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MARTINGALE PROPERTY OF EXPONENTIAL SEMIMARTINGALES: A NOTE ON EXPLICIT CONDITIONS AND APPLICATIONS TO FINANCIAL MODELS

DAVID CRIENS, KATHRIN GLAU, AND ZORANA GRBAC

ABSTRACT. We give a collection of explicit sufficient conditions for the true martingale property of a wide class of exponentials of semimartingales. We express the conditions in terms of semimartingale characteristics. This turns out to be very convenient in financial modeling in general. Especially it allows us to carefully discuss the question of well-definedness of semimartingale Libor models, whose construction crucially relies on a sequence of measure changes.

1. INTRODUCTION

Local martingales are the core object of stochastic integration. Thus they provide a natural access to time evolutionary stochastic modeling, which is a cornerstone of mathematical finance. The fundamental theorem of asset pricing states that the absence of arbitrage is essentially equivalent to the local martingale property of discounted asset prices under some equivalent probability measure. One important benefit of the true martingale property of discounted asset price processes is their use for density processes of a change of measure. In financial terms this corresponds to a change of numeraire. Since the seminal work of Geman et al. (1995) this concept became indispensable for both computational and modeling aspects. Often a change of numeraire facilitates option pricing by reducing complexity of computations. Moreover, it is a building stone of the construction of Libor market models introduced by Brace et al. (1997) and Miltersen et al. (1997). More fundamentally, a change of measure connects historical and risk-neutral probability measures. On the other hand if the discounted asset price process is a strict local martingale, i.e. a local martingale which is not a true martingale, this is sometimes interpreted as financial bubble. However, the definition and existence of financial bubbles critically depends on the specific notion of the market price, arbitrage and admissible strategies, see for example Cox and Hobson (2005) and Jarrow et al. (2010). In a typical modeling situation it is enjoyable to work with true martingales.

Usually price processes are non-negative and therefore are modeled as exponentials of semimartingales, which form a wide and flexible class of positive processes. One can characterize the local martingales in this class by a drift condition. It is, however, more involved to identify conditions for their true martingale property. In order to formulate the problem more precisely denote by X an \mathbb{R}^d -valued semimartingale and by λ an \mathbb{R}^d -valued

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predictable process which is integrable with respect to X . Then $\lambda \cdot X := \sum_{i \leq d} \int_0^\cdot \lambda^i dX^i$ denotes the real-valued stochastic integral process of λ with respect to X . Moreover, let V be a predictable process with finite variation. We pose the following question: *Under which conditions on the characteristics of X is a real-valued semimartingale Z of the form*

$$Z := e^{\lambda \cdot X - V}$$

a (uniformly integrable) martingale?

If $e^{\lambda \cdot X}$ is a special semimartingale, there exists a unique predictable process of finite variation V such that Z is a local martingale. In this case, V is called the *exponential compensator* of $\lambda \cdot X$, see Section 2 for details. Various criteria for the more delicate true martingale property of Z have been proposed. The seminal paper by Novikov (1972) treats the continuous semimartingale case. Sufficient conditions for general semimartingales are provided for example in Lepingle and Mémín (1978), Kallsen and Shiryaev (2002), Jacod (1979), Cheridito et al. (2005) and Protter and Shimbo (2008), see also a recent paper by Larsson and Ruf (2014) for further generalizations of Novikov-Kazamaki type conditions based on convergence results for local supermartingales. Moreover, we refer to Section 1 and Section 3 of Kallsen and Shiryaev (2002) for an exhaustive literature overview. In the special case when X is a process with independent increments and absolutely continuous characteristics and λ deterministic, Eberlein et al. (2005) show that if Z is a local martingale, it is also a true martingale. Deterministic conditions ensuring the martingale property of an exponential of an affine process are given in Kallsen and Muhle-Karbe (2010). The conditions for more general semimartingales are not as explicit.

Our contribution is to give explicit conditions for the martingale property of an exponential quasi-left continuous semimartingale in terms of its characteristics. In Section 2 we introduce the notation and describe the general semimartingale setting following Jacod and Shiryaev (2003). Section 3 contains the main results. The advantage of the explicit conditions is their convenience for applications. We illustrate this by investigating the true martingale property of asset prices in semimartingale stochastic volatility models in Section 4.1. Finally, in Section 4.2 we prove the well-definedness of the backward construction of Lévy Libor models. More precisely, we show that the candidate density processes for the measure changes are indeed true martingales which has not been rigorously proved earlier. Moreover, we present a natural extension to the semimartingale Libor model.

2. SEMIMARTINGALE NOTATION AND PRELIMINARIES

In this section we introduce the notation and summarize the basic notions and facts from the semimartingale theory in order to keep the paper self-contained. Our main reference is Jacod and Shiryaev (2003), whose notation we use throughout the paper. Other standard references for stochastic calculus and semimartingales are e.g. Jacod (1979), Métivier (1982) and Protter (2004).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ denote a stochastic basis, i.e. a filtered probability space with right-continuous filtration. For a class of processes \mathcal{C} , we say that a process X is in the localized class \mathcal{C}_{loc} if there exists a sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that a.s. $\tau_n \uparrow \infty$ as $n \rightarrow \infty$ and $X^{\tau_n} \in \mathcal{C}$. Denote by \mathcal{M} the class of càdlàg uniformly integrable martingales. The processes in the localized class \mathcal{M}_{loc} are called *local martingales*. We denote by \mathcal{V}^+ (resp. \mathcal{V}) the set of all real-valued càdlàg processes starting from zero that have *non-decreasing paths* (resp. *paths with finite variation* over each finite interval $[0, t]$). Let \mathcal{A}^+ denote the set of all processes $A \in \mathcal{V}^+$ that are integrable, i.e. such that $\mathbb{E}[A_\infty] < \infty$, where $A_\infty(\omega) := \lim_{t \rightarrow \infty} A_t(\omega) \in \overline{\mathbb{R}}_+$ for every $\omega \in \Omega$. Moreover, let \mathcal{A} denote the set of all

$A \in \mathcal{V}$ that have integrable variation, i.e. $\text{Var}(A) \in \mathcal{A}^+$, where for every $t \geq 0$ and every $\omega \in \Omega$, $\text{Var}(A)_t(\omega)$ is defined as the total variation of the function $s \mapsto A_s(\omega)$ on $[0, t]$. A process X is called a *semimartingale* if it has a decomposition of the form

$$X = X_0 + M + A, \quad (2.1)$$

where X_0 is finite-valued and \mathcal{F}_0 -measurable, $M \in \mathcal{M}_{\text{loc}}$ with $M_0 = 0$ and $A \in \mathcal{V}$. If A in decomposition (2.1) is predictable, X is called a *special semimartingale* and the decomposition is unique. A semimartingale is called *quasi-left continuous* if a.s. $\Delta X_\tau = 0$ on the set $\{\tau < \infty\}$ for all predictable times τ .

Let X be an \mathbb{R}^d -valued semimartingale, i.e. each component of X satisfies (2.1). Denoting by ε_a the Dirac measure at point a , the random measure of jumps μ^X of X is an integer-valued random measure of the form

$$\mu^X(\omega; dt, dx) := \sum_{s \geq 0} \mathbf{1}_{\{\Delta X_s(\omega) \neq 0\}} \varepsilon_{(s, \Delta X_s(\omega))}(dt, dx).$$

There is a version of the predictable compensator of μ^X , denoted by ν , such that the \mathbb{R}^d -valued semimartingale X is quasi-left continuous if and only if $\nu(\omega, \{t\} \times \mathbb{R}^d) = 0$ for all $\omega \in \Omega$, c.f. Jacod and Shiryaev (2003), Corollary II.1.19.

In general, ν satisfies

$$(|x|^2 \wedge 1) * \nu \in \mathcal{A}_{\text{loc}}. \quad (2.2)$$

The semimartingale X admits a *canonical representation*

$$X = X_0 + B(h) + X^c + (x - h(x)) * \mu^X + h(x) * (\mu^X - \nu),$$

where $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a *truncation function*, i.e. a function that is bounded and behaves like $h(x) = x$ around 0, $B(h)$ is a predictable \mathbb{R}^d -valued process with components in \mathcal{V} , and X^c is the continuous martingale part of X .

Denote by C the predictable $\mathbb{R}^d \otimes \mathbb{R}^d$ -valued covariation process defined as $C^{ij} := \langle X^{i,c}, X^{j,c} \rangle$. Then the triplet $(B(h), C, \nu)$ is called the *triplet of predictable characteristics* of X (or simply the *characteristics* of X). It can be shown (see Proposition II.2.9 in Jacod and Shiryaev (2003)) that there exists a predictable process $A \in \mathcal{A}_{\text{loc}}^+$ such that

$$B(h) = b(h) \cdot A, \quad C = c \cdot A, \quad \nu = A \times F,$$

where $b(h)$ is a d -dimensional predictable process, c is a predictable process taking values in the set of symmetric non-negative definite $d \times d$ -matrices and F is a transition kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Here \mathcal{P} denotes the predictable σ -field on $\Omega \times \mathbb{R}_+$. We call $(b(h), c, F; A)$ the *triplet of differential (or local) characteristics* of X . If X admits the choice $A_t = t$ above, we say that X has *absolutely continuous* characteristics (or shortly AC) and call X an *Itô semimartingale*.

An important subclass of semimartingales is the class of Itô semimartingales with independent increments. These processes are known as time-inhomogeneous Lévy processes or as Processes with Independent Increments and Absolutely Continuous characteristics (PIIAC), see e.g. Section 2 in Eberlein et al. (2005). The differential characteristics $(b(h), c, F)$ of a PIIAC X , for every truncation function h , are deterministic and satisfy the following integrability assumption: For every $T > 0$

$$\int_0^T \left(|b(h)_s| + \|c_s\| + \int_{\mathbb{R}^d} (|x|^2 \wedge 1) F_s(dx) \right) ds < \infty, \quad (2.3)$$

where $\|\cdot\|$ denotes any norm on the set of $d \times d$ -matrices. For every $t > 0$, the law of X_t is characterized by a Lévy-Khintchine type formula for its characteristic function, see again Section 2 in Eberlein et al. (2005). This property makes the class of PIIAC

particularly suitable for applications. The following definition and results on exponentials of semimartingales are given in Definition 2.12, Lemma 2.13 and Lemma 2.15 in Kallsen and Shiryaev (2002).

Definition 2.1. *A real-valued semimartingale Y is called exponentially special if $\exp(Y - Y_0)$ is a special semimartingale.*

Remark 2.2. Let Y be a real-valued semimartingale and denote by ν^Y the compensator of the random measure of jumps of Y and h a truncation function.

- (a) The following statements are equivalent:
 - (i) Y is an exponentially special semimartingale.
 - (ii) $(e^y - 1 - h(y)) * \nu^Y \in \mathcal{V}$.
 - (iii) $e^y \mathbf{1}_{\{y > 1\}} * \nu^Y \in \mathcal{V}$.
- (b) If Y is exponentially special, then it admits an *exponential compensator*, i.e. there exists a predictable process $V \in \mathcal{V}$ such that $\exp(Y - Y_0 - V) \in \mathcal{M}_{\text{loc}}$.

Let X be an \mathbb{R}^d -valued semimartingale with differential characteristics $(b(h), c, F; A)$ and $\lambda \in L(X)$, where $L(X)$ denotes the set of predictable processes integrable with respect to X , c.f. Jacod and Shiryaev (2003), page 207. Moreover, assume that $\lambda \cdot X$ is exponentially special. Following Jacod and Shiryaev (2003), Section III.7.7a we define the *Laplace cumulant process*

$$\tilde{K}^X(\lambda) := \tilde{\kappa}^X(\lambda) \cdot A, \quad (2.4)$$

where

$$\tilde{\kappa}_s^X(\lambda) := \langle \lambda_s, b_s \rangle + \frac{1}{2} \langle \lambda_s, c_s \lambda_s \rangle + \int (e^{\langle \lambda_s, x \rangle} - 1 - \langle \lambda_s, h(x) \rangle) F_s(dx), \quad (2.5)$$

and the *modified Laplace cumulant process* $K^X(\lambda) := \ln(\mathcal{E}(\tilde{K}^X(\lambda)))$, where \mathcal{E} denotes the stochastic exponential, and

$$K^X(\lambda) = \tilde{K}^X(\lambda) + \sum_{s \leq \cdot} (\ln(1 + \Delta \tilde{K}_s^X(\lambda)) - \Delta \tilde{K}_s^X(\lambda)). \quad (2.6)$$

The following results are proved in Proposition III.7.14 and Theorem III.7.4 in Jacod and Shiryaev (2003):

Proposition 2.3. *Let X be an \mathbb{R}^d -valued semimartingale and $\lambda \in L(X)$ such that $\lambda \cdot X$ is exponentially special.*

- (i) *The modified Laplace cumulant process $K^X(\lambda)$ is the exponential compensator of $\lambda \cdot X$, i.e. the process Z defined by*

$$Z := \exp(\lambda \cdot X - K^X(\lambda))$$

is a local martingale.

- (ii) *If X is quasi-left continuous, the Laplace cumulant process $\tilde{K}^X(\lambda)$ and the modified Laplace cumulant process $K^X(\lambda)$ coincide, i.e. $K^X(\lambda) = \tilde{K}^X(\lambda)$.*

In the following section we give sufficient conditions for the martingale property of exponential semimartingales.

3. THE MARTINGALE PROPERTY OF EXPONENTIAL SEMIMARTINGALES

Integrability conditions ensuring the (UI) martingale property of a non-negative or positive local martingale were studied from many perspectives and in various levels of generality. It started with the classical conditions of Novikov (1972) which applies to continuous exponential local martingales. A natural generalization included jumps was given in the seminal paper of Lepingle and Mémmin (1978). Various related conditions are given by Kallsen and Shiryaev (2002). A profound overview on Novikov-type conditions as well as boundedness conditions is given in the monograph of Jacod (1979). In this section we collect conditions for exponential semimartingales and express them in terms of semimartingale characteristics. Thanks to these expression we usually call these type of conditions *predictable conditions*. Let us start with a Novikov-type integrability condition which is based on the main result of Lepingle and Mémmin (1978). We follow its statement given by Jacod (1979) as Corollary 8.44.

Proposition 3.1. *Let Y be a real-valued quasi-left continuous semimartingale with characteristics $(B(h), C, \nu)$. If*

$$(A1) \quad \mathbb{E} \left(\exp \left\{ \frac{1}{2} C_T + ((y-1)e^y + 1) * \nu_T \right\} \right) < \infty \text{ for every } T \geq 0,$$

the process $M := e^{Y-K^Y(1)}$ is a true martingale. Moreover, replacing condition (A1) with

$$(A2) \quad \mathbb{E} \left(\exp \left\{ \frac{1}{2} C_\infty + ((y-1)e^y + 1) * \nu_\infty \right\} \right) < \infty,$$

M is a UI martingale.

Proof: Note that the characteristics of X^T , for any $T > 0$, are given by (B^T, C^T, ν^T) , where $\nu^T(dt, dx) := \mathbf{1}_{[0, T] \times \mathbb{R}^d} \nu(dt, dx)$. Now, since local martingales whose localizing sequence is deterministic are martingales, the first claim follows immediately from the second. Note that (A1) implies that X is exponentially special and hence that M is a local martingale. In view of Theorem 2.19 in Kallsen and Shiryaev (2002), the second claim follows from Jacod (1979), Corollary 8.44. \square

Remark 3.2. If Y is continuous, i.e. $\nu = 0$, then (A2) reduces to the classical Novikov condition as presented in Section 3.5.D in Karatzas and Shreve (1991).

As an immediate corollary we derive the follows sufficient conditions for the case where X is given as a stochastic integral.

Corollary 3.3. *Let X be an \mathbb{R}^d -valued quasi-left continuous semimartingale with differential characteristics $(b(h), c, F; A)$ and $\lambda \in L(X)$. If*

$$(B1) \text{ for every } T \geq 0,$$

$$\mathbb{E} \left(\exp \left\{ \frac{1}{2} \int_0^T \langle \lambda_s, c_s \lambda_s \rangle dA_s + \int_0^T \int_{\mathbb{R}^d} ((\langle \lambda_s, x \rangle - 1) e^{\langle \lambda_s, x \rangle} + 1) F_s(dx) dA_s \right\} \right) < \infty$$

the process $M := e^{\lambda \cdot X - K^X(\lambda)}$ is a true martingale. Moreover, replacing (B1) with

$$(B2) \quad \mathbb{E} \left(\exp \left\{ \frac{1}{2} \int_0^\infty \langle \lambda_s, c_s \lambda_s \rangle dA_s + \int_0^\infty \int_{\mathbb{R}^d} ((\langle \lambda_s, x \rangle - 1) e^{\langle \lambda_s, x \rangle} + 1) F_s(dx) dA_s \right\} \right) < \infty,$$

then M is a UI martingale.

Proof: The characteristics of $\lambda \cdot X$ are given by Proposition IX.5.3 in Jacod and Shiryaev (2003). Now the claim follows by an application of Proposition 3.1. \square

Remark 3.4. Obviously, by considering an real-valued semimartingale X and $\lambda = 1$ in Corollary 3.3 we recover Proposition 3.1.

For applications the following boundedness condition turns out to be useful, see for instance Corollary 4.1, Proposition 4.3 and 4.2 in the sections below.

Proposition 3.5. *Let X be as in Corollary 3.3 and let $\lambda \in L(X)$. If*

(C1) *for every $T \geq 0$, there exists a non-negative constant $\kappa(T)$ such that a.s.*

$$\int_0^T \langle \lambda_s, c_s \lambda_s \rangle dA_s + \int_0^T \int_{\mathbb{R}^d} \left(1 - \sqrt{e^{\langle \lambda_s, x \rangle}}\right)^2 F_s(dx) dA_s \leq \kappa(T)$$

the process $M := e^{\lambda \cdot X - K^X(\lambda)}$ is a true martingale. Moreover, replacing (C1) with

(C2) *there exists a non-negative constant κ such that a.s.*

$$\int_0^\infty \langle \lambda_s, c_s \lambda_s \rangle dA_s + \int_0^\infty \int_{\mathbb{R}^d} \left(1 - \sqrt{e^{\langle \lambda_s, x \rangle}}\right)^2 F_s(dx) dA_s \leq \kappa$$

then M is a UI martingale.

Proof: Again, the first part is an immediate consequence of the second. Note that (C1) implies that $\lambda \cdot X$ is exponentially special. Hence, we can deduce the claim from Theorem 2.19 in Kallsen and Shiryaev (2002) together with Lemma 8.8 and Theorem 8.25 in Jacod (1979). \square

Remark 3.6. Clearly, thanks to Corollary 3.3, the condition (C1) could be replaced by the following condition: for every $T \geq 0$ there exists a constant $\kappa(T)$ such that a.s.

$$\int_0^T \langle \lambda_s, c_s \lambda_s \rangle dA_s + \int_0^T \int_{\mathbb{R}^d} \left((\langle \lambda_s, x \rangle - 1)e^{\langle \lambda_s, x \rangle} + 1 \right) F_s(dx) dA_s \leq \kappa(T). \quad (3.1)$$

The elementary inequality

$$0 \leq (1 - \sqrt{x})^2 \leq x \log(x) - (x - 1), \text{ for all } x > 0,$$

as for instance noted in Esche (2004), Lemma 2.13, shows that condition (C1) is an improvement to (3.1).

Let us shortly turn to the subclass of semimartingales with independent increments (SII processes), for which the situation is slightly different than in the more general case. For exponential SII processes the local martingale property is equivalent to the true martingale property. From a mathematical finance perspective this interesting fact for instance implies that exponential SII models cannot include bubbles which are modeled as strict local martingales.

Let us formalize this observation and add some simple deterministic conditions for the martingale property. The main implication (ii) \Rightarrow (i) is essentially thanks to Kallsen and Muhle-Karbe (2010), Proposition 3.12. Note that the assertion does not require quasi-left continuity.

Proposition 3.7. *Let X be an \mathbb{R}^d -valued semimartingale with deterministic characteristics (B^X, C^X, ν^X) , $\lambda \in L(X)$ be deterministic and $M := e^{\lambda \cdot X - K^X(\lambda)}$. The following are equivalent*

- (i) *M is a martingale.*
- (ii) *M is a local martingale.*
- (iii) *$\lambda \cdot X$ is exponentially special.*
- (iv) *$(e^{\langle \lambda, x \rangle} - 1 - h(\langle \lambda, x \rangle)) * \nu^X \in \mathcal{V}$.*
- (v) *$e^{\langle \lambda, x \rangle} \mathbf{1}_{\{\langle \lambda, x \rangle > 1\}} * \nu^X \in \mathcal{V}$.*

Proof: The implication $(i) \Rightarrow (ii)$ is trivial and equivalence $(ii) \Leftrightarrow (iii)$ holds by definition. The equivalences of (iii) , (iv) and (v) are due to Remark 2.2 and Jacod and Shiryaev (2003), Proposition IX.5.3, which shows that $\lambda \cdot X$ has deterministic characteristics with

$$\nu^{\lambda \cdot X}(A, ds) = \int_{\mathbb{R}^d} \mathbf{1}_A(\langle \lambda_s, x \rangle) \nu^X(dx, ds), \quad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}).$$

It is left to show the implication $(iii) \Rightarrow (i)$. In view of (2.6) and since $\lambda \cdot X$ has deterministic characteristics, $K^X(\lambda)$ is a deterministic process of finite variation and hence also has deterministic characteristics. Define $f(x, y) := x - y$, then $Y := \lambda \cdot X - K^X(\lambda) = f(\lambda \cdot X, K^X(\lambda))$. It follows from Goll and Kallsen (2000), Corollary 5.6 applied to $f(\lambda \cdot X, K^X(\lambda))$ that Y has also deterministic characteristics. From the relationship of ordinary and stochastic exponentials given in Jacod and Shiryaev (2003), Theorem II.8.10, we obtain that $M = e^Y = \mathcal{E}(\overline{Y})$, where \overline{Y} is a semimartingale with $\Delta \overline{Y} = (e^{\Delta Y} - 1) > -1$. We deduce from Jacod and Shiryaev (2003), Equation II.8.14 that \overline{Y} inherits the property of deterministic characteristics from Y . Due to Remark 2.2(b), the condition $e^{\langle \lambda, x \rangle} \mathbf{1}_{\{\langle \lambda, x \rangle > 1\}} * \nu \in \mathcal{V}$ yields that $\lambda \cdot X$ is exponentially special. Thus, since $K^X(\lambda)$ is the exponential compensator of $\lambda \cdot X$, c.f. Proposition 2.3 (i), M is a local martingale. The claim now follows from Proposition 3.12 in Kallsen and Muhle-Karbe (2010). \square

4. APPLICATIONS TO FINANCIAL MODELS

In this section we present two applications of the results from Section 3 to financial modeling. A detailed overview concerning applications of general semimartingales in finance is for instance provided by the monographs of Shiryaev (1999), Cont and Tankov (2003), Musiela and Rutkowski (2005) and Jeanblanc et al. (2009).

4.1. Stochastic Volatility Asset Price Model. Here, we illustrate how the conditions of Section 3 can be used to facilitate pricing in arbitrage-free models driven by semimartingales. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a stochastic basis, where $T > 0$ denotes a finite time horizon. We model the asset price S and a bank account B with stochastic interest rate r by

$$S := S_0 e^{\sigma^S \cdot X^S - V}, \quad B := e^{\sigma^r \cdot X^r} \quad (4.1)$$

with $S_0 > 0$, a d -dimensional semimartingale $X := (X^S, X^r)$ with X^S d_1 -dimensional and X^r d_2 -dimensional such that $d_1 + d_2 = d$, and a d -dimensional predictable process $\sigma := (\sigma^S, -\sigma^r)$ with $\sigma^S \in L(X^S)$ and $\sigma^r \in L(X^r)$ such that $\sigma \cdot X$ is exponentially special. We assume that the process V is the exponential compensator of $\sigma^S \cdot X^S - \sigma^r \cdot X^r$, c.f. Proposition 2.3. Thanks to this assumption, the discounted stock price $\tilde{S} := B^{-1}S$ is a local martingale, i.e. in other words \mathbb{P} is a risk-neutral probability measure. According to the fundamental theorem of asset pricing for general semimartingales in Delbaen and Schachermayer (1998), the No Free Lunch With Vanishing Risk (NFLVR) holds in this case.

Note that in general the risk-neutral probability measure may not be unique and the model is incomplete. Let us now consider a European call option with strike $K > 0$ with payoff $(S_T - K)^+$ at maturity $T > 0$. Its fundamental price under \mathbb{P} , denoted by C_t^* for any $t \in [0, T]$, is given by

$$C_t^* := B_t \mathbb{E}_{\mathbb{P}}(B_T^{-1}(S_T - K)^+ | \mathcal{F}_t) \leq B_t \mathbb{E}_{\mathbb{P}}(\tilde{S}_T | \mathcal{F}_t) \leq B_t \tilde{S}_t = S_t < \infty, \quad (4.2)$$

which is well-defined and finite a.s. The inequality $\mathbb{E}(\tilde{S}_T|\mathcal{F}_t) \leq \tilde{S}_t$ is a consequence of \tilde{S} being a positive local martingale and hence a supermartingale. The price C^* is an arbitrage-free price, which – even in the case of a complete model – might be non-unique. This subtle issue is closely related to financial bubbles, see for example Definition 3.6 in Jarrow et al. (2010) and Definition 2.10 in Biagini et al. (2014). For a detailed mathematical treatment we refer to Protter (2013).

When \tilde{S} is a true martingale, these delicate issues do not appear: The asset price has no bubble and the market prices coincide with the fundamental prices – this was proved for example in the setting of Jarrow et al. (2010). Thus, our Section 3 provides convenient conditions to exclude ambiguities in the pricing due to the possible presence of bubbles.

To illustrate a further benefit of the explicit martingale conditions from Section 3 we use the true martingale property of the discounted asset price to perform a change of numeraire which reduces the complexity of a pricing problem at hand. Considering for example a call option as above, in order to compute the expectation in (4.2) directly, information on the joint distribution of S and B is required. Here the true martingale property of \tilde{S} allows to facilitate the computation of the expectation by a change of numeraire. More precisely, we can express the call price as a conditional expectation of a function of the asset value S_T solely. Defining a probability measure $\tilde{\mathbb{P}}$ via $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t} := S_0^{-1}\tilde{S}_t$ for $0 \leq t \leq T$, and denoting by $\mathbb{E}_{\tilde{\mathbb{P}}}$ the expectation under $\tilde{\mathbb{P}}$, Bayes formula yields

$$C_t = S_t \mathbb{E}_{\tilde{\mathbb{P}}}((1 - KS_T^{-1})^+ | \mathcal{F}_t). \quad (4.3)$$

Compared with the original pricing formula $C_t = B_t \mathbb{E}_{\mathbb{P}}(B_T^{-1}(S_T - K)^+ | \mathcal{F}_t)$, the random variable B_T does not appear in the conditional expectation in (4.3). This typically facilitates the computation since the semimartingale characteristics of S are known under the new probability measure.

By combining Corollary 3.3 and Proposition 3.5 the following characterization of the true martingale property for the semimartingale asset price model defined above.

Corollary 4.1. *Assume that X is quasi-left continuous and denote its local characteristics by $(b, c, F; A)$. If $(b, c, F; A)$ and σ satisfy (B1) of Corollary 3.3, resp. condition (C1) of Proposition 3.5, the discounted asset price process \tilde{S} is a true martingale.*

Under the conditions of Corollary 4.1 the fair price at time t of the call option with maturity T and strike K is therefore given by C_t in (4.3).

4.2. Semimartingale Libor model. In this subsection we apply the results from Section 3 to Libor models. These are models for discretely compounded forward interest rates known as Libor rates, where the term Libor stems from the London Interbank Offered Rate. The Libor models were introduced in Brace et al. (1997) and Miltersen et al. (1997) and later further developed and studied by many authors. We refer to Musiela and Rutkowski (2005), Section 12.4, for a detailed overview.

The challenge in modeling Libor rates is to simultaneously define the rates for different maturities as local martingales under different equivalent measures which ensures the absence of arbitrage. These measures are in fact forward measures and they are interconnected via the Libor rates themselves. A convenient way to obtain such a model is by backward construction, following the pioneering work of Musiela and Rutkowski (1997). This construction relies on the martingale property of Libor rates (under the corresponding forward measures), which allows to define changes of measure. In the backward construction the Libor rates thus have to be not only local, but true martingales under their

corresponding forward measures. When the model is driven by a continuous semimartingale this is standard by using Novikov type conditions, but verifying that the Libor rates are true martingales in general semimartingale models including jumps is more involved and has not been properly addressed in the financial literature. Using explicit conditions from Section 3, we study this issue in detail below to close this gap.

Let us begin by describing a general semimartingale Libor model. Assume that $T^* > 0$ is a fixed finite time horizon and we are given a pre-determined collection of maturities $0 = T_0 < T_1 < \dots < T_n = T^*$, with $\delta_k := T_{k+1} - T_k$ for $k = 0, \dots, n-1$. Moreover, let $(\Omega, \mathcal{F}_{T^*}, (\mathcal{F}_t)_{0 \leq t \leq T^*}, \mathbb{P}_{T^*})$ be a stochastic basis. A general semimartingale Libor model consists of a family of semimartingales modeling the Libor rates $(L(\cdot, T_k))_{1 \leq k \leq n-1}$ for lending periods $([T_k, T_{k+1}])_{1 \leq k \leq n-1}$ and a family of probability measures $(\mathbb{P}_{T_k})_{1 \leq k \leq n}$, where $L(\cdot, T_k)$ and \mathbb{P}_{T_k} are defined on $(\Omega, \mathcal{F}_{T_k}, (\mathcal{F}_t)_{0 \leq t \leq T_k})$ and $\mathbb{P}_{T_n} = \mathbb{P}_{T^*}$, such that

(SML1) $L(\cdot, T_k)$ is a $\mathbb{P}_{T_{k+1}}$ -martingale for all $k = 1, \dots, n-1$.

(SML2) For $k = n-1, \dots, 1$

$$\frac{d\mathbb{P}_{T_k}}{d\mathbb{P}_{T_{k+1}}} = \frac{1 + \delta_k L(T_k, T_k)}{1 + \delta_k L(0, T_k)}.$$

For each k the probability measure \mathbb{P}_{T_k} is called the *forward Libor measure* for maturity T_k , cf. Musiela and Rutkowski (2005), Definition 12.4.1. The measure \mathbb{P}_{T_k} is in fact the forward martingale measure associated with maturity T_k and the density process above is a forward price process. This can be seen from the link between forward Libor rates and zero-coupon bond prices, see Musiela and Rutkowski (2005), Sections 12.1.1 and 12.4.4.

Below we present the main ideas of the backward construction of the Libor model in a semimartingale framework. We start by modeling the Libor rate with the most distant maturity under a given probability measure and then proceed backwards. We define in each step the next forward measure via a density process based on the previously modeled Libor rates and model the next Libor rate under this measure.

Let X be an \mathbb{R}^d -valued semimartingale on the stochastic basis $(\Omega, \mathcal{F}_{T^*}, (\mathcal{F}_t)_{0 \leq t \leq T^*}, \mathbb{P}_{T^*})$. We start by modeling the Libor rate $L(\cdot, T_{n-1})$ for maturity T_{n-1} by

$$L(t, T_{n-1}) := L(0, T_{n-1}) \exp \left\{ \lambda(\cdot, T_{n-1}) \cdot X_t - K_t^X(\mathbb{P}_{T^*}, \lambda(\cdot, T_{n-1})) \right\}, \quad (4.4)$$

for $t \leq T_{n-1}$, where $L(0, T_{n-1}) > 0$ and $\lambda(\cdot, T_{n-1}) \in L(X)$ is a volatility process such that the stochastic integral $\lambda(\cdot, T_{n-1}) \cdot X$ is \mathbb{P}_{T^*} -exponentially special with $K^X(\mathbb{P}_{T^*}, \lambda(\cdot, T_{n-1}))$ its \mathbb{P}_{T^*} -exponential compensator. Hence, $L(\cdot, T_{n-1})$ is a \mathbb{P}_{T^*} -local martingale. Assuming that $L(\cdot, T_{n-1})$ is a true \mathbb{P}_{T^*} -martingale, we can define the probability measure $\mathbb{P}_{T_{n-1}}$ on $(\Omega, \mathcal{F}_{T_{n-1}})$ by the Radon-Nikodym derivative

$$\frac{d\mathbb{P}_{T_{n-1}}}{d\mathbb{P}_{T^*}} = \frac{1 + \delta_{n-1} L(T_{n-1}, T_{n-1})}{1 + \delta_{n-1} L(0, T_{n-1})}. \quad (4.5)$$

Moreover, we obtain for $t \leq T_{n-1}$

$$\left. \frac{d\mathbb{P}_{T_{n-1}}}{d\mathbb{P}_{T^*}} \right|_{\mathcal{F}_t} = \frac{1 + \delta_{n-1} L(t, T_{n-1})}{1 + \delta_{n-1} L(0, T_{n-1})}. \quad (4.6)$$

Now we recursively model the Libor rates $L(\cdot, T_k)$ for $k = n-2, \dots, 1$ by

$$L(t, T_k) := L(0, T_k) \exp \left\{ \lambda(\cdot, T_k) \cdot X_t - K_t^X(\mathbb{P}_{T_{k+1}}, \lambda(\cdot, T_k)) \right\}, \quad (4.7)$$

for $t \leq T_k$, where $L(0, T_k) > 0$ and $\lambda(\cdot, T_k) \in L(X)$ is a volatility process such that $\lambda(\cdot, T_k) \cdot X$ is $\mathbb{P}_{T_{k+1}}$ -exponentially special with $\mathbb{P}_{T_{k+1}}$ -exponential compensator $K^X(\mathbb{P}_{T_{k+1}}, \lambda(\cdot, T_k))$. As above, this means that $L(\cdot, T_k)$ is a $\mathbb{P}_{T_{k+1}}$ -local martingale. Note that the Libor rate

for the interval starting at $T_0 = 0$ and ending at T_1 is simply the given spot Libor rate $L(0, T_0) > 0$. The probability measure \mathbb{P}_{T_k} is defined on $(\Omega, \mathcal{F}_{T_k})$ by the Radon-Nikodym derivative

$$\frac{d\mathbb{P}_{T_k}}{d\mathbb{P}_{T_{k+1}}} = \frac{1 + \delta_k L(T_k, T_k)}{1 + \delta_k L(0, T_k)}, \quad (4.8)$$

where it has to be assumed that $L(\cdot, T_k)$ is a true $\mathbb{P}_{T_{k+1}}$ -martingale. Then we have for $t \leq T_k$

$$\left. \frac{d\mathbb{P}_{T_k}}{d\mathbb{P}_{T_{k+1}}} \right|_{\mathcal{F}_t} = \frac{1 + \delta_k L(t, T_k)}{1 + \delta_k L(0, T_k)}. \quad (4.9)$$

Furthermore, we obtain that the probability measure $\mathbb{P}_{T_{k+1}}$ is related to \mathbb{P}_{T^*} via

$$\left. \frac{d\mathbb{P}_{T_{k+1}}}{d\mathbb{P}_{T^*}} \right|_{\mathcal{F}_t} = \prod_{i=k+1}^{n-1} \frac{1 + \delta_i L(t, T_i)}{1 + \delta_i L(0, T_i)}, \quad t \leq T_{k+1}. \quad (4.10)$$

Note that the construction is well-defined if the Libor rates $L(\cdot, T_k)$ are $\mathbb{P}_{T_{k+1}}$ -martingales for all $k = 1, \dots, n-1$.

To justify the backward construction (4.4) – (4.10) of the measures $(\mathbb{P}_{T_k})_{1 \leq k \leq n-1}$, we prove the required martingale property of the Libor rates in the proposition below.

Proposition 4.2. *Let X in equation (4.7) be an \mathbb{R}^d -valued quasi-left continuous semimartingale with differential characteristics $(b^{T^*}, c, F^{T^*}; A)$ with respect to P_{T^*} , and non-negative $\lambda(\cdot, T_k) \in L(X)$. Assume*

(SL) *for all $i = 1, \dots, n-1$ there exists a non-negative constant κ such that a.s.*

$$\begin{aligned} & \int_0^{T^*} \langle \lambda(t, T_i), c_t \lambda(t, T_i) \rangle dA_t \\ & + \int_0^{T^*} \int_{\mathbb{R}^d} \left(1 - \sqrt{e^{\langle \lambda(t, T_i), x \rangle}} \right)^2 e^{\langle \sum_{k=i+1}^{n-1} \lambda(t, T_k), x \rangle} F_t^{T^*}(dx) dA_t \leq \kappa, \end{aligned}$$

where we use the convention $\sum_{\emptyset} = 0$.

Then for each $k = 1, \dots, n-1$, the process $L(\cdot, T_k)$ from (4.7) is a martingale with respect to $\mathbb{P}_{T_{k+1}}$ given by (4.10).

Proof: For $k = n-1$, the assertion follows directly from assumption (SL) and Proposition 3.5.

For $k \leq n-2$, denote the semimartingale characteristics of X with respect to $\mathbb{P}_{T_{k+1}}$ by $(B^{T_{k+1}}, C^{T_{k+1}}, \nu^{T_{k+1}})$. Next we compute these characteristics by backward induction and Girsanov's theorem as given by Theorem III.3.24 in Jacod and Shiryaev (2003). We shortly give some details on the application of Girsanov's theorem. Denote $d\mathbb{P}_{T_{n-1}}/d\mathbb{P}_{T^*}|_{\mathcal{F}} =: Z^{T_{n-1}}$ and note that

$$L(\cdot, T^*) = L(0, T_{n-1}) \mathcal{E} \left(\lambda(\cdot, T_{n-1}) \cdot X^{c, T^*} + \left(e^{\langle \lambda(\cdot, T_{n-1}), x \rangle} - 1 \right) * \left(\mu^X - \nu^{T^*} \right) \right),$$

where X^{c, T^*} denotes the continuous local \mathbb{P}_{T^*} -martingale part of X . We have

$$M_{\mu^X}^{\mathbb{P}_{T^*}}(Z^{T_{n-1}}|\tilde{\mathcal{P}}) = Z_-^{T_{n-1}} + \frac{\delta_{n-1} L(\cdot, T_{n-1})}{1 + \delta_{n-1} L(0, T_{n-1})} \left(e^{\langle \lambda(\cdot, T_{n-1}), x \rangle} - 1 \right).$$

Now Girsanov's theorem yields that

$$C^{T_{n-1}} = C, \quad \nu^{T_{n-1}}(dt, dx) = \beta(t, x, T_{n-1}) F_t^{T^*}(dx) dA_t,$$

with

$$\beta(t, x, T_l) := \frac{\delta_l L(t-, T_l)}{1 + \delta_l L(t-, T_l)} \left(e^{\langle \lambda(t, T_l), x \rangle} - 1 \right) + 1,$$

for $l = 1, \dots, n-1$. Repeating these steps, by backward induction we obtain

$$C^{T_{k+1}} = C, \quad \nu^{T_{k+1}}(dt, dx) = \prod_{l=k+1}^{n-1} \beta(t, x, T_l) F_t^{T^*}(dx) dA_t.$$

Noting that $\beta(t, x, T_l) \leq e^{\langle \lambda(t, T_l), x \rangle}$, the claim follows from Proposition 3.5. \square

Let us link our discussion to the Lévy Libor model of Eberlein and Özkan (2005) in which the driving process X is assumed to be an \mathbb{R}^d -valued PIIAC with differential characteristic $(0, c, F^{T^*})$ under \mathbb{P}_{T^*} . Eberlein and Özkan impose the following assumptions: *For some $M, \varepsilon > 0$ and every $k = 1, \dots, n-1$ we have*

$$(L1) \quad \int_0^{T^*} \int_{|x|>1} e^{\langle u, x \rangle} F_t^{T^*}(dx) dt < \infty \text{ for every } u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d,$$

$$(L2) \quad \lambda(\cdot, T_k) : [0, T^*] \rightarrow \mathbb{R}_+^d \text{ is a bounded, nonnegative function such that for } t > T_k, \\ \lambda(t, T_k) = 0 \text{ and } \sum_{k=1}^{n-1} \lambda^j(t, T_k) \leq M, \text{ for all } t \in [0, T^*] \text{ and every coordinate } \\ j \in \{1, \dots, d\},$$

$$(L3) \quad \lambda(\cdot, T_k) : [0, T^*] \rightarrow \mathbb{R}_+^d \text{ is deterministic.}$$

Let us point that even when the driving process has deterministic characteristics under \mathbb{P}_{T^*} and λ is deterministic (as in the case above), the characteristics of X under \mathbb{P}_{T_k} for $k = 1, \dots, n-1$ are stochastic.

We obtain the following sufficient conditions for the Lévy Libor model, where we also allow λ to be stochastic.

Corollary 4.3. *Assume that $\sum_{j=1}^n |\lambda(\cdot, T^j)| \leq N$ for a non-negative constant N , and that there exists a non-negative constant κ such that*

$$\int_0^{T^*} \int_{|x|>1} e^{N|x|} F_t^{T^*}(dx) dt \leq \kappa.$$

Then for each $k = 1, \dots, n-1$ the process $L(\cdot, T_k)$ defined in (4.7) is a martingale with respect to $\mathbb{P}_{T_{k+1}}$ given by (4.10).

Proof: It suffices to show that (SL) is satisfied. Note that we find a non-negative constant K^* such that for any $i = 1, \dots, n-1$ for all $x \in \mathbb{R}^d$ with $|x| \leq 1$

$$\left(1 - \sqrt{e^{\langle \lambda(t, T_i), x \rangle}} \right)^2 e^{\langle \sum_{k=i+1}^{n-1} \lambda(t, T_k), x \rangle} \leq K^* |x|^2.$$

Next we bound the large jumps. Using the fact that $(1 - \sqrt{x})^2 \leq 1 + x$ for $x > 0$ and some non-negative constant K , and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \int_0^{T^*} \int_{|x|>1} \left(1 - \sqrt{e^{\langle \lambda(t, T_i), x \rangle}} \right)^2 e^{\langle \sum_{k=i+1}^{n-1} \lambda(t, T_k), x \rangle} F_t^{T^*}(dx) dt \\ & \leq \int_0^{T^*} \int_{|x|>1} \left(e^{\langle \sum_{k=i+1}^{n-1} \lambda(t, T_k), x \rangle} + e^{\langle \sum_{k=i}^{n-1} \lambda(t, T_k), x \rangle} \right) F_t^{T^*}(dx) dt \\ & \leq 2 \int_0^{T^*} \int_{|x|>1} e^{N|x|} F_t^{T^*}(dx) dt. \end{aligned}$$

Finally, since

$$\int_0^{T^*} \langle \lambda(t, T_i), c_t \lambda(t, T_i) \rangle dt \leq N^2 \int_0^{T^*} \|c_t\| dt,$$

where $\|\cdot\|$ denotes the operator norm of c , we conclude that (SL) holds. This concludes the proof. \square

As mentioned in the introduction of the section, the martingale property of Libor rates under their corresponding measures is crucial for the validity of the backward construction of Libor models. Therefore, Proposition 4.2 and Corollary 4.3 provide a theoretical justification of the construction of the Lévy Libor model by Eberlein and Özkan (2005), and more generally of Libor models driven by quasi-left continuous semimartingales.

REFERENCES

- Biagini, F., H. Föllmer, and S. Nedelcu (2014). Shifting martingale measures and the birth of a bubble as a submartingale. *Finance and Stochastics* 18(2), 297–326.
- Brace, A., D. Gatarek, and M. Musiela (1997). The market model of interest rate dynamics. *Mathematical Finance* 7, 127–155.
- Cheridito, P., D. Filipović, and M. Yor (2005). Equivalent and absolutely continuous measure changes for jump-diffusion processes. *The Annals of Applied Probability* 15(3), 1713–1732.
- Cont, R. and P. Tankov (2003). *Financial Modelling with Jump Processes*. Chapman and Hall/CRC Press.
- Cox, A. and D. Hobson (2005). Local martingales, bubbles and option prices. *Finance and Stochastics* 9, 477–492.
- Delbaen, F. and W. Schachermayer (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen* 312, 215–250.
- Eberlein, E., J. Jacod, and S. Raible (2005). Lévy term structure models: no-arbitrage and completeness. *Finance Stochastics* 9, 67–88.
- Eberlein, E. and F. Özkan (2005). The Lévy Libor model. *Finance Stochastics* 9, 327–348.
- Esche, F. (2004). *Two essays on incomplete markets*. Ph. D. thesis.
- Geman, H., N. El Karoui, and J.-C. Rochet (1995). Changes of numéraire, changes of probability measures and option pricing. *Journal of Applied Probability* 32, 443–458.
- Goll, T. and J. Kallsen (2000). Optimal portfolios for logarithmic utility. *Stochastic Processes and their Applications* 89, 31–48.
- Jacod, J. (1979). *Calcul Stochastique et Problèmes de Martingales*. Lecture Notes in Mathematics 714. Springer.
- Jacod, J. and A. N. Shiryaev (2003). *Limit Theorems for Stochastic Processes* (2nd ed.). Springer.
- Jarrow, R. A., P. Protter, and K. Shimbo (2010). Asset price bubbles in incomplete markets. *Mathematical Finance* 20(2), 145–185.
- Jeanblanc, M., M. Yor, and M. Chesney (2009). *Mathematical Methods for Financial Markets*. Springer.
- Kallsen, J. and J. Muhle-Karbe (2010). Exponentially affine martingales, affine measure changes and exponential moments of affine processes. *Stochastic Processes and their Applications* 120, 163–181.
- Kallsen, J. and A. N. Shiryaev (2002). The cumulant process and Esscher’s change of measure. *Finance Stochastics* 6, 397–428.

- Karatzas, I. and S. E. Shreve (1991). *Brownian motion and stochastic calculus* (Second ed.), Volume 113 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.
- Larsson, M. and J. Ruf (2014). Convergence of local supermartingales and Novikov-Kazamaki type conditions for processes with jumps. Preprint, [arXiv:1411.6229](#).
- Lepingle, D. and J. Mémin (1978). Sur l'intégrabilité uniforme des martingales exponentielles. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 42, 175–203.
- Métivier, M. (1982). *Semimartingales*, Volume 2 of *de Gruyter Studies in Mathematics*. Berlin: Walter de Gruyter & Co. A course on stochastic processes.
- Miltersen, K. R., K. Sandmann, and D. Sondermann (1997). Closed form solutions for term structure derivatives with log-normal interest rates. *The Journal of Finance* 52, 409–430.
- Musiela, M. and M. Rutkowski (1997). Continuous-time term structure models: forward measure approach. *Finance Stochastics* 1, 261–291.
- Musiela, M. and M. Rutkowski (2005). *Martingale Methods in Financial Modelling* (2nd ed.). Springer.
- Novikov, A. A. (1972). A certain identity for stochastic integrals. *Theory of Probability and its Applications* 17, 761–765.
- Protter, P. (2004). *Stochastic Integration and Differential Equations* (3rd ed.). Springer.
- Protter, P. (2013). A Mathematical Theory of Financial Bubbles. In V. Henderson and R. Sircar (Eds.), *Paris-Princeton Lectures on Mathematical Finance*, Lecture Notes in Mathematics, vol. 2081, pp. 1–108. Springer, Berlin.
- Protter, P. and K. Shimbo (2008). No arbitrage and general semimartingales. In S. Ethier, J. Feng, and R. Stockbridge (Eds.), *Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz*, IMS Lecture Notes - Monograph Series 4, pp. 267–283.
- Shiryaev, A. N. (1999). *Essentials of Stochastic Finance: Facts, Models, Theory*. World Scientific.

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