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METRIC PROPERTIES OF BOUNDARY MAPS, HILBERT ENTROPY AND NON-DIFFERENTIABILITY

BEATRICE POZZETTI AND ANDRÉS SAMBARINO

Abstract. We interpret the Hilbert entropy of a convex projective structure on a closed higher-genus surface as the Hausdorff dimension of the non-differentiability points of the limit set in the full flag space $\mathcal{F}(\mathbb{R}^3)$. Generalizations for regularity properties of boundary maps between locally conformal representations are also discussed. An ingredient for the proofs is the concept of *hyperplane conicality* that we introduce for a θ-Anosov representation into a reductive real-algebraic Lie group G. In contrast with directional conicality, hyperplane-conical points always have full mass for the corresponding Patterson-Sullivan measure.

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1. Introduction

Consider a closed connected orientable surface S of genus at least two, and let $\rho : \pi_1 S \to \mathsf{PSL}(3,\mathbb{R})$ be a faithful representation preserving an open convex set $\Omega = \Omega_\rho \subset \mathbb{P}(\mathbb{R}^3)$, properly contained in an affine chart. The group $\rho(\pi_1 S)$ is necessarily discrete and acts co-compactly on Ω: one says that ρ *divides* Ω.

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The geometry of such convex set Ω is well studied, by Benoist [5] it is strictly convex with $\mathrm{C}^{1+\nu}$ boundary $\partial\Omega$ (that is not C^2 unless it is an ellipse), and the Hilbert metric of Ω is Gromov-hyperbolic. The geodesic flow of $\Omega/\rho(\pi_1 S)$ is an Anosov flow and its topological entropy, the *Hilbert entropy* $\hbar_{\mathsf{H}} = (\hbar_{\mathsf{H}})_\rho$, satisfies

$$\hbar_{\mathsf{H}} \leqslant 1,$$

an inequality proved by Crampon [18] that is strict if Ω is not an ellipse.

A consequence of Theorem B below is a new geometric interpretation of the Hilbert entropy which we now explain. For each $x \in \partial\Omega$ let $\Xi(x) \in \mathrm{Gr}_2(\mathbb{R}^3)$ be the unique plane whose projectivisation is tangent to $\partial\Omega$ at x. By [5], the image curve $\Xi(\partial\Omega) \subset \mathrm{Gr}_2(\mathbb{R}^3) \simeq \mathbb{P}((\mathbb{R}^3)^*)$ is also the boundary of a strictly convex divisible set Ω^* and is thus again a $\mathrm{C}^{1+\nu}$-circle. The *full-flag-curve*

$$\{(x, \Xi(x)) : x \in \partial\Omega\} \subset \mathcal{F}(\mathbb{R}^3),$$

is the graph of a monotone map between C^1 circles and thus is a Lipschitz submanifold that is therefore differentiable almost everywhere. We establish the following:

Corollary A. *Let $\rho : \pi_1 S \to \mathsf{PSL}(3,\mathbb{R})$ divide a strictly convex set that is not an ellipse. Then, the set of non-differentiability points of the full flag curve has Hausdorff dimension $(\hbar_{\mathsf{H}})_\rho$.*

Throughout the paper the Hausdorff dimension is computed with respect to a(ny) Riemannian metric on the flag space. When Ω is an ellipse the result does not apply as the associated curve is differentiable everywhere while $\hbar_{\mathsf{H}} = 1$.

A classical result by Choi-Goldman [16] states that the space of representations dividing a convex set forms a connected component of the character variety $\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}(3,\mathbb{R})\big)$ of homomorphisms up to conjugation. This component is known today as *the Hitchin component* of $\mathsf{PSL}(3,\mathbb{R})$ and is diffeomorphic to a ball of dimension $-8\chi(S)$. Nie [39] and Zhang [53] have found paths (ρ_t) in this Hitchin component such that $(\hbar_{\mathsf{H}})_{\rho_t} \to 0$ as $t \to \infty$. Together with Corollary A this suggest that the closer Ω is to being an ellipse (the *Fuchsian locus*), the less differentiable the flag curve is whilst the furthest away from Fuchsian locus, the more regular the flag curve becomes.

The proof of Corollary A is outlined in § 1.4 and serves as a guide path for the strategy on the general case (Theorems A and B).

1.1. **Locally conformal representations and concavity properties.** Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$ or the non-commutative field of Hamilton's quaternions $\mathbb{H}$. Denote by

$$\mathfrak{a} = \big\{(a_1, \dots, a_d) \in \mathbb{R}^d : \sum_i a_i = 0\big\}$$

the Cartan subspace of the real-algebraic group $\mathsf{SL}(d,\mathbb{K})$, by

$$\tau_i(a_1, \dots, a_d) = a_i - a_{i+1} \tag{1.1}$$

the i-th simple root and by $\mathfrak{a}^+ \subset \mathfrak{a}$ the Weyl chamber whose associated set of simple roots is $\Delta = \{\tau_i : i \in [\![1, d-1]\!]\}$. Let $a : \mathsf{SL}(d,\mathbb{K}) \to \mathfrak{a}^+$ be the *Cartan projection* with respect to the choice of an inner (or Hermitian) product on $\mathbb{K}^d$. The $e^{a_i(g)}$'s are the *singular values* of the matrix g, namely the square roots of the modulus of the eigenvalues of the matrix gg^*. We also let $d_{\mathbb{P}}$ denote the distance on $\mathbb{P}(\mathbb{K}^d)$ induced by the chosen Hermitian product.

Let Γ be a finitely generated word-hyperbolic group, consider a finite symmetric generating set and let $|\,|$ be the associated word-length. For $k \in [\![1, d-1]\!]$, a representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{K})$ is $\{\tau_k\}$-*Anosov* if there exist positive constants μ and c such that for all $\gamma \in \Gamma$ one has

$$\tau_k\big(a(\rho(\gamma))\big) \geqslant \mu|\gamma| - c.$$

A $\{\tau_k\}$-Anosov representation is also $\{\tau_{d-k}\}$-Anosov. Under such assumption there exists an equivariant Hölder-continuous map

$$\xi^k_\rho : \partial\Gamma \to \mathrm{Gr}_k(\mathbb{K}^d),$$

called the *limit map* in the Grassmannian $\mathrm{Gr}_k(\mathbb{K}^d)$ of k-dimensional subspaces of $\mathbb{K}^d$, which is a homeomorphism onto its image. If $k \leqslant l \in [\![1, d-1]\!]$ and ρ is also $\{\tau_l\}$-Anosov then the limit maps are compatible, i.e. $\xi^k_\rho(x) \subset \xi^l_\rho(x)$ $\forall x$, see §4 for references and details.

Definition 1.1. Fix $p \in [\![2, d-1]\!]$. A $\{\tau_1, \tau_{d-p}\}$-Anosov representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{K})$ is $(1, 1, p)$-*hyperconvex* if for every pairwise distinct triple $x, y, z \in \partial\Gamma$ one has

$$\big(\xi^1_\rho(x) + \xi^1_\rho(y)\big) \cap \xi^{d-p}_\rho(z) = \{0\}. \tag{1.2}$$

If in addition one has $a_2(\rho(\gamma)) = a_p(\rho(\gamma))$ $\forall\gamma$, we say that ρ is *locally conformal*.

Hyperconvex representations form an open subset of the character variety

$$\mathfrak{X}\big(\Gamma, \mathsf{SL}(d, \mathbb{K})\big) = \hom\big(\Gamma, \mathsf{SL}(d, \mathbb{K})\big)/\,\mathsf{SL}(d, \mathbb{K})$$

and appear naturally. For example, when $\mathbb{K} = \mathbb{R}$, strictly convex divisible sets give rise to $(1, 1, d-1)$-hyperconvex representations, while higher rank Teichmüller theory provides many examples of $(1, 1, 2)$-hyperconvex representations of surface groups, see Example 1.4.

When $p = 2$ the second part of the definition is trivially true, so $(1, 1, 2)$-hyperconvex representations over $\mathbb{K}$ *are* locally conformal, when $p > 2$ the assumption constrains the Zariski closure of $\rho(\Gamma)$. However, Zariski-dense locally conformal representations exist (and form open sets) for the groups locally isomorphic to $\mathsf{SL}(n, \mathbb{R})$, $\mathsf{SL}(n, \mathbb{C})$, $\mathsf{SL}(n, \mathbb{H})$, $\mathsf{SU}(1, n)$, $\mathsf{Sp}(1, n)$, $\mathsf{SO}(p, q)$, see P.-S.-Wienhard [43, § 8] for details, and, of course, $\mathsf{SO}(1, n)$ where every convex co-compact representation is locally conformal.

A concrete example in $\mathsf{SU}(1, n)$ consist on considering a convex co-compact group in $\mathbb{H}^n_{\mathbb{C}}$ whose limit set intersects the projectivization of any complex line in at most 2 points. These subgroups are locally conformal ([43, Proposition 8.3]) and their limit set (though fractal) is tangent to the contact distribution of $\partial\mathbb{H}^n_{\mathbb{C}}$.

Consider also $\overline{\mathbb{K}} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and positive integers d and $\overline{d}$. Throughout the paper we mainly deal with a pair of locally conformal representations

$$\rho : \Gamma \to \mathsf{SL}(d, \mathbb{K}) \text{ and } \overline{\rho} : \Gamma \to \mathsf{SL}(\overline{d}, \overline{\mathbb{K}}),$$

with equivariant maps $\xi = \xi^1_\rho$ and $\overline{\xi} = \xi^1_{\overline{\rho}}$, and we study regularity properties of the equivariant Hölder-continuous homeomorphism

$$\Xi = \overline{\xi} \circ \xi^{-1} : \xi(\partial\Gamma) \to \overline{\xi}(\partial\Gamma).$$

To avoid confusion we denote the simple roots of $\mathsf{SL}(\overline{d}, \overline{\mathbb{K}})$ by $\big\{\overline{\tau}_i : i \in [\![1, \overline{d}-1]\!]\big\}$, and to simplify notation we identify γ with $\rho(\gamma)$ and we let $\overline{\gamma} = \overline{\rho}(\gamma)$. We consider

also the graph of Ξ, or equivalently the *graph map*,

$$\mathscr{G} : \big(\xi, \overline{\xi}\big) : \partial\Gamma \to \mathbb{P}(\mathbb{K}^d) \times \mathbb{P}(\overline{\mathbb{K}}^d).$$

Definition 1.2. Fix $\mathscr{b} \in (0,1]$. We will say that that Ξ is *$\mathscr{b}$-concave* at $x \in \partial\Gamma$, or that x is a *$\mathscr{b}$-concavity point for* Ξ, if there exists a sequence (y_k) converging to x as $k \to \infty$ such that the incremental quotient

$$\frac{d_{\mathbb{P}}\big(\overline{\xi}(x), \overline{\xi}(y_k)\big)}{d_{\mathbb{P}}\big(\xi(x), \xi(y_k)\big)^{\mathscr{b}}} \tag{1.3}$$

is bounded away from $\{0, \infty\}$. The set of $\mathscr{b}$-concavity points is denoted by $\mathcal{H}^{\mathscr{b}}_{\rho,\overline{\rho}}$.

Observe that Ξ can be $\mathscr{b}$-concave at x for several $\mathscr{b}$'s and that it is a 1-concave point if one has $y_k \to x$ such that $d(\xi(x), \xi(y_k))$ and $d_{\mathbb{P}}(\overline{\xi}(x), \overline{\xi}(y_k))$ are comparable.

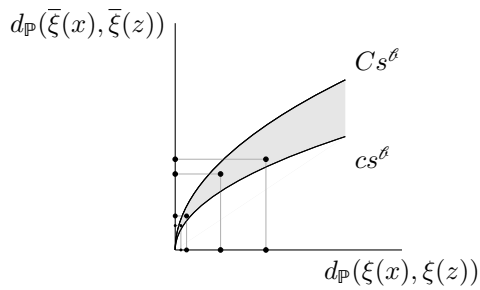

FIGURE 1. A $\mathscr{b}$-concave point x. The marked points on the axis' represent $d_{\mathbb{P}}(\xi(x), \xi(y_k))$ and $d_{\mathbb{P}}(\overline{\xi}(x), \overline{\xi}(y_k))$ respectively.

In what follows we will compute the Hausdorff dimension of $\mathscr{G}(\mathcal{H}^{\mathscr{b}}_{\rho,\overline{\rho}})$ with respect to the product metric on $\mathbb{P}(\mathbb{K}^d) \times \mathbb{P}(\overline{\mathbb{K}}^d)$ for $\mathscr{b}$ lying on an interval that we now define. The *dynamical intersection* between ρ and $\overline{\rho}$ with respect to $\overline{\tau}_1$ and τ_1 is defined by

$$\mathbf{I}_{\overline{\tau}_1}(\tau_1) = \lim_{t\to\infty} \frac{1}{\#\mathsf{R}_t(\overline{\tau}_1)} \sum_{\gamma \in \mathsf{R}_t(\overline{\tau}_1)} \frac{\tau_1(\lambda(\gamma))}{\overline{\tau}_1(\lambda(\overline{\gamma}))},$$

where $\mathsf{R}_t(\overline{\tau}_1) = \big\{[\gamma] \in [\Gamma] : \overline{\tau}_1\big(\lambda(\overline{\gamma})\big) \leqslant t\big\}$ and $\lambda : \mathsf{SL}(d, \mathbb{K}) \to \mathfrak{a}^+$ is the Jordan projection. This concept (from Bridgeman-Canary-Labourie-S. [11], Burger [13], Knieper [34], among others) generalizes Bonahon's intersection number between two elements in Teichmüller space.

Let us say that ρ and $\overline{\rho}$ are *gap-isospectral* if for all $\gamma \in \Gamma$ one has

$$\tau_1\big(\lambda(\gamma)\big) = \overline{\tau}_1\big(\lambda(\overline{\gamma})\big).$$

Corollary 6.6 (a consequence of [11] together with Proposition 6.3) implies that if ρ and $\overline{\rho}$ are not gap-isospectral, then $\mathbf{I}_{\tau_1}(\overline{\tau}_1) > \big(\mathbf{I}_{\overline{\tau}_1}(\tau_1)\big)^{-1}$. We will study $\mathscr{b}$-concavity for any $\mathscr{b} \in (0,1]$ with

$$\mathbf{I}_{\tau_1}(\overline{\tau}_1) > \mathscr{b} > \big(\mathbf{I}_{\overline{\tau}_1}(\tau_1)\big)^{-1}.$$

Finally, consider the critical exponents

$$\hbar^{\tau_1} = \lim_{t\to\infty} \frac{1}{t} \log \#\{\gamma \in \Gamma : \tau_1\big(a(\gamma)\big) \leqslant t\},$$
$$\hbar^{\infty,\ell} = \lim_{t\to\infty} \frac{1}{t} \log \#\{\gamma \in \Gamma : \max\big\{\ell\tau_1\big(a(\gamma)\big), \overline{\tau}_1\big(a(\overline{\gamma})\big)\big\} \leqslant t\}.$$

Theorem A (Theorem 6.1)**.** *Let $\{\mathbb{K}, \overline{\mathbb{K}}\} \subset \{\mathbb{R}, \mathbb{C}\}$ and let $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{K})$ and $\overline{\rho} : \Gamma \to \mathsf{SL}(\overline{d}, \overline{\mathbb{K}})$ be locally conformal, $\mathbb{R}$-irreducible and not gap-isospectral. Then for any $\ell \in (0, 1]$ with $\mathbf{I}_{\tau_1}(\overline{\tau}_1) > \ell > \big(\mathbf{I}_{\overline{\tau}_1}(\tau_1)\big)^{-1}$, one has*

$$\begin{aligned}\ell\hbar^{\infty,\ell} \leqslant \dim_{\mathrm{Hff}}(\mathscr{G}(\mathcal{H}^{\ell}_{\rho,\overline{\rho}})) &\leqslant \min\{\hbar^{\infty,\ell}, \ell\hbar^{\infty,\ell} + 1 - \ell\} \\ &< \min\{\hbar^{\overline{\tau}_1}, \hbar^{\tau_1}/\ell\} \\ &\leqslant \dim_{\mathrm{Hff}}(\mathscr{G}(\partial\Gamma)) \\ &= \max\{\hbar^{\tau_1}, \hbar^{\overline{\tau}_1}\}.\end{aligned}$$

If $\mathbb{K} = \mathbb{H}$ (resp. $\overline{\mathbb{K}} = \mathbb{H}$) we further assume that the Zariski closure if ρ (resp. $\overline{\rho}$) does not have compact factors, then the same conclusion holds.

The proof of the above Theorem is completed in § 6.3. For representations in $\mathsf{PSL}(2, \mathbb{C})$ we can furthermore give a geometric interpretation of the 1-weakly-bi-Hölder points, see § 8.4.

1.2. **Surface-group representations.** Observe that the first line of inequalities in Theorem A becomes an equality when $\ell = 1$. We pursue now this situation while further restricting the source and ambient groups.

Let then $\mathbb{K} = \mathbb{R}$ and assume $\partial\Gamma$ is homeomorphic to a circle. Real representations of Γ that are $(1, 1, 2)$-hyperconvex are necessarily locally conformal and form the prototype example of Anosov representations with C^1 limit sets: indeed we have the following result from P.-S.-Wienhard [43] and Zhang-Zimmer [54].

Theorem 1.3. *Assume $\partial\Gamma$ is homeomorphic to a circle and let $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{R})$ be $\{\tau_1\}$-Anosov.*

[43],[54]*: If ρ is $(1, 1, 2)$-hyperconvex, then $\xi^1(\partial\Gamma) \subset \mathbb{P}(\mathbb{R}^d)$ is a C^1 submanifold tangent at $\xi^1(x)$ to $\xi^2(x)$.*

[54]*: If ρ is irreducible and $\xi(\partial\Gamma)$ is a C^1 circle then ρ is $(1, 1, 2)$-hyperconvex.*

The graph map $\mathscr{G} = \big(\xi, \overline{\xi}\big) : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{\overline{d}})$ has image contained in the $\mathrm{C}^{1+\nu}$ torus $\xi(\partial\Gamma) \times \overline{\xi}(\partial\Gamma)$ and $\mathscr{G}(\partial\Gamma)$ is the graph of Ξ, a Hölder-continuous homeomorphism between $\mathrm{C}^{1+\nu}$-circles. By monotonicity of Ξ, $\mathscr{G}(\partial\Gamma)$ is a Lipschitz curve and is thus differentiable almost everywhere. We let

$$\mathsf{NDiff}_{\rho,\overline{\rho}} \subset \mathscr{G}(\partial\Gamma)$$

be the subset of points where the curve $\mathscr{G}(\partial\Gamma)$ is not differentiable. The combination of Lemma 6.2 and Corollary 8.1 establishes that in the current situation (with mild additional assumptions)

$$\mathscr{G}\big(\mathcal{H}^1_{\rho,\overline{\rho}}\big) = \mathsf{NDiff}_{(\rho,\overline{\rho})},$$

whence with Theorem A one obtains the following:

Theorem B. *Assume $\partial\Gamma$ is homeomorphic to a circle and let $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ and $\overline{\rho} : \Gamma \to \mathsf{SL}(\overline{d}, \mathbb{R})$ be $(1, 1, 2)$-hyperconvex and not gap-isospectral. Then,*

$$\dim_{\mathrm{Hff}}\big(\mathsf{NDiff}_{\rho,\overline{\rho}}\big) = \hbar^{\infty,1} < 1.$$

We emphasize that no irreducibility assumption is made on the representations ρ and $\overline{\rho}$. On the other hand, if the representations are irreducible and gap-isospectral, we show that there exists an isomorphism between the Zariski closures of $\rho(\Gamma)$ and of $\overline{\rho}(\Gamma)$ intertwining the two representations. It follows then that $\mathscr{G}(\partial\Gamma)$ is the diagonal of the $\mathrm{C}^{1+\nu}$ torus, and thus differentiable everywhere. To prove this we give the following preliminary classification of Zariski-closures, established in § 7.3.

Recall that if G is a semi-simple real-algebraic group of non-compact type, then irreducible proximal representations $\Phi : \mathsf{G} \to \mathsf{PGL}(V)$ are determined by their highest restricted weight χ_Φ. A special subset of dominant weights are the so-called *fundamental weights* $\{\varpi_{\mathsf{a}} : \mathsf{a} \in \Delta\}$, and are indexed by the set of simple roots Δ of G (see § 2.3 for definitions and details).

Theorem C. *Assume $\partial\Gamma$ is homeomorphic to a circle and let $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{R})$ be irreducible and $(1, 1, 2)$-hyperconvex. Then the Zariski closure G of $\rho(\Gamma)$ is simple and the highest weight of the induced representation $\Phi : \mathsf{G} \to \mathsf{PGL}(d, \mathbb{R})$ is a multiple of a fundamental weight associated to a root whose root-space is one-dimensional.*

In light of the following examples it is unclear if further restrictions can occur.

Example 1.4. Any pair of representations $\rho : \pi_1 S \to \mathsf{G}$ and $\Phi : \mathsf{G} \to \mathsf{PGL}(V)$ in each of the following classes (and small deformations), gives rise to a $(1, 1, 2)$-hyperconvex representation via post-composition $\Phi \circ \rho$. In particular the limit set of ρ in the specified flag manifold of G is a $\mathrm{C}^{1+\nu}$ curve:

- G is split, $\rho : \pi_1 S \to \mathsf{G}$ is Hitchin, and Φ satisfies $\chi_\Phi = n\varpi_{\mathsf{a}}$ for any $\mathsf{a} \in \Delta$ and $n \in \mathbb{N}_{>0}$. This is non-trivial and requires results from Fock-Goncharov [20] and Labourie [35] together with Lusztig's canonical basis [36, Proposition 3.2] (see S. [50, § 5.8] for details). As a result the limit set of ρ in any maximal flag manifold $\mathcal{F}_{\{\mathsf{a}\}}$ of G is a $\mathrm{C}^{1+\nu}$ curve.
- $\rho : \pi_1 S \to \mathsf{PO}(p, q)$ is Θ-positive and Φ has highest weight ϖ_{a} for any root a in the interior[1] of Θ (P.-S.-Wienhard [42, Theorem 10.3], see also Beyrer-P. [8, Remark 4.6]). In particular the limit set in any flag manifold of the form $\mathsf{Is}_k(\mathbb{R}^{p,q})$ for $k \leqslant p - 2$ is a $\mathrm{C}^{1+\nu}$-curve. When ρ is moreover Zariski-dense, we can consider any Φ with $\chi_\Phi^+ = n\varpi_{\mathsf{a}}$ for any $\mathsf{a} \in \operatorname{int}\Theta$ and $n \in \mathbb{N}_{>0}$.
- for all $k \geqslant 1$, k-positive representations $\rho : \pi_1 S \to \mathsf{PSL}(d, \mathbb{R})$ introduced in Beyrer-P. [7] are $(1, 1, 2)$-hyperconvex.

For these examples also the following applies:

Corollary B. *Assume $\partial\Gamma$ is homeomorphic to a circle, let G be a simple Lie group and let $\rho : \Gamma \to \mathsf{G}$ have Zariski-dense image. Assume there exist $\{\mathsf{a}, \mathsf{b}\} \subset \Delta$ distinct such that both $\Phi_{\mathsf{a}} \circ \rho$ and $\Phi_{\mathsf{b}} \circ \rho$ are $(1, 1, 2)$-hyperconvex. Then:*

(i) *The image of $\xi^{\{\mathsf{a},\mathsf{b}\}} : \partial\Gamma \to \mathcal{F}_{\{\mathsf{a},\mathsf{b}\}}$ is Lipschitz and the Hausdorff dimension of the points where it is non-differentiable is $\hbar^{\max\{\mathsf{a},\mathsf{b}\}}$.*

(ii) *If the opposition involution i on $\mathfrak{g}$ is non-trivial and $\mathsf{b} = \mathsf{ia}$ then*

$$\hbar^{\max\{\mathsf{a},\mathsf{b}\}} = \hbar^{(\mathsf{a}+\mathsf{b})/2}.$$

Remark 1.5. A different approach to Theorem B, relying on Theorem C and Theorem 1.3, would be to code the action of $\pi_1 S$ on $\partial\pi_1 S$ via Bowen-Series and apply Jordan-Kesseböhmer-Pollicott-Stratmann [29, Theorem 1.1]. This method, followed by Pollicott-Sharp [40] for two representations in the Teichmüller space of S,

[1] i.e. a is only connected to roots in Θ in the Dynkin diagram of Δ

is not applicable for groups other than $\pi_1 S$, in particular this approach cannot be used in the generality of Theorem A.

1.3. **Hyperplane vs directional conicality.** To prove Theorems A and B we introduce the concept of *hyperplane conicality*, a generalization of directional conicality from Burger-Landesberg-Lee-Oh [14].

Let G be a real-algebraic semi-simple Lie group of non-compact type, $\mathfrak{a} \subset \mathfrak{g}$ a Cartan subspace, $\Phi \subset \mathfrak{a}^*$ the associated root system and $\Delta \subset \Phi$ a choice of simple roots with associated Weyl chamber $\mathfrak{a}^+$.

Consider a non-empty $\theta \subset \Delta$ and let $\mathfrak{a}_\theta$ be the associated Levi space. Fix a θ-Anosov representation $\rho : \Gamma \to \mathsf{G}$ and denote by $\mathcal{L}_{\theta,\rho} \subset \mathfrak{a}_\theta$ its θ-limit cone. We will recall in § 4.3 that, when $\rho(\Gamma)$ is Zariski-dense, there are natural bijections

$$\begin{aligned} \operatorname{int} \mathbb{P}(\mathcal{L}_{\theta,\rho}) &\leftrightarrow \mathcal{Q}_{\theta,\rho} = \{\varphi \in (\mathfrak{a}_\theta)^* : \hbar_\varphi = 1\} \\ &\leftrightarrow \big\{\text{Patterson-Sullivan measures supported on } \xi^\theta(\partial\Gamma)\big\}. \end{aligned}$$

For $\varphi \in \mathcal{Q}_{\theta,\rho}$ we let $\mathsf{u}_\varphi \in \operatorname{int} \mathbb{P}(\mathcal{L}_{\theta,\rho})$ be the associated direction and μ^φ the associated Patterson-Sullivan measure.

Consider now a hyperplane $\mathsf{W} \subset \mathfrak{a}_\theta$ and assume, for the notion to be interesting, that W intersects the relative interior of $\mathcal{L}_{\theta,\rho}$. Then $x \in \partial\Gamma$ is W-*conical* if there exists a conical sequence $(\gamma_n)_0^\infty \subset \Gamma$ converging to x, a constant K and a sequence $(w_n)_0^\infty \in \mathsf{W}$ such that for all n one has

$$\big\|a_\theta\big(\rho(\gamma_n)\big) - w_n\big\| \leqslant K,$$

where $a_\theta : \mathsf{G} \to \mathfrak{a}_\theta^+$ is the θ-Cartan projection. The set of W-conical points will be denoted by $\partial_{\mathsf{W},\rho}\Gamma = \partial_{\mathsf{W}}\Gamma$. Inspired by [14], in Theorem 4.16 we show the following.

Theorem D. *Let $\rho : \Gamma \to \mathsf{G}$ be a Zariski-dense θ-Anosov representation and W be a hyperplane of $\mathfrak{a}_\theta$ intersecting non-trivially the interior of $\mathcal{L}_{\theta,\rho}$. Then for every $\varphi \in \mathcal{Q}_{\theta,\rho}$ with $\mathsf{u}_\varphi \in \mathbb{P}(\mathsf{W})$ one has $\mu^\varphi(\partial_{\mathsf{W}}\Gamma) = 1$.*

1.4. **Strategy of the proof of Corollary A.** Corollary A is a consequence of Theorem B where $\overline{\rho}$ is the dual representation of ρ. We sketch a direct proof of Corollary A serving as a guide-path for the general result.

Let $\rho : \pi_1 S \to \mathsf{SL}(3,\mathbb{R})$ be the holonomy of a strictly convex projective structure dividing the convex set Ω. We consider the L^∞ distance on the product $(\mathbb{P}(\mathbb{R}^3), d_{\mathbb{P}}) \times (\mathbb{P}((\mathbb{R}^3)^*), d_{\mathbb{P}})$, which is equivalent to the Riemannian distance, and thus induces the same Hausdorff dimension.

As a replacement of Sullivan's shadows we use *coarse cone type at infinity*, inspired by Cannon's work on *cone types* [15] (see also §4.1). Fix a finite symmetric generating set on $\pi_1 S$ and let $|\,|$ be the associated word length. For $\gamma \in \pi_1 S$ and $c > 0$, the *coarse cone type at infinity* $\mathcal{C}_\infty^c(\gamma)$ of γ is the set of endpoints at infinity of (c,c)-quasi geodesic rays based at γ^{-1} passing through the identity. See Figure 2.

We let $\xi : \partial\pi_1 S \to \partial\Omega$ be the natural identification via the action of $\rho(\pi_1 S)$ on Ω, and analogously $\overline{\xi} : \pi_1 S \to \partial\Omega^*$. We denote by $\mathscr{G} := (\xi, \overline{\xi}) : \pi_1 S \to \partial\Omega \times \partial\Omega^*$ the flag curve. Consider $x \in \partial\pi_1 S$ and let $\alpha_i \to x$ be a geodesic ray on $\pi_1 S$. The following fact is a consequence of Proposition 5.6.

Fact. *For big enough i, the subset $\xi\big(\alpha_i \mathcal{C}_\infty^c(\alpha_i)\big) \subset \partial\Omega$ is coarsely the intersection of a ball of radius $e^{-\tau_1(\alpha_i)}$ about $\xi(x)$ with $\partial\Omega$. By duality, one has $\overline{\xi}\big(\alpha_i \mathcal{C}_\infty^c(\alpha_i)\big) \subset \partial\Omega^*$ is coarsely the intersection of a ball of radius $e^{-\tau_2(\alpha_i)}$ about $\overline{\xi}(x)$ with $\partial\Omega^*$.*

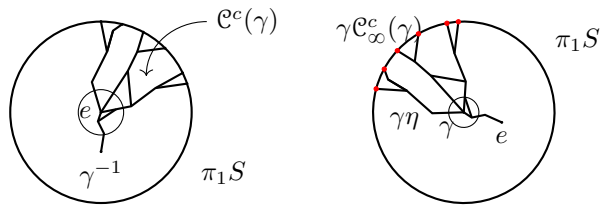

FIGURE 2. The coarse cone type of $\gamma \in \Gamma$ (left). The set $\gamma \cdot \mathcal{C}^c_\infty(\gamma)$ (right). Pictures from P.-S.-Wienhard [43].

The coarse constants and the minimal length i required in the above statement depend only on the representation and not on the point x.

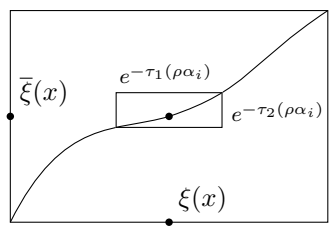

FIGURE 3. The image of the cone type $\alpha_i\mathcal{C}^c_\infty(\alpha_i)$ by the graph curve $\mathscr{G}$ in the $\mathrm{C}^{1+\nu}$-torus $\partial\Omega \times \partial\Omega^*$.

For any point $x \in \partial\pi_1 S$ we distinguish two complementary situations that don't depend on the choice of the geodesic ray $(\alpha_i)_{i\in\mathbb{N}}$ converging to x:

i) For all $R > 0$ there exists $N \in \mathbb{N}$ with $|\tau_1(a(\alpha_i)) - \tau_2(a(\alpha_i))| \geqslant R$ for all $i \geqslant N$;
ii) There exists $R > 0$ and an infinite set of indices $\mathbb{I} \subset \mathbb{N}$ such that for all $k \in \mathbb{I}$ one has $|\tau_1(a(\alpha_k)) - \tau_2(a(\alpha_k))| \leqslant R$. We say in this case that x is b-*conical* (b stands for 'barycenter of the chamber').

In the first case one is easily convinced by looking at Figure 3 that the rectangle becomes flatter along one of its sides (see § 8 for details in the general case). Furthermore, since $\tau_1(a(\alpha_i)) - \tau_1(a(\alpha_{i+1}))$ is uniformly bounded, its sign is eventually constant, and thus the longer side only depends on the point. As a result x is necessarily a differentiability point of the graph curve $\mathscr{G}$, with either horizontal or vertical derivative.

We are thus bound to understand the set of b-conical points. We show (see Corollary 8.1):

Fact. *The non-differentiabilty points of the curve $\mathscr{G}(\partial\pi_1 S)$ and the b-conical points coincide.*

The main idea for this is to show that if a b-conical point x were a differentiability point, then the derivative could not be horizontal nor vertical, and thus (by Proposition 7.2) Ξ would be bi-Lipschitz. In turn, this would force the periods of the two roots to agree, which in turn would imply that the representation is Fuchsian, contradicting the assumption that Ω is not an ellipse.

It remains to understand the Hausdorff dimension of the set of b-conical points. The upper bound (Proposition 5.11)

$$\dim_{\mathrm{Hff}}\big(\{\mathsf{b}-\text{conical}\}\big) \leqslant \hbar^{\max\{\tau_1,\tau_2\}} \tag{1.4}$$

follows readily: since for a b-conical point the lengths $e^{-\tau_1(\alpha_k)}$ and $e^{-\tau_2(\alpha_k)}$ are comparable independently on $k \in \mathbb{I}$, one can replace the rectangle in Figure 3 by the (smaller) square of length

$$e^{-\max\{\tau_1(a(\alpha_k)),\tau_2(a(\alpha_k))\}}$$

and still get a covering[2] (this time by balls on the L^∞ metric) of the set $\{\mathsf{b}-\text{conical}\}$. Standard arguments on Hausdorff dimension give Equation (1.4).

Finding a lower bound for the Hausdorff dimension is more subtle; we use here an appropriate Patterson-Sullivan measure to study how the mass of a ball of radius r scales with r.

Since $\mathscr{G}(\partial\pi_1 S)$ is a subset the full flag space $\mathcal{F}(\mathbb{R}^3)$ and

$$\|v\|_\infty := \max\{|\tau_1(v)|, |\tau_2(v)|\}$$

is a norm on $\mathfrak{a}_{\mathsf{PSL}(3,\mathbb{R})}$, we can apply results by Quint [44] to determine a linear form $\varphi_{\mathsf{b}}^\infty \in \mathfrak{a}^*$ whose associated growth direction is the barycenter $\mathscr{b} = \ker(\tau_1 - \tau_2)$. By Quint [44, Proposition 3.3.3]

$$\hbar^{\max\{\tau_1,\tau_2\}} = \|\varphi_{\mathsf{b}}^\infty\|^1,$$

where $\|\ \|^1$ is the operator norm on $\mathfrak{a}^*$ defined by $\|\ \|_\infty$, which turns out to be the L^1 norm $\|a\tau_1 + b\tau_2\|^1 = |a| + |b|$. The form $\varphi_{\mathsf{b}}^\infty$ additionally admits an associated *Patterson-Sullivan* probability measure, namely a measure μ^∞ such that for all $\gamma \in \pi_1 S$ one has (see Corollary 4.14)

$$\mu^\infty\big(\mathscr{G}(\gamma\mathcal{C}_\infty^c(\gamma))\big) \leqslant Ce^{-\varphi_{\mathsf{b}}^\infty(a(\gamma))}. \tag{1.5}$$

A key extra information available in the case of $\mathsf{PSL}(3,\mathbb{R})$ is that the form $\varphi_{\mathsf{b}}^\infty$ is explicit and doesn't depend on ρ. For this we need a small parenthesis on the *critical hypersurface* $\mathcal{Q}_\rho$ of ρ, depicted in Figure 4, and characterized by

$$\mathcal{Q}_\rho = \big\{\varphi \in \mathfrak{a}^* : \hbar^\varphi = 1\big\},$$

where the *critical exponent* of a functional $\varphi \in \mathfrak{a}^*$ is

$$\hbar^\varphi := \lim_{t\to\infty} \frac{1}{t} \log \#\big\{\gamma \in \pi_1 S : \varphi(a(\gamma)) \leqslant t\big\} \in (0,\infty].$$

The critical hypersurface $\mathcal{Q}_\rho \subset \mathfrak{a}^*$ is a closed analytic curve that bounds a strictly convex set (S. [46] and Potrie-S. [41]), and thus by Quint [44], the linear form $\varphi_{\mathsf{b}}^\infty$ is uniquely determined by

$$\|\varphi_{\mathsf{b}}^\infty\|^1 = \inf\big\{\|\varphi\|^1 : \varphi \in \mathcal{Q}_\rho\big\}. \tag{1.6}$$

Again by [41] one has $\{\tau_1,\tau_2\} \subset \mathcal{Q}_\rho$. Since both $\mathcal{Q}_\rho$ and the norm $\|\ \|^1$ are invariant by the opposition involution i (see again Figure 4) we deduce that, if we let $\mathsf{H} = (\tau_1+\tau_2)/2$, then

$$\varphi_{\mathsf{b}}^\infty = \hbar^{\mathsf{H}} \cdot \mathsf{H} \geqslant \hbar^{\mathsf{H}} \min\{\tau_1,\tau_2\}. \tag{1.7}$$

[2]Choosing the longer side $e^{-\min\{\tau_1(a(\alpha_k)),\tau_2(a(\alpha_k))\}}$ gives the bound $\dim_{\mathrm{Hff}}\mathscr{G}(\partial\pi_1 S) \leqslant 1$.

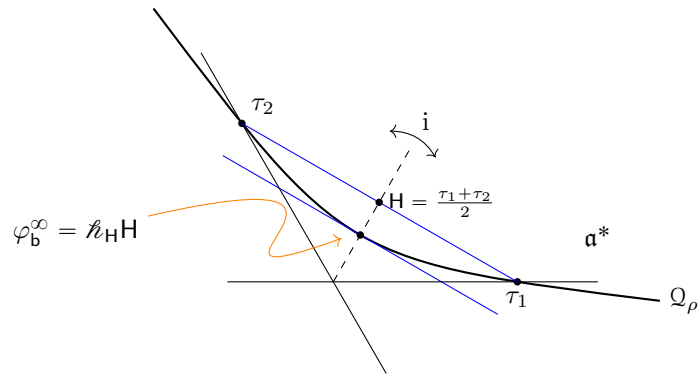

FIGURE 4. The critical hypersurface of a strictly convex projective structure on S. Since H is a convex combination of $\{\tau_1, \tau_2\}$ one has $\|\mathsf{H}\|^1 = 1$ and thus $\|\varphi_{\mathsf{b}}^{\infty}\|^1 = \hbar^{\mathsf{H}}$.

In particular, using Equation (1.6), we obtain that $\hbar^{\max\{\tau_1,\tau_2\}} = \hbar^{\mathsf{H}}$. Moreover, since the geodesic flow is Anosov (by Benoist [5]) we can apply Bowen's characterization of entropy [10] (and Remark 4.13), to obtain that the Hilbert entropy $\hbar_{\mathsf{H}} = \hbar^{\mathsf{H}}$.

After this small parenthesis on the critical hypersurface, we come back to the lower bound on the Hausdorff dimension. Since $\mathscr{G}$ is a graph, $\mathscr{G}(\partial\pi_1 S)$ has the same intersection with the rectangle in Figure 3 than with the larger square of size

$$e^{-\min\{\tau_1(a(\alpha_i)),\tau_2(a(\alpha_i))\}};$$

this square is now a ball (for the L^∞ metric) of radius $e^{-\min\{\tau_1(a(\alpha_i)),\tau_2(a(\alpha_i))\}}$. Thus for all i, $\mathscr{G}\big(\alpha_i\mathcal{C}^c_\infty(\alpha_i)\big)$ is coarsely a ball of the latter radius and one has

$$\begin{aligned}\mu^\infty\big(B(\mathscr{G}(x), e^{-\min\{\tau_1(a(\alpha_i)),\tau_2(a(\alpha_i))\}}\big) &\leqslant \mu^\infty\big(\mathscr{G}(\alpha_i\mathcal{C}^c_\infty(\alpha_i)\big) \leqslant Ce^{-\varphi_{\mathsf{b}}^\infty(a(\alpha_i))}\\ &\leqslant C\big(e^{-\min\{\tau_1(a(\alpha_i)),\tau_2(a(\alpha_i))\}}\big)^{\hbar^{\mathsf{H}}},\end{aligned}$$

where the last inequalities follow from Equations (1.5) and (1.7). This gives a possibly bigger constant C' such that, for all r,

$$\mu^\infty\big(B(\mathscr{G}(x), r)\big) \leqslant C' r^{\hbar^{\mathsf{H}}}.$$

Again, classical Hausdorff dimension arguments (c.f. Corollary 5.8 below) give that, for any measurable subset $E \subset \mathscr{G}(\partial\pi_1 S)$ with full μ^∞ mass, one has $\dim_{\mathrm{Hff}}(E) \geqslant \hbar^{\mathsf{H}}$.

Since $\mathsf{PSL}(3,\mathbb{R})$ has rank smaller than 3 and ρ is Δ-Anosov we can apply Burger-Landesberg-Lee-Oh [14, Theorem 1.6] to obtain that $\mu^\infty(\{\mathsf{b}\text{-conical}\}) = 1$ and thus we have the desired lower bound

$$\dim_{\mathrm{Hff}}\big(\{\mathsf{b}\text{-conical}\}\big) \geqslant \hbar^{\mathsf{H}},$$

which combined with the upper bound (1.4) and the equality $\hbar^{\max\{\tau_1,\tau_2\}} = \hbar^{\mathsf{H}}$, gives the proof of Corollary A.

In the general case [14, Theorem 1.6] is not applicable and we replace it with Theorem D. □

Structure of the paper. The preliminaries of the paper are standard facts about linear algebraic groups, recalled in §2, the work of S. [49] about linear cocycles over the boundary of a hyperbolic group (in §3), as well as basic facts about Anosov representations and their Patterson-Sullivan theory recalled from [25, 9, 43, 49] in the first part of §4. In the rest of §4 we prove Theorem 4.16 a more precise statement than Theorem D, discussing the Patterson-Sullivan measure of (W, φ)-conical points. The heart of the proof is to construct and study a rank 2 flow whose recurrence set is related to (W, φ)-conical points.

In §5 we consider two locally conformal representations. We prove Theorem 5.3, stating that for such a pair the Hausdorff dimension of the set of $\mathscr{b}$-conical points belongs to

$$[\mathscr{b}\hbar^{\infty,\mathscr{b}}, \min\{\hbar^{\infty,\mathscr{b}}, \mathscr{b}\hbar^{\infty,\mathscr{b}} + 1 - \mathscr{b}\}].$$

The lower bound is obtained by analyzing properties of the linear form $\varphi^{\infty}_{\mathscr{b}}$ whose associated growth direction is $(\mathscr{b}, 1)$; its Patterson-Sullivan measure $\mu^{\varphi^{\infty}_{\mathscr{b}}}$ gives full mass to the set of $\mathscr{b}$-conical points thanks to Theorem 4.16. Using cone-types we can show that for a fine set of balls $\mu^{\varphi^{\infty}_{\mathscr{b}}}(B(x,r)) \leqslant Cr^{-\mathscr{b}\hbar^{\infty,\mathscr{b}}}$. The upper bound uses results of [43] to construct a fine covering of the set of $\mathscr{b}$-conical points with balls of radius $e^{-\max\{\mathscr{b}\tau, \overline{\tau}\}}$. In §6 we prove Theorem A.

In §7 we prove that if the graph map between $\mathbb{R}$-hyperconvex representations has an oblique derivative, then the map is bi-Lipschitz (Proposition 7.2). This only relies on basic properties of hyperconvex representations, and is crucial for the proof of Theorem B, achieved in §8, as it allows the identification of b-conical points and points of non-differentiability.

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2. Linear algebraic groups

Throughout the text G will denote a real-algebraic semi-simple Lie group of non-compact type and $\mathfrak{g}$ its Lie algebra.

2.1. **Linear algebraic groups.** Fix a Cartan involution $o : \mathfrak{g} \to \mathfrak{g}$ with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and let $\Phi \subset \mathfrak{a}^*$ be the set of restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$. For $\mathsf{a} \in \Phi$, we denote by

$$\mathfrak{g}_{\mathsf{a}} = \{u \in \mathfrak{g} : [a, u] = \mathsf{a}(a)u \ \forall a \in \mathfrak{a}\}$$

its associated root space. The (restricted) root space decomposition is $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\mathsf{a}\in\Phi} \mathfrak{g}_{\mathsf{a}}$, where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$. Fix a Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$ and let Φ^+ and Δ be, respectively, the associated sets of positive and simple roots. Let $\mathcal{W}$ be the Weyl group of Φ and $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ be the opposition involution: if $u : \mathfrak{a} \to \mathfrak{a}$ is the unique element in $\mathcal{W}$ with $u(\mathfrak{a}^+) = -\mathfrak{a}^+$ then $\mathrm{i} = -u$.

We denote by $(\cdot, \cdot)$ both the Killing form of $\mathfrak{g}$, its restriction to $\mathfrak{a}$, and its associated dual form on $\mathfrak{a}^*$, the dual of $\mathfrak{a}$. For $\chi, \psi \in \mathfrak{a}^*$ let

$$\langle \chi, \psi \rangle = 2\frac{(\chi, \psi)}{(\psi, \psi)}.$$

The *restricted weight lattice* is defined by

$$\Pi = \{\varphi \in \mathfrak{a}^* : \langle \varphi, \mathsf{a} \rangle \in \mathbb{Z} \ \forall \mathsf{a} \in \Phi\}.$$

It is spanned by the *fundamental weights* $\{\varpi_{\mathsf{a}} : \mathsf{a} \in \Delta\}$, defined by

$$\langle \varpi_{\mathsf{a}}, \mathsf{b} \rangle = d_{\mathsf{a}} \delta_{\mathsf{ab}} \tag{2.1}$$

for every $\mathsf{a}, \mathsf{b} \in \Delta$, where $d_{\mathsf{a}} = 1$ if $2\mathsf{a} \notin \Phi^+$ and $d_{\mathsf{a}} = 2$ otherwise.

A subset $\theta \subset \Delta$ determines a pair of opposite parabolic subgroups P_θ and $\check{\mathsf{P}}_\theta$ whose Lie algebras are

$$\mathfrak{p}_\theta = \bigoplus_{\mathsf{a} \in \Phi^+ \cup \{0\}} \mathfrak{g}_{\mathsf{a}} \oplus \bigoplus_{\mathsf{a} \in \langle \Delta - \theta \rangle} \mathfrak{g}_{-\mathsf{a}},$$
$$\check{\mathfrak{p}}_\theta = \bigoplus_{\mathsf{a} \in \Phi^+ \cup \{0\}} \mathfrak{g}_{-\mathsf{a}} \oplus \bigoplus_{\mathsf{a} \in \langle \Delta - \theta \rangle} \mathfrak{g}_{\mathsf{a}}.$$

The group $\check{\mathsf{P}}_\theta$ is conjugated to the parabolic group $\mathsf{P}_{\mathrm{i}\theta}$. We denote the *flag space* associated to θ by $\mathcal{F}_\theta = \mathsf{G}/\mathsf{P}_\theta$. The G orbit of the pair $([\mathsf{P}_\theta], [\check{\mathsf{P}}_\theta])$ is the unique open orbit for the action of G in the product $\mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}\theta}$ and is denoted by $\mathcal{F}_\theta^{(2)}$.

2.2. **Cartan and Jordan projection.** Denote by $\mathsf{K} = \exp \mathfrak{k}$ and $\mathsf{A} = \exp \mathfrak{a}$. The *Cartan decomposition* asserts the existence of a continuous map $a : \mathsf{G} \to \mathfrak{a}^+$, called the *Cartan projection*, such that every $g \in \mathsf{G}$ can be written as $g = ke^{a(g)}l$ for some $k, l \in \mathsf{K}$.

We will need the following uniform continuity of the Cartan projection:

Proposition 2.1 (Benoist [2, Proposition 5.1])**.** *For any compact $L \subset \mathsf{G}$ there exists a compact set $H \subset \mathfrak{a}$ such that, for every $g \in \mathsf{G}$, one has*

$$a(LgL) \subset a(g) + H.$$

By the Jordan's decomposition, every element $g \in \mathsf{G}$ can be uniquely written as a commuting product $g = g_e g_{ss} g_u$ where g_e is conjugate to an element in K, g_{ss} is conjugate to an element in $\exp(\mathfrak{a}^+)$ and g_u is unipotent. The *Jordan projection* $\lambda = \lambda_{\mathsf{G}} : \mathsf{G} \to \mathfrak{a}^+$ is the unique map such that g_{ss} is conjugated to $\exp\big(\lambda(g)\big)$.

Definition 2.2. Let $\Gamma \subset \mathsf{G}$ be a discrete subgroup, then its *limit cone* $\mathcal{L}_\Gamma$ is the smallest closed cone of the closed Weyl chamber $\mathfrak{a}^+$ that contains $\{\lambda(g) : g \in \Gamma\}$.

We will need the following results by Benoist.

Theorem 2.3 (Benoist [3, 4])**.** *Let $\Gamma \subset \mathsf{G}$ be a Zariski-dense sub-semigroup, then its limit cone $\mathcal{L}_\Gamma$ has non-empty interior. Moreover, the group generated by the Jordan projections $\{\lambda(g) : g \in \Gamma\}$ is dense in $\mathfrak{a}$.*

2.3. **Representations of G.** The standard references for the following are Fulton-Harris [21], Humphreys [27] and Tits [51].

Let $\Phi : \mathsf{G} \to \mathsf{PGL}(V)$ be a finite dimensional rational[3] irreducible representation and denote by $\phi_\Phi : \mathfrak{g} \to \mathfrak{sl}(V)$ the Lie algebra homomorphism associated to Φ. The *weight space* associated to $\chi \in \mathfrak{a}^*$ is the vector space

$$V_\chi = \{v \in V : \phi_\Phi(a)v = \chi(a)v \ \forall a \in \mathsf{A}\}.$$

We say that $\chi \in \mathfrak{a}^*$ is a *restricted weight* of Φ if $V_\chi \neq 0$. Tits [51, Theorem 7.2] states that the set of weights has a unique maximal element with respect to the partial order $\chi \succ \psi$ if $\chi - \psi$ is a $\mathbb{N}$-linear combination of positive roots. This is

[3]Namely a rational map between algebraic varieties.

called *the highest weight* of Φ and denoted by χ_Φ. By definition, for every $g \in \mathsf{G}$ one has

$$\lambda_1\big(\Phi(g)\big) = \chi_\Phi(\lambda(g)), \tag{2.2}$$

where λ_1 is the logarithm of the spectral radius of $\Phi(g)$.

We denote by $\Pi(\phi)$ the set of restricted weights of the representation ϕ_Φ

$$\Pi(\phi) = \big\{\chi \in \mathfrak{a}^* : V_\chi \neq \{0\}\big\},$$

these are all bounded above by χ_Φ (see for example Humphreys [27, §13.4 Lemma B]), namely every weight $\chi \in \Pi(\phi)$ has the form

$$\chi_\Phi - \sum_{\mathsf{a}\in\Delta} n_{\mathsf{a}}\mathsf{a} \text{ for } n_{\mathsf{a}} \in \mathbb{N}.$$

The *level* of a weight χ is the integer $\sum_{\mathsf{a}} n_{\mathsf{a}}$, the highest weight is thus the only weight of level zero. Additionally, if $\chi \in \Pi(\phi_\Phi)$ and $\mathsf{a} \in \Phi^+$ then the elements of $\Pi(\phi_\Phi)$ of the form $\chi + j\mathsf{a}$, $j \in \mathbb{Z}$ form an unbroken string

$$\chi + j\mathsf{a},\ j \in [\![-r, q]\!]$$

and $r - q = \langle \chi, \mathsf{a}\rangle$. One can then recover algorithmically the set $\Pi(\phi_\Phi)$ level by level starting from χ_Φ, as follows:

- Assume the set of weights of level at most k is known and consider a weight χ of level k.
- For each $\mathsf{a} \in \Delta$ compute $\langle \chi, \mathsf{a}\rangle$, this gives the length $r - q$ of the a-string through χ. The weights of the form $\chi + j\mathsf{a}$, for positive j, have level smaller than k and are thus known, thus we can decide whether $\chi - \mathsf{a}$ is a weight or not, determining the set of weights of level $k + 1$.

The following lemma follows at once from the algorithmic description above. Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be the decomposition in simple factors of a semi-simple real Lie algebra of non-compact type. Recall that if $\mathfrak{a}_i \subset \mathfrak{g}_i$ is a Cartan subspace, then $\mathfrak{a} = \bigoplus_i \mathfrak{a}_i$ is a Cartan subspace of $\mathfrak{g}$. Any $\varphi \in (\mathfrak{a}_i)^*$ extends to a functional on $\mathfrak{a}$, still denoted φ, by vanishing on the remaining factors. The restricted root system of $\mathfrak{g}$ is then $\Delta_{\mathfrak{g}} = \bigcup \Delta_{\mathfrak{g}_i}$. The *associated simple factor* to $\mathsf{a} \in \Delta_{\mathfrak{g}}$ is $\mathfrak{g}_i$ such that $\mathsf{a} \in \Delta_i$.

Lemma 2.4. *Let $\mathfrak{g}$ be a semi-simple real Lie algebra of non-compact type and ϕ be an irreducible representation of $\mathfrak{g}$ whose highest restricted weight is a multiple of a fundamental weight, $\chi_\phi = k\varpi_{\mathsf{a}}$ for some $\mathsf{a} \in \Delta$. Then ϕ factors as a representation of the simple factor associated to a.*

Proof. Proceeding by induction on the levels of ϕ, one readily sees that for every $\tau \in \Delta_j$ for $j \neq i$ and all $\chi \in \Pi(\phi)$ one has $\langle \chi, \tau\rangle = 0$. Thus the associated root space $(\mathfrak{g}_j)_{-\tau}$ acts trivially on every weight space of ϕ and so the whole factor $\mathfrak{g}_j$ acts trivially. □

The following set of simple roots plays a special role in representation theory.

Definition 2.5. Let $\Phi : \mathsf{G} \to \mathsf{PGL}(V)$ be a representation. We denote by θ_Φ the set of simple roots $\mathsf{a} \in \Delta$ such that $\chi_\Phi - \mathsf{a}$ is still a weight of Φ. Equivalently

$$\theta_\Phi = \big\{\mathsf{a} \in \Delta : \langle \chi_\Phi, \mathsf{a}\rangle \neq 0\big\}. \tag{2.3}$$

The following lemma will be needed in the proof of Theorem C.

Lemma 2.6. *Let $\mathfrak{g}$ be semi-simple of non-compact type and $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ an irreducible representation. For $\mathsf{a} \in \theta_\phi$ and $v \in V_{\chi_\phi} - \{0\}$, the map $n \mapsto \phi(n)v$ is injective when defined on $\mathfrak{g}_{-\mathsf{a}}$.*

Proof. By definition of χ_ϕ every $n \in \mathfrak{g}_{\mathsf{a}}$ acts trivially on V_{χ_ϕ}. For $y \in \mathfrak{g}_{-\mathsf{a}} - \{0\}$, there exists $x \in \mathfrak{g}_{\mathsf{a}}$ such that $\{x, y, h_{\mathsf{a}}\}$ spans a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, where h_{a} is defined by $\varphi(h_{\mathsf{a}}) = \langle \varphi, \mathsf{a} \rangle$ for all $\varphi \in \mathfrak{a}^*$. If $\phi(y)v = 0$ then, since $\phi(x)V_{\chi_\phi} = 0$ one concludes $\phi(h_{\mathsf{a}})v = 0$ and since V_{χ_ϕ} is a weight-space one has $\phi(h_{\mathsf{a}})V_{\chi_\phi} = 0$. This in turn implies that

$$\langle \chi_\phi, \mathsf{a} \rangle = \chi_\phi(h_{\mathsf{a}}) = 0,$$

contradicting that $\mathsf{a} \in \theta_\phi$. □

We denote by $\| \, \|_\Phi$ an Euclidean norm on V invariant under $\Phi\mathsf{K}$ and such that $\Phi\mathsf{A}$ is self-adjoint, see for example Benoist-Quint's book [6, Lemma 6.33]. By definition of χ_Φ and $\| \, \|_\Phi$, and Equation (2.2) one has, for every $g \in \mathsf{G}$, that

$$\log \|\Phi g\|_\Phi = \chi_\Phi(a(g)). \tag{2.4}$$

Here, with a slight abuse of notation, we denote by $\| \, \|_\Phi$ also the induced operator norm, which doesn't depend on the scale of $\| \, \|_\Phi$.

Denote by W_{χ_Φ} the $\Phi\mathsf{A}$-invariant complement of V_{χ_Φ}. The stabilizer in G of W_{χ_Φ} is $\check{\mathsf{P}}_{\theta_\Phi}$, and thus one has a map of flag spaces

$$(\zeta_\Phi, \zeta_\Phi^*) : \mathcal{F}^{(2)}_{\theta_\Phi}(\mathsf{G}) \to \mathrm{Gr}^{(2)}_{\dim V_{\chi_\Phi}}(V). \tag{2.5}$$

This is a proper embedding which is an homeomorphism onto its image. Here, as above, $\mathrm{Gr}^{(2)}_{\dim V_{\chi_\Phi}}(V)$ is the open $\mathsf{PGL}(V)$-orbit in the product of the Grassmannian of $(\dim V_{\chi_\Phi})$-dimensional subspaces and the Grassmannian of $(\dim V - \dim V_{\chi_\Phi})$-dimensional subspaces. One has the following proposition (see also Humphreys [28, Chapter XI]).

Proposition 2.7 (Tits [51])**.** *For each $\mathsf{a} \in \Delta$ there exists a finite dimensional rational irreducible representation $\Phi_{\mathsf{a}} : \mathsf{G} \to \mathsf{PSL}(V_{\mathsf{a}})$, such that $\chi_{\Phi_{\mathsf{a}}}$ is an integer multiple $l_{\mathsf{a}}\varpi_{\mathsf{a}}$ of the fundamental weight and $\dim V_{\chi_{\Phi_{\mathsf{a}}}} = 1$.*

We will fix from now on such a set of representations and call them, for each $\mathsf{a} \in \Delta$, the *Tits representation associated to* a.

2.4. **The center of the Levi group $\mathsf{P}_\theta \cap \check{\mathsf{P}}_\theta$.** We now consider the vector subspace

$$\mathfrak{a}_\theta = \bigcap_{\mathsf{a} \in \Delta - \theta} \ker \mathsf{a}.$$

Denoting by $\mathcal{W}_\theta = \{w \in \mathcal{W} : w(v) = v \quad \forall v \in \mathfrak{a}_\theta\}$ the subgroup of the Weyl group generated by reflections associated to roots in $\Delta - \theta$, there is a unique projection $\pi_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ invariant under $\mathcal{W}_\theta$.

The dual $(\mathfrak{a}_\theta)^*$ is canonically identified with the subspace of $\mathfrak{a}^*$ of π_θ-invariant linear forms. Such space is spanned by the fundamental weights of roots in θ,

$$(\mathfrak{a}_\theta)^* = \{\varphi \in \mathfrak{a}^* : \varphi \circ \pi_\theta = \varphi\} = \langle \varpi_{\mathsf{a}} | \mathfrak{a}_\theta : \mathsf{a} \in \theta \rangle.$$

We will denote, respectively, by

$$a_\theta = \pi_\theta \circ a : \mathsf{G} \to \mathfrak{a}_\theta$$
$$\lambda_\theta = \pi_\theta \circ \lambda : \mathsf{G} \to \mathfrak{a}_\theta,$$

the compositions of the Cartan and Jordan projections with π_θ.

2.5. **The Buseman-Iwasawa cocycle.** The *Iwasawa decomposition* of G states that every $g \in \mathsf{G}$ can be written uniquely as a product lzu with $l \in \mathsf{K}$, $z \in \mathsf{A}$ and $u \in \mathsf{U}_\Delta$, where U_Δ is the unipotent radical of P_Δ.

The *Buseman-Iwasawa cocycle* of G is the map $\mathrm{b} : \mathsf{G} \times \mathcal{F} \to \mathfrak{a}$ such that, for all $g \in \mathsf{G}$ and $k[\mathsf{P}_\Delta] \in \mathcal{F}$,

$$\mathrm{b}(g, k[\mathsf{P}_\Delta]) = \log(z)$$

where $\log : \mathsf{A} \to \mathfrak{a}$ denotes the inverse of the exponential map, and $gk = lzu$ is the Iwasawa decomposition of gk. Quint [45, Lemmes 6.1 and 6.2] proved that the function $\mathrm{b}_\theta = \pi_\theta \circ \mathrm{b}$ factors as a cocycle $\mathrm{b}_\theta : \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta$.

The Buseman-Iwasawa cocycle can also be read from the representations of G. Indeed, Quint [45, Lemme 6.4] shows that for every $g \in \mathsf{G}$ and $x \in \mathcal{F}_\theta$ one has

$$l_{\mathsf{a}} \varpi_{\mathsf{a}}(\mathrm{b}(g, x)) = \log \frac{\|\Phi_{\mathsf{a}}(g)v\|_\Phi}{\|v\|_\Phi}, \tag{2.6}$$

where $v \in \zeta_{\Phi_{\mathsf{a}}}(x) \in \mathbb{P}(\mathsf{V}_{\mathsf{a}})$ is non-zero, and l_{a} is as in Proposition 2.7.

2.6. **Gromov product and Cartan attractors.** Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{R}$. For a decomposition $\mathbb{K}^d = \ell \oplus V$ into a line ℓ and a hyperplane V together with an inner (Hermitian) product o on $\mathbb{K}^d$, one defines the *Gromov product* by

$$\mathcal{G}(V, \ell) = \mathcal{G}^o(V, \ell) := \log \frac{|\varphi(v)|}{\|\varphi\|\|v\|} = \log \sin \measuredangle_o(\ell, V),$$

for any non-vanishing $v \in \ell$ and $\varphi \in (\mathbb{K}^d)^*$ with $\ker \varphi = V$.

This induces, for any semisimple Lie group G and subset $\theta < \Delta$, a *Gromov product* $\mathcal{G}_\theta : \mathcal{F}_\theta^{(2)} \to \mathfrak{a}_\theta$ defined, for every $(x, y) \in \mathcal{F}_\theta^{(2)}$ and $\mathsf{a} \in \theta$, by

$$l_{\mathsf{a}} \varpi_{\mathsf{a}}\big(\mathcal{G}_\theta(x, y)\big) = \mathcal{G}^{\Phi_{\mathsf{a}}}(\zeta^*_{\Phi_{\mathsf{a}}} x, \zeta_{\Phi_{\mathsf{a}}} y) = \log \sin \measuredangle_o\big(\zeta_{\Phi_{\mathsf{a}}} y, \zeta^*_{\Phi_{\mathsf{a}}} x\big),$$

where $\zeta^*_{\Phi_{\mathsf{a}}}$ and $\zeta_{\Phi_{\mathsf{a}}}$ are the equivariant maps from Equation (2.5), and the Hermitian product o is induced by an Euclidean norm $\| \, \|_{\Phi_{\mathsf{a}}}$ invariant under $\Phi_{\mathsf{a}}\mathsf{K}$.

From S. [47, Lemma 4.12] one has, for all $g \in \mathsf{G}$ and $(x, y) \in \mathcal{F}_\theta^{(2)}$,

$$\mathcal{G}_\theta(gx, gy) - \mathcal{G}_\theta(x, y) = -\big(\mathrm{i}\, \mathrm{b}_{\mathrm{i}\theta}(g, x) + \mathrm{b}_\theta(g, y)\big). \tag{2.7}$$

If $g = k \exp(a(g)) l$ is a Cartan decomposition of $g \in \mathsf{G}$ we define its θ-*Cartan attractor* (resp. *repeller*) by

$$U_\theta(g) = k[\mathsf{P}_\theta] \in \mathcal{F}_\theta \quad \text{and} \quad U_{\mathrm{i}\theta}(g^{-1}) = l^{-1}[\check{\mathsf{P}}_\theta] \in \mathcal{F}_{\mathrm{i}\theta}.$$

The *Cartan basin of* g is defined, for $\alpha > 0$, by

$$B_{\theta,\alpha}(g) = \big\{x \in \mathcal{F}_\theta : \varpi_{\mathsf{a}} \mathcal{G}_\theta\left(U_{\mathrm{i}\theta}(g^{-1}), x\right) > -\alpha, \ \forall \mathsf{a} \in \theta\big\}. \tag{2.8}$$

Remark 2.8. Observe that a statement of the form $\varpi_{\mathsf{a}} \mathcal{G}_\theta(x, y) \geqslant -\kappa$ for all $\mathsf{a} \in \theta$ is a quantitative version (depending on the choice of K) of the transversality between x and y; in particular it implies that x and y are transverse.

Neither the Cartan attractor nor its basin are uniquely defined unless for all $\mathsf{a} \in \theta$ one has $\mathsf{a}\big(a(g)\big) > 0$, regardless one has the following:

Remark 2.9. Given $\alpha > 0$ there exists a constant K_α such that if $y \in \mathcal{F}_\theta$ belongs to $B_{\theta,\alpha}(g)$ then one has

$$\big\|a_\theta(g) - \mathrm{b}_\theta(g,y)\big\| \leqslant K_\alpha. \tag{2.9}$$

Indeed, using Tits's representations of G and Equations (2.4) and (2.6) this boils down to the elementary fact that if $A \in \mathsf{GL}_d(\mathbb{R})$ verifies[4] $\tau_1(a(A)) > 0$ then for every $v \in \mathbb{R}^d$ one has

$$\log \frac{\|Av\|}{\|v\|} \geqslant \log \|A\| + \log \sin \measuredangle\big(\mathbb{R}\cdot v, U_{d-1}(A^{-1})\big)$$

(see for example [9, Lemma A.3]).

3. Hölder cocycles on $\partial\Gamma$

Let Γ be a finitely generated group, and fix a finite generating set S. A group Γ is *Gromov hyperbolic* if its Cayley graph $\mathsf{Cay}(\Gamma, S)$ is a Gromov hyperbolic geodesic metric space. In this case we denote by $\partial\Gamma$ its Gromov boundary, namely the equivalence classes of (quasi)-geodesic rays. It is well known that, up to Hölder homeomorphism, $\partial\Gamma$ doesn't depend on the choice of the generating set S. We will denote by $\partial^2\Gamma$ the set of distinct pairs in $\partial\Gamma$:

$$\partial^2\Gamma := \{(x,y) \in \partial\Gamma \times \partial\Gamma \,|\, x \neq y\}.$$

For a finitely generated, non-elementary, word-hyperbolic group Γ we denote by $\mathsf{g} = \big(\mathsf{g}_t : \mathsf{U}\Gamma \to \mathsf{U}\Gamma\big)_{t\in\mathbb{R}}$ the *Gromov-Mineyev geodesic* flow of Γ (see Gromov [23] and Mineyev [38]). Throughout this section we will have the same assumptions as in S. [49, § 3], namely that g is metric-Anosov and that the lamination induced on the quotient by $\widetilde{\mathcal{W}}^{cu} = \{(x,\cdot,\cdot) \in \widetilde{\mathsf{U}\Gamma}\}$ is the central-unstable lamination of g.

Since we will mostly recall needed results from S. [49, § 3] we do not overcharge the paper with the definitions of metric-Anosov and central-unstable lamination: by Bridgeman-Canary-Labourie-S. [11], word-hyperbolic groups admitting an Anosov representation verify the required assumptions.

Definition 3.1. Let V be a finite dimensional real vector space. A *Hölder cocycle* is a function $c : \Gamma \times \partial\Gamma \to V$ such that:

- for all $\gamma, h \in \Gamma$ one has $c\big(\gamma h, x\big) = c\big(h, x\big) + c\big(\gamma, h(x)\big)$,
- there exists $\alpha \in (0,1]$ such that for every $\gamma \in \Gamma$ the map $c(\gamma,\cdot)$ is α-Hölder continuous.

Recall that every *hyperbolic element*[5] $\gamma \in \Gamma$ has two fixed points on $\partial\Gamma$, the attracting γ_+ and the repelling γ_-. If $x \in \partial\Gamma - \{\gamma_-\}$ then $\gamma^n x \to \gamma_+$ as $n \to \infty$. The *period* of a Hölder cocycle for a hyperbolic $\gamma \in \Gamma$ is $\ell_c(\gamma) := c\big(\gamma, \gamma^+\big)$. A cocycle $c^* : \Gamma \times \partial\Gamma \to \mathbb{R}$ is *dual to* c if for every hyperbolic $\gamma \in \Gamma$ one has

$$\ell_{c^*}(\gamma) = \ell_c\big(\gamma^{-1}\big).$$

[4]Recall from Equation (1.1) that we denote by τ_i the simple roots of $\mathsf{GL}_d(\mathbb{R})$

[5]Namely an infinite order element

3.1. **Real-valued coycles.** Assume now $V = \mathbb{R}$ and consider a cocycle κ with non-negative (and not all vanishing) periods. For $t > 0$ we let

$$\mathsf{R}_t(\kappa) = \big\{[\gamma] \in [\Gamma] \text{ hyperbolic} : \ell_\kappa(\gamma) \leqslant t\big\}$$

and define the *entropy* of κ by

$$\hbar_\kappa = \limsup_{t\to\infty} \frac{1}{t} \log \#\mathsf{R}_t(\kappa) \in (0, \infty].$$

For such a cocycle consider the action of Γ on $\partial^2\Gamma \times \mathbb{R}$ via κ:

$$\gamma \cdot (x, y, t) = (\gamma x, \gamma y, t - \kappa(\gamma, y)). \tag{3.1}$$

The following is a straightforward consequence of S. [49, Theorem 3.2.2].

Proposition 3.2. *Let κ be a Hölder cocycle with non-negative periods and finite entropy. Then, the above action of Γ on $\partial^2\Gamma \times \mathbb{R}$ is properly-discontinuous and co-compact. If moreover c is another Hölder cocycle with non-negative periods and finite entropy then there exists a Γ-equivariant bi-Hölder-continuous homeomorphism $E : \partial^2\Gamma \times \mathbb{R} \to \partial^2\Gamma \times \mathbb{R}$ which is an orbit equivalence between the $\mathbb{R}$-translation actions.*

We recall the notion of *dynamical intersection*, a concept from Bridgeman-Canary-Labourie-S. [11] for Hölder functions over a metric-Anosov flow, that can be pulled back to this setting via the existence of the *Ledrappier potential* of κ from S. [49, § 3.1].

The *dynamical intersection* of two real valued cocycles κ, c is

$$\mathbf{I}(\kappa, c) = \lim_{t\to\infty} \frac{1}{\mathsf{R}_t(\kappa)} \sum_{\gamma\in\mathsf{R}_t(\kappa)} \frac{\ell_c(\gamma)}{\ell_\kappa(\gamma)}. \tag{3.2}$$

We record in the following proposition various needed facts about $\mathbf{I}$:

Proposition 3.3 ([11, § 3])**.** *The dynamical intersection defined above is well defined, linear in the second variable and for all positive s satisfies $\mathbf{I}(s\kappa, c) = \mathbf{I}(\kappa, c)/s$. If also c has non-negative periods and finite entropy then $\mathbf{I}(\kappa, c) \geqslant \hbar_\kappa/\hbar_c$. Moreover, if $\mathbf{I}(\kappa, c) = \hbar_\kappa/\hbar_c$ then for every $\gamma \in \Gamma$ one has $\hbar_\kappa \ell_\kappa(\gamma) = \hbar_c \ell_c(\gamma)$.*

We will also need the following definitions.

Definition 3.4.

- A *Patterson-Sullivan measure for κ of exponent $\delta \in \mathbb{R}_+$* is a probability measure μ on $\partial\Gamma$ such that for every $\gamma \in \Gamma$ one has

$$\frac{d\gamma_*\mu}{d\mu}(\cdot) = e^{-\delta\cdot\kappa(\gamma^{-1},\,\cdot\,)}. \tag{3.3}$$

- Let κ^* be a cocycle dual to κ, then a *Gromov product* for the ordered pair (κ^*, κ) is a function $[\cdot,\cdot] : \partial^2\Gamma \to \mathbb{R}$ such that for all $\gamma \in \Gamma$ and $(x, y) \in \partial^2\Gamma$ one has

$$[\gamma x, \gamma y] - [x, y] = -\big(\kappa^*(\gamma, x) + \kappa(\gamma, y)\big).$$

3.2. **The critical hypersurface and intersection.** Let now $c : \Gamma \times \partial\Gamma \to V$ be a Hölder cocycle. Its *limit cone* is denoted by

$$\mathcal{L}_c = \overline{\bigcup_{\gamma\in\Gamma} \mathbb{R}_+ \cdot \ell_c(\gamma)}$$

and its *dual cone* by $\left(\mathcal{L}_c\right)^* = \{\psi \in V^* : \psi|_{\mathcal{L}_c} \geqslant 0\}$. Observe that for every $\varphi \in \operatorname{int}\left(\mathcal{L}_c\right)^*$, $\varphi \circ c$ is a real-valued cocycle, so the concepts from Section 3.1 apply. We denote by

$$\begin{aligned} \mathcal{Q}_c &= \left\{\varphi \in \operatorname{int}\left(\mathcal{L}_c\right)^* : \hbar_{\varphi\circ c} = 1\right\}, \\ \mathcal{D}_c &= \left\{\varphi \in \operatorname{int}\left(\mathcal{L}_c\right)^* : \hbar_{\varphi\circ c} \in (0,1)\right\}, \end{aligned} \tag{3.4}$$

respectively the *critical hypersurface* and the *convergence domain* of c.

For $\varphi \in \operatorname{int}\left(\mathcal{L}_c\right)^*$ we consider the linear map $\mathbf{I}_\varphi = \mathbf{I}^c_\varphi : V^* \to \mathbb{R}$ defined by

$$\mathbf{I}^c_\varphi(\psi) := \mathbf{I}(\varphi \circ c, \psi \circ c),$$

as in Equation (3.2). The natural identification between the set of hyperplanes in V^* and $\mathbb{P}(V)$ is used in the next proposition.

Corollary 3.5 (S. [49, Cor. 3.4.3])**.** *Assume $\mathcal{L}_c$ has non-empty interior and that there exists $\psi \in \left(\mathcal{L}_c\right)^*$ such that $\hbar_\psi < \infty$. Then $\mathcal{D}_c$ is a strictly convex set with boundary $\mathcal{Q}_c$. The latter is an analytic co-dimension one sub-manifold of V. The map $\mathsf{u}^c : \mathcal{Q}_c \to \mathbb{P}(V)$ defined by*

$$\varphi \mapsto \mathsf{u}^c_\varphi := \mathsf{T}_\varphi \mathcal{Q}_c = \ker \mathbf{I}_\varphi$$

is an analytic diffeomorphism between $\mathcal{Q}_c$ and $\operatorname{int}\left(\mathbb{P}(\mathcal{L}_c)\right)$.

3.3. **Ergodicity of directional flows.** It follows from Proposition 3.2 that if there exists $\psi \in \left(\mathcal{L}_c\right)^*$ with $\hbar_\psi < \infty$ then the Γ-action $\partial^2\Gamma \times V$

$$\gamma(x, y, v) = \left(\gamma x, \gamma y, v - c(\gamma, y)\right)$$

is properly discontinuous.

Definition 3.6. A Hölder cocycle c is *non-arithmetic* if the periods of c generate a dense subgroup in V.

We fix $\varphi \in \mathcal{Q}_c$ and denote by $u_\varphi \in \mathsf{u}_\varphi$ the unique vector in $\mathcal{L}_c \cap \mathsf{u}_\varphi$ with $\varphi(u_\varphi) = 1$. We define then the *directional flow* $\omega^\varphi = \left(\omega^\varphi_t : \Gamma\backslash\left(\partial^2\Gamma \times V\right) \to \Gamma\backslash\left(\partial^2\Gamma \times V\right)\right)_{t\in\mathbb{R}}$ by

$$t \cdot (x, y, v) = (x, y, v - tu_\varphi).$$

Assumption 3.7. We assume there exists:

- a dual cocycle $(\varphi \circ c)^*$,
- a Gromov product $[\,,\,]_\varphi$ for such a pair,
- Patterson-Sullivan measures, μ^φ and $\overline{\mu}^\varphi$, respectively for each of the cocycles; (the exponent of both measures is then necessarily $\hbar_\varphi = 1$ S. [49, Proposition 3.3.2]).

Consider then the φ-*Bowen-Margulis* measure Ω^φ on $\Gamma\backslash\left(\partial^2\Gamma \times V\right)$ defined as the measure induced on the quotient by the measure

$$e^{-[\cdot,\cdot]_\varphi}\overline{\mu}^\varphi \otimes \mu^\varphi \otimes \operatorname{Leb}_V, \tag{3.5}$$

for a V-invariant Lebesgue measure on V. We denote by $\mathcal{K}(\omega^\varphi)$ the *recurrence set* of the directional flow ω^φ:

$$\mathcal{K}(\omega^\varphi) := \{p \in \Gamma\backslash\big(\partial^2\Gamma \times V\big) \big|\ \exists B \text{ open bounded}, t_n \to \infty \text{ with } \omega^\varphi_{t_n}(p) \in B\}.$$

Corollary 3.8 (S. [49, Cor. 3.6.1])**.** *Assume that c is non-arithmetic, and that there exists $\varphi \in \mathcal{Q}_c$ satisfying Assumptions 3.7. If $\dim V \leqslant 2$ then the directional flow ω^φ is Ω^φ-ergodic, and $\mathcal{K}(\omega^\varphi)$ has total mass. If $\dim V \geqslant 4$ then $\mathcal{K}(\omega^\varphi)$ has measure zero.*

4. Subspace conicality for Anosov representations: Theorem D

4.1. **Gromov hyperbolic groups and cone types.** Let $\Gamma = \langle S\rangle$ be a finitely generated non-elementary Gromov hyperbolic group, and recall from §3 that we denote by $\partial^2\Gamma$ the set of distinct pairs in its Gromov boundary $\partial\Gamma$.

Definition 4.1. A divergent sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset \Gamma$ converges to a point $x \in \partial\Gamma$ *conically* if for every $y \in \partial\Gamma - \{x\}$ the sequence $(\gamma_n^{-1}y, \gamma_n^{-1}x)$ remains on a compact set of $\partial^2\Gamma$.

Remark 4.2. It is easy to verify that a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ converges conically to $x \in \partial\Gamma$ if and only if it lies in an uniform neighborhood of any geodesic ray $(\alpha_n)_0^\infty$ converging to x, namely there exists $K > 0$ and a subsequence $\{\alpha_{n_k}\}$ such that for all k one has $d_\Gamma(\alpha_{n_k}, \gamma_k) < K$.

Given $\gamma \in \Gamma$ we denote by $\mathcal{C}(\gamma)$ the *cone type* of $\gamma \in \Gamma$, namely

$$\mathcal{C}(\gamma) := \{h \in \Gamma |\ d(e, \gamma h) = d(e, \gamma) + d(e, h)\}.$$

Cannon showed [15] the set of cone types of a Gromov hyperbolic group is finite, see for example Bridson-Haefliger's book [12, P. 455]. We denote by $\mathcal{C}_\infty(\gamma) \subset \partial\Gamma$ the set of points x that can be represented by a geodesic ray contained in $\mathcal{C}(\gamma)$.

We will also need a coarse version of these. Recall that a sequence $(\alpha_j)_0^\infty$ is a (c, C)-quasigeodesic if for every pair j, l it holds

$$\frac{1}{c}|j - l| - C \leqslant d_\Gamma(\alpha_j, \alpha_l) \leqslant c|j - l| + C.$$

The *coarse cone type at infinity* of an element γ is the set of endpoints at infinity of quasi-geodesic rays based at γ^{-1} passing through the identity:

$$\mathcal{C}^c_\infty(\gamma) = \Big\{[(\alpha_j)_0^\infty] \in \partial\Gamma |\ (\alpha_i)_0^\infty \text{ is a } (c, c)\text{-quasi-geodesic}, \alpha_0 = \gamma^{-1}, e \in \{\alpha_j\}\Big\}.$$

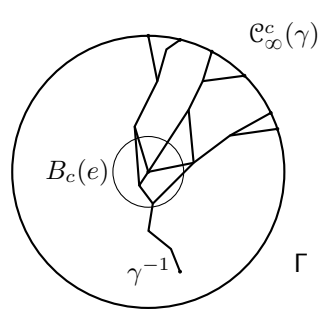

Figure 5. The coarse cone type at infinity, picture from P.-S.-Wienhard [42].

4.2. **Anosov representations.** Fix a subset $\theta \subset \Delta$. Let Γ be a finitely generated group and denote by $|\,|$ the word-length associated to a finite generating set S.

Definition 4.3. Following[6] Kapovich-Leeb-Porti [30], a representation $\rho : \Gamma \to \mathsf{G}$ is *θ-Anosov* if there exist positive constants C and μ such that for all $\gamma \in \Gamma$ and $\mathsf{a} \in \theta$ one has

$$\mathsf{a}(a(\rho\gamma)) \geqslant \mu|\gamma| - C.$$

The constants μ and C are usually referred to as *the domination constants* of ρ. If $\mathsf{G} = \mathsf{PGL}(d, \mathbb{R})$ and $\theta = \{\tau_1\}$ we say that ρ is *projective Anosov*. In order to easy the notation we ill identify in what follows γ with $\rho(\gamma)$.

Anosov representations were introduced by Labourie [35] and further developed by Guichard-Wienhard [25]. They have played a central role in understanding the Hitchin component of split groups (see below) and are considered nowadays as the higher-rank generalization of convex co-compact groups. We refer the reader to the surveys by Kassel [31] and Wienhard [52] for further information.

Remark 4.4. A Zariski-dense representation $\rho : \Gamma \to \mathsf{G}$ is θ-Anosov if and only if ρ is a quasi-isometric embedding and its limit cone $\mathcal{L}_\rho$ does not meet any wall $\ker \mathsf{a}$ for $\mathsf{a} \in \theta$: this follows from the definition since by Benoist [3], if $\rho(\Gamma)$ is Zariski-dense then the limit cone $\mathcal{L}_\rho$ equals the asymptotic cone.

A useful property of θ-Anosov representations is that their limit set $\Lambda_\Gamma \subset \mathcal{F}_\theta$, namely the minimal Γ-invariant subset in $\mathcal{F}_\theta$, is parametrized by the Gromov boundary of the group Γ, see Kapovich-Leeb-Porti [30], Guéritaud-Guichard-Kassel-Wienhard [24]. We will need the following precise statement.

Proposition 4.5 (Bochi-Potrie-S. [9, Proposition 4.9])**.** *If $\rho : \Gamma \to \mathsf{G}$ is θ-Anosov, then for any geodesic ray $(\alpha_n)_0^\infty$ with endpoint x, the limits*

$$\xi_\rho^\theta(x) := \lim_{n\to\infty} U_\theta(\alpha_n) \quad \xi_\rho^{\mathrm{i}\theta}(x) := \lim_{n\to\infty} U_{\mathrm{i}\theta}(\alpha_n)$$

exist and do not depend on the ray; they define continuous ρ-equivariant transverse maps $\xi^\theta : \partial\Gamma \to \mathcal{F}_\theta$, $\xi^{\mathrm{i}\theta} : \partial\Gamma \to \mathcal{F}_{\mathrm{i}\theta}$. If $\gamma \in \Gamma$ is hyperbolic, then γ is θ-proximal with attracting point $\xi^\theta(\gamma^+) = (\gamma)_\theta^+$.

We conclude the section with a number of quantitative results that will be needed in the paper. For an Anosov representation ρ there exists a constant δ_ρ quantifying transversality of Cartan-attractors along (quasi)-geodesic rays:

Proposition 4.6 (Bochi-Potrie-S. [9, Lemma 2.5])**.** *If $\rho : \Gamma \to \mathsf{G}$ is θ-Anosov and $c > 0$ is given, then there exist $\overline{L} \in \mathbb{N}$ and $\delta_{\rho,c} > 0$, depending only c and the domination constants of ρ, such that for every (c, c)-quasi-geodesic segment through the identity $\{\alpha_i\}_{-m}^k$ with $k, m \geqslant L$ one has, for all $\mathsf{a} \in \theta$, that*

$$\varpi_{\mathsf{a}} \mathcal{G}_\theta\big(U_{\mathrm{i}\theta}(\alpha_{-m}), U_\theta(\rho\alpha_k)\big) \geqslant \log \delta_{\rho,c}.$$

Combining Proposition 4.5 and Proposition 4.6 we obtain:

Corollary 4.7. *Up to decreasing $\delta_{\rho,c}$, for every $\gamma \in \Gamma$ and every $x \in \mathcal{C}_\infty^c(\gamma)$ one has*

$$\varpi_{\mathsf{a}} \mathcal{G}_\theta\big(U_{\mathrm{i}\theta}(\gamma^{-1}), \xi_\rho^\theta(x)\big) \geqslant \log \delta_{\rho,c}.$$

In particular, if we let $\alpha = -\log \delta_{\rho,c}$ then (recall Equation (2.8))

$$\xi_\rho^\theta(\mathcal{C}_\infty^c(\gamma)) \subset B_{\theta,\alpha}(\gamma). \tag{4.1}$$

[6]See also Bochi-Potrie-S. [9] and Guéritaud-Guichard-Kassel-Wienhard [24].

Definition 4.8. Let $\rho : \Gamma \to \mathsf{G}$ be θ-Anosov and $c > 0$, then the constant $\delta_{\rho,c}$ verifying both Proposition 4.6 and Corollary 4.7 will be called *the fundamental constant* of ρ and c. If we consider geodesics instead of quasi-geodesics (i.e. $(c, C) = (1, 0)$) we let δ_ρ be the *fundamental constant* associated to ρ.

The following two results will be needed in Section 7.1.

Proposition 4.9 (cfr. P.-S.-Wienhard [43, §5.1])**.** *Let $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{K})$ be projective Anosov and consider $c > 0$. Then there exists a constant K, depending on c and on ρ such that for every large enough $\gamma \in \Gamma$ one has*

$$\xi^1_\rho\big(\gamma \mathcal{C}^c_\infty(\gamma)\big) \subset B\big(U_1(\gamma), Ke^{-\tau_1(a(\gamma))}\big).$$

Proof. Using Corollary 4.7 for $\theta = \{\tau_1\}$, the result follows as in P.-S.-Wienhard [43, §5.1]. □

Proposition 4.10. *Let $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{K})$ be projective Anosov. For every $\alpha > 0$ there exist C and $\mu > 0$ such that for every $\ell_1, \ell_2 \in \mathbb{P}(\mathbb{K}^d)$ with*

$$\mathcal{G}\big(\ell_i, U_{d-1}(\gamma^{-1})\big) \geqslant -\alpha, \quad i = 1, 2$$

it holds $d_{\mathbb{P}}(\rho(\gamma)\ell_1, \rho(\gamma)\ell_2) \leqslant Ce^{-\mu|\gamma|} d(\ell_1, \ell_2)$.

Proof. For an Hermitian product on $\mathbb{C}^d$, and every $\alpha > 0$ there exists $C > 0$ such that if $h \in \mathsf{GL}(d, \mathbb{C})$ is such that $\tau_1(a(h)) > 0$, then for all $\ell_1, \ell_2 \in \mathbb{P}(\mathbb{C}^d)$ with $\measuredangle(\ell_i, U_{d-1}(h^{-1})) > \alpha$ one has

$$d_{\mathbb{P}}(h\ell_1, h\ell_2) \leqslant Ce^{-\tau_1(a(h))} d_{\mathbb{P}}(\ell_1, \ell_2),$$

(a proof follows, for instance, by applying [43, Lemma 2.8] to $g = h^{-1}$, $P = U_1(h)$ and $Q = hU_{d-1}(h)$). The result then follows by applying Definition 4.3. □

The following technical result will be useful in the proof of Proposition 4.23. Given an Anosov representation, we can use the Gromov product to determine the endpoint of a conical sequence (recall Definition 4.1):

Lemma 4.11. *Let $\rho : \Gamma \to \mathsf{G}$ be θ-Anosov. If $\{\gamma_n\} \subset \Gamma$ is a conical sequence, $x \in \partial\Gamma$, and there exists $\mathsf{a} \in \theta$ such that $\varpi_{\mathsf{a}} \mathcal{G}_\theta\big(U_{\mathrm{i}\theta}(\gamma_n), \xi^\theta(x)\big) \to -\infty$, then $\gamma_n \to x$.*

Proof. We denote by y the endpoint of the conical sequence γ_n. Proposition 4.5 and Remark 4.2 imply that $U_{\mathrm{i}\theta}(\gamma_n) \to \xi^{\mathrm{i}\theta}(y)$. Since, however, $\varpi_{\mathsf{a}} \mathcal{G}_\theta\big(U_{\mathrm{i}\theta}(\gamma_n), \xi^\theta(x)\big) \to -\infty$, we deduce that $\xi^{\mathrm{i}\theta}(y)$ is not transverse to $\xi^\theta(x)$ (recall Remark 2.8). Since ξ^θ is transverse, we deduce that $x = y$. □

It will be useful in the proof of Proposition 4.23 to know that the endpoints of conical sequences belong to pushed Cartan basins:

Lemma 4.12. *Let $\rho : \Gamma \to G$ be θ-Anosov, $x \in \partial\Gamma$. If $\gamma_n \to x$ conically, then there exists α only depending on the sequence and the representation ρ such that for every n, $\xi^\theta(x) \in \gamma_n B_{\theta,\alpha}(\gamma_n)$.*

Proof. We know from Remark 4.2 that γ_n is contained in a neighbourhood of a geodesic ray to x, or equivalently there exist a constant c such that $\gamma^{-1}x \in \mathcal{C}^c_\infty(\gamma_n)$. The result is then a consequence of Equation (4.1). □

4.3. **Patterson-Sullivan theory of Anosov representations.** If ρ is a θ-Anosov representation, then we can pullback the Buseman-Iwasawa cocycle of G using the equivariant maps: the *refraction cocycle* associated to a θ-Anosov representation $\rho : \Gamma \to \mathsf{G}$ is $\beta : \Gamma \times \partial\Gamma \to \mathfrak{a}_\theta$ given by

$$\beta(\gamma, x) = \beta_{\theta,\rho}(\gamma, x) = \mathrm{b}_\theta\left(\rho(\gamma), \xi^\theta_\rho(x)\right).$$

Bridgeman-Canary-Labourie-S. [11, Theorem 1.10] show that the Mineyev geodesic flow of a group Γ admitting an Anosov representations is metric-Anosov, and thus § 3 applies to β. Moreover, the following fact places β in the assumptions required in § 3.1 and § 3.2, see S. [49] for details.

Fact. *The periods of the refraction cocycle equal the θ-Jordan projection: $\beta(\gamma, \gamma^+) = \lambda_\theta(\gamma)$. For any $\mathsf{a} \in \theta$ the real valued cocycle $\varpi_{\mathsf{a}}\beta$ has finite entropy.*

We import the following concepts of cocycles to the setting of Anosov representations:

- The limit cone of β will be denoted by $\mathcal{L}_{\theta,\rho}$ and referred to as *the θ-limit cone of ρ*; it is the smallest closed cone that contains the projected Jordan projections $\{\lambda_\theta(\gamma) : \gamma \in \Gamma\}$.
- The *interior of the dual cone* $\operatorname{int}\left(\mathcal{L}_{\theta,\rho}\right)^* \subset \mathfrak{a}^*_\theta$ consists of linear forms whose *entropy*
$$\hbar_\varphi = \lim_{t\to\infty} \frac{1}{t} \log \#\big\{[\gamma] \in [\Gamma] : \varphi(\lambda_\theta(\gamma)) \leqslant t\big\}$$
is finite.
- The *θ-critical hypersurface*, resp. *θ-convergence domain*, of β will be denoted by
$$\mathcal{Q}_{\theta,\rho} = \Big\{\varphi \in \operatorname{int}\left(\mathcal{L}_{\theta,\rho}\right)^* : \hbar_\varphi = 1\Big\},$$
$$\mathcal{D}_{\theta,\rho} = \Big\{\varphi \in \operatorname{int}\left(\mathcal{L}_{\theta,\rho}\right)^* : \hbar_\varphi \in (0,1)\Big\}.$$
- If $\mathcal{L}_{\theta,\rho}$ has non-empty interior, then we have a *duality* diffeomorphism between $\mathcal{Q}_{\theta,\rho}$ and $\operatorname{int}\mathbb{P}(\mathcal{L}_{\theta,\rho})$ given by
$$\varphi \mapsto \mathsf{u}_\varphi = \mathsf{T}_\varphi \mathcal{Q}_\rho.$$

More information on these objets can be found on S. [49, § 5.9].

Remark 4.13. It it proven in Glorieux-Monclair-Tholozan [22, Theorem 2.31 (2)] (see also S. [49, Corollary 5.5.3]) that if ρ is θ-Anosov then for every $\varphi \in \operatorname{int}\left(\mathcal{L}_{\theta,\rho}\right)^*$ the entropy $\hbar_\varphi$ equals the critical exponent

$$\hbar^\varphi := \lim_{t\to\infty} \frac{1}{t} \log \#\big\{\gamma \in \Gamma : \varphi(a(\gamma)) \leqslant t\big\}.$$

In particular the θ-convergence domain is also given by

$$\mathcal{D}_{\theta,\rho} = \Big\{\varphi \in (\mathfrak{a}_\theta)^* : \sum_{\gamma\in\Gamma} e^{-\varphi(a(\gamma))} < \infty\Big\},$$

see S. [49, § 5.7.2].

We observe that for $\varphi \in \operatorname{int}\left(\mathcal{L}_{\theta,\rho}\right)^*$ Assumptions 3.7 are guaranteed for $\beta_\varphi := \varphi \circ \beta$. Indeed the cocycle

$$\overline{\beta}(\gamma, x) = \mathrm{i}\,\mathrm{b}_{\mathrm{i}\theta}\left(\gamma, \xi^{\mathrm{i}\theta}(x)\right)$$

is dual to β, from Equation (2.7) the function $[\cdot,\cdot]_\varphi : \partial^2\Gamma \to \mathbb{R}$

$$[x,y]_\varphi = \varphi\Big(\mathcal{G}_\theta\big(\xi^{\mathrm{i}\theta}(x), \xi^\theta(y)\big)\Big)$$

is a Gromov product for the pair $(\overline{\beta}_\varphi, \beta_\varphi)$, and we have the following result guaranteeing existence of Patterson-Sullivan measures μ^φ and $\overline{\mu}^\varphi$, as well as their values on Cartan basins defined in Equation (2.8).

Corollary 4.14 (S. [49, Cor. 5.5.3+Lemma 5.7.1])**.** *For every $\varphi \in \operatorname{int}\big(\mathcal{L}_{\theta,\rho}\big)^*$ there exists a β_φ-Patterson-Sullivan measure μ^φ of exponent $\hbar_\varphi$, moreover for every α there exists a constant C such that for every $\gamma \in \Gamma$ one has*

$$\mu^\varphi\big((\xi^\theta)^{-1}(\gamma B_{\theta,\alpha}(\gamma)\big) \leqslant Ce^{-\hbar_\varphi \varphi\big(a(\gamma)\big)}.$$

4.4. **Subspace-conicality.** In this section we are interested in a notion of conicality along higher dimensional subspaces of the ambient Levi space.

Definition 4.15. Let $\rho : \Gamma \to \mathsf{G}$ be θ-Anosov and consider a subspace $\mathsf{W} \subset \mathfrak{a}_\theta$. A point $x \in \partial\Gamma$ is W-*conical* if there exists a conical sequence $\{\gamma_n\}_0^\infty \subset \Gamma$ converging to x, a constant K and $\{w_n\}_0^\infty \subset \mathsf{W}$ such that for all n one has

$$\big\|a_\theta(\gamma_n) - w_n\big\| \leqslant K.$$

The set of such points will be denoted by $\partial_{\mathsf{W},\rho}\Gamma = \partial_{\mathsf{W}}\Gamma$.

Assume from now on that W intersects the relative interior of $\mathcal{L}_{\theta,\rho}$, and consider $\varphi \in \operatorname{int}\big(\mathcal{L}_{\theta,\rho}\big)^*$ with $\mathsf{u}_\varphi \subset \mathsf{W}$. The intersection $\mathsf{W}_\varphi = \mathsf{W} \cap \ker\varphi$ has co-dimension 1 in W and has trivial intersection with the limit cone $\mathcal{L}_{\theta,\rho}$. Consider the quotient space

$$V = \mathfrak{a}_\theta/\mathsf{W}_\varphi$$

equipped with the quotient projection $\Pi : \mathfrak{a}_\theta \to V$. We say that ρ is (W,φ)-*non-arithmetic* if the group spanned by $\big\{\Pi(\lambda_\theta(\gamma)) : \gamma \in \Gamma\big\}$ is dense in V. In this section we prove the following.

Theorem 4.16. *Let $\rho : \Gamma \to \mathsf{G}$ be θ-Anosov, W be a subspace of $\mathfrak{a}_\theta$ intersecting non-trivially the relative interior of $\mathcal{L}_{\theta,\rho}$, and $\varphi \in (\mathfrak{a}_\theta)^*$ with $\mathsf{u}_\varphi \subset \mathsf{W}$. Assume ρ is (W,φ)-non-arithmetic, then:*

- *if W has codimension 1 then $\mu^\varphi(\partial_{\mathsf{W}}\Gamma) = 1$;*
- *if $\operatorname{codim}\mathsf{W} \geqslant 3$ then $\mu^\varphi(\partial_{\mathsf{W}}\Gamma) = 0$.*

Remark 4.17. If ρ is Zariski-dense then Theorem 2.3 (Benoist [4]) guarantees (W,φ)-non-arithmeticity for every $\varphi \in (\mathfrak{a}_\theta)^*$ with $\mathsf{u}_\varphi \in \mathbb{P}(\mathsf{W})$, thus Theorem 4.16 readily implies Theorem D.

The remainder of the section is devoted to the proof of Theorem 4.16. Let

$$V^* = \operatorname{Ann}(\mathsf{W}_\varphi) = \{\psi \in (\mathfrak{a}_\theta)^* : \psi|\mathsf{W}_\varphi \equiv 0\},$$

with a slight abuse of notation we will identify the dual of V with $V^* \subset (\mathfrak{a}_\theta)^* \subset \mathfrak{a}^*$ (recall from Section 2.4 that we are identifying $(\mathfrak{a}_\theta)^*$ with the subspace of $\mathfrak{a}^*$ consisting of π_θ-invariant linear forms).

The composition of the refraction cocycle of ρ with Π is a V-valued Hölder cocycle $\upsilon : \Gamma \times \partial\Gamma \to V$,

$$\upsilon(\gamma, x) = \Pi\big(\beta(\gamma, x)\big).$$

Its periods are $\upsilon(\gamma,\gamma_+) = \Pi\big(\lambda_\theta(\gamma)\big)$, and thus its limit cone is $\mathcal{L}_\upsilon = \Pi(\mathcal{L}_{\theta,\rho})$. By (W,φ)-non-arithmeticity, $\mathcal{L}_\upsilon \subset V$ has non-empty interior.

The heart of the proof of Theorem 4.16 consits on relating (W,φ)-conical points with elements of $\widetilde{\mathcal{K}}\big(\omega^\varphi\big)$, where ω^φ is the directional flow on $\Gamma\backslash\partial^2\Gamma \times V$ associated to the cocycle υ as in § 3.3. The first step is thus to observe that we can apply Corollary 3.8 to υ, a task we enter at this point.

Since $\varphi \in \mathcal{Q}_{\theta,\rho}$, it has in particular finite entropy. Moreover, by definition of V^* one has $\varphi \in V^*$. Consequently, the cocycle υ verifies assumptions in Corollary 3.5. One can moreover transfer existence properties from β to υ, indeed one has the following.

Proposition 4.18. *The cocycle $\overline{\upsilon} = \Pi \circ \overline{\beta}$ is a dual cocycle for υ. For each $\psi \in \mathcal{Q}_\upsilon$ there exist Paterson-Sullivan measures for υ and $\overline{\upsilon}$ and the projection $\psi\big(\Pi\big([\cdot,\cdot]\big)\big)$ is a Gromov product for the pair $\psi\circ\upsilon, \psi\circ\overline{\upsilon}$.*

Proof. Since $\psi \in \mathcal{Q}_\upsilon = \mathcal{Q}_{\theta,\rho} \cap V^*$ we can apply Corollary 4.14 to ψ to obtain the desired Patterson-Sullivan measure, the remaining statements follow trivially as the equalities are linear. □

Since we are assuming (W,φ)-non-arithmeticity, the cocycle υ is non-arithmetic and thus Corollary 3.8 gives the following dynamical information, observe that $\dim V = \operatorname{codim}\mathsf{W} + 1$.

Corollary 4.19. *If $\operatorname{codim}\mathsf{W} \leqslant 1$ then the directional flow ω^φ is Ω^φ-ergodic, in particular $\mathcal{K}(\omega^\varphi)$ has total mass. If $\operatorname{codim}\mathsf{W} \geqslant 3$ then $\mathcal{K}(\omega^\varphi)$ has measure zero.*

Observe that modulo the understood identifications $\mathcal{Q}_\upsilon = \mathcal{Q}_{\theta,\rho} \cap V^*$, hence

$$\mathsf{T}_\varphi \mathcal{Q}_\upsilon = (\mathsf{T}_\varphi \mathcal{Q}_{\theta,\rho}) \cap V^*$$

and thus the map $\mathsf{u}^\upsilon : \mathcal{Q}_\upsilon \to \operatorname{int}\mathbb{P}(\mathcal{L}_\upsilon)$ from Corollary 3.5 verifies $\mathsf{u}^\upsilon_\varphi = \Pi(\mathsf{u}_\varphi)$. So measuring W-conicality with respect to μ^φ translates to directional conicality along the direction $\mathsf{u}^\upsilon_\varphi$, which we now recall. We fix an arbitrary norm $\|\,\|$ on V and define, for $\ell \in \mathbb{P}(V)$ and $r > 0$, the *r-tube about ℓ* by

$$\mathbb{T}_r(\ell) := \{v \in V \,|\, \exists w \in \ell, \|v - w\| < r\}.$$

Definition 4.20. A point $y \in \partial\Gamma$ is *$\mathsf{u}^\upsilon_\varphi$-conical* if there exists $r > 0$ and a conical sequence $\{\gamma_n\}_0^\infty \subset \Gamma$ with $\gamma_n \to y$ such that for all n one has $\Pi\big(a_\theta(\rho(\gamma_n))\big) \in \mathbb{T}_r(\mathsf{u}^\upsilon_\varphi)$.

The next statement follows from the definitions.

Lemma 4.21. *A point $y \in \partial\Gamma$ is W-conical if and only if it is $\mathsf{u}^\upsilon_\varphi$-conical.*

If we are allowed to worsen the constants, we can replace, in Definition 4.20, the conical sequence (γ_n) with an infinite subset of a geodesic ray:

Lemma 4.22. *A point $y \in \partial\Gamma$ is $\mathsf{u}^\upsilon_\varphi$-conical if and only if there exists $r > 0$, a geodesic ray $(\alpha_i)_0^\infty$ converging to y and an infinite set of indices $\mathbb{I} \subset \mathbb{N}$ such that for all $k \in \mathbb{I}$ one has*

$$\Pi\big(a_\theta(\alpha_k)\big) \in \mathbb{T}_r(\mathsf{u}^\upsilon_\varphi).$$

Proof. Assume y is $\mathsf{u}^\upsilon_\varphi$-conical, then since $\{\gamma_n\}_0^\infty$ is conical, for any geodesic ray $(\alpha_n)_0^\infty$ converging to y there exists $K > 0$ and a subsequence $\{\alpha_{n_k}\}$ such that for

all k one has $d_\Gamma(\alpha_{n_k}, \gamma_k) < K$ (Remark 4.2). Proposition 2.1 implies then that for all k one has

$$\|a(\alpha_{n_k}) - a(\gamma_k)\|$$

is bounded independently of k. This implies the result. □

We now relate $\mathsf{u}^{\upsilon}_{\varphi}$-conicality with the recurrence set $\mathcal{K}(\omega^\varphi)$. By definition of $\mathcal{K}(\omega^\varphi)$, a point $(x, y, v) \in \partial^2\Gamma \times V$ projects to $\mathcal{K}(\omega^\varphi)$ if and only if there exist divergent sequences $(\gamma_n) \subset \Gamma$ and $t_n \to +\infty$ in $\mathbb{R}$ such that

$$\omega^{\varphi}_{t_n}\gamma_n^{-1}(x, y, v) = \big(\gamma_n^{-1}x, \gamma_n^{-1}y, v - \upsilon(\gamma_n^{-1}, y) - t_n u_\varphi\big) \tag{4.2}$$

is contained in a subset of the form $\{(z, w) \in \partial^2\Gamma : d(z, w) \geqslant \kappa\} \times B(v, K)$ for some distance d on $\partial\Gamma$. One has the following

Proposition 4.23. *A point $y \in \partial\Gamma$ is $\mathsf{u}^{\upsilon}_{\varphi}$-conical if and only if for every $x \in \partial\Gamma - \{y\}$ and $v \in V$ the point (x, y, v) projects to $\mathcal{K}(\omega^\varphi)$.*

Proof. The implication ($\Rightarrow$) follows exactly as in the proof of S. [49, Proposition 5.13.4]. The other implication also follows similarly but with a minor difference we now explain.

Assume that (x, y, v) projects to $\mathcal{K}(\omega^\varphi)$ and consider sequences $\{\gamma_n\}$ and t_n as in Equation (4.2). Since $\big(\gamma_n^{-1}x, \gamma_n^{-1}y\big)$ remains in a compact subset of $\partial^2\Gamma$, the sequence $\{\gamma_n\}$ is conical, we will show now that $\gamma_n \to y$. Indeed, since $t_n \to +\infty$ necessarily $\upsilon(\gamma_n^{-1}, y) \to -\infty$.

Consider now any root $\mathsf{a} \in \theta$, with associated fundamental weight $\varpi_{\mathsf{a}} \in \big(\mathcal{L}_{\theta,\rho}\big)^*$, and Tits representation $\Phi_{\mathsf{a}} : \mathsf{G} \to V$. Since ρ is θ-Anosov, the Hölder cocycle $\beta_{\varpi_{\mathsf{a}}}$ has positive periods and finite entropy. Since $\upsilon(\gamma_n^{-1}, y) \to -\infty$ Proposition 3.2 implies that

$$\beta_{\varpi_{\mathsf{a}}}(\gamma_n^{-1}, y) \to -\infty.$$

By definition of the cocycle $\beta_{\varpi_{\mathsf{a}}}$ and Equation (2.6) we have

$$\frac{\|\Phi_{\mathsf{a}}(\gamma_n^{-1})v\|}{\|v\|} \to 0 \tag{4.3}$$

for a non-zero $v \in \zeta_{\mathsf{a}}(\xi(y))$, (recall that the map $\zeta_{\mathsf{a}} : \mathcal{F}_{\mathsf{a}}(\mathsf{G}) \to \mathbb{P}(V)$ was defined in Equation (2.5)). Setting $\dim V = d$, a standard linear algebra computation (for example in Bochi-Potrie-S. [9, Lemma A.3]) gives

$$\begin{aligned}\frac{\|\Phi_{\mathsf{a}}(\gamma_n^{-1})v\|}{\|v\|} &\geqslant \|\Phi_{\mathsf{a}}(\gamma_n^{-1})\| \sin \measuredangle\big(\zeta_{\mathsf{a}}\xi(y), U_{d-1}(\Phi_{\mathsf{a}}\gamma_n)\big)\\ &\geqslant e^{l_{\mathsf{a}}\varpi_{\mathsf{a}}\mathcal{G}_\theta\left(U_\theta(\gamma_n), y\right)}\end{aligned}$$

and thus, by Equation (4.3) and Lemma 4.11 one has $\gamma_n \to y$, as desired.

The point $\xi(y)$ lies then in the pushed Cartan basin $\gamma_n B_{\theta,\alpha}(\gamma_n)$ for an α independent of n (Lemma 4.12), and thus Equation (2.9) gives a constant K such that for all n one has

$$K \geqslant \big\|a_\theta(\gamma_n) - \beta\big(\gamma_n, \gamma_n^{-1}y\big)\big\| = \big\|a_\theta(\gamma_n) + \beta(\gamma_n^{-1}, y)\big\|$$

implying, by Equation (4.2), that y is $\mathsf{u}^{\upsilon}_{\varphi}$-conical, as desired. □

The proof of Theorem 4.16 follows now along the same lines as in S. [49, Theorem 5.13.3]. We include the arguments here for completeness.

For $y \in \partial_{\mathsf{W},\rho}\Gamma, x \in \partial\Gamma - \{y\}$ we fix neighbourhoods A^- and A^+ of x and y respectively and $T > 0$ small enough so that the quotient projection $\mathsf{p} : \partial^2\Gamma \times V \to \Gamma\backslash\partial^2\Gamma \times V$ is injective on $\tilde{\mathrm{B}} = A^- \times A^+ \times B(0,T)$. We can thus use Equation (3.5) to compute the measure of $\mathrm{B} = \mathsf{p}(\tilde{\mathrm{B}})$.

For $\tilde{\mathcal{K}}(\omega^\varphi) = \mathsf{p}^{-1}\big(\mathcal{K}(\omega^\varphi)\big)$, Proposition 4.23 asserts

$$A^- \times (A^+ \cap \partial_{\mathsf{W},\rho}\Gamma) \times B(0,T) = \tilde{\mathcal{K}}(\omega^\varphi) \cap \tilde{\mathrm{B}}.$$

If $\operatorname{codim}\mathsf{W} = 1$ by Corollary 4.19 $\Omega^\varphi(\tilde{\mathrm{B}}) = \Omega^\varphi\big(\tilde{\mathcal{K}}(\omega^\varphi) \cap \tilde{\mathrm{B}}\big)$, which implies that $\mu^\varphi(A^+\backslash\partial_{\mathsf{W},\rho}\Gamma) = 0$ and thus $\mu^\varphi(\partial_{\mathsf{W},\rho}\Gamma) = 1$. On the other hand, if $\operatorname{codim}\mathsf{W} \geqslant 3$, then we have $\Omega^\varphi\big(\tilde{\mathcal{K}}(\omega^\varphi)\big) = 0$ so $\mu^\varphi(A^+ \cap \partial_{\mathsf{W},\rho}\Gamma) = 0$ and the theorem is proved.

5. Locally conformal representations: Hausdorff dimension of $\mathscr{C}$-conical points

In this section we let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, the non-commutative field of Hamilton's quaternions. A Cartan subspace $\mathfrak{a}$ of $\mathsf{SL}(d,\mathbb{K})$ is the subspace of $\mathbb{R}^d$ consisting of vectors whose coordinates sum 0. For $g \in \mathsf{SL}(d,\mathbb{K})$ we denote by

$$a(g) = \big(a_1(g), \cdots, a_d(g)\big) \in \mathfrak{a}^+$$

the coordinates of the Cartan projection. We recall Definition 1.1.

Definition 5.1. Let $p \in [\![2, d-1]\!]$. A $\{\tau_1, \tau_{d-p}\}$-Anosov representation $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{K})$ is $(1,1,p)$-*hyperconvex* if, for every pairwise distinct triple $(x,y,z) \in \partial\Gamma^{(3)}$, one has

$$\big(\xi^1(x) + \xi^1(y)\big) \cap \xi^{d-p}(z) = \{0\}.$$

If in addition one has $a_2(\rho(\gamma)) = a_p(\rho(\gamma))\ \forall\gamma$, we say that ρ is *locally conformal*. As before, we identify from now on γ and $\rho(\gamma)$.

The terminology is justified by Proposition 5.6 below stating that for such representations pushed coarse cone types are coarsely balls, a small refinement of an analogous result from P.-S.-Wienhard [43].

In this section we will study conicality from § 4.4 on a specific situation that we now explain. Later, in § 6, we will relate this section to the notion of $\mathscr{C}$-concavity and in § 8 to differentiability properties of the map $\overline{\xi} \circ \xi^{-1}$.

Consider $\overline{\mathbb{K}} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and two locally conformal representations $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{K})$ and $\overline{\rho} : \Gamma \to \mathsf{SL}(\overline{d},\overline{\mathbb{K}})$, with projective equivariant maps

$$\xi : \partial\Gamma \to \mathbb{P}(\mathbb{K}^d)$$
$$\overline{\xi} : \partial\Gamma \to \mathbb{P}(\overline{\mathbb{K}}^{\overline{d}}).$$

The product representation $(\rho, \overline{\rho}) : \Gamma \to \mathsf{SL}(d,\mathbb{K}) \times \mathsf{SL}(\overline{d},\overline{\mathbb{K}})$ is θ-Anosov for $\theta = \{\tau_1, \tau_p, \overline{\tau}_1, \overline{\tau}_p\}$ with $\{\tau_1, \overline{\tau_1}\}$-limit map the "graph map"

$$\mathscr{G} = \big(\xi, \overline{\xi}\big) : \partial\Gamma \to \mathbb{P}(\mathbb{K}^d) \times \mathbb{P}(\overline{\mathbb{K}}^{\overline{d}}).$$

We consider a Cartan subspace of the product group $\mathsf{SL}(d,\mathbb{K}) \times \mathsf{SL}(\overline{d},\overline{\mathbb{K}})$ and let $\mathfrak{a}_\theta$ be the associated Levi space. Its dual $(\mathfrak{a}_\theta)^*$ is spanned by the fundamental weights

of roots in θ. We let

$$\tau := \frac{p\varpi_{\tau_1} - \varpi_{\tau_p}}{p-1},$$
$$\overline{\tau} := \frac{p\varpi_{\overline{\tau}_1} - \varpi_{\overline{\tau}_p}}{p-1}.$$

Both $\tau, \overline{\tau} \in (\mathfrak{a}_\theta)^*$ and under the assumption $a_2(\gamma) = a_p(\gamma)$ for all γ of Definition 5.1, it holds on $\mathcal{L}_\rho$ that $\tau_1 = \tau$ and $\overline{\tau} = \overline{\tau}_1$ (if $p = 2$ the equality holds on $\mathfrak{a}$).

Definition 5.2. Fix $\mathscr{b} \in (0,1]$. A point $x \in \partial\Gamma$ is *$\mathscr{b}$-conical* if it is conical as in Definition 4.15 for the product representation $(\rho, \overline{\rho})$ with respect to the hyperplane

$$\{v \in \mathfrak{a}_\theta : \mathscr{b}\tau(v) = \overline{\tau}(v)\} = \ker(\mathscr{b}\tau - \overline{\tau}).$$

Equivalently, there exist R, a geodesic ray $(\alpha_n)_0^\infty \subset \Gamma$ with $\alpha_n \to x$, and a subsequence $\{n_k\}$ such that for all k one has

$$\left|\mathscr{b}\tau\big(a(\alpha_{n_k})\big) - \overline{\tau}\big(a(\overline{\alpha}_{n_k})\big)\right| \leqslant R.$$

Consider also the critical exponent

$$\hbar^{\infty,\mathscr{b}} = \lim_{t\to\infty} \frac{1}{t} \log \#\big\{\gamma \in \Gamma : \max\big\{\mathscr{b}\tau(a(\gamma)), \overline{\tau}(a(\overline{\gamma}))\big\} \leqslant t\big\},$$

and recall from Equation (3.2) the dynamical intersection defined by

$$\mathbf{I}_\tau(\overline{\tau}) = \lim_{t\to\infty} \frac{1}{\#\mathsf{R}_t(\tau)} \sum_{\gamma \in \mathsf{R}_t(\tau)} \frac{\overline{\tau}(\lambda(\overline{\gamma}))}{\tau(\lambda(\gamma))}, \tag{5.1}$$

where $\mathsf{R}_t(\tau) = \big\{[\gamma] \in [\Gamma] : \tau\big(\lambda(\gamma)\big) \leqslant t\big\}$.

In this section we compute the Hausdorff dimension of the image under the graph map $\mathscr{G}$ of the set of $\mathscr{b}$-conical points with respect to a Riemannian metric:

Theorem 5.3. *Let $\rho, \overline{\rho}$ be locally conformal representations over $\mathbb{K}$ and $\overline{\mathbb{K}}$ respectively. Assume the group generated by $\{(\tau(\lambda(\gamma)), \overline{\tau}(\lambda(\overline{\gamma}))) : \gamma \in \Gamma\}$ is dense in $\mathbb{R}^2$. Then, for every $\mathscr{b} \in (0,1]$ with*

$$\mathbf{I}_\tau(\overline{\tau}) > \mathscr{b} > 1/\mathbf{I}_{\overline{\tau}}(\tau),$$

one has

$$\begin{aligned}\mathscr{b}\hbar^{\infty,\mathscr{b}} \leqslant \dim_{\mathrm{Hff}} \mathscr{G}\big(\{\mathscr{b}\text{−conical points}\}\big) &\leqslant \min\{\hbar^{\infty,\mathscr{b}}, \mathscr{b}\hbar^{\infty,\mathscr{b}} + (1-\mathscr{b})\} \\ &< \min\{\hbar_{\overline{\tau}}, \hbar_\tau/\mathscr{b}\} \\ &\leqslant \dim_{\mathrm{Hff}}(\mathscr{G}(\partial\Gamma)) \\ &= \max\{\hbar_\tau, \hbar_{\overline{\tau}}\}.\end{aligned}$$

The proof of the above result is completed in § 5.5.

Recall that if $\hbar^{\tau_1} = \hbar^{\overline{\tau}_1}$ and the representations are not gap-isospectral, then Proposition 3.3 gives $\mathbf{I}_{\overline{\tau}_1}(\tau_1) > 1$. Theorem 5.3 studies then $\mathscr{b}$-conical points for any $\mathscr{b}$ with $\mathbf{I}_{\overline{\tau}_1}(\tau_1) > 1/\mathscr{b} \geqslant 1$. As the following result shows, the equality between entropies is rather natural for $\mathbb{K} = \mathbb{R}$.

Theorem 5.4 (P.-S.-Wienhard [43])**.** *Let $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{K})$ be locally conformal, then*

$$\hbar_\tau = \dim_{\mathrm{Hff}}\big(\xi(\partial\Gamma)\big).$$

Moreover, when $\mathbb{K} = \mathbb{R}$ and $\partial\Gamma$ is homeomorphic to a $p-1$-dimensional sphere, $\hbar_\tau = p-1$.

When Γ is a surface group we can also weaken the assumption on the density of periods:

Corollary 5.5. *Assume $\partial\Gamma$ is homeomorphic to a circle and let ρ and $\overline{\rho}$ be non-gap-isospectral real $(1,1,2)$-hyperconvex representations of Γ. Then*

$$\dim_{\mathrm{Hff}} \mathscr{G}\big(\{1-\text{conical points}\}\big) = \hbar^{\infty} < 1.$$

Proof. Proposition 6.3 below states that under our assumptions the group generated by $\{(\tau(\lambda(\gamma)), \overline{\tau}(\lambda(\overline{\gamma}))) : \gamma \in \Gamma\}$ is dense in $\mathbb{R}^2$. Theorem 5.4 guarantees that $\mathbf{I}_\tau(\overline{\tau}) \geqslant 1$. The equality then follows from Theorem 5.3. □

Kim-Minsky-Oh [32] have established realted Hausdorff dimension computations when ρ and $\overline{\rho}$ are convex-co-compact representations in $\mathsf{SO}(n,1)$ without any assumption on $\mathbf{I}$.

5.1. **Cone types are coarsely balls.** In [43] P.-S.-Wienhard gave a concrete description of the images under the boundary map of the cone types at infinity. We discuss here a slight extension of that result adapted to our needs. We denote by $d_{\mathbb{P}}$ the distance on $\mathbb{P}(\mathbb{K}^d)$ induced by the choice of an inner (Hermitian) product on $\mathbb{K}^d$ and by $B(\ell, r) \subset \mathbb{P}(\mathbb{K}^d)$ the associated ball of radius r about ℓ.

Proposition 5.6. *Let $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{K})$ be locally conformal. Then there exist positive constants c, k_1, k_2 and $L \in \mathbb{N}$ such that for every $x \in \partial\Gamma$, every geodesic ray $(\alpha_n)_0^\infty$ with endpoint x and every $n > L$ one has*

$$B\Big(\xi(x), k_1 e^{-\tau_1(a(\alpha_n))}\Big) \cap \xi(\partial\Gamma) \subset \xi\Big(\alpha_n \mathcal{C}^c_\infty(\alpha_n)\Big) \subset B\Big(\xi(x), k_2 e^{-\tau_1(a(\alpha_n))}\Big).$$

Proof. The desired inclusions are proven in [43] for thickened cone types at infinity. We briefly explain here how to deduce from it the result we need.

Following [43] we denote by $X_\infty(\gamma)$, for $\gamma \in \Gamma$, the *thickened cone type at infinity*, namely the tubular neighborhood in $\mathbb{P}(\mathbb{K}^d)$ of $\xi\big(\mathcal{C}_\infty(\gamma)\big)$ of radius $\delta_\rho/2$, where δ_ρ is the fundamental constant from Definition 4.8. In [43, Corollary 5.10] it is established that there exists $c_1 > 0$ and $L_0 > 0$ only depending on the domination constants of ρ such that for all $i \geqslant L_0$ one has

$$B\Big(\xi(x), c_1 e^{-\tau_1(a(\alpha_i))}\Big) \cap \xi(\partial\Gamma) \subset \alpha_i X_\infty(\alpha_i).$$

By definition the thickened cone type $X_\infty(\gamma)$ is contained in the Cartan basin $B_{\{\tau_1\},\alpha}(\gamma)$ for $\alpha = -2\log\delta_\rho$. So P.-S.-Wienhard [42, Proposition 3.3] provides the existence of c and L_0 such that for $\gamma \in \Gamma$ with $|\gamma| > L_0$, one has

$$X_\infty(\gamma) \cap \xi(\partial\Gamma) \subset \xi\big(\mathcal{C}^c_\infty(\gamma)\big).$$

Combining both equations one has, for all $i \geqslant L_0$ that

$$B\Big(\xi(x), c_1 e^{-\tau_1(a(\alpha_i))}\Big) \cap \xi(\partial\Gamma) \subset \xi\Big(\alpha_i \mathcal{C}^c_\infty(\alpha_i)\Big) \subset B\Big(\xi(x), K e^{-\tau_1(a(\alpha_i))}\Big), \quad (5.2)$$

the second inclusion following from Proposition 4.9. This concludes the proof. □

5.2. **Hausdorff dimension and related concepts.** Recall that, given a metric space (X,d) and a real number $s>0$, the *s-capacity* of X is

$$\mathcal{H}^s(X,d)=\lim_{\varepsilon\to 0}\inf\left\{\sum_{U\in\mathcal{U}}\operatorname{diam}U^s\,\middle|\,\mathcal{U}\text{ open covering of }\Lambda,\ \sup_{U\in\mathcal{U}}\operatorname{diam}U<\varepsilon\right\}$$

and that

$$\dim_{\mathrm{Hff}}(X)=\inf\{s|\,\mathcal{H}^s(X)=0\}=\sup\{s|\,\mathcal{H}^s(X)=\infty\}. \tag{5.3}$$

The following can be verified directly from the definition:

Lemma 5.7. *If $X=\bigcup_{n\in\mathbb{N}}X_n$ then*

$$\dim_{\mathrm{Hff}}(X)=\sup\dim_{\mathrm{Hff}}(X_n).$$

We will use the following consequence of Theorem 1.5.14 from Edgar's book [19]:

Corollary 5.8. *Let $E\subset\mathbb{R}^d$ be a measurable subset equipped with a probability measure ν. If the upper density*

$$\overline{D}^{\alpha}(x)=\limsup_{r\to 0}\frac{\nu\big(B(x,r)\cap E\big)}{r^{\alpha}}$$

is ν-essentially bounded above, then $\dim_{\mathrm{Hff}}(E)\geqslant\alpha$.

5.3. **The lower bound $\dim_{\mathrm{Hff}}(\mathcal{G}\{\mathscr{b}\text{−conical points}\})\geqslant\mathscr{b}\mathscr{h}^{\infty,\mathscr{b}}$.** We import some tools from the proof of Theorem 4.16. Consider the vector space

$$V^*:=\operatorname{span}\{\tau,\overline{\tau}\}$$

together with its radical $\operatorname{Ann}(V^*)=\ker\tau\cap\ker\overline{\tau}$ and the quotient vector space $V=\mathfrak{a}_\theta/\operatorname{Ann}(V^*)$. Any element of V^* vanishes on $\operatorname{Ann}(V^*)$ and thus V^* is naturally identified with the dual space of V. Using the preferred basis $\{\tau,\overline{\tau}\}$ of V^* we identify V and $\mathbb{R}^2$ via the isomorphism $v\mapsto\big(\tau(v),\overline{\tau}(v)\big)$ and we let

$$\Pi:\mathfrak{a}_\theta\to\mathbb{R}^2$$

be the quotient projection (composed with the above isomorphism). The image of the hyperplane $\ker\mathscr{b}\tau-\overline{\tau}$ under the composition of Π and the identification of V with $\mathbb{R}^2$ is the line passing through $(1,\mathscr{b})$,

$$\Pi\big(\ker(\mathscr{b}\tau-\overline{\tau})\big)=\big\{v\in V:\mathscr{b}\tau(v)=\overline{\tau}(v)\big\}.$$

We consider the quadrant

$$V^+=\{\tau\geqslant 0\}\cap\{\overline{\tau}\geqslant 0\}.$$

Let $\upsilon=\upsilon_{(\rho,\overline{\rho})}:\Gamma\times\partial\Gamma\to V$ be the composition of the refraction cocycle $\beta_{(\rho,\overline{\rho})}$ of the pair with Π. Its periods are

$$\upsilon(\gamma,\gamma_+)=\Big(\tau\big(\lambda(\gamma)\big),\overline{\tau}\big(\lambda(\overline{\gamma})\big)\Big),$$

so by assumption υ is non-arithmetic. As in § 4.4 one has $\Omega_\upsilon=V^*\cap\Omega_{\theta,\rho}$; by non-arithmeticity, the cone $\mathcal{L}_\upsilon$ has non-empty interior and thus Corollary 3.5 gives that Ω_υ is a strictly convex curve. We consider the max norm $\|v\|_{\infty,\mathscr{b}}=\max\{\mathscr{b}|\tau(v)|,|\overline{\tau}(v)|\}$ on V, and its dual (operator) norm on V^* denoted by $\|\ \|^{1,\mathscr{b}}$. Let $\varphi_{\mathscr{b}}^{\infty}\in\Omega_\upsilon$ be the unique form such that

$$\|\varphi_{\mathscr{b}}^{\infty}\|^{1,\mathscr{b}}=\inf\{\|\varphi\|^{1,\mathscr{b}}:\varphi\in\Omega_\upsilon\}.$$

In the following lemma the role of the assumptions on dynamical intersection in Theorem 5.3 becomes clear:

Lemma 5.9. *The functional* $\varphi^\infty_\ell/\|\varphi^\infty_\ell\|^{1,\ell}$ *is a convex combination* $s\ell\tau + (1-s)\overline{\tau}$ *with* $s \in (0,1)$ *if and only if*

$$\mathbf{I}_\tau(\overline{\tau}) > \ell > 1/\mathbf{I}_{\overline{\tau}}(\tau). \tag{5.4}$$

In this case one has $\mathsf{T}_{\varphi^\infty_\ell}\mathcal{Q}_v = \operatorname{span}\{\ell\tau - \overline{\tau}\}$.

Proof. Recall from Corollary 3.5 that $\mathsf{T}_{\hbar_\tau\tau}\mathcal{Q}_v = \ker \mathbf{I}_{\hbar_\tau\tau}$ and $\mathcal{Q}_v$ is strictly convex. Furthermore, by definition the functional φ^∞_ℓ is the point of $\mathcal{Q}_v$, that minimizes the norm $\|\ \|^{1,\ell}$. The level set $\{\|\varphi\|^{1,\ell} = 1\}$ is a rhombus with vertices $(\ell\tau, \overline{\tau})$ (in blue in Figure 6), the tangent to $\mathcal{Q}_v$ at $\hbar_\tau\tau$, in red in Figure 6, is the level set $\mathbf{I}_{\hbar_\tau\tau}(\cdot) = 1$, whence its intersection with the $\overline{\tau}$-axis is $\overline{\tau}/\mathbf{I}_{\hbar_\tau\tau}(\overline{\tau})$, and the the tangent to to $\mathcal{Q}_v$ at $\hbar_{\overline{\tau}}\overline{\tau}$ is the level set $\mathbf{I}_{\hbar_{\overline{\tau}}\overline{\tau}}(\cdot) = 1$, and it intersects the τ-axis is $\tau/\mathbf{I}_{\hbar_{\overline{\tau}}\overline{\tau}}(\tau)$.

Equation (5.4) is thus equivalent to the fact that the slope of the side of the rhombus, equal to $-1/\ell$, is between the slope of the tangent at $\hbar_\tau\tau$, which is equal to $-\hbar_\tau/\mathbf{I}_{\hbar_\tau\tau}(\overline{\tau}) = -1/\mathbf{I}_\tau(\overline{\tau})$, and the slope of the tangent at $\hbar_{\overline{\tau}}\overline{\tau}$, which is equal to $-\mathbf{I}_{\hbar_{\overline{\tau}}\overline{\tau}}(\tau)/\hbar_{\overline{\tau}} = -\mathbf{I}_{\overline{\tau}}(\tau)$.

Strict convexity of $\mathcal{Q}_v$ ensures that this is equivalent to having a unique point in $\mathcal{Q}_v \cap \{t\overline{\tau} : t > 0\} \times \{s\tau : s > 0\}$ tangent to the side of the rhombus, which is the desired functional φ^∞_ℓ. □

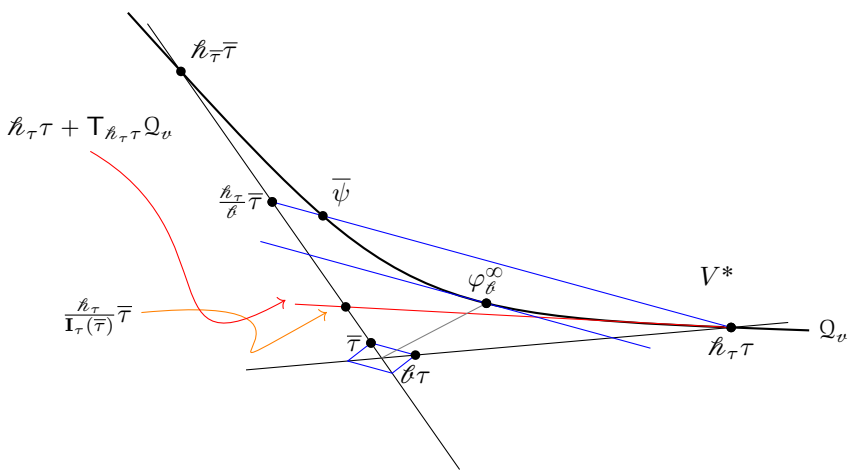

FIGURE 6. The situation of Lemma 5.9.

We thus obtain the following key properties of φ^∞_ℓ:

Lemma 5.10. *Under the assumptions of Theorem 5.3 one has*

(i) $\mathsf{u}^v_{\varphi^\infty_\ell} = \Pi(\ker(\ell\tau - \overline{\tau}))$;

(ii) *for any* $v \in V^+$ *one has*

$$\varphi^\infty_\ell(v) \geqslant \hbar^{\infty,\ell}\ell \min\{\tau(v), \overline{\tau}(v)\}.$$

Moreover one has $\hbar^{\infty,\ell} < \min\{\hbar_{\overline{\tau}}, \hbar_\tau/\ell\}$.

Proof. Lemma 5.9 implies that

(i) $\mathsf{T}_{\varphi^\infty_\ell}\mathcal{Q}_v = \operatorname{span}\{\ell\tau - \overline{\tau}\}$ and thus

$$\mathsf{u}^v_{\varphi^\infty_\ell} = \operatorname{Ann}\big(\mathbb{R}\cdot(\ell\tau - \overline{\tau})\big) = \Pi(\ker(\ell\tau - \overline{\tau})).$$

(ii) $\varphi_\ell^\infty/\|\varphi_\ell^\infty\|^{1,\ell} = s\ell\tau + (1-s)\overline{\tau}$ for some $s \in (0,1)$ and hence[7], since $\ell \in (0,1]$,

$$\varphi_\ell^\infty \geqslant \|\varphi_\ell^\infty\|^{1,\ell}\ell \min\{\tau, \overline{\tau}\}.$$

In order to prove item (ii), we need to show that $\hbar^{\infty,\ell} \leqslant \|\varphi_\ell^\infty\|^{1,\ell}$. Since $\varphi_\ell^\infty\big(a_\theta((\rho,\overline{\rho})\gamma)\big) \leqslant \|\Pi(a_\theta((\rho,\overline{\rho})\gamma))\|_{\infty,\ell}\|\varphi_\ell^\infty\|^{1,\ell}$, we deduce, for all $s > \|\varphi_\ell^\infty\|^{1,\ell}$,

$$\sum_{\gamma\in\Gamma} e^{-s\|\Pi(a_\theta((\rho,\overline{\rho})\gamma))\|_{\infty,\ell}} \leqslant \sum_{\gamma\in\Gamma} e^{-(s/\|\varphi_\ell^\infty\|^{1,\ell})\varphi_\ell^\infty\left(a_\theta((\rho,\overline{\rho})\gamma)\right)} < \infty$$

where last inequality holds as $\hbar^{\varphi_\ell^\infty} = 1$ (by Equation (3.4) and Remark 4.13).

The last assertion follows directly from the definitions:

$$\begin{aligned}\hbar^{\infty,\ell} &= \lim_{t\to\infty}\frac{1}{t}\log\#\big\{\gamma\in\Gamma : \max\big\{\ell\tau(a(\gamma)), \overline{\tau}(a(\overline{\gamma}))\big\} \leqslant t\big\}\\ &\leqslant \lim_{t\to\infty}\frac{1}{t}\log\#\big\{\gamma\in\Gamma : \ell\tau(a(\gamma))\big\} \leqslant t\big\} = \hbar^\tau/\ell = \hbar_\tau/\ell,\end{aligned}$$

where the last equality follows from Remark 4.13. The inequality $\hbar^{\infty,\ell} \leqslant \hbar_{\overline{\tau}}$ is analogous. □

Let $\mu^{\varphi_\ell^\infty}$ be the Patterson-Sullivan measure associated to φ_ℓ^∞ by Corollary 4.14. Combining Equation (2.8), Equation (4.1) and Corollary 4.14 we deduce that, for every $\gamma \in \Gamma$,

$$\mu^{\varphi_\ell^\infty}\big(\gamma\mathcal{C}_\infty^c(\gamma)\big) \leqslant Ce^{-\varphi_\ell^\infty\left(a_\theta((\rho,\overline{\rho})\gamma)\right)} \leqslant Ce^{-\hbar^{\infty,\ell}\ell\min\left\{\tau(a(\gamma)),\overline{\tau}(a(\overline{\gamma}))\right\}}, \tag{5.5}$$

where the last inequality comes from Lemma 5.10.

By Proposition 5.6 there exist constants c, k_1 and $\overline{k}_1$ such that if $(\alpha_i)_0^\infty$ is a geodesic ray from id to x then for all i the subsets

$$\xi\big(\alpha_i\mathcal{C}_\infty^c(\alpha_i)\big) \text{ and } \overline{\xi}\big(\alpha_i\mathcal{C}_\infty^c(\alpha_i)\big)$$

contain balls on the corresponding projective spaces of radi

$$k_1e^{-\tau(a(\alpha_i))} \text{ and } \overline{k}_1e^{-\overline{\tau}(a(\overline{\alpha}_i))}$$

respectively where $k_1, \overline{k}_1$ depend on the representations but not on i. Since $\mathcal{G}(\partial\Gamma)$ is a graph, the preceding radius computation implies that the image of the cone type $\mathcal{G}\big(\alpha_i\mathcal{C}_\infty^c(\alpha_i)\big)$ contains the intersection of $\partial\Gamma \times \partial\overline{\Gamma}$ with a ball, for the product metric on $\mathbb{P}(\mathbb{K}^d) \times \mathbb{P}(\mathbb{K}^{\overline{d}})$, of radius

$$ke^{-\min\left\{\tau(a(\alpha_i)),\overline{\tau}(a(\overline{\alpha}_i))\right\}}, \tag{5.6}$$

for some uniform constant k. This set of balls forms a fine set of neighbourhoods around any point $x \in \partial\Gamma$. Combining this with Equation (5.5) and the fact that $\mu^{\varphi_\ell^\infty}$ is supported on $\partial\Gamma$, one has, possibly enlarging the constant C, that for all r the measure of the ball of radius r about $\mathcal{G}(x)$ is

$$\mu^{\varphi_\ell^\infty}\big(B(x,r)\big) \leqslant Cr^{-\hbar^{\infty,\ell}\ell}.$$

[7]Indeed, if $x, y \geqslant 0$, $s \in (0,1)$ and $\ell \in (0,1]$ one has: $s\ell x + (1-s)y \geqslant \ell\min(x,y)$: Assume for example that $y \geqslant x$ (the other case follows smilarly), then

$$s\ell x + (1-s)y - \ell x \geqslant (1-s)(1-\ell)x \geqslant 0.$$

Since $\dim V^* = 2$ and $\upsilon_{(\rho,\overline{\rho})}$ is assumed non-arithmetic, Theorem 4.16 states that the subset of $\mathscr{b}$-conical points has full $\mu^{\varphi_{\mathscr{b}}^{\infty}}$ measure. Applying Corollary 5.8 one concludes that

$$\dim_{\mathrm{Hff}}\big(\mathscr{G}\{\mathscr{b}-\text{conical points}\}\big) \geqslant \mathscr{b}\mathscr{h}^{\infty,\mathscr{b}}.$$

5.4. **The upper bound.** We now prove the second inequality.

Proposition 5.11. *Let $\rho,\overline{\rho}$ be locally conformal representations over $\mathbb{K}$ and $\overline{\mathbb{K}}$. For every $\mathscr{b}\leqslant 1$,*

$$\dim_{\mathrm{Hff}}\big(\mathscr{G}\{\mathscr{b}-\text{conical points}\}\big) \leqslant \min\{\mathscr{h}^{\infty,\mathscr{b}}, \mathscr{b}\mathscr{h}^{\infty,\mathscr{b}}+(1-\mathscr{b})\}.$$

Proof. We say that a point x is $(R,\mathscr{b})$-*conical* if there exists a geodesic ray $(\alpha_i)_{i\in\mathbb{N}}$ converging to x and such that for an infinite subset $\mathbb{I}\subset\mathbb{N}$ of indices and for every $k\in\mathbb{I}$

$$\Big|\mathscr{b}\tau\big(a(\alpha_k)\big)-\overline{\tau}\big(a(\overline{\alpha}_k)\big)\Big| \leqslant R. \tag{5.7}$$

We denote by $\mathsf{C}_{\mathscr{b}}^R$ the set of $(R,\mathscr{b})$-conical points. By Lemma 4.22 one has

$$\bigcup_{R>0}\mathsf{C}_{\mathscr{b}}^R = \{x\in\partial\Gamma : x \text{ is } \mathscr{b}-\text{conical}\},$$

and thus by Lemma 5.7 it suffices to show that for every R one has

$$\dim_{\mathrm{Hff}}\big(\mathsf{C}_{\mathscr{b}}^R\big) \leqslant \mathscr{h}^{\infty}.$$

For any constant $K>0$ and any $\gamma\in\Gamma$ we denote by $B_\gamma^{\max,K}$ the open ball of $\mathbb{P}(\mathbb{K}^d)\times\mathbb{P}(\mathbb{K}^{\overline{d}})$ given by:

$$B_\gamma^{\max,K} := B\Big(\big(U_1(\gamma),U_1(\overline{\gamma})\big), Ke^{-\max\{\mathscr{b}\tau(a(\gamma)),\overline{\tau}(a(\overline{\gamma}))\}}\Big),$$

and denote by

$$\mathcal{U}_T^K := \big\{B_\gamma^{\max,K} \,|\, |\gamma|\geqslant T\big\}.$$

Let K, resp. $\overline{K}$, be the constants given by Proposition 4.9 for the representation ρ (resp. $\overline{\rho}$).

We first observe that for $C=2e^R\max\{K,\overline{K}\}$ and every $T>0$, the set $\mathcal{U}_T^C$ covers $\mathscr{G}(\mathsf{C}_{\mathscr{b}}^R)$. Indeed, if $x\in\mathsf{C}_{\mathscr{b}}^R$ consider the geodesic ray $(\alpha_i)_{i\in\mathbb{N}}$ converging to x, and the set $\mathbb{I}$ of indices for which Equation (5.7) holds. Then for every $k\in\mathbb{I}$ one has, since $\mathscr{b}\leqslant 1$, that

$$\tau\big(a(\rho\alpha_k)\big) \geqslant \mathscr{b}\tau\big(a(\alpha_k)\big) > \max\big\{\mathscr{b}\tau\big(a(\alpha_k)\big),\overline{\tau}\big(a(\overline{\alpha}_k)\big)\big\}-R, \tag{5.8}$$

$$\overline{\tau}\big(a(\overline{\alpha}_k)\big) > \max\big\{\mathscr{b}\tau\big(a(\alpha_k)\big),\overline{\tau}\big(a(\overline{\alpha}_k)\big)\big\}-R. \tag{5.9}$$

Let now T be fixed and choose $k\in\mathbb{I}$, $k>T$. Since $x\in\alpha_k\mathcal{C}_\infty^c(\alpha_k)$, Proposition 4.9 together with Equation (5.9) give

$$d\big(\xi(x),U_1(\alpha_k)\big) \leqslant Ce^{-\max\big\{\mathscr{b}\tau\big(a(\alpha_k)\big),\overline{\tau}\big(a(\overline{\alpha}_k)\big)\big\}}$$

$$d\big(\overline{\xi}(x),U_1(\overline{\alpha}_k)\big) \leqslant Ce^{-\max\big\{\mathscr{b}\tau\big(a(\alpha_k)\big),\overline{\tau}\big(a(\overline{\alpha}_k)\big)\big\}},$$

as desired.

Furthermore, by definition of $\mathscr{h}^{\infty,\mathscr{b}}$, for every $s>\mathscr{h}^{\infty,\mathscr{b}}$,

$$\sum_{U\in\mathcal{U}_T^C}\operatorname{diam}U^s \leqslant 2^sC^s\sum_{|\gamma|\geqslant T}e^{-s\max\{\mathscr{b}\tau(a(\gamma)),\overline{\tau}(a(\overline{\gamma}))\}} < +\infty,$$

whence, Equation (5.3) yields $\dim_{\mathrm{Hff}}(\mathsf{C}^R_\ell) \leqslant \hbar^{\infty,\ell}$. In order to obtain the second upper bound we observe that, if $\alpha \in \Gamma$ satisfies Equation (5.7), the set $\mathscr{G}(\alpha\mathcal{C}^c_\infty(\alpha))$ can be covered with $e^{(1-\ell)\tau(a(\alpha))}$ balls of radius $2Ce^{-\tau(a(\alpha))}$. We denote by $\mathcal{U}_T$ the collection of open balls, that only take into account elements $\alpha \in \Gamma$ with $|\alpha| > T$ that verify (5.7), which in particular covers the set C^R_ℓ. Using Equation (5.8) we obtain

$$\begin{aligned}\sum_{U\in\mathcal{U}_T} \operatorname{diam} U^s &\leqslant 2^s C^s \sum_{|\gamma|\geqslant T} e^{(1-\ell)\tau(a(\gamma))} e^{-s\tau(a(\gamma))}\\ &\leqslant 2^s C^s \sum_{|\gamma|\geqslant T} e^{-(s-(1-\ell))\tau(a(\gamma))}\\ &\leqslant 2^s C^s e^{\frac{R(s-1+\ell)}{\ell}} \sum_{|\gamma|\geqslant T} e^{-\frac{(s-(1-\ell))}{\ell}\max\{\ell\tau(a(\gamma)),\overline{\tau}(a(\overline{\gamma}))\}}.\end{aligned}$$

Since the latter quantity is finite whenever $\frac{(s-(1-\ell))}{\ell} > \hbar^{\infty,\ell}$, we deduce

$$\dim_{\mathrm{Hff}}(\mathsf{C}^R_\ell) < \ell\hbar^{\infty,\ell} + (1-\ell).$$

□

We conclude this subsection computing the Hausdorff dimension of the image of the whole boundary through the graph map. See [17] for examples of homeomorphisms between Cantor sets for which the Hausdorff dimension of the graph exceeds the maximal Hausdorff dimension of the factors.

Proposition 5.12. *Let $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{K})$, $\overline{\rho} : \Gamma \to \mathsf{SL}(\overline{d},\overline{\mathbb{K}})$ be locally conformal. Then*

$$\dim_{\mathrm{Hff}}(\mathscr{G}(\partial\Gamma)) = \max\{\hbar^{\tau}, \hbar^{\overline{\tau}}\}$$

Proof. This follows as in the proof of Proposition 5.11 considering the covers of $\mathscr{G}(\partial\Gamma)$ given by $\mathcal{U}^C_T := \left\{B^{\min,C}_\gamma \,|\, |\gamma| \geqslant T\right\}$ with

$$B^{\min,K}_\gamma := B\Big(\big(U_1(\gamma), U_1(\overline{\gamma})\big), Ke^{-\min\{\tau(a(\gamma)),\overline{\tau}(a(\overline{\gamma}))\}}\Big),$$

and $C = 2\max\{K,\overline{K}\}$ where K (resp. $\overline{K}$) is the constant given by Proposition 4.9 for the representations ρ (resp. $\overline{\rho}$). To conclude it is enough to observe that

$$\hbar^{\min\{\tau,\overline{\tau}\}} = \max\{\hbar^{\tau}, \hbar^{\overline{\tau}}\},$$

a fact proven for example in P.-S.-Wienhard [42, Lemma 5.1]. □

It is easy to generalize Proposition 5.12 to an arbitrary number of factors. as an application we get.

Corollary 5.13. *Let $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{K})$ and $\theta \subset \Delta$ be such that for all $\tau_i \in \theta$, $\Phi_{\tau_i} \circ \rho$ is $(1,1,2)$-hyperconvex. Then*

$$\dim_{\mathrm{Hff}}(\xi^\theta_\rho(\partial\Gamma)) = \max_{\tau_i\in\theta} \hbar^{\tau_i}.$$

5.5. **Proof of Theorem 5.3.** The first inequality is established in § 5.3, the second inequality is proven in Proposition 5.11, the third inequality follows from Lemma 5.10 and the fourth from Theorem 5.4.The last equality was stablished in Proposition 5.12.

6. $\mathscr{b}$-concavity and $\mathscr{b}$-conicality: Final steps for the proof of Theorem A

The goal of this section is to prove the following more general version of Theorem A. As before, fix $\{\mathbb{K},\overline{\mathbb{K}}\}\subset\{\mathbb{R},\mathbb{C},\mathbb{H}\}$ together with locally conformal representations $\rho:\Gamma\to\mathsf{SL}(d,\mathbb{K})$ and $\overline{\rho}:\Gamma\to\mathsf{SL}(\overline{d},\overline{\mathbb{K}})$ of an arbitrary word-hyperbolic group Γ. For $\mathscr{b}\in(0,1]$ recall that $\Xi:\xi(\partial\Gamma)\to\overline{\xi}(\partial\Gamma)$ is *$\mathscr{b}$-concave* at $x\in\partial\Gamma$ if there exists $y_k\to x$ such that the incremental quotients

$$\frac{d_{\mathbb{P}}\big(\overline{\xi}(x),\overline{\xi}(y_k)\big)}{d_{\mathbb{P}}\big(\xi(x),\xi(y_k)\big)^{\mathscr{b}}} \tag{6.1}$$

are bounded away from 0 and ∞ (independently of k). We also let $\mathcal{H}^{\mathscr{b}}_{(\rho,\overline{\rho})}$ be the set of $x\in\partial\Gamma$ that are $\mathscr{b}$-concavity points of Ξ. Finally, recall that ρ and $\overline{\rho}$ are *not gap-isospectral* if there exists $\gamma\in\Gamma$ such that $\tau(a(\gamma))\neq\overline{\tau}(a(\overline{\gamma}))$.

Theorem 6.1. *Let $\rho,\overline{\rho}$ be locally conformal representations acting irreducibly, on $\mathbb{K}^d$ and $\overline{\mathbb{K}}^{\overline{d}}$ respectively, as real vector spaces, and that are not gap-isospectral. Consider any $\mathscr{b}\in(0,1]$ with $\mathbf{I}_\tau(\overline{\tau})>\mathscr{b}>(\mathbf{I}_{\overline{\tau}}(\tau))^{-1}$, then*

- *if $\{\mathbb{K},\overline{\mathbb{K}}\}\subset\{\mathbb{R},\mathbb{C}\}$ one has*

$$\begin{aligned}\mathscr{b}\hbar^{\infty,\mathscr{b}}\leqslant\dim_{\mathrm{Hff}}(\mathcal{H}^{\mathscr{b}}_{\rho,\overline{\rho}})&\leqslant\min\{\hbar^{\infty,\mathscr{b}},\mathscr{b}\hbar^{\infty,\mathscr{b}}+(1-\mathscr{b})\}\\&<\min\{\hbar_{\overline{\tau}},\hbar_\tau/\mathscr{b}\}\\&\leqslant\dim_{\mathrm{Hff}}(\mathscr{G}(\partial\Gamma))\qquad(6.2)\\&=\max\{\hbar_\tau,\hbar_{\overline{\tau}}\};\qquad(6.3)\end{aligned}$$

- *if $\mathbb{K}=\mathbb{H}$ (resp. $\overline{\mathbb{K}}=\mathbb{H}$), Equation (6.2) holds if we further assume that the real Zariski closure of $\rho(\Gamma)$ (resp. of $\overline{\rho}(\Gamma)$) does not have compact factors.*

6.1. Hyperplane conicality and the concavity condition.

We commence with a lemma relating $\mathscr{b}$-conicality to the desired concavity properties of the equivariant map $\Xi:\xi(\partial\Gamma)\to\overline{\xi}(\partial\Gamma)$.

Lemma 6.2. *Let ρ and $\overline{\rho}$ be locally conformal representations over $\mathbb{K}$ and $\overline{\mathbb{K}}$ respectively, and $\mathscr{b}\in(0,1]$. Then one has $\{\mathscr{b}-\text{conical points of }(\rho,\overline{\rho})\}=\mathcal{H}^{\mathscr{b}}_{\rho,\overline{\rho}}$.*

Proof. Let $(\alpha_i)_{i\in\mathbb{N}}$ denote a geodesic ray converging to x. Proposition 5.6 gives constants $C_1,C_2,\overline{C}_1,\overline{C}_2$ and $L\in\mathbb{N}$ such that, for every $n\in\mathbb{N}$ and every $y_n\in\alpha_n\mathcal{C}^c_\infty(\alpha_n)\backslash\alpha_{n+L}\mathcal{C}^c_\infty(\alpha_{n+L})$, it holds

$$\begin{aligned}C_1e^{-\tau(a(\alpha_n))}<d_{\mathbb{P}}\big(\xi(y_n),\xi(x)\big)&<C_2e^{-\tau(a(\alpha_n))},\\\overline{C}_1e^{-\overline{\tau}(a(\overline{\alpha}_n))}<d_{\mathbb{P}}\big(\overline{\xi}(y_n),\overline{\xi}(x)\big)&<\overline{C}_2e^{-\overline{\tau}(a(\overline{\alpha}_n))}.\end{aligned}\tag{6.4}$$

Assume first that x is $\mathscr{b}$-conical. By Definition 5.2 we obtain a geodesic ray $(\alpha_i)_0^\infty$, an infinite set of indices $\mathbb{I}\subset\mathbb{N}$ and a number R, such that for all $k\in\mathbb{I}$ one has

$$|\mathscr{b}\tau(a(\alpha_k))-\overline{\tau}(a(\overline{\alpha}_k))|<R. \tag{6.5}$$

For each such k we choose a point $y_k\in\alpha_k\mathcal{C}^c_\infty(\alpha_k)\backslash\alpha_{k+L}\mathcal{C}^c_\infty(\alpha_{k+L})$. By construction y_k converges to x. Combining both equations, for every $k\in\mathbb{I}$ it holds

$$e^{-R}\frac{\overline{C}_1}{{C_2}^{\mathscr{b}}}\leqslant\frac{d_{\mathbb{P}}\big(\overline{\xi}(y_k),\overline{\xi}(x)\big)}{d_{\mathbb{P}}\big(\xi(y_k),\xi(x)\big)^{\mathscr{b}}}\leqslant e^{R}\frac{\overline{C}_2}{{C_1}^{\mathscr{b}}},$$

so the incremental quotient (6.1) is uniformly far from 0 and ∞. Whence $\{\mathscr{E}-\text{conical points}\} \subset \mathcal{H}^{\mathscr{E}}_{\rho,\overline{\rho}}$.

Conversely, assume that x is not $\mathscr{E}$-conical. The Cartan projections of two consecutive elements α_i and α_{i+1} make uniformly bounded gaps (Proposition 2.1), and thus there exists C such that for all $n \in \mathbb{N}$ one has

$$\left|\tau\big(a(\alpha_{n+1})\big) - \tau\big(a(\alpha_n)\big)\right| < C.$$

As a consequence, we can assume, up to switching the roles of ρ and $\overline{\rho}$, that for any R there exists n_R such that for every $n > n_R$ one has

$$\mathscr{E}\tau\big(a(\alpha_n)\big) - \overline{\tau}\big(a(\overline{\alpha}_n)\big) > R.$$

In turn this implies, thanks to Equation (6.4), that for every $y \in \alpha_{n_R}\mathcal{C}^c_\infty(\alpha_{n_R})$,

$$\frac{d_{\mathbb{P}}\big(\overline{\xi}(y), \overline{\xi}(x)\big)}{d_{\mathbb{P}}\big(\xi(y), \xi(x)\big)^{\mathscr{E}}} \leqslant e^{-R}\frac{\overline{C}_2}{{C_1}^{\mathscr{E}}}.$$

Since R is arbitrary, and the sets $\alpha_{n_R}\mathcal{C}^c_\infty(\alpha_{n_R})$ form a system of neighborhoods of the point x, we deduce that the limit in Equation (6.1) exists and equals 0. This concludes the proof. □

6.2. **Non-arithmeticity of periods.** In this section we establish a non-arithmeticity condition, necessary to apply later Theorem 5.3. This is established in a rather general setting. Recall that a subgroup $\Lambda < \mathsf{SL}(d, \mathbb{K})$ is $\mathbb{K}$-*proximal* if it contains a $\mathbb{K}$-proximal element, i.e. there exists $g \in \Lambda$ such that $\tau_1(\lambda(g)) > 0$.

Proposition 6.3. *Let Λ be a finitely generated group. Let $\rho : \Lambda \to \mathsf{SL}(d, \mathbb{K})$ and $\overline{\rho} : \Lambda \to \mathsf{SL}(\overline{d}, \overline{\mathbb{K}})$ be two $\mathbb{K}$-proximal representations that act irreducibly on $\mathbb{K}^d$ and $\overline{\mathbb{K}}^{\overline{d}}$ respectively, as real vector spaces. Assume there exists $\gamma \in \Lambda$ such that $\tau_1(\lambda(\rho\gamma)) \neq \overline{\tau}_1(\lambda(\overline{\rho}\gamma))$. If $\{\mathbb{K}, \overline{\mathbb{K}}\} \subset \{\mathbb{R}, \mathbb{C}\}$, then the group generated by the pairs*

$$\Big\{\Big(\tau_1\big(\lambda(\rho\gamma)\big), \overline{\tau}_1\big(\lambda(\overline{\rho}\gamma)\big)\Big) : \gamma \in \Lambda\Big\}$$

is dense in $\mathbb{R}^2$. If $\mathbb{K} = \mathbb{H}$ we further assume that the Zariski closure over $\mathbb{R}$ of $\rho(\Lambda)$ has no compact factors, and the same for $\overline{\rho}(\Lambda)$ if moreover $\overline{\mathbb{K}} = \mathbb{H}$, then the same conclusion holds.

To prove the proposition we need Lemmas 6.4 and 6.5 below.

Lemma 6.4. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $\Lambda < \mathsf{SL}(d, \mathbb{K})$ be a subgroup acting irreducibly on $\mathbb{K}^d$ as a real vector space and assume Λ contains a $\mathbb{K}$-proximal element. Then the real Zariski closure of Λ is semi-simple, has finite center and without compact factors.*

Proof. If $\mathbb{K} = \mathbb{R}$ the Lemma is the content of S. [48, Lemma 8.6] and the proof over $\mathbb{C}$ is a slight modification of the latter. Indeed, let G be the Zariski closure of $\rho(\Lambda)$ over the reals, by the irreducibility assumption it is a reductive (real-algebraic) group. By Schur's Lemma the elements commuting with Λ consist only on homotheties, but since we're in special linear group one has that the center of G is finite.

The group G is then semi-simple and we let K be the identity component of the product of all the compact simple factors of G. We also let H be the identity component of the product of all the non-compact simple factors of G. The groups H and K commute and one has HK has finite index in G.

Consider a proximal $g \in \mathsf{G}$, up to a fixed power we may write $g = kh$ with $k \in K$ and $h \in H$. Since K is compact, its eigenvalues have modulus one so we conclude that h is proximal and that $g_+ = h_+$. The attracting line of h is thus invariant under K. Since K is connected, an element of K acts on h_+ as multiplication by some element of $\mathbb{S}^1$.

By irreducibility we may find a basis of $\mathbb{C}^d$ consisting on fixed attracting lines of proximal elements of H. This basis simultaneously diagonalizes K, so we get an injective map from K to a compact group isomorphic to a d-dimensional torus. Consequently K is abelian, and since it commutes with H we conclude that K is contained in the identity component of the center of G, which we proved earlier to be trivial. □

Lemma 6.5. *Let G be a semi-simple real-algebraic Lie group with finite center and no compact factors. Fix $\vartheta, \overline{\vartheta} \subset \Delta_G$ two non-empty subsets with $\vartheta \cap \overline{\vartheta} = \varnothing$. Let Λ be a group and $\varkappa : \Lambda \to G$ a representation with Zariski-dense image. Then, for every closed cone with non-empty interior $\mathscr{C} \subset \operatorname{int} \mathcal{L}_{\varkappa(\Lambda)}$, the group spanned by the pairs*

$$\Big\{ \big(\min_{\sigma \in \vartheta} \sigma\big(\lambda(\varkappa g)\big), \min_{\overline{\sigma} \in \overline{\vartheta}} \overline{\sigma}\big(\lambda(\varkappa g)\big)\big) : g \in \Lambda \text{ and } \lambda(\varkappa g) \in \mathscr{C} \Big\}$$

is dense in $\mathbb{R}^2$.

Proof. Define the piecewise linear maps $\tau, \overline{\tau} : \mathfrak{a}^+ \to \mathbb{R}$ by:

$$\tau(v) = \min \big\{ \sigma(v) : \sigma \in \vartheta \big\}$$
$$\overline{\tau}(v) = \min \big\{ \overline{\sigma}(v) : \overline{\sigma} \in \overline{\vartheta} \big\}.$$

The vanishing set of the difference $\tau - \overline{\tau}$ is contained the union of $\ker(\mathsf{a} - \mathsf{b})$ for arbitrary $\mathsf{a} \in \vartheta$ and $\mathsf{b} \in \overline{\vartheta}$. Since ϑ and $\overline{\vartheta}$ are disjoint, this is a union of hyperplanes of $\mathfrak{a}$, from which we deduce that the set of zeroes of $\tau - \overline{\tau}$ has empty interior.

Since $\mathscr{C} \subset \operatorname{int} \mathcal{L}_{\varkappa(\Lambda)}$ has non-empty interior, the difference $\tau - \overline{\tau}$ does not identically vanish on $\mathscr{C}$. Since τ and $\overline{\tau}$ are piecewise linear, we can choose a possibly smaller closed cone with non-empty interior

$$\mathscr{C}' \subset \mathscr{C},$$

and $\mathsf{a} \in \vartheta$, $\overline{\mathsf{b}} \in \overline{\vartheta}$ such that for all $v \in \mathscr{C}'$ one has

$$\tau \times \overline{\tau}(v) := (\tau(v), \overline{\tau}(v)) = (\mathsf{a}(v), \overline{\mathsf{b}}(v)).$$

Since a and b are distinct simple roots the map $(\mathsf{a}, \mathsf{b}) : \mathfrak{a} \to \mathbb{R}^2$ is surjective.

By Benoist [3, Proposition 5.1] there exists a sub-semigroup $\Lambda' < \Lambda$ such that $\varkappa(\Lambda')$ is a Zariski-dense Schottky semi-group with $\mathcal{L}_{\varkappa(\Lambda')} = \mathscr{C}'$. In particular, for all $\gamma \in \Lambda'$ one has

$$\tau \times \overline{\tau}\big(\lambda(\varkappa\gamma)\big) = \big(\mathsf{a}\big(\lambda(\varkappa\gamma)\big), \overline{\mathsf{b}}\big(\lambda(\varkappa\gamma)\big)\big).$$

By Benoist's Theorem 2.3, stating that the group generated by the Jordan projections $\lambda(\varkappa\gamma)$, for $\gamma \in \Lambda'$, is dense in $\mathfrak{a}$, we conclude that the group spanned by

$$\Big\{ \Big(\big(\mathsf{a}\big(\lambda(\varkappa\gamma)\big), \overline{\mathsf{b}}\big(\lambda(\varkappa\gamma)\big)\big) \Big) : \gamma \in \Lambda' \Big\}$$

is dense in $\mathbb{R}^2$, giving in turn the desired conclusion. □

Proof of Proposition 6.3. Denote by G and $\overline{\mathsf{G}}$ the Zariski closures of $\rho(\Lambda)$ and $\overline{\rho}(\Lambda)$ respectively. Both G and $\overline{\mathsf{G}}$ are semi-simple, have finite center, and don't have compact factors: if $\{\mathbb{K}, \overline{\mathbb{K}}\} \subset \{\mathbb{R}, \mathbb{C}\}$ then this is the content of Lemma 6.4, if either $\mathbb{K}$ and/or $\overline{\mathbb{K}}$ equals $\mathbb{H}$ then this is an assumption. We let $\iota : \Lambda \to \mathsf{G}$ and $\overline{\iota} : \Lambda \to \overline{\mathsf{G}}$ be the respective inclusions.

If we let $\phi : \mathsf{G} \to \mathsf{SL}(d, \mathbb{K})$ and $\overline{\phi} : \overline{\mathsf{G}} \to \mathsf{SL}(\overline{d}, \overline{\mathbb{K}})$ be the associated real representations, so that $\rho = \phi \circ \iota$ and $\overline{\rho} = \overline{\phi} \circ \overline{\iota}$, we have from §2.3 two subsets of simple roots $\theta := \theta_\phi$ and $\overline{\theta} := \theta_{\overline{\phi}}$ such that for all $a \in \mathfrak{a}_{\mathsf{G}}^+$ and $b \in \mathfrak{a}_{\overline{\mathsf{G}}}^+$ one has

$$\begin{aligned} \tau(a) &:= \tau_1(\phi(a)) = \min\left\{\mathsf{a}(a) : \mathsf{a} \in \theta\right\} \\ \overline{\tau}(b) &:= \overline{\tau}_1(\overline{\phi}(b)) = \min\left\{\overline{\mathsf{a}}(b) : \overline{\mathsf{a}} \in \overline{\theta}\right\}. \end{aligned} \tag{6.6}$$

In particular, for every $\gamma \in \Lambda$ one has $\tau_1(\lambda(\gamma)) = \tau(\lambda_{\mathsf{G}}(\iota\gamma))$, and similarly for $\overline{\rho}$.

Since ϕ and $\overline{\phi}$ are faithful, θ and $\overline{\theta}$ contain at least one root of each factor of, respectively, G and $\overline{\mathsf{G}}$. If $\vartheta \subset \theta$ then we let

$$\tau^{\vartheta}(v) = \min_{\sigma \in \vartheta} \sigma(v), \ \ v \in \mathfrak{a}_{\mathsf{G}}.$$

If H is a non-trivial product of simple factors of G then we let $\iota_{\mathsf{H}} : \Lambda \to \mathsf{H}$ be the composition of ι with the projection of G onto H. By Zarisk-density of $\iota(\Lambda)$, each representation ι_{H} has Zariski-dense image (though unlikely to be discrete). We also let

$$\theta_{\mathsf{H}} = \theta \cap \Delta_{\mathsf{H}}.$$

Each θ_{H} is non-empty. We analogously define $\overline{\iota}_{\overline{\mathsf{H}}}$, $\theta^{\overline{\mathsf{H}}}$ and $\overline{\tau}^{\overline{\mathsf{H}}}$.

We now let L be the largest product of simple factors, simultaneously of G and $\overline{\mathsf{G}}$, so that ι_{L} is conjugated (up to finite index) to $\overline{\iota}_{\mathsf{L}}$. Let H and $\overline{\mathsf{H}}$ be the remaining factors of G and $\overline{\mathsf{G}}$ respectively, i.e.

$$\mathsf{G} = \mathsf{L} \times \mathsf{H} \text{ and } \overline{\mathsf{G}} = \mathsf{L} \times \overline{\mathsf{H}},$$

and moreover, by definition of L, the representation $\varkappa : \Lambda \to \mathsf{L} \times \overline{\mathsf{H}} \times \mathsf{H}$

$$\varkappa : g \mapsto \left(\iota_{\mathsf{L}}(g), \overline{\iota}_{\overline{\mathsf{H}}}(g), \iota_{\mathsf{H}}(g)\right) \tag{6.7}$$

has Zariski-dense image, see for example Bridgeman-Canary-Labourie-S. [11, Corollary 11.6]. We remark that we are not assuming that any of L, $\overline{\mathsf{H}}$ or H is non-trivial (they can't, of course, be all trivial).

If $(u, v, w) \in \mathfrak{a}_{\mathsf{L}} \times \mathfrak{a}_{\overline{\mathsf{H}}} \times \mathfrak{a}_{\mathsf{H}}$ we naturally think of (u, v) as an element of $\mathfrak{a}_{\overline{\mathsf{G}}}$ and of (u, w) as an element of $\mathfrak{a}_{\mathsf{G}}$. We now write

$$\begin{aligned} \Theta &= \theta_{\mathsf{L}} \cap \overline{\theta}_{\mathsf{L}}, \\ \Theta_{\mathsf{L}} &= \theta_{\mathsf{L}} \backslash \Theta, \\ \overline{\Theta}_{\mathsf{L}} &= \overline{\theta}_{\mathsf{L}} \backslash \Theta. \end{aligned}$$

One has, for all $(u, v, w) \in \mathfrak{a}_{\mathsf{L}} \times \mathfrak{a}_{\overline{\mathsf{H}}} \times \mathfrak{a}_{\mathsf{H}}$ that

$$\begin{aligned} \tau(u, w) &= \min\left\{\tau^{\Theta_{\mathsf{L}}}(u), \tau^{\Theta}(u), \tau^{\theta_{\mathsf{H}}}(w)\right\} \\ \overline{\tau}(u, v) &= \min\left\{\tau^{\overline{\Theta}_{\mathsf{L}}}(u), \tau^{\Theta}(u), \overline{\tau}^{\theta_{\overline{\mathsf{H}}}}(v)\right\}. \end{aligned} \tag{6.8}$$

By assumption, there exists $g \in \Lambda$ such that $\rho(g)$ and $\overline{\rho}(g)$ are proximal and $\tau(\lambda_{\mathsf{G}}(\iota g)) \neq \overline{\tau}(\lambda_{\overline{\mathsf{G}}}(\overline{\iota} g))$. Assume, without loss of generality, that

$$\tau(\lambda_{\mathsf{G}}(\iota g)) < \overline{\tau}(\lambda_{\overline{\mathsf{G}}}(\overline{\iota} g)). \tag{6.9}$$

By means of Equations (6.8) we see that in this situation one has

$$\tau^{\Theta_{\mathsf{L}}\cup\theta_{\mathsf{H}}}(\lambda_{\mathsf{G}}(\iota g)) = \tau(\lambda_{\mathsf{G}}(\iota g)) < \overline{\tau}(\lambda_{\overline{\mathsf{G}}}(\overline{\iota} g)),$$

in particular the union $\Theta_{\mathsf{L}} \cup \theta_{\mathsf{H}}$ must be non-empty. Moreover, this strict inequality yields the existence of a small closed cone with non-empty interior $\mathscr{C}_0 \subset \mathcal{L}_\rho \subset \mathfrak{a}_{\mathsf{G}}^+$ about $\mathbb{R}_+\lambda_{\mathsf{G}}(\rho g)$ such that

$$\tau^{\Theta_{\mathsf{L}}\cup\theta_{\mathsf{H}}}(a) = \tau_1(a)\,\forall a \in \mathscr{C}_0. \tag{6.10}$$

Consider now the representation $\mathscr{r} : \Lambda \to \mathsf{L} \times \overline{\mathsf{H}} \times \mathsf{H}$ from (6.7) and a closed cone with non-empty interior $\mathscr{C} \subset \mathcal{L}_{\mathscr{r}(\Lambda)} \subset \mathfrak{a}_{\mathsf{L}}^+ \times \mathfrak{a}_{\overline{\mathsf{H}}}^+ \times \mathfrak{a}_{\mathsf{H}}^+$ whose natural projection onto $\mathfrak{a}_{\mathsf{G}}^+ = \mathfrak{a}_{\mathsf{L}}^+ \times \mathfrak{a}_{\mathsf{H}}^+$ is $\mathscr{C}_0$.

Lemma 6.5 applied to the group $G = \mathsf{L} \times \overline{\mathsf{H}} \times \mathsf{H}$, the representation $\mathscr{r}$, the disjoint non-empty subsets $\vartheta = \Theta_{\mathsf{L}} \cup \theta_{\mathsf{H}}$ and $\overline{\vartheta} = \overline{\theta}_{\mathsf{L}} \cup \theta_{\overline{\mathsf{H}}}$ and the cone $\mathscr{C}$, provides the desired conclusion. □

We conclude with the following Corollary that we don't need but is of independent interest.

Corollary 6.6. *Let $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{K})$ and $\overline{\rho} : \Gamma \to \mathsf{SL}(\overline{d}, \overline{\mathbb{K}})$ be $\mathbb{R}$-irreducible and $\{\tau_1, \tau_2\}$-Anosov and $\{\overline{\tau}_1, \overline{\tau}_2\}$-Anosov respectively. If $\mathbb{K} = \mathbb{H}$ assume moreover the Zariski closure of $\rho(\Gamma)$ does not contain compact factors, and analogously for $\overline{\rho}$. If ρ and $\overline{\rho}$ are not gap-isospectral then*

$$\mathbf{I}_{\overline{\tau}_1}(\tau_1) > \hbar_{\overline{\tau}_1}/\hbar_{\tau_1}.$$

Proof. Since both representations are projective-Anosov they are $\mathbb{K}$-proximal. Proposition 6.3 implies then that, since they are not gap-isospectral, the group spanned by the pairs $\left\{\left(\tau_1(\lambda(\gamma)), \overline{\tau}_1(\lambda(\overline{\gamma}))\right) : \gamma \in \Gamma\right\}$ is dense. Since both representations are also Anosov with respect to 2-dimensional stabilizers, the functionals τ_1 and $\overline{\tau}_1$ lie in the Anosov-Levy space of ρ and $\overline{\rho}$ respectively, we can apply Proposition 3.3 to obtain the desired strict inequality. □

6.3. **Proof of Theorem 6.1.** Theorem 6.1 follows from Proposition 6.3 giving the desired non-arithmecity of periods, Lemma 6.2 identifying the set $\mathcal{H}^{\mathscr{b}}_{\rho,\overline{\rho}}$ with the set of $\mathscr{b}$-conical points of $(\rho, \overline{\rho})$ and Theorem 5.3 computing the Hausdorff dimension of the latter when the periods are non-arithmetic. The last equality is a direct consequence of Proposition 5.12. □

7. Theorem C: Zariski closures of real-hyperconvex surface-group representations

In this section we prove Theorem C giving a preliminary classification of Zariski closures of irreducible real $(1, 1, 2)$-hyperconvex representations of surface groups. For most of the section we work with a pair of $(1, 1, 2)$-hyperconvex representations and eventually reduce the proof of Theorem C to a situation like this; we will crucially use Theorem 1.3.

7.1. **When Ξ has oblique derivative.** We prove here a result of independent interest, albeit possibly known to experts. This subsection only requires § 4.1 and § 4.2 and will be needed not only for Theorem C but also for Theorems B and 8.6.

Either we let Γ have boundary homeomorphic to a circle, either we let it be a Kleinian group. In the first case we let

$$\rho, \overline{\rho} : \Gamma \to \mathrm{Diff}^{1+\nu}(\mathbb{S}^1)$$

be Hölder conjugated to action of Γ on its boundary; if instead $\Gamma < \mathsf{PSL}(2, \mathbb{C})$ is a Kleinian group we let $\rho, \overline{\rho} : \Gamma \to \mathsf{PSL}(2, \mathbb{C})$ be two convex co-compact representations that lie in the same connected component of the subset of the character variety $\mathfrak{X}(\Gamma, \mathsf{PSL}(2, \mathbb{C}))$ consisting of convex cocompact representations.

We let X be either the circle or $\partial\mathbb{H}^3$. To simplify notation we will denote the action of $\gamma \in \Gamma$ on X via ρ by γ, the action via $\overline{\rho}$ by $\overline{\gamma}$, and the limit sets of ρ and $\overline{\rho}$ by $\partial\Gamma, \overline{\partial\Gamma} \subset X$ respectively.

In both situations there exists a Hölder-continuous map

$$\Xi : X \to X$$

conjugating ρ and $\overline{\rho}$. Indeed while in the surface case this holds by definition, in the Kleinian case this is a theorem by Marden [37], see also Anderson's survey [1, page 32]: the equivariant limit map $\Xi : \partial\Gamma \to \partial\Gamma$ conjugating the actions ρ and $\overline{\rho}$ on their respective limit sets extends to a Γ-equivariant, Hölder continuous homeomorphism of the whole Riemann sphere $\partial\mathbb{H}^3$. We study differentiability points of Ξ with oblique derivative.

We let d be either a visual distance on X (in the complex case) or a distance inducing the chosen C^1 structure on the circle $\mathbb{S}^1$.

Definition 7.1. An action ρ admits a *Lipschitz-compatible cover* if there exists a finite open cover $\mathcal{B}$ of X and a map $\Gamma \to \mathcal{B}$, $\gamma \mapsto \mathcal{B}_\infty(\gamma)$ such that

(i) for any $a, b \in \Gamma$ so that $|ab| = |a| + |b|$ one has
  (a) $b\mathcal{B}_\infty(ab) \subset \mathcal{B}_\infty(a)$,
  (b) $\mathcal{B}_\infty(ab) \subset \mathcal{B}_\infty(b)$;

(ii) there exist $\lambda > 0$, C and $L \in \mathbb{N}$ such that if $|\gamma| \geqslant L$ and $x, y \in \mathcal{B}_\infty(\gamma)$ then

$$d(\gamma x, \gamma y) \leqslant Ce^{-|\gamma|\lambda} d(x, y);$$

(iii) there exist constants r_1, r_2 and a function $\tau : \Gamma \to \mathbb{R}$ with $\tau(\gamma) \geqslant \lambda|\gamma|$ such that for every $\gamma \in \Gamma$ and every $x \in \mathcal{C}_\infty(\gamma)$,

$$B(x, r_1 e^{-\tau(\gamma)}) \subset \gamma\mathcal{B}_\infty(\gamma) \subset B(x, r_2 e^{-\tau(\gamma)}).$$

The goal of the subsection is to prove the following result, similar arguments can be found in Guizhen [26] in the context of conjugacies of expanding circle maps.

Proposition 7.2. *Let $\rho, \overline{\rho}$ be as above and assume both admit a Lipschitz compatible cover. If there exists $p \in \partial\Gamma$ such that Ξ has a finite non-vanishing derivative (complex derivative in the Kleinian case) at p then $\Xi|\partial\Gamma$ is bi-Lipschitz.*

We work under the assumptions of Proposition 7.2 and begin its proof with the following lemma. For $\gamma \in \Gamma$ we denote its derivative at $x \in X$ by $\gamma'(x) \in \mathbb{K}$ defined, according our two situations, by

$X = \mathbb{S}^1$: the derivative $\tilde{\gamma}'(\tilde{x})$ of a lift of γ to the universal cover $\mathbb{R}$ of $\mathbb{S}^1$, and a lift $\tilde{x} \in \mathbb{R}$ of x, the number $\tilde{\gamma}'(\tilde{x})$ is independent of these choices;

$X = \partial\mathbb{H}^3$: we fix an arbitrary point $\infty \notin \partial\Gamma$, identify $X - \{\infty\}$ with $\mathbb{K}$ via the stereographic projection and let $\gamma'(x)$ be the standard complex derivative.

Lemma 7.3. *Let $\rho : \Gamma \to \mathrm{Diff}^{1+\nu}(X)$ admit a Lipschitz compatible cover. There exists a constant $\kappa > 0$ and $N \in \mathbb{N}$ such that for all $\gamma \in \Gamma$ with $|\gamma| \geqslant N$ and $x, y \in \mathcal{B}_\infty(\gamma)$ one has*

$$\big|\log|\gamma'(x)| - \log|\gamma'(y)|\big| \leqslant \kappa d(x,y)^\nu.$$

Proof. We consider L from Definition 7.1, so that for every $\eta \in \Gamma$ with $|\eta| \geqslant L$ and $x, y \in \mathcal{B}_\infty(\eta)$ one has

$$d(\eta x, \eta y) \leqslant C e^{-|\eta|\lambda} d(x,y). \tag{7.1}$$

Since the action is $\mathrm{C}^{1+\nu}$ we can find a positive K such that for every β with $|\beta| \leqslant L$ and $u, w \in X$ one has

$$\big|\log|\beta'(u)| - \log|\beta'(w)|\big| \leqslant K d(u,w)^\nu. \tag{7.2}$$

We let then $K' = \max\{K, KC^\nu\}$. We begin by showing, by induction on k, that if $|\gamma| = kL$ then for all $x, y \in \mathcal{B}_\infty(\gamma)$, one has

$$\big|\log|\gamma'(x)| - \log|\gamma'(y)|\big| \leqslant K'\big(\sum_{i=0}^{k-1} e^{-\nu\lambda L i}\big) d(x,y)^\nu. \tag{7.3}$$

Equation (7.2) gives the base case, so assume that the result holds up to $k-1$. We write $\gamma = \beta\eta$ with $|\beta| = L$, $|\eta| = (k-1)L$. By Definition 7.1 (ib) we have

$$\mathcal{B}_\infty(\gamma) \subseteq \mathcal{B}_\infty(\eta). \tag{7.4}$$

Applying the chain rule gives that for every $u \in X$ one has

$$\log|\gamma'(u)| = \log|(\beta)'(\eta u)| + \log|(\beta)'(u)|$$

and thus, when $x, y \in \mathcal{B}_\infty(\gamma)$,

$$\begin{aligned}
\big|\log|\gamma'(x)| - \log|\gamma'(y)|\big| &\leqslant \big|\log|\beta'(\eta x)| - \log|\beta'(\eta y)|\big| + \big|\log|\eta'(x)| - \log|\eta'(y)|\big| \\
&\leqslant K d(\eta x, \eta y)^\nu + K'\big(\sum_{i=0}^{k-2} e^{-\nu\lambda L i}\big) d(x,y)^\nu \quad \text{(by (7.2) and induction)} \\
&\leqslant K C^\nu e^{-|\eta|\nu\lambda} d(x,y)^\nu + K'\big(\sum_{i=0}^{k-2} e^{-\nu\lambda L i}\big) d(x,y)^\nu \quad \text{(by (7.4) and (7.1))}.
\end{aligned}$$

This shows Equation (7.3) which implies that for $\kappa_0 = K'/(1 - e^{-\nu\lambda L})$, every $\gamma \in \Gamma$ whose word-length is an integer multiple of L, and $x, y \in \mathcal{B}_\infty(\gamma)$ one has

$$\big|\log|\gamma'(x)| - \log|\gamma'(y)|\big| \leqslant \kappa_0 d(x,y)^\nu.$$

To conclude the lemma we consider an arbitrary γ with $|\gamma| = mL + t$ and $t < L$. We write $\gamma = \beta\eta$ with $|\beta| = mL$. By Definition 7.1 (ia) it holds

$$\eta\mathcal{B}_\infty(\gamma) \subset \mathcal{B}_\infty(\beta). \tag{7.5}$$

Applying the chain rule gives then

$$\begin{aligned}
\big|\log|\gamma'(x)| - \log|\gamma'(y)|\big| &\leqslant \big|\log|\beta'(\eta x)| - \log|\beta'(\eta y)|\big| + \big|\log|\eta'(x)| - \log|\eta'(y)|\big| \\
&\leqslant \kappa_0 d(\eta x, \eta y)^\nu + K d(x,y)^\nu \qquad \text{(by (7.2) and (7.5))} \\
&\leqslant (\kappa_0 C^\nu e^{-mL\lambda} + K) d(x,y)^\nu \qquad \text{(by (7.1))}
\end{aligned}$$

so taking $\kappa = K + \kappa_0 C^\nu e^{-L\lambda}$ we conclude the proof. □

Proof of Proposition 7.2. Let $p \in \partial\Gamma$ be such that Ξ has a derivative at p that is neither horizontal nor vertical. Fix a geodesic ray $(\alpha_n)_0^\infty$ through the identity with $\alpha_n \to p$. By definition for all n one has $p \in \alpha_n \mathcal{C}_\infty(\alpha_n)$. Without loss of generality we may also assume that

$$p = 0 = \Xi(0)$$

and we may write the derivative as the incremental limit

$$\Xi'(0) = \lim_{y \to 0} \frac{\Xi(y)}{y} \in \mathbb{K} - \{0\}.$$

For each n we let $s_n = r_1 e^{-\tau(\alpha_n)}$, so that by Definition 7.1 (iii),

$$B(0, s_n) \subset \alpha_n \mathcal{B}_\infty(\alpha_n).$$

We consider the scaling map

$$g_n : B(0,1) \to \alpha_n \mathcal{B}_\infty(\alpha_n)$$

defined by $g_n(z) = s_n z$.

Let a_n be an arbitrary point at distance s_n from 0 and let $\tilde{s}_n = \Xi(a_n)$. Observe that since Ξ is differentiable at zero, for n big enough the image $\Xi(B(0, s_n))$ is coarsely a ball around zero of size comparable to that of $\overline{\alpha}_n \mathcal{C}_\infty(\overline{\alpha}_n)$, and in particular we can assume, since the cover $\{\mathcal{B}_\infty(\overline{\gamma})\}$ is Lipschitz compatible (Definition 7.1 (iii)), that $\Xi(B(0, s_n))$ is contained in $\overline{\alpha}_n \mathcal{B}_\infty(\overline{\alpha}_n)$. Furthermore we deduce that there exist positive constants d, D such that for every n

$$d < \frac{\overline{r}_2 e^{-\overline{\tau}(\overline{\alpha}_n)}}{|\tilde{s}_n|} < D.$$

Here we denote by $\overline{r}_i, \overline{\lambda}, \overline{C}, \overline{\tau}$ the constants and function associated to the Lipschitz compatible cover $\{\mathcal{B}_\infty(\overline{\gamma})\}$ for the action $\overline{\rho}$. We consider the scaling map

$$\tilde{g}_n : B(0, D) \to B(0, |\tilde{s}_n| D)$$

by $z \mapsto z\tilde{s}_n$.

Since $s_n \to 0$ and $\Xi'(0) \notin \{0, \infty\}$ exists, the composition

$$\tilde{g}_n^{-1} \Xi g_n(z) = \frac{\Xi(z s_n)}{\tilde{s}_n} \cdot \frac{s_n z}{s_n z} = \frac{\Xi(z s_n)}{s_n z} \cdot \frac{s_n}{\tilde{s}_n} \cdot z = \frac{\Xi(z s_n)}{s_n z} \cdot \frac{s_n}{\Xi(s_n)} \cdot z$$

converges uniformly on compact subsets to the identity map.

On the other hand, one has

$$\tilde{g}_n^{-1} \Xi g_n = \tilde{g}_n^{-1} \overline{\alpha}_n \Xi \alpha_n^{-1} g_n.$$

We now study the maps $f_n := \alpha_n^{-1} \circ g_n$ and $\tilde{f}_n := \tilde{g}_n^{-1} \circ \overline{\alpha}_n$. Since the coverings $\mathcal{B}$ and $\overline{\mathcal{B}}$ are finite, we can assume, up to extracting a subsequence that there exists sets $\mathcal{B}_\infty \in \mathcal{B}$, $\overline{\mathcal{B}}_\infty \in \overline{\mathcal{B}}$ so that, for every n, $\mathcal{B}_\infty(\alpha_n) = \mathcal{B}_\infty$ (resp. $\mathcal{B}_\infty(\overline{\alpha}_n) = \overline{\mathcal{B}}_\infty$).

Observe that for every $x \in B(0,1)$ one has

$$\log |f_n'(x)| = \log |(\alpha_n^{-1})'(g_n x)| + \log |s_n| = -\log |\alpha_n'(\alpha_n^{-1} g_n x)| + \log |s_n|.$$

Now by definition of g_n, we have that $g_n x \in \alpha_n \mathcal{B}_\infty(\alpha_n)$ and thus $\alpha_n^{-1}(g_n x) \in \mathcal{B}_\infty(\alpha_n)$. For n large enough we can apply Lemma 7.3 to α_n to obtain κ so that for every pair $x, y \in B(0,1)$ it holds

$$\big| \log |f_n'(x)| - \log |f_n'(y)| \big| \leqslant \kappa d(x,y)^\nu.$$

We conclude that the family of maps (f_n) is uniformly bi-Lipschitz on $B(0,1)$ and thus, since $(f_n 0)$ is bounded, Arzela-Ascoli's Theorem applies to give a subsequence (still denoted by f_n) that converges to a bi-Lipschitz map f defined on $B(0,1)$.

A similar reasoning applies to the maps $\tilde{f}$ defined on $\overline{\mathcal{B}}_\infty$, and we obtain that, about 0, Ξ can be written as a composition of bi-Lipschitz maps and is thus bi-Lipschitz. Using the action of Γ we extend the Lipschitz property of Ξ to the whole $\partial\Gamma$, concluding the proof. □

The following Lemma guarantees we can later apply the results of this section to the situation of our interest.

Lemma 7.4.

- *Assume $\partial\Gamma$ is homeomorphic to a circle and let $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ be $(1,1,2)$-hyperconvex. Then the induced action of $\rho(\Gamma)$ on the $\mathrm{C}^{1+\nu}$ circle $\xi(\partial\Gamma)$ admits a Lipschitz compatible covering.*
- *If Γ is a convex-co-compact Kleinian group then the action of Γ on $\partial_\infty\mathbb{H}^3$ admits a Lipschitz compatible cover.*

Proof. Recall from Section 4.1 that we have fixed a word metric on Γ and we denote by $\mathcal{C}_\infty(\gamma) \subset \partial\Gamma \subset X$ the set of endpoints of geodesic rays contained in the cone type $\mathcal{C}(\gamma)$.

Let δ_ρ be the fundamental constant of ρ from Definition 4.8, and let $\mathcal{B}_\infty(\gamma) = X_\infty(\gamma)$ be the $\delta_\rho/2$-neighbourhood of $\mathcal{C}_\infty(\gamma)$ inside $\mathbb{S}^1$. This is the thickened cone type at infinity considered in [43, Section 5] (see also the proof of Proposition 5.6). It is a proper subset of $\mathbb{S}^1$ by Corollary 4.7. The cover $\mathcal{B}$ is finite since there are only finitely many cone types [12, p. 455].

Property (i) holds since the same property holds for $\mathcal{C}_\infty(\gamma)$, Property (ii) is a consequence of Proposition 4.10. Finally, Property (iii) was proven in [43, Corollary 5.10] choosing $\tau(\gamma) := \tau_1(a\rho(\gamma))$ (see also the proof of Proposition 5.6). Observe that in the real case by considering $X = \mathbb{S}^1$ we are implicitly considering only the intersection with the limit set, while in the Kleinian group case it is not necessary to intersect with the limit set since the Γ-action on the whole X is conformal. □

We now establish the following corollary that will be used in the sequel.

Corollary 7.5. *Assume $\partial\Gamma$ is homeomorphic to a circle. Let $\rho : \pi_1 S \to \mathsf{PGL}(d,\mathbb{R})$ and $\overline{\rho} : \pi_1 S \to \mathsf{PGL}(\overline{d},\mathbb{R})$ be $(1,1,2)$-hyperconvex, consider the map between $\mathrm{C}^{1+\nu}$ circles*

$$\Xi = \overline{\xi} \circ \xi^{-1} : \xi(\partial\pi_1 S) \to \overline{\xi}(\partial\pi_1 S).$$

If Ξ has a differentiability point with finite non-vanishing derivative then ρ and $\overline{\rho}$ are gap-isospectral.

Proof. By Lemma 7.4 we can apply Proposition 7.2 to obtain that Ξ is bi-Lipschitz. The following standard lemma from linear algebra (see for example Benoist [5] and S. [46, Lemma 3.4]) gives the period computation completing the proof. □

Lemma 7.6. *Let $g \in \mathsf{PGL}(d,\mathbb{R})$ be proximal with attracting point $g_+ \in \mathbb{P}(\mathbb{R}^d)$ and repelling hyperplane $g_- \in \mathbb{P}((\mathbb{R}^d)^*)$. Let $V_{\lambda_2(g)}$ be the sum of the characteristic spaces of g whose associated eigenvalue is of modulus $\exp\lambda_2(g)$, Then for every $v \notin \mathbb{P}(g_-)$, with non-zero component in $V_{\lambda_2(g)}$, one has*

$$\lim_{n\to\infty} \frac{\log d_{\mathbb{P}}(g^n(v), g_+)}{n} = -\tau_1(\lambda(g)).$$

7.2. **Limit curves in non-maximal flags.** We proceed with another intermediate step for the proof of Theorem C describing differentiability points of boundary maps in partial flag manifolds $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}}$ for $\{\mathsf{a},\mathsf{b}\}$-Anosov representations. This step follows from the combination of Theorem 1.3 and Corollary 7.5.

Let G be real-algebraic and semi-simple. Let $\{\mathsf{a},\mathsf{b}\} \subset \Delta$ be two distinct simple roots. The partial flag space $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}}$ carries two transverse foliations that are the level sets of the natural projections $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}} \to \mathcal{F}_{\{\mathsf{a}\}}$ and $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}} \to \mathcal{F}_{\{\mathsf{b}\}}$. We will refer to these as the *canonical foliations* of $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}}$.

Corollary 7.7. *Let G be real-algebraic and semi-simple and let $\{\mathsf{a},\mathsf{b}\} \subset \Delta$ distinct. Let $\rho : \pi_1 S \to \mathsf{G}$ be Zariski-dense and $\{\mathsf{a},\mathsf{b}\}$-Anosov. If both curves $\xi^{\mathsf{a}}(\partial\pi_1 S)$ and $\xi^{\mathsf{b}}(\partial\pi_1 S)$ are C^1 then every differentiability point of $\xi^{\{\mathsf{a},\mathsf{b}\}}(\partial\pi_1 S)$ is tangent to one of the canonical foliations of $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}}$.*

Proof. By Benoist's Theorem 2.3 the limit cone of ρ has non-empty interior, in particular there exists $\gamma \in \pi_1 S$ such that

$$\mathsf{a}(\lambda(\gamma)) \neq \mathsf{b}(\lambda(\gamma)). \tag{7.6}$$

Consider the Tits representations Φ_{a} and Φ_{b} associated to a and b. Since $\rho(\pi_1 S)$ is Zariski-dense, both representation $\Phi_{\mathsf{a}}\rho$ and $\Phi_{\mathsf{b}}\rho$ are irreducible and since ρ is $\{\mathsf{a},\mathsf{b}\}$-Anosov both representation $\Phi_{\mathsf{a}}\rho$ and $\Phi_{\mathsf{b}}\rho$ are projective Anosov. Recall that by definition of Φ_{a}, for every $g \in \mathsf{G}$ one has

$$\tau_1\big(\lambda\big(\Phi_{\mathsf{a}}(g)\big)\big) = \mathsf{a}\big(\lambda(g)\big),$$

so by Equation (7.6) the representations $\Phi_{\mathsf{a}}\rho$ and $\Phi_{\mathsf{b}}\rho$ are not gap-isospectral.

Since the maps ζ_{a} and ζ_{b} are analytic, both projective curves $\zeta_{\mathsf{a}}\xi^{\mathsf{a}}(\partial\pi_1 S)$ and $\zeta_{\mathsf{b}}\xi^{\mathsf{b}}(\partial\pi_1 S)$ are C^1 and thus by Zhang-Zimmer's Theorem 1.3 the representations $\Phi_{\mathsf{a}}\rho$ and $\Phi_{\mathsf{b}}\rho$ are $(1,1,2)$-hyperconvex.

The natural embedding $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}} \to \mathbb{P}(V_{\mathsf{a}}) \times \mathbb{P}(V_{\mathsf{b}})$ sends $\xi^{\{\mathsf{a},\mathsf{b}\}}$ to the graph of the map Ξ from Corollary 7.5 and thus the corollary implies the result. □

7.3. **Proof of Theorem C.** The goal of the section is to prove Theorem C, stating that the Zariski closure G of the image of an irreducible $(1,1,2)$-hyperconvex representation $\rho : \pi_1 S \to \mathsf{PGL}(d,\mathbb{R})$ is simple and the highest weight of the induced representation $\Phi : \mathsf{G} \to \mathsf{PGL}(d,\mathbb{R})$ is a multiple of a fundamental weight associated to a root whose root-space is one-dimensional.

It is known that an irreducible subgroup $\mathsf{G} < \mathsf{PGL}(d,\mathbb{R})$ containing a proximal element is semi-simple without compact factors (see S. [48, Lemma 8.6] for an explicit argument following a suggestion by Quint).

We consider the induced representation $\rho_0 : \Gamma \to \mathsf{G}$ and denote by $\Phi : \mathsf{G} \to \mathsf{PGL}(d,\mathbb{R})$ the linear representation so that $\rho = \Phi\rho_0$. Let $\chi = \chi_\Phi \in \mathfrak{a}^*$ be the highest weight of Φ. As in Definition 2.5 we consider

$$\theta = \theta_\Phi = \{\mathsf{a} \in \Delta : \chi - \mathsf{a} \text{ is a weight of } \Phi\} = \{\mathsf{a} \in \Delta : \langle \chi, \mathsf{a}\rangle \neq 0\}.$$

It is enough to show that θ is reduced to a single root $\{\mathsf{a}_0\}$; indeed, if this is the case, upon writing χ in the basis of fundamental weights $\{\varpi_{\mathsf{a}} : \mathsf{a} \in \Delta\}$ (recall their defining Equation (2.1)) one has

$$\chi = \sum_{\mathsf{a}\in\Delta} \langle \chi, \mathsf{a}\rangle \varpi_{\mathsf{a}} = \langle \chi, \mathsf{a}_0\rangle \varpi_{\mathsf{a}_0},$$

Moreover this gives:

- G is simple by Lemma 2.4;
- the weights on the first level consist solely on $\chi - \mathsf{a}$ and its associated weight space is $\phi(\mathfrak{g}_{-\mathsf{a}})V_{\chi_\Phi}$. Since $\rho(\Gamma)$ is $\{\tau_2\}$-Anosov one has that $\phi(\mathfrak{g}_{-\mathsf{a}})V_{\chi_\Phi}$ is one-dimensional, but by Lemma 2.6 no element of $\mathfrak{g}_{-\mathsf{a}}$ acts trivially on V_{χ_Φ} so $\mathfrak{g}_{-\mathsf{a}}$ is 1-dimensional, as desired.

We proceed now to show that in the present situation θ consists of only one element. By definition of θ one has, for every $g \in \mathsf{G}$, that

$$\tau_1\big(\lambda\big(\Phi(g)\big) = \min_{\mathsf{a}\in\theta}\big\{\mathsf{a}(\lambda_{\mathsf{G}}(g))\big\}.$$

Consequently, the limit cone $\mathcal{L}_{\rho_0} \subset \mathfrak{a}_{\mathsf{G}}^+$ does not intersect the walls of elements in θ and, since $\rho_0 : \Gamma \to \mathsf{G}$ is a quasi-isometry, Remark 4.4 implies that the representation ρ_0 is θ-Anosov.

Recall from Equation (2.5) that we have a Φ-equivariant analytic embedding $\zeta_\theta : \mathsf{G}/\mathsf{P}_\theta \to \mathbb{P}(\mathbb{R}^d)$. One has moreover that $\xi^1_\rho = \zeta_\theta \circ \xi^\theta_{\rho_0}$. In particular the boundary map ξ^θ has C^1-image. Composing with the projections $\mathcal{F}_\theta \to \mathcal{F}_{\theta'}$ one sees that, for any $\theta' \subset \theta$ the curve $\xi^{\theta'}_{\rho_0}(\partial\Gamma)$ is a C^1 circle.

Assume now there exists two distinct roots a, b in θ. By the previous paragraph the curve $\xi^{\{\mathsf{a},\mathsf{b}\}}(\partial\Gamma)$ is C^1 . Corollary 7.7 gives then that $\xi^{\{\mathsf{a},\mathsf{b}\}}(\partial\Gamma)$ is necessarily contained in one of the leaves of the canonical foliations of $\mathcal{F}_{\{\mathsf{a},\mathsf{b}\}}$, thus giving that one of the maps ξ^{a} or ξ^{b} is constant, achieving a contradiction. □

8. Non-differentiability and 1-conicality: The proof of Theorem B

8.1. **Non-differentiability and 1-conicality.** By means of § 7.1 we can improve Lemma 6.2 when we deal with a pair of real hyperconvex representations of surface groups, this is the missing ingredient for Theorem B:

Corollary 8.1. *Assume $\partial\Gamma$ is homeomorphic to a circle. Let $\rho, \overline{\rho}$ two $(1,1,2)$-hyperconvex representations over $\mathbb{R}$ of Γ that are not gap-isospectral. Then, the set of non-differentiability points of Ξ coincides with the set of 1-conical points.*

Proof. We choose a C^1 identification of the C^1 torus $\xi(\partial\Gamma) \times \overline{\xi}(\partial\Gamma) \subset \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{\overline{d}})$ with the quotient of the square $[-1,1] \times [-1,1]$ preserving the product structure, and such that the point $(x, \Xi(x))$ corresponds to $(0,0)$. In these coordinates the graph of Ξ is a monotone curve $[-1,1] \to [-1,1]$ passing through the origin. Since the chosen identification is C^1, it is in particular K-bi-Lipschitz for some K, so we can write (coarsely in a small neighbourhood of x) $d(\xi(y), \xi(x)) = |y|$ and $d(\overline{\xi}(y), \overline{\xi}(x)) = |\Xi(y)|$.

From Lemma 6.2 we know that x is 1-conical if and only if either $\lim_{y\to x} \frac{|\Xi(y)|}{|y|}$ exists and is far from 0 and ∞, either it does not exist. The proposition is settled if we show that the first situation cannot happen, so let's assume it does. However, since Ξ is monotone we can remove the $|\,|$ and we get that x is a differentiability point of Ξ with oblique derivative. Corollary 7.5 implies then that for all $\gamma \in \Gamma$ one has $\tau_1(\lambda(\gamma)) = \overline{\tau}_1(\lambda(\overline{\gamma}))$, contradicting our assumption. □

8.2. **Proof of Theorem B and an analogous for Kleinian groups.** We begin with the proof of Theorem B by recalling the following result from Beyrer-P. [7]

Corollary 8.2 (Beyrer-P. [7]). *Assume $\partial\Gamma$ is homeomorphic to a circle and let $\rho : \Gamma \to \mathsf{PGL}(d,\mathbb{R})$ be $(1,1,2)$-hyperconvex. Then there exists an irreducible $(1,1,2)$-hyperconvex representation $\rho_0 : \Gamma \to \mathsf{PGL}(m,\mathbb{R})$ such that, for every $\gamma \in \Gamma$ one has*

$$\tau_1\big(\lambda(\gamma)\big) = \tau_1\big(\lambda(\rho_0\gamma)\big).$$

We now prove Theorem B. Since there exists $\gamma \in \Gamma$ with $\tau_1\big(\lambda(\gamma)\big) \neq \overline{\tau}_1\big(\lambda(\overline{\gamma})\big)$, Corollary 8.2 allows us to apply Proposition 6.3 to obtain the density assumption in Theorem 5.3, so one has

$$\dim_{\mathrm{Hff}} \Xi\big(\{1\text{-conical points}\}\big) = \hbar^{\max\{\tau,\overline{\tau}\}}.$$

Corollary 8.1 states that the set of 1-conical points coincides with the set of non-differentiability points of Ξ. The inequality $\hbar^{\infty} < 1$ follows from the strict convexity of the critical hypersurface $\mathcal{Q}_{\upsilon}$, where υ is the cocycle studied in Section 5.3. This completes the proof of Theorem B.

8.3. **Proof of Corollary B.** We conclude the paper proving Corollary B. Recall from Section 2.3 that for every simple root a of G we chose a Tits representation $\Phi_{\mathsf{a}} : \mathsf{G} \to \mathsf{PSL}(V_{\mathsf{a}})$.

Corollary 8.3. *Assume $\partial\Gamma$ is homeomorphic to a circle and let G be a simple Lie group. Let $\rho : \Gamma \to \mathsf{G}$ have Zariski-dense image. If for $\mathsf{a},\mathsf{b} \in \Delta$ the representations $\Phi_{\mathsf{a}} \circ \rho$ and $\Phi_{\mathsf{b}} \circ \rho$ are $(1,1,2)$-hyperconvex, then*

(i) *the image of the limit curve $\xi^{\{\mathsf{a},\mathsf{b}\}} : \partial\Gamma \to \mathcal{F}_{\{\mathsf{a},\mathsf{b}\}}$ is Lipschitz and the Hausdorff dimension of the points where it is non-differentiable is $\hbar^{\max\{\mathsf{a},\mathsf{b}\}}$.*

(ii) *If the opposition involution i on $\mathfrak{g}$ is non-trivial and $\mathsf{b} = \mathrm{i}\mathsf{a}$ then*

$$\hbar^{\max\{\mathsf{a},\mathsf{b}\}} = \hbar^{(\mathsf{a}+\mathsf{b})/2}.$$

Proof.

(i) Since the map $\Phi_{\mathsf{a}} : \mathcal{F}_{\mathsf{a}} \to \mathbb{P}(V_{\mathsf{a}})$ is analytic, and $\Phi_{\mathsf{a}} \circ \xi^{\mathsf{a}}(\partial\Gamma)$ is a C^1-submanifold (Theorem 1.3), $\xi^{\mathsf{a}}(\partial\Gamma)$ is a C^1 submanifold as well. The curve $\xi^{\{\mathsf{a},\mathsf{b}\}} := \mathcal{F}_{\{\mathsf{a},\mathsf{b}\}} \cap (\xi^{\mathsf{a}}(\partial\Gamma) \times \xi^{\mathsf{b}}(\partial\Gamma))$ is the graph of the homeomorphism Ξ and is thus a Lipschitz curve. The second claim is then a direct consequence of Theorem B.

(ii) Assume the opposition involution i of $\mathfrak{g}$ is non-trivial and that $\mathsf{b} = \mathrm{i}\mathsf{a}$. Using notation from Section 5.3 with $\mathsf{a} = \tau$ and $\mathsf{b} = \mathrm{i}\mathsf{a} = \overline{\tau}$ we let $V^* = \mathrm{span}\{\mathsf{a},\mathsf{b}\}$, $V = \mathfrak{a}_\theta/\mathrm{Ann}(V^*)$, $\Pi : \mathfrak{a}_\theta \to V$ the quotient projection, $\|\ \|_\infty = \max\{|\mathsf{a}|,|\mathsf{b}|\}$, $\|\ \|^1$ its dual norm on V^* and $\varphi^{\infty}_{\hat{\theta}} \in \mathcal{Q}_{\upsilon}$ the only form minimizing $\|\ \|^1$.

Since $\mathrm{i}\mathsf{a} = \mathsf{b}$, the space V^* is preserved by i and the fact that $\lambda(g^{-1}) = \mathrm{i}\lambda(g)$ (for all $g \in \mathsf{G}$) implies that $\mathcal{Q}_{\upsilon}$ is i-invariant. Moreover, the norm $\|\ \|^1$ is i-invariant and by definition of $\varphi^{\infty}_{\hat{\theta}}$ one has $\mathrm{i}\varphi^{\infty}_{\hat{\theta}} = \varphi^{\infty}_{\hat{\theta}}$. However, $(\mathsf{a}+\mathsf{b})/2$ is also i-invariant and $\hbar^{(\mathsf{a}+\mathsf{b})/2}(\mathsf{a}+\mathsf{b})/2 \in \mathcal{Q}_{\upsilon}$ whence

$$\varphi^{\infty}_{\hat{\theta}} = \hbar^{(\mathsf{a}+\mathsf{b})/2}(\mathsf{a}+\mathsf{b})/2.$$

In order to prove the result it is thus enough to show that

$$\hbar^{\max\{\mathsf{a},\mathsf{b}\}} = \|\varphi^{\infty}_{\hat{\theta}}\|^1. \tag{8.1}$$

We conclude the proof deducing this equality from Quint's [44, Proposition 3.3.3].

We consider the counting measure

$$\nu = \sum_{\gamma\in\Gamma} \delta_{\Pi a_\theta(\gamma)}$$

on the vector space $\mathcal{E} = V$ and the norm $N = \| \|_\infty$. We then have, in the notation of [44, §3], that $\tau_\nu^N = \hbar^{\max\{\mathsf{a},\mathsf{b}\}}$ and, by Remark 4.13, $\sigma_\nu^N = \inf_{\varphi \in \mathcal{Q}_\nu} \|\varphi\|^1$. Thus, in order to deduce Equation (8.1) from [44, Proposition 3.3.3] it is enough to verify that the counting measure ν is of concave growth as in [44, §3.2]. In turn this is a consequence of Lemma 8.4 below, an adaptation of [44, Proposition 2.3.1] (see also Kim-Oh-Wang [33, Lemma 3.8] where similar arguments are explained for the $\mathfrak{a}_\theta$ counting measure). □

Lemma 8.4. *Let $\| \|$ be a norm on V. Let $\Lambda < \mathsf{G}$ be Zariski-dense and $\{\mathsf{a},\mathsf{b}\}$-Anosov. Then there exists a product map $m : \Lambda \times \Lambda \to \Lambda$ with the following properties:*

(i) *there exists a real number $\kappa \geqslant 0$ such that, for all $\gamma_1, \gamma_2 \in \Lambda$,*

$$\|\Pi a_\theta(m(\gamma_1,\gamma_2)) - \Pi a_\theta(\gamma_1) - \Pi a_\theta(\gamma_2)\| \leqslant \kappa;$$

(ii) *for every real $R \geqslant 0$ there exists a finite subset H of Λ such that, for $\gamma_1, \gamma_2 \gamma_1', \gamma_2'$ in Λ with $\|\Pi a_\theta(\gamma_i) - \Pi a_\theta(\gamma_i')\| < R$ for $i = 1, 2$, then*

$$m(\gamma_1,\gamma_2) = m(\gamma_1',\gamma_2') \Rightarrow \gamma_i' \in \gamma_i H, \text{ for } i = 1, 2.$$

Proof. It is enough to consider the *generic product* map $\pi : \Lambda \times \Lambda \to \Lambda$ constructed in [44, Proposition 2.3.1], which satisfies the analogous properties with respect to the Cartan projection $a : \mathsf{G} \to \mathfrak{a}$ and a norm $\| \|$ on $\mathfrak{a}$. The first property is satisfied since we can assume that the projection $\Pi \circ \pi_\theta : \mathfrak{a} \to V$ is norm non-increasing. The second follows from the Anosov property: by the construction in [44, Proposition 2.3.1] one can choose H to be the set of elements γ such that $\|\Pi a_\theta(\gamma)\| < R'$ for some R' depending on R. Such set is finite because, by definition of Π, there exists R'' depending on R' and the norm $\| \|$ such that if $\|\Pi a_\theta(\gamma)\| < R'$ then $\mathsf{a}(a(\gamma)) < R''$, which in turn implies by Definition 4.3 that $|\gamma| < R''/\mu + C$, and thus γ belongs to a finite subset. □

8.4. **The $\mathsf{PSL}(2,\mathbb{C})$-case.** If $\rho, \overline{\rho} : \Gamma \to \mathsf{PSL}(2,\mathbb{C})$ are convex co-compact representations that are connected by convex-co-compact representations, it was proven by Marden [37] that the natural map $\Xi : \Lambda_\rho \to \Lambda_{\overline{\rho}}$ conjugating the respective actions extends to a Hölder homeomorphism $\Xi : \mathbb{CP}^1 \to \mathbb{CP}^1$ that is $(\rho, \overline{\rho})$-equivariant. We consider in this case the complex derivative of such an extension Ξ and say that Ξ is $\mathbb{C}$-*differentiable* at a given $x \in \Lambda_\rho$ if, conformally identifying $\partial\mathbb{H}^3 - \{\text{point}\}$ to $\mathbb{C}$, the limit

$$\Xi'(x) := \lim_{y \to x} \frac{\Xi(x) - \Xi(y)}{x - y}$$

exists or is infinite. We let now $\mathsf{NDiff}_{\rho,\overline{\rho}}$ be the set of points $x \in \Lambda_\rho$ where the extended conjugating map Ξ is not $\mathbb{C}$-differentiable and let

The proof of the following works verbatim as in Corollary 8.1.

Proposition 8.5. *Let $\rho, \overline{\rho} : \Gamma \to \mathsf{PSL}(2,\mathbb{C})$ be non-gap-isospectral and in the same connected component of*

$$\big\{\varrho : \Gamma \to \mathsf{PSL}(2,\mathbb{C}) : \ \varrho \text{ is convex co} - \text{compact}\big\}.$$

Then, the set of non-$\mathbb{C}$-differentiability points of Ξ coincides with the set of 1-conical points.

Density of the group generated by the pairs $\{(\lambda(\gamma), \lambda(\bar{\gamma})) : \gamma \in \Gamma\}$ follows readily from Benoist [4] (see Theorem 2.3), from this point on the exact same proof of Theorem B gives the following.

Theorem 8.6. *Let $\rho, \overline{\rho} : \Gamma \to \mathsf{PSL}(2, \mathbb{C})$ be non-gap-isospectral convex co-compact representations that are connected by convex co-compact representations. Assume without loss of generality that $\hbar_{\overline{\tau}} \geqslant \hbar_{\tau}$. If $\mathbf{I}_{\tau}(\overline{\tau}) > 1$, then $\dim_{\mathrm{Hff}}(\mathsf{NDiff}_{\rho,\overline{\rho}}) = \hbar^{\infty}$.*

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