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# VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS IN PROPER CAT(0) SPACES

OTHMANE JERHAOUI AND HASNAA ZIDANI

ABSTRACT. In this article, we develop a novel notion of viscosity solutions for first order Hamilton-Jacobi equations in proper CAT(0) spaces. The notion of viscosity is defined by taking test functions that are directionally differentiable and can be represented as a difference of two semiconvex functions. Under mild assumptions on the Hamiltonian, we recover the main features of viscosity theory for both the stationary and the time-dependent cases in this setting: the comparison principle and Perron's method. Finally, we show that this notion of viscosity coincides with classical one in  $\mathbb{R}^N$  and we give several examples of Hamilton-Jacobi equations in more general CAT(0) spaces covered by this setting.

## 1 INTRODUCTION

First order Hamilton-Jacobi equations have been extensively studied in the Euclidean space or more generally in infinite dimensional Banach spaces that enjoy the Radon Nikodym property. A substantial literature exists on the subject going back to several decades ago [14, 15, 16, 5]. More recently, there has been an increasing interest in studying first order Hamilton-Jacobi equations posed in more general metric spaces. Typical examples include topological networks, the space of Borel probability measures, or more generally any generic metric space. This problem involves many challenging mathematical obstacles and has a wide range of applications in various fields such as data transmission, social network problems, traffic management problems, fluid mechanics, optimal control of multi-agent systems and mean field control problems.

Several new notions of viscosity were proposed for first order Hamilton-Jacobi equations in metric spaces. Since a notion of a differential for real valued functions defined in a general metric space is not well defined, the Hamiltonians studied in this case depend on the differential of the unknown function only through its local Lipschitz constant, called the local slope. In [20, 28], the authors studied a class of Hamilton-Jacobi equations of Eikonal-type in a general metric space. The notion of viscosity used by the authors is defined via optimal control interpretations along absolutely continuous curves. This has the advantage to reduce the viscosity notion into a one dimensional problem and requires no structure on the space considered. In [19, 3], the authors proposed a different notion of viscosity for a similar class of Hamilton-Jacobi equations defined in a complete geodesic metric space using local slopes and suitable test functions. In [30], the authors provide a comparison between these notions of viscosity. In particular, they show that the notions coincide in the case of the Eikonal equation defined in a complete geodesic space.

On the other hand, there is a growing interest in studying Hamilton-Jacobi equations on a simpler structure in the form of a network. The latter is defined as a finite collection of isometric

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half-spaces glued together along their boundary. For example in the one dimensional case, a network is the result of gluing a finite number of half-lines along their origin. The subset where the gluing operation occurs is called the *junction*. On each branch of the network, one defines a Hamiltonian, the Hamiltonians are a priori independent from one another and a discontinuity occurs at the junction. Thanks to the smooth structure that each branch of the network possesses, one can define more general Hamilton-Jacobi equations than the Eikonal-type equations. The notion of viscosity solution is defined here using test functions that are continuous on the network and continuously differentiable on each branch. First, the special case of the Eikonal equation on networks has been considered in [31, 11]. Later in [24] the authors treated the case of convex Hamiltonians on each branch in a one dimensional network. In their work, an additional junction condition is considered, called the flux-limiter, in order to guarantee well-posedness of the problem. These results have been extended to the case of quasi-convex Hamiltonians in [21, 22]. The case of a higher dimensional network was treated in [23]. In [27] the authors studied the case of a one dimensional network with Hamiltonians that are not necessarily convex nor quasi-convex. They introduced a junction condition called the Kirchoff condition and proved well-posedness of the problem using the same notion of viscosity as in [21, 22]. Furthermore, they proved that the flux-limiter type condition at the junction is a special case of the Kirchoff-type condition. The book [7] offers a detailed discussion on the different junction conditions considered on networks and the notion of viscosity solutions adopted in this space.

Despite the progress made regarding the study of Hamilton-Jacobi equations in metric spaces, we propose in this paper to answer some questions that are still open in current state of the art. Indeed, the techniques developed for the treatment of Hamilton-Jacobi equations in a general geodesic metric space are restricted to a certain class of Hamilton-Jacobi equations such as Eikonal equations. On the other hand, the setting of a network allows to study more general Hamilton-Jacobi equations but the techniques used in this setting do not take advantage of the metric structure of the network and focus more on the differential structure that exists on each branch. Furthermore, extending those results to a network where the branches have different Hausdorff dimensions is still a challenging question. The purpose of this paper is to define a viscosity notion for first order Hamilton-Jacobi equations in a class of metric spaces general enough that includes Euclidean spaces and networks and can allow to treat more general equations than the Eikonal-type ones. Furthermore, this viscosity notion should ideally coincide with the classical one developed in Euclidean spaces. Therefore, we focus our attention in this manuscript on developing a theory of first order viscosity notion in a subclass of metric spaces called CAT(0) spaces.

A metric space  $(X, d)$ , is said to be a CAT(0) space if, roughly speaking, it is a geodesic space and its geodesic triangles are “thinner” than the triangles of the Euclidean plane  $\mathbb{R}^2$  (see Definition 2.2). This method of comparing geodesic triangles of a geodesic space with triangles from a model space, such as the Euclidean plane, was first introduced by Alexandrov [2, 9]. It gives a synthetic definition of curvature for geodesic spaces. In particular, CAT(0) spaces are spaces of curvature not greater than 0 in the sense of Alexandrov. Typical examples of CAT(0) spaces are Hilbert spaces, convex sets of Hilbert spaces, simply connected Riemannian manifolds with nonpositive sectional curvature and the space of Borel probability measures over the real line [26]. Although CAT(0) spaces do not possess any smooth structure, they carry a solid first order calculus similar to smooth manifolds with sectional curvature not greater than 0. For example, a notion of tangent cone  $T_x X$  is well defined at each point of  $X$ . The tangent cone is the metric counterpart of the tangent space in Riemannian geometry or the Bouligand tangent cone in convex analysis. Furthermore, a notion

of differential is well defined for any function  $u : X \rightarrow \mathbb{R}$  that is Lipschitz and DC. By DC functions we intend real valued functions that can be represented as a difference of two semiconvex functions. The exact definition of this class of functions is given in Definition 2.16. We refer to [10, 8, 9, 2, 4] for a more detailed discussion on the topic.

In this paper, we propose to study first order Hamilton-Jacobi equations in proper CAT(0) spaces, i.e., CAT(0) spaces whose closed bounded sets are compact. More specifically, we consider the following stationary problem,

$$(1.1) \quad \begin{cases} H(u(x), x, D_x u) = 0, & \forall x \in \Omega, \\ u(x) = \ell(x), & \forall x \in \partial\Omega, \end{cases}$$

and its time-dependent counterpart,

$$(1.2) \quad \begin{cases} \partial_t u + H(x, D_x u) = 0, & \forall (t, x) \in (0, +\infty) \times X, \\ u(0, x) = \ell(x), & x \in X, \end{cases}$$

where  $\Omega$  is an open set,  $\ell$  is a real valued continuous and bounded function on its domain, and  $u$  is a Lipschitz and DC function on its domain. The differential  $D_x u : T_x X \rightarrow \mathbb{R}$  is defined in the tangent cone of  $X$  at a point  $x$ . The differential  $D_x u$  is itself a Lipschitz, DC and positively homogeneous function (see Proposition 2.18). The Hamiltonian  $H : \mathbb{R} \times X \times \text{DC}_1(T_x X) \rightarrow \mathbb{R}$  is a real valued function. The set  $\text{DC}_1(T_x X)$  represents Lipschitz, positively homogeneous and DC functions on  $T_x X$ .

The viscosity notion we adopt here generalizes what is currently present in the literature. We define the notion of viscosity using subsets of the class of Lipschitz and DC functions. More precisely, we test upper semicontinuous subsolutions with Lipschitz semiconvex functions and we test lower semicontinuous supersolutions with Lipschitz semiconcave functions. We prove comparison results that apply for any upper semicontinuous subsolution and any lower semicontinuous supersolution using the same techniques as in the classical theory of viscosity. In particular, we apply the variable doubling technique using the squared distance function in the same way as in [6, 13]. Comparison results guarantee the uniqueness of the solution. Furthermore, we prove existence of the solution by virtue of Perron's method in a similar way as originally developed in [6, 13]. We would like to also mention that the same notion of viscosity was used by the authors in the case of a Hamilton-Jacobi equation defined on the space of probability measures over a compact Riemannian manifold in [25].

We give several examples showing the degree of generality of our setting. Namely, we show that the setting developed in this paper coincides with the classical setting when  $X = \mathbb{R}^N$  by treating the examples of Hamilton-Jacobi-Bellman equations and Isaacs equations defined in  $\mathbb{R}^N$ . Furthermore, we give several examples of Hamilton-Jacobi equations defined in more general proper CAT(0) spaces, covered by this setting.

The paper is organized in the following way. In Section 2 we give the definition of CAT(0) spaces, we describe the gluing operation of a collection of CAT(0) spaces, we define the central notion of the tangent cone and we give the definition of DC functions and their differential in CAT(0) spaces. In Section 3 and Section 4, we define the notion of viscosity solutions and the general form of the Hamiltonian we are going to work with for the stationary case and the time-dependent cases respectively. We show that we recover the main features of viscosity theory in this setting: the comparison principle and Perron's method. Finally, we give several examples showing the degree

of generality of our setting. In particular, we treat classical examples of Hamilton-Jacobi equations when the space  $X = \mathbb{R}^N$  to demonstrate that our setting coincides with the classical one in  $\mathbb{R}^N$  and we treat other examples of Hamilton-Jacobi equations defined in more general CAT(0) spaces.

## 2 CALCULUS IN CAT(0) SPACES

Let us briefly recall some facts in metric geometry. Classical references are [10, 8, 9, 2, 4]. Let  $(X, d)$  be a metric space. For  $x \in X$  and  $r > 0$  we denote by  $B(x, r)$  and  $\bar{B}(x, r)$  the open and closed balls of center  $x$  and radius  $r$  respectively. The metric space  $(X, d)$  is said to be *proper* if all of its closed bounded sets are compact sets.

Let  $l > 0$ . A metric space  $(X, d)$  is said to be a *geodesic space* if any two points  $x, y \in X$  are connected by at least one *unit speed geodesic*, i.e. a map  $\gamma : [0, l] \rightarrow X$  such that  $\gamma_0 = x$ ,  $\gamma_l = y$  and

$$d(\gamma_t, \gamma_s) = |t - s|, \quad \forall t, s \in [0, l].$$

In particular, we necessarily have  $l = d(x, y)$ . The image of  $\gamma$  is called the *geodesic segment* with endpoints  $x$  and  $y$ .

Let  $I \subset \mathbb{R}$  be an interval. A map  $\gamma : I \rightarrow X$  is said to be a *constant speed geodesic* if there exists a constant  $D \geq 0$  such that

$$\forall s, t \in I, \quad d(\gamma_s, \gamma_t) = D|s - t|.$$

In what follows, we will refer to constant speed geodesics simply by *geodesics*.

*Remark 2.1.* In Section 3, we will also need another property on geodesic spaces (see Example 3.9). A geodesic space  $(X, d)$  is said to be *geodesically extendable* if for every geodesic  $\gamma : [a, b] \rightarrow X$  with  $a < b \in \mathbb{R}$ , there exists a geodesic  $\tilde{\gamma} : (-\infty, +\infty) \rightarrow X$  such that

$$\tilde{\gamma}|_{[a, b]} = \gamma.$$

**2.1 CAT(0) spaces.** Let  $(X, d)$  be a geodesic space. In order to define the notion of curvature of  $X$ , we shall first introduce the notion of *geodesic triangles*. A geodesic triangle  $\Delta(x, y, z)$  in a geodesic space is the result of three points  $x, y, z \in X$ , called the *vertices*, together with a choice of three corresponding geodesics, the *edges*, linking the vertices. A *comparison triangle* for the geodesic triangle  $\Delta(x, y, z)$  is a geodesic triangle built in the Euclidean plane  $(\mathbb{R}^2, d_{\mathbb{R}^2})$ , denoted by  $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$ , with  $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^2$ , such that

$$d_{\mathbb{R}^2}(\bar{x}, \bar{y}) = d(x, y), \quad d_{\mathbb{R}^2}(\bar{y}, \bar{z}) = d(y, z), \quad d_{\mathbb{R}^2}(\bar{x}, \bar{z}) = d(x, z).$$

The choice of the comparison triangle is unique up to an isometry [8, Lemma I.2.14]. A point  $a \in X$  is said to be between  $y$  and  $z$  provided that we have

$$d(y, a) + d(z, a) = d(y, z).$$

This means that the point  $a$  lies in a geodesic segment of  $y$  and  $z$ . The *comparison point* of  $a$  is the unique point  $\bar{a} \in \mathbb{R}^2$ , once the comparison triangle  $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$  is fixed, such that

$$d_{\mathbb{R}^2}(\bar{y}, \bar{a}) + d_{\mathbb{R}^2}(\bar{z}, \bar{a}) = d_{\mathbb{R}^2}(\bar{y}, \bar{z}).$$

**Definition 2.2** (CAT(0) spaces). A metric space  $(X, d)$  is called a CAT(0) space if it is a geodesic space and satisfies the following comparison triangle inequality: for any  $x, y, z \in X$  and any point  $a \in X$  between  $y$  and  $z$ , the comparison points  $\bar{x}, \bar{y}, \bar{z}, \bar{a} \in \mathbb{R}^2$  satisfy

$$(2.1) \quad d(x, a) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{a}).$$

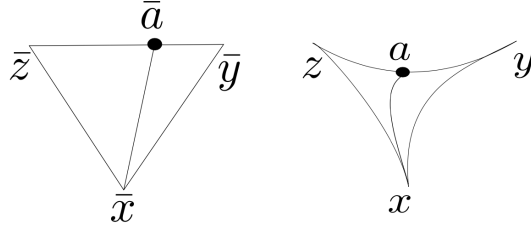


FIGURE 1. The comparison triangle on the left and the geodesic triangle on the right.

*Remark 2.3.* The comparison triangle inequality (2.1) is equivalent to the following one: for any  $x, y, z \in X$  and any comparison points  $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^2$ ,

$$(2.2) \quad d^2(\gamma_t, x) \leq d_{\mathbb{R}^2}^2((1-t)\bar{y} + t\bar{z}, \bar{x}), \quad \forall t \in [0, 1],$$

where  $\gamma : [0, 1] \rightarrow X$  is the geodesic joining  $\gamma_0 = y$  and  $\gamma_1 = z$ . By expanding the right hand side of (2.2) using the elementary properties of the inner product in  $\mathbb{R}^2$ , it becomes

$$(2.3) \quad d^2(\gamma_t, x) \leq (1-t)d^2(\gamma_0, x) + td^2(\gamma_1, x) - t(1-t)d^2(\gamma_0, \gamma_1), \quad \forall t \in [0, 1].$$

Inequality (2.3) can be used in an equivalent way as a definition of CAT(0) spaces. It can be understood as a synthetic inequality that quantifies the deficit of the curvature of  $X$  with respect to the Euclidean space  $\mathbb{R}^2$ , where inequality (2.3) is an equality. In other words, inequality (2.3) quantifies how much the triangle  $\Delta(x, y, z)$  in  $X$  is thinner with respect to the triangle  $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$  in  $\mathbb{R}^2$ .

**Example 2.4.** Here are some examples of CAT(0) spaces [8, Example II.1.15].

- Euclidean spaces, Hilbert spaces (the only Banach spaces which are CAT(0)).
- Convex subsets of Hilbert spaces.
- Convex subsets of other CAT(0) spaces.
- The  $n$ -dimensional hyperbolic space, denoted  $\mathbb{H}^n$ . It is the unique simply connected,  $n$ -dimensional complete Riemannian manifold with a constant negative sectional curvature equal to  $-1$ .
- Simply connected Riemannian manifolds with sectional curvature not greater than 0.
- Metric  $\mathbb{R}$ -trees, i.e. any metric space  $T$  such that:
  - there exists a unique geodesic segment joining each pair of points  $x, y \in T$ ; we denote it by  $[x, y]$ ;
  - if  $[x, y] \cap [y, z] = \{y\}$ , then  $[x, y] \cup [y, z] = [x, z]$ .
- The 2-Wasserstein space over the real line, denoted  $\mathcal{P}_2(\mathbb{R})$  [26, Proposition 4.1].

Following [8, Proposition II.1.4], an important result which is a consequence of Definition 2.2 is that in a CAT(0) space  $(X, d)$ , any two points  $x, y \in X$  are connected by a *unique* geodesic joining  $x$  and  $y$ .

Let  $(X, d)$  be a CAT(0) space. A subset  $\mathcal{C} \subset X$  is said to be *convex* if for every  $x, y \in \mathcal{C}$ , the geodesic segment connecting  $x$  and  $y$  lies entirely in  $\mathcal{C}$ .

In the Euclidean plane  $\mathbb{R}^2$ , the open balls are convex. Hence, from Definition 2.2, it is straightforward to prove that the open balls of  $(X, d)$  are convex (see [8, Proposition II.1.4-(3)] for a detailed proof

of this fact). Furthermore, any convex subset of  $X$  equipped with the distance  $d$  is also a CAT(0) space [8, Examples II.1.15].

Another useful result concerning CAT(0) spaces is that any product of two CAT(0) spaces is a CAT(0) space when equipped with the product distance, as the following lemma shows.

**Lemma 2.5.** ([8, Exercice II.1.16]). *Let  $(Y, d_Y)$  and  $(Z, d_Z)$  be two CAT(0) spaces. Then the product space  $(Y \times Z, d_{Y \times Z})$ , equipped with the distance*

$$d_{Y \times Z}^2\left((y_1, z_1), (y_2, z_2)\right) := d_Y^2(y_1, y_2) + d_Z^2(z_1, z_2),$$

*is a CAT(0) space. Moreover, if  $Y$  and  $Z$  are proper spaces then the product space is also a proper space.*

**2.2 Gluing constructions.** In this section, we will discuss the most obvious way of gluing metric spaces, which is to attach them along isometric subsets. Furthermore, we will see that when the underlying metric spaces are CAT(0) spaces, and the isometric subsets are complete CAT(0) subspaces, then the resulting space by the gluing operation is a CAT(0) space. In this subsection, the set  $I$  will denote an arbitrary index set (countable or uncountable). The following definition can be found in [8, Definition I.5.23].

**Definition 2.6.** (Gluing operation). Let  $I$  be an index set. Let  $(X_\lambda, d_\lambda)_{\lambda \in I}$  be a family of metric spaces. Let  $A_\lambda \subset X_\lambda$  be fixed closed subsets. Let  $A$  be a metric space and suppose that for each  $\lambda \in I$ , there exists an isometry  $i_\lambda : A \rightarrow A_\lambda$ . Let  $\cup_{\lambda \in I} X_\lambda$  be the disjoint union of the metric spaces  $X_\lambda$ ,  $\lambda \in I$ . We define the space  $X$  as the quotient space of  $\cup_I X_\lambda$  by the equivalence relation  $\mathcal{R}$  defined the following way

$$\begin{aligned} \forall x, y \in \cup_I X_\lambda, x \mathcal{R} y &\Leftrightarrow \exists a \in A, \lambda, \lambda' \in I : x \in A_\lambda, y \in A_{\lambda'} \\ &\text{and } i_\lambda^{-1}(\{x\}) = i_{\lambda'}^{-1}(\{y\}) = a, \end{aligned}$$

where we identify each  $X_\lambda$  with its image in  $X$ . The space  $X$  is called *the glued space* along  $A$  and is denoted

$$X := \bigsqcup_A X_\lambda.$$

Some examples of glued spaces will be given below. The following theorem shows how to define a distance on the glued space  $X$  and summarizes its main properties.

**Theorem 2.7.** ([8, Lemma I.5.24]). *Let  $X = \bigsqcup_A X_\lambda$ . Let  $x \in X_\lambda$  and  $y \in X_{\lambda'}$ . we define the following function:*

$$d(x, y) := \begin{cases} d_\lambda(x, y) & \text{if } \lambda = \lambda', \\ \inf_{a \in A} \{d_\lambda(x, i_\lambda(a)) + d_{\lambda'}(i_{\lambda'}(a), y)\} & \text{if } \lambda \neq \lambda'. \end{cases}$$

We have

- (1)  $d$  is a distance on  $X$ ;
- (2) if  $I$  is finite and each  $(X_\lambda, d_\lambda)$  is proper, then  $(X, d)$  is proper;
- (3) if each space  $(X_\lambda, d_\lambda)$  is a geodesic space and  $A$  is proper, then  $(X, d)$  is a geodesic space.

For CAT(0) spaces, we have a stronger result that we give in the next proposition.

**Proposition 2.8.** ([8, Theorem II.11.3] Gluing families of CAT(0) spaces). Let  $I$  be an index set. Let  $(X_\lambda, d_\lambda)_{\lambda \in I}$  be a family of CAT(0) spaces. Let  $A_\lambda \subset X_\lambda$  be closed subsets. Let  $A$  be a metric space and suppose that for all  $\lambda \in I$ , there exist isometries  $i_\lambda : A \rightarrow A_\lambda$ . Let  $X = \sqcup_A X_\lambda$  be the resulting glued space along  $A$ .

If  $A$  is a complete CAT(0) space, then the glued space  $X$  is a CAT(0) space, endowed with the distance defined in Theorem 2.7.

**Example 2.9.** Let  $X_1$  and  $X_2$  be the following two proper CAT(0) spaces:

$$\begin{cases} X_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}, \\ X_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0\}. \end{cases}$$

Let  $A := \{0\}$ . We consider the following glued space along  $A$

$$X := X_1 \bigsqcup_A X_2.$$

The resulting glued space  $X$  is a proper CAT(0) space represented in figure 2

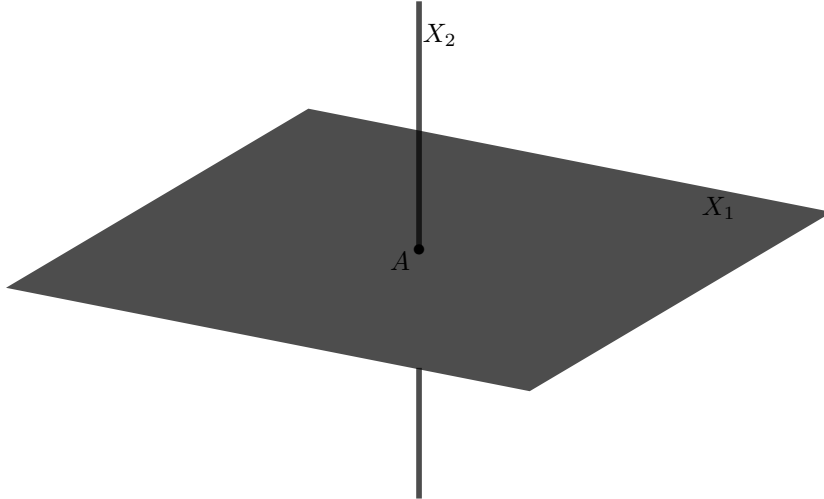


FIGURE 2. (Example 2.9) - The space obtained by gluing  $X_1$  and  $X_2$  along  $A$ .

Indeed,  $X_1$  and  $X_2$  are Euclidean subspaces of  $\mathbb{R}^3$ . Hence they are proper CAT(0) spaces when endowed with the induced Euclidean distances from  $\mathbb{R}^3$ . Furthermore,  $A$  is a complete CAT(0) space since it is reduced to one point. Hence, according to Theorem 2.7 and Proposition 2.8,  $X$  is a proper CAT(0) space when endowed with its geodesic distance. The resulting distance is obtained thanks to Theorem 2.7 in the following way

$$(2.4) \quad \forall x, y \in X, d(x, y) := \begin{cases} |x - y|, & \text{if } \exists i \in \{1, 2\} : x, y \in X_i, \\ |x| + |y|, & \text{otherwise,} \end{cases}$$

where  $|\cdot|$  denotes the Euclidean norm on  $\mathbb{R}^3$ .



**Example 2.10.** Let  $X_1$  and  $X_2$  be the following two proper CAT(0) spaces:

$$\begin{cases} X_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}, \\ X_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0, x_3 \geq 0\}. \end{cases}$$

Let  $A := \{0\}$ . We consider the following glued space along  $A$

$$X := X_1 \bigsqcup_A X_2.$$

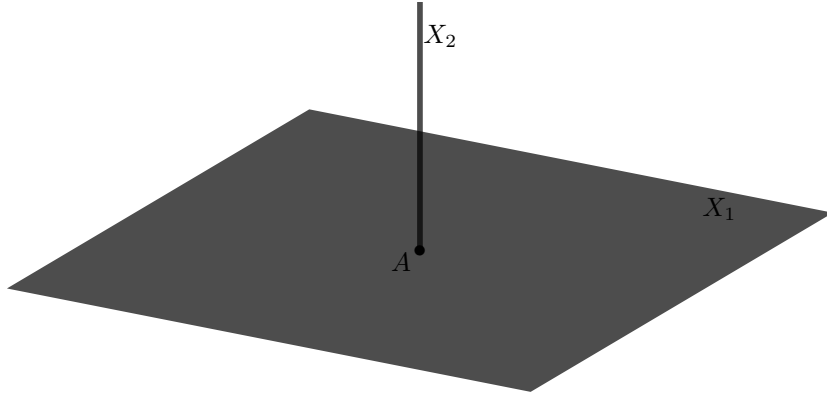


FIGURE 3. (Example 2.10) - The space  $X$  obtained by gluing  $X_1$  and  $X_2$  along  $A$ .

Figure 3 shows a representation of the space  $X$ . In this example,  $X_2$  is not an Euclidean space. However,  $X_2$  is a closed convex set of an Euclidean space, then it is still a proper CAT(0) space when endowed with the Euclidean distance of  $\mathbb{R}$ . By similar arguments as in Example 2.9, the resulting glued space  $X$  is a proper CAT(0) space endowed with its geodesic distance which is similar to (2.4).

**Example 2.11.** Here are more examples covered by this setting:

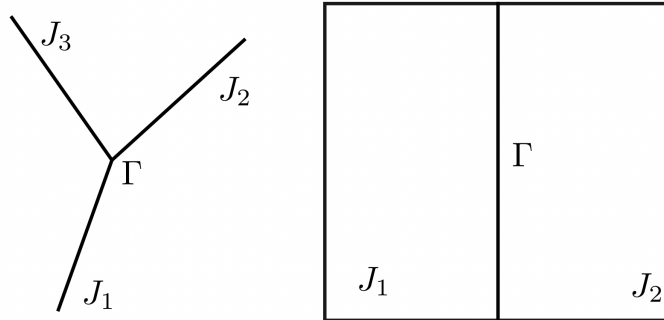


FIGURE 4. (Example 2.11) - More examples of glued CAT(0) spaces.

In Figure 4(left-hand side), the space

$$J = \bigsqcup_{\Gamma} J_i$$

is the result of gluing three copies of the half-line  $[0, +\infty)$  along the subset  $\Gamma = \{0\}$ . In Figure 4(right-hand side), the space  $J$  is isometric to the Euclidean plane  $\mathbb{R}^2$  obtained by gluing two copies of the half-plane  $\{(x, y) \in \mathbb{R}^2 : x \geq 0\}$ , along the subset  $\Gamma = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ .

**2.3 Tangent cone.** In this section, we recall the notion of the tangent cone on geodesic spaces and give its main properties in the case of CAT(0) spaces. The tangent cone is a central notion in metric geometry, similar to the tangent space for differentiable manifolds or the Bouligand tangent cone in convex analysis. We refer to the bibliography mentioned at the beginning of this section for a more detailed discussion.

Let  $(X, d)$  be a geodesic space and let  $x \in X$ . We denote by  $\text{Geo}_x(X)$  the set of geodesics emanating from  $x$  and defined in some neighborhood of the form  $[0, \varepsilon]$ , with  $\varepsilon > 0$ . Let  $\eta, \gamma \in \text{Geo}_x(X)$ . Then the following quantity

$$(2.5) \quad d_x(\eta, \gamma) := \limsup_{t \downarrow 0} \frac{d(\eta_t, \gamma_t)}{t}$$

is a pseudo-distance on the space  $\text{Geo}_x(X)$ . Moreover,  $d_x$  defines an equivalence relation on  $\text{Geo}_x(X)$  in the following way:

$$\forall \eta, \gamma \in \text{Geo}_x(X), \eta \sim \gamma \quad \text{if and only if} \quad d_x(\eta, \gamma) = 0.$$

The quotient space  $\text{Geo}_x(X)/\sim$  endowed with quotient distance, still denoted by  $d_x$ , is a metric space. The equivalence class of a geodesic  $\gamma \in \text{Geo}_x(X)$  under the equivalence relation  $\sim$  is denoted by  $\gamma'_0 \in \text{Geo}_x(X)/\sim$ . It represents the initial *velocity* or *direction* of  $\gamma$ .

**Definition 2.12** (Tangent cone). Let  $(X, d)$  be a geodesic space and  $x \in X$ . The *tangent cone* at  $x$  is the metric space  $(T_x X, d_x)$ , where  $T_x X$  is the abstract completion of  $(\text{Geo}_x(X)/\sim, d_x)$ , i.e.

$$T_x X := \overline{\text{Geo}_x(X)/\sim}^{d_x}.$$

We denote by  $0_x \in T_x X$  the equivalence class of the geodesic with speed equal to 0 in  $\text{Geo}_x(X)/\sim$ . It is called *the origin* or *the apex* of the tangent cone  $T_x X$ .

**Example 2.13.** If  $(X, d)$  is a simply connected manifold with sectional curvature not greater than 0, then the tangent cone at a point  $x \in X$  is isometric to the usual tangent space.

When  $(X, d)$  is a general geodesic space, the structure of the tangent cone can be very wild, and little can be said about it. However, when  $(X, d)$  is a CAT(0) space, then the tangent cone behaves nicely. This fact is exploited to build a first order calculus in  $(X, d)$ . First, we have the following key result. If  $(X, d)$  is a CAT(0) space and  $x \in X$ , then the supremum limit in (2.5) is actually a limit. In fact, we have a stronger result. The tangent cone at a given point  $x$  of a CAT(0) space is a complete CAT(0) space when endowed with the distance  $d_x$  [8, Theorem II-3.19]. Furthermore, the tangent cone of a CAT(0) space has a structure resembling a Hilbert space. This is due to the fact that it is a complete CAT(0) space and it has a cone structure.

To make the latter statement clearer, first notice that for any  $\lambda \geq 0$ , the map sending the geodesics  $(\gamma_t) : t \mapsto \gamma_t \in \text{Geo}_x(X)$  to the geodesics  $(\gamma_{\lambda t}) : t \mapsto \gamma_{\lambda t} \in \text{Geo}_x(X)$  can be passed to the quotient  $\text{Geo}_x(X)/\sim$  and the resulting quotient map sending the equivalence class of  $(\gamma_t)$  to the equivalence

class of  $(\gamma_{\lambda t})$  is  $\lambda$ -Lipschitz on  $\text{Geo}_x(X)/\sim$ . Indeed, for any two geodesics  $(\gamma_t)_t, (\eta_t)_t$  that belong to  $\text{Geo}_x(X)$ , we have

$$\lim_{t \downarrow 0} \frac{d(\gamma_{\lambda t}, \eta_{\lambda t})}{t} = \lambda \lim_{t \downarrow 0} \frac{d(\gamma_{\lambda t}, \eta_{\lambda t})}{\lambda t} = \lambda \lim_{s \downarrow 0} \frac{d(\gamma_s, \eta_s)}{s}.$$

Therefore, by passing to the quotient, the map is  $\lambda$ -Lipschitz from  $\text{Geo}_x(X)/\sim$  to itself. Hence it can be extended by continuity to  $T_x X$  and can be seen as the operation of *multiplication by a positive scalar*. We denote it the following way:

$$\forall v \in T_x X, \forall \lambda \geq 0, \quad \lambda v \in T_x X.$$

Thus  $T_x X$  has a structure of a cone. Moreover, for any  $v, w \in T_x X$  and  $\lambda \in \mathbb{R}^+$ , we define the norm and the scalar product on  $T_x X$  the following way:

$$(2.6a) \quad \text{Norm : } |v|_x := d_x(v, 0_x),$$

$$(2.6b) \quad \text{Scalar product : } \langle v, w \rangle_x := \frac{1}{2}(|v|_x^2 + |w|_x^2 - d_x^2(v, w)).$$

Furthermore, we have the following results on the norm and scalar product.

**Proposition 2.14.** ([17, Proposition 2.11] Calculus on the tangent cone). Let  $(X, d)$  be a CAT(0) space, let  $x \in X$  be a fixed point and  $T_x X$  be the tangent cone of  $X$  at  $x$ . Then the operations (2.6a) and (2.6b) are continuous in their variables. The operation (2.6b) is symmetric. Furthermore, we have

$$(2.7a) \quad |\lambda v|_x = \lambda |v|_x,$$

$$(2.7b) \quad \langle \lambda v, w \rangle_x = \langle v, \lambda w \rangle_x = \lambda \langle v, w \rangle_x,$$

$$(2.7c) \quad |\langle v, w \rangle_x| \leq |v|_x |w|_x \text{ and } \langle v, w \rangle_x = |v|_x |w|_x \text{ if and only if } |w|_x v = |v|_x w,$$

for all  $v, w \in T_x X$  and  $\lambda \in \mathbb{R}^+$ .

Since CAT(0) spaces are uniquely geodesic, meaning that any two points are connected by a unique unit speed geodesic, we introduce the following notation which is going to be useful throughout this paper.

**Notation 2.15.** Let  $(X, d)$  be a CAT(0) space, and let  $x, y \in X$ . The unique unit speed geodesic connecting  $x$  and  $y$  is denoted by

$$t \mapsto G_t^{x,y}, \quad \forall t \in [0, d(x, y)].$$

Furthermore, we denote by

$$\uparrow_x^y := (G_0^{x,y})' \in T_x X$$

the *direction* of  $G^{x,y}$  at  $x$ . The direction  $\uparrow_x^y$  has a norm equal to 1, meaning that

$$|\uparrow_x^y|_x = 1.$$

**2.4 DC calculus.** In this section, we introduce the notion of real valued *directionally differentiable* functions in CAT(0) spaces. Special attention will be given to Lipschitz functions that are semiconvex or semiconcave since they are differentiable at every point according to this definition. These functions are going to serve us as test functions in the definition of viscosity notion in the next section.

Let  $(X, d)$  be a CAT(0) space and  $x \in X$ . Let  $f : X \rightarrow \mathbb{R}$  be a function. We say that  $f$  has a *directional derivative* at  $x$  along the geodesic  $\gamma : [0, \varepsilon] \rightarrow X$  emanating from  $x$ , with  $\varepsilon > 0$ , if the limit

$$\left. \frac{d}{dt} \right|_{t=0} f(\gamma_t) = \lim_{t \downarrow 0} \frac{f(\gamma_t) - f(\gamma_0)}{t}$$

exists and is finite.

**Definition 2.16.** Let  $f : X \rightarrow \mathbb{R}$  be a function.

- We say that  $f$  is *semiconvex* if there exists  $\lambda \in \mathbb{R}$  such that for every geodesic  $\gamma : [0, 1] \rightarrow X$  the following inequality holds:

$$(2.8) \quad f(\gamma_t) \leq (1-t)f(\gamma_0) + tf(\gamma_1) - \frac{\lambda}{2}t(1-t)d^2(\gamma_0, \gamma_1),$$

or equivalently, if the real-to-real function

$$t \mapsto f(\gamma_t) - \frac{\lambda}{2}d^2(\gamma_0, \gamma_1)t^2$$

is convex. We also say that  $f$  is  $\lambda$ -convex. If  $\lambda = 0$  then we simply say that  $f$  is convex.

- We say that  $f$  is *semiconcave* (or  $\lambda$ -concave for some  $\lambda \in \mathbb{R}$ ) if and only if  $-f$  is semiconvex (or  $(-\lambda)$ -convex).
- Finally, we say that  $f$  is a DC function if it can be represented as a difference of two semiconvex functions.

In particular, every semiconvex function is a DC function and every semiconcave function is also a DC function. Furthermore, we can define locally semiconvex and locally semiconcave functions as well.

**Definition 2.17.** Let  $\Omega \subset X$  be an open subset. A function  $f : \Omega \rightarrow \mathbb{R}$  is said to be *locally semiconvex* if for any point  $x \in \Omega$  there exists a neighborhood  $U_x$  of  $x$  such that for all geodesics  $\gamma : [0, 1] \rightarrow \Omega$  with endpoints in  $U_x$  we have that inequality (2.8) holds. Similarly, a function  $f : \Omega \rightarrow \mathbb{R}$  is said to be *locally semiconcave* if and only if  $-f$  is locally semiconvex. Finally, a function  $f : \Omega \rightarrow \mathbb{R}$  is said to be a *locally DC function* if it can be locally represented as a difference of two semiconvex functions.

Let  $\Omega \subset X$  be an open subset and  $x \in \Omega$ . Let  $f : \Omega \rightarrow \mathbb{R}$  be a locally Lipschitz and locally semiconvex function. Then the directional derivative of  $f$  along any geodesic emanating from  $x$  exists and is finite by [17, Proposition 2.16]. Furthermore, we define the *differential function* of  $f$  at  $x$  from its directional derivatives as the map  $D_x f : (\text{Geo}_x X / \sim) \rightarrow \mathbb{R}$  defined as

$$D_x f \cdot \gamma'_0 := \left. \frac{d}{dt} \right|_{t=0} f(\gamma_t) = \lim_{t \downarrow 0} \frac{f(\gamma_t) - f(\gamma_0)}{t}, \quad \forall \gamma \in \text{Geo}_x X, \quad \gamma'_0 \in \text{Geo}_x X / \sim.$$

Notice that the above definition does not depend on the choice of the geodesic  $\gamma$  whose velocity is  $\gamma'_0$ . Moreover, the differential function is Lipschitz, convex and positively homogeneous. Thus it can be uniquely extended to the whole tangent cone  $T_x X$  by density. These properties are collected in the next proposition.

**Proposition 2.18.** ([17, Proposition 2.16] Differential of semiconvex functions). Let  $f : \Omega \rightarrow \mathbb{R}$  be a locally Lipschitz and locally semiconvex function around  $x \in \Omega$ . Then  $f$  is directionally differentiable at  $x$  and the differential function  $D_x f : T_x X \rightarrow \mathbb{R}$  is Lipschitz, convex and positively homogeneous, i.e.,

$$D_x f \cdot (\lambda v) = \lambda D_x f \cdot v, \quad \forall v \in T_x X \text{ and } \lambda \geq 0.$$

Similarly, if  $f : \Omega \rightarrow \mathbb{R}$  is locally Lipschitz and locally semiconcave, then it is directionally differentiable at any  $x \in \Omega$  and its differential function is Lipschitz, concave, positively homogeneous and defined by

$$D_x f := -D_x(-f).$$

Finally, if  $f : \Omega \rightarrow \mathbb{R}$  is a locally Lipschitz and locally DC function, then  $f$  is directionally differentiable at any  $x \in \Omega$  and its differential function is Lipschitz, DC, and positively homogeneous.

In what follows we will refer to directionally differentiable functions simply by *differentiable* functions.

We denote by  $\text{DC}_{\text{lip}}(\Omega)$  the class of locally Lipschitz and locally DC functions on  $\Omega$ . We also denote by  $\text{DC}_1(T_x X)$  the class of Lipschitz, DC, and positively homogeneous functions on the tangent cone  $T_x X$  at some point  $x \in X$ . Finally we denote by  $\text{DC}_1(TX)$  the set

$$(2.9) \quad \text{DC}_1(TX) := \{(x, p_x) \in X \times \text{DC}_1(T_x X)\},$$

which is the metric analog of the cotangent bundle in this setting.

Next, we give several examples of locally Lipschitz and locally DC functions in  $\text{CAT}(0)$  spaces to demonstrate how abundant these functions are in this class of metric spaces. Moreover, we will give the explicit expression of their differential function at every point.

In the Euclidean plane  $\mathbb{R}^2$ , for  $\bar{y} \in \mathbb{R}^2$  fixed, the Euclidean distance function  $\bar{x} \mapsto d_{\mathbb{R}^2}(\bar{x}, \bar{y})$  is Lipschitz continuous and convex and the squared Euclidean distance function  $\bar{x} \mapsto d_{\mathbb{R}^2}^2(\bar{x}, \bar{y})$ , is locally Lipschitz continuous and 2-convex. These properties are preserved in any  $\text{CAT}(0)$  space  $(X, d)$ . Indeed, it follows directly from Definition 2.2 that for any given  $y \in X$ , the distance function  $x \mapsto d(x, y)$  is Lipschitz continuous and convex [8, Proposition II.2.2] and from Remark 2.3 that the squared distance function  $x \mapsto d^2(x, y)$  is locally Lipschitz continuous and 2-convex. Furthermore, their differential functions are given explicitly in the next proposition, whose proof can be found in [17, Proposition 2.17].

**Proposition 2.19.** Let  $(X, d)$  be a  $\text{CAT}(0)$  space. Let  $y \in X$  be a fixed point. The following properties hold true.

- For all  $x \in X$ , we have

$$\forall v \in T_x X, \quad D_x d(\cdot, y) \cdot v = \begin{cases} -\langle v, \uparrow_x^y \rangle_x, & \text{if } x \neq y, \\ |v|_x, & \text{if } x = y, \end{cases}$$

where  $\uparrow_x^y$  is given in Notation 2.15.

- For all  $x \in X$ , we have

$$\forall v \in T_x X, \quad D_x d^2(\cdot, y) \cdot v = -2d(x, y)\langle v, \uparrow_x^y \rangle_x.$$

More generally, if  $(X, d)$  is a  $\text{CAT}(0)$  space and  $\mathcal{C}$  is a closed convex subset of  $X$ , then the distance function to  $\mathcal{C}$  is also Lipschitz and convex. We summarize the main properties of the distance function to a closed convex subset in the next proposition.

**Proposition 2.20.** (Distance function to a convex set [8, Proposition II.2.4, Corollary II.2.5]). Let  $(X, d)$  be a  $\text{CAT}(0)$  space. Let  $\mathcal{C}$  be a closed convex subset of  $X$ . Then the following holds:

- (1) for every  $x \in X$ , there exists a unique point  $\pi(x)$  called the *projection* of  $x$  onto  $\mathcal{C}$  such that

$$d(x, \pi(x)) = d(x, \mathcal{C}) := \inf_{y \in \mathcal{C}} d(x, y);$$

- (2) for all  $x, y \in X$ , we have  $|d(x, \mathcal{C}) - d(y, \mathcal{C})| \leq d(x, y)$ ;
- (3) the function  $x \mapsto d(x, \mathcal{C})$  is convex.
- (4) the function  $x \mapsto \pi(x)$  is a retraction from  $X$  to  $\mathcal{C}$ . In particular, it is continuous.

Since the distance to a closed convex subset in a CAT(0) space is Lipschitz and convex, then according to Proposition 2.18, it is differentiable at every point. The next theorem is the first result of this paper where we give the explicit expression of the differential of the distance function to a closed convex subset in a CAT(0) space.

**Theorem 2.21** (Differential of the distance function to a closed convex set). *Let  $(X, d)$  be a CAT(0) space. Let  $\mathcal{C}$  be a closed convex subset of  $X$ . Then the following holds:*

$$\forall x \in X, \forall v \in T_x X, \quad D_x d(\cdot, \mathcal{C}) \cdot v = \begin{cases} -\langle \uparrow_x^{\pi(x)}, v \rangle_x, & \text{if } x \notin \mathcal{C}, \\ d_x(v, T_x \mathcal{C}), & \text{if } x \in \mathcal{C}, \end{cases}$$

where,  $\pi(x)$  is the projection of  $x$  onto  $\mathcal{C}$ ,  $T_x \mathcal{C}$  is the tangent cone of  $x \in \mathcal{C}$ , when  $(\mathcal{C}, d)$  is seen as a complete CAT(0) space and

$$d_x(v, T_x \mathcal{C}) := \inf_{w \in T_x \mathcal{C}} d_x(v, w).$$

The tangent cone  $T_x \mathcal{C}$  is a closed convex subset of the CAT(0) space  $(T_x X, d_x)$ .

*Proof.* The proof is decomposed into two steps. Let  $x \in X$ .

Step 1. If  $x \notin \mathcal{C}$ , then we have

$$\forall y \in X, \quad d(y, \mathcal{C}) \leq d(y, \pi(x)),$$

which implies that

$$\forall \gamma(\cdot) \in \text{Geo}_x, \quad \lim_{t \downarrow 0} \frac{d(\gamma_t, \mathcal{C}) - d(x, \mathcal{C})}{t} \leq \lim_{t \downarrow 0} \frac{d(\gamma_t, \pi(x)) - d(x, \pi(x))}{t}.$$

The last inequality is equivalent to

$$\forall v \in \text{Geo}_x / \sim, \quad D_x d(\cdot, \mathcal{C}) \cdot v \leq D_x d(\cdot, \pi(x)) \cdot v.$$

By Lipschitz continuity of the differential functions, we get

$$\forall v \in T_x X, \quad D_x d(\cdot, \mathcal{C}) \cdot v \leq D_x d(\cdot, \pi(x)) \cdot v,$$

and by Proposition 2.19 we have

$$\forall v \in T_x X, \quad D_x d(\cdot, \pi(x)) \cdot v = -\langle \uparrow_x^{\pi(x)}, v \rangle_x.$$

Therefore, we get

$$\forall v \in T_x X, \quad D_x d(\cdot, \mathcal{C}) \cdot v \leq -\langle \uparrow_x^{\pi(x)}, v \rangle_x.$$

For the other inequality, let  $v \in \text{Geo}_x / \sim$  and let  $\gamma : [0, r] \rightarrow X$  be a geodesic such that  $\gamma'_0 = v$ . First, by [8, Lemma II.3.20] we have

$$\lim_{s \rightarrow 0} d_x(\uparrow_x^{\pi(\gamma_s)}, \uparrow_x^{\pi(x)}) = 0.$$

Moreover, by Proposition 2.19 we have

$$D_x d(\cdot, \pi(x)) \cdot v = -\langle v, \uparrow_x^{\pi(x)} \rangle_x.$$

Therefore, by the continuity of the scalar product asserted in Proposition 2.14 we get

$$D_x d(\cdot, \pi(x)) \cdot v = -\langle v, \uparrow_x^{\pi(x)} \rangle_x = \lim_{s \downarrow 0} -\langle v, \uparrow_x^{\pi(\gamma_s)} \rangle_x.$$

Furthermore, we have

$$\begin{aligned} D_x d(\cdot, \pi(x)) \cdot v &= \lim_{s \downarrow 0} -\langle v, \uparrow_x^{\pi(\gamma_s)} \rangle_x = \lim_{s \downarrow 0} \lim_{t \downarrow 0} \frac{d(\gamma_t, \pi(\gamma_s)) - d(x, \pi(\gamma_s))}{t} \\ &= \lim_{s \downarrow 0} \inf_{0 < t < r} \frac{d(\gamma_t, \pi(\gamma_s)) - d(x, \pi(\gamma_s))}{t}. \end{aligned}$$

The last equality is true since the real-to-real function

$$t \mapsto d(\gamma_t, \pi(\gamma_s))$$

is convex, so the incremental ratio

$$(0, r] \ni t \mapsto \frac{d(\gamma_t, \pi(\gamma_s)) - d(x, \pi(\gamma_s))}{t}$$

is non-decreasing. The monotonicity property of the above incremental ratio also gives us

$$\begin{aligned} \lim_{s \downarrow 0} \inf_{0 < t < r} \frac{d(\gamma_t, \pi(\gamma_s)) - d(x, \pi(\gamma_s))}{t} &\leq \lim_{s \downarrow 0} \inf_{s \leq t < r} \frac{d(\gamma_t, \pi(\gamma_s)) - d(x, \pi(\gamma_s))}{t} \\ &= \lim_{s \downarrow 0} \frac{d(\gamma_s, \pi(\gamma_s)) - d(x, \pi(\gamma_s))}{s}. \end{aligned}$$

Moreover, we have

$$\forall s \in [0, r], \quad d(x, \pi(\gamma_s)) \geq d(x, \mathcal{C}), \quad \text{and} \quad d(\gamma_s, \pi(\gamma_s)) = d(\gamma_s, \mathcal{C}).$$

Therefore, we get

$$D_x d(\cdot, \pi(x)) \cdot v \leq \lim_{s \downarrow 0} \frac{d(\gamma_s, \pi(\gamma_s)) - d(x, \pi(\gamma_s))}{s} \leq \lim_{s \downarrow 0} \frac{d(\gamma_s, \mathcal{C}) - d(x, \mathcal{C})}{s} = D_x d(\cdot, \mathcal{C}) \cdot v.$$

This is true for any  $v \in \text{Geo}_x / \sim$ . Lastly, by the Lipschitz continuity of the differential of the distance function, we get

$$\forall v \in T_x X, \quad D_x d(\cdot, \pi(x)) \cdot v \leq D_x d(\cdot, \mathcal{C}) \cdot v,$$

which is the desired inequality.

**Step 2.** Let  $x \in \mathcal{C}$ . By the Lipschitz continuity of the differentials, it is enough to consider only geodesic directions in  $T_x X$ . Let  $\gamma : [0, r] \rightarrow X$  be a geodesic such that  $\gamma'_0 = v \in \text{Geo}_x / \sim$ . Then we have

$$D_x d(\cdot, \mathcal{C}) \cdot v = \lim_{t \downarrow 0} \frac{d(\gamma_t, \mathcal{C})}{t} = \inf_{0 < t < r} \frac{d(\gamma_t, \mathcal{C})}{t},$$

where the last equality holds since the real-to-real function  $t \mapsto d(\gamma_t, \mathcal{C})$  is convex. Consequently, we have

$$D_x d(\cdot, \mathcal{C}) \cdot v = \inf_{0 < t < r} \frac{d(\gamma_t, \mathcal{C})}{t} = \inf_{0 < t < r} \inf_{y \in \mathcal{C}} \frac{d(\gamma_t, y)}{t}.$$

On the other hand, we have

$$\begin{aligned} d_x(v, T_x \mathcal{C}) &:= \inf_{w \in T_x \mathcal{C}} d_x(v, w) = \inf_{\beta \in \text{Geo}_x(\mathcal{C})} \lim_{t \downarrow 0} \frac{d(\gamma_t, \beta_t)}{t} = \inf_{y \in \mathcal{C}} \lim_{t \downarrow 0} \frac{d(\gamma_t, y)}{t} \\ &= \inf_{y \in \mathcal{C}} \inf_{0 < t < r} \frac{d(\gamma_t, y)}{t}, \end{aligned}$$

where the last two equalities hold because

$$\{\beta_t \mid \beta : [0, r'] \rightarrow X \in \text{Geo}_x(\mathcal{C}), \text{ for some } r' \geq 0, \text{ and } t \in [0, r']\} = \mathcal{C},$$

and the real-to-real function  $t \mapsto d(\gamma_t, y)$  is convex. Finally, notice that we have

$$\inf_{0 < t < r} \inf_{y \in \mathcal{C}} \frac{d(\gamma_t, y)}{t} = \inf_{y \in \mathcal{C}} \inf_{0 < t < r} \frac{d(\gamma_t, y)}{t} = \inf_{y \in \mathcal{C}} \frac{d(\gamma_r, y)}{r},$$

which concludes that

$$D_x d(\cdot, \mathcal{C}) \cdot v = d_x(v, T_x \mathcal{C}).$$

□

### 3 STATIONARY HAMILTON-JACOBI EQUATIONS IN PROPER CAT(0) SPACES

In this section, we study stationary first order Hamilton-Jacobi equations in proper CAT(0) spaces. We recall that a metric space is proper if its closed bounded sets are compact. We use subsets of Lipschitz and DC functions as test functions to define the viscosity notion. More precisely, we use subsets of Lipschitz and semiconvex functions to test subsolutions and subsets of Lipschitz and semiconcave functions to test supersolutions. With this class of test functions, we will see that we can define a notion of viscosity for first order Hamilton-Jacobi equations in proper CAT(0) spaces and recover the main features of the theory: the comparison principle and Perron's method. Throughout this section,  $(X, d)$  is a proper CAT(0) space and  $\Omega$  is an open subset of  $X$ . We denote by  $\bar{\Omega}$  its closure and we set  $\partial\Omega := \bar{\Omega} \setminus \Omega$ . The boundary  $\partial\Omega$  might be empty.

First, we define the notion of viscosity used throughout this section. Let

$$H : \mathbb{R} \times \text{DC}_1(TX) \rightarrow \mathbb{R}$$

be a function called the *Hamiltonian* and let  $\ell : \partial\Omega \rightarrow \mathbb{R}$  be a given bounded continuous function. We consider the following Hamilton-Jacobi equation

$$(3.1a) \quad H(u(x), x, D_x u) = 0, \quad \forall x \in \Omega,$$

$$(3.1b) \quad u(x) = \ell(x), \quad \forall x \in \partial\Omega,$$

where  $u : \bar{\Omega} \rightarrow \mathbb{R}$  is a Lipschitz and DC function which is the unknown of equation (3.1). We give the following definition of *classical solutions* of equation (3.1).

**Definition 3.1** (Classical solution). A continuous function  $u : \bar{\Omega} \rightarrow \mathbb{R}$  such that  $u|_{\Omega} \in \text{DC}_{\text{lip}}(\Omega)$  is said to be a *classical solution* of (3.1) if for every  $x \in \Omega$  we have

$$H(u(x), x, D_x u) = 0,$$

and  $u = \ell$  on  $\partial\Omega$ .

Let  $\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-$  and  $\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_+$  be two subsets of  $\text{DC}_{\text{lip}}(\Omega)$ .  $\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-$  and  $\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_+$  verify the following definition.

**Definition 3.2.** (Test functions).

Let  $\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-$  be a subset of  $\text{DC}_{\text{lip}}(\Omega)$  such that

- constant functions belong to  $\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-$ ;
- for all  $\phi(\cdot), \psi(\cdot) \in \mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-$  and  $a, b \geq 0$ ,  $a\phi(\cdot) + b\psi(\cdot) \in \mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-$ ;
- let  $y \in X$  be fixed. Then the function  $x \mapsto d^2(x, y)$  belongs to  $\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-$ .

Let  $\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_+$  be a subset of  $\text{DC}_{\text{lip}}(\Omega)$  such that

- $\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_+ = -\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_- := \{-\phi(\cdot) : \phi(\cdot) \in \mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-\}$ .



**Example 3.3.** For example, one can take the following test functions

$$\mathcal{TEST}_- := \{\text{real valued locally Lipschitz and locally semiconvex functions}\},$$

and

$$\mathcal{TEST}_+ = \{\text{real valued locally Lipschitz and locally semiconcave functions}\}.$$

We are now ready to define the notion of viscosity solutions. This definition is dependent upon the choice of  $\mathcal{TEST}_-$  and  $\mathcal{TEST}_+$  from Definition 3.2.

**Definition 3.4.** (Viscosity solution).

- An upper semicontinuous function  $u : \Omega \rightarrow \mathbb{R}$  is said to be a *viscosity subsolution* of (3.1a) if, for any  $\phi \in \mathcal{TEST}_-$  such that  $u - \phi$  attains a local maximum at  $x$ , we have

$$H(u(x), x, D_x \phi) \leq 0.$$

- Similarly, a lower semicontinuous function  $u : \Omega \rightarrow \mathbb{R}$  is said to be a *viscosity supersolution* of (3.1a) if, for any  $\phi \in \mathcal{TEST}_+$  such that  $u - \phi$  attains a local minimum at  $x$ , we have

$$H(u(x), x, D_x \phi) \geq 0.$$

- A continuous function  $u : \bar{\Omega} \rightarrow \mathbb{R}$  is said to be a viscosity solution of (3.1) if it is both a viscosity supersolution and a viscosity subsolution and satisfies the boundary condition

$$u = \ell, \quad \text{in } \partial\Omega.$$

In the classical viscosity solutions theory in an Euclidean space, the set of test functions used to define the subsolution and the supersolution is the same, that is  $\mathcal{TEST}_- = \mathcal{TEST}_+$ . In our setting, we allow the test functions to belong to the same set or to two different sets that satisfy Definition 3.2.

**3.1 Comparison principle.** Let  $H : \mathbb{R} \times \text{DC}_1(TX) \rightarrow \mathbb{R}$  be a Hamiltonian and  $\Omega$  be an open subset of  $X$  (recall that  $\text{DC}_1(TX)$  is defined in (2.9)). We consider the following Hamilton-Jacobi equation:

$$(3.2) \quad H(u(x), x, D_x u) = 0, \quad \forall x \in \Omega.$$

We test subsolutions of (3.2) with  $\mathcal{TEST}_-$  functions and we test supersolutions of (3.2) with  $\mathcal{TEST}_+$  functions. Next, we prove the comparison principle for the Hamilton-Jacobi equation (3.2). We assume the following hypotheses on the Hamiltonian.

*Hypothesis 3.5.* The Hamiltonian  $H$  is such that there exists  $K_{db} > 0$  such that for all  $\alpha > 0$ , for all  $r \in \mathbb{R}$  and for all  $x, y \in \Omega$ , we have

$$H(r, x, D_x(-\alpha d^2(\cdot, y))) - H(r, y, D_y(\alpha d^2(x, \cdot))) \leq K_{db} d(x, y)(1 + \alpha d(x, y)).$$

*Hypothesis 3.6.* The Hamiltonian  $H$  is such that there exists  $\gamma > 0$  such that

$$\gamma(r - s) \leq H(r, x, p) - H(s, x, p) \quad \text{for all } r \geq s, x \in \Omega, \text{ and } p \in \text{DC}_1(T_x X).$$

Now, we prove the following key lemma. It allows using the variable doubling technique to prove comparison type results. It was first proven in [13, Proposition 3.7] in the particular case of Euclidean spaces. We prove it here for every metric space.

**Lemma 3.7.** *Let  $\mathcal{O}$  be a subset of a metric space  $(Z, d_Z)$ . Let  $\Phi : \mathcal{O} \rightarrow \mathbb{R}$  be an upper semicontinuous function and  $\Psi : \mathcal{O} \rightarrow \mathbb{R}$  be a lower semicontinuous function such that  $\Psi \geq 0$ , and*

$$M_{\alpha_n} = \sup_{z \in \mathcal{O}} \{ \Phi(z) - \alpha_n \Psi(z) \},$$

*with  $(\alpha_n)_n \subset \mathbb{R}^+ \setminus \{0\}$  is an increasing sequence such that  $\alpha_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Suppose that  $\lim_{\alpha_n \rightarrow +\infty} M_{\alpha_n}$  exists and*

$$-\infty < \lim_{\alpha_n \rightarrow +\infty} M_{\alpha_n} < +\infty.$$

*Let  $z_{\alpha_n} \in \mathcal{O}$  be chosen such that*

$$\lim_{\alpha_n \rightarrow +\infty} (M_{\alpha_n} - (\Phi(z_{\alpha_n}) - \alpha_n \Psi(z_{\alpha_n}))) = 0.$$

*Then the following holds:*

$$\left\{ \begin{array}{l} (i) \quad \lim_{\alpha_n \rightarrow +\infty} \alpha_n \Psi(z_{\alpha_n}) = 0, \\ (ii) \quad \Psi(\hat{z}) = 0 \text{ and } \Phi(\hat{z}) = \sup_{\{\Psi(z)=0\}} \Phi(z) = \lim_{\alpha_n \rightarrow +\infty} M_{\alpha_n}, \\ \quad \text{whenever } \hat{z} \in \mathcal{O} \text{ is an accumulation point of } (z_{\alpha_n})_{\alpha_n}. \end{array} \right.$$

*Proof.* The proof is exactly the same as in [13, Proposition 3.7] even though it was asserted for Euclidean spaces. We give it hereafter for the sake of completeness. Let

$$\delta_{\alpha_n} = M_{\alpha_n} - (\Phi(z_{\alpha_n}) - \alpha_n \Psi(z_{\alpha_n})),$$

so that  $\lim_{\alpha_n \rightarrow +\infty} \delta_{\alpha_n} = 0$ . Since  $\Psi \geq 0$ ,  $M_{\alpha_n}$  decreases as  $\alpha_n$  increases and  $\lim_{\alpha_n \rightarrow +\infty} M_{\alpha_n}$  exists and is finite by assumption. Furthermore, we have:

$$M_{\frac{\alpha_n}{2}} \geq \Phi(z_{\alpha_n}) - \frac{\alpha_n}{2} \Psi(z_{\alpha_n}) = \Phi(z_{\alpha_n}) - \alpha_n \Psi(z_{\alpha_n}) + \frac{\alpha_n}{2} \Psi(z_{\alpha_n}) = M_{\alpha_n} - \delta_{\alpha_n} + \frac{\alpha_n}{2} \Psi(z_{\alpha_n}),$$

which implies that  $\alpha_n \Psi(z_{\alpha_n}) \leq 2(\delta_{\alpha_n} + M_{\frac{\alpha_n}{2}} - M_{\alpha_n})$  and therefore

$$\lim_{\alpha_n \rightarrow +\infty} \alpha_n \Psi(z_{\alpha_n}) = 0.$$

Suppose now that there exists a subsequence of  $(z_{\alpha_n})_{\alpha_n}$ , not relabeled here, that converges to  $\hat{z} \in \mathcal{O}$ . Then  $\lim_{\alpha_n \rightarrow +\infty} \Psi(z_{\alpha_n}) = 0$  and by lower semicontinuity and positivity of  $\Psi$  we also get  $\Psi(\hat{z}) = 0$ . Moreover, since

$$\Phi(z_{\alpha_n}) - \alpha_n \Psi(z_{\alpha_n}) = M_{\alpha_n} - \delta_{\alpha_n} \geq \sup_{\{\Psi(z)=0\}} \Phi(z) - \delta_{\alpha_n},$$

and  $\Phi$  is upper semicontinuous, we get by letting  $\alpha_n \rightarrow \infty$

$$\sup_{\{\Psi(z)=0\}} \Phi(z) \geq \Phi(\hat{z}) \geq \lim_{\alpha_n \rightarrow \infty} M_{\alpha_n} \geq \sup_{\{\Psi(z)=0\}} \Phi(z),$$

which forces equality everywhere. This ends the proof.  $\square$

In the next theorem, we start by proving the comparison principle on a bounded open subset of  $X$ . The proof is similar to the proof of the comparison principle in the classical theory of viscosity. The main differences here are that we use test functions that verify Definition 3.2 and we use the differential of the test functions instead of their gradient as it is the case for the classical theory of viscosity solutions.

**Theorem 3.8** (Comparison on bounded domains). *Assume  $H$  satisfies Hypotheses 3.5 and 3.6, and assume that  $\Omega$  is bounded.*

*Consider  $u : \bar{\Omega} \rightarrow \mathbb{R}$  a bounded from above upper semicontinuous subsolution of (3.2), and  $v : \bar{\Omega} \rightarrow \mathbb{R}$  a bounded from below lower semicontinuous supersolution of (3.2).*

*Then  $u \leq v$  in  $\partial\Omega$  implies  $u \leq v$  in  $\bar{\Omega}$ .*

*Proof.* Let  $M := \sup_{\bar{\Omega}}(u(x) - v(x))$ . Assume by contradiction that  $u \leq v$  in  $\partial\Omega$  and  $M > 0$ .

For every  $\alpha > 0$ , define  $\psi_\alpha : \bar{\Omega} \times \bar{\Omega} \rightarrow \mathbb{R}$  as

$$\psi_\alpha(x, y) = u(x) - v(y) - \frac{\alpha}{2}d^2(x, y), \quad \forall (x, y) \in \bar{\Omega} \times \bar{\Omega}.$$

Since  $u$  and  $-v$  are bounded from above and  $u - v$  is upper semicontinuous, the supremum  $M_\alpha := \sup_{\bar{\Omega} \times \bar{\Omega}} \psi_\alpha$  is reached. Let  $(x_\alpha, y_\alpha)$  be such that  $M_\alpha = \psi_\alpha(x_\alpha, y_\alpha)$ . We have

$$\lim_{\alpha \rightarrow +\infty} (M_\alpha - \psi_\alpha(x_\alpha, y_\alpha)) = 0, \quad \text{and} \quad -\infty < M \leq M_\alpha \leq \sup_{\bar{\Omega}}(u) + \sup_{\bar{\Omega}}(-v) < +\infty.$$

Since  $\bar{\Omega}$  is closed and bounded, then it is compact by the assumption of  $X$  being proper. Hence we can take a subsequence  $((x_{\alpha_n}, y_{\alpha_n}))_{\alpha_n}$  that converges as  $\alpha_n \rightarrow +\infty$ . We have

$$\lim_{\alpha_n \rightarrow +\infty} (M_{\alpha_n} - \psi_{\alpha_n}(x_{\alpha_n}, y_{\alpha_n})) = 0, \quad \text{and} \quad -\infty < \lim_{\alpha_n \rightarrow +\infty} M_{\alpha_n} < +\infty.$$

Therefore, we can apply Lemma 3.7 via the correspondences

$$Z = X \times X, \quad \mathcal{O} = \bar{\Omega} \times \bar{\Omega}, \quad \Phi(z) = u(x) - v(y), \quad \Psi(z) = \frac{1}{2}d^2(x, y),$$

and we get

$$\begin{cases} (i) & \lim_{\alpha_n \rightarrow +\infty} \frac{\alpha_n}{2}d^2(x_{\alpha_n}, y_{\alpha_n}) = 0, \\ (ii) & \lim_{\alpha_n \rightarrow +\infty} M_{\alpha_n} = M. \end{cases}$$

It follows that for  $\alpha_n$  big enough we have  $x_{\alpha_n}, y_{\alpha_n} \in \Omega$  since  $u \leq v$  in  $\partial\Omega$ . Thus we get

$$(3.3) \quad H\left(v(y_{\alpha_n}), y_{\alpha_n}, D_{y_{\alpha_n}}\left(-\frac{\alpha_n}{2}d^2(x_{\alpha_n}, \cdot)\right)\right) \geq 0 \geq H\left(u(x_{\alpha_n}), x_{\alpha_n}, D_{x_{\alpha_n}}\left(\frac{\alpha_n}{2}d^2(\cdot, y_{\alpha_n})\right)\right).$$

Hence, using Hypotheses 3.5 and 3.6 and inequality (3.3), we get

$$\begin{aligned} \gamma(u(x_{\alpha_n}) - v(y_{\alpha_n})) &\stackrel{3.6}{\leq} H\left(u(x_{\alpha_n}), x_{\alpha_n}, D_{x_{\alpha_n}}\left(\frac{\alpha_n}{2}d^2(\cdot, y_{\alpha_n})\right)\right) \\ &\quad - H\left(v(y_{\alpha_n}), x_{\alpha_n}, D_{x_{\alpha_n}}\left(\frac{\alpha_n}{2}d^2(\cdot, y_{\alpha_n})\right)\right) \\ &\stackrel{(3.3)}{\leq} H\left(v(y_{\alpha_n}), y_{\alpha_n}, D_{y_{\alpha_n}}\left(-\frac{\alpha_n}{2}d^2(x_{\alpha_n}, \cdot)\right)\right) \\ &\quad - H\left(v(y_{\alpha_n}), x_{\alpha_n}, D_{x_{\alpha_n}}\left(\frac{\alpha_n}{2}d^2(\cdot, y_{\alpha_n})\right)\right) \\ &\stackrel{3.5}{\leq} K_{db}d(x_{\alpha_n}, y_{\alpha_n}) \left(1 + \frac{\alpha_n}{2}d(x_{\alpha_n}, y_{\alpha_n})\right). \end{aligned}$$

By letting  $\alpha_n \rightarrow +\infty$ , we get

$$\gamma M \leq 0,$$

a contradiction with  $M > 0$ . □

If  $\Omega$  is an unbounded open set of  $X$ , then we need the following additional hypothesis to prove the comparison principle.

*Hypothesis 3.9.* The Hamiltonian  $H$  is such that there exists  $K_L > 0$  such that, for every  $x \in \Omega$  and  $r \in \mathbb{R}$ , we have

$$\forall p_x, q_x \in \text{DC}_1(T_x X), \quad \left| H(r, x, p_x) - H(r, x, q_x) \right| \leq K_L \sup_{|v|_x=1} |p_x \cdot v - q_x \cdot v|.$$

*Remark 3.10.* Note that the mapping

$$\text{DC}_1(T_x X) \ni p_x \mapsto \sup_{|v|_x=1} |p_x \cdot v|$$

verifies all the axioms of a norm on  $\text{DC}_1(T_x X)$ .

*Remark 3.11.* Hypothesis 3.9 asserts that the Hamiltonian  $H$  is Lipschitz continuous with respect to the variable  $p_x$ . When  $X = \mathbb{R}^N$  and the test functions are continuously differentiable, then Hypothesis 3.9 is the same as the Lipschitz assumption on  $p_x$  usually required for the Hamiltonian in the classical theory of viscosity.

**Theorem 3.12** (Comparison on unbounded domains). *Assume  $H$  satisfies Hypotheses 3.5, 3.6 and 3.9. Let  $u : \overline{\Omega} \rightarrow \mathbb{R}$  be a bounded from above upper semicontinuous subsolution of (3.2), and  $v : \overline{\Omega} \rightarrow \mathbb{R}$  a bounded from below lower semicontinuous supersolution of (3.2). Then*

$$u \leq v \text{ in } \partial\Omega \text{ implies } u \leq v \text{ in } \overline{\Omega}.$$

*Proof.* Let  $M := \sup_{\overline{\Omega}}(u(x) - v(x))$ . Assume by contradiction that  $M > 0$  and  $u \leq v$  in  $\partial\Omega$ .

Let

$$\varepsilon \in \left( 0, \min \left\{ M, \left( \frac{\gamma}{\gamma + 4K_L} M \right)^2, 1 \right\} \right),$$

where  $\gamma$  and  $K_L$  are given in Hypotheses 3.6 and 3.9. Let  $x_\varepsilon \in \Omega$  be such that

$$u(x_\varepsilon) - v(x_\varepsilon) \geq M - \varepsilon > 0.$$

For  $\alpha > 0$ , set

$$\psi_\alpha(x, y) = u(x) - v(y) - (d^2(x, x_\varepsilon) + d^2(y, x_\varepsilon)) - \frac{\alpha}{2} d^2(x, y), \quad \forall (x, y) \in \overline{\Omega} \times \overline{\Omega}.$$

It is clear that  $\psi_\alpha$  is upper semicontinuous and bounded from above. Set  $M_\alpha := \sup_{\overline{\Omega} \times \overline{\Omega}} \psi_\alpha$ . We have

$$0 < \psi_\alpha(x_\varepsilon, x_\varepsilon), \quad \text{and } \psi_\alpha(x, y) \leq 0 \quad \forall x, y \in \overline{\Omega} \setminus \overline{B}(x_\varepsilon, \delta),$$

where  $\delta = \sqrt{|\sup_{\overline{\Omega}}(u)| + |\sup_{\overline{\Omega}}(-v)|}$ . Hence, the supremum of  $\psi_\alpha$  is reached on the compact set  $\overline{B}(x_\varepsilon, \delta)$  and we will denote by  $(x_\alpha, y_\alpha)$  a maximum of  $\psi_\alpha$ . We have

$$\lim_{\alpha \rightarrow +\infty} (M_\alpha - \psi_\alpha(x_\alpha, y_\alpha)) = 0 \quad \text{and} \quad -\infty < M - \varepsilon \leq M_\alpha \leq \sup_{\overline{\Omega}}(u) + \sup_{\overline{\Omega}}(-v) < +\infty.$$

Since  $x_\alpha, y_\alpha$  are in a compact set, then there exists a subsequence  $(x_{\alpha_n}, y_{\alpha_n})$  that converges as  $\alpha_n \rightarrow +\infty$  and such that

$$\lim_{\alpha_n \rightarrow +\infty} (M_{\alpha_n} - \psi_{\alpha_n}(x_{\alpha_n}, y_{\alpha_n})) = 0, \quad \text{and} \quad -\infty < \lim_{\alpha_n \rightarrow +\infty} M_{\alpha_n} < +\infty.$$

Therefore, we can apply Lemma 3.7 via the correspondences

$$Z = X \times X, \quad \mathcal{O} = \overline{\Omega} \times \overline{\Omega}, \quad \Phi(z) = u(x) - v(y) - (d^2(x, x_\varepsilon) + d^2(y, x_\varepsilon)), \quad \Psi(z) = \frac{1}{2} d^2(x, y),$$

and we get

$$\begin{cases} (i) & \lim_{\alpha_n \rightarrow +\infty} \frac{\alpha_n}{2} d^2(x_{\alpha_n}, y_{\alpha_n}) = 0, \text{ and } x_{\alpha_n}, y_{\alpha_n} \rightarrow \hat{x} \in \bar{\Omega}, \\ (ii) & \lim_{\alpha_n \rightarrow +\infty} M_{\alpha_n} = \sup_{x \in \bar{\Omega}} u(x) - v(x) - 2d^2(x, x_\varepsilon) = u(\hat{x}) - v(\hat{x}) - 2d^2(\hat{x}, x_\varepsilon) > 0. \end{cases}$$

On the other hand, notice first that  $\hat{x} \in \Omega$  since we have  $u(\hat{x}) - v(\hat{x}) > 0$ . It follows that for  $\alpha_n$  big enough we have  $x_{\alpha_n}, y_{\alpha_n} \in \Omega$  since  $\hat{x} \in \Omega$ . Furthermore, we have

$$M - \varepsilon \leq u(\hat{x}) - v(\hat{x}) - 2d^2(\hat{x}, x_\varepsilon) \implies 2d^2(\hat{x}, x_\varepsilon) \leq \varepsilon \implies d(\hat{x}, x_\varepsilon) \leq \sqrt{\varepsilon},$$

and

$$(3.4) \quad H\left(v(y_{\alpha_n}), y_{\alpha_n}, D_{y_{\alpha_n}}\left(-\frac{\alpha_n}{2}d^2(x_{\alpha_n}, \cdot) - d^2(\cdot, x_\varepsilon)\right)\right) \geq 0 \\ 0 \geq H\left(u(x_{\alpha_n}), x_{\alpha_n}, D_{x_{\alpha_n}}\left(\frac{\alpha_n}{2}d^2(\cdot, y_{\alpha_n}) + d^2(\cdot, x_\varepsilon)\right)\right).$$

Hence, it follows from Hypotheses 3.5, 3.6, 3.9 and the above inequality

$$\begin{aligned} \gamma(u(x_{\alpha_n}) - v(y_{\alpha_n})) &\stackrel{3.6}{\leq} H\left(u(x_{\alpha_n}), x_{\alpha_n}, D_{x_{\alpha_n}}\left(\frac{\alpha_n}{2}d^2(y_{\alpha_n}, \cdot) + d^2(\cdot, x_\varepsilon)\right)\right) - \\ &\quad H\left(v(y_{\alpha_n}), y_{\alpha_n}, D_{y_{\alpha_n}}\left(\frac{\alpha_n}{2}d^2(\cdot, y_{\alpha_n}) + d^2(\cdot, x_\varepsilon)\right)\right) \\ &\stackrel{(3.4)}{\leq} H\left(v(y_{\alpha_n}), y_{\alpha_n}, D_{y_{\alpha_n}}\left(-\frac{\alpha_n}{2}d^2(\cdot, x_{\alpha_n}) - d^2(\cdot, x_\varepsilon)\right)\right) - \\ &\quad H\left(v(y_{\alpha_n}), x_{\alpha_n}, D_{x_{\alpha_n}}\left(\frac{\alpha_n}{2}d^2(\cdot, y_{\alpha_n}) + d^2(\cdot, x_\varepsilon)\right)\right) \\ &\leq K_{db}d(x_{\alpha_n}, y_{\alpha_n})\left(1 + \frac{\alpha_n}{2}d(x_{\alpha_n}, y_{\alpha_n})\right) + 2K_L(d(x_{\alpha_n}, x_\varepsilon) + d(y_{\alpha_n}, x_\varepsilon)), \end{aligned}$$

where the last inequality is obtained thanks to Proposition 2.19 and Hypotheses 3.5 and 3.9. Furthermore, notice that we have for all  $\alpha_n$

$$\gamma(M - \varepsilon) \leq \gamma(u(x_\varepsilon) - v(x_\varepsilon)) \leq \gamma(u(x_{\alpha_n}) - v(y_{\alpha_n})).$$

Whence, by letting  $\alpha_n \rightarrow +\infty$ , we get

$$\gamma(M - \varepsilon) \leq 4K_L\sqrt{\varepsilon}.$$

Moreover, we have

$$\gamma(M - \sqrt{\varepsilon}) \leq \gamma(M - \varepsilon),$$

since  $0 < \varepsilon < 1$  by assumption. We get

$$\gamma(M - \sqrt{\varepsilon}) \leq \gamma(M - \varepsilon) \leq 4K_L\sqrt{\varepsilon} \implies \sqrt{\varepsilon} \geq \frac{\gamma}{\gamma + 4K_L}M.$$

This is a contradiction with  $\sqrt{\varepsilon} < \frac{\gamma}{\gamma + 4K_L}M$ , which ends the proof.  $\square$

**3.2 Existence of a viscosity solution: Perron's method.** In this subsection we suppose that  $\Omega$  is an arbitrary open subset of  $X$  (bounded or unbounded). Let  $H : \mathbb{R} \times \text{DC}_1(TX) \rightarrow \mathbb{R}$  be a Hamiltonian and  $\ell : \partial\Omega \rightarrow \mathbb{R}$  be a bounded and continuous function. We consider the following Hamilton-Jacobi equation with Dirichlet boundary condition

$$(3.5) \quad \begin{cases} H(u(x), x, D_x u) = 0, & \forall x \in \Omega, \\ u(x) = \ell(x), & \forall x \in \partial\Omega. \end{cases}$$

We consider the following hypotheses on the Hamiltonian  $H$ .

*Hypothesis 3.13.* The Hamiltonian  $H$  is such that:

- (i)– For every  $\phi : \Omega \rightarrow \mathbb{R}$  such that  $\phi \in \mathcal{TEST}_-$ , the function

$$(r, x) \mapsto H(r, x, D_x \phi)$$

is lower semicontinuous;

- (ii)– For every  $\phi : \Omega \rightarrow \mathbb{R}$  such that  $\phi \in \mathcal{TEST}_+$ , the function

$$(r, x) \mapsto H(r, x, D_x \phi)$$

is upper semicontinuous.

*Hypothesis 3.14.* The Hamiltonian  $H$  is such that for every  $\phi_1, \phi_2 \in \text{DC}_1(TX)$ , and every  $(x, r) \in \Omega \times \mathbb{R}$ , we have

$$\forall \eta \in T_x X, \quad D_x \phi_2 \cdot \eta \leq D_x \phi_1 \cdot \eta \implies H(r, x, D_x \phi_1) \leq H(r, x, D_x \phi_2).$$

*Remark 3.15.* Hypothesis 3.13 on the Hamiltonian depends on the choice of the test functions adopted in the definition of viscosity. It is a generalization of the continuity assumption usually required for  $H(r, \cdot, \cdot)$  when  $X = \mathbb{R}^N$  and

$$\mathcal{TEST}_- = \mathcal{TEST}_+ = \{\text{Twice continuously differentiable functions}\}.$$

Indeed, when  $X = \mathbb{R}^N$  and the test functions are twice continuously differentiable, Hypothesis 3.13 is automatically verified as a consequence of the continuity of the Hamiltonian and the regularity of the test functions. In a general CAT(0) space, Hypothesis 3.13 is a more suitable assumption.

Hypothesis 3.14 is needed in the case of general proper CAT(0) spaces in order to generalize Perron's method in this setting. More precisely, Hypothesis 3.14 gives us the following useful result given below, which asserts that any classical subsolution is a viscosity subsolution, and any classical supersolution is a viscosity supersolution.

**Lemma 3.16.** *Assume that the Hamiltonian  $H$  verifies Hypothesis 3.14. Let  $\phi : \Omega \rightarrow \mathbb{R}$  be a  $\text{DC}_{\text{lip}}(\Omega)$  function.*

- *If the inequality*

$$H(\phi(x_0), x_0, D_{x_0} \phi) \leq 0,$$

*is verified at every  $x_0 \in \Omega$ , then  $\phi$  is a viscosity subsolution in  $\Omega$  in the sense of Definition 3.4.*

- *Similarly, if the inequality*

$$H(\phi(x_0), x_0, D_{x_0} \phi) \geq 0,$$

*is verified at every  $x_0 \in \Omega$ , then  $\phi$  is a viscosity supersolution in  $\Omega$  in the sense of Definition 3.4.*

*Proof.* We will only prove the first part of the lemma. The other part is done in the exact same way.

Let  $x_0 \in \Omega$  and let  $\phi_{\text{test}} \in \mathcal{TEST}_-$  such that  $\phi - \phi_{\text{test}}$  attains a local maximum at  $x_0$ . Then in a small neighborhood  $V$  of  $x_0$ , we have

$$\forall y \in V, \quad \phi(y) - \phi(x_0) \leq \phi_{\text{test}}(y) - \phi_{\text{test}}(x_0).$$

The last inequality implies that for any geodesic  $\beta : [0, r'] \rightarrow X$  emanating from  $x_0$  we have

$$\lim_{t \downarrow 0} \frac{\phi(\beta(t)) - \phi(x_0)}{t} \leq \lim_{t \downarrow 0} \frac{\phi_{\text{test}}(\beta(t)) - \phi_{\text{test}}(x_0)}{t} \iff D_{x_0} \phi \cdot \beta'_0 \leq D_{x_0} \phi_{\text{test}} \cdot \beta'_0.$$

Consequently, by Proposition 2.18, the Lipschitz continuity of the differentials gives us

$$\forall \eta \in T_{x_0}X, \quad D_{x_0}\phi \cdot \eta \leq D_{x_0}\phi_{test} \cdot \eta.$$

Finally, by Hypothesis 3.14, we get

$$H(\phi(x_0), x_0, D_{x_0}\phi_{test}) \leq H(\phi(x_0), x_0, D_{x_0}\phi) \leq 0.$$

This ends the proof.  $\square$

We will derive the existence of the solution of (3.5) from the comparison result proven in Theorem 3.12. First, we define the *half-relaxed limits* of a family of locally bounded functions.

**Definition 3.17.** (Half-relaxed limits).

Let  $(u_\varepsilon)_{\varepsilon>0}$  be a family of uniformly locally bounded functions such that  $u_\varepsilon : \Omega \rightarrow \mathbb{R}$ . We define the following *half-relaxed limits* of the family  $(u_\varepsilon)_\varepsilon$  as:

$$\begin{aligned} \limsup^* u_\varepsilon(x) &= \limsup_{\substack{\varepsilon \rightarrow 0 \\ \Omega \ni z \rightarrow x}} u_\varepsilon(z); \\ \liminf_* u_\varepsilon(x) &= \liminf_{\substack{\varepsilon \rightarrow 0 \\ \Omega \ni z \rightarrow x}} u_\varepsilon(z). \end{aligned}$$

It is clear from the above definition that  $\limsup^* u_\varepsilon$  is an upper semicontinuous function and that  $\liminf_* u_\varepsilon$  is a lower semicontinuous function. Before getting to Perron's method, we need two key lemmas. They are classical results when  $X = \mathbb{R}^N$  (see for example [6]).

**Lemma 3.18.** *Let  $(v_\varepsilon)_{\varepsilon>0}$  be a family of uniformly locally bounded upper semicontinuous functions on  $\Omega$  and  $\bar{v} := \limsup^* v_\varepsilon$ . Let  $y \in \Omega$  be a strict local maximum point of  $\bar{v}$  on  $\Omega$ . Then there exists a subsequence  $(v_{\varepsilon_n})_{\varepsilon_n}$  and a sequence  $(y_{\varepsilon_n})_{\varepsilon_n}$  such that for all  $\varepsilon_n$ ,  $y_{\varepsilon_n}$  is a local maximum point of  $v_{\varepsilon_n}$  in  $\Omega$ , the sequence  $(y_{\varepsilon_n})_{\varepsilon_n}$  converges to  $y$  and  $v_{\varepsilon_n}(y_{\varepsilon_n})$  converges to  $\bar{v}(y)$  as  $\varepsilon_n \rightarrow 0$ .*

*Proof.* Since  $y$  is a strict local maximum point of  $\bar{v}$  on  $\Omega$ , there exists  $r > 0$  such that  $\bar{B}(y, r) \subset \Omega$  and

$$\forall z \in \bar{B}(y, r) \setminus \{y\}, \quad \bar{v}(z) < \bar{v}(y).$$

On the other hand,  $\bar{B}(y, r)$  is compact and  $v_\varepsilon$  is upper semicontinuous bounded from above on  $\bar{B}(y, r)$ , therefore for any  $\varepsilon > 0$  there exists a maximum point  $y_\varepsilon$  of  $v_\varepsilon$  on  $\bar{B}(y, r)$ , i.e.,

$$\forall z \in \bar{B}(y, r), \quad v_\varepsilon(z) \leq v_\varepsilon(y_\varepsilon).$$

Hence, by taking the limsup for  $z \rightarrow y$  and  $\varepsilon \rightarrow 0$ , we get:

$$\bar{v}(y) \leq \limsup_{\varepsilon} v_\varepsilon(y_\varepsilon).$$

Next, we consider the right-hand side of the last inequality. By extracting a subsequence of  $(y_\varepsilon)_\varepsilon$ , denoted by  $(y_{\varepsilon_n})_{\varepsilon_n}$ , we have  $\limsup_{\varepsilon \rightarrow 0} v_\varepsilon(y_\varepsilon) = \lim_{\varepsilon_n \rightarrow 0} v_{\varepsilon_n}(y_{\varepsilon_n})$ . Furthermore, since  $\bar{B}(y, r)$  is compact, we may assume that  $(y_{\varepsilon_n})_{\varepsilon_n}$  converges to some  $\bar{y}$ . But using again the definition of  $\limsup^*$  at  $\bar{y}$  we get,

$$\bar{v}(y) \leq \limsup_{\varepsilon \rightarrow 0} v_\varepsilon(y_\varepsilon) = \lim_{\varepsilon_n \rightarrow 0} v_{\varepsilon_n}(y_{\varepsilon_n}) \leq \bar{v}(\bar{y}).$$

Since  $y$  is a strict maximum point of  $\bar{v}$ , we get  $\bar{y} = y$  and  $v_{\varepsilon_n}(y_{\varepsilon_n}) \rightarrow \bar{v}(y)$ .  $\square$

**Lemma 3.19.** *Assume  $H$  satisfies Hypothesis 3.13. Let  $(u_\varepsilon)_{\varepsilon>0}$  be a family of uniformly locally bounded upper semicontinuous functions on  $\Omega$ , and set  $u = \limsup^* u_\varepsilon$ . If for all  $\varepsilon > 0$ ,  $u_\varepsilon$  is a subsolution of (3.5), then  $u$  is a subsolution of (3.5).*

*Proof.* Let  $\phi \in \mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-$  be a function such that  $u - \phi$  attains a local maximum at  $x \in \Omega$ . Then  $\psi(\cdot) = \phi(\cdot) + d^2(\cdot, x)$  is also a  $\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-$  function such that  $u - \psi$  attains a strict local maximum at  $x$ . Since  $\psi$  is continuous we have  $u - \psi = \limsup^*(u_\varepsilon - \psi)$ . Applying the lemma above, there exists a subsequence  $(x_{\varepsilon_n})_{\varepsilon_n}$  such that  $x_{\varepsilon_n} \rightarrow x$ ,  $u_{x_{\varepsilon_n}}(x_{\varepsilon_n}) \rightarrow u(x)$  and  $u_{\varepsilon_n} - \psi$  attains a local maximum at  $x_{\varepsilon_n}$ . So we get

$$H(u_{\varepsilon_n}(x_{\varepsilon_n}), x_{\varepsilon_n}, D_{x_{\varepsilon_n}}(\phi(\cdot) + d^2(\cdot, x))) = H(u_{\varepsilon_n}(x_{\varepsilon_n}), x_{\varepsilon_n}, D_{x_{\varepsilon_n}}\psi) \leq 0.$$

On the other hand, Proposition 2.19 implies

$$D_x d^2(\cdot, x) = 0.$$

Hence, by Hypothesis 3.13 we get

$$H(u(x), x, D_x\phi) \leq \liminf_{\varepsilon_n \rightarrow 0} H(u_{\varepsilon_n}(x_{\varepsilon_n}), x_{\varepsilon_n}, D_{x_{\varepsilon_n}}\psi) \leq 0,$$

which is the required result.  $\square$

*Remark 3.20.* The above lemma indicates a stability result for subsolutions of equation (3.5). A similar stability result can also be proved in the exact same way for a family of locally bounded lower semicontinuous supersolutions.

In the next theorem, we derive the existence of the solution from the comparison principle asserted in Theorem 3.12. The proof is similar to the one in the classical theory of viscosity. The difficulties here are the lack of continuity of the Hamiltonian and the fact that test functions are not continuously differentiable. However, with the use of Hypotheses 3.13 and 3.14, we are able to recover the same result as in the classical setting.

**Theorem 3.21** (Perron's method). *Let  $\Omega$  be an open set of  $X$  and set  $\partial\Omega = \bar{\Omega} \setminus \Omega$ . Suppose that the Hamiltonian  $H$  satisfies Hypotheses 3.5, 3.6, 3.9, 3.13 and 3.14. Assume that there exist  $\underline{u} : \bar{\Omega} \rightarrow \mathbb{R}$  a bounded upper semicontinuous subsolution of (3.5) and  $\bar{u} : \bar{\Omega} \rightarrow \mathbb{R}$  a bounded lower semicontinuous supersolution of (3.5). If*

$$\liminf_* \underline{u}(x) \geq \ell(x) \geq \limsup^* \bar{u}(x), \quad \forall x \in \partial\Omega,$$

*then there exists a unique continuous and bounded viscosity solution of (3.5).*

*Proof.* We define the set

$$\mathcal{S} = \{h : \bar{\Omega} \rightarrow \mathbb{R} : \underline{u} \leq h \leq \bar{u} \text{ and } h \text{ is a subsolution of (3.5)}\}.$$

The set  $\mathcal{S}$  is nonempty since  $\underline{u} \in \mathcal{S}$ . For  $x \in \bar{\Omega}$ , we set

$$u(x) = \sup\{h(x), h \in \mathcal{S}\}.$$

We will show that  $v := \limsup^* \{u\}$  is the viscosity solution of (3.5). First, we show that  $v$  is a subsolution. Notice that  $v$  is obviously upper semicontinuous. Let  $\phi : \Omega \rightarrow \mathbb{R}$  be a  $\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-$  function such that  $v - \phi$  attains a local maximum at  $x_0 \in \Omega$ . Without loss of generality, we can suppose that

$$v(x_0) = \phi(x_0).$$

By definition of  $v$ , there exists a sequence of points  $x_j \rightarrow x_0$  and a sequence of functions  $u_j \in \mathcal{S}$  such that

$$v(x_0) = \lim_{j \rightarrow \infty} u_j(x_j).$$



In particular  $\limsup^* \{u_j\}(x_0) \geq v(x_0)$ . On the other hand, by construction we have  $v \geq \limsup^* \{u_j\}$ . Therefore, at  $x_0$  we have  $\limsup^* \{u_j\}(x_0) = v(x_0)$ . For the other points on a small enough neighborhood of  $x_0$ , we have

$$\phi \geq v \geq \limsup^* \{u_j\}.$$

Therefore, by using Lemma 3.19 on a small enough open neighborhood of  $x_0$ , we get that  $\limsup^* \{u_j\}$  is a subsolution at  $x_0$ , since Hypothesis 3.13 holds. Therefore, by definition we have

$$H(v(x_0), x_0, D_{x_0}\phi) \leq 0.$$

This shows that  $v$  is a subsolution at  $x_0$ .

Now we show that  $v_* := \liminf_* v$  is a supersolution. We argue by contradiction. Suppose that there exists a point  $x_0 \in \Omega$  and a function  $\psi \in \mathcal{TEST}_+$  such that  $v_* - \psi$  attains a local minimum at  $x_0$ , and

$$H(v_*(x_0), x_0, D_{x_0}\psi) < 0.$$

Without loss of generality, we can suppose that  $\psi(x_0) = v_*(x_0)$ . Thus we have

$$H(\psi(x_0), x_0, D_{x_0}\psi) < 0.$$

So, by Hypothesis 3.14,  $\psi$  is a strict viscosity subsolution of (3.5) at  $x_0$ . Indeed, let  $\psi_{test} \in \mathcal{TEST}_-$  such that  $\psi - \psi_{test}$  attains a local maximum at  $x_0$ . Then for all  $y \in X$  in a small enough open neighborhood of  $x_0$ , we have

$$\psi(y) - \psi(x_0) \leq \psi_{test}(y) - \psi_{test}(x_0).$$

This implies that for any geodesic  $\beta : [0, r'] \rightarrow X$  emanating from  $x_0$  we have

$$\lim_{t \downarrow 0} \frac{\psi(\beta(t)) - \psi(x_0)}{t} \leq \lim_{t \downarrow 0} \frac{\psi_{test}(\beta(t)) - \psi_{test}(x_0)}{t} \iff D_{x_0}\psi \cdot \beta'_0 \leq D_{x_0}\psi_{test} \cdot \beta'_0.$$

Consequently, by Proposition 2.18, the Lipschitz continuity of the differentials gives us

$$\forall \eta \in T_{x_0}X, \quad D_{x_0}\psi \cdot \eta \leq D_{x_0}\psi_{test} \cdot \eta.$$

Hence by Hypothesis 3.14, we get

$$H(\psi(x_0), x_0, D_{x_0}\psi_{test}) \leq H(\psi(x_0), x_0, D_{x_0}\psi) < 0.$$

Furthermore from Hypothesis 3.13, it is also a strict subsolution of (3.5) in a small enough open neighborhood of  $x_0$  by upper semicontinuity of the Hamiltonian. Indeed, the function

$$g : x \mapsto H(\psi(x), x, D_x\psi)$$

is upper semicontinuous. So the set  $\{x \in \Omega : g(x) < 0\}$  is open. In particular, there exists a small enough neighborhood of  $x_0$  such that for all  $x \in X$  that belong to this neighborhood, we have

$$H(\psi(x), x, D_x\psi) < 0.$$

Therefore, by Lemma 3.4,  $\psi$  is a strict viscosity subsolution, i.e. for any  $\psi_{test} \in \mathcal{TEST}_-$  such that  $\psi - \psi_{test}$  attains a local maximum at  $x \in X$  belonging to a small enough neighborhood of  $x_0$ , we get

$$H(\psi(x), x, D_x\psi_{test}) < 0.$$

Moreover, for  $\delta > 0$  small enough,  $\tilde{\psi} = \psi + \delta$  is a subsolution on a small enough open neighborhood of  $x_0$  denoted by  $B(x_0, r) \subset \Omega$ , with  $r > 0$  small enough, since the function

$$s \mapsto H(s, x, D_x\psi),$$

is upper semicontinuous also by Hypothesis 3.13.

We have  $\tilde{\psi}(x_0) > v_*(x_0)$ . This implies that there are points at every neighborhood of  $x_0$  such that  $\tilde{\psi}(x) > v(x)$ . Let

$$w(x) := \begin{cases} \max\{v, \tilde{\psi}\}(x), & \text{if } x \in \overline{B}(x_0, \frac{r}{2}), \\ v(x), & \text{otherwise.} \end{cases}$$

By Lemma 3.19,  $w$  a subsolution of (3.5). Consequently, we have

$$v, w \in \mathcal{S}.$$

However,  $w > v$  at some points, which is a contradiction. Therefore,  $v_*$  is a viscosity supersolution of (3.5). Finally, observe that

$$\liminf_* \underline{u}(x) \leq \liminf_* v(x) \leq \limsup^* v(x) \leq \limsup^* \bar{u}(x), \quad \forall x \in \partial\Omega,$$

implies that  $v(x) = \ell(x)$  on  $\partial\Omega$ . In the end, by Theorem 3.8,  $v$  is continuous, bounded, and is the unique viscosity solution to equation (3.5).  $\square$

**3.3 Examples.** We give hereafter some examples showing the degree of generality of this new setting. The first two examples treat the Hamilton-Jacobi-Bellman equation and the Hamilton-Jacobi-Isaacs equation in the case where the state space is  $\mathbb{R}^N$ , with the change being that we use different sets of test functions from the classical theory. The remaining examples are taken in more general CAT(0) spaces.

**Example 3.22.** For  $(X, d) = (\mathbb{R}^N, d_{\mathbb{R}^N})$ , with  $d_{\mathbb{R}^N}$  the Euclidean distance, consider the Hamiltonian

$$H(u(x), x, D_x u) := \gamma u(x) + \sup_{\alpha \in \mathcal{A}} \{-D_x u \cdot f(x, \alpha) + b(x, \alpha)\}, \quad x \in \mathbb{R}^N,$$

where  $\gamma > 0$ ,  $\mathcal{A}$  is a compact metric space and  $f : \mathbb{R}^N \times \mathcal{A} \rightarrow \mathbb{R}^N$  is a Lipschitz bounded function. The function  $b : \mathbb{R}^N \times \mathcal{A} \rightarrow \mathbb{R}$  is a Lipschitz bounded function. We consider

$$(3.6a) \quad \mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_- = \{\text{Locally Lipschitz and locally semiconvex functions of } \mathbb{R}^N\},$$

$$(3.6b) \quad \mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_+ = \{\text{Locally Lipschitz and locally semiconcave functions of } \mathbb{R}^N\}.$$

$\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-$  and  $\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_+$  satisfy all the requirements of Definition 3.2. Furthermore, it is straightforward to check that the Hamiltonian  $H$  satisfies Hypotheses 3.5, 3.6, 3.9 and 3.14.

It remains to prove that the Hamiltonian  $H$  satisfies Hypothesis 3.13. We start by Hypothesis 3.13-(i).

Let  $\phi \in \mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-$ .  $\phi$  is a locally Lipschitz and semiconvex function. Hence it is Clarke regular [12, Definition 10.12]. Therefore, the function

$$(x, v) \mapsto D_x \phi \cdot v$$

is upper semicontinuous [12, Proposition 10.2]. Consequently, the function

$$(x, \alpha) \mapsto -D_x \phi \cdot f(x, \alpha)$$

is lower semicontinuous. Finally the function

$$x \mapsto \sup_{\alpha \in \mathcal{A}} \{-D_x \phi \cdot f(x, \alpha) + b(x, \alpha)\}$$

is lower semicontinuous as the pointwise supremum of a family of lower semicontinuous functions, which implies Hypothesis 3.13-(i).

Now we turn our attention to Hypothesis 3.13-(ii). Let  $\phi \in \mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_+$ .  $\phi$  is a locally Lipschitz and

locally semiconcave function. So  $-\phi$  is a locally Lipschitz and locally semiconvex function. Hence it is Clarke regular, i.e. the function

$$(x, v) \mapsto -D_x \phi \cdot v$$

is upper semicontinuous. Consequently, the function

$$(x, \alpha) \mapsto -D_x \phi \cdot f(x, \alpha) + b(x, \alpha)$$

is also upper semicontinuous. Now, let  $x \in \mathbb{R}^N$  and  $(x_n)_n \subset \mathbb{R}^N$  be a sequence converging to  $x$ . Let  $(\alpha_n)_n \subset \mathcal{A}$  be a sequence such that

$$\forall n \in \mathbb{N}, \quad \sup_{\alpha \in \mathcal{A}} \{-D_{x_n} \phi \cdot f(x_n, \alpha) + b(x_n, \alpha)\} = -D_{x_n} \phi \cdot f(x_n, \alpha_n) + b(x_n, \alpha_n).$$

Since  $\mathcal{A}$  is a compact metric space, then we can assume, without loss of generality, that the sequence  $(\alpha_n)_n$  converges to some  $\bar{\alpha} \in \mathcal{A}$ . Finally, we have

$$\begin{aligned} \sup_{\alpha \in \mathcal{A}} \{-D_x \phi \cdot f(x, \alpha) + b(x, \alpha)\} &\geq -D_x \phi \cdot f(x, \bar{\alpha}) + b(x, \bar{\alpha}) \geq \\ &\limsup_{\substack{x_n \rightarrow x \\ \alpha_n \rightarrow \bar{\alpha}}} -D_{x_n} \phi \cdot f(x_n, \alpha_n) + b(x_n, \alpha_n) \geq \limsup_{x_n \rightarrow x} \sup_{\alpha \in \mathcal{A}} \{-D_{x_n} \phi \cdot f(x_n, \alpha) + b(x_n, \alpha)\}. \end{aligned}$$

This implies Hypothesis 3.13-(ii).

On the other hand,  $\underline{u}(x) = -C$ ,  $\bar{u}(x) = C$  are a bounded upper semicontinuous subsolution and a bounded lower semicontinuous supersolution respectively for  $C > 0$  big enough. Consequently, Theorem 3.21 applies and there exists a unique continuous and bounded viscosity solution to equation (3.5), with the Hamiltonian defined above.

**Example 3.23.** In the previous example, we considered a Bellman-type Hamiltonian. In the classical theory of viscosity theory in Euclidean spaces, the Bellman Hamiltonian depends on the gradient and enjoys a convexity structure. In our setting, the Hamiltonian depends on the differential, and the convexity of the Hamiltonian is never required. To support further this remark, we consider now a Hamiltonian in Isaacs form. We will show that the assumptions are satisfied for this Hamiltonian in a similar way it is for Bellman's Hamiltonian.

For  $(X, d) = (\mathbb{R}^N, d_{\mathbb{R}^N})$ , with  $d_{\mathbb{R}^N}$  the Euclidean distance, consider the Hamiltonian

$$H(u(x), x, D_x u) := \gamma u(x) + \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \{-D_x u \cdot f(x, \alpha, \beta) + b(x, \alpha, \beta)\}, \quad x \in \mathbb{R}^N,$$

where  $\gamma > 0$ ,  $\mathcal{A}, \mathcal{B}$  are compact metric spaces and  $f : \mathbb{R}^N \times \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}^N$  is a Lipschitz bounded function. The function  $b : \mathbb{R}^N \times \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}$  is a Lipschitz bounded function. In this example, we consider again the same sets of test functions defined in (3.6).

It is straightforward to check that the Hamiltonian  $H$  satisfies Hypotheses 3.5, 3.6, 3.9 and 3.14. It remains to prove that the Hamiltonian  $H$  satisfies Hypotheses 3.13. We start by Hypothesis 3.13-(i). Let  $\phi \in \mathcal{TETS}_-$  a locally Lipschitz and semiconvex function. Hence  $\phi$  is Clarke regular [12, Proposition 10.2 and Definition 10.12]. As we have seen from Example 3.22, we have

$$(x, \alpha) \mapsto \sup_{\beta \in \mathcal{B}} \{-D_x \phi \cdot f(x, \alpha, \beta) + b(x, \alpha, \beta)\},$$

is lower semicontinuous. Now, let  $(x_n)_n \subset \mathbb{R}^N$  be a sequence converging to  $x \in \mathbb{R}^N$ . Let  $(\alpha_n)_n \subset \mathcal{A}$  be a sequence such that for all  $n \in \mathbb{N}$  we have

$$\inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \{-D_{x_n} \phi \cdot f(x_n, \alpha, \beta) + b(x_n, \alpha, \beta)\} = \sup_{\beta \in \mathcal{B}} -D_{x_n} \phi \cdot f(x_n, \alpha_n, \beta) + b(x_n, \alpha_n, \beta).$$

Since  $\mathcal{A}$  is a compact metric space, then we can assume, without loss of generality, that the sequence  $(\alpha_n)_n$  converges to some  $\bar{\alpha} \in \mathcal{A}$ . Finally, we have

$$\begin{aligned} \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \{-D_x \phi \cdot f(x, \alpha) + b(x, \alpha)\} &\leq \sup_{\beta \in \mathcal{B}} \{-D_x \phi \cdot f(x, \bar{\alpha}, \beta) + b(x, \bar{\alpha}, \beta)\} \\ &\leq \liminf_{\substack{x_n \rightarrow x \\ \alpha_n \rightarrow \bar{\alpha}}} \sup_{\beta \in \mathcal{B}} \{-D_{x_n} \phi \cdot f(x_n, \alpha_n, \beta) + b(x_n, \alpha_n, \beta)\} \\ &\leq \liminf_{x_n \rightarrow x} \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \{-D_{x_n} \phi \cdot f(x_n, \alpha, \beta) + b(x_n, \alpha, \beta)\}. \end{aligned}$$

This implies Hypothesis 3.13-(i).

Next, we turn our attention to Hypothesis 3.13-(ii). Let  $\phi \in \mathcal{TETS}_+$ .  $\phi$  is a locally Lipschitz and semiconcave function. Therefore, from Example 3.22, we have

$$x \mapsto \sup_{\beta \in \mathcal{B}} \{-D_x \phi \cdot f(x, \alpha, \beta) + b(x, \alpha, \beta)\},$$

is upper semicontinuous. Hence, the function

$$x \mapsto \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} -D_x \phi \cdot f(x, \alpha, \beta),$$

is also upper semicontinuous since it is the pointwise infimum of a family of upper semicontinuous functions, which implies Hypothesis 3.13-(ii).

On the other hand,  $\underline{u}(x) = -C$  and  $\bar{u}(x) = C$  are a bounded upper semicontinuous subsolution and a bounded lower semicontinuous supersolution respectively for  $C > 0$  big enough. Consequently, Theorem 3.21 applies and there exists a unique continuous viscosity solution to equation (3.5).

*Remark 3.24.* In the above two examples, the only change we made from the classical theory of viscosity solutions, was to change the sets of test functions in  $\mathbb{R}^N$ . The interest here is limited, as we could also have chosen the test functions to be twice continuously differentiable functions for both the supersolution and the subsolution in the current setting as in the classical theory of viscosity. Indeed, a result due to Alexandrov [1] shows that twice continuously differentiable functions are locally DC functions. Actually they are locally both semiconvex and semiconcave functions. In other words, twice continuously differentiable functions constitute a subset of the intersection between the sets of locally semiconvex and semiconcave functions in  $\mathbb{R}^N$ . So the present setting and the classical theory of viscosity coincide in  $\mathbb{R}^N$ . Therefore, The present setting can be applied to more general Hamilton-Jacobi equations defined in  $\mathbb{R}^N$ , by taking test functions that are twice continuously differentiable functions to test both the supersolutions and the subsolutions.

**Example 3.25** (Eikonal-type equation in proper geodesically extendable CAT(0) spaces). As mentioned in Section 1, Eikonal equations have been studied, in the literature, in very general metric spaces. We describe here a stationary Eikonal-type equation in a CAT(0) setting. To prove the existence and uniqueness of the viscosity solution in the sense of Definition 3.4, we need first to show that the Hamiltonian function satisfies the hypotheses 3.5, 3.6, 3.9, 3.13 and 3.14. As we will see hereafter, most Hypotheses are satisfied in any proper CAT(0) space. However, Hypothesis 3.13-(ii) requires an additional assumption on the geodesic extendability of the space (see Remark 2.1).

Let  $(X, d)$  be a proper, *geodesically extendable* CAT(0) space (see Remark 2.1 for the precise definition of *geodesically extendable*). Consider the Hamiltonian

$$(3.7) \quad H(u(x), x, D_x u) := \gamma u(x) + \sup_{\substack{v \in T_x X \\ |v|_x = 1}} \{-D_x u \cdot v\} - b(x), \quad x \in X,$$

where  $\gamma > 0$  and  $b : X \rightarrow \mathbb{R}$  is a Lipschitz function of constant  $Lip(b)$  and bounded. We consider

$$(3.8a) \quad \mathcal{TEST}_- = \{ \text{Locally Lipschitz and locally semiconvex functions of } X \},$$

$$(3.8b) \quad \mathcal{TEST}_+ = \{ \text{Locally Lipschitz and locally semiconcave functions of } X \}.$$

The sets  $\mathcal{TEST}_-$  and  $\mathcal{TEST}_+$  satisfy all the requirements of Definition 3.2.

First, we prove that the Hamiltonian defined in (3.7) verifies Hypothesis 3.5. Let  $\alpha > 0$ ,  $r \in \mathbb{R}$  and  $x, y \in X$ . By Proposition 2.19, it comes that

$$\begin{aligned} H(r, x, -D_x(\alpha d^2(\cdot, y))) - H(r, y, D_y(\alpha d^2(x, \cdot))) = \\ d(x, y) \left( \sup_{\substack{v \in T_x X \\ |v|_x=1}} \{-2\alpha \langle v, \uparrow_x^y \rangle\} - \sup_{\substack{v \in T_y X \\ |v|_y=1}} \{2\alpha \langle v, \uparrow_y^x \rangle\} \right) + b(y) - b(x). \end{aligned}$$

By inequality (2.7c), we have

$$\sup_{\substack{v \in T_x X \\ |v|_x=1}} \{-2\alpha \langle v, \uparrow_x^y \rangle\} \leq \sup_{\substack{v \in T_x X \\ |v|_x=1}} \{2\alpha |v|_x | \uparrow_x^y | \} = 2\alpha,$$

and

$$\sup_{\substack{v \in T_y X \\ |v|_y=1}} \{2\alpha \langle v, \uparrow_y^x \rangle\} = 2\alpha, \quad \text{reached at } v = \uparrow_y^x.$$

Hence, we get

$$H(r, x, -D_x(\alpha d^2(\cdot, y))) - H(r, y, D_y(\alpha d^2(x, \cdot))) \leq b(y) - b(x) \leq Lip(b)d(x, y),$$

which implies the result.

Hypotheses 3.6, 3.9 and 3.14 are straightforward. It remains to prove that the Hamiltonian verifies Hypotheses 3.13.

We start by Hypothesis 3.13-(i). The proof is inspired from [29, Lemma 1.3.4]. Let  $\phi \in \mathcal{TEST}_-$ . So  $\phi$  is a locally Lipschitz and locally semiconvex function which implies that  $\psi := -\phi$  is a locally Lipschitz and locally semiconcave function. Let  $x \in X$ . Suppose that  $\psi$  is  $2\lambda$ -concave in a neighborhood of  $x$ , for some  $\lambda \in \mathbb{R}$ . Let  $\varepsilon > 0$  and let  $y \in X$  near  $x$  such that

$$y \neq x, \quad |\lambda|d(x, y) < \varepsilon, \quad \text{and} \quad \frac{\psi(y) - \psi(x)}{d(x, y)} \geq \sup_{\substack{v \in T_x X \\ |v|_x=1}} \{D_x \psi \cdot v\} - \varepsilon.$$

Let  $(x_n)_n$  and  $(y_n)_n$  be two sequences converging to  $x$  and  $y$  respectively. Let  $[0, d(x_n, y_n)] \ni t \mapsto G_t^{x_n, y_n}$  be the unit speed geodesic connecting  $x_n$  and  $y_n$ . By definition, the  $2\lambda$ -concavity of  $\psi$  implies that the real-to-real function

$$[0, d(x_n, y_n)] \ni t \mapsto \psi(G_t^{x_n, y_n}) - \lambda t^2$$

is concave. Therefore, the incremental ratio

$$(0, d(x_n, y_n)] \ni t \mapsto \frac{\psi(G_t^{x_n, y_n}) - \lambda t^2 - \psi(x_n)}{t}$$

is non-increasing. Hence, the  $2\lambda$ -concavity of  $\psi$  gives

$$D_{x_n} \psi \cdot \uparrow_{x_n}^{y_n} \geq \frac{\psi(y_n) - \psi(x_n) - \lambda d^2(x_n, y_n)}{d(x_n, y_n)} \geq \frac{\psi(y) - \psi(x)}{d(x, y)} - \varepsilon,$$

where the last inequality is obtained when  $n$  is large enough. Hence we get

$$\sup_{\substack{v \in T_{x_n} X \\ |v|_{x_n}=1}} \{D_{x_n} \psi \cdot v\} \geq D_{x_n} \psi \cdot \uparrow_{x_n}^{y_n} \geq \sup_{\substack{v \in T_x X \\ |v|_x=1}} \{D_x \psi \cdot v\} - 2\varepsilon.$$

By taking the infimum limit in the left-hand side of the last inequality, we get

$$\liminf_{\substack{x_n \rightarrow x \in X \\ r_n \rightarrow r \in \mathbb{R}}} \gamma r_n + \sup_{\substack{v \in T_{x_n} X \\ |v|_{x_n}=1}} \{-D_{x_n} \phi \cdot v\} - b(x_n) \geq \gamma r + \sup_{\substack{v \in T_x X \\ |v|_x=1}} \{-D_x \phi \cdot v\} - b(x) - 2\varepsilon.$$

Since  $\varepsilon$  is arbitrary, we get that Hypothesis 3.13-(i) is verified.

Now, we prove Hypothesis 3.13-(ii). Let  $\phi \in \mathcal{TES}T_+$ . So  $\phi$  is a locally Lipschitz and locally semiconcave function which implies that  $\psi := -\phi$  is a locally Lipschitz and locally semiconvex function.

Let  $x \in X$ . Suppose that  $\psi$  is  $2\lambda$ -convex around  $x$  for some  $\lambda \in \mathbb{R}$ . Let  $0 < \varepsilon < M$  be two strictly positive constants,  $(x_n)_n \subset X$  be a sequence converging to  $x$  and  $(y_n)_n \subset X$  be a sequence such that

$$d(x, y_n) \leq M, \quad \text{and} \quad \forall n \in \mathbb{N}, \quad D_{x_n} \psi \cdot \uparrow_{x_n}^{y_n} \geq \sup_{\substack{v \in T_{x_n} X \\ |v|_{x_n}=1}} \{D_{x_n} \psi \cdot v\} - \varepsilon.$$

Since  $(X, d)$  is geodesically extendable, we can also always choose  $y_n$  such that

$$d(x, y_n) \geq \varepsilon, \quad \forall n \in \mathbb{N}.$$

Since  $(y_n)_n \subset \overline{B}(x, M)$ , then we can suppose, without loss of generality, that it converges to some  $y \in X$ . Moreover  $y \neq x$  since we have  $d(x, y_n) \geq \varepsilon$  for all  $n \in \mathbb{N}$ . Let  $t \mapsto G_t^{x_n, y_n}$  be the unique unit speed geodesic between  $x_n$  and  $y_n$ . By the extendability property of  $X$ , we can extend all geodesics  $G^{x_n, y_n}$  to be defined in the same interval  $[0, K]$ , with  $K > M$  large enough. Then by Arzela–Ascoli theorem [9, Theorem 2.5.14] there exists a converging subsequence (not relabeled here) of the sequence of curves  $(G^{x_n, y_n})_n$ . Moreover, by [9, Proposition 2.5.17], the limit curve is  $G^{x, y}$ ; the unit speed geodesic starting from  $x$  and passing through  $y$ , and defined in  $[0, K]$ . Furthermore, the  $2\lambda$ -convexity of  $\psi$  gives

$$\begin{aligned} \sup_{\substack{v \in T_{x_n} X \\ |v|_{x_n}=1}} \{D_{x_n} \psi \cdot v\} - \varepsilon &\leq D_{x_n} \psi \cdot \uparrow_{x_n}^{y_n} = D_{x_n} \psi \cdot \uparrow_{x_n}^{G_t^{x_n, y_n}} \\ &\leq \frac{\psi(G_t^{x_n, y_n}) - \psi(x_n) - \lambda d^2(x_n, G_t^{x_n, y_n})}{d(x_n, G_t^{x_n, y_n})} \leq \frac{\psi(G_t^{x, y}) - \psi(x)}{d(x, G_t^{x, y})} + \varepsilon, \end{aligned}$$

where the last inequality holds when taking  $t$  small enough and  $n$  big enough. Hence we get

$$\begin{aligned} \limsup_{x_n \rightarrow x} \sup_{\substack{v \in T_{x_n} X \\ |v|_{x_n}=1}} \{D_{x_n} \psi \cdot v\} &\leq \lim_{t \downarrow 0} \frac{\psi(G_t^{x, y}) - \psi(x)}{d(x, G_t^{x, y})} + 2\varepsilon = \lim_{t \downarrow 0} \frac{\psi(G_t^{x, y}) - \psi(x)}{t} + 2\varepsilon \\ &\leq \sup_{\substack{v \in T_x X \\ |v|_x=1}} \{D_x \psi \cdot v\} + 2\varepsilon. \end{aligned}$$

Finally, we get

$$\limsup_{\substack{x_n \rightarrow x \in X \\ r_n \rightarrow r \in \mathbb{R}}} \gamma r_n + \sup_{\substack{v \in T_{x_n} X \\ |v|_{x_n}=1}} \{D_{x_n} \psi \cdot v\} - b(x_n) \leq \gamma r + \sup_{\substack{v \in T_x X \\ |v|_x=1}} \{D_x \psi \cdot v\} - b(x) + 2\varepsilon,$$

which is equivalent to

$$\limsup_{\substack{x_n \rightarrow x \in X \\ r_n \rightarrow r \in \mathbb{R}}} \gamma r_n + \sup_{\substack{v \in T_{x_n} X \\ |v|_{x_n} = 1}} \{-D_{x_n} \phi \cdot v\} - b(x_n) \leq \gamma r + \sup_{\substack{v \in T_x X \\ |v|_x = 1}} \{-D_x \phi \cdot v\} - b(x) + 2\varepsilon.$$

Since  $\varepsilon$  is arbitrary, we get the result.

In summary, the Hamiltonian  $H$  verifies all Hypotheses 3.5, 3.6, 3.9, 3.13 and 3.14. Furthermore, the functions  $\underline{u}(x) = -C$ ,  $\bar{u}(x) = C$  are bounded upper semicontinuous subsolution and bounded lower semicontinuous supersolution respectively for  $C > 0$  big enough. Hence by Theorem 3.21, there exists a unique bounded and continuous viscosity solution to the Hamilton-Jacobi equation

$$H(u(x), x, D_x u) = 0, \quad \forall x \in X,$$

in the sense of Definition 3.4 and with the test functions given in (3.8).

**Example 3.26.** Let us analyze the Eikonal equation in a particular proper, geodesically extendable CAT(0) space  $(X, d)$  given in Example 2.9. The space  $(X, d)$  is obtained by gluing the two proper CAT(0) spaces

$$\begin{cases} X_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}, \\ X_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0\}, \end{cases}$$

along the closed convex subset  $A := \{0\}$ . The distance  $d$  is given in (2.4). The tangent cone at a point  $x \in X$  is:

$$T_x X = \begin{cases} X_1 & \text{if } x \in X_1 \setminus A, \\ X_2 & \text{if } x \in X_2 \setminus A, \\ X & \text{if } x \in A. \end{cases}$$

We will consider the Eikonal equation as in Example 3.25. Let  $\gamma > 0$  and  $b : X \rightarrow \mathbb{R}$  be a Lipschitz bounded function. As discussed in Example 3.25, the Eikonal equation

$$(3.9) \quad \gamma u(x) + \sup_{\substack{v \in T_x X \\ |v|_x = 1}} \{-D_x u \cdot v\} - b(x) = 0, \quad x \in X,$$

admits a unique solution of a viscosity solution associated with the sets of test functions defined in (3.8). Now, we consider smaller sets of test functions defined as follows.

$$(3.10a) \quad \mathcal{TETS}_- = \{ \phi : X \rightarrow \mathbb{R} : \text{Locally Lipschitz and locally semiconvex functions of } X \text{ such that } \phi|_{X_i \setminus A} \text{ is twice continuously differentiable with } i = 1, 2 \},$$

$$(3.10b) \quad \mathcal{TETS}_+ = \{ \phi : X \rightarrow \mathbb{R} : \text{Locally Lipschitz and locally semiconcave functions of } X \text{ such that } \phi|_{X_i \setminus A} \text{ is twice continuously differentiable, with } i = 1, 2 \}.$$

One can check that  $\mathcal{TETS}_-$  and  $\mathcal{TETS}_+$  satisfy all the requirements of Definition 3.2. Furthermore, it is straightforward to see that the unique viscosity solution of (3.9) with the sets of test functions defined in (3.8) is also the unique viscosity solution with the test functions (3.10). Taking into account the structure of the set  $X$ , the expression of the tangent cone at each point  $x \in X$ , and the structure of the sets of test functions, equation (3.9) is equivalent to

$$\begin{cases} \gamma u(x) + |\nabla_x u| - b(x) = 0, & \text{if } x \in X_1 \setminus A, \\ \gamma u(x) + |\nabla_x u| - b(x) = 0, & \text{if } x \in X_2 \setminus A, \\ \gamma u(x) + \sup_{\substack{v \in X \\ |v| = 1}} \{-D_x u \cdot v\} - b(x) = 0, & \text{if } x \in A, \end{cases}$$

where  $\nabla_x u$  denotes the classical gradient of  $u$  at  $x \in X \setminus \{A\}$ .

This example shows that Hamilton-Jacobi equations can be considered in more general networks than the ones considered so far in the literature (see [?, ?, 11, 24, 27] and the references therein). Indeed, in the existing literature on this topic, the network is a collection of manifolds of the same dimension  $d$  glued on a manifold (called junction or interface) of a lower dimension  $d - 1$ . Here, we claim that our setting allows more general situations where the network may be constituted by manifolds of different dimensions.

Finally, let us stress on the fact that the property of *geodesically extendable* is satisfied by all the examples presented in Exemples 2.9, 2.10 and 2.11. Therefore, the viscosity solution associated with the set of the test functions (3.8) exists and is unique for different structures of the proper CAT(0) geodesically extendable space  $X$ .

**Example 3.27** (Nonconvex Hamiltonian). Let  $(X, d)$  be a proper, geodesically extendable CAT(0) space. An interesting remark, is that in our setting, nonconvex Hamiltonian functions can be considered in the same framework as the convex functions. A simple example is the Hamiltonian

$$H(u(x), x, D_x u) := \min \left\{ \gamma u(x) + \sup_{\substack{v \in T_x X \\ |v|_x = 1}} \{-D_x u \cdot v\} - b_1(x), u(x) - b_2(x) \right\}, \quad x \in X,$$

where  $\gamma > 0$  and  $b_1 : X \rightarrow \mathbb{R}$  and  $b_2 : X \rightarrow \mathbb{R}$  are Lipschitz and bounded functions. From Example 3.25, the Hamiltonian

$$H_1(u(x), x, D_x u) := \gamma u(x) + \sup_{\substack{v \in T_x X \\ |v|_x = 1}} \{-D_x u \cdot v\} - b_1(x),$$

verifies all Hypotheses 3.5, 3.6, 3.9, 3.13 and 3.14 with the test function defined in (3.8). Furthermore, it is easy to check that the Hamiltonian

$$H_2(u(x), x, D_x u) := u(x) - b_2(x),$$

also verifies all Hypotheses 3.5, 3.6, 3.9, 3.13 and 3.14 with the same sets of test functions. Consequently, the Hamiltonian  $H$  verifies all the mentioned Hypotheses as well since it is in the form of a minimum of two Hamiltonians that verify the same Hypotheses. Furthermore, the functions  $\underline{u}(x) = -C$ ,  $\bar{u}(x) = C$  are bounded upper semicontinuous subsolution and bounded lower semicontinuous supersolution respectively for  $C > 0$  big enough. Hence, by Theorem 3.21, there exists a unique bounded and continuous viscosity solution to the equation

$$H(u(x), x, D_x u) = 0, \quad \forall x \in X.$$

*Remark 3.28.* The above example shows that by taking test functions that are piecewise twice continuously differentiable and in the class of DC functions on the whole space, then we recover the same setting considered in [24, 27, 31, 11].

**Example 3.29.** In this examples, we treat an example of a Hamilton-Jacobi equation more general than the Eikonal-type equations. Now, let us take the proper, geodesically extendable CAT(0) space  $(X, d)$  given in Example 2.9. The space  $(X, d)$  is obtained by gluing two proper CAT(0) spaces

$$\begin{cases} X_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}, \\ X_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0\}, \end{cases}$$



along the closed convex subset  $A := \{0\}$ . We define the function  $g : X \rightarrow \mathbb{R}$  the following way

$$\forall x \in X, \quad g(x) = \begin{cases} 1 + |x - e_3|^2 e^{-|x - e_3|^2}, & \text{if } x \in X_2 \setminus B(A, 1), \\ 1, & \text{otherwise.} \end{cases}$$

Consider the following Hamiltonian  $H : \mathbb{R} \times DC_1(TX) \rightarrow \mathbb{R}$

$$H(u(x), x, D_x u) = \gamma u(x) + \sup_{\substack{v \in T_x X \\ |v|_x = 1}} \{-D_x u \bullet (g(x)v)\} - b(x),$$

where  $\gamma > 0$ , and  $b : X \rightarrow \mathbb{R}$  is a Lipschitz bounded function.

Notice that if  $x \notin B(A, 1)$ , then the above Hamiltonian is equal to the Hamiltonian defined in Example 3.25. Furthermore, on  $X_1 \setminus A$  or  $X_2 \setminus A$ , the Hamiltonian can be seen as a Hamiltonian defined in a Euclidean space and is of the form given in Example 3.22. Consequently, with similar arguments as in Examples 3.25 and 3.22, the Hamiltonian  $H$  verifies all Hypotheses 3.5, 3.6, 3.9, 3.13 and 3.14 by taking the same sets of test functions as in (3.8). Furthermore, the functions  $\underline{u}(x) = -C$ ,  $\bar{u}(x) = C$  are bounded upper semicontinuous subsolution and bounded lower semicontinuous supersolution respectively for  $C > 0$  big enough. Hence by Theorem 3.21, there exists a unique bounded and continuous viscosity solution to the Hamilton-Jacobi equation

$$H(u(x), x, D_x u) = 0, \quad \forall x \in X.$$

*Remark 3.30.* The above example is of fundamental importance. Indeed the Hamilton-Jacobi equation of this example cannot be treated by any notion of viscosity solutions developed in [19, 3, 20, 28, 21, ?, 11, 27]. This demonstrates that the current setting is different and sometimes more general than what exists currently in the literature.

#### 4 TIME-DEPENDENT HAMILTON-JACOBI EQUATIONS IN PROPER CAT(0) SPACES

In this section, we discuss time-dependent Hamilton-Jacobi equations in a proper CAT(0) space  $(X, d)$ . First, notice that by means of Lemma 2.5, the product space  $[0, +\infty) \times X$  is also a CAT(0) space. One could consider the time variable as being one of the state variables and use the setting developed for the stationary case. Although it is possible to do it, we choose to treat the time-dependent case separately, as it has its own specificities.

Let  $H : DC_1(TX) \rightarrow \mathbb{R}$  be a Hamiltonian and  $\ell : X \rightarrow \mathbb{R}$  be a bounded and continuous function. We consider the following Hamilton-Jacobi equation:

$$(4.1) \quad \begin{cases} \partial_t u + H(x, D_x u) = 0, & \forall (t, x) \in (0, +\infty) \times X, \\ u(0, x) = \ell(x), & x \in X, \end{cases}$$

where  $u : [0, +\infty) \times X$  is a Lipschitz and DC function. The term  $\partial_t u$  is the usual right derivative with respect to time, i.e.

$$\partial_t u = \lim_{r \downarrow 0} \frac{u(t+r, x) - u(t, x)}{r}.$$

Notice that in this section, for simplicity, we consider that the Hamiltonian is defined on  $DC_1(TX)$ , however all the results of this section can be extended to the case when the Hamiltonian depends also on the value of the solution  $u(x)$ . Let  $C^2((0, +\infty))$  denotes the space of twice continuously differentiable functions of  $(0, +\infty)$ . We will take test functions that are  $C^2((0, +\infty))$  with respect to the time variable and in the class of DC functions with respect to the space variable and verify Definition 3.2.

**Definition 4.1.** Let  $\mathcal{TEST}_-^x$  and  $\mathcal{TEST}_+^x$  be two subsets of  $\text{DC}_{\text{lip}}(X)$  as in Definition 3.2. Let  $\mathcal{TEST}_-$  and  $\mathcal{TEST}_+$  be two subsets of  $\text{DC}_{\text{lip}}((0, \infty) \times X)$  such that:

$$\mathcal{TEST}_- := \{(t, x) \mapsto \phi_1(t) + \phi_2(x) : \phi_1 \in C^2((0, +\infty)) \text{ and } \phi_2 \in \mathcal{TEST}_-^x\},$$

and

$$\mathcal{TEST}_+ := \{(t, x) \mapsto \phi_1(t) + \phi_2(x) : \phi_1 \in C^2((0, +\infty)) \text{ and } \phi_2 \in \mathcal{TEST}_+^x\}.$$

*Remark 4.2.* In the above definition, we considered test functions that are twice continuously differentiable with respect to time since the time variable belongs to the open set  $(0, +\infty)$  of the Euclidean space  $\mathbb{R}$ .

**4.1 Comparison principle.** Next, we prove a comparison result in the time-dependent case. Since the Hamiltonian in (4.1) does not depend on  $u(x)$ , there is no need to assume Hypothesis 3.6 on the Hamiltonian. By assuming only Hypotheses 3.5 and 3.9, we can prove the comparison principle for the time-dependent case, as the following theorem shows.

**Theorem 4.3.** *Assume  $H$  satisfies Hypotheses 3.5 and 3.9. Let  $u : [0, +\infty) \times X \rightarrow \mathbb{R}$  be a bounded from above upper semicontinuous subsolution of (4.1), and  $v : [0, +\infty) \times X \rightarrow \mathbb{R}$  a bounded from below lower semicontinuous supersolution of equation (4.1). Then it holds:*

$$\sup_{[0, +\infty) \times X} (u - v)_+ \leq \sup_{\{0\} \times X} (u - v)_+,$$

where  $(r)_+ = \max(r, 0)$ .

*Proof.* Without loss of generality, we can suppose that  $\sup_{\{0\} \times X} (u - v)_+ = 0$ .

Let  $M := \sup_{[0, +\infty) \times X} (u(t, x) - v(t, x))$ . It suffices to prove that  $M \leq 0$ . Assume by contradiction that  $M > 0$ . Let  $\lambda > 0$  sufficiently small so that

$$\sup_{[0, +\infty) \times X} (u(t, x) - v(t, x) - \lambda t) > 0.$$

Let  $(t_0, x_0) \in [0, +\infty) \times X$  be such that

$$u(t_0, x_0) - v(t_0, x_0) - \lambda t_0 > 0.$$

Let  $\varepsilon \in (0, M)$ . For every  $\alpha > 0$ , define  $\psi_\alpha : [0, +\infty)^2 \times X^2 \rightarrow \mathbb{R}$  as

$$\psi_\alpha(t, s, x, y) = u(t, x) - v(s, y) - \frac{\lambda}{2}(t + s) - \frac{\varepsilon}{2}(d^2(x, x_0) + d^2(y, x_0)) - \frac{\alpha}{2}|t - s|^2 - \frac{\alpha}{2}d^2(x, y).$$

It is clear that  $\psi_\alpha$  is upper semicontinuous and bounded from above. We denote by  $M_\alpha := \sup \psi_\alpha$ , where the supremum is taken over  $[0, +\infty)^2 \times X^2$ . Furthermore, we have

$$0 < \psi_\alpha(t_0, t_0, x_0, x_0),$$

and for all  $x, y \notin B\left(x_0, \sqrt{2 \frac{|\sup(u)| + |\sup(-v)|}{\varepsilon}}\right)$  or  $t, s \geq 2 \frac{|\sup(u)| + |\sup(-v)|}{\lambda}$  we have

$$\psi_\alpha(t, s, x, y) \leq 0.$$

Hence, the supremum of  $\psi_\alpha$  is reached in a compact set independent of  $\alpha$ . Let  $(t_\alpha, s_\alpha, x_\alpha, y_\alpha)$  be such that  $M_\alpha = \psi_\alpha(t_\alpha, s_\alpha, x_\alpha, y_\alpha)$ . We have

$$\lim_{\alpha \rightarrow +\infty} (M_\alpha - \psi_\alpha(t_\alpha, s_\alpha, x_\alpha, y_\alpha)) = 0$$

and

$$-\infty < u(t_0, x_0) - v(t_0, x_0) - \lambda t_0 \leq M_\alpha \leq \sup_{[0, +\infty) \times X} (u) + \sup_{[0, +\infty) \times X} (-v) < +\infty.$$

Since  $(t_\alpha, s_\alpha, x_\alpha, y_\alpha)$  is in a compact set, we take a subsequence  $(t_{\alpha_n}, s_{\alpha_n}, x_{\alpha_n}, y_{\alpha_n})$  such that  $(t_{\alpha_n}, s_{\alpha_n}, x_{\alpha_n}, y_{\alpha_n})$  converges as  $\alpha_n \rightarrow +\infty$  and

$$\lim_{\alpha_n \rightarrow +\infty} (M_{\alpha_n} - \psi_{\alpha_n}(t_{\alpha_n}, s_{\alpha_n}, x_{\alpha_n}, y_{\alpha_n})) = 0, \quad \text{and} \quad -\infty < \lim_{\alpha_n \rightarrow +\infty} M_{\alpha_n} < +\infty.$$

Therefore, we can apply Lemma 3.7 via the correspondences

$$Z = \mathcal{O} = [0, +\infty)^2 \times X^2, \quad \Phi(z) = u(t, x) - v(s, y) - \frac{\lambda}{2}(t + s) - \frac{\varepsilon}{2}(d^2(x, x_0) + d^2(y, x_0)),$$

$$\Psi(z) = \frac{1}{2}|t - s|^2 + \frac{1}{2}d^2(x, y),$$

and we get

$$\begin{cases} (i) & \lim_{\alpha_n \rightarrow +\infty} \frac{\alpha_n}{2}d^2(x_{\alpha_n}, y_{\alpha_n}) + \frac{\alpha_n}{2}|t_{\alpha_n} - s_{\alpha_n}|^2 = 0, \\ & \text{and } x_{\alpha_n}, y_{\alpha_n} \rightarrow \hat{x} \in X, t_{\alpha_n}, s_{\alpha_n} \rightarrow \hat{t} \in [0, +\infty), \\ (ii) & \lim_{\alpha_n \rightarrow +\infty} M_{\alpha_n} \geq u(t_0, x_0) - v(t_0, x_0) - \lambda t_0 > 0. \end{cases}$$

Moreover, we have

$$(4.2) \quad 0 < u(t_0, x_0) - v(t_0, x_0) - \lambda t_0 \leq u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{x}) - \lambda \hat{t} - \varepsilon d^2(\hat{x}, x_0).$$

This implies

$$\varepsilon d^2(\hat{x}, x_0) \leq M \implies \varepsilon d(\hat{x}, x_0) \leq \sqrt{M\varepsilon}.$$

On the other hand, notice that  $\hat{t} \neq 0$  since  $u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{x}) > 0$ . It follows that for  $\alpha_n$  big enough we have  $t_{\alpha_n} \neq 0$ . Furthermore, we have

$$(4.3) \quad -\frac{\lambda}{2} + \alpha_n(t_{\alpha_n} - s_{\alpha_n}) + H\left(y_{\alpha_n}, D_{y_{\alpha_n}}\left(-\frac{\alpha_n}{2}d^2(x_{\alpha_n}, \cdot) - \frac{\varepsilon}{2}d^2(\cdot, x_\varepsilon)\right)\right) \geq 0 \geq$$

$$\frac{\lambda}{2} + \alpha_n(t_{\alpha_n} - s_{\alpha_n}) + H\left(x_{\alpha_n}, D_{x_{\alpha_n}}\left(\frac{\alpha_n}{2}d^2(\cdot, y_{\alpha_n}) + \frac{\varepsilon}{2}d^2(\cdot, x_\varepsilon)\right)\right).$$

Hence, it follows from Hypotheses 3.5 and 3.9 and the inequality above

$$\begin{aligned} \lambda &\stackrel{(4.3)}{\leq} H\left(y_{\alpha_n}, -D_{y_{\alpha_n}}\left(\frac{\alpha_n}{2}d^2(x_{\alpha_n}, \cdot) + \frac{\varepsilon}{2}d^2(\cdot, x_\varepsilon)\right)\right) - H\left(x_{\alpha_n}, D_{x_{\alpha_n}}\left(\frac{\alpha_n}{2}d^2(\cdot, y_{\alpha_n}) + \frac{\varepsilon}{2}d^2(\cdot, x_\varepsilon)\right)\right) \\ &\stackrel{3.5, 3.9}{\leq} K_{db}d(x_{\alpha_n}, y_{\alpha_n})\left(1 + \frac{\alpha_n}{2}d(x_{\alpha_n}, y_{\alpha_n})\right) + K_L\varepsilon(d(y_{\alpha_n}, x_0) + d(x_{\alpha_n}, x_0)). \end{aligned}$$

By letting  $\alpha_n \rightarrow +\infty$ , we get

$$\lambda \leq 2K_L\sqrt{M\varepsilon}.$$

The last inequality is valid for any  $0 < \varepsilon < M$ , a contradiction.  $\square$

**4.2 Perron's method.** Next, we prove Perron's method on the product space  $[0, +\infty) \times X$ . First, notice that if we assume that the Hamiltonian  $H$  verifies Hypothesis 3.13, then the same hypothesis is verified by the full Hamiltonian

$$\partial_t u + H(x, D_x u),$$

in the product space  $[0, +\infty) \times X$  since it is a CAT(0) space and the test functions chosen in Definition 4.1 are continuously differentiable with respect to the time variable.

**Theorem 4.4.** *Let  $\Omega = (0, +\infty) \times X$  and set  $\partial\Omega = \{0\} \times X$ . Assume  $H$  satisfies Hypotheses 3.5 and 3.9. Suppose that there exist  $\underline{u} : \overline{\Omega} \rightarrow \mathbb{R}$  a locally bounded and bounded from above upper semicontinuous subsolution of (4.1) and  $\overline{u} : \overline{\Omega} \rightarrow \mathbb{R}$  a locally bounded and bounded from below lower semicontinuous supersolution of (4.1) such that*

$$\liminf_* \underline{u}(t, x) \geq \ell(x) \geq \limsup^* \overline{u}(t, x), \quad \forall (t, x) \in \partial\Omega.$$

*Then there exists a unique continuous viscosity solution of (4.1).*

*Proof.* We define the set

$$\mathcal{S} = \{h : \overline{\Omega} \rightarrow \mathbb{R} : \underline{u} \leq h \leq \overline{u} \text{ and } h \text{ is a subsolution of (4.1)}\}.$$

The set  $\mathcal{S}$  is nonempty since  $\underline{u} \in \mathcal{S}$ . For  $x \in \overline{\Omega}$ , we set

$$u(t, x) = \sup\{h(t, x), h \in \mathcal{S}\}.$$

We will show that  $v := \limsup^* \{u\}$  is the viscosity solution of (4.1). First, we show that  $v$  is a subsolution. Notice that  $v$  is obviously upper semicontinuous. Let  $\phi : \Omega \rightarrow \mathbb{R}$  be a  $\mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_-$  function such that  $v - \phi$  attains a local maximum at  $(t_0, x_0) \in \Omega$ . Without loss of generality, we can suppose that

$$v(t_0, x_0) = \phi(t_0, x_0).$$

By definition of  $v$ , there exists a sequence of points  $(t_j, x_j) \rightarrow x_0$  and a sequence of functions  $u_j \in \mathcal{S}$  such that

$$v(t_0, x_0) = \lim_{j \rightarrow \infty} u_j(t_j, x_j).$$

In particular,  $\limsup^* \{u_j\}(t_0, x_0) \geq v(t_0, x_0)$ . On the other hand, by construction we have  $v \geq \limsup^* \{u_j\}$ . Therefore, at  $(t_0, x_0)$  we have  $\limsup^* \{u_j\}(t_0, x_0) = v(t_0, x_0)$ . For the other points on a small enough neighborhood of  $(t_0, x_0)$ , we have

$$\phi \geq v \geq \limsup^* \{u_j\}.$$

Therefore, by using Lemma 3.19 on a small enough bounded open neighborhood of  $(t_0, x_0)$ , we get that  $\limsup^* \{u_j\}$  is a subsolution at  $(t_0, x_0)$ , since Hypothesis 3.13 holds. Therefore, by definition, we have

$$\partial_{t_0} \phi + H(x_0, D_{x_0} \phi) \leq 0.$$

This shows that  $v$  is a subsolution at  $(t_0, x_0)$ . Now we show that  $v_* := \liminf_* v$  is a supersolution. We argue by contradiction.

Suppose that there exists a point  $(t_0, x_0) \in \Omega$  and a function  $\psi \in \mathcal{T}\mathcal{E}\mathcal{S}\mathcal{T}_+$  such that  $v_* - \psi$  attains a local minimum at  $(t_0, x_0)$ , but

$$\partial_{t_0} \psi + H(x_0, D_{x_0} \psi) < 0.$$

By Hypothesis 3.13 and the continuity of  $t \mapsto \partial_t \psi$ , we get that the function

$$(t, x) \mapsto \partial_t \psi + H(x, D_x \psi)$$

is upper semicontinuous. Hence, on a small open neighborhood of  $(t_0, x_0)$ , denoted by  $(t_0 - r, t_0 + r) \times B(x_0, r)$  we have

$$\forall (t, x) \in (t_0 - r, t_0 + r) \times B(x_0, r), \quad \partial_t \psi + H(D_x \psi) < 0.$$

Furthermore, by Hypothesis 3.14,  $\psi$  is a strict viscosity subsolution of (4.1) on  $(t_0 - r, t_0 + r) \times B(x_0, r)$ .

Indeed, let  $\psi_- \in \mathcal{TEST}_-$  such that  $\psi - \psi_-$  attains a local maximum at  $(t, x) \in (t_0 - r, t_0 + r) \times B(x_0, r)$ . Notice that since  $\psi \in \mathcal{TEST}_+$  and  $\psi_- \in \mathcal{TEST}_-$ , then they are of the form

$$\psi(t, x) = f(t) + g(x), \quad \psi_-(t, x) = f_-(t) + g_-(x),$$

where  $f$  and  $f_-$  are twice continuously differentiable functions, and  $g$  and  $g_-$  are locally Lipschitz and locally semiconcave and semiconvex respectively. We have for all  $(s, y)$  in a small neighborhood of  $(t, x)$ :

$$f(t) + g(x) - (f_-(t) + g_-(x)) \geq f(s) + g(y) - (f_-(s) + g_-(y)).$$

It follows from the above inequality that we have for all  $s$  in a small neighborhood of  $t$ :

$$f(t) - f_-(t) \geq f(s) - f_-(s),$$

and for all  $y$  in a small neighborhood of  $x$  we get

$$g(x) - g_-(x) \geq g(y) - g_-(y).$$

This implies that

$$\partial_t f = \partial_t f_- \quad \text{and} \quad \partial_x g \leq \partial_x g_-.$$

Therefore, by Hypothesis 3.14 we get

$$\partial_t \psi_- + H(x, D_x \psi_-) \leq \partial_t \psi + H(x, D_x \psi) < 0.$$

On the other hand, without loss of generality, we can suppose that  $\psi(t_0, x_0) = v_*(t_0, x_0)$ .

Moreover, for  $\delta > 0$ ,  $\tilde{\psi} = \psi + \delta$  is also a viscosity subsolution on a small enough neighborhood  $(t_0 - r, t_0 + r) \times B(x_0, r)$ . We have  $\tilde{\psi}(t_0, x_0) > v_*(t_0, x_0)$ . This implies that there are points at every neighborhood of  $(t_0, x_0)$  such that  $\tilde{\psi}(t, x) > v(t, x)$ . Let

$$w(t, x) := \begin{cases} \max\{v, \tilde{\psi}\}(t, x), & \text{if } (t, x) \in (t_0 - \frac{r}{2}, t_0 + \frac{r}{2}) \times \bar{B}(x_0, \frac{r}{2}), \\ v(t, x), & \text{otherwise.} \end{cases}$$

By Lemma 3.19,  $w$  is a subsolution of (4.1). Consequently, we have

$$v, w \in \mathcal{S}.$$

However,  $w > v$  at some points, which is a contradiction. Therefore,  $v_*$  is a viscosity supersolution of (4.1).

Finally, observe that

$$\liminf_* \underline{u}(t, x) \leq \liminf_* v(x) \leq \limsup^* v(t, x) \leq \limsup^* \bar{u}(t, x), \quad \forall (t, x) \in \partial\Omega,$$

implies that  $v(t, x) = \ell(x)$  on  $\partial\Omega$ . In the end, by Theorem 4.3,  $v$  is continuous, bounded and is the unique viscosity solution to equation (4.1).  $\square$

*Remark 4.5.* A sufficient condition to guarantee the existence of a locally bounded and bounded from above upper semicontinuous subsolution  $\underline{u}$  and a locally bounded and bounded from below lower semicontinuous supersolution  $\bar{u}$  of (4.1), that verify all the conditions of Theorem 4.4, is to suppose the following condition on the Hamiltonian  $H$ .

*Hypothesis 4.6.* The Hamiltonian  $H$  is such that

$$X \ni x \mapsto H(x, 0_{\text{DC}_1(T_x X)}) \text{ is bounded.}$$

Indeed if Hypothesis 4.6 holds, let

$$C := \sup_{x \in X} |H(x, 0_{\text{DC}_1(T_x X)})|, \quad \text{and} \quad M := \sup_{x \in X} |\ell(x)|.$$

Then for  $(t, x) \in [0, +\infty) \times X$ , the following two functions

$$\underline{u}(t, x) := M - Ct, \quad \text{and} \quad \bar{u}(t, x) := -M + Ct$$

are respectively a locally bounded and bounded from above upper semicontinuous subsolution and a locally bounded and bounded from below lower semicontinuous supersolution of (4.1) on  $(0, +\infty) \times X$  and

$$M = \liminf_{*} \underline{u}(t, x) \geq \ell(x) \geq \limsup^{*} \bar{u}(t, x) = -M, \quad \forall (t, x) \in \{0\} \times X.$$

**Example 4.7** (time-dependent Eikonal equation in proper geodesically extendable CAT(0) spaces). Let  $(X, d)$  be a proper, geodesically extendable CAT(0) space. All spaces given in Examples, 2.9, 2.10 and 2.11 verify this condition. Consider the Hamiltonian

$$H(x, D_x u) := \sup_{\substack{v \in T_x X \\ |v|_x = 1}} \{-D_x u \cdot v\}, \quad x \in X.$$

We consider the Hamilton-Jacobi equation (4.1) with the Hamiltonian  $H$  defined above:

$$\begin{cases} \partial_t u + \sup_{\substack{v \in T_x X \\ |v|_x = 1}} \{-D_x u \cdot v\} = 0, & (t, x) \in (0, +\infty) \times X, \\ u(0, x) = \ell(x), & \text{if } x \in X. \end{cases}$$

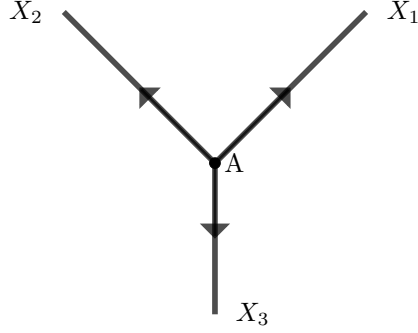
This equation is the *time-dependent Eikonal equation*. We consider the test functions given in Definition 4.1 for the viscosity notion. From Example 3.25, we know that the Hamiltonian  $H$  verifies Hypotheses 3.5 and 3.9. Hence we can apply Theorem 4.3 for any bounded from above upper semicontinuous subsolution and any bounded from below lower semicontinuous supersolution. Furthermore, from Example 3.25, we know that the Hamiltonian verifies Hypotheses 3.13 and 3.14. Finally, we have

$$\forall x \in X, \quad H(x, 0_{\text{DC}_1(T_x X)}) = 0.$$

Consequently, Hypothesis 4.6 is also verified by the Hamiltonian. In conclusion, Theorem 4.4 applies and there exists a unique bounded and continuous viscosity solution to equation (4.1) with the Hamiltonian  $H$  defined above.

**Example 4.8.** Let us analyze the Eikonal equation in a particular proper, geodesically extendable CAT(0) space  $(X, d)$ , in the form of a simple one dimensional network. Let  $e_1, e_2$  and  $e_3$  be three unit vectors of  $\mathbb{R}^2$ . Let us take the proper, geodesically extendable CAT(0) space obtained by gluing together three half-lines, denoted by  $X_1, X_2$  and  $X_3$  along the origin point  $A = \{0\}$ :

$$X_1 := [0, +\infty)e_1, \quad X_2 := [0, +\infty)e_2, \quad X_3 := [0, +\infty)e_3.$$



The glued space

$$X := \bigsqcup_A X_i,$$

along  $A$  is a proper, geodesically extendable CAT(0) space when endowed with the following distance:

$$\forall x, y \in X, d(x, y) := \begin{cases} |x - y|, & \text{if } \exists i \in \{1, 2, 3\} : x, y \in X_i, \\ |x| + |y|, & \text{otherwise.} \end{cases}$$

Notice that this distance is similar to the one introduced in [21, 24]. The tangent cone at a point  $x \in X$  is:

$$T_x X = \begin{cases} \mathbb{R}e_i & \text{if } x \in X_i \setminus A, \text{ and } i = 1, 2, 3, \\ X & \text{if } x \in A. \end{cases}$$

We will consider the same Hamilton-Jacobi equation as in Example 4.7. However, we will take smaller sets of test functions this time, in the following way.

$$\mathcal{TEST}_- = \{ (t, x) \mapsto \psi(t) + \phi(x) : \psi \in C^2((0, +\infty)) \text{ and } \phi \text{ Locally Lipschitz and locally semi-convex functions of } X \text{ such that } \phi|_{X_i \setminus A} \text{ is twice continuously differentiable with } i = 1, 2, 3 \}.$$

$$\mathcal{TEST}_- = \{ (t, x) \mapsto \psi(t) + \phi(x) : \psi \in C^2((0, +\infty)) \text{ and } \phi \text{ Locally Lipschitz and locally semi-concave functions of } X \text{ such that } \phi|_{X_i \setminus A} \text{ is twice continuously differentiable with } i = 1, 2, 3 \}.$$

$\mathcal{TEST}_-$  and  $\mathcal{TEST}_+$  satisfy all the requirements of Definition 4.1.

The equation studied in Example 4.7 has the following expression, when considering the above test functions.:

$$\begin{cases} \partial_t u + |\nabla_x u| = 0, & \text{if } x \in X_i \setminus A, \\ \partial_t u + \sup_{\substack{v \in X \\ |v|=1}} \{-D_x u \cdot v\} = 0, & \text{if } x \in A, \end{cases}$$

where  $\nabla_x u$  denotes the classical gradient of  $u$  at  $x$  in  $\mathbb{R}e_i$ , with  $i = 1, 2, 3$ . From Example 4.7, the above equation admits a unique continuous and bounded viscosity solution.

## 5 CONCLUSION

To conclude, we have used real-valued Lipschitz and DC functions of a proper CAT(0) space to define a notion of viscosity solutions for first order Hamilton-Jacobi equations defined in this space. With this new notion of viscosity, we proved the existence and uniqueness of the solution using the comparison principle and Perron's method in the same way as in the classical theory of viscosity originally developed in  $\mathbb{R}^N$ . We have tried to keep the presentation leisurely and easy to follow. In particular, the comparison results can be sharpened in order to get local comparison results and one could consider a more general Hamilton-Jacobi equation in the time-dependent case. We showed through several examples that the notion of viscosity defined here and the classical one coincide in  $\mathbb{R}^N$ . Furthermore, several other examples of Hamilton-Jacobi equations are given in a general proper, geodesically extendable CAT(0) space. In the future, we want to develop an optimal control theory in this space that will allow us to give an optimal control interpretation of the solutions of some examples of Hamilton-Jacobi equations given here. Furthermore, we want to tackle the problem of well-posedness of discontinuous Hamilton-Jacobi equations defined on networks and stratified domains, studied in [24, 21, 22, 27, ?, 7], using the framework developed in this paper.

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