inventories can be cancelled out due to the FIFO dominance rule
- ▶ Total inventory $\sum_{i \in \mathcal{C} \cup \{0\}} I_i^t$ is known after each period $t \in \mathcal{T}$

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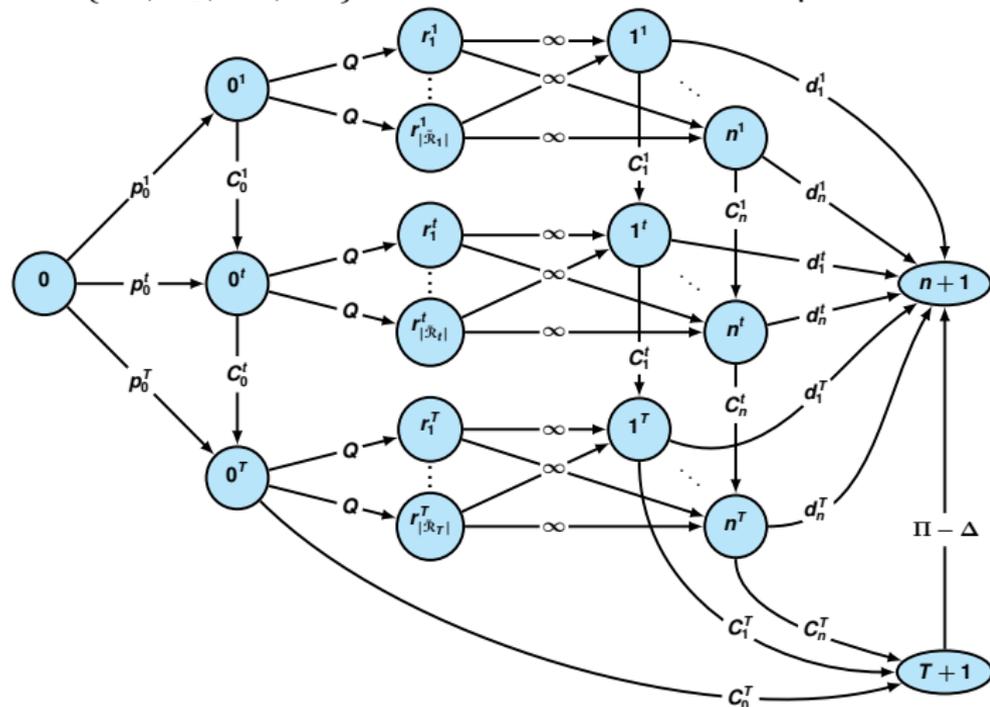
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Case with fixed inventory cost

The inventory cost is a constant if, in every period, holding costs are the same for all customers and the supplier

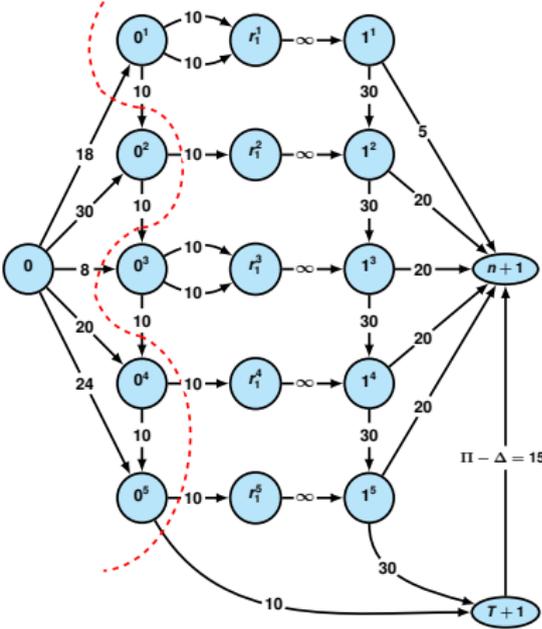
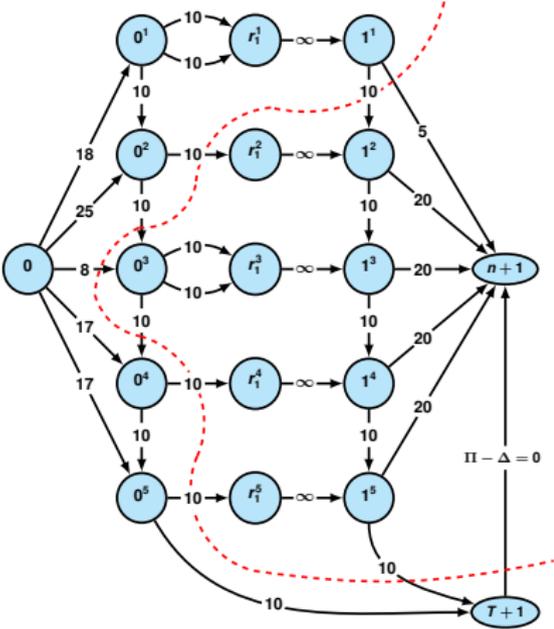
The flow graph for the case with fixed inventory cost

$\tilde{\mathcal{R}} = \{\tilde{\mathcal{R}}_1, \tilde{\mathcal{R}}_2, \dots, \tilde{\mathcal{R}}_T\}$ — sets of routes for each period.



$\tilde{\mathcal{R}}$ forms a feasible IRP solution iff the max-flow value in the flow graph is equal to $\Pi = \sum_{t \in \mathcal{T}} p_0^t$.

Examples of infeasible sets of routes



A cut with node $T + 1$ on the source side

(U_0, U_1, \dots, U_n) — set of nodes on the sink side

For a route $r \in \mathcal{R}_t$ to cross the cut, it should visit at least one client in set

$$S(t, U_0, \dots, U_n) = \begin{cases} \{i \in \mathcal{C} : t \in U_i\}, & t \notin U_0, \\ \emptyset, & t \in U_0. \end{cases}$$

In order to have value of the cut $\geq \Pi$, the total capacity of route arcs which cross the cut should not be smaller than

$$\mu(U_0, U_1, \dots, U_n) = \sum_{i \in \mathcal{C}} \sum_{t \in U_i} d_i^t - \sum_{t \in U_0} p_0^t - \sum_{i \in \mathcal{C} \cup \{0\}} \sum_{\substack{t \notin U_i: \\ \{t+1\} \in U_i}} c_i^t$$

Thus,
$$\sum_{t \in \mathcal{T}} \sum_{r \in \mathcal{R}_t} h_{r, S(t, U_0, \dots, U_n)} \theta_r \geq \left\lceil \frac{\mu(U_0, U_1, \dots, U_n)}{Q} \right\rceil$$

A cut with node $T + 1$ on the sink side

(U_0, U_1, \dots, U_n) — characterisation of a cut (set of nodes in the flow graph in the sink side)

In order to have value of the cut $\geq \Pi$, the total capacity of route arcs which cross the cut should not be smaller than

$$\nu(U_0, U_1, \dots, U_n) = \sum_{t \notin U_0} p_0^t - \sum_{i \in C} \sum_{t \notin U_i} d_i^t - \sum_{i \in C \cup \{0\}} \sum_{\substack{t \notin U_i: \\ \{t+1\} \in U_i \cup \{T+1\}}} c_i^t$$

Thus,
$$\sum_{t \in T} \sum_{t \in R_t} h_{r,S(t,U_0,\dots,U_n)} \theta_r \geq \left\lceil \frac{\nu(U_0, U_1, \dots, U_n)}{Q} \right\rceil$$

Formulation for the case with fixed inventory cost

$$\text{Min } \sum_{t \in \mathcal{T}} \sum_{r \in \mathcal{R}_t} c^r \theta_r + \text{fixed inventory cost,}$$

$$\text{s.t. } \sum_{t \in \mathcal{T}} \sum_{r \in \mathcal{R}_t} h_{r,S(t,U_0,\dots,U_n)} \theta_r \geq \left\lceil \frac{\mu(U_0, U_1, \dots, U_n)}{Q} \right\rceil, \quad \forall (U_0, U_1, \dots, U_n),$$

$$\sum_{t \in \mathcal{T}} \sum_{r \in \mathcal{R}_t} h_{r,S(t,U_0,\dots,U_n)} \theta_r \geq \left\lceil \frac{\nu(U_0, U_1, \dots, U_n)}{Q} \right\rceil, \quad \forall (U_0, U_1, \dots, U_n),$$

$$\sum_{r \in \mathcal{R}_t} h_{r,\{i\}} \theta_r \leq 1, \quad \forall t \in \mathcal{T}, i \in \mathcal{C},$$

$$\sum_{r \in \mathcal{R}_t} \theta_r \leq H, \quad \forall t \in \mathcal{T},$$

$$\theta_r \in \{0, 1\}, \quad \forall t \in \mathcal{T}, r \in \mathcal{R}_t.$$

The work has just started...

- ▶ Extension to the case with general holding costs.
- ▶ How do we separate generalized (strong) k -path inequalities?
- ▶ How do we make the formulation stronger by considering delivery quantities in paths?
- ▶ Which known and new cutting planes can we use?
- ▶ What is the performance of “cut-based” formulations?
- ▶ Extension to the production-routing problems?

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