LIPSCHITZ LIMIT SETS REVISITED: HILBERT ENTROPY AND NON-DIFFERENTIABILITY

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Abstract. We interpret the Hilbert entropy of a convex projective structure on a closed surface of higher genus as the Hausdorff dimension of the non-differentiability points of the limit set in the full flag space $\mathcal{F}(\mathbb{R}^3)$. Generalizations of this for hyperconvex representations are also discussed. An ingredient for the proofs is the concept of *hyperplane conicality* that we introduce for a θ-Anosov representation into a reductive real-algebraic Lie group G. In contrast with directional conicality, hyperplane-conical points always have full mass for the corresponding Patterson-Sullivan measures.

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1. Introduction

Consider a closed connected orientable surface S of genus at least two, and let $\rho : \pi_1 S \to \mathsf{PSL}(3,\mathbb{R})$ be a faithful representation preserving an open convex set $\Omega = \Omega_\rho \subset \mathbb{P}(\mathbb{R}^3)$, properly contained in an affine chart. The group $\rho(\pi_1 S)$ is necessarily discrete and acts co-compactly on Ω and one says that ρ *divides* Ω.

A classical result by Choi-Goldman [14] states that the space of all such representations forms a connected component of the character variety $\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}(3,\mathbb{R})\big)$ of homomorphisms up to conjugation. This component is known today as *the Hitchin component* of $\mathsf{PSL}(3,\mathbb{R})$ and is diffeomorphic to a ball of dimension $-8\chi(S)$.

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A lot of information on (the geometry of) such convex set Ω is known (see Benoist [5]). It is strictly convex with $\mathrm{C}^{1+\alpha}$ boundary $\partial\Omega$ (that is not C^2 unless it is an ellipse). Moreover, the Hilbert metric of Ω is Gromov-hyperbolic and the geodesic flow of $\Omega/\rho(\pi_1 S)$ is a smooth Anosov flow whose topological entropy, the *Hilbert entropy*, computed[1] by

$$\hbar^{\mathsf{H}} = \hbar^{\mathsf{H}}_{\rho} = \lim_{t\to\infty} \frac{1}{t} \log \#\big\{[g] \in [\pi_1 S] : (\lambda_1(g) + \lambda_1(g^{-1}))/2 \leqslant t\big\},$$

is bounded above: $\hbar^{\mathsf{H}} \leqslant 1$, see Crampon [15] who also showed that the inequality is furthermore strict if Ω is not an ellipse.

A consequence of Theorem A below is a new geometric interpretation of the Hilbert entropy which we now explain. For each $x \in \partial\Omega$ let $\Xi(x) \in \mathrm{Gr}_2(\mathbb{R}^3)$ be the unique plane whose projectivisation is tangent to $\partial\Omega$ at x. By Benoist [5], the image curve $\Xi(\partial\Omega) \subset \mathrm{Gr}_2(\mathbb{R}^3) \simeq \mathbb{P}((\mathbb{R}^3)^*)$ is also the boundary of a strictly convex divisible set Ω^* and is thus again a $\mathrm{C}^{1+\nu}$-circle.

The map $\Xi : \partial\Omega \to \partial\Omega^*$ is an equivariant homeomorphism between C^1-circles, so it is monotone and thus differentiable almost everywhere with respect to the Lebesgue measure associated to this C^1-structure. Equivalently one may consider the *full flag curve*

$$\{(x, \Xi(x)) : x \in \partial\Omega\} \subset \mathcal{F}(\mathbb{R}^3).$$

Differentiability of this full flag curve corresponds to points $x \in \partial\Omega$ where the derivative of Ξ exists or is infinite. We then establish the following:

Corollary A. *Let $\rho : \pi_1 S \to \mathsf{PSL}(3,\mathbb{R})$ divide a convex set Ω that is not an ellipse. Then the set of non-differentiability[2] points of Ξ has Hausdorff dimension $\hbar^{\mathsf{H}}_{\rho}$.*

The Hausdorff dimension in the above Corollary, and throughout this paper, is computed with respect to a Riemannian metric on the flag space. When Ω is an ellipse the result does not apply as the associated curve is actually algebraic, so the non-differentiability points is the empty set, but $\hbar^{\mathsf{H}} = 1$. On the other hand Nie [34] and Zhang [48] have found paths $\big(\rho_t : \pi_1 S \to \mathsf{PSL}(3,\mathbb{R})\big)_{t\geqslant 0}$ such that $\hbar^{\mathsf{H}}_{\rho_t} \to 0$ as $t \to \infty$. These facts together with Corollary A suggest that the closer Ω is to being an ellipse (the *Fuchsian locus*), the less differentiable the flag curve is whilst the furthest away from Fuchsian locus, the more regular the flag curve becomes.

The proof of Corollary A is outlined in § 1.4 and serves as a guide path for the strategy on the general case (Theorem's A and C below).

1.1. **Hyperconvex representations.** Denote by

$$\mathfrak{a} = \big\{\underline{a} = (a_1, \dots, a_d) \in \mathbb{R}^d : \sum_i a_i = 0\big\}$$

the Cartan subspace of the Lie group $\mathsf{PGL}(d,\mathbb{R})$, by

$$\tau_i(\underline{a}) = a_i - a_{i+1}$$

the i-th simple root and by $\mathfrak{a}^+ \subset \mathfrak{a}$ the Weyl chamber whose associated set of simple roots is $\Delta = \{\tau_i : i \in [\![1, d-1]\!]\}$. Let $a : \mathsf{PGL}(d,\mathbb{R}) \to \mathfrak{a}^+$ be the *Cartan projection* with respect to the choice of a inner product. Concretely the numbers $e^{a_i(g)}$ are the *singular values* of the matrix g, i.e. the square roots of the eigenvalues of the

[1] for $g \in \mathsf{PSL}(d,\mathbb{R})$ we let $\lambda_1(g)$ be the logarithm of the spectral radius of a lift of g to $\mathsf{SL}(d,\mathbb{R})$.

[2] We convene that 'infinite derivative' is a differentiability point.

matrix gg^*, where g^* is the adjoint operator of g with respect to the chosen inner product.

Let Γ be a finitely generated word-hyperbolic group, consider a finite symmetric generating set and let $|\,|$ be the associated word-length. For $k \in [\![1, d-1]\!]$; a representation $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{R})$ is $\{\tau_k\}$-*Anosov* if there exist positive constants μ and c such that for all $\gamma \in \Gamma$ one has[3]

$$\tau_k\big(a(\rho\gamma)\big) \geqslant \mu|\gamma| - c.$$

Of course a $\{\tau_k\}$-Anosov representation is also $\{\tau_{d-k}\}$-Anosov. Under such assumption it is also known the existence of an equivariant Hölder-continuous map

$$\xi^k_\rho : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{R}^d),$$

called the *limit map* in $\mathrm{Gr}_k(\mathbb{R}^d)$, which is an homeomorphism onto its image. If $k \leqslant l \in [\![1, d-1]\!]$ and ρ is also $\{\tau_l\}$-Anosov then the limit maps are *compatible*, meaning that that for all x one has $\xi^k_\rho(x) \subset \xi^l_\rho(x)$, see §4 for references and details. A $\{\tau_1\}$-Anosov representation is more commonly called a *projective Anosov* representation.

Definition 1.1. A $\{\tau_1, \tau_{d-2}\}$-Anosov representation $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{R})$ is $(1,1,2)$-*hyperconvex* if for every pairwise distinct triple $x, y, z \in \partial\Gamma$ one has

$$\xi^1_\rho(x) + \xi^1_\rho(y) + \xi^{d-2}_\rho(z) = \mathbb{R}^d. \tag{1.1}$$

Hyperconvex representations form an open subset of the character variety

$$\mathfrak{X}\big(\Gamma, \mathsf{PGL}(d, \mathbb{R})\big) = \hom\big(\Gamma, \mathsf{PGL}(d, \mathbb{R})\big)/\,\mathsf{PGL}(d, \mathbb{R})$$

and appear very naturally in higher rank Teichmüller theory; they are moreover the prototype example of Anosov representations with regular limit sets, indeed one has the following result from P.-S.-Wienhard [38] and Zhang-Zimmer [49].

Theorem 1.2. *Assume $\partial\Gamma$ is homeomorphic to a circle and let $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{R})$ be projective Anosov.*

[38],[49]*: If ρ is $(1,1,2)$-hyperconvex, then $\xi^1(\partial\Gamma) \subset \mathbb{P}(\mathbb{R}^d)$ is a C^1 submanifold tangent at $\xi^1(x)$ to $\xi^2(x)$.*

[49]*: If ρ is irreducible and $\xi(\partial\Gamma)$ is a C^1 circle then ρ is $(1,1,2)$-hyperconvex.*

Example 1.3. Recall that if G is a semi-simple real algebraic group of the non-compact type, then irreducible proximal representations $\Phi : \mathsf{G} \to \mathsf{PGL}(V)$ are determined by their highest restricted weight χ^+_Φ. A special subset of dominant weights are the so called *fundamental weights* $\{\varpi_\mathsf{a} : \mathsf{a} \in \Delta\}$, indexed by the set of simple roots Δ of G (see § 2.3 for definitions and details). Any pair of representations $\rho : \pi_1 S \to \mathsf{G}$ and $\Phi : \mathsf{G} \to \mathsf{PGL}(V)$ in each of the following classes (and small deformations of these), gives rise to a $(1,1,2)$-hyperconvex representation of a surface group via post-composition $\Phi \circ \rho$. In particular the limit set of ρ in the specified flag manifold of G is a $\mathrm{C}^{1+\alpha}$ curve:

- if G is split and $\rho : \pi_1 S \to \mathsf{G}$ is Hitchin, then we can take any Φ with $\chi^+_\Phi = n\varpi_\mathsf{a}$ for any $\mathsf{a} \in \Delta$and $n \in \mathbb{N}_{>0}$. This is non-trivial and requires results from Fock-Goncharov [17] and/or Labourie [30] together with Lusztig's canonical basis [31, Proposition 3.2] (see S. [43, § 5.8] for details). As a result the limit set of ρ in any maximal flag manifold $\mathcal{F}_{\{\mathsf{a}\}}$ of G is a $\mathrm{C}^{1+\alpha}$ curve.

[3] In order to lighten the notation we let here, and in the following, $\rho\gamma$ denote $\rho(\gamma)$

- if $\rho : \pi_1 S \to \mathsf{PO}(p,q)$ is Θ-positive then by P.-S.-Wienhard [37, Theorem 10.3] we can take Φ to have highest weight ϖ_{a} for any root a in the interior[4] of Θ (see also Beyrer-P. [8, Remark 4.6] for general Θ-positive representations). In particular the limit set in any flag manifold of the form $\mathsf{ls}_k(\mathbb{R}^{p,q})$ for $k \leqslant p-2$ is a $\mathrm{C}^{1+\alpha}$-curve. Zhang-Zimmer's result (Theorem 1.2 above) implies then that when ρ is moreover Zariski-dense, we can consider any Φ with $\chi^+_\Phi = n\varpi_{\mathsf{a}}$ for any $\mathsf{a} \in \operatorname{int}\Theta$ and $n \in \mathbb{N}_{>0}$.
- for all $k \geqslant 1$, k-positive representations $\rho : \pi_1 S \to \mathsf{PSL}(d,\mathbb{R})$ are $(1,1,2)$-hyperconvex (see Beyrer-P. [7]). They include small deformations of representations of the form $\iota_{s_1,\dots,s_r} \circ \rho_h : \pi_1 S \to \mathsf{PSL}(d,\mathbb{R})$ where $\rho_h : \pi_1 S \to \mathsf{PSL}(2,\mathbb{R})$ is discrete and faithful, $s_1 - 2k > s_2 > \dots > s_r$ and $\iota_{s_1,\dots,s_r} : \mathsf{PSL}(2,\mathbb{R}) \to \mathsf{PSL}(d,\mathbb{R})$ has an irreducible factor decomposition $\mathbb{R}^d = \mathbb{R}^{s_1} \oplus \dots \oplus \mathbb{R}^{s_r}$ (see [38, Proposition 6.16]). In particular the limit set in any Grassmanian of the form $\mathrm{Gr}_s(\mathbb{R}^d)$ for $s \leqslant k$ is a $\mathrm{C}^{1+\alpha}$-curve.

The main result of this paper deals with a pair of $(1,1,2)$-hyperconvex representations

$$\rho : \pi_1 S \to \mathsf{PGL}(d,\mathbb{R}) \text{ and } \overline{\rho} : \pi_1 S \to \mathsf{PGL}(\overline{d},\mathbb{R}),$$

for positive integers d and $\overline{d}$. In Section § 1.2 we deal also with Kleinian groups. To avoid confusion we denote the simple roots of $\mathsf{PGL}(\overline{d},\mathbb{R})$ by $\{\overline{\tau}_i : i \in [\![1, d-1]\!]\}$, and to simplify notation we let

$$\xi = \xi^1_\rho \text{ and } \overline{\xi} = \xi^1_{\overline{\rho}}.$$

The *graph map*

$$\mathscr{G} = \big(\xi, \overline{\xi}\big) : \partial\pi_1 S \to \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{\overline{d}}),$$

has image contained in the $\mathrm{C}^{1+\mathrm{Höl}}$ torus $\xi(\partial\pi_1 S) \times \overline{\xi}(\partial\pi_1 S)$ and, as the name suggests, the curve $\mathscr{G}(\partial\pi_1 S)$ is the graph of the Hölder-continuous homeomorphism

$$\Xi = \overline{\xi} \circ \xi^{-1} : \xi(\partial\pi_1 S) \to \overline{\xi}(\partial\pi_1 S)$$

between $\mathrm{C}^{1+\mathrm{Höl}}$-circles. By monotonicity of Ξ, $\mathscr{G}(\partial\pi_1 S)$ is a Lipschitz curve and is thus differentiable almost everywhere. We let $\mathsf{NDiff}_{\rho,\overline{\rho}} \subset \mathscr{G}(\partial\pi_1 S)$ be the subset of points where the curve $\mathscr{G}(\partial\pi_1 S)$ is not differentiable.

Remark 1.4. We record that a differentiability point $\mathscr{G}(x)$ of the curve $\mathscr{G}(\partial\pi_1 S)$ with oblique tangent, corresponds to a differentiability point $\xi(x)$ of the map Ξ, whose derivative is finite and non-zero.

Consider also the exponential rate

$$\hbar^\infty = \hbar^{\max\{\tau_1,\overline{\tau}_1\}} = \lim_{t\to\infty} \frac{1}{t} \log \#\big\{\gamma \in \Gamma : \max\big\{\tau_1\big(a(\rho\gamma)\big), \overline{\tau}_1\big(a(\overline{\rho}\gamma)\big)\big\} \leqslant t\big\}.$$

The main result of this paper is the following:

Theorem A. *Let $\rho : \pi_1 S \to \mathsf{PGL}(d,\mathbb{R})$ and $\overline{\rho} : \pi_1 S \to \mathsf{PGL}(\overline{d},\mathbb{R})$ be $(1,1,2)$-hyperconvex representations. Assume there exists $\gamma \in \pi_1 S$ such that $\tau_1\big(\lambda(\rho\gamma)\big) \neq \overline{\tau}_1\big(\lambda(\overline{\rho}\gamma)\big)$, then*

$$\dim_{\mathrm{Hff}}\big(\mathsf{NDiff}_{\rho,\overline{\rho}}\big) = \hbar^\infty < 1.$$

[4] only connected to roots in Θ in the Dynkin diagram of Δ

As before, the Hausdorff dimension is computed with respect to a Riemannian metric. We emphasize also that no irreducibility assumption is made on the representations ρ and $\overline{\rho}$. Furthermore, if the assumption is not satisfied, namely if for ever $\gamma \in \pi_1 S$, $\tau_1\big(\lambda(\rho\gamma)\big) = \overline{\tau}_1\big(\lambda(\overline{\rho}\gamma)\big)$, we show that there exists an isomorphism between the Zariski closures of $\rho(\pi_1 S)$ and of $\overline{\rho}(\pi_1 S)$ intertwining the two representations. It follows then that $\mathscr{G}(\partial\pi_1 S)$ is the diagonal of the $\mathrm{C}^{1+\mathrm{Höl}}$ torus, and is in particular differentiable everywhere. To prove this we give the following preliminary classification of Zariski-closures over $\mathbb{R}$ of irreducible $(1,1,2)$-hyperconvex representations, established in § 6.3.

Theorem B. *Consider an irreducible $(1,1,2)$-hyperconvex representation $\rho : \pi_1 S \to \mathsf{PGL}(d,\mathbb{R})$. Then the Zariski closure G of $\rho(\pi_1 S)$ is simple and the highest weight of the induced representation $\Phi : \mathsf{G} \to \mathsf{PGL}(d,\mathbb{R})$ is a multiple of a fundamental weight associated to a root that has one-dimensional root-space.*

In light of Examples 1.3 it is unclear which further restrictions can occur on G.

Remark 1.5. We observe that, relying on Theorem B and Theorem 1.2, a different approach to Theorem A would be to use a Bowen-Series coding for the action of $\pi_1 S$ on $\partial\pi_1 S$ and apply Jordan-Kesseböhmer-Pollicott-Stratmann [25, Theorem 1.1]. This is indeed the method followed by Pollicott-Sharp [35] when dealing with two elements of the Teichmüller space of S. This method however is not applicable for groups other than $\pi_1 S$ and we have thus decided to use a unified approach based on the theory of Anosov representations, leading at once results over $\mathbb{R}$ and $\mathbb{C}$.

1.2. **Kleinian groups.** The same strategy of proof of Theorem A gives the following result for Kleinian groups. Let Γ be a word-hyperbolic group and $\rho : \Gamma \to \mathsf{PSL}(2,\mathbb{C})$ be a convex co-compact action with limit set $\mathsf{L}_\rho \subset \partial\mathbb{H}^3$. We denote by $\lambda(g)$ the translation length on $\mathbb{H}^3$ of an element $g \in \mathsf{PSL}(2,\mathbb{C})$, and we set

$$\hbar_\rho = \lim_{t\to\infty} \frac{1}{t} \#\big\{[\gamma] \in [\Gamma] : \lambda(\rho\gamma) \leqslant t\big\}.$$

By Sullivan [45], the Hausdorff dimension of L_ρ coincides with $\hbar_\rho$.

If $\overline{\rho} : \Gamma \to \mathsf{PSL}(2,\mathbb{C})$ is again convex co-compact we let $\Xi : \mathsf{L}_\rho \to \mathsf{L}_{\overline{\rho}}$ be the map conjugating the respective actions. We consider in this case the complex derivative and say that Ξ is $\mathbb{C}$-*differentiable* at a given $x \in \mathsf{L}_\rho$ if, conformally identifying $\partial\mathbb{H}^3 - \{\text{point}\}$ to $\mathbb{C}$, the limit

$$\Xi'(x) := \lim_{\partial\Gamma \ni y \to x} \frac{\Xi(x) - \Xi(y)}{x - y}$$

exists or is infinite. We let now $\mathsf{NDiff}_{\rho,\overline{\rho}}$ be the set of points $x \in \mathsf{L}_\rho$ where Ξ is not $\mathbb{C}$-differentiable and let

$$\hbar^\infty = \lim_{t\to\infty} \frac{1}{t} \log \#\big\{[\gamma] \in [\Gamma] : \max\big\{\lambda(\rho\gamma), \lambda(\overline{\rho}\gamma)\big\} \leqslant t\big\}.$$

Theorem C. *Let $\rho, \overline{\rho} : \Gamma \to \mathsf{PSL}(2,\mathbb{C})$ be two convex co-compact representations that lie in the same connected component of*

$$\big\{\varrho : \Gamma \to \mathsf{PSL}(2,\mathbb{C}) : \ \varrho \text{ is convex co-compact}\big\}.$$

Assume moreover that $\hbar_\rho = \hbar_{\overline{\rho}}$, then $\dim_{\mathrm{Hff}}(\mathsf{NDiff}_{\rho,\overline{\rho}}) = \hbar^\infty$.

1.3. **Hyperplane vs directional conicality.** As the reader may check in the proof of Corollary A in § 1.4, a key ingredient in this particular case is a recent result by Burger-Landesberg-Lee-Oh [12], concerning directional conical points for Borel-Anosov representations, which we now explain. This result is used by Kim-Minsky-Oh [29] to find bounds on the Hausdorff dimension of directional conical limit points on the ambient group $\mathsf{PSO}(1,n)\times\mathsf{PSO}(1,n)$.

Let G be a real-algebraic semi-simple Lie group of the non-compact type, $\mathfrak{a}\subset\mathfrak{g}$ a Cartan subspace, $\Phi\subset\mathfrak{a}^*$ the associated root system and $\Delta\subset\Phi$ a choice of simple roots with associated Weyl chamber $\mathfrak{a}^+$. In [12] the authors introduce the notion of *directional conicality*, i.e. conicality along a specific direction of the Weyl chamber for Δ-Anosov representations, a notion later studied by S. [44] for directions on the Levi-subspace $\mathfrak{a}_\theta$ for a θ-Anosov representation $\rho:\Gamma\to\mathsf{G}$, for an arbitrary non-empty subset θ of Δ, see § 4.2 for the definitions.

If we let $\mathcal{L}_{\theta,\rho}\subset\mathfrak{a}_\theta$ be the θ-limit cone of ρ (see § 4.3) and $\mathsf{u}\in\mathbb{P}(\mathcal{L}_{\theta,\rho})$ be a direction on the relative interior of $\mathcal{L}_{\theta,\rho}$, then a point $x\in\partial\Gamma$ is said to be u-*conical* if there exists a conical sequence $(\gamma_n)_{n=0}^\infty\subset\Gamma$ converging to x such that the associated Cartan projections $a(\rho\gamma_n)$ lie in a tubular neighborhood of $\mathsf{u}\subset\mathfrak{a}_\theta$. To be consistent with the notation introduced later in the paper, recall that when $\rho(\Gamma)$ is Zariski-dense there are natural bijections (see § 4.3 for details)

$$\begin{aligned}\operatorname{int}\mathbb{P}(\mathcal{L}_{\theta,\rho})&\leftrightarrow\mathcal{Q}_{\theta,\rho}=\{\varphi\in(\mathfrak{a}_\theta)^*:\hbar_\varphi=1\}\\&\leftrightarrow\big\{\text{Patterson-Sullivan measures supported on }\xi^\theta(\partial\Gamma)\big\}.\end{aligned}$$

For $\varphi\in\mathcal{Q}_{\theta,\rho}$ we let $\mathsf{u}_\varphi\in\operatorname{int}\mathbb{P}(\mathcal{L}_{\theta,\rho})$ be the associated direction and μ^φ the associated Patterson-Sullivan measure. Burger-Landesberg-Lee-Oh [12] show then that, when $\theta=\Delta$ and $\rho(\Gamma)$ is Zariski-dense, one has

$$\mu^\varphi\big(\{x\in\partial\Gamma:x\text{ is }\mathsf{u}_\varphi\text{-conical}\}\big)=\begin{cases}1&\operatorname{rank}\mathsf{G}\leqslant 3\\0&\operatorname{rank}\mathsf{G}\geqslant 4\end{cases}.$$

Since the groups we are interested in have arbitrary rank we are taken to replace directional conicality by *hyperplane conicality*, these points have the advantage of always having full mass, as Theorem D below shows. To be precise, let $\rho:\Gamma\to\mathsf{G}$ be a θ-Anosov representation, consider a hyperplane $\mathsf{W}\subset\mathfrak{a}_\theta$ and assume, for the notion to be interesting, that W intersects the relative interior of $\mathcal{L}_{\theta,\rho}$. Then $x\in\partial\Gamma$ is W-*conical* if there exists a conical sequence $(\gamma_n)_0^\infty\subset\Gamma$ converging to x, a constant K and $w_n\in\mathsf{W}$ such that for all n one has

$$\big\|a(\rho\gamma_n)-w_n\big\|\leqslant K,$$

where $a:\mathsf{G}\to\mathfrak{a}^+$ is the Cartan projection. The set of such points will be denoted by $\partial_{\mathsf{W},\rho}\Gamma=\partial_{\mathsf{W}}\Gamma$. In § 4.4 we show the following, see Theorem 4.12 for more information.

Theorem D. *Let $\rho:\Gamma\to\mathsf{G}$ be a Zariski-dense θ-Anosov representation and W be a hyperplane of $\mathfrak{a}_\theta$ intersecting non-trivially the relative interior of $\mathcal{L}_{\theta,\rho}$. Then for every $\varphi\in\mathcal{Q}_{\theta,\rho}$ with $\mathsf{u}_\varphi\in\mathbb{P}(\mathsf{W}\cap\mathcal{L}_{\theta,\rho})$ one has $\mu^\varphi(\partial_{\mathsf{W}}\Gamma)=1$.*

1.4. **Strategy of the proof of Theorem A for strictly convex projective structures on surfaces.** Corollary A is indeed a consequence of Theorem A, where $\overline{\rho}=\rho^*$ is the dual representation of ρ. With this in mind, we sketch a direct proof of Corollary A serving as a guide-path for the general result.

Proof of Corollary A. Let $\rho:\pi_1S\to\mathsf{PSL}(3,\mathbb{R})$ be the holonomy of a strictly convex projective structure with invariant convex set Ω. In order to simplify notation we

identify γ and $\rho(\gamma)$. We consider the distances $d_{\mathbb{P}}$ on $\mathbb{P}(\mathbb{R}^3)$ and $d_{\mathbb{P}*}$ on $\mathbb{P}((\mathbb{R}^3)^*) = \mathrm{Gr}_2(\mathbb{R}^3)$, induced by an inner product on $\mathbb{R}^3$. We consider also the L^∞ distance on the product $(\mathbb{P}(\mathbb{R}^3), d_{\mathbb{P}}) \times (\mathbb{P}((\mathbb{R}^3)^*), d_{\mathbb{P}*})$, which is equivalent to the Riemannian distance, and thus induces the same Hausdorff dimension as a Riemannian metric.

We will use the *cone types* of $\pi_1 S$, introduced by Cannon [13] (for an arbitrary word-hyperbolic group). Fix a finite symmetric generating set on $\pi_1 S$ and let $|\,|$ be the associated word length. For $\gamma \in \pi_1 S$ we let its *cone type* be

$$\mathcal{C}(\gamma) = \{\eta \in \pi_1 S : |\gamma\eta| = |\eta| + |\gamma|\},$$

equivalently η belongs to $\mathcal{C}(\gamma)$ if there exists a geodesic on the Cayley graph of $\pi_1 S$, between γ^{-1} and η passing through the identity. The endpoints of rays in $\mathcal{C}(\gamma)$ form an open subset of $\partial\pi_1 S$ denoted by $\mathcal{C}_\infty(\gamma)$. See Figure 1.

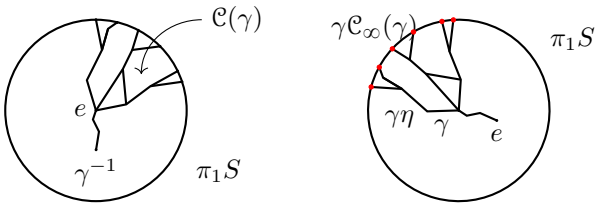

FIGURE 1. The cone type of $\gamma \in \Gamma$ (left). The set $\gamma \cdot \mathcal{C}_\infty(\gamma)$ (right). Pictures from P.-S.-Wienhard [38]

We let $\xi : \partial\pi_1 S \to \partial\Omega$ be the natural identification via the action of $\rho(\pi_1 S)$ on Ω, and analogously $\overline{\xi} : \pi_1 S \to \partial\Omega^*$. We denote by $\mathcal{G} := (\xi, \overline{\xi}) : \pi_1 S \to \partial\Omega \times \partial\Omega^*$ the flag curve. Consider $x \in \partial\pi_1 S$ and let $\alpha_i \to x$ be a geodesic ray on $\pi_1 S$. The following fact is a consequence of Proposition 5.7.

Fact. *For big enough i, the subset $\xi\big(\alpha_i \mathcal{C}_\infty(\alpha_i)\big) \subset \partial\Omega$ is coarsely the intersection of a ball of radius $e^{-\tau_1(\alpha_i)}$ about $\xi(x)$, with $\partial\Omega$. By duality one has $\overline{\xi}\big(\alpha_i \mathcal{C}_\infty(\alpha_i)\big) \subset \partial\Omega^*$ is coarsely the intersection of a ball of radius $e^{-\tau_2(\alpha_i)}$ about $\overline{\xi}(x)$, with $\partial\Omega^*$.*

The coarse constants and the minimal length i required in the above statement depend only on the representation and not on the point x.

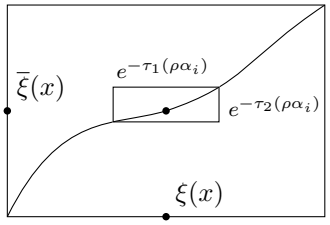

FIGURE 2. The image of the cone type $\alpha_i \mathcal{C}_\infty(\alpha_i)$ by the graph curve $\mathcal{G}$ in the $\mathrm{C}^{1+\mathrm{Höl}}$-torus $\partial\Omega \times \partial\Omega^*$.

For any point $x \in \partial\pi_1 S$ we distinguish two complementary situations that don't depend on the choice of the geodesic ray $(\alpha_i)_{i\in\mathbb{N}}$ converging to x:

i) For all $R > 0$ there exists $N \in \mathbb{N}$ with $|\tau_1(a(\alpha_i)) - \tau_2(a(\alpha_i))| \geqslant R$ for all $i \geqslant N$;

ii) There exists $R > 0$ and an infinite set of indices $\mathbb{I} \subset \mathbb{N}$ such that for all $k \in \mathbb{I}$ one has $|\tau_1(a(\alpha_k)) - \tau_2(a(\alpha_k))| \leqslant R$. We say in this case that x is $\flat$-*conical*.

In the first case one is easily convinced by looking at Figure 2 that the rectangle becomes flatter along one of its sides (see § 7 for details in the general case). Furthermore, since $\tau_1(a(\alpha_i)) - \tau_1(a(\alpha_{i+1}))$ is uniformly bounded, its sign is eventually constant, and thus the flatter side only depend on the point. As a result x is necessarily a differentiability point of the graph curve $\mathscr{G}$, with either horizontal or vertical derivative.

We are thus bound to understand the set of $\flat$-conical points. Indeed we show (see Proposition 7.1):

Fact. *The non-differentiabilty points of the curve $\mathscr{G}(\partial\pi_1 S)$ and the $\flat$-conical points coincide.*

The main idea for this is to show that if a $\flat$-conical point x were a differentiability point, then the derivative cannot be horizontal nor vertical, and thus (by Proposition 6.2) Ξ would be bi-Lipschitz. In turn, this would force the periods of the two roots to agree, contradicting the Zariski-density assumption.

We have to understand then the Hausdorff dimension of the set of $\flat$-conical points. The upper bound (Proposition 5.11)

$$\dim_{\mathrm{Hff}}\big(\{\flat - \text{conical}\}\big) \leqslant \hbar^{\max\{\tau_1,\tau_2\}} \tag{1.2}$$

follows readily: since for a $\flat$-conical point the lengths $e^{-\tau_1(\alpha_k)}$ and $e^{-\tau_2(\alpha_k)}$ are comparable independently on $k \in \mathbb{I}$, one can replace the rectangle in Figure 2 by the (smaller) square of length

$$e^{-\max\{\tau_1(a(\alpha_k)),\tau_2(a(\alpha_k))\}}$$

and still get a covering[5] (this time by balls on the L^∞ metric) of the set $\{\flat-\text{conical}\}$. Standard arguments on Hausdorff dimension give Equation (1.2).

Finding a lower bound for the Hausdorff dimension is more subtle, and usually requires a probability measure to get hands on how the mass of a ball of radius r scales with r.

In this case, since $\mathscr{G}(\partial\pi_1 S)$ is a subset the full flag space $\mathcal{F}(\mathbb{R}^3)$ and

$$\|v\|_\infty := \max\{|\tau_1(v)|, |\tau_2(v)|\}$$

is a norm on $\mathfrak{a}_{\mathsf{PSL}(3,\mathbb{R})}$, we can apply results by Quint [39] to obtain a linear form $\varphi_\infty \in \mathfrak{a}^*$ with

$$\hbar^{\max\{\tau_1,\tau_2\}} = \|\varphi_\infty\|^1,$$

where $\|\,\|^1$ is the operator norm on $\mathfrak{a}^*$ defined by $\|\,\|_\infty$, which turns out to be the L^1 norm $\|a\tau_1 + b\tau_2\|^1 = |a| + |b|$. The form φ_∞ additionally admits an associated *Patterson-Sullivan* probability measure, namely a measure μ^∞ such that for all $\gamma \in \pi_1 S$ one has (see Lemma 4.2)

$$\mu^\infty\big(\mathscr{G}(\gamma\mathcal{C}_\infty(\gamma))\big) \leqslant Ce^{-\varphi_\infty(a(\gamma))}. \tag{1.3}$$

A key extra information available in the case of $\mathsf{PSL}(3,\mathbb{R})$ is that we can determine φ_∞ explicitly. For this we need a small parenthesis on the *critical hypersurface* $\mathcal{Q}_\rho$ of ρ, depicted in Figure 3, and characterized by

$$\mathcal{Q}_\rho = \big\{\varphi \in \mathfrak{a}^* : \hbar_\varphi = 1\big\},$$

[5] choosing the longer side $e^{-\min\{\tau_1(a(\alpha_k)),\tau_2(a(\alpha_k))\}}$ gives the bound $\dim_{\mathrm{Hff}} \mathscr{G}(\partial\pi_1 S) \leqslant 1$.

where, the *entropy* of a functional $\varphi \in \mathfrak{a}^*$ is

$$\hbar_\varphi := \lim_{t\to\infty} \frac{1}{t} \log \#\{\gamma \in \pi_1 S : \varphi(a(\gamma)) \leqslant t\} \in (0, \infty].$$

The interest of the critical hypersurface lies in the fact that $\Omega_\rho \subset \mathfrak{a}^*$ is a closed analytic curve that bounds a strictly convex set (S. [40] and Potrie-S. [36]), and thus by Quint [39], the linear form φ_∞ is uniquely determined by

$$\|\varphi_\infty\|^1 = \inf\{\|\varphi\|^1 : \varphi \in \Omega_\rho\}. \tag{1.4}$$

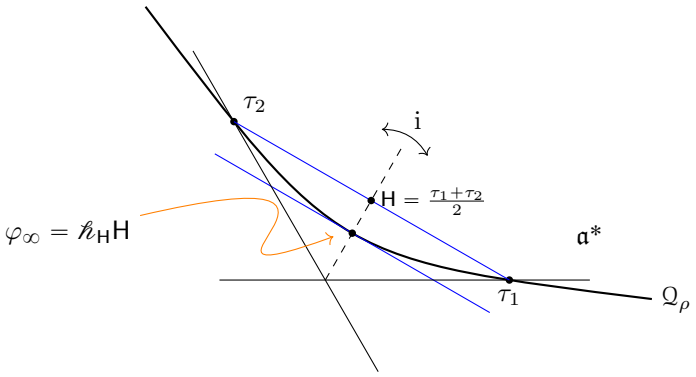

FIGURE 3. The critical hypersurface of a strictly convex projective structure on S. Since H is a convex combination of $\{\tau_1, \tau_2\}$ one has $\|\mathsf{H}\|^1 = 1$ and thus $\|\varphi_\infty\|^1 = \hbar_\mathsf{H}$.

Again by [36] one has $\{\tau_1, \tau_2\} \subset \Omega_\rho$. Together with i-invariance of the picture (see again Figure 3) we deduce that, if we let $\mathsf{H} = (\tau_1 + \tau_2)/2$, then

$$\varphi_\infty = \hbar_\mathsf{H} \cdot \mathsf{H} \geqslant \hbar_\mathsf{H} \min\{\tau_1, \tau_2\}. \tag{1.5}$$

In particular, using Equation (1.4), we obtain that $\hbar^{\max\{\tau_1,\tau_2\}} = \hbar_\mathsf{H}$.

After this small parenthesis on the critical hypersurface, we come back to the lower bound on the Hausdorff dimension. Since $\mathscr{G}$ is a graph, $\mathscr{G}(\partial\pi_1 S)$ has the same intersection with the rectangle in Figure 2 than with the larger square of size

$$e^{-\min\{\tau_1(a(\alpha_i)),\tau_2(a(\alpha_i))\}};$$

this square is now a ball (for the L^∞ metric) of radius $e^{-\min\{\tau_1(a(\alpha_i)),\tau_2(a(\alpha_i))\}}$. Thus for all i, $\mathscr{G}\big(\alpha_i \mathcal{C}_\infty(\alpha_i)\big)$ is coarsely a ball of the latter radius and one has

$$\begin{aligned}\mu^\infty\big(B(\mathscr{G}(x), e^{-\min\{\tau_1(a(\alpha_i)),\tau_2(a(\alpha_i))\}}\big) &\leqslant \mu^\infty\big(\mathscr{G}(\alpha_i \mathcal{C}_\infty(\alpha_i)\big) \leqslant C e^{-\varphi_\infty(a(\alpha_i))} \\ &\leqslant C\big(e^{-\min\{\tau_1(a(\alpha_i)),\tau_2(a(\alpha_i))\}}\big)^{\hbar_\mathsf{H}},\end{aligned}$$

where the last inequalities follow from Equations (1.3) and (1.5). This gives a coarser constant C' such that, for all r,

$$\mu^\infty\big(B(\mathscr{G}(x), r)\big) \leqslant C' r^{\hbar_\mathsf{H}}.$$

Again, classical Hausdorff dimension arguments (c.f. Corollary 5.9 below) give that, for any measurable subset $E \subset \mathscr{G}(\partial\pi_1 S)$ with full μ^∞ mass, one has $\dim_{\mathrm{Hff}}(E) \geqslant \hbar_\mathsf{H}$.

A recent result by Burger-Landesberg-Lee-Oh [12, Theorem 1.6], implies, since $\mathsf{PSL}(3,\mathbb{R})$ has rank $2 \leqslant 3$ and ρ is Δ-Anosov, that $\mu^{\infty}(\{\mathfrak{b}\text{-conical}\}) = 1$ and thus we have the desired lower bound

$$\dim_{\mathrm{Hff}}\left(\{\mathfrak{b}\text{-conical}\}\right) \geqslant \hbar_{\mathsf{H}},$$

which combined with the upper bound (1.2) and the equality $\hbar^{\max\{\tau_1,\tau_2\}} = \hbar_{\mathsf{H}}$, gives the proof of Corollary A. □

We end this introduction with the following consequence of Theorem A for Hitchin representations.

Corollary B. *Let G be an adjoint real split group and $\rho : \pi_1 S \to \mathsf{G}$ a Zariski-dense Hitchin representation. For $\{\mathsf{a},\mathsf{b}\} \subset \Delta$ consider the limit curve $\xi^{\{\mathsf{a},\mathsf{b}\}} : \partial\pi_1 S \to \mathcal{F}_{\{\mathsf{a},\mathsf{b}\}}$, then it is a Lipschitz curve and the Hausdorff dimension of the points where it is non-differentiable coincides with $\hbar^{\max\{\mathsf{a},\mathsf{b}\}}$. If the Dynkin diagram of $\mathfrak{g}$ carries a non-trivial involution i and $\mathsf{b} = \mathsf{ia}$ then $\hbar^{\max\{\mathsf{a},\mathsf{b}\}} = \hbar^{(\mathsf{a}+\mathsf{b})/2}$.*

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2. Linear algebraic groups

Throughout the text G will denote a real-algebraic semi-simple Lie group of non-compact type and $\mathfrak{g}$ its Lie algebra.

2.1. Linear algebraic groups.

Fix a Cartan involution $o : \mathfrak{g} \to \mathfrak{g}$ with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and let $\Phi \subset \mathfrak{a}^*$ be the set of restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$. For $\mathsf{a} \in \Phi$, we denote by

$$\mathfrak{g}_{\mathsf{a}} = \{u \in \mathfrak{g} : [a,u] = \mathsf{a}(a)u \ \forall a \in \mathfrak{a}\}$$

its associated root space. The (restricted) root space decomposition is $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\mathsf{a}\in\Phi} \mathfrak{g}_{\mathsf{a}}$, where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$. Fix a Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$ and let Φ^+ and Δ be, respectively, the associated sets of positive and simple roots. Let $\mathcal{W}$ be the Weyl group of Φ and $\mathsf{i} : \mathfrak{a} \to \mathfrak{a}$ be the opposition involution: if $u : \mathfrak{a} \to \mathfrak{a}$ is the unique element in $\mathcal{W}$ with $u(\mathfrak{a}^+) = -\mathfrak{a}^+$ then $\mathsf{i} = -u$.

We denote by $(\cdot,\cdot)$ both the Killing form of $\mathfrak{g}$, its restriction to $\mathfrak{a}$, and its associated dual form on $\mathfrak{a}^*$, the dual of $\mathfrak{a}$. For $\chi, \psi \in \mathfrak{a}^*$ let

$$\langle \chi, \psi \rangle = 2\frac{(\chi,\psi)}{(\psi,\psi)}.$$

The *restricted weight lattice* is defined by

$$\Pi = \{\varphi \in \mathfrak{a}^* : \langle \varphi, \mathsf{a} \rangle \in \mathbb{Z} \ \forall \mathsf{a} \in \Phi\}.$$

It is spanned by the *fundamental weights* $\{\varpi_{\mathsf{a}} : \mathsf{a} \in \Delta\}$, defined by

$$\langle \varpi_{\mathsf{a}}, \mathsf{b} \rangle = d_{\mathsf{a}}\delta_{\mathsf{ab}} \tag{2.1}$$

for every $\mathsf{a},\mathsf{b} \in \Delta$, where $d_{\mathsf{a}} = 1$ if $2\mathsf{a} \notin \Phi^+$ and $d_{\mathsf{a}} = 2$ otherwise.

A subset $\theta \subset \Delta$ determines a pair of opposite parabolic subgroups P_θ and $\check{\mathsf{P}}_\theta$ whose Lie algebras are defined by

$$\mathfrak{p}_\theta = \bigoplus_{\mathsf{a}\in\Phi^+\cup\{0\}} \mathfrak{g}_\mathsf{a} \oplus \bigoplus_{\mathsf{a}\in\langle\Delta-\theta\rangle} \mathfrak{g}_{-\mathsf{a}},$$
$$\check{\mathfrak{p}}_\theta = \bigoplus_{\mathsf{a}\in\Phi^+\cup\{0\}} \mathfrak{g}_{-\mathsf{a}} \oplus \bigoplus_{\mathsf{a}\in\langle\Delta-\theta\rangle} \mathfrak{g}_\mathsf{a}.$$

The group $\check{\mathsf{P}}_\theta$ is conjugated to the parabolic group $\mathsf{P}_{\mathrm{i}\theta}$. We denote the *flag space* associated to θ by $\mathcal{F}_\theta = \mathsf{G}/\mathsf{P}_\theta$. The G orbit of the pair $([\mathsf{P}_\theta], [\check{\mathsf{P}}_\theta])$ is the unique open orbit for the action of G in the product $\mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}\theta}$ and is denoted by $\mathcal{F}^{(2)}_\theta$.

2.2. **Cartan and Jordan projection.** Denote by $\mathsf{K} = \exp\mathfrak{k}$ and $\mathsf{A} = \exp\mathfrak{a}$. The *Cartan decomposition* asserts the existence of a continuous map $a : \mathsf{G} \to \mathfrak{a}^+$, called the *Cartan projection*, such that every $g \in \mathsf{G}$ can be written as $g = ke^{a(g)}l$ for some $k, l \in \mathsf{K}$.

We will need the following standard fact.

Proposition 2.1 (Benoist [2, Proposition 5.1])**.** *For any compact $L \subset \mathsf{G}$ there exists a compact set $H \subset \mathfrak{a}$ such that, for every $g \in \mathsf{G}$ one has*

$$a(LgL) \subset a(g) + H.$$

By the Jordan's decomposition, every element $g \in \mathsf{G}$ can be uniquely written as a commuting product $g = g_e g_{ss} g_u$ where g_e is conjugate to an element in K, g_{ss} is conjugate to an element in $\exp(\mathfrak{a}^+)$ and g_u is unipotent. The *Jordan projection* $\lambda = \lambda_\mathsf{G} : \mathsf{G} \to \mathfrak{a}^+$ is the unique map such that g_{ss} is conjugated to $\exp\big(\lambda(g)\big)$.

Definition 2.2. Let $\Gamma \subset \mathsf{G}$ be a discrete subgroup, then its *limit cone* $\mathcal{L}_\Gamma$ is the smallest closed cone of the closed Weyl chamber $\mathfrak{a}^+$ that contains $\{\lambda(g) : g \in \Gamma\}$.

We will need the following result.

Theorem 2.3 (Benoist [3, 4])**.** *Let $\Gamma \subset \mathsf{G}$ be a Zariski-dense subgroup, then its limit cone $\mathcal{L}_\Gamma$ has non-empty interior. Moreover, the group spanned by the Jordan projections $\{\lambda(g) : g \in \Gamma\}$ is dense in $\mathfrak{a}$.*

2.3. **Representations of G.** The standard references for the following are Fulton-Harris [18], Humphreys [23] and Tits [46].

Let $\Phi : \mathsf{G} \to \mathsf{PGL}(V)$ be a finite dimensional rational[6] irreducible representation and denote by $\phi_\Phi : \mathfrak{g} \to \mathfrak{sl}(V)$ the Lie algebra homomorphism associated to Φ. The *weight space* associated to $\chi \in \mathfrak{a}^*$ is the vector space

$$V_\chi = \{v \in V : \phi_\Phi(a)v = \chi(a)v \ \forall a \in \mathsf{A}\}.$$

We say that $\chi \in \mathfrak{a}^*$ is a *restricted weight* of Φ if $V_\chi \neq 0$. Tits [46, Theorem 7.2] states that the set of weights has a unique maximal element with respect to the partial order $\chi \succ \psi$ if $\chi - \psi$ is a $\mathbb{N}$-linear combination of positive roots. This is called *the highest weight* of Φ and denoted by χ_Φ. By definition, for every $g \in \mathsf{G}$ one has

$$\lambda_1\big(\Phi(g)\big) = \chi_\Phi\big(\lambda(g)\big). \tag{2.2}$$

[6] i.e. a rational map between algebraic varieties.

Moreover (see for example Humphreys [23, §13.4 Lemma B]), the set of restricted weights $\Pi(\phi) = \{\chi \in \mathfrak{a}^* : V_\chi \neq \{0\}\}$ of the representation ϕ_Φ, is exactly the set of weights bounded above by χ_Φ,

$$\Pi(\phi_\Phi) = \{\chi \in \mathfrak{a}^* : \chi \prec \chi_\Phi\},$$

namely every weight $\chi \in \Pi(\phi_\Phi)$ has the form

$$\chi_\Phi - \sum_{\mathsf{a}\in\Delta} n_\mathsf{a}\mathsf{a} \text{ for } n_\mathsf{a} \in \mathbb{N}.$$

The *level* of a weight χ is the integer $\sum_\mathsf{a} n_\mathsf{a}$, the highest weight is thus the only weight of level zero.

One other fact we will need concerns the a-*string* of χ. If $\chi \in \Pi(\phi_\Phi)$ and $\mathsf{a} \in \Phi^+$ then the elements of $\Pi(\phi_\Phi)$ of the form $\chi + j\mathsf{a}$, $j \in \mathbb{Z}$ form an unbroken string

$$\chi + j\mathsf{a},\ j \in [\![-r, q]\!]$$

and $r - q = \langle \chi, \mathsf{a}\rangle$. One can then recover algorithmically the set $\Pi(\phi_\Phi)$ level by level starting from χ_Φ, as follows:

- assume the set of weights of level $\leqslant k$ is known and consider a weight χ of level k.
- For each $\mathsf{a} \in \Delta$ compute $\langle \chi, \mathsf{a}\rangle$, this gives the length $r - q$ of the a-string through χ. The weights of the form $\chi + j\mathsf{a}$, for positive j, have level $\leqslant k$ and are thus known, thus we can decide whether $\chi - \mathsf{a}$ is a weight or not, determining the set of weights of level $k + 1$.

We record the following Lemma that follows at once from the algorithmic description above.

Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be the decomposition in simple factors of a semi-simple real Lie algebra of the non-compact type. Recall that if $\mathfrak{a}_i \subset \mathfrak{g}_i$ is a Cartan subspace, then $\mathfrak{a} = \bigoplus_i \mathfrak{a}_i$ is a Cartan subspace of $\mathfrak{g}$. Any $\varphi \in (\mathfrak{a}_i)^*$ extends to a functional, still denoted φ, on $\mathfrak{a}$ vanishing on the remaining factors. The restricted root system of $\mathfrak{g}$ is then $\Delta_\mathfrak{g} = \bigcup \Delta_{\mathfrak{g}_i}$. The *associated simple factor* to $\mathsf{a} \in \Delta_\mathfrak{g}$ is $\mathfrak{g}_i$ such that $\mathsf{a} \in \Delta_i$.

Lemma 2.4. *Let $\mathfrak{g}$ be a semi-simple real Lie algebra of non-compact type and ϕ be an irreducible representation of $\mathfrak{g}$ whose highest restricted weight is a multiple of a fundamental weight, $\chi_\phi = k\varpi_\mathsf{a}$ for some $\mathsf{a} \in \Delta$. Then ϕ factors as a representation of the simple factor associated to a.*

Proof. Proceeding by induction on the levels of ϕ, one readily sees that for every $\tau \in \Delta_j$ for $j \neq i$ and all $\chi \in \Pi(\phi)$ one has $\langle \chi, \tau\rangle = 0$. Thus the associated root space $(\mathfrak{g}_j)_{-\tau}$ acts trivially on every weight space of ϕ and so the whole factor $\mathfrak{g}_j$ acts trivially. □

The following set of simple roots plays a special role in representation theory.

Definition 2.5. The set of simple roots $\mathsf{a} \in \Delta$ such that $\chi_\Phi - \mathsf{a}$ is still a weight of Φ is denoted by θ_Φ. Equivalently (see Humphreys [23]), one has

$$\theta_\Phi = \{\mathsf{a} \in \Delta : \langle \chi_\Phi, \mathsf{a}\rangle \neq 0\}. \tag{2.3}$$

The following Lemma will be needed in Theorem B.

Lemma 2.6. *Let $\mathfrak{g}$ be semi-simple of the non-compact type and $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ an irreducible representation. Consider $\mathsf{a} \in \theta_\phi$, then no element of $\mathfrak{g}_{-\mathsf{a}}$ acts trivially on V^+.*

Proof. By definition of χ_ϕ every $n \in \mathfrak{g}_{\mathsf{a}}$ acts trivially on V^+. Consider then $y \in \mathfrak{g}_{-\mathsf{a}} - \{0\}$, there exists then $x \in \mathfrak{g}_{\mathsf{a}}$ such that $\{x, y, h_{\mathsf{a}}\}$ spans a Lie algebra isomorphic to ${}_2(\mathbb{R})$, where h_{a} is defined by $\varphi(h_{\mathsf{a}}) = \langle \varphi, \mathsf{a} \rangle$ for all $\varphi \in \mathfrak{a}^*$. If $\phi(y)V^+ = 0$ then, since $\phi(x)V^+ = 0$ one concludes $\phi(h_{\mathsf{a}})V^+ = 0$. This in turn implies that

$$\langle \chi_\phi, \mathsf{a} \rangle = \chi_\phi(h_{\mathsf{a}}) = 0$$

contradicting that $\mathsf{a} \in \theta_\phi$. □

We denote by $\| \, \|_\Phi$ an Euclidean norm on V invariant under $\Phi\mathsf{K}$ and such that $\Phi\mathsf{A}$ is self-adjoint, see for example Benoist-Quint's book [6, Lemma 6.33]. By definition of χ_Φ and $\| \, \|_\Phi$ one has, for every $g \in \mathsf{G}$, that

$$\log \|\Phi g\|_\Phi = \chi_\Phi(a(g)). \tag{2.4}$$

Here, with a slight abuse of notation, we denote by $\| \, \|_\Phi$ also the induced operator norm, which doesn't depend on the scale of $\| \, \|_\Phi$.

Denote by W_{χ_Φ} the $\Phi\mathsf{A}$-invariant complement of V_{χ_Φ}. The stabilizer in G of W_{χ_Φ} is $\check{\mathsf{P}}_{\theta_\Phi}$, and thus one has a map of flag spaces

$$(\zeta_\Phi, \zeta_\Phi^*) : \mathcal{F}^{(2)}_{\theta_\Phi}(\mathsf{G}) \to \mathrm{Gr}^{(2)}_{\dim V_{\chi_\Phi}}(V). \tag{2.5}$$

This is a proper embedding which is an homeomorphism onto its image. Here $\mathrm{Gr}^{(2)}_{\dim V_{\chi_\Phi}}(V)$ is the open $\mathsf{PGL}(V)$-orbit in the product of the Grassmannian of $(\dim V_{\chi_\Phi})$-dimensional subspaces and the Grassmannian of $(\dim V - \dim V_{\chi_\Phi})$-dimensional subspaces. One has the following proposition (see also Humphreys [24, Chapter XI]).

Proposition 2.7 (Tits [46])**.** *For each $\mathsf{a} \in \Delta$ there exists a finite dimensional rational irreducible representation $\Phi_{\mathsf{a}} : \mathsf{G} \to \mathsf{PSL}(V_{\mathsf{a}})$, such that $\chi_{\Phi_{\mathsf{a}}}$ is an integer multiple $l_{\mathsf{a}}\varpi_{\mathsf{a}}$ of the fundamental weight and $\dim V_{\chi_{\Phi_{\mathsf{a}}}} = 1$.*

We will fix from now on such a set of representations and call them, for each $\mathsf{a} \in \Delta$, the *Tits representation associated to* a.

2.4. **The center of the Levi group $\mathsf{P}_\theta \cap \check{\mathsf{P}}_\theta$.** We now consider the vector subspace

$$\mathfrak{a}_\theta = \bigcap_{\mathsf{a} \in \Delta - \theta} \ker \mathsf{a}.$$

It is equipped with a projection $\pi_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ uniquely determined by being invariant under the subgroup $\mathcal{W}_\theta$ of the Weyl group spanned by reflections associated to roots in $\Delta - \theta$: $\mathcal{W}_\theta = \{w \in \mathcal{W} : w(v) = v \quad \forall v \in \mathfrak{a}_\theta\}$.

Its dual $(\mathfrak{a}_\theta)^*$ is canonically identified with the subspace of $\mathfrak{a}^*$ of π_θ-invariant linear forms. Such space is spanned by the fundamental weights of roots in θ

$$(\mathfrak{a}_\theta)^* = \{\varphi \in \mathfrak{a}^* : \varphi \circ \pi_\theta = \varphi\} = \langle \varpi_{\mathsf{a}} | \mathfrak{a}_\theta : \mathsf{a} \in \theta \rangle.$$

We will denote by $\lambda_\theta = \pi_\theta \circ \lambda : \mathsf{G} \to \mathfrak{a}_\theta \cap \mathfrak{a}^+$ the composition of the Jordan projection and the projection π_θ.

2.5. **The Buseman-Iwasawa cocycle.** The *Iwasawa decomposition* of G states that every $g \in \mathsf{G}$ can be written uniquely as a product lzu with $l \in \mathsf{K}$, $z \in \mathsf{A}$ and $u \in \mathsf{U}_\Delta$, where U_Δ is the unipotent radical of P_Δ.

The *Buseman-Iwasawa cocycle* of G is the map $\mathrm{b} : \mathsf{G} \times \mathcal{F} \to \mathfrak{a}$ such that, for all $g \in \mathsf{G}$ and $k[\mathsf{P}_\Delta] \in \mathcal{F}$,

$$\mathrm{b}(g, k[\mathsf{P}_\Delta]) = \log(z)$$

where $\log : \mathsf{A} \to \mathfrak{a}$ denotes the inverse of the exponential map, and $gk = lzu$ is the Iwasawa decomposition of gk. Quint [39, Lemmes 6.1 and 6.2] proved that the function $\mathrm{b}_\theta = p_\theta \circ \mathrm{b}$ factors as a cocycle $\mathrm{b}_\theta : \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$.

The Buseman-Iwasawa cocycle can also be read from the representations of G. Indeed, Quint [39, Lemme 6.4] shows that for every $g \in \mathsf{G}$ and $x \in \mathcal{F}_\theta$ one has

$$l_{\mathsf{a}} \varpi_{\mathsf{a}}(\mathrm{b}(g, x)) = \log \frac{\|\Phi_{\mathsf{a}}(g)v\|_\Phi}{\|v\|_\Phi}, \tag{2.6}$$

where $v \in \zeta_{\Phi_{\mathsf{a}}}(x) \in \mathbb{P}(\mathsf{V}_{\mathsf{a}})$ is non-zero, and l_{a} is as in Proposition 2.7.

2.6. **Gromov product and Cartan attractors.** For a decomposition $\mathbb{K}^d = \ell \oplus V$ into a line ℓ and a hyperplane V together with an inner (Hermitian) product o on $\mathbb{K}^d$, one defines the *Gromov product* by

$$\mathcal{G}(V, \ell) = \mathcal{G}^o(V, \ell) := \log \frac{|\varphi(v)|}{\|\varphi\|\|v\|} = \log \sin \measuredangle_o(\ell, V),$$

for any non-vanishing $v \in \ell$ and $\varphi \in (\mathbb{K}^d)^*$ with $\ker \varphi = V$.

On then considers the *Gromov product* $\mathcal{G}_\theta : \mathcal{F}_\theta^{(2)} \to \mathfrak{a}_\theta$ defined, for every $(x, y) \in \mathcal{F}_\theta^{(2)}$ and $\mathsf{a} \in \theta$, by

$$l_{\mathsf{a}} \varpi_{\mathsf{a}}\big(\mathcal{G}_\theta(x, y)\big) = \mathcal{G}^{\Phi_{\mathsf{a}}}(\zeta^*_{\Phi_{\mathsf{a}}} x, \zeta_{\Phi_{\mathsf{a}}} y) = \log \sin \measuredangle_{\Phi_{\mathsf{a}}}\big(\zeta_{\Phi_{\mathsf{a}}} y, \zeta^*_{\Phi_{\mathsf{a}}} x\big),$$

where $\zeta^*_{\Phi_{\mathsf{a}}}$ and $\zeta_{\Phi_{\mathsf{a}}}$ are the equivariant maps from Equation (2.5). From S. [41, Lemma 4.12] one has, for all $g \in \mathsf{G}$ and $(x, y) \in \mathcal{F}_\theta^{(2)}$,

$$\mathcal{G}_\theta(gx, gy) - \mathcal{G}_\theta(x, y) = -\big(\mathrm{i}\, \mathrm{b}_{\mathrm{i}\theta}(g, x) + \mathrm{b}_\theta(g, y)\big). \tag{2.7}$$

If $g = k \exp(a(g)) l$ is a Cartan decomposition of $g \in \mathsf{G}$ we define its θ-*Cartan attractor* (resp. *repeller*) by

$$U_\theta(g) = k[\mathsf{P}_\theta] \in \mathcal{F}_\theta \text{ and } U_{\mathrm{i}\theta}(g^{-1}) = l^{-1}[\check{\mathsf{P}}_\theta] \in \mathcal{F}_{\mathrm{i}\theta}.$$

The *Cartan basin of* g is defined, for $\alpha > 0$, by

$$B_{\theta,\alpha}(g) = \big\{x \in \mathcal{F}_\theta : \varpi_{\mathsf{a}} \mathcal{G}_\theta\left(U_{\mathrm{i}\theta}(g^{-1}), x\right) > -\alpha,\ \forall \mathsf{a} \in \theta\big\}.$$

Remark 2.8. Observe that a statement of the form $\varpi_{\mathsf{a}} \mathcal{G}_\theta(x, y) \geqslant -\kappa$ for all $\mathsf{a} \in \theta$ is a quantitative version (depending on the choice of K) of the transversality between x and y; in particular it implies that x and y are transverse.

Neither the Cartan attractor nor its basin are uniquely defined unless for all $\mathsf{a} \in \theta$ one has $\mathsf{a}\big(a(g)\big) > 0$, regardless one has the following:

Remark 2.9. Given $\alpha > 0$ there exists a constant K_α such that if $y \in \mathcal{F}_\theta$ belongs to $B_{\theta,\alpha}(g)$ then one has

$$\big\|a_\theta(g) - \mathrm{b}_\theta(g, y)\big\| \leqslant K_\alpha. \tag{2.8}$$

Indeed, using Tits's representations of G and Equations (2.4) and (2.6) this boils down to the elementary fact that if $A \in \mathsf{GL}_d(\mathbb{R})$ verifies $\tau_1(a(A)) > 0$ then for every $v \in \mathbb{R}^d$ one has

$$\log \frac{\|Av\|}{\|v\|} \geqslant \log \|A\| + \log \sin \measuredangle\big(\mathbb{R} \cdot v, U_{d-1}(A^{-1})\big)$$

(see for example [9, Lemma A.3]).

3. Hölder cocycles on $\partial\Gamma$

Let Γ be a finitely generated, non-elementary, word-hyperbolic group. Denote by $\mathsf{g} = \big(\mathsf{g}_t : \mathsf{U}\Gamma \to \mathsf{U}\Gamma\big)_{t\in\mathbb{R}}$ the *Gromov-Mineyev geodesic* flow of Γ (see Gromov [19] and Mineyev [33]). Throughout this section we will have the same assumptions as in S. [44, § 3], namely that g is metric-Anosov and that the lamination induced on the quotient by $\widetilde{\mathcal{W}}^{cu} = \{(x, \cdot, \cdot) \in \widetilde{\mathsf{U}\Gamma}\}$ is the central-unstable lamination of g.

Since we will mostly recall needed results from S. [44, § 3] we do not overcharge the paper with the definitions of metric-Anosov and central-unstable lamination: by Bridgeman-Canary-Labourie-S. [10], word-hyperbolic groups admitting an Anosov representation verify the required assumptions.

Definition 3.1. Let V be a finite dimensional real vector space. A *Hölder cocycle* is a function $c : \Gamma \times \partial\Gamma \to V$ such that:

- for all $\gamma, h \in \Gamma$ one has $c\big(\gamma h, x\big) = c\big(h, x\big) + c\big(\gamma, h(x)\big)$,
- there exists $\alpha \in (0, 1]$ such that for every $\gamma \in \Gamma$ the map $c(\gamma, \cdot)$ is α-Hölder continuous.

Recall that every *hyperbolic element*[7] $\gamma \in \Gamma$ has two fixed points on $\partial\Gamma$, the attracting γ_+ and the repelling γ_-. If $x \in \partial\Gamma - \{\gamma_-\}$ then $\gamma^n x \to \gamma_+$ as $n \to \infty$. The *period* of a Hölder cocycle for a hyperbolic $\gamma \in \Gamma$ is $\ell_c(\gamma) := c\big(\gamma, \gamma^+\big)$. A cocycle $c^* : \Gamma \times \partial\Gamma \to \mathbb{R}$ is *dual to* c if for every hyperbolic $\gamma \in \Gamma$ one has

$$\ell_{c^*}(\gamma) = \ell_c\big(\gamma^{-1}\big).$$

3.1. **Real-valued coycles.** Assume now $V = \mathbb{R}$ and consider a cocycle κ with non-negative (and not all vanishing) periods. For $t > 0$ we let

$$\mathsf{R}_t(\kappa) = \big\{[\gamma] \in [\Gamma] \text{ hyperbolic} : \ell_\kappa(\gamma) \leqslant t\big\}$$

and define the *entropy* of κ by

$$\hbar_\kappa = \limsup_{t\to\infty} \frac{1}{t} \log \#\mathsf{R}_t(\kappa) \in (0, \infty].$$

For such a cocycle consider the action of Γ on $\partial\Gamma \times \mathbb{R}$ via κ:

$$\gamma \cdot (x, y, t) = \big(\gamma x, \gamma y, t - \kappa\,(\gamma, y)\big)\,. \tag{3.1}$$

The following is a straightforward consequence of S. [44, Theorem 3.2.2].

Proposition 3.2. *Let κ be a Hölder cocycle with non-negative periods and finite entropy. The above action of Γ on $\partial\Gamma \times \mathbb{R}$ is properly-discontinuous and co-compact.*

If moreover c is another Hölder cocycle with non-negative periods and finite entropy then there exists a Γ-equivariant bi-Hölder-continuous homeomorphism $E : \partial\Gamma \times \mathbb{R} \to \partial\Gamma \times \mathbb{R}$ which is an orbit equivalence between the $\mathbb{R}$-translation actions.

[7] i.e. an infinite order element

We recall the notion of *dynamical intersection*, a concept from Bridgeman-Canary-Labourie-S. [10] for Hölder functions over a metric-Anosov flow, that can be pulled back to this setting via the existence of the *Ledrappier potential* of κ from S. [44, § 3.1].

For a second real valued cocycle c, the *dynamical intersection*

$$\mathbf{I}(\kappa, c) = \lim_{t\to\infty} \frac{1}{\mathsf{R}_t(\kappa)} \sum_{\gamma\in\mathsf{R}_t(\kappa)} \frac{\ell_c(\gamma)}{\ell_\kappa(\gamma)}. \tag{3.2}$$

We record in the following Proposition various needed facts about $\mathbf{I}$:

Proposition 3.3 ([10, § 3.4])**.** *The dynamical intersection defined above is well defined, linear in the second variable and for all positive s satisfies $\mathbf{I}(s\kappa, c) = \mathbf{I}(\kappa, c)/s$. If also c has non-negative periods and finite entropy then $\mathbf{I}(\kappa, c) \geqslant \hbar_\kappa/\hbar_c$.*

We will also need the following definitions.

Definition 3.4.

- A *Patterson-Sullivan measure for κ of exponent $\delta \in \mathbb{R}_+$* is a probability measure μ on $\partial\Gamma$ such that for every $\gamma \in \Gamma$ one has
$$\frac{d\gamma_*\mu}{d\mu}(\cdot) = e^{-\delta\cdot\kappa(\gamma^{-1},\,\cdot\,)}. \tag{3.3}$$
- Let κ^* be a cocycle dual to κ, then a *Gromov product* for the ordered pair (κ^*, κ) is a function $[\cdot,\cdot] : \partial^2\Gamma \to \mathbb{R}$ such that for all $\gamma \in \Gamma$ and $(x, y) \in \partial^2\Gamma$ one has
$$[\gamma x, \gamma y] - [x, y] = -\big(\kappa^*(\gamma, x) + \kappa(\gamma, y)\big).$$

3.2. **The critical hypersurface and intersection.** Let now $c : \Gamma \times \partial\Gamma \to V$ be a Hölder cocycle. Its *limit cone* is denoted by

$$\mathcal{L}_c = \overline{\bigcup_{\gamma\in\Gamma} \mathbb{R}_+ \cdot \ell_c(\gamma)}$$

and its *dual cone* by $\big(\mathcal{L}_c\big)^* = \{\psi \in V^* : \psi|_{\mathcal{L}_c} \geqslant 0\}$. Observe that for every $\varphi \in \operatorname{int}\big(\mathcal{L}_c\big)^*$, $\varphi \circ c$ is a real-valued cocycle, so the concepts from Section 3.1 apply. We denote by

$$\begin{aligned} \mathcal{Q}_c &= \Big\{\varphi \in \operatorname{int}\big(\mathcal{L}_c\big)^* : \hbar_{\varphi\circ c} = 1\Big\} \\ \mathcal{D}_c &= \Big\{\varphi \in \operatorname{int}\big(\mathcal{L}_c\big)^* : \hbar_{\varphi\circ c} \in (0,1)\Big\} \subset \Big\{\varphi \in V^* : \sum_{[\gamma]\in[\Gamma]} e^{-\varphi(\ell_c(\gamma))} < \infty\Big\} \end{aligned} \tag{3.4}$$

respectively the *critical hypersurface* and the *convergence domain* of c.

For $\varphi \in \operatorname{int}\big(\mathcal{L}_c\big)^*$ we consider the linear map $\mathbf{I}_\varphi = \mathbf{I}^c_\varphi : V^* \to \mathbb{R}$ defined by

$$\mathbf{I}^c_\varphi(\psi) := \mathbf{I}(\varphi \circ c, \psi \circ c),$$

as in Equation (3.2). The natural identification between hyperplanes in V^* and $\mathbb{P}(V)$ is used in the next proposition.

Corollary 3.5 (S. [44, Cor. 3.4.3])**.** *Assume $\mathcal{L}_c$ has non-empty interior and that there exists $\psi \in \big(\mathcal{L}_c\big)^*$ such that $\hbar_\psi < \infty$. Then $\mathcal{D}_c$ is a strictly convex set with boundary $\mathcal{Q}_c$. The latter is an analytic co-dimension-one sub-manifold of V. The map $\mathsf{u}^c : \mathcal{Q}_c \to \mathbb{P}(V)$ defined by*

$$\varphi \mapsto \mathsf{u}^c_\varphi := \mathsf{T}_\varphi\mathcal{Q}_c = \ker \mathbf{I}_\varphi$$

is an analytic diffeomorphism between $\mathcal{Q}_c$ *and* $\operatorname{int}\big(\mathbb{P}(\mathcal{L}_c)\big)$.

We end this section with the following technical Lemma needed in Lemma 5.10.

Lemma 3.6. *Assume* V *is* 2*-dimensional. Fix linearly independent* $\tau, \psi \in \operatorname{int}\big(\mathcal{L}_c\big)^*$, *and consider the operator norm* $\|\ \|^1$ *on* V^* *associated to the* L^∞ *the norm defined, for* $v \in V$, *by* $\|v\|_\infty = \max\{|\tau(v)|, |\psi(v)|\}$. *Assume without less of generality that* $\hbar_\psi \geqslant \hbar_\tau$. *If*

$$\mathbf{I}_\tau(\psi) \geqslant 1 \tag{3.5}$$

and we let φ_∞ *be the unique form on* $\mathcal{Q}_c$ *minimizing* $\|\ \|^1$, *then* $\varphi_\infty/\|\varphi_\infty\|^1$ *is a convex combination* $s\tau + (1-s)\psi$ *with* $s \in (0,1)$.

Observe that if $\hbar_\psi = \hbar_\tau$ then Proposition 3.3 implies Equation (3.5) is satisfied. The fact that $\mathcal{Q}_c$ is strictly convex implies uniqueness of a form in $\mathcal{Q}_c$ minimizing $\|\ \|^1$. The Lemma states that this unique linear form lies in the cone $\{t\psi : t \geqslant 0\} \times \{s\tau : s \geqslant 0\}$.

Proof. Using the identification $\mathsf{T}_\tau \mathcal{Q}_c = \ker \mathbf{I}_\tau$ and strict convexity of $\mathcal{Q}_c$, both facts from Corollary 3.5, the Lemma readily follows as in Figure 4 which we now explain. The blue rhombus in the middle is the sphere $\{\|\varphi\|^1 = 1\}$, the red line, tangent to $\mathcal{Q}_c$ at $h_\tau\tau$, is the level set $\mathbf{I}_{\hbar_\tau\tau}(\cdot) = 1$, whence its intersection with the ψ-axis is $\psi/\mathbf{I}_{\hbar_\tau\tau}(\psi)$. Equation (3.5) implies the ordering

$$0 < \frac{1}{\mathbf{I}_{\hbar_\tau\tau}(\psi)} \leqslant h_\tau \leqslant \hbar_\psi,$$

giving that the sphere $\{\|\ \|^1 = \hbar_\tau\}$ intersects $\mathcal{Q}_c$ in $\{\hbar_\tau\tau\}$ and some other point $\overline{\psi}$ in the segment $[\hbar_\tau\tau, \hbar_\tau\psi]$, an application of Lagrange's classical result provides a unique tangent to $\mathcal{Q}_c$ in the cone $\{t\psi : t \geqslant 0\} \times \{s\tau : s \geqslant 0\}$ as desired. □

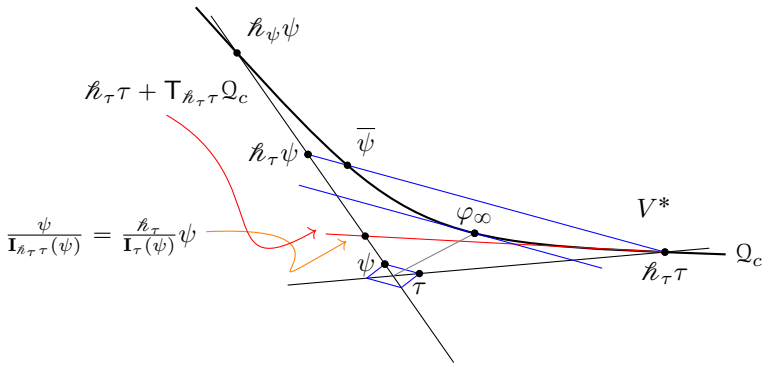

FIGURE 4. The situation of Lemma 3.6.

3.3. **Ergodicity of directional flows.** It follows from Proposition 3.2 that if there exists $\psi \in \big(\mathcal{L}_c\big)^*$ with $\hbar_\psi < \infty$ then the Γ-action $\partial^2\Gamma \times V$

$$\gamma(x, y, v) = \big(\gamma x, \gamma y, v - c(\gamma, y)\big)$$

is properly discontinuous.

Definition 3.7. A real valued cocycle c is *non-arithmetic* if the periods of c span a dense subgroup in V.

We fix $\varphi \in \mathcal{Q}_c$ and denote by $u_\varphi \in \mathsf{u}_\varphi$ the unique vector in $\mathcal{L}_c \cap \mathsf{u}_\varphi$ with $\varphi(u_\varphi) = 1$. We define then the *directional flow* $\omega^\varphi = \left(\omega^\varphi_t : \Gamma\backslash\left(\partial^2\Gamma \times V\right) \to \Gamma\backslash\left(\partial^2\Gamma \times V\right)\right)_{t\in\mathbb{R}}$ by

$$t \cdot (x, y, v) = (x, y, v - tu_\varphi).$$

Assumption 3.8. We assume there exists:

- a dual cocycle $(\varphi \circ c)^*$,
- a Gromov product $[\,,\,]_\varphi$ for such a pair,
- Patterson-Sullivan measures, μ^φ and $\overline{\mu}^\varphi$, respectively for each of the cocycles; (the exponent of both measures is then necessarily $\hbar_\varphi = 1$ S. [44, Proposition 3.3.2]).

Consider then the *φ-Bowen-Margulis* measure Ω^φ on $\Gamma\backslash\left(\partial^2\Gamma \times V\right)$ defined as the measure induced on the quotient by the measure

$$e^{-[\cdot,\cdot]_\varphi}\overline{\mu}^\varphi \otimes \mu^\varphi \otimes \mathrm{Leb}_V, \tag{3.6}$$

for a V-invariant Lebesgue measure on V. We denote by $\mathcal{K}(\omega^\varphi)$ the *recurrent set* of the directional flow ω^φ:

$$\mathcal{K}(\omega^\varphi) := \{p \in \Gamma\backslash\left(\partial^2\Gamma \times V\right) \big|\ \exists B \text{ open bounded}, t_n \to \infty \text{ with } \omega^\varphi_{t_n}(p) \in B\}.$$

Corollary 3.9 (S. [44, Cor. 3.6.1])**.** *Assume that c is non-arithmetic, that there exists $\psi \in \left(\mathcal{L}_c\right)^*$ with $\hbar_\psi < \infty$, and that the existence assumptions in 3.8 hold. If $\dim V \leqslant 2$ then the directional flow ω^φ is Ω^φ-ergodic, in particular $\mathcal{K}(\omega^\varphi)$ has total mass. If $\dim V \geqslant 4$ then $\mathcal{K}(\omega^\varphi)$ has measure zero.*

4. Subspace conicality for Anosov representations: Theorem D

4.1. **Gromov hyperbolic groups and cone types.** Let Γ be a finitely generated group, and fix a finite generating set S. A group Γ is *Gromov hyperbolic* if its Cayley graph $\mathsf{Cay}(\Gamma, S)$ is a Gromov hyperbolic geodesic metric space. In this case we denote by $\partial\Gamma$ its Gromov boundary, namely the equivalence classes of (quasi)-geodesic rays. It is well known that, up to Hölder homeomorphism $\partial\Gamma$ doesn't depend on the choice of the generating set S. We will furthermore denote by $\partial^2\Gamma$ the set of distinct pairs in $\partial\Gamma$:

$$\partial^2\Gamma := \{(x, y) \in \partial\Gamma \times \partial\Gamma |\, x \neq y\}.$$

Definition 4.1. A divergent sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset \Gamma$ converges to a point $x \in \partial\Gamma$ *conically* if for every $y \in \partial\Gamma - \{x\}$ the sequence $(\gamma_n^{-1}y, \gamma_n^{-1}x)$ remains on a compact set of $\partial^2\Gamma$.

Recall from the introduction that, given $\gamma \in \Gamma$ we denote by $\mathcal{C}(\gamma)$ the *cone type* of $\gamma \in \Gamma$, namely

$$\mathcal{C}(\gamma) := \{h \in \Gamma |\, d(e, \gamma h) = d(e, \gamma) + d(e, h)\}.$$

Cannon showed [13] the set of cone types of a Gromov hyperbolic group is finite, see for example Bridson-Haefliger's book [11, P. 455]. We denote by $\mathcal{C}_\infty(\gamma) \subset \partial\Gamma$ the set of points x that can be represented by a geodesic ray contained in $\mathcal{C}_\infty(\gamma)$,

We will require a coarse version of these. Recall that a sequence $(\alpha_j)_0^\infty$ is a (ν, c)-quasigeodesic if for every pair j, l it holds

$$\frac{1}{\nu}|j-l| - c \leqslant d_\Gamma(\alpha_j, \alpha_l) \leqslant \nu|j-l| + c.$$

We associate to every element γ a *coarse cone type at infinity*, consisting of endpoints at infinity of quasi geodesic rays based at γ^{-1} passing through the identity:

$$\mathcal{C}_\infty^{\nu,c}(\gamma) = \left\{ [(\alpha_j)_0^\infty] \in \partial\Gamma \,|\, (\alpha_i)_0^\infty \text{ is a } (\nu,c)\text{-quasigeodesic, } \alpha_0 = \gamma^{-1}, e \in \{\alpha_j\} \right\}.$$

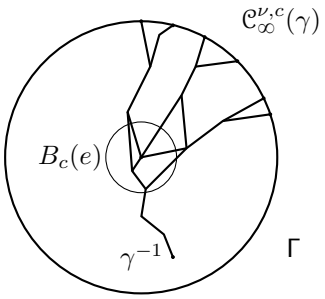

FIGURE 5. The coarse cone type at infinity, picture borrowed from P.-S.-Wienhard [37].

We record the following lemma for later use, the first inclusion follows from the definitions, the second one by hyperbolicity of Γ.

Lemma 4.2. *Given ν, c there exists $N > 0$ such that if $(\alpha_i)_0^\infty$ is a geodesic ray through e then for all $i \in \mathbb{N}$ one has*

$$\alpha_{i+N}\mathcal{C}_\infty(\alpha_{i+N}) \subset \alpha_{i+N}\mathcal{C}_\infty^{\nu,c}(\alpha_{i+N}) \subset \alpha_i\mathcal{C}_\infty(\alpha_i).$$

4.2. **Anosov representations.** Fix a subset $\theta \subset \Delta$. Let Γ be a finitely generated group and denote by $|\,|$ the word-length associated to the finite generating set S of Γ.

Definition 4.3. Following[8] Kapovich-Leeb-Porti [26], a representation $\rho : \Gamma \to \mathsf{G}$ is *θ-Anosov* if there exists positive constants C and μ such that for all $\gamma \in \Gamma$ and $\mathsf{a} \in \theta$ one has

$$\mathsf{a}(a(\rho\gamma)) \geqslant \mu|\gamma| - C.$$

If $\mathsf{G} = \mathsf{PGL}(d, \mathbb{R})$ and $\theta = \{\tau_1\}$ we say that ρ is *projective Anosov*.

Anosov representations were introduced by Labourie [30] and further developed by Guichard-Wienhard [21]. They have played a central role in understanding the Hitchin component of split groups (see below) and are considered nowadays as the higher-rank generalization of convex co-compact groups. We refer the reader to the surveys by Kassel [27] and Wienhard [47] for further information.

Remark 4.4. A Zariski-dense representation $\rho : \Gamma \to \mathsf{G}$ is θ-Anosov if and only if ρ is a quasi-isometric embedding and its limit cone $\mathcal{L}_\rho$ does not meet any wall $\ker \mathsf{a}$ for $\mathsf{a} \in \theta$: this follows from the definition since by Benoist [3], for Zariski dense representations, the limit cone $\mathcal{L}_\rho$ equals the asymptotic cone.

[8] see also Bochi-Potrie-S. [9] and Guéritaud-Guichard-Kassel-Wienhard [20]

A useful property of θ-Anosov representations is that their limit set $\Lambda_\Gamma \subset \mathcal{F}_\theta$, namely the minimal Γ-invariant subset in $\mathcal{F}_\theta$, is parametrized by the Gromov boundary of the group Γ, see Kapovich-Leeb-Porti [26], Guéritaud-Guichard-Kassel-Wienhard [20]. We will need the following precise statement.

Proposition 4.5 (Bochi-Potrie-S. [9, Proposition 4.9])**.** *If $\rho : \Gamma \to \mathsf{G}$ is θ-Anosov, then for any geodesic ray $(\alpha_n)_0^\infty$ with endpoint x, the limits*

$$\xi^\theta_\rho(x) := \lim_{n\to\infty} U_\theta(\rho\alpha_n) \quad \xi^{\mathrm{i}\theta}_\rho(x) := \lim_{n\to\infty} U_{\mathrm{i}\theta}(\rho\alpha_n)$$

exist and do not depend on the ray; they define continuous ρ-equivariant transverse maps $\xi^\theta : \partial\Gamma \to \mathcal{F}_\theta$, $\xi^{\mathrm{i}\theta} : \partial\Gamma \to \mathcal{F}_{\mathrm{i}\theta}$. If $\gamma \in \Gamma$ is hyperbolic, then γ_ρ is θ-proximal with attracting point $\xi^\theta(\gamma^+) = (\gamma_\rho)^+_\theta$.

The Gromov product gives the following criterion to understand limit points of conical sequences (recall Definition 4.1) in case of Anosov representations, it will be useful in the proof of Proposition 4.18.

Lemma 4.6. *Let $\rho : \Gamma \to \mathsf{G}$ be θ-Anosov. If $\{\gamma_n\} \subset \Gamma$ is a conical sequence, $x \in \partial\Gamma$, and there exists $\mathsf{a} \in \theta$ such that $\varpi_\mathsf{a}\mathcal{G}_\theta\big(U_{\mathrm{i}\theta}(\rho\gamma_n), \xi^\theta(x)\big) \to -\infty$, then $\gamma_n \to x$.*

Proof. We denote by y the endpoint of the conical sequence γ_n. Proposition 4.5 implies that $U_{\mathrm{i}\theta}(\rho\gamma_n) \to \xi^{\mathrm{i}\theta}(y)$. Since, however, $\varpi_\mathsf{a}\mathcal{G}_\theta\big(U_{\mathrm{i}\theta}(\rho\gamma_n), \xi^\theta(x)\big) \to -\infty$, we deduce that $\xi^{\mathrm{i}\theta}(y)$ is not transverse to $\xi^\theta(x)$ (recall Remark 2.8). Since ξ^θ is transverse, we deduce that $x = y$. □

Another application of the Gromov product is to quantify the transversality of the Cartan attractors along images of geodesic segments through Anosov representations:

Proposition 4.7 (Bochi-Potrie-S. [9, Lemma 2.5])**.** *If $\rho : \Gamma \to \mathsf{G}$ is θ-Anosov then there exist $\overline{L} \in \mathbb{N}$ and $\delta_\rho > 0$ such that for every geodesic segment through the identity $\{\alpha_i\}_{-m}^k$ with $k, m \geqslant L$ one has, for all $\mathsf{a} \in \theta$,that*

$$\varpi_\mathsf{a}\mathcal{G}_\theta\big(U_{\mathrm{i}\theta}(\rho\alpha_{-m}), U_\theta(\rho\alpha_k)\big) \geqslant -\delta_\rho.$$

The following standard linear algebra computation allows to obtain precise estimates for the action of projective Anosov representations on the projective space.

Lemma 4.8. *Fix an Hermitian product on $\mathbb{C}^n$, then given $\alpha > 0$ there exists $C > 0$ such that if $h \in \mathsf{GL}(d, \mathbb{C})$ is such that $\tau_1(a(h)) > 0$, then for all $\ell_1, \ell_2 \in \mathbb{P}(\mathbb{C}^d)$ with $\measuredangle(\ell_i, U_{d-1}(h^{-1})) > -\alpha$ one has*

$$d_\mathbb{P}(h\ell_1, h\ell_2) \leqslant Ce^{-\tau_1(a(h))} d_\mathbb{P}(\ell_1, \ell_2).$$

Proof. This is very standard, we add some comments for completeness. Indeed the result follows by applying, for example, [38, Lemma 2.8] to $g = h^{-1}$, $P = U_1(h)$ and $Q = hU_{d-1}(h)$. □

One has thus the following consequences of Lemma 4.8 and Proposition 4.5, see for example P.-S.-Wienhard [38, §4.1] for details.

Proposition 4.9. *Let $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ be projective Anosov and consider positive ν and c. Then there exists a constant K, depending on ν, c and ρ such that for every large enough $\gamma \in \Gamma$ one has*

$$\xi^1\big(\gamma \mathcal{C}^{\nu,c}_\infty(\gamma)\big) \subset B\big(U_1(\rho\gamma), Ke^{-\tau_1(a(\rho\gamma))}\big).$$

Moreover, there exist C *and* $\mu > 0$ *given* $\alpha > 0$ *and* $\ell_1, \ell_2 \in \mathbb{P}(\mathbb{K}^d)$ *with*

$$\mathcal{G}\big(\ell_1, U_{d-1}(\rho\gamma^{-1})\big) \geqslant -\alpha \text{ and } \mathcal{G}\big(\ell_2, U_{d-1}(\rho\gamma^{-1})\big) \geqslant -\alpha$$

one has $d_{\mathbb{P}}(\rho(\gamma)x, \rho(\gamma)y) \leqslant Ce^{-\mu|\gamma|}d(x,y)$.

Proof. Follows directly from Proposition 4.7 and Lemma 4.8. □

4.3. **Patterson-Sullivan Theory of Anosov representations.** If ρ is a θ-Anosov representation, then we can pullback the Buseman-Iwasawa cocycle of G using the equivariant maps. To be more precise, the *refraction cocycle* associated to a θ-Anosov representation $\rho : \Gamma \to \mathsf{G}$ is $\beta : \Gamma \times \partial\Gamma \to \mathfrak{a}_\theta$ given by

$$\beta(\gamma, x) = \beta_{\theta,\rho}(\gamma, x) = \mathrm{b}_\theta\left(\rho(\gamma), \xi^\theta_\rho(x)\right).$$

Bridgeman-Canary-Labourie-S. [10, Theorem 1.10] show that the Mineyev geodesic flow of a group Γ admitting an Anosov representations is metric-Anosov, and thus § 3 applies to β. Moreover, the following fact places β in the assumptions required in § 3.1 and § 3.2, see S. [44] for details.

Fact. *The periods of the refraction cocycle equal the* θ*-Jordan projection:* $\beta(\gamma, \gamma^+) = \lambda_\theta(\rho\gamma)$. *For any* $\mathsf{a} \in \theta$ *the real valued cocycle* $\varpi_{\mathsf{a}}\beta$ *has finite entropy.*

We import the following concepts of cocycles to the setting of Anosov representations:

- The limit cone of β will be denoted by $\mathcal{L}_{\theta,\rho}$ and referred to as *the* θ*-limit cone of* ρ; it is the smallest closed cone that contains the projected Jordan projections $\{\lambda_\theta(\rho\gamma) : \gamma \in \Gamma\}$.
- The *interior of the dual cone* $\operatorname{int}\left(\mathcal{L}_{\theta,\rho}\right)^* \subset \mathfrak{a}_\theta^*$ consists of linear forms whose *entropy*

$$\hbar_\varphi = \lim_{t\to\infty} \frac{1}{t} \log \#\big\{[\gamma] \in [\Gamma] : \varphi(\lambda_\theta(\rho\gamma)) \leqslant t\big\}$$

  is finite.
- The θ*-critical hypersurface*, resp. θ*-convergence domain*, of β will be denoted by

$$\Omega_{\theta,\rho} = \Big\{\varphi \in \operatorname{int}\left(\mathcal{L}_{\theta,\rho}\right)^* : \hbar_\varphi = 1\Big\}$$

$$\mathcal{D}_{\theta,\rho} = \Big\{\varphi \in \operatorname{int}\left(\mathcal{L}_{\theta,\rho}\right)^* : \hbar_\varphi \in (0,1)\Big\} = \Big\{\varphi \in (\mathfrak{a}_\theta)^* : \sum_{\gamma\in\Gamma} e^{-\varphi(a(\rho\gamma))} < \infty\Big\}.$$

  The second equality in this case follows from S. [44, § 5.7.2].
- If $\mathcal{L}_{\theta,\rho}$ has non-empty interior, then we have a *duality* diffeomorphism between $\Omega_{\theta,\rho}$ and $\operatorname{int}\mathbb{P}(\mathcal{L}_{\theta,\rho})$ given by

$$\varphi \mapsto \mathsf{u}_\varphi = \mathsf{T}_\varphi\Omega_\rho.$$

More information on these objets can be found on S. [44, § 5.9].

We observe that for $\varphi \in \operatorname{int}\left(\mathcal{L}_{\theta,\rho}\right)^*$ the existence assumptions of § 3.3 are also guaranteed for $\beta_\varphi := \varphi \circ \beta$. Indeed the cocycle

$$\overline{\beta}(\gamma, x) = \mathrm{i}\,\mathrm{b}_{\mathrm{i}\theta}\left(\gamma_\rho, \xi^{\mathrm{i}\theta}(x)\right)$$

is dual to β, from Equation (2.7) the function $[\cdot,\cdot]_\varphi : \partial^2\Gamma \to \mathbb{R}$

$$[x,y]_\varphi = \varphi\Big(\mathcal{G}_\theta\big(\xi^{\mathrm{i}\theta}(x), \xi^\theta(y)\big)\Big) \tag{4.1}$$

is a Gromov product for the pair $(\overline{\beta}_\varphi, \beta_\varphi)$, and we have the following result guaranteeing existence of Patterson-Sullivan measures μ^φ and $\overline{\mu}^\varphi$ (the last assertion follows from the inclusion in Proposition 4.9 and the cited Lemma).

Corollary 4.10 (S. [44, Cor. 5.5.3+Lemma 5.7.1])**.** *For every $\varphi \in \operatorname{int}\big(\mathcal{L}_{\theta,\rho}\big)^*$ there exists a β^φ-Patterson-Sullivan measure μ^φ of exponent $\hbar_\varphi$, moreover there exists a constant C such that for every $\gamma \in \Gamma$ one has*

$$\mu^\varphi\big(\gamma \mathcal{C}_\infty(\gamma)\big) \leqslant C e^{-\hbar_\varphi \varphi\left(a(\rho\gamma)\right)}. \tag{4.2}$$

4.4. **Subspace-conicality.** In this section we are interested on a notion of conicality along higher dimensional subspaces of the ambient Levi space. Let $\rho : \Gamma \to \mathsf{G}$ be a θ-Anosov representation.

Definition 4.11. Consider a subspace $\mathsf{W} \subset \mathfrak{a}_\theta$ and assume, for the notion to be interesting, that W intersects the relative interior of $\mathcal{L}_{\theta,\rho}$, then $x \in \partial\Gamma$ is W-*conical* if there exists a conical sequence $\{\gamma_n\}_0^\infty \subset \Gamma$ converging to x, a constant K and $\{w_n\}_0^\infty \subset \mathsf{W}$ such that for all n one has

$$\big\|a\big(\rho(\gamma_n)\big) - w_n\big\| \leqslant K.$$

The set of such points will be denoted by $\partial_{\mathsf{W},\rho}\Gamma = \partial_{\mathsf{W}}\Gamma$.

Consider $\varphi \in (\mathfrak{a}_\theta)^*$ with $\mathsf{u}_\varphi \subset \mathsf{W}$. The intersection $\mathsf{W}_\varphi = \mathsf{W} \cap \ker\varphi$ has codimension 1 in W and has trivial intersection with the limit cone $\mathcal{L}_{\theta,\rho}$. Consider the quotient space $V = \mathfrak{a}_\theta/\mathsf{W}_\varphi$ equipped with the quotient projection $\Pi : \mathfrak{a}_\theta \to V$. We say that ρ is (W,φ)-*non-arithmetic* if the group spanned by $\big\{\Pi(\lambda_\theta(\rho\gamma)) : \gamma \in \Gamma\big\}$ is dense in V. In this section we prove the following.

Theorem 4.12. *Let $\rho : \Gamma \to \mathsf{G}$ be θ-Anosov representation, W be a subspace of $\mathfrak{a}_\theta$ intersecting non-trivially the relative interior of $\mathcal{L}_{\theta,\rho}$, and $\varphi \in (\mathfrak{a}_\theta)^*$ with $\mathsf{u}_\varphi \subset \mathsf{W}$. Assume ρ is (W,φ)-non-arithmetic, then:*

- *if W has codimension 1 then $\mu^\varphi(\partial_{\mathsf{W}}\Gamma) = 1$;*
- *if $\operatorname{codim}\mathsf{W} \geqslant 3$ then $\mu^\varphi(\partial_{\mathsf{W}}\Gamma) = 0$.*

Observe that if ρ is Zariski-dense then Benoist [4] (Theorem 2.3) guarantees (W,φ)-non-arithmeticity for every $\varphi \in (\mathfrak{a}_\theta)^*$ with $\mathsf{u}_\varphi \in \mathbb{P}(\mathsf{W})$, thus Theorem 4.12 readily implies Theorem D.

The remainder of the section is devoted to the proof of Theorem 4.12. Let

$$V^* = \operatorname{Ann}(\mathsf{W}_\varphi) = \{\psi \in (\mathfrak{a}_\theta)^* : \psi|\mathsf{W}_\varphi \equiv 0\},$$

with a slight abuse of notation we will identify the dual of V with $V^* \subset (\mathfrak{a}_\theta)^* \subset \mathfrak{a}^*$ (recall from Section 2.4 that we are identifying $(\mathfrak{a}_\theta)^*$ with the subspace of $\mathfrak{a}^*$ consisting of π_θ-invariant linear forms).

The composition of the refraction cocycle of ρ with Π is a V-valued Hölder cocycle $\upsilon : \Gamma \times \partial\Gamma \to V$,

$$\upsilon(\gamma, x) = \Pi\big(\beta(\gamma, x)\big).$$

Its periods are $\upsilon(\gamma,\gamma_+) = \Pi\big(\lambda_\theta(\rho\gamma)\big)$, and thus its limit cone is $\mathcal{L}_\upsilon = \Pi(\mathcal{L}_{\theta,\rho})$. By (W,φ)-non-arithmeticity, $\mathcal{L}_\upsilon \subset V$ has non-empty interior.

The heart of the proof of Theorem 4.12 consits on relating (W,φ)-conical points with elements of $\widetilde{\mathcal{K}}\big(\omega^\varphi\big)$, where ω^φ is the directional flow on $\Gamma\backslash\partial^2\Gamma \times V$ associated to the cocycle υ as in § 3.3. The first step is thus to observe that we can apply Corollary 3.9 to υ, a task we enter at this point.

Since $\varphi \in \Omega_{\theta,\rho}$, it has in particular finite entropy. Moreover, by definition of V^* one has $\varphi \in V^*$. Consequently, the cocycle υ verifies assumptions in Corollary 3.5. One can moreover transfer existence properties from β to υ, indeed one has the following.

Proposition 4.13. *The cocycle $\overline{\upsilon} = \Pi \circ \overline{\beta}$ is a dual cocycle for υ. For each $\psi \in \Omega_\upsilon$ there exist Paterson-Sullivan measures for υ and $\overline{\upsilon}$ and the projection $\psi\big(\Pi\big([\cdot,\cdot]\big)\big)$ is a Gromov product for the pair $\psi \circ \upsilon, \psi \circ \overline{\upsilon}$.*

Proof. Since $\psi \in \Omega_\upsilon = \Omega_{\theta,\rho} \cap V^*$ we can apply Corollary 4.10 to ψ to obtain the desired Patterson-Sullivan measure, the remaining statements follow trivially as the equalities are linear. □

Since we are assuming (W, φ)-non-arithmecity, the cocycle υ is non-arithmetic and thus Corollary 3.9 gives the following dynamical information, observe that $\dim V = \operatorname{codim} \mathsf{W} + 1$.

Corollary 4.14. *If $\operatorname{codim} \mathsf{W} \leqslant 1$ then the directional flow ω^φ is Ω^φ-ergodic, in particular $\mathcal{K}(\omega^\varphi)$ has total mass. If $\operatorname{codim} \mathsf{W} \geqslant 3$ then $\mathcal{K}(\omega^\varphi)$ has measure zero.*

Observe that modulo the understood identifications $\Omega_\upsilon = \Omega_{\theta,\rho} \cap V^*$, hence

$$\mathsf{T}_\varphi \Omega_\upsilon = (\mathsf{T}_\varphi \Omega_{\theta,\rho}) \cap V^*$$

and thus the map from Corollary 3.5 $\mathsf{u}^\upsilon : \Omega_\upsilon \to \operatorname{int} \mathbb{P}(\mathcal{L}_\upsilon)$ verifies $\mathsf{u}^\upsilon_\varphi = \Pi(\mathsf{u}_\varphi)$. So measuring W-conicality with respect to μ^φ translates to directional conicality along the direction $\mathsf{u}^\upsilon_\varphi$, which we now recall. We fix an arbitrary norm $\| \, \|$ on V and denote, for $\ell \in \mathbb{P}(V)$ and $r > 0$, the *r-tube about ℓ* by $\mathbb{T}_r(\ell)$ and define it as the r-tubular neighborhood in V of ℓ.

Definition 4.15. A point $y \in \partial\Gamma$ is *$\mathsf{u}^\upsilon_\varphi$-conical* if there exists $r > 0$ and a conical sequence $\{\gamma_n\}_0^\infty \subset \Gamma$ with $\gamma_n \to y$ such that for all n one has $\Pi\big(a_\theta(\rho(\gamma_n))\big) \in \mathbb{T}_r(\mathsf{u}^\upsilon_\varphi)$.

The next statement follows from the definitions.

Lemma 4.16. *A point $y \in \partial\Gamma$ is W-conical if and only if it is $\mathsf{u}^\upsilon_\varphi$-conical.*

If we are allowed to worsen the constants, we can replace, in Definition 4.15, the conical sequence (γ_n) with an infinite subset of a geodesic ray:

Lemma 4.17. *A point $y \in \partial\Gamma$ is $\mathsf{u}^\upsilon_\varphi$-conical if and only if there exists $r > 0$, a geodesic ray $(\alpha_i)_0^\infty$ converging to y and an infinite set of indices $\mathbb{I} \subset \mathbb{N}$ such that for all $k \in \mathbb{I}$ one has*

$$\Pi\Big(a(\rho\alpha_k)\Big) \in \mathbb{T}_r(\mathsf{u}^\upsilon_\varphi).$$

Proof. Assume y is $\mathsf{u}^\upsilon_\varphi$-conical, then since $\{\gamma_n\}_0^\infty$ is conical, given $x \in \partial\Gamma - \{y\}$ there exists a subsequence (still denoted by $\{\gamma_n\}$) such that $(\gamma_n^{-1}x, \gamma_n^{-1}y)$ converges on $\partial^2\Gamma$. Equivalently, for any geodesic ray $(\alpha_n)_0^\infty$ converging to y there exists $K > 0$ and a subsequence $\{\alpha_{n_k}\}$ such that for all k one has $d_\Gamma(\alpha_{n_k}, \gamma_k) < K$. Proposition 2.1 implies then that for all k one has

$$\|a(\rho\alpha_{n_k}) - a(\rho\gamma_k)\|$$

is bounded independently of k. This implies the result. □

We now relate u^v_φ-conicality with the recurrence set $\mathcal{K}(\omega^\varphi)$. By definition of $\mathcal{K}(\omega^\varphi)$, a point $(x,y,v) \in \partial^2\Gamma \times V$ projects to $\mathcal{K}(\omega^\varphi)$ if and only if there exists divergent sequences $(\gamma_n) \subset \Gamma$ and $t_n \to +\infty$ in $\mathbb{R}$ such that

$$\omega^\varphi_{t_n}\gamma_n^{-1}(x,y,v) = \big(\gamma_n^{-1}x, \gamma_n^{-1}y, v - \mathfrak{v}(\gamma_n^{-1},x) - t_n u_\varphi\big) \tag{4.3}$$

is contained in a subset of the form $\{(z,w) \in \partial^2\Gamma : d(z,w) \geqslant \kappa\} \times B(v,K)$ for some distance d on $\partial\Gamma$. One has the following

Proposition 4.18. *A point $y \in \partial\Gamma$ is u^v_φ-conical if and only if for every $x \in \partial\Gamma - \{y\}$ and $v \in V$ the point (x,y,v) projects to $\mathcal{K}(\omega^\varphi)$.*

Proof. The implication ($\Rightarrow$) follows exactly as in the proof of S. [44, Proposition 5.12.4]. The other implication also follows similarly but with a minor difference we now explain. Assume that (x,y,v) projects to $\mathcal{K}(\omega^\varphi)$ and consider sequences $\{\gamma_n\}$ and t_n as in Equation (4.3).

Since $\big(\gamma_n^{-1}x, \gamma_n^{-1}y\big)$ remains in a compact subset of $\partial^2\Gamma$, the sequence $\{\gamma_n\}$ is conical, we will show now that $\gamma_n \to y$. Indeed, since $t_n \to +\infty$ one readily sees that necessarily $\mathfrak{v}(\gamma_n^{-1},y) \to -\infty$.

Consider now any root $\mathsf{a} \in \theta$, with associated fundamental weight $\varpi_\mathsf{a} \in \big(\mathcal{L}_{\theta,\rho}\big)^*$, and Tits representation $\Phi_\mathsf{a} : \mathsf{G} \to V$. Since ρ is θ-Anosov, the Hölder cocycle $\beta^{\varpi_\mathsf{a}}$ has positive periods and finite entropy. Since $\mathfrak{v}(\gamma_n^{-1},y) \to -\infty$ Proposition 3.2 implies that

$$\beta_{\varpi_\mathsf{a}}(\gamma_n^{-1},y) \to -\infty.$$

By definition of the cocycle β and Equation (2.6) we have

$$\frac{\|\Phi_\mathsf{a}\rho(\gamma_n^{-1})v\|}{\|v\|} \to 0 \tag{4.4}$$

for a non-zero $v \in \zeta_\mathsf{a}(\xi(y))$, (recall that the map $\zeta_\mathsf{a} : \mathcal{F}_\mathsf{a}(\mathsf{G}) \to \mathbb{P}(V)$ was defined in Equation (2.5)). Setting $\dim V = d$, a standard linear algebra computation (for example in Bochi-Potrie-S. [9, Lemma A.3]) gives

$$\begin{aligned}\frac{\|\Phi_\mathsf{a}\rho(\gamma_n^{-1})v\|}{\|v\|} &\geqslant \|\Phi_\mathsf{a}\rho(\gamma_n^{-1})\| \sin \measuredangle\big(\zeta_\mathsf{a}\xi(y), U_{d-1}(\Phi_\mathsf{a}\rho\gamma_n)\big)\\ &\geqslant e^{l_\mathsf{a}\varpi_\mathsf{a}\mathcal{G}_\theta\left(U_\theta(\rho\gamma_n),y\right)}\end{aligned}$$

and thus, by Equation (4.4) and Lemma 4.6 one has $\gamma_n \to y$, as desired.

The point $\xi(y)$ lies then in the pushed Cartan basin $\rho(\gamma_n)B_{\theta,\alpha}(\rho(\gamma_n))$ for an α independent of n, and thus Equation (2.8) gives a constant K such that for all n one has

$$K \geqslant \big\|a_\theta\big(\rho,\overline{\rho}\big)(\gamma_n)\big) - \beta\big(\gamma_n,\gamma_n^{-1}x\big)\big\| = \big\|a_\theta\big((\rho,\overline{\rho})(\gamma_n)\big) + \beta(\gamma_n^{-1},x)\big\|$$

implying that y is u_φ-conical, as desired. □

The proof of Theorem 4.12 follows now along the same lines as in S. [44, Theorem 5.13.3]. We include the arguments here for completeness.

Consider a positive ε. Fix $y \in \partial_{\mathsf{W},\rho}\Gamma, x \in \partial\Gamma - \{y\}$ and two neighborhoods A^- and A^+ of x and y respectively so that for all $(z,w) \in A^-, A^+$ one has

$$\big|\Pi\big([z,w]_\varphi - [x,y]_\varphi\big)\big| < \varepsilon.$$

Pick also an arbitrary $T > 0$ so that the quotient projection $\mathsf{p} : \partial^2\Gamma \times V \to \Gamma\backslash\partial^2\Gamma \times V$ is injective on $\tilde{\mathrm{B}} = A^- \times A^+ \times B(0,T)$. We can thus compute the measure of $\mathrm{B} = \mathsf{p}(\tilde{\mathrm{B}})$ by the formula (3.6).

If we let $\tilde{\mathcal{K}}(\omega^\varphi) = \mathsf{p}^{-1}\big(\mathcal{K}(\omega^\varphi)\big)$, then Proposition 4.18 asserts that

$$A^- \times (A^+ \cap \partial_{\mathsf{W},\rho}\Gamma) \times B(0,T) = \tilde{\mathcal{K}}(\omega^\varphi) \cap \tilde{\mathrm{B}}.$$

If $\operatorname{codim}\mathsf{W} \leqslant 1$ Corollary 4.14 states that $\Omega^\varphi(\tilde{\mathrm{B}}) = \Omega^\varphi\big(\tilde{\mathcal{K}}(\omega^\varphi) \cap \tilde{\mathrm{B}}\big)$, which implies,

$$|\mu^\varphi(A^+) - \mu^\varphi(A^+ \cap \partial_{\mathsf{W},\rho}\Gamma)| < e^{-\varepsilon}.$$

Since ε is arbitrary one concludes $\mu^\varphi(\partial_{\mathsf{W},\rho}\Gamma) = 1$. On the other hand, if $\operatorname{codim}\mathsf{W} \geqslant 3$ then we have $\Omega^\varphi\big(\tilde{\mathcal{K}}(\omega^\varphi)\big) = 0$ so $\mu^\varphi(A^+ \cap \partial_{\mathsf{W},\rho}\Gamma) = 0$ and the theorem is proved.

5. Locally conformal representations: Hausdorff dimension of ♭-conical points

In this section we let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, the non-commutative field of Hamilton's quaternions. The Cartan subspace $\mathfrak{a}$ of $\mathsf{PGL}(d,\mathbb{K})$ is the subspace of $\mathbb{R}^d$ consisting of vectors of sum 0. For $g \in \mathsf{PGL}(d,\mathbb{K})$ we denote by

$$a(g) = \big(a_1(g), \cdots, a_d(g)\big)$$

the coordiates of the Cartan projection.

Definition 5.1. Consider $p \in [\![2, d-1]\!]$. Recall from the introduction (Definition 1.1) that a $\{\tau_1, \tau_{d-p}\}$-Anosov representation $\rho : \Gamma \to \mathsf{PGL}(d,\mathbb{K})$ is $(1,1,p)$-*hyperconvex* if, for every pairwise distinct triple $(x,y,z) \in \partial\Gamma^{(3)}$, one has

$$\big(\xi^1(x) + \xi^1(y)\big) \cap \xi^{d-p}(z) = \{0\}.$$

We say moreover that ρ is *locally conformal* if for every $\gamma \in \Gamma$ one has $a_2(\rho(\gamma)) = a_p(\rho(\gamma))$.

The terminology is justified by Proposition 5.7 below stating that for such representations pushed cone types are coarsely balls, this is a small refinement of an analogous result from P.-S.-Wienhard [38].

Remark 5.2. Observe that when $p = 2$ the second part of the definition is trivially true, so $(1,1,2)$-hyperconvex representations over $\mathbb{K}$ *are* locally conformal. We refer the reader to P.-S.-Wienhard [38, § 8] for more examples of locally conformal representations.

For two locally conformal representations over $\mathbb{K}$, ρ and $\overline{\rho}$, with equivariant maps ξ and $\overline{\xi}$ we want to introduce the notion of ♭-conicality. Roughly speaking, a point $x \in \partial\Gamma$ is ♭-conical, if the geometry of the limit sets $\xi(\partial\Gamma)$ and $\overline{\xi}(\partial\Gamma)$ about $\xi(x)$ and $\overline{\xi}(x)$ respectively looks similar (in a metric sense). In § 7 we relate ♭-conicality with non-differentiability points of the map $\overline{\xi} \circ \xi^{-1}$ for $p = 2$, and $\mathbb{K} = \mathbb{R}$.

In order to be more precise, we remark that the product representation $(\rho, \overline{\rho}) : \Gamma \to \mathsf{PGL}(d,\mathbb{K}) \times \mathsf{PGL}(\overline{d},\mathbb{K})$ is θ-Anosov for $\theta = \{\tau_1, \tau_p, \overline{\tau}_1, \overline{\tau}_p\}$ with limit map the "graph map"

$$\mathscr{G} = \big(\xi, \overline{\xi}\big) : \partial\Gamma \to \mathbb{P}(\mathbb{K}^d) \times \mathbb{P}(\mathbb{K}^{\overline{d}}).$$

We consider a Cartan subspace of the product group $\mathsf{PGL}(d,\mathbb{K}) \times \mathsf{PGL}(\overline{d},\mathbb{K})$, we let $\mathfrak{a}_\theta$ be the associated Levi space and $(\mathfrak{a}_\theta)^*$ its dual. Since $(\mathfrak{a}_\theta)^*$ is spanned by the

fundamental weights of roots in θ, one sees that $(p-1)\tau_1 = p\varpi_{\tau_1} - \varpi_{\tau_p} \in (\mathfrak{a}_\theta)^*$ and analogously for $\overline{\tau}_1$.

To simplify notation we let $\tau := \tau_1$ and $\overline{\tau} := \overline{\tau}_1$.

Definition 5.3. A point $x \in \partial\Gamma$ is $\mathfrak{b}$-*conical* if it is conical as in Definition 4.11 for the product representation $(\rho, \overline{\rho})$ with respect to the hyperplane

$$\mathfrak{b} = \{v \in \mathfrak{a}_\theta : \tau(v) = \overline{\tau}(v)\} = \ker(\tau - \overline{\tau}).$$

Equivalently, there exist R, a geodesic ray $(\alpha_n)_0^\infty \subset \Gamma$ with $\alpha_n \to x$, and a subsequence $\{n_k\}$ such that for all k one has

$$\big|\tau\big(a(\rho\alpha_{n_k})\big) - \overline{\tau}\big(a(\overline{\rho}\alpha_{n_k})\big)\big| \leqslant R.$$

Consider also the exponential rate

$$\hbar^\infty = \lim_{t\to\infty} \frac{1}{t} \log \#\big\{\gamma \in \Gamma : \max\big\{\tau(a(\rho\gamma)), \overline{\tau}(\overline{a}(\overline{\rho}\gamma))\big\} \leqslant t\big\}.$$

and recall the dynamical intersection defined by

$$\mathbf{I}_\tau(\overline{\tau}) = \lim_{t\to\infty} \frac{1}{\#\mathsf{R}_t(\tau)} \sum_{\gamma\in\mathsf{R}_t(\tau)} \frac{\overline{\tau}(\lambda(\overline{\rho}\gamma))}{\tau(\lambda(\rho\gamma))},$$

where $\mathsf{R}_t(\tau) = \big\{[\gamma] \in [\Gamma] : \tau\big(\lambda(\rho\gamma)\big) \leqslant t\big\}$.

In this section we compute the Hausdorff dimension of the image under the graph map $\mathscr{G}$ of the set of $\mathfrak{b}$-conical points with respect to a Riemannian metric on $\mathbb{P}(\mathbb{K}^d) \times \mathbb{P}(\mathbb{K}^{\overline{d}})$:

Theorem 5.4. *Let $\rho, \overline{\rho}$ be two locally conformal representations over $\mathbb{K}$ as above. Assume the group spanned by $\{(\tau(\lambda(\rho\gamma)), \overline{\tau}(\lambda(\overline{\rho}\gamma))) : \gamma \in \Gamma\}$ is dense in $\mathbb{R}^2$. Without loss of generality we assume that $\hbar_{\overline{\tau}} \geqslant \hbar_\tau$. If $\mathbf{I}_\tau(\overline{\tau}) \geqslant 1$ then*

$$\dim_{\mathrm{Hff}} \mathscr{G}\big(\{\mathfrak{b}\text{-conical points}\}\big) = \hbar^\infty.$$

Recall that by Proposition 3.3 one has $\mathbf{I}_\tau(\overline{\tau}) \geqslant \hbar_\tau/\hbar_{\overline{\tau}}$, so the hypothesis is automatically satisfied if, for example, $\hbar_\tau = \hbar_{\overline{\tau}}$. This is the case if one is working with $\mathbb{K} = \mathbb{R}$ and $\partial\Gamma$ is homeomorphic to a $(p-1)$-dimensional sphere, indeed one has the following.

Theorem 5.5 (P.-S.-Wienhard [38])**.** *Let $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ be locally conformal, then*

$$\hbar_\tau = \dim_{\mathrm{Hff}} \big(\xi(\partial\Gamma)\big),$$

in particular, when $\mathbb{K} = \mathbb{R}$, $\Gamma = \pi_1 S$ and $p = 2$ one has $\hbar_\tau = 1$.

In the surface group situation the other assumption (density of periods) can also be weakened:

Corollary 5.6. *Let $\rho, \overline{\rho}$ be two real $(1,1,2)$-hyperconvex representations of a closed surface group $\pi_1 S$. Assume there exists $\gamma \in \pi_1 S$ such that $\tau(\lambda(\rho\gamma)) \neq \overline{\tau}(\lambda(\rho\gamma))$, then*

$$\dim_{\mathrm{Hff}} \mathscr{G}\big(\{\mathfrak{b}\text{-conical points}\}\big) = \hbar^\infty.$$

Proof. Follows from the combination of Theorems 5.4, 5.5 and Proposition 6.9. □

We end the introduction of this section by mentioning that Kim-Minsky-Oh [28] have proved better Hausdorff dimension computations when ρ and $\overline{\rho}$ are convex-co-compact representations in $\mathsf{PSO}(n,1)$ not requiring the assumption on $\mathbf{I}$ that we do.

5.1. **Cone types are coarsely balls.** In [38] we give a concrete description of the images under the boundary map of the cone types at infinity. We denote by $d_{\mathbb{P}}$ the distance on $\mathbb{P}(\mathbb{K}^d)$ induced by the choice of a inner (Hermitian) product on $\mathbb{K}^d$ and by $B(\ell, r) \subset \mathbb{P}(\mathbb{K}^d)$ the associated ball of radius r about ℓ.

Proposition 5.7. *Let $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{K})$ be locally conformal. Then there exist positive constants k_1, k_2 and $L \in \mathbb{N}$ such that for every $x \in \partial\Gamma$, every geodesic ray $(\alpha_n)_0^\infty$ with endpoint x and every $n \in \mathbb{N}$ greater then L one has*

$$B\Big(\xi(x), k_1 e^{-\tau_1(a(\rho\alpha_n))}\Big) \cap \xi(\partial\Gamma) \subset \xi\Big(\alpha_i \mathcal{C}_\infty(\alpha_n)\Big) \subset B\Big(\xi(x), k_2 e^{-\tau_1(a(\rho\alpha_n))}\Big).$$

Proof. The desired inclusions are proven in [38] for thickened cone types at infinity. We briefly explain here how to deduce from it the result we need.

Following [38] we denote by $X_\infty(\gamma)$, for $\gamma \in \Gamma$, the *thickened cone type at infinity*, namely the tubular neighborhood in $\mathbb{P}(\mathbb{K}^d)$ of $\xi\big(\mathcal{C}_\infty(\gamma)\big)$ of radius $\delta_\rho/2$, where δ_ρ is the constant from Lemma 4.7. In [38, Corollary 5.10] it is established that there exists $c_1 > 0$ and $L_0 > 0$ only depending on the domination constants of ρ such that for all $i \geqslant L_0$ one has

$$B\Big(\xi(x), c_1 e^{-\tau_1(a(\rho\alpha_i))}\Big) \cap \xi(\partial\Gamma) \subset \rho(\alpha_i) X_\infty(\alpha_i).$$

The thickened cone type $X_\infty(\gamma)$ is contained in the Cartan basin $B_{\{\tau_1\},\alpha}(\rho\gamma)$ for a well chosen α depending on δ_ρ. So P.S.-Wienhard [37, Proposition 3.3] provides the existence of ν and c such that for all large enough $\gamma \in \Gamma$ one has

$$X_\infty(\gamma) \cap \xi(\partial\Gamma) \subset \xi\big(\mathcal{C}_\infty^{\nu,c}(\gamma)\big).$$

Combining both equations one has, for all $i \geqslant L_0$ that

$$B\Big(\xi(x), c_1 e^{-\tau_1(a(\rho\alpha_i))}\Big) \cap \xi(\partial\Gamma) \subset \xi\Big(\alpha_i \mathcal{C}_\infty^{\nu,c}(\alpha_i)\Big) \subset B\Big(\xi(x), \overline{K} e^{-\tau_1(a(\rho\alpha_i))}\Big), \quad (5.1)$$

where the second inclusion comes readily from Proposition 4.9, and $\overline{K}$ only depends on δ_ρ and the constant K from that Proposition. This is to say, the pushed coarse cone types $\xi\big(\alpha_i \mathcal{C}_\infty^{\nu,c}(\alpha_i)\big)$ are coarsely balls of radius $e^{-\tau_1(a(\rho\alpha_i))}$. Using Lemma 4.2 one replaces, up to modifying the constants c_1 and $\overline{K}$, the coarse cone types with actual cone types $\alpha_i \mathcal{C}_\infty(\alpha_i)$. □

5.2. **Hausdorff dimension and related concepts.** Recall that, given a metric space (X, d) and a real number $s > 0$, the *s-capacity* of X is

$$\mathcal{H}^s(X, d) = \inf_\varepsilon \left\{ \sum_{U \in \mathcal{U}} \operatorname{diam} U^s \,\middle|\, \mathcal{U} \text{ is an open covering of } \Lambda \text{ with } \sup_{U \in \mathcal{U}} \operatorname{diam} U < \varepsilon \right\}$$

and that

$$\dim_{\mathrm{Hff}}(X) = \inf\{s \,|\, \mathcal{H}^s(X) = 0\} = \sup\{s \,|\, \mathcal{H}^s(X) = \infty\}. \quad (5.2)$$

The following can be verified directly from the definition:

Lemma 5.8. *If $X = \bigcup_{n \in \mathbb{N}} X_n$ then*

$$\dim_{\mathrm{Hff}}(X) = \sup \dim_{\mathrm{Hff}}(X_n).$$

We will use the following consequence of Theorem 1.5.14 from Edgar's book [16]:

Corollary 5.9. *Let $E \subset \mathbb{R}^d$ be a measurable subset equipped with a probability measure ν. If the upper density*

$$\overline{D}^{\alpha}(x) = \limsup_{r\to 0} \frac{\nu\big(B(x,r)\big)}{r^{\alpha}}$$

is ν-essentially bounded above, then $\dim_{\mathrm{Hff}}(E) \geqslant \alpha$.

5.3. **The lower bound $\dim_{\mathrm{Hff}}(\mathscr{G}\{\mathfrak{b}$-conical points$\}) \geqslant \hbar^{\infty}$.** We import some tools from the proof of Theorem 4.12.

We consider the vector space

$$V^* := \mathrm{span}\{\tau, \overline{\tau}\}$$

together with its annihilator $\mathrm{Ann}(V^*) = \ker \tau \cap \overline{\tau}$ and the quotient vector space $V = \mathfrak{a}_\theta / \mathrm{Ann}(V^*)$. Any element of V^* vanishes on $\mathrm{Ann}(V^*)$ and thus V^* is naturally identified with the dual space of V. Since we have the preferred basis $\{\tau, \overline{\tau}\}$ we naturally identify V and $\mathbb{R}^2$ via the isomorphism $v \mapsto \big(\tau(v), \overline{\tau}(v)\big)$ and we let

$$\Pi : \mathfrak{a}_\theta \to \mathbb{R}^2$$

be the quotient projection (composed with the above isomorphism). The hyperplane $\mathfrak{b}$ is sent to the diagonal

$$\Pi\big(\ker(\tau - \overline{\tau})\big) = \big\{v \in V : \tau(v) = \overline{\tau}(v)\big\}$$

of the quadrant

$$V^+ = \{\tau \geqslant 0\} \cap \{\overline{\tau} \geqslant 0\}$$

on $\mathbb{R}^2$. Let $\upsilon = \upsilon_{(\rho,\overline{\rho})} : \Gamma \times \partial\Gamma \to V$ be the composition of the refraction cocycle $\beta_{(\rho,\overline{\rho})}$ of the pair with Π. Its periods are

$$\upsilon(\gamma, \gamma_+) = \Big(\tau\big(\lambda(\rho\gamma)\big), \overline{\tau}\big(\lambda(\overline{\rho}\gamma)\big)\Big),$$

so by assumption υ is non-arithmetic. As in § 4.4 one has $\Omega_\upsilon = V^* \cap \Omega_{\theta,\rho}$; by non-atihmeticity the cone $\mathcal{L}_\upsilon$ has non-empty interior and thus Corollary 3.5 gives that Ω_υ is a strictly convex curve. We consider the max norm on V $\|v\|_\infty = \max\{|\tau(v)|, |\psi(v)|\}$, and its dual (operator) norm on V^* denoted by $\|\,\|^1$. Let $\varphi_\infty \in \Omega_\upsilon$ be the unique form such that

$$\|\varphi_\infty\|^1 = \inf\{\|\varphi\|^1 : \varphi \in \Omega_\upsilon\}.$$

In Corollary 3.5 we introduced the map $\varphi \mapsto \mathsf{u}^\upsilon_\varphi$ defined on Ω_υ with values on $\mathbb{P}(V)$.

Lemma 5.10. *One has $\mathsf{u}^\upsilon_{\varphi_\infty} = \Pi(\mathfrak{b})$ and for any $v \in V^+$ one has*

$$\varphi_\infty(v) \geqslant \hbar^{\infty} \min\{\tau(v), \overline{\tau}(v)\}.$$

Proof. We apply Lemma 3.6 to τ and $\psi = \overline{\tau}$, Equation (**??**) guarantees the hypothesis of the Lemma. One then has (recall Figure 4)

i) $\mathsf{T}_{\varphi_\infty}\Omega_\upsilon = \mathrm{span}\{\tau - \overline{\tau}\}$ and thus

$$\mathsf{u}^\upsilon_{\varphi_\infty} = \mathrm{Ann}\big(\mathbb{R}\cdot(\tau - \overline{\tau})\big) = \Pi(\mathfrak{b})$$

ii) $\varphi_\infty/\|\varphi_\infty\|^1 = s\tau + (1-s)\overline{\tau}$ for some $s \in [0,1]$ and hence

$$\varphi_\infty \geqslant \|\varphi_\infty\|^1 \min\{\tau, \overline{\tau}\}$$

on V^+.

To conclude the lemma we need to show that $\hbar^\infty \leqslant \|\varphi_\infty\|^1$. Since $\varphi_\infty\big(a_\theta((\rho,\overline{\rho})\gamma)\big) \leqslant \|\Pi(a_\theta((\rho,\overline{\rho})\gamma))\|_\infty \|\varphi_\infty\|^1$, we deduce, for all $s > \|\varphi_\infty\|^1$,

$$\sum_{\gamma\in\pi_1 S} e^{-s\|\Pi(a_\theta((\rho,\overline{\rho})\gamma))\|_\infty} \leqslant \sum_{\gamma\in\pi_1 S} e^{-(s/\|\varphi_\infty\|^1)\varphi_\infty\left(a_\theta((\rho,\overline{\rho})\gamma)\right)} < \infty$$

where in the last inequality we used that $\hbar_{\varphi_\infty} = 1$ by Equation (3.4). □

Let μ^{φ_∞} be the Patterson-Sullivan measure associated to φ_∞ by Corollary 4.10. One has, for every $\gamma \in \Gamma$, that,

$$\mu^{\varphi_\infty}\big(\gamma\mathcal{C}_\infty(\gamma)\big) \leqslant Ce^{-\varphi_\infty\left(a_\theta((\rho,\overline{\rho})\gamma)\right)} \leqslant Ce^{-\hbar^\infty \min\left\{\tau(a(\rho\alpha_i)),\overline{\tau}(a(\overline{\rho}\alpha_i))\right\}}, \tag{5.3}$$

where the last inequality comes from Lemma 5.10.

By Proposition 5.7 there exist constants C_1 and C_2 such that if $(\alpha_i)_0^\infty$ is a geodesic ray from id to x then for all i the subsets

$$\xi\big(\alpha_i\mathcal{C}_\infty(\alpha_i)\big) \text{ and } \overline{\xi}\big(\alpha_i\mathcal{C}_\infty(\alpha_i)\big)$$

are coarsely balls on the corresponding projective spaces of radi

$$C_1e^{-\tau(a(\rho\alpha_i))} \text{ and } C_2e^{-\overline{\tau}(a(\overline{\rho}\alpha_i))}$$

respectively. Since $\mathscr{G}(\partial\Gamma)$ is a graph, the preceding radius computation implies that the image of the cone type $\mathscr{G}\big(\alpha_i\mathcal{C}_\infty(\alpha_i)\big)$ is also coarsely a ball (for the product metric on $\mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{\overline{d}})$) that can be chosen of radius

$$Ce^{-\min\left\{\tau(a(\rho\alpha_i)),\overline{\tau}(a(\overline{\rho}\alpha_i))\right\}}, \tag{5.4}$$

for some uniform constant C. Recall that this set of balls forms a fine set of neighbourhoods. Combining this with Equation (5.3) one has, possibly modifying the constant C, that for all r the measure of the ball of radius r about $\mathscr{G}(x)$ is

$$\mu^{\varphi_\infty}\big(B(x,r)\big) \leqslant Cr^{-\hbar^\infty}.$$

Since $\dim V^* = 2$ and $\upsilon_{(\rho,\overline{\rho})}$ is assumed non-arithmetic, Theorem 4.12 states that the subset of $\mathfrak{b}$-conical points has full μ^{φ_∞} measure. Applying Corollary 5.9 one concludes that

$$\dim_{\mathrm{Hff}}\big(\mathscr{G}\{\mathfrak{b}-\text{conical points}\}\big) \geqslant \hbar^\infty.$$

5.4. **The upper bound.** We now prove the other inequality.

Proposition 5.11. *One has* $\dim_{\mathrm{Hff}}\big(\mathscr{G}\{\mathfrak{b}-\text{conical points}\}\big) \leqslant \hbar^\infty$.

Proof. A point x is $(R,\mathfrak{b})$-*conical* if there exists a geodesic ray $(\alpha_i)_{i\in\mathbb{N}}$ such that for an infinite subset $\mathbb{I} \subset \mathbb{N}$ of indices and for every $k \in \mathbb{I}$

$$\Big|\tau\big(a(\rho\alpha_n)\big) - \overline{\tau}_1\big(a(\overline{\rho}\alpha_n)\big)\Big| \leqslant R.$$

We denote by $\mathsf{C}_\mathfrak{b}^R$ the set of $(R,\mathfrak{b})$-conical points. By Lemma 4.17 one has

$$\bigcup_{R>0} \mathsf{C}_\mathfrak{b}^R = \{x \in \partial\Gamma : x \text{ is } \mathfrak{b}-\text{conical}\},$$

and thus by Lemma 5.8 it suffices to show that for every R one has

$$\dim_{\mathrm{Hff}}\big(\mathsf{C}_\mathfrak{b}^R\big) \leqslant \hbar^\infty.$$

For any constant $K > 0$ and any $\gamma \in \pi_1 S$ we denote by $B_\gamma^{\max,K}$ the open ball of $\mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{\overline{d}})$ given by:

$$B_\gamma^{\max,K} := B\Big(\big(U_1(\rho\gamma), U_1(\overline{\rho}\gamma)\big), Ke^{-\max\{\tau(a(\rho\gamma)),\overline{\tau}_1(a(\overline{\rho}\gamma))\}}\Big)$$

And denote by

$$\mathcal{U}_T^K := \big\{B_\gamma^{\max,K} \,|\, |\gamma| \geqslant T\big\}.$$

Let C, resp. $\overline{C}$, be the constants given by Proposition 4.9 for the representation ρ (resp. $\overline{\rho}$). Observe that for $K = 2e^R \max\{C, \overline{C}\}$ and for every T the set $\mathcal{U}_T^K$ covers $\mathsf{C}_\flat^R$. Indeed, if $x \in \mathsf{C}_\flat^R$ there exists a geodesic $(\alpha_i)_{i\in\mathbb{N}}$ converging to x, such that for infinitely many $n \in \mathbb{N}$,

$$\big|\tau\big(a(\rho\alpha_n)\big) - \overline{\tau}\big(a(\overline{\rho}\alpha_n)\big)\big| < R,$$

and thus

$$\tau\big(a(\rho\alpha_n)\big) > \max\{\tau\big(a(\rho\alpha_n)\big), \overline{\tau}\big(a(\overline{\rho}\alpha_n)\big)\} - R.$$

In order to conclude we observe that the covers $\mathcal{U}_T^K$ form arbitrarily fine covers of the set $\mathsf{C}_\flat^R$ and have the property that, by definition of $\hbar^\infty$, for every $s > \hbar^\infty$,

$$\sum_{U\in\mathcal{U}_T^K} \operatorname{diam} U^s < +\infty.$$

This implies that for every $s > \hbar^\infty$ the s-capacity of $\mathsf{C}_\flat^R$ is finite, and thus Equation (5.2) gives

$$\dim_{\mathrm{Hff}}(\mathsf{C}_\flat^R) \leqslant \hbar^\infty.$$

□

6. Theorem B: Zariski closures of real-hyperconvex surface-group representations

In this section we prove Theorem B that classifies possible Zariski closures of real $(1,1,2)$-hyperconvex representations of surface groups. For most of the section we work with two $(1,1,2)$-hyperconvex representations and eventually reduce the proof of Theorem B to a situation like this; we will crucially use that a representation is $(1,1,2)$-hyperconvex if and only if its limit set is C^1 (Theorem 1.2).

6.1. When Ξ has oblique derivative.

We prove here a result that we believe to be of independent interest. This subsection only requires § 4.1 and § 4.2 and will be needed for Theorem B but also for Theorems A and C.

We let Γ be either a closed surface group or a Kleinian group. In the surface group case we let

$$\rho, \overline{\rho} : \Gamma \to \mathrm{Diff}^{1+\alpha}(\mathbb{S}^1)$$

be Hölder conjugated to a Fuchsian action, if $\Gamma < \mathsf{PSL}(2,\mathbb{C})$ is a Kleinian group we let $\rho, \overline{\rho} : \Gamma \to \mathsf{PSL}(2,\mathbb{C})$ be two convex co-compact representations that lie in the same connected component of

$$\big\{\varrho : \Gamma \to \mathsf{PSL}(2,\mathbb{C}) : \ \varrho \text{ is convex co-compact}\big\}.$$

We let X be either the circle or $\partial\mathbb{H}^3$. To simplify notation we will denote, for $\gamma \in \Gamma$, its action on X via ρ simply by γ, and its action via $\overline{\rho}$ by $\overline{\gamma}$ and by $\partial\Gamma, \overline{\partial\Gamma} \subset X$ the limit sets of ρ and $\overline{\rho}$ respectively.

In both situations there exists a Hölder-continuous map

$$\Xi : X \to X$$

conjugating ρ and $\overline{\rho}$; in the surface case this holds by definition, in the Kleinian case this is a Theorem by Marden [32], see also Anderson's survey [1, page 32]. The point in this case is that the equivariant limit map $\Xi : \partial\Gamma \to \partial\Gamma$ conjugating the actions ρ and $\overline{\rho}$ on their respective limit sets extends to the whole Riemann sphere $\partial\mathbb{H}^3$. We study differentiability points of this map with oblique derivative.

We let d be either a visual distance on X (in the complex case) or a distance inducing the chosen C^1 structure on the circle $\mathbb{S}^1$.

Definition 6.1. A representation ρ is *Lipschitz-compatible* (with the C^1 structure) if there exist positive C, λ and $L \in \mathbb{N}$ such that if $\gamma \in \Gamma$ has word-length $|\gamma| \geqslant L$ and $x, y \in \mathcal{C}_\infty(\gamma)$ then

$$d(\gamma x, \gamma y) \leqslant Ce^{-|\gamma|\lambda} d(x, y).$$

Observe that in the Kleinian case we actually have more, contraction does not only hold on $\mathcal{C}_\infty(\gamma)$ but also on the complementary of a(ny) disk about $U_1(\gamma^{-1})$, indeed Proposition 4.9 states that for all x, y with

$$\begin{aligned} &\mathcal{G}(x, U_1(\gamma^{-1})) \geqslant -\delta_\rho \text{ and } \mathcal{G}(y, U_1(\gamma^{-1})) \geqslant -\delta_\rho \\ \Rightarrow \quad &d(\gamma x, \gamma y) \leqslant Ce^{-|\gamma|\lambda} d(x, y). \end{aligned} \tag{6.1}$$

The goal of the subsection is to prove the following result which is probably know to experts. Similar arguments can be found in Guizhen [22] in the context of conjugacies of expanding circle maps.

Proposition 6.2. *Let $\rho, \overline{\rho}$ be as above and assume both are Lipschitz compatible. If there exists $p \in \partial\Gamma$ such that Ξ has a finite non-vanishing derivative (complex derivative in the Kleinian case) then $\Xi|\partial\Gamma$ is bi-Lipschitz.*

We work under the assumptions of Proposition 6.2 and begin its proof with the following lemma. For $\gamma \in \Gamma$ we denote it's derivative at $x \in X$ by $\gamma'(x) \in \mathbb{K}$ defined, according our two situations, by

$X = \mathbb{S}^1$: the derivative $\tilde{\gamma}'(\tilde{x})$ of a lift of γ to the universal cover $\mathbb{R}$ of $\mathbb{S}^1$, and a lift $\tilde{x} \in \mathbb{R}$ of x, the number $\tilde{\gamma}'(\tilde{x})$ is independent of these choices;

$X = \partial\mathbb{H}^3$: we fix an arbitrary point $\infty \notin \partial\Gamma$, identify $X - \{\infty\}$ with $\mathbb{K}$ via the stereographic projection and let $\gamma'(x)$ be the standard complex derivative.

In order to simplify notation in the sequel we let $\mathcal{B}_\infty(\gamma) \subset X$ be the smallest disc that contains $\mathcal{C}_\infty(\gamma)$. If $X = \mathbb{S}^1$ then $\mathcal{B}_\infty(\gamma) = \mathcal{C}_\infty(\gamma)$, if $X = \partial\mathbb{H}^3$ then one has

$$\mathcal{B}_\infty(\gamma) \subset \left\{x \in \mathbb{P}(\mathbb{C}^2) : \mathcal{G}(x, U_1(\gamma^{-1})) \geqslant -\delta_\rho\right\}.$$

Lemma 6.3. *There exists constants $\kappa, \nu > 0$ and $N \in \mathbb{N}$ such that for all $\gamma \in \Gamma$ with $|\gamma| \geqslant N$ and $x, y \in \mathcal{B}_\infty(\gamma)$ one has*

$$\left|\log|\gamma'(x)| - \log|\gamma'(y)|\right| \leqslant \kappa d(x, y)^\nu.$$

Proof. We consider L from Definition 6.1 (or Equation (6.1)), so that for every $\eta \in \Gamma$ with $|\eta| \geqslant L$ and $x, y \in \mathcal{B}_\infty(\eta)$ one has

$$d(\eta x, \eta y) \leqslant Ce^{-|\eta|\lambda} d(x, y). \tag{6.2}$$

Since the action is $\mathrm{C}^{1+\alpha}$ we can find positive K, ν such that for every β with $|\beta| \leqslant L$ and $u, w \in X$ one has

$$\big|\log|\beta'(u)| - \log|\beta'(w)|\big| \leqslant K d(u,w)^{\nu}. \tag{6.3}$$

We let then $K' = \max\{K, KC^{\nu}\}$. We begin by showing, by induction on k, that if $|\gamma| = kL$ then for all $x, y \in \mathcal{C}_{\infty}(\gamma)$, one has

$$\big|\log|\gamma'(x)| - \log|\gamma'(y)|\big| \leqslant K'\big(\sum_{i=0}^{k-1} e^{-\nu\lambda L i}\big) d(x,y)^{\nu}. \tag{6.4}$$

Equation (6.3) gives the base case, so assume that the result holds up to $k-1$. We let then $\gamma = \beta_1 \cdots \beta_{kL}$ be a reduced word on the fixed generating set of Γ. We let also $\gamma_i = \beta_1 \cdots \beta_i$ and $\eta = \beta_{L+1} \cdots \beta_{kL}$ so that $|\eta| = (k-1)L$ and $\gamma = \gamma_L \eta$. By definition of cone type one has

$$\mathcal{B}_{\infty}(\gamma) \subseteq \mathcal{B}_{\infty}(\eta). \tag{6.5}$$

Applying the chain rule gives that for every $u \in X$ one has

$$\log|\gamma'(u)| = \log|(\gamma_L)'(\eta u)| + \log|(\eta)'(u)|$$

and thus, when $x, y \in \mathcal{B}_{\infty}(\gamma)$,

$$\begin{aligned}
\big|\log|\gamma'(x)| - \log|\gamma'(y)|\big| &\leqslant \big|\log|\gamma_L'(\eta x)| - \log|\gamma_L'(\eta y)|\big| + \big|\log|\eta'(x)| - \log|\eta'(y)|\big| \\
&\leqslant K d(\eta x, \eta y)^{\nu} + K'\big(\sum_{i=0}^{k-2} e^{-\nu\lambda L i}\big) d(x,y)^{\nu} \quad \text{(by (6.3) and induction)} \\
&\leqslant K C^{\nu} e^{-|\eta|\nu\lambda} d(x,y)^{\nu} + K'\big(\sum_{i=0}^{k-2} e^{-\nu\lambda L i}\big) d(x,y)^{\nu} \quad \text{(by Eqs. (6.5) and (6.2))}.
\end{aligned}$$

We have proven thus that for $\kappa_0 = K'/(1 - e^{-\nu\lambda L})$ and every $\gamma \in \Gamma$ whose word-length is an integer multiple of L that if $x, y \in \mathcal{B}_{\infty}(\gamma)$ one has

$$\big|\log|\gamma'(x)| - \log|\gamma'(y)|\big| \leqslant \kappa_0 d(x,y)^{\nu}.$$

To conclude the lemma we consider an arbitrary γ with $|\gamma| = mL + n$ and $n < L$. We write $\gamma = \gamma_{mL}\alpha$ with $\alpha = \beta_{mL+1} \cdots \beta_{|\gamma|}$. Observe that by definition of cone-type one has

$$\alpha \mathcal{B}_{\infty}(\gamma) \subset \mathcal{B}_{\infty}(\gamma_{mL}).$$

Applying the chain rule gives then

$$\begin{aligned}
\big|\log|\gamma'(x)| - \log|\gamma'(y)|\big| &\leqslant \big|\log|\gamma_{mL}'(\alpha x)| - \log|\gamma_{mL}'(\alpha y)|\big| + \big|\log|\alpha'(x)| - \log|\alpha'(y)|\big| \\
&\leqslant \kappa_0 d(x,y)^{\nu} + K d(x,y)^{\nu} \qquad (\text{ by } \alpha\mathcal{B}_{\infty}(\gamma) \subset \mathcal{B}_{\infty}(\gamma_{mL}) \text{ and (6.3)}) \\
&\leqslant (\kappa_0 + K) d(x,y)^{\nu},
\end{aligned}$$

so taking $\kappa = \kappa_0 + K$ we conclude the proof. □

Proof of Proposition 6.2. Let $p \in \partial\Gamma$ be such that Ξ has non-horizontal nor vertical derivative at p. Fix a geodesic ray $(\alpha_n)_0^{\infty}$ through the identity with $\alpha_n \to p$. By definition for all n one has $p \in \alpha_n \mathcal{C}_{\infty}(\alpha_n)$. Without loss of generality we may also assume that

$$p = 0 = \Xi(0)$$

and we may write the derivative as the incremental limit

$$\Xi'(0) = \lim_{y\to 0} \frac{\Xi(y)}{y} \in \mathbb{K} - \{0\}.$$

For each n we let s_n be the radius of the ball $\alpha_n \mathcal{B}_\infty(\alpha_n)$, by Lipschitz compatibility s_n is coarsely $e^{-\lambda n}$.

Let $D(0,r)$ be the ball of radius r about 0 and consider the scaling map

$$g_n : D(0,1) \to \alpha_n \mathcal{B}_\infty(\alpha_n)$$

defined by $g_n(z) = s_n z$. Let a_n be an arbitrary point at distance s_n from 0 and let $\tilde{s}_n = \Xi(a_n)$. Define

$$\tilde{g}_n : D(0,1) \to D(0, |\tilde{s}_n|)$$

by $z \mapsto z\tilde{s}_n$. Since $s_n \to 0$ and $\Xi'(0) \notin \{0, \infty\}$ exists, the composition

$$\tilde{g}_n^{-1} \Xi g_n(z) = \frac{\Xi(zs_n)}{\tilde{s}_n} \cdot \frac{s_n z}{s_n z} = \frac{\Xi(zs_n)}{s_n z} \cdot \frac{s_n}{\tilde{s}_n} \cdot z = \frac{\Xi(zs_n)}{s_n z} \cdot \frac{s_n}{\Xi(s_n)} \cdot z$$

converges uniformly on compact subsets to the identity map.

On the other hand one has

$$\tilde{g}_n^{-1} \Xi g_n = \tilde{g}_n^{-1} \overline{\alpha_n} \Xi \alpha_n^{-1} g_n;$$

we now study the maps $f_n := \alpha_n^{-1} \circ g_n$ and $\tilde{f}_n := \tilde{g}_n^{-1} \circ \overline{\alpha_n}$.

Observe that for every $x \in D(0,1)$ one has

$$\log |f_n'(x)| = \log |(\alpha_n^{-1})'(g_n x)| + \log |s_n| = -\log |\alpha_n'(\alpha_n^{-1} g_n x)| + \log |s_n|.$$

Now by definition of g_n, we have that $g_n x \in \alpha_n \mathcal{B}_\infty(\alpha_n)$ and thus $\alpha_n^{-1}(g_n x) \in \mathcal{B}_\infty(\alpha_n)$. For n large enough we can apply Lemma 6.3 to α_n to obtain κ and ν so that for every pair it holds $x, y \in D(0,1)$

$$\big|\log |f_n'(x)| - \log |f_n'(y)|\big| \leqslant \kappa d(x,y)^\nu.$$

We conclude that the family of maps (f_n) is uniformly bi-Lipschitz on $D(0,1)$ and thus, since $(f_n p)$ is bounded, Arzela-Ascoli's Theorem applies to give a subsequence (still denoted by f_n) that converges to a bi-Lipschitz map f defined on $D(0,1)$.

Analog reasoning applies to $\tilde{f}$, and we obtain that, about 0, Ξ can be written as a composition of bi-Lipschitz maps and is thus bi-Lipschitz. Using the action of Γ we extend the Lipschitz property of Ξ to the whole $\partial\Gamma$, concluding the proof. □

We now establish the following Corollary that will be used in the sequel.

Corollary 6.4. *Let S be a closed surface of genus $\geqslant 2$ and let $\rho : \pi_1 S \to \mathsf{PGL}(d,\mathbb{R})$ and $\overline{\rho} : \pi_1 S \to \mathsf{PGL}(\overline{d},\mathbb{R})$ be $(1,1,2)$-hyperconvex, consider the map between $\mathrm{C}^{1+\alpha}$ circles*

$$\Xi = \overline{\xi} \circ \xi^{-1} : \xi(\partial\pi_1 S) \to \overline{\xi}(\partial\pi_1 S).$$

If Ξ has a differentiability point with finite non-vanishing derivative then for all $\gamma \in \pi_1 S$ one has $\tau(\lambda(\rho\gamma)) = \overline{\tau}(\lambda(\overline{\rho}\gamma))$.

Proof. Theorem 1.2 implies that a $(1,1,2)$-hyperconvex representation of a surface group induces a $\mathrm{C}^{1+\mathrm{Höl}}$ action on the $\mathrm{C}^{1+\mathrm{Höl}}$ circle $\xi(\partial\pi_1 S)$. The distance $d_{\mathbb{P}}$ induced on this circle is compatible with its C^1-structure and thus Proposition 4.9 implies that the action on the circle is Lipschitz-compatible. We can thus apply Proposition 6.2 to obtain that Ξ is bi-Lipschitz. The following standard lemma from linear algebra (see for example Benoist [5] and S. [40, Lemma 3.4]) gives the period computation completing the proof. □

Lemma 6.5. *Let $g \in \mathsf{PGL}(d,\mathbb{R})$ be proximal and let $V_{\lambda_2(g)}$ be the sum of the characteristic spaces of g whose associated eigenvalue is of modulus $\exp\lambda_2(g)$. Then for every $v \notin \mathbb{P}(g_-)$, with non-zero component in $V_{\lambda_2(g)}$, one has*

$$\lim_{n\to\infty} \frac{\log d_{\mathbb{P}}(g^n(v), g_+)}{n} = -\tau_1(\lambda(g)).$$

6.2. **Limit curves in non-maximal flags.** We proceed with another intermediate step for the proof of Theorem B, this step follows from the combination of Theorem 1.2 and Corollary 6.4.

Let G be real-algebraic and semi-simple. Let $\{\mathsf{a},\mathsf{b}\} \subset \Delta$ be two distinct simple roots. The partial flag space $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}}$ carries two transverse foliations that are the level sets of the natural projections $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}} \to \mathcal{F}_{\{\mathsf{a}\}}$ and $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}} \to \mathcal{F}_{\{\mathsf{b}\}}$. We will refer to these as the *canonical foliations* of $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}}$.

Corollary 6.6. *Let G be real-algebraic and semi-simple and let $\{\mathsf{a},\mathsf{b}\} \subset \Delta$. Let $\rho : \pi_1 S \to \mathsf{G}$ be Zariski-dense and $\{\mathsf{a},\mathsf{b}\}$-Anosov. If both curves $\xi^{\mathsf{a}}(\partial\pi_1 S)$ and $\xi^{\mathsf{b}}(\partial\pi_1 S)$ are C^1 then every differentiability point of $\xi^{\{\mathsf{a},\mathsf{b}\}}(\partial\pi_1 S)$ is tangent to one of the canonical foliations of $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}}$.*

Proof. By Benoist's Theorem 2.3 the limit cone of ρ has non-empty interior, in particular there exists $\gamma \in \pi_1 S$ such that

$$\mathsf{a}(\lambda(\rho\gamma)) \neq \mathsf{b}(\lambda(\rho\gamma))).$$

Consider the Tits representations Φ_{a} and Φ_{b} associated to a and b. Since $\rho(\pi_1 S)$ is Zariski-dense, both representation $\Phi_{\mathsf{a}}\rho$ and $\Phi_{\mathsf{b}}\rho$ are irreducible and projective Anosov. Recall that by definition of Φ_{a}, for every $g \in \mathsf{G}$ one has

$$\tau_1\big(\lambda\big(\Phi_{\mathsf{a}}(g)\big)\big) = \mathsf{a}\big(\lambda(g)\big).$$

Since the maps ζ_{a} and ζ_{b} are analytic, both projective curves $\zeta_{\mathsf{a}}\xi^{\mathsf{a}}(\partial\pi_1 S)$ and $\zeta_{\mathsf{b}}\xi^{\mathsf{b}}(\partial\pi_1 S)$ are C^1 and thus by Zhang-Zimmer's Theorem 1.2 the representations $\Phi_{\mathsf{a}}\rho$ and $\Phi_{\mathsf{b}}\rho$ are $(1,1,2)$-hyperconvex.

The natural embedding $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}} \to \mathbb{P}(V_{\mathsf{a}}) \times \mathbb{P}(V_{\mathsf{b}})$ sends $\xi^{\{\mathsf{a},\mathsf{b}\}}$ to the graph of the map Ξ from Corollary 6.4 and thus the Corollary implies the result. □

6.3. **Proof of Theorem B.** The goal of the section is to prove Theorem B, stating that the Zariski closure G of the image of an irreducible $(1,1,2)$-hyperconvex representation $\rho : \pi_1 S \to \mathsf{PGL}(d,\mathbb{R})$ is simple and the highest weight of the induced representation $\Phi : \mathsf{G} \to \mathsf{PGL}(d,\mathbb{R})$ is a multiple of a fundamental weight.

It is proven in S. [42, Lemma 8.6] that an irreducible subgroup G containing a proximal element is semi-simple without compact factors.

We consider the induced representation $\rho_0 : \pi_1 S \to \mathsf{G}$ and denote by $\Phi : \mathsf{G} \to \mathsf{PGL}(d,\mathbb{R})$ the linear representation so that $\rho = \Phi\rho_0$. Let $\chi = \chi_\Phi \in \mathfrak{a}^*$ be the highest weight of Φ. As in Definition 2.5 we consider

$$\theta = \theta_\Phi = \{\mathsf{a} \in \Delta : \chi - \mathsf{a} \text{ is a weight of } \Phi\} = \{\mathsf{a} \in \Delta : \langle\chi, \mathsf{a}\rangle \neq 0\}.$$

It is enough to show that θ is reduced to a single root $\{\mathsf{a}_0\}$; indeed, if this is the case, upon writing χ in the basis of fundamental weights $\{\varpi_{\mathsf{a}} : \mathsf{a} \in \Delta\}$ (recall their defining Equation (2.1)) one has

$$\chi = \sum_{\mathsf{a}\in\Delta} \langle\chi, \mathsf{a}\rangle\varpi_{\mathsf{a}} = \langle\chi, \mathsf{a}_0\rangle\varpi_{\mathsf{a}_0},$$

Moreover this gives:

- G is simple by Lemma 2.4;
- the weights on the first level consist solely on $\chi - \mathsf{a}$ and its associated weight space is $\phi(\mathfrak{g}_{-\mathsf{a}})V^+$. Since $\rho(\pi_1 S)$ is $\{\tau_2\}$-Anosov one has that $\phi(\mathfrak{g}_{-\mathsf{a}})V^+$ is one-dimensional, but by Lemma 2.6 no element of $\mathfrak{g}_{-\mathsf{a}}$ acts trivially on V^+ so $\mathfrak{g}_{-\mathsf{a}}$ is 1-dimensional, as desired.

We proceed now to show that in the present situation θ consists of only one element. By definition of θ one has, for every $g \in \mathsf{G}$, that

$$\tau_1\big(\lambda\big(\Phi(g)\big) = \min_{\mathsf{a}\in\theta}\big\{\mathsf{a}(\lambda_{\mathsf{G}}(g))\big\}.$$

Consequently, the limit cone $\mathcal{L}_{\rho_0} \subset \mathfrak{a}^+_{\mathsf{G}}$ does not intersect the walls of elements in θ and, since $\rho_0 : \pi_1 S \to \mathsf{G}$ is a quasi-isometry, Remark 4.4 implies that the representation ρ_0 is θ-Anosov.

Recall from Equation (2.5) that we have a Φ-equivariant analytic embedding $\zeta_\theta : \mathsf{G}/\mathsf{P}_\theta \to \mathbb{P}(\mathbb{R}^d)$. One has moreover that $\xi^1_\rho = \zeta_\theta \circ \xi^\theta_{\rho_0}$. In particular the boundary map ξ^θ has C^1-image. Composing with the projections $\mathcal{F}_\theta \to \mathcal{F}_{\theta'}$ one sees that, for any $\theta' \subset \theta$ the curve $\xi^{\theta'}_{\rho_0}(\partial\pi_1 S)$ is a C^1 circle.

Assume now there exists two distinct roots a, b in θ. By the previous paragraph the curve $\xi^{\{\mathsf{a},\mathsf{b}\}}(\partial\pi_1 S)$ is C^1 . Corollary 6.6 gives then that $\xi^{\{\mathsf{a},\mathsf{b}\}}(\partial\pi_1 S)$ is necessarily contained in one of the leaves of the canonical foliations of $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}}$, thus giving that one of the maps ξ^{a} or ξ^{b} is constant, achieving a contradiction. This completes the proof.

The following immediate corollary will be useful in the next section

Corollary 6.7. *Let $\rho : \pi_1 S \to \mathsf{PGL}(d,\mathbb{R})$ be an irreducible $(1,1,2)$-hyperconvex representation. Then there exists $\mathsf{a} \in \Delta_{\mathsf{G}}$ such that for every $a \in \mathfrak{a}^+_{\mathsf{G}}$ one has*

$$\mathsf{a}(a) = \tau_1\big(\phi(a)\big),$$

where $\phi : \mathfrak{g} \to \mathfrak{sl}(d,\mathbb{R})$ is the associated representation of G.

6.4. **A useful consequence of Theorem B: Non-arithmeticity of periods.** As an application of Theorem B we show that the periods with respect to the first simple root of a pair of real $(1,1,2)$-hyperconvex representations either agree or span a dense subgroup of $\mathbb{R}^2$. The following corollary from Beyrer-P. [7] allows us to bypass the irreducibility assumption in Theorem B.

Corollary 6.8 (Beyrer-P. [7])**.** *Let $\rho : \pi_1 S \to \mathsf{PGL}(d,\mathbb{R})$ be $(1,1,2)$-hyperconvex. Then there exists an irreducible $(1,1,2)$-hyperconvex representation $\rho_0 : \pi_1 S \to \mathsf{PGL}(m,\mathbb{R})$ such that, for every $\gamma \in \pi_1 S$ one has*

$$\tau_1\big(\lambda(\rho\gamma)\big) = \tau_1\big(\lambda(\rho_1\gamma)\big).$$

Proof. It follows from [7, Corollary 5.5] that if $\rho : \pi_1 S \to \mathsf{PGL}(d,\mathbb{R})$ is $(1,1,2)$-hyperconvex, then the semisemplification ρ^{ss} is also $(1,1,2)$-hyperconvex, and the first two weights of such representations belong to the same irreducible factor. The result follows by defining ρ_0 to be the restriction of ρ^{ss} to such an irreducible factor. □

We can now prove the following *non-arithmeticity of periods*:

Proposition 6.9. *Let $\rho : \pi_1 S \to \mathsf{PGL}(d,\mathbb{R})$ and $\overline{\rho} : \pi_1 S \to \mathsf{PGL}(\overline{d},\mathbb{R})$ be $(1,1,2)$-hyperconvex. If there exists $\gamma \in \pi_1 S$ with $\tau_1(\lambda(\rho\gamma)) \neq \overline{\tau}_1(\lambda(\overline{\rho}\gamma))$, then the group spanned by*

$$\left\{(\tau_1(\lambda(\rho\gamma)), \overline{\tau}_1(\lambda(\overline{\rho}\gamma))) : \gamma \in \pi_1 S\right\}$$

is dense in $\mathbb{R}^2$.

Proof. Observe first that thanks to Corollary 6.8 we can assume without loss of generality that ρ and $\overline{\rho}$ are irreducible. Let then G and $\overline{\mathsf{G}}$ be the Zariski closures respectively of $\rho(\pi_1 S)$ and $\overline{\rho}(\pi_1 S)$. By Theorem B both groups are simple and Corollary 6.7 guarantees that the restrictions of τ to $\overline{\mathfrak{a}}^+_{\mathsf{G}}$ and of $\overline{\tau}$ to $\overline{\mathfrak{a}}^+_{\overline{\mathsf{G}}}$ are simple roots of G and $\overline{\mathsf{G}}$ respectively, still denoted by τ and $\overline{\tau}$. We can furthermore assume without loss of generality that G and $\overline{\mathsf{G}}$ are adjoint. We distinguish two cases.

If $\rho \times \overline{\rho}(\pi_1 S)$ is Zariski dense in $\mathsf{G} \times \overline{\mathsf{G}}$, then the result now follows from Benoist's Theorem 2.3 and the fact that $\{\tau, \overline{\tau}\} \subset (\mathfrak{a}_{\mathsf{G}\times\overline{\mathsf{G}}})^*$.

Otherwise, it is well known that there exists an isomorphism $\Phi : \mathsf{G} \to \overline{\mathsf{G}}$ so that $\overline{\rho} = \Phi \circ \rho$ (see for example Bridgeman-Canary-Labourie-S. [10, Corollary 11.6]). In this case $\Phi^*\overline{\tau}$ is a root of G and we distinguish two cases: either $\Phi^*\overline{\tau} = \tau$, but then $\tau(\lambda(\rho\gamma)) = \overline{\tau}(\lambda(\overline{\rho}\gamma))$ for all $\gamma \in \pi_1 S$ contradicting our assumption, or $\Phi^*\overline{\tau}$ and τ are distinct roots of G. Since $\rho : \pi_1 S \to \mathsf{G}$ is Zariski-dense, then again Benoist's Theorem 2.3 implies the result. □

7. Non-differentiability and ♭-conicality: The proofs of Theorems A and C

The following Proposition relates non-differentiability with ♭-conicality, the missing piece for the proofs of Theorems A and C.

Proposition 7.1. *If there exists $\gamma \in \Gamma$ such that $\tau\big(\lambda(\rho\gamma)\big) \neq \overline{\tau}\big(\lambda(\overline{\rho}\gamma)\big)$, then the set of non-differentiability points of Ξ coincides with the set of ♭-conical points.*

The inclusion {non-diff (non-$\mathbb{C}$-diff) points} $\subset$ {$\flat-$conical} is rather general, the other inclusion requires Proposition 6.2.

Proof. We deal with the real case, which illustrates the main ideas, the Kleinian case works analogously.

We choose a C^1 identification of the C^1 torus $\xi(\partial\pi_1 S) \times \overline{\xi}(\partial\pi_1 S) \subset \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{\overline{d}})$ with the quotient of the square $[-1,1] \times [-1,1]$ preserving the product structure, and such that the point $(x, \Xi(x))$ corresponds to $(0,0)$. In these coordinates the graph of Ξ is a monotone curve $[-1,1] \to [-1,1]$ passing through the origin. Given a point $y \in \partial\pi_1 S$, we denote by $(y^1, y^2) = (y^1, \Xi(y^1))$ its image in $[-1,1] \times [-1,1]$.

Let now $(\alpha_i)_{i\in\mathbb{N}}$ denote a geodesic ray converging to x. Since the chosen identification is C^1, it is in particular k-bi-Lipschitz for some k, as a result we deduce from Proposition 5.7 that there exist constants $C_1, C_2, \overline{C}_1, \overline{C}_2$ and $L \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ and every $y_n \in \alpha_n \mathcal{C}_\infty(\alpha_n) \backslash \alpha_{n+L} \mathcal{C}_\infty(\alpha_{n+L})$, it holds

$$\begin{aligned} C_1 e^{-\tau_1(a(\rho(\alpha_n)))} < |y_n^1| < C_2 e^{-\tau_1(a(\rho(\alpha_n)))}, \\ \overline{C}_1 e^{-\overline{\tau}_1(a(\overline{\rho}(\alpha_n)))} < |y_n^2| < \overline{C}_2 e^{-\overline{\tau}(a(\overline{\rho}(\alpha_n)))}. \end{aligned} \tag{7.1}$$

Assume first that x is ♭-conical, and assume, by contradiction, that x is a differentiability point of Ξ. Using Definition 5.3 we obtain a geodesic ray $(\alpha_i)_0^\infty$, an

infinite set of indices $\mathbb{I} \subset \mathbb{N}$ and a number R, such that for all $n \in \mathbb{I}$ one has

$$|\tau_1(a(\rho\alpha_n))) - \overline{\tau}_1(a(\overline{\rho}\alpha_n)))| < R. \tag{7.2}$$

For each such n we choose a point $y_n \in \alpha_n \mathcal{C}_\infty(\alpha_n) \backslash \alpha_{n+L} \mathcal{C}_\infty(\alpha_{n+L})$. By construction y_n converges to x. Now one has,

$$\Xi'(0) = \lim_{n\to\infty} \frac{\Xi(y_n^1)}{y_n^1}. \tag{7.3}$$

Restricting the limit to the infinite set of indices for which Equation (7.2) holds, we deduce from Equation (7.1) that

$$e^{-R}\frac{\overline{C}_1}{C_2} \leqslant |\Xi'(0)| \leqslant e^{R}\frac{\overline{C}_2}{C_1}.$$

Which implies that the derivative is neither horizontal nor vertical. Corollary 6.4 implies then that for all $\gamma \in \Gamma$ one has $\tau(\lambda(\rho\gamma)) = \overline{\tau}(\lambda(\overline{\rho}\gamma))$, since by assumption there exists $\gamma \in \Gamma$ with $\tau\big(\lambda(\rho\gamma)\big) \neq \overline{\tau}\big(\lambda(\overline{\rho}\gamma)\big)$, we obtain a contradiction.

Conversely, assume that x is not $\flat$-conical. The Cartan projections of two consecutive elements α_i, α_{i+1} make uniformly bounded gaps (Proposition 2.1), and thus there exists C such that for all $n \in \mathbb{N}$ one has

$$\Big|\tau\big(a(\rho\alpha_{n+1})\big) - \tau\big(a(\rho\alpha_n)\big)\Big| < C.$$

As a consequence, we can assume, up to switching the roles of ρ and $\overline{\rho}$, that for any R there exists n_R such that for every $n > n_R$ one has

$$\tau\big(a(\rho\alpha_n)\big) - \overline{\tau}\big(a(\overline{\rho}\alpha_n)\big) > R.$$

In turn this implies, thanks to Equation (7.1) that, for every $y \in \alpha_{n_R}\mathcal{C}_\infty(\alpha_{n_R})$,

$$\frac{|y^2|}{|y^1|} \leqslant e^{-R}\frac{\overline{C}_2}{C_1}.$$

Since R is arbitrary, and the sets $\alpha_{n_R}\mathcal{C}_\infty(\alpha_{n_R})$ form a system of neighborhoods of the point x, we deduce that the limit in Equation (7.3) exists and is a vertical vector. This concludes the proof. □

7.1. **Proof of Theorems A and C.** Since there exists $\gamma \in \Gamma$ with $\tau_1\big(\lambda(\rho\gamma)\big) \neq \overline{\tau}_1\big(\lambda(\overline{\rho}\gamma)\big)$, Proposition 6.9 implies the density assumption in Theorem 5.4, so one has

$$\dim_{\mathrm{Hff}} \Xi\big(\{\flat\text{-conical points}\}\big) = \hbar^{\max\{\tau,\overline{\tau}\}}.$$

Proposition 7.1 states that the set of $\flat$-conical points coincides with the set of non-differentiability points of Ξ, thus completing the proof of Theorem A. In the Kleinian case, density of the group spanned by the pairs $\{(\lambda(\rho\gamma), \lambda(\overline{\rho}\gamma)) : \gamma \in \Gamma\}$ follows from Benoist [4] (Theorem 2.3), from this point on the proof works verbatim.

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