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LOOP GROUP METHODS FOR THE NON-ABELIAN HODGE CORRESPONDENCE ON A 4-PUNCTURED SPHERE

LYNN HELLER, SEBASTIAN HELLER, AND MARTIN TRAIZET

Abstract. The non-abelian Hodge correspondence is a real analytic map between the moduli space of stable Higgs bundles and the deRham moduli space of irreducible flat connections mediated by solutions of the self-duality equation. In this paper we construct such solutions for strongly parabolic $\mathfrak{sl}(2,\mathbb{C})$ Higgs fields on a 4-punctured sphere with parabolic weights $t \sim 0$ using loop groups methods through an implicit function theorem argument at the trivial connection for $t = 0$. We identify the rescaled limit hyper-Kähler moduli space at the singular point at $t = 0$ to be the completion of the nilpotent orbit in $\mathfrak{sl}(2,\mathbb{C})$ equipped the Eguchi-Hanson metric (modulo a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action). Our methods and computations are based on the twistor approach to the self-duality equations using Deligne and Simpson's λ-connections interpretation. Due to the implicit function theorem, Taylor expansions of these quantities can be computed at $t = 0$. By construction they have closed form expressions in terms of Multiple-Polylogarithms and their geometric properties lead to some identities of Ω-values which we believe deserve further investigations.

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1. Introduction

Hitchin's self-duality equations [21] on a degree zero and rank two hermitian vector bundle $V \to \Sigma$ over a compact Riemann surface Σ are equations on a pair (∇, Ψ) consisting of a special unitary connection ∇ and an (trace free) endomorphism valued $(1,0)$-form Ψ satisfying

$$\bar{\partial}^{\nabla}\Psi = 0 \quad \text{and} \quad F^{\nabla} + [\Psi, \Psi^*] = 0. \tag{1}$$

These equations are equivalent to the flatness of the whole associated family of connections

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$$\nabla^\lambda = \nabla + \lambda^{-1}\Psi + \lambda\Psi^* \tag{2}$$

parametrized by $\lambda \in \mathbb{C}^*$ – the spectral parameter. Solutions to (1) give rise to equivariant harmonic maps from the Riemann surface into the hyperbolic 3-space $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ by considering the Higgs field Ψ as the $(1,0)-$part of the differential of the harmonic map [7]. An application of a classical result by Eells-Sampson [9] then shows the existence of a unique solution in each homotopy class of equivariant maps, i.e., when prescribing the (totally reducible) monodromy of the harmonic map [7].

As first recognized by Hitchin [21], the moduli space of solutions to the self-duality equations (modulo gauge transformations) $\mathcal{M}_{SD}$ has two very different incarnations as complex analytic spaces. Firstly as the moduli space of Higgs bundles $\mathcal{M}_{Higgs}$, i.e., as the moduli space of pairs $(\bar\partial_V, \Psi)$ satisfying $\bar\partial_V\Psi = 0$, and secondly as the moduli space of flat connections $\mathcal{M}_{dR}$. The non-abelian Hodge correspondence is the map between $\mathcal{M}_{Higgs}$ and $\mathcal{M}_{dR}$ obtained through solutions to Hitchin's equations. Both complex analytic spaces $\mathcal{M}_{Higgs}$ and $\mathcal{M}_{dR}$ induce anti-commuting complex structures on $\mathcal{M}_{SD}$ turning it into a hyper-Kähler manifold when equipped with its natural L^2-metric. Since the mapping is mediated by the harmonic map this correspondence is not explicit and it is not possible to see every facet of the geometry within one framework only.

Every hyper-Kähler manifold can be described using complex analytic data (subject to additional reality conditions) via twistor theory [22]. For $\mathcal{M}_{SD}$, the twistor space has been identified with the so-called Deligne-Hitchin moduli $\mathcal{M}_{DH}$ space introduced by Deligne and Simpson [30]. The space $\mathcal{M}_{DH}$ is obtained by gluing the moduli space of λ-connections on the Riemann surface with the moduli space of λ-connections on the complex conjugate Riemann surface, and the associated family of flat connections (2) naturally extends to a special real holomorphic section – a twistor line. The main subject of the paper is to construct twistor lines and the hyper-Kähler structure of $\mathcal{M}_{SD}$ *'entirely complex analytically, so we bypass the nonlinear elliptic theory necessary to define the harmonic metrics'* [1].

The self-duality equations (1) generalize to punctured Riemann surfaces by imposing first order poles of the Higgs fields and a growth condition of the harmonic metric determined by their parabolic weights, called tameness, see Simpson [29]. In the following, we restrict to *strongly* parabolic case, where the Higgs fields have nilpotent residues. Then the associated family of flat connections has (up to conjugation) the same local monodromies for all spectral parameter $\lambda \in \mathbb{C}^*$ at the punctures [29, table on page 720]. The moduli space of solutions has then again a hyper-Kähler structure which was first studied by Konno [24]. The complex structure I is hereby the one of the moduli space of (polystable) parabolic Higgs fields, and the complex structure J is that of the moduli space of logarithmic connections with prescribed local monodromy conjugacy classes.

In this paper, we study the simplest non-trivial case where the underlying Riemann surface is a 4-punctured sphere and restrict to Fuchsian systems, i.e., logarithmic connections on the trivial holomorphic (rank 2) bundle. On the Higgs side, we assume that the parabolic Higgs bundles are strongly stable with the same parabolic weights $\pm t$ with $t \in (0, \frac{1}{4})$ at all singular points. For $t \to 0$ the space of polystable parabolic structures degenerates to a point given by the trivial parabolic Higgs pair. When rescaling by t its blow-up limit at $t = 0$ consists of the moduli space of parabolic Higgs fields Φ on the trivial holomorphic bundle. Using an

[1] C. Simpson in [30, Section 4]

implicit theorem argument we construct for $t \sim 0$ twistor lines near this singular limit, i.e., where the Higgs fields $\Psi \sim t\Phi$ is small enough. On the corresponding rescaled moduli spaces we then obtain an explicit version of the non-abelian Hodge correspondence. Moreover, we study the hyper-Kähler structure of $\mathcal{M}(t)$ via twistor theory and identify the limit metric as the Eguchi-Hanson metric modulo a $\mathbb{Z}_2 \times \mathbb{Z}_2$-action. It would be interesting to compare our work with the results of [12] where the moduli space is studied for fixed weights and large Higgs fields in the regular locus.

The paper is organized as follows. In Section 2 we introduce Higgs bundles, the associated moduli spaces as well as its hyper-Kähler structure for compact Riemann surfaces. Their complex geometric properties are encoded in the Deligne-Hitchin moduli space, see [30], and we introduce twistor lines as special real holomorphic sections induced by self-duality solutions. Furthermore, we identify the twisted holomorphic symplectic structure on the Deligne-Hitchin moduli space with (a version of) Goldmann's symplectic structure. Thereafter, we give the ansatz in terms of Fuchsian systems for constructing real holomorphic sections through the implicit function theorem and loop group methods in Section 3. In Section 4 we conduct the actual implicit function theorem argument to obtain real holomorphic sections of the Deligne-Hitchin moduli space. To find appropriate coordinates, we first consider the four-fold covering of the Higgs bundles moduli space $\mathcal{M}_{Higgs}$ and we rescale it by by the factor $\frac{1}{t}$. The limit space at $t = 0$ is then given by the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ (with coordinates (u, v)) at the origin. To perform the implicit theorem argument, we thus first consider the regular case in Theorem 4 and then when $(u, v) \to (0, 0)$ in Theorem 5. It turns out that in the appropriate $(u, v) \to (0, 0)$ limit the Higgs field vanishes and the corresponding sections are flat unitary connections yielding twistor lines. Due to the fact that twistor lines form a connected component of the space of real holomorphic sections, all constructed sections must be twistor lines. Interestingly, we find a Lax pair type equation describing the deformation given by the implicit function theorem. In Section 5, we compute the limit non-abelian Hodge correspondence at $t = 0$, and identify the rescaled limit hyper-Kähler metric.

Theorem 1. *For $t \to 0$ the blow-up limit of the moduli space $\mathcal{M}(t)$ of strongly parabolic Higgs bundles on the 4-punctured sphere at the singular point given by the trivial parabolic structure is the Eguchi-Hanson space modulo a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action.*

The proof uses the twisted holomorphic symplectic form of $\mathcal{M}_{SD}$ on the 4-punctured sphere in the Fuchsian ansatz. One advantage of our construction through the implicit function theorem is that we can compute Taylor expansion of all geometric quantities. As an example, we compute first order derivatives of the parameters in Section 5.4 and put them together to obtain the first order expansion of the non-abelian Hodge correspondence, the twisted holomorphic symplectic structure and the hyper-Kähler metric. In the last Section 6 we analyze the structure of the higher order derivatives of the twistor sections with respect to t. As in [20] for minimal surfaces in the 3-sphere, the n-th order derivatives of the parameters are polynomial in λ of order $(n + 1)$.

Theorem 2. *The hyper-Kähler metric of the moduli space $\mathcal{M}(t)$ of strongly parabolic Higgs bundles on the 4-punctured sphere is real analytic in its weight t and there exist an explicit algorithm to computing its Taylor expansion at the trivial parabolic Higgs pair in $t = 0$. More precisely, when computing the Taylor expansion of the twistor lines in t at the trivial connection, the n-th order coefficients are polynomials in λ of degree $n+1$ and can be expressed explicitly in terms of multiple polylogarithms of depth and weight at most $n + 1$.*

Since the twisted holomorphic symplectic form is a Laurent polynomial of degree one in λ, evaluating it on meromorphic 1-forms which represent tangent vectors to $\mathcal{M}(t)$ yields infinitely many cancelations for higher order terms in λ leading to identities for some iterated integrals, called Ω-*values*, which can be expressed in terms of multiple polylogarithms. In Section 6.3 we give some identities of depth three Ω-values obtained from this idea which are non-trivial in the sense that they cannot be derived from the shuffle and stuffle relations alone.

2. Preliminaries

2.1. **The hyper-Kähler structures and their twistor spaces.** The moduli space of Hitchin's self-duality equations $\mathcal{M}_{SD}$ has through the non-abelian Hodge correspondence three complex structures $I, J, K = IJ$ which are Kähler with respect to the same Riemannian metric g. In fact there exist a whole $\mathbb{C}P^1$ worth of complex structures defined by

$$I_\lambda = \frac{1-|\lambda|^2}{1+|\lambda|^2} I + \frac{\lambda+\bar{\lambda}}{1+|\lambda|^2} J - \frac{i(\lambda-\bar{\lambda})}{1+|\lambda|^2} K$$

for $\lambda \in \mathbb{C} \subset \mathbb{C}P^1$. Within this family we find the complex structure I at $\lambda = 0$, the complex structure J at $\lambda = 1$, and the complex structure K at $\lambda = i$.

We use the sign convention

$$\omega_I = -g(.,I.) \quad \omega_J = -g(.,J.) \quad \text{and} \quad \omega_K = -g(.,K.)$$

for the associated Käher forms so that $h = g + i\omega$ is the corresponding hermitian metric. The twistor space of a hyper-Kähler manifold, introduced in [22], is the smooth manifold

$$\mathcal{P} := \mathcal{M}_{SD} \times \mathbb{C}P^1$$

equipped with the (integrable) complex structure

$$\mathbb{I} = (I_\lambda, \mathrm{i}),$$

at the point $x \in \mathcal{M}_{SD}$ and $\lambda \in \mathbb{C} \subset \mathbb{C}P^1$, where i is the standard complex structure of $\mathbb{C}P^1$. Furthermore, the twistor space has a natural anti-holomorphic involution $\mathcal{T}$ given by $(x, \lambda) \mapsto (x, -\bar{\lambda}^{-1})$. By construction the twistor space has a holomorphic projection π to $\mathbb{C}P^1$ and a twisted relative holomorphic symplectic form given by

$$\hat{\varpi} \in H^0(P, \Lambda^2 V^* \otimes \mathcal{O}(2))$$

where $V = \ker d\pi$ (is the complex tangent bundle to the fibers) and $\mathcal{O}(2)$ denotes the pull-back bundle of the canonical bundle over $T^*\mathbb{C}P^1$. In terms of the Kähler forms the twisted relative holomorphic symplectic form has the following explicit expression

$$\hat{\varpi} = \varpi \otimes \lambda \frac{\partial}{\partial \lambda}, \tag{3}$$

where

$$\varpi = \lambda^{-1}(\omega_J + i\omega_K) - 2\omega_I - \lambda(\omega_J - i\omega_K), \tag{4}$$

see [22, Equation (3.87)]. (Be aware of the sign difference, due to a -1 factor in the stereographic projections used to identify S^2 with $\mathbb{C}P^1$.)

Sections of $\mathcal{P}$ are holomorphic maps $s\colon \mathbb{C}P^1 \to \mathcal{P}$ with $\pi \circ s = \mathrm{Id}$. The simplest sections are *constant* sections

$$\lambda \mapsto ((\nabla, \Psi), \lambda) \tag{5}$$

for a solution of the self-duality equations (∇, Ψ). By dimension count, these twistor lines give rise to an open subspace of the space of real holomorphic sections [22]. The following well known theorem then follows directly from the completeness (when adding reducible solutions) of the moduli space of self duality solutions [21].

Theorem 3. *Twistor lines form an open and closed subset of the space of real sections.*

A convenient set-up for studying associated families of flat connections obtained from solutions to self-duality equations is to consider them as real sections of the Deligne-Hitchin moduli space $\mathcal{M}_{DH}$.

2.2. **The Deligne-Hitchin moduli space $\mathcal{M}_{DH}$.** The Deligne-Hitchin moduli space was first introduced by Deligne (see [30, 31]) as a complex analytic way of viewing the associated twistor space of the moduli space of solutions to the self-duality equations. As such it interpolates between the moduli space of Higgs bundles $\mathcal{M}_{Higgs}$ and the moduli space of flat connections $\mathcal{M}_{dR}$.

Definition 1. *Let Σ be a Riemann surface and $\lambda \in \mathbb{C}$ fixed. A (integrable) λ-connection on a $\mathcal{C}^\infty-$complex vector bundle $V \to \Sigma$ is a pair $(\bar\partial_V, D)$ consisting of a holomorphic structure on V and a linear first order differential operator*

$$D\colon \Gamma(\Sigma, V) \to \Omega^{(1,0)}(\Sigma, V)$$

satisfying the λ-Leibniz rule

$$D(fs) = \lambda \partial f \otimes s + fDs$$

for functions f and sections s, and the integrability condition

$$D\bar\partial_V + \bar\partial_V D = 0. \tag{6}$$

Remark 2. *The operators D and $\bar\partial_V$ also act on $(0,1)$-forms and $(1,0)$-forms respectively. For $\lambda = 0$ the integrability condition (6) is equivalent to*

$$D = \Psi \in H^0(M, K_\Sigma \mathrm{End}(V))$$

being a holomorphic endomorphism-valued 1-form, and for $\lambda \neq 0$ we have that

$$\nabla = \tfrac{1}{\lambda} D + \bar\partial$$

is a flat connection.

Example 3. *Consider on the hermitian bundle $V \to \Sigma$ a solution $(\nabla = \partial^\nabla + \bar\partial^\nabla, \Psi)$ of the self-duality equations. Then, the pair*

$$(\bar\partial^\nabla + \lambda\Psi^*, \lambda\partial^\nabla + \Psi)$$

defines a λ-connection on V for every $\lambda \in \mathbb{C}$ which coincides with the Higgs pair $(\bar\partial^\nabla, \Psi)$ at $\lambda = 0$ and with the flat connection $\nabla^1 = \nabla + \Psi + \Psi^*$ at $\lambda = 1$.

Definition 4. *A $\mathrm{SL}(2,\mathbb{C})$ λ-connection is a λ-connection on a rank 2 vector bundle $V \to \Sigma$, such that the induced λ-connection on the determinant bundle $\Lambda^2 V$ is trivial.*

Definition 5. *A* $\mathrm{SL}(2,\mathbb{C})$ *λ-connection* $(\bar\partial_V, D)$ *is called stable, if every* $\bar\partial_V$*-holomorphic subbundle* $L \subset V$ *with*

$$D(\Gamma(\Sigma, L)) \subset \Omega^{(1,0)}(\Sigma, L)$$

is of negative degree and semi-stable if its degree is non-positive. All other λ*-connections are called unstable. A* $\mathrm{SL}(2,\mathbb{C})$ *λ-connection is called polystable if it is either stable or the direct sum of dual* λ*-connections on degree zero line bundles.*

For $\lambda \neq 0$, every $\bar\partial$-holomorphic and D-invariant line subbundle $L \subset V$ must be parallel with respect to the flat connection $\nabla = \frac{1}{\lambda}D + \bar\partial$. Therefore, the degree of L is 0 and the λ-connection $(\bar\partial, D)$ is semi-stable. Moreover, $(\bar\partial, D)$ is stable if and only if the flat connection $\nabla = \frac{1}{\lambda}D + \bar\partial$ is irreducible. The situation is different at $\lambda = 0$ and we need to restrict to polystable λ-connections to obtain a well-behaved moduli space.

Definition 6. *The Hodge moduli space* $\mathcal{M}_{Hod} = \mathcal{M}_{Hod}(\Sigma)$ *is the space of all polystable* $\mathrm{SL}(2,\mathbb{C})$ *λ-connections on* $V = \Sigma \times \mathbb{C}^2 \to \Sigma$ *modulo gauge transformations, i.e.,* $\mathcal{M}_{Hod}$ *consists of gauge classes of triples* $(\lambda, \bar\partial, D)$ *for* $\lambda \in \mathbb{C}$ *and* $(\bar\partial, D)$ *a* λ*-connection.*

The gauge-equivalence class of $(\lambda, \bar\partial, D)$ *is denoted by*

$$[\lambda, \bar\partial, D] \in \mathcal{M}_{Hod}$$

or by

$$[\lambda, \bar\partial, D]_\Sigma \in \mathcal{M}_{Hod}(\Sigma)$$

to emphasis its dependence on the Riemann surface.

The Hodge moduli space admits a natural holomorphic map

$$\pi_\Sigma \colon \mathcal{M}_{Hod} \longrightarrow \mathbb{C}; \quad [\lambda, \bar\partial, D] \longmapsto \lambda$$

whose fiber at $\lambda = 0$ is the (polystable) Higgs moduli space $\mathcal{M}_{Higgs}$, and at $\lambda = 1$ it is the deRham moduli space of flat (and totally reducible) $\mathrm{SL}(2,\mathbb{C})$-connections $\mathcal{M}_{dR}$, which we consider as complex analytic spaces endowed with their natural complex structures I and J respectively.

The next step is then to compactify the λ-plane $\mathbb{C}$ to $\mathbb{C}P^1$. For a Riemann surface Σ let $\overline{\Sigma}$ be its complex conjugate, i.e., the Riemann surface with conjugate complex structure. As differentiable manifolds we have $\Sigma \cong \overline{\Sigma}$ and thus their deRham moduli spaces of flat $\mathrm{SL}(2,\mathbb{C})$-connections are naturally isomorphic. Then the two Hodge moduli spaces $\mathcal{M}_{Hod}(\Sigma)$ and $\mathcal{M}_{Hod}(\overline{\Sigma})$ can be glued together via Deligne gluing [31]

$$\mathcal{G} \colon \mathcal{M}_{Hod}(\Sigma) \setminus \pi_\Sigma^{-1}(0) \to \mathcal{M}_{Hod}(\overline{\Sigma}) \setminus \pi_{\overline{\Sigma}}^{-1}(0); \; [\lambda, \bar\partial, D]_\Sigma \mapsto [\tfrac{1}{\lambda}, \tfrac{1}{\lambda}D, \tfrac{1}{\lambda}\bar\partial]_{\overline{\Sigma}}$$

along $\mathbb{C}^*$ to give Deligne-Hitchin moduli space

$$\mathcal{M}_{DH} = \mathcal{M}_{Hod}(\Sigma) \cup_{\mathcal{G}} \mathcal{M}_{Hod}(\overline{\Sigma}).$$

The natural fibration π_Σ on $\mathcal{M}_{Hod}$ extends holomorphically to the whole Deligne-Hitchin moduli space to give $\pi \colon \mathcal{M}_{DH} \to \mathbb{C}P^1$ whose restriction to $\mathcal{M}_{Hod}(\overline{\Sigma})$ is $1/\pi_{\overline{\Sigma}}$.

Remark 7. *Note that the Deligne gluing map* $\mathcal{G}$ *maps stable* λ*-connections over* Σ *to stable* $\frac{1}{\lambda}$*-connections on* $\overline{\Sigma}$*. Hence, it maps the smooth locus of* $\mathcal{M}_{Hod}(M)$ *(consisting of stable* λ*-connections) to the smooth locus of* $\mathcal{M}_{Hod}(\overline{M})$*, and thus* $\mathcal{M}_{DH}$ *is equipped with a complex manifold structure at all stable points.*

Definition 8. *A section of $\mathcal{M}_{DH}$ is a holomorphic map*

$$s : \mathbb{C}P^1 \to \mathcal{M}_{DH}$$

such that $\pi \circ s = Id$.

Example 9. *The associated family of flat connections ∇^λ to a solution of the self-duality equations* (1) *gives rise to a section of $\mathcal{M}_{DH} \to \mathbb{C}P^1$ via*

$$s(\lambda) = [\lambda, \bar\partial^\nabla + \lambda\Psi^*, \lambda\partial^\nabla + \Psi]_\Sigma \in \mathcal{M}_{Hod}(\Sigma) \subset \mathcal{M}_{DH}. \tag{7}$$

When identifying the Deligne-Hitchin moduli space with the twistor space $\mathcal{P} \to \mathbb{C}P^1$ of the hyper-Kähler space $\mathcal{M}_{SD}$ (at the smooth points), the section given by (7) *is identified with the 'constant' twistor line* (5)*, see* [30].

Definition 10. *A holomorphic section s of $\mathcal{M}_{DH}$ is called stable, if the λ-connection $s(\lambda)$ is stable for all $\lambda \in \mathbb{C}^*$ and if the Higgs pairs $s(0)$ on Σ and $s(\infty)$ on $\overline{\Sigma}$ are stable.*

It follows from Hitchin [21] and Donaldson [7] that every stable point in $\mathcal{M}_{DH}$ uniquely determines a twistor line. Therefore, a twistor line s is already stable if $s(\lambda_0)$ is stable for one $\lambda_0 \in \mathbb{C}$. Moreover, twistor lines are in one-to-one correspondence with self-duality solutions (1). Hence the following characterization of twistor lines as particular negative real holomorphic sections of $\mathcal{M}_{DH}$ is useful to decide when certain real sections are in fact global solutions to the self-duality equations.

2.3. **Automorphisms of the Deligne-Hitchin moduli space.** To define a real structure on $\mathcal{M}_{DH}$ we need to look at some natural automorphisms of Deligne-Hitchin moduli space. First of all, for every $\mu \in \mathbb{C}^*$ the (multiplicative) action of μ on $\mathbb{C}P^1$ has a natural lift to $\mathcal{M}_{DH}$ by

$$\mu([\lambda, \bar\partial, D]) = [\mu\lambda, \bar\partial, \mu D].$$

Definition 11. *We denote by $N : \mathcal{M}_{DH} \to \mathcal{M}_{DH}$ the map given by multiplication with $\mu = -1$, namely*

$$[\lambda, \bar\partial, D] \longmapsto [-\lambda, \bar\partial, -D].$$

Furthermore, we have a natural anti-holomorphic automorphism denoted by C.

Definition 12. *Let $C\colon \mathcal{M}_{DH} \longrightarrow \mathcal{M}_{DH}$ be the continuation of the map*

$$\tilde C : \mathcal{M}_{Hod}(\Sigma) \longrightarrow \mathcal{M}_{Hod}(\overline{\Sigma})$$

given by

$$[\lambda, \bar\partial, D] \longmapsto [\bar\lambda, \bar{\bar\partial}, \bar D]_{\overline{\Sigma}}. \tag{8}$$

To be more concrete, for

$$\bar\partial = \bar\partial^0 + \eta \quad \text{and} \quad D = \lambda(\partial^0) + \omega$$

where $d = \partial^0 + \bar\partial^0$ is the trivial connection, $\eta \in \Omega^{0,1}(\Sigma, \mathfrak{sl}(2,\mathbb{C}))$, and $\omega \in \Omega^{1,0}(\Sigma, \mathfrak{sl}(2,\mathbb{C}))$, we define the complex conjugate on the trivial $\mathbb{C}^2$-bundle over $\overline{\Sigma}$ to be

$$\bar{\bar\partial} = \partial^0 + \bar\eta \quad \text{and} \quad \bar D = \bar\lambda(\bar\partial^0) + \bar\omega.$$

The map C covers the map

$$\lambda \in \mathbb{C}P^1 \longmapsto \bar{\lambda}^{-1} \in \mathbb{C}P^1.$$

Since C and N commute and both maps are involutive, their composition

$$\mathcal{T} = CN$$

is an involution as well, which covers the fixed-point free involution $\lambda \mapsto -\bar{\lambda}^{-1}$ on $\mathbb{C}P^1$.

2.4. **Real sections.** Consider the anti-holomorphic involution of the associated Deligne-Hitchin moduli space

$$\mathcal{T} = CN \colon \mathcal{M}_{DH} \longrightarrow \mathcal{M}_{DH}$$

covering the antipodal involution

$$\lambda \longmapsto -\bar{\lambda}^{-1}$$

of $\mathbb{C}P^1$. We call a holomorphic section s of $\mathcal{M}_{DH}$ real (with respect to $\mathcal{T}$) if

$$\mathcal{T}(s(\lambda)) = s(-\bar{\lambda}^{-1}) \tag{9}$$

holds for all $\lambda \in \mathbb{C}P^1$.

Example 13. *Twistor lines* (7) *are real holomorphic sections with respect to* $\mathcal{T}$. *Let* (∇, Ψ) *be a solution of the self-duality equations on* Σ *with respect to the standard hermitian metric on* $\underline{\mathbb{C}}^2 \to \Sigma$. *Because we are dealing with* $\mathfrak{sl}(2,\mathbb{C})$*-matrices, the unitary connection* ∇ *satisfies*

$$\nabla = \nabla^* = \bar{\nabla}. \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which is equivalent to

$$\bar{\partial}^\nabla = \overline{\partial^\nabla}. \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \textit{and} \quad \partial^\nabla = \overline{\bar{\partial}^\nabla}. \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and analogously we have

$$\Psi = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{\Psi^*} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \textit{and} \quad \Psi^* = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{\Psi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Therefore, the twistor line (7) *satisfies*

$$\begin{aligned} \mathcal{T}(s(\lambda)) &= \mathcal{T}([\lambda, \bar{\partial}^\nabla + \lambda\Psi^*, \Psi + \lambda\partial^\nabla]_M) \\ &= [-\bar{\lambda}^{-1}, \bar{\lambda}^{-1}(\overline{\Psi + \lambda\partial^\nabla}), -\bar{\lambda}^{-1}(\overline{\bar{\partial}^\nabla + \lambda\Psi^*})]_M \\ &= [-\bar{\lambda}^{-1}, \overline{\partial^\nabla} + \bar{\lambda}^{-1}\overline{\Psi}, -\overline{\Psi^*} - \bar{\lambda}^{-1}\overline{\bar{\partial}^\nabla}]_M \\ &= [(-\bar{\lambda}^{-1}, \overline{\partial^\nabla} + \bar{\lambda}^{-1}\overline{\Psi}, -\overline{\Psi^*} - \bar{\lambda}^{-1}\overline{\bar{\partial}^\nabla}). \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}]_M \\ &= [(-\bar{\lambda}^{-1}, \bar{\partial}^\nabla - \bar{\lambda}^{-1}\Psi^*, \Psi - \bar{\lambda}^{-1}\partial^\nabla)]_M = s(-\bar{\lambda}^{-1}). \end{aligned} \tag{10}$$

Let $s(\lambda)$ be a real holomorphic section with $\nabla^\lambda \sim \lambda^{-1}\Psi + \nabla + \dots$ being a lift to the space of flat connections, and such that $(\bar{\partial}^\nabla, \Psi)$ is stable. For the existence of such lifts see [5, Lemma 2.2]. Then the reality condition (9) gives rise to a family of gauge transformations $g(\lambda)$ satisfying

$$\overline{\nabla^{-\bar{\lambda}^{-1}}} = \nabla^\lambda . g(\lambda).$$

Applying this equation twice we obtain

$$\nabla^\lambda . g(\lambda) \overline{g(-\bar{\lambda}^{-1})} = \nabla^\lambda.$$

Because the section s is stable, the connections ∇^λ are irreducible for all $\lambda \in \mathbb{C}^*$. Therefore $g(\lambda)\overline{g(-\bar{\lambda}^{-1})}$ is a constant multiple of the identity for every $\lambda \in \mathbb{C}^*$. By [16, Lemma 1.18] we can choose g to be $\mathrm{SL}(2,\mathbb{C})$-valued yielding

$$g(\lambda)\overline{g(-\bar{\lambda}^{-1})} = \pm \mathrm{Id} \tag{11}$$

and the sign on the right hand side is independent of the lift ∇^λ of s and is preserved in a connected component of real sections motivating the following definition.

Definition 14. [5, Definition 2.16] *A stable real holomorphic section s of $\mathcal{M}_{DH}$ is called positive or negative depending on the sign of* (11).

Example 15. *Twistor lines are always negative sections. In fact, the associated family of flat connections is a canonical lift of the twistor line to the space of flat connections, and with respect to the standard hermitian structure on $\mathbb{C}^2$ the gauge*

$$g(\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

as in (11)*, is constant in λ and to $-Id$.*

The space of real sections of $\mathcal{M}_{DH}$ has multiple connected components. A negative real section lying in the connected component of the twistor lines must be a twistor line itself by Theorem 3. It corresponds therefore to a global solution of the self-duality equation.

2.5. **Reconstruction of self-duality solutions from admissible negative real sections.** A section s of the Deligne-Hitchin moduli space has lifts to families of flat connections in both Hodge moduli spaces. Let

$$\tilde{\nabla}^\lambda = \lambda^{-1}\Psi_1 + \nabla + \text{higher order terms in } \lambda$$

be a lift around $\lambda = 0$. Due to s being real, there exists a family of gauges $g(\lambda)$ satisfying

$$\overline{\tilde{\nabla}^{-\bar{\lambda}^{-1}}} = \tilde{\nabla}^\lambda . g(\lambda).$$

Assume that the Birkhoff factorization

$$g(\lambda) = g^+(\lambda)g^-(\lambda)$$

such that g^+ extends to $\lambda = 0$ and g^- extends to $\lambda = \infty$ exist for every $z \in \Sigma$, see [27] for details about loop groups. This factorization is unique up to the multiplication with a λ-independent B, i.e., let $g(\lambda) = \tilde{g}^+(\lambda)\tilde{g}^-(\lambda)$ be a second splitting, then there exist a $B\colon \Sigma \to \mathrm{SL}(2,\mathbb{C})$ such that

$$\tilde{g}^+(\lambda) = g^+(\lambda)B^{-1} \quad \text{and} \quad \tilde{g}^-(\lambda) = Bg^-(\lambda).$$

Moreover, assume that the section s is negative, i.e,

$$\overline{g(-\bar{\lambda}^{-1})}^{-1} = \overline{g^-(-\bar{\lambda}^{-1})}^{-1}\overline{g^+(-\bar{\lambda}^{-1})}^{-1} = -g(\lambda),$$

and since $\lambda \to -\bar{\lambda}^{-1}$ interchanges the $()^+$ and $()^-$ parts of the Birkhoff factorization, the uniqueness assertion gives

$$g^+(\lambda) = -\overline{g^-(-\bar{\lambda}^{-1})}^{-1}B^{-1} \quad \text{and} \quad g^-(\lambda) = B\overline{g^+(-\bar{\lambda}^{-1})}^{-1}$$

for some $B\colon \Sigma \to \mathrm{SL}(2,\mathbb{C})$ with $\bar{B}B = -\mathrm{Id}$. This gives that B lies in the same conjugacy class as $\delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, i.e., there exist G satisfying $\delta = G^{-1}B\overline{G}$. Then

$$\nabla^\lambda = \tilde{\nabla}^\lambda.(g^+(\lambda)G)$$

satisfies $\overline{\nabla^{-\bar{\lambda}^{-1}}} = \nabla^\lambda.\delta$ and is therefore the associated family of a self-duality solution with respect to the standard hermitian metric.

Remark 16. *In this paper we construct real holomorphic sections via loop group methods, i.e, we write down a particular lift of a real section s of the Deligne-Hitchin moduli space around $\lambda = 0$ in terms of a Fuchsian potential η (i.e., a λ-dependent Fuchsian connection 1-form). To obtain the actual harmonic map into the hyperbolic 3-space $\mathbb{H}^3$ (or the associated self-duality solution), we need thus to perform a global Birkhoff factorization and we require negativeness of the section. This needs further conditions, as examples for which the harmonic map into $\mathbb{H}^3$ becomes singular and intersects the boundary at infinity of $\mathbb{H}^3$ exists* [16] *as well as positive real sections with global Birkhoff factorization that give rise to harmonic maps into the de-Sitter 3-space* [5, Theorem 3.4].

By starting at appropriate initial conditions, the constructed solutions lie in the same connected component as unitary connections, i.e., solutions to the self-duality equations with $\Psi = 0$ implying negativeness of the constructed solutions. The global factorization follows (after a positive gauge near the poles) from the fact that the set of loops factoring into positive and negative parts is an open dense subset of the whole loop group, containing the identity, see [27, Chapter 8]. *Therefore, for small weights, our approach gives an entirely complex analytic proof of existence of solutions of the self-duality equations. By using the real analytic dependence of the solutions on the weights by* [23] *and the completeness of the moduli space (Theorem 3), one can actually deduce that the global Birkhoff factorization exists for all t.*

2.6. **Goldman's symplectic form on the moduli space of λ-connections.** Fix $\lambda \in \mathbb{C}$ and consider the space of λ-connections modulo gauge transformations. Let $(\bar{\partial}, D)$ be a λ-connection. A tangent vector to the (infinite dimensional) space of λ-connection is given by

$$(A, B) \in \Omega^{(0,1)}(\mathfrak{sl}(2,\mathbb{C})) \oplus \Omega^{(1,0)}(\mathfrak{sl}(2,\mathbb{C}))$$

satisfying the linearized compatibility (flatness) condition

$$0 = \bar{\partial}B + DA. \tag{12}$$

Tangent vectors at $(\bar{\partial}, D)$ which are generated by the infinitesimal gauge transformation $\xi \in \Gamma(\Sigma, \mathfrak{sl}(2,\mathbb{C})$ are given by

$$(A, B) = (\bar{\partial}\xi, D\xi).$$

The Goldman symplectic structure O^λ on the moduli space of λ-connections [10, Section 1] is defined to be

$$O^\lambda((A_1, B_1), (A_2, B_2)) = 4\int_\Sigma \mathrm{trace}(A_1 \wedge B_2 - A_2 \wedge B_1) \tag{13}$$

for $(A, B) \in \Omega^{(0,1)}(\mathfrak{sl}(2,\mathbb{C})) \oplus \Omega^{(1,0)}(\mathfrak{sl}(2,\mathbb{C}))$ representing tangent vectors. On a compact Riemann surface, the symplectic structure O^λ is gauge invariant, and thus can be computed using arbitrary representatives of the tangent vectors, which makes it well-defined on the moduli space of λ-connections. Since we are not aware of any explicit reference (except for

$\lambda = 0$ and $\lambda = 1$), we give the simple proof (which is of course an instance of an infinite-dimensional symplectic reduction for the gauge group action with the curvature as moment map, see [1]).

Lemma 17. *Let $\lambda \in \mathbb{C}$ be fixed. The holomorphic symplectic form O^λ is well-defined on the moduli space of λ-connections.*

Proof. A gauge transformation acts on tangent vectors to the space of connections by conjugation, and since the trace is conjugation invariant we obtain

$$O^\lambda((A_1, B_1).g, (A_2, B_2).g) = O^\lambda((A_1, B_1), (A_2, B_2)).$$

Let (A_1, B_1) satisfy $0 = \bar\partial B + DA$ and let $(A_2, B_2) = (\bar\partial\xi, D\xi)$ be a tangent vector that is also tangent to the gauge orbit. Then,

$$\begin{aligned} O^\lambda((A_1, B_1), (A_2, B_2)) &= 4\int_\Sigma \operatorname{trace}(A_1 \wedge D\xi - \bar\partial\xi \wedge B_1) \\ &= -4\int_\Sigma d\operatorname{trace}(A_1\xi + \xi B_1) - 4\int_\Sigma \operatorname{trace}(DA_1\xi + \xi\bar\partial B_1) = 0. \end{aligned} \tag{14}$$

Therefore, O^λ is well-defined on the quotient space (the moduli space of λ-connections). By construction the form is closed and of type $(2, 0)$ as required for a holomorphic symplectic form. □

The Goldmann symplectic form in fact coincides (up to scaling) with the twisted holomorphic symplectic form (4) on the twistor space $\mathcal{P} = \mathcal{M}_{SD} \times \mathbb{C}P^1$ of the moduli space of self-duality solutions, also compare with [14, Equation (4.10)].

Proposition 18. *Let $\lambda \in \mathbb{C} \subset \mathbb{C}P^1$ Then, the fiber $\mathcal{P}_\lambda = \pi^{-1}(\lambda)$ is the moduli space of λ-connections, and*

$$\lambda\varpi_{|\mathcal{P}_\lambda}$$

is the Goldman symplectic form O^λ on the moduli space of λ-connections.

Proof. We have discussed that $\mathcal{P}_\lambda = \pi^{-1}(\{\lambda\})$ is the moduli space of λ-connections by [31, Theorem 4.2].

In order to evaluate the corresponding symplectic forms, we need to first find appropriate representatives of tangent vectors. For doing so, we fix h to be a standard metric (even though this is not necessary by [13]). The tangent space of $\mathcal{M}_{SD}$ at (the representative) (∇, Ψ) is given by those

$$(\xi, \phi) \in \Omega^{(0,1)}(\mathfrak{sl}(2, \mathbb{C})) \oplus \Omega^{(1,0)}(\mathfrak{sl}(2, \mathbb{C}))$$

satisfying

$$\begin{aligned} 0 &= d^\nabla(\xi - \xi^*) + \tfrac{1}{2}[\phi \wedge \Psi^*] + \tfrac{1}{2}[\Psi \wedge \phi^*] \\ 0 &= \bar\partial^\nabla + [\xi, \Psi] \end{aligned} \tag{15}$$

modulo infinitesimal gauge deformation (see [21, (6.1)]). These equations mean that we have an infinitesimal deformation

$$t \mapsto (\nabla + t(\xi - \xi^*), \Psi + t\phi)$$

of a solution to the self-duality equation (for the fixed hermitian metric), and we can choose harmonic representatives which are orthogonal to the (unitary) gauge orbit, which is equivalent to

$$\mathcal{D}^*(\xi,\phi)=0 \tag{16}$$

where

$$\mathcal{D}\colon \Gamma(\Sigma,\mathfrak{su}(2))\to\Omega^{(0,1)}(\Sigma,\mathfrak{sl}(2))\oplus\Omega^{(1,0)}(\Sigma,\mathfrak{sl}(2));\quad \psi\mapsto\big((d^\nabla\psi)'',[\Psi,\psi]\big)$$

see [21, 13]. Provided (15), (16) hold, the complex structures $I,J,K=IJ$ are given by

$$I(\xi,\phi)=(i\xi,i\phi)\quad J(\xi,\phi)=(i\phi^*,-i\xi^*)\quad\text{and}\quad K(\xi,\phi)=(-\phi^*,\xi^*), \tag{17}$$

see [21, page 109]. Provided (15), (16) hold for $k=1,2$, the Hitchin metric (see [21, (6.2)]) is defined to be

$$g\left((\xi_1,\phi_1),(\xi_2,\phi_2)\right):=2i\int_\Sigma \operatorname{trace}\left(\xi_1^*\wedge\xi_2+\xi_2^*\wedge\xi_1+\phi_1\wedge\phi_2^*+\phi_2\wedge\phi_1^*\right).$$

Thus

$$\begin{aligned}\omega_I((\xi_1,\phi_1),(\xi_2,\phi_2))&=-g((\xi_1,\phi_1),(i\xi_2,i\phi_2))\\&=2\int_\Sigma\operatorname{trace}(\xi_1^*\wedge\xi_2-\xi_2^*\wedge\xi_1-\phi_1\wedge\phi_2^*+\phi_2\wedge\phi_1^*)\\ \omega_J((\xi_1,\phi_1),(\xi_2,\phi_2))&=2\int_\Sigma\operatorname{trace}(\xi_1^*\wedge\phi_2^*-\phi_2\wedge\xi_1+\phi_1\wedge\xi_2-\xi_2^*\wedge\phi_1^*)\\ \omega_K((\xi_1,\phi_1),(\xi_2,\phi_2))&=-2i\int_\Sigma\operatorname{trace}(-\xi_1^*\wedge\phi_2^*-\phi_2\wedge\xi_1+\phi_1\wedge\xi_2+\xi_2^*\wedge\phi_1^*)\end{aligned}$$

which gives

$$(\omega_J+i\omega_K)(\xi_1,\phi_1),(\xi_2,\phi_2)=-4\int_\Sigma\operatorname{trace}(\phi_2\wedge\xi_1-\phi_1\wedge\xi_2).$$

Recall that twistor lines corresponding to the self duality solution (∇,Ψ) are given by

$$\lambda\mapsto(\lambda,\bar\partial^\nabla+\lambda\Psi^*,\Psi+\lambda\partial^\nabla)_\Sigma$$

or equivalently by the associated family of flat connections

$$\lambda\in\mathbb{C}^*\mapsto\nabla+\lambda^{-1}\Psi+\lambda\Psi^*. \tag{18}$$

Thus, the tangent vectors $X=(\xi_1,\phi_1)$ and $Y=(\xi_2,\phi_2)$ correspond to the normal bundle sections given by

$$X:\lambda\mapsto(0,\xi_1+\lambda\phi_1^*,\phi_1-\lambda\xi_1^*)$$

and

$$Y:\lambda\mapsto(0,\xi_2+\lambda\phi_2^*,\phi_2-\lambda\xi_2^*).$$

Therefore, when fixing $\lambda\in\mathbb{C}$, we obtain from (13)

$$\begin{aligned}O^\lambda(X,Y)=&O^\lambda((\xi_1+\lambda\phi_1^*,\phi_1-\lambda\xi_1^*),(\xi_2+\lambda\phi_2^*,\phi_2-\lambda\xi_2^*))\\=&(\omega_J+i\omega_K)(X,Y)-2\lambda\omega_I(X,Y)-\lambda^2(\omega_J-i\omega_K)(X,Y).\end{aligned}$$

□

2.7. **The Goldman symplectic form for the space of logarithmic connections.** See [3] or [2] for details about Poison and symplectic structures on moduli spaces of flat connections on punctured Riemann surfaces. We include the following discussion only for convenience to the reader.

Let Σ be a compact Riemann surface, $p_1, \dots, p_n \in \Sigma$ be pairwise disjoint. Fix $\mathrm{SL}(2,\mathbb{C})$ conjugacy classes

$$\mathcal{C}_1, \dots, \mathcal{C}_n$$

of (non-zero) diagonal matrices $C_1, \dots, C_n \in \mathfrak{sl}(2,\mathbb{C})$. Consider the infinite dimensional space $\mathcal{A}$ of flat $\mathrm{SL}(2,\mathbb{C})$-connections ∇ on

$$\Sigma^0 := \Sigma \setminus \{p_1, \dots, p_n\}$$

which are of the form

$$\nabla = A_j \frac{dz_j}{z_j} + \text{smooth connection} \tag{19}$$

with $A_j \in \mathcal{C}_j$, and with respect to centered holomorphic coordinate z_j at p_j.

Lemma 19. *Let $X \in \Omega^1(\Sigma^0, \mathfrak{sl}(2,\mathbb{C}))$ be a tangent vector to $\nabla \in \mathcal{A}$. Then, there exists smooth $\xi \in \Gamma(\Sigma, \mathfrak{sl}(2,\mathbb{C}))$ and $\hat{X} \in \Omega^1(\Sigma, \mathfrak{sl}(2,\mathbb{C}))$ such that*

$$X = d^\nabla \xi + \hat{X} \quad \textit{and} \quad d^\nabla \hat{X} = 0$$

on the punctured surface Σ^0.

Proof. Since X is a tangent vector at ∇, we have in particular $d^\nabla X = 0$. Moreover, X preserves the form of Equation (19) therefore we can write

$$X = [A_j, \xi_j] \frac{dz_j}{z_j} + \tilde{X}$$

around p_j for appropriate $\xi_j \in \mathfrak{sl}(2,\mathbb{C})$ and smooth $\tilde{X}$. Now, consider any section $\xi \in \Gamma(\Sigma, \mathfrak{sl}(2,\mathbb{C}))$ with $\xi(p_j) = \xi_j$. Then the splitting

$$X = d^\nabla \xi + (X - d^\nabla \xi)$$

is of the required form. □

Lemma 20. *Let $\nabla \in \mathcal{A}$ with A_j as in (19) and $X = d^\nabla \xi + \hat{X}$, $Y = d^\nabla \mu + \hat{Y} \in T_\nabla \mathcal{A}$ for $\xi, \mu \in \Gamma(\Sigma, \mathfrak{sl}(2,\mathbb{C}))$ and $\hat{X}, \hat{Y} \in \Omega^1(\Sigma, \mathfrak{sl}(2,\mathbb{C}))$. Then*

$$\int_{\Sigma^0} \mathrm{trace}\,(X \wedge Y) = \int_\Sigma \mathrm{trace}\left(\hat{X} \wedge \hat{Y}\right) - 2\pi i \sum_{j=1}^n \mathrm{trace}\,(A_j[\xi(p_j), \mu(p_j)])\,.$$

Proof. Consider the punctured Riemann surface Σ and a centered holomorphic coordinate z_j around a puncture p_j. For $t > 0$ small let

$$\gamma_t \colon S^1 \to \Sigma^0; \quad e^{2\pi i \varphi} \mapsto (z_j)^{-1}(te^{2\pi i \varphi}).$$

Then we have with respect to the splittings of X and Y

$$\int_{\Sigma^0} \mathrm{trace}\,(X \wedge Y) = \int_{\Sigma^0} \mathrm{trace}\left(\hat{X} \wedge \hat{Y}\right) + \mathrm{trace}\left(d^\nabla \xi \wedge \hat{Y}\right) + \mathrm{trace}\left(\hat{X} \wedge d^\nabla \mu\right) + \mathrm{trace}\left(d^\nabla \xi \wedge d^\nabla \mu\right).$$

We need to compute the second till the fourth term in the above expression.

$$\begin{aligned}\int_{\Sigma^0} \operatorname{trace}\left(d^\nabla \xi \wedge \hat{Y}\right) &= -\int_{\Sigma^0} d\operatorname{trace}(\xi \hat{Y}) + \int_{\Sigma^0} \operatorname{trace}(\xi (d^\nabla \hat{Y})) \\ &= \lim_{t\to 0} \sum_j \int_{\gamma_t} \operatorname{trace}(\xi \hat{Y}) = 0,\end{aligned} \tag{20}$$

since $d^\nabla \hat{Y} = 0$ and $\hat{Y}, \xi$ are both smooth on Σ. Analogously, we have

$$\int_{\Sigma^0} \operatorname{trace}(\hat{X} \wedge d^\nabla \mu) = 0.$$

For the last term we have

$$\begin{aligned}\int_{\Sigma^0} \operatorname{trace}(d^\nabla \xi \wedge d^\nabla \mu) &= \int_{\Sigma^0} d\operatorname{trace}(\xi d^\nabla \mu) = -\lim_{t\to 0} \sum_j \int_{\gamma_t} \operatorname{trace}(\xi d^\nabla \mu) \\ &= -\lim_{t\to 0} \sum_j \int_{\gamma_t} \operatorname{trace}(\xi [A_j, \mu] \frac{dz_j}{z_j}) - \lim_{t\to 0} \sum_j \int_{\gamma_t} \operatorname{trace}(\text{something smooth}) \\ &= -2\pi i \sum_j \operatorname{trace}(\xi(p_j)[A_j, \mu(p_j)]) = -2\pi i \sum_j \operatorname{trace}(A_j[\xi(p_j), \mu(p_j)])\end{aligned} \tag{21}$$

□

This Lemma allows to define a holomorphic complex bilinear and skew-symmetric form on the (infinite dimensional) tangent space $T_\nabla \mathcal{A}$ by

$$\mathcal{O} \colon T_\nabla \mathcal{A} \times T_\nabla \mathcal{A} \to \mathbb{C}$$

$$(X, Y) \mapsto \int_{\Sigma^0} \operatorname{trace}(X \wedge Y) + 2\pi i \sum_j \operatorname{trace}(\operatorname{Res}_{p_j}(\nabla)[\operatorname{Res}_{p_j}(X), \operatorname{Res}_{p_j}(Y)]).$$

Corollary 21. *For $d^\nabla \xi, Y \in T_\nabla \mathcal{A}$ we have*

$$O(d^\nabla \xi, Y) = 0.$$

Therefore, the bilinear form O descents to a well defined holomorphic 2-form on the quotient of $\mathcal{A}$ by gauge transformations.

Remark 22. *When the infinitesimal deformation of the flat connection preserves the holomorphic structure of Σ^0, i.e., the tangent vector fields X, Y are meromorphic and the first term when evaluating the bilinear form O vanishes, and only the residue term remains. For $A \in \mathcal{O}_j$, the complex skew bilinear form*

$$[A, X], [A, Y] \in T_A \mathcal{O}_j \mapsto \operatorname{trace}(A[X, Y])$$

is well-defined and known as the Kirillov symplectic form on the adjoint orbit $\mathcal{O}_j$.

3. Ansatz and initial properties at $t = 0$.

In this section we write down an ansatz for potentials (λ-dependent connection 1-forms) on the 4-puncture sphere $\Sigma = \Sigma_p$ depending on the parabolic weight $t \in (-\frac{1}{2}, \frac{1}{2})$. We then explicitly compute the initial conditions at $t = 0$ which we will deform via an implicit function theorem argument in Section 4 to obtain equivariant harmonic maps from Σ_p into $\mathbb{H}^3$ for $t \neq 0$.

3.1. **The potential.** To construct real holomorphic sections explicitly, we restrict to the case where the underlying holomorphic structure is trivial for all $\lambda \in \mathbb{D}_a = \{\lambda \mid |\lambda|^2 < a^2\}$ for some $a > 1$. Then the connection 1-form is a holomorphic and complex linear

$$\eta \in \Omega^{1,0}(\Sigma, \Lambda\mathfrak{sl}(2,\mathbb{C}))$$

with

$$(\lambda\eta) \in \Omega^{1,0}(M, \Lambda_+\mathfrak{sl}(2,\mathbb{C})),$$

where

$$\begin{aligned} \Lambda\mathfrak{sl}(2,\mathbb{C}) &:= \{\eta\colon \mathbb{S}^1 \xrightarrow{C^\infty} \mathfrak{sl}(2,\mathbb{C})\} \\ \Lambda_+\mathfrak{sl}(2,\mathbb{C}) &:= \{\eta \in \Lambda_+\mathfrak{sl}(2,\mathbb{C}) \mid \eta \text{ extends holomorphically to } \mathbb{D}_a\} \end{aligned} \tag{22}$$

with $(\lambda\eta)$ denoting pointwise multiplication. Its residue at $\lambda = 0$

$$\eta_{-1} := \mathrm{Res}_{\lambda=0}(\eta)$$

is the (parabolic) Higgs field. Such an η will be referred to as a potential. On the 4-punctured sphere we can further specialize to η being a Fuchsian system for every $\lambda \in \mathbb{D}_a \setminus \{0\}$, i.e., we consider potentials of the form

$$\eta = \sum_{j=1}^{4} B_j \frac{dz}{z - p_j}, \qquad \text{and} \qquad \sum_{j=0}^{4} B_j = 0$$

with z being the homogenous coordinate of $\mathbb{C}P^1$, $B_j \in \Lambda_+\mathfrak{sl}(2,\mathbb{C})$. To be more explicit, the coefficients of λB_j, as functions of λ, are in the functional space

$$\mathcal{W}_a^{\geq 0} := \left\{ h = \sum_{k=0}^{\infty} h_k \lambda^k \mid \sum_{k=0}^{\infty} |h_k| a^k < \infty \right\}$$

so have convergent power series in the disk of radius $a > 1$. We denote the subspace of convergent power series with vanishing constant term h_0 by $\mathcal{W}_a^+$. Whenever the dependence on a particular a is not important, we will omit this index. More generally, $\mathcal{W}_a$ denote the space of convergent Laurent series on the annulus $\mathbb{A}_a := \{\frac{1}{a^2} < |\lambda|^2 < a^2\}$. Elements of $h \in \mathcal{W}$ will be splitted into its positive part $h^+ \in \mathcal{W}^+$ and the constant part $h^0 = h_0$ and the negative part $h^- = h - h^+ - h^0$.

To fix further notations let $\mathbb{C}^+$ denote the upper-right quadrant of the plane

$$\mathbb{C}^+ = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$$

and let $p \in \mathbb{C}^+$ and consider the four-punctured sphere

$$\Sigma = \Sigma_p = \mathbb{C}P^1 \setminus \{p_1, p_2, p_3, p_4\}$$

with

$$p_1 = p, \quad p_2 = -1/p, \quad p_3 = -p \quad \text{and} \quad p_4 = 1/p. \tag{23}$$

Up to Möbius transformations, every four-punctured sphere is of this form. By definition Σ is invariant under the holomorphic involutions $\delta(z) = -z$ and $\tau(z) = 1/z$. Moreover, consider the Pauli matrices

$$\mathfrak{m}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathfrak{m}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathfrak{m}_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and the holomorphic 1-forms on Σ

$$\begin{aligned}\omega_1 &= \frac{dz}{z-p_1} - \frac{dz}{z-p_2} + \frac{dz}{z-p_3} - \frac{dz}{z-p_4}\\ \omega_2 &= \frac{dz}{z-p_1} - \frac{dz}{z-p_2} - \frac{dz}{z-p_3} + \frac{dz}{z-p_4}\\ \omega_3 &= \frac{dz}{z-p_1} + \frac{dz}{z-p_2} - \frac{dz}{z-p_3} - \frac{dz}{z-p_4}.\end{aligned} \tag{24}$$

Then the ω_i have the symmetries

$$\begin{cases}\delta^*\omega_1 = \omega_1, \quad \delta^*\omega_2 = -\omega_2, \quad \delta^*\omega_3 = -\omega_3\\ \tau^*\omega_1 = -\omega_1, \quad \tau^*\omega_2 = \omega_2, \quad \tau^*\omega_3 = -\omega_3.\end{cases} \tag{25}$$

In the following we will consider a potential of the form

$$\eta_t = t\sum_{j=1}^{3} x_j(\lambda)\mathfrak{m}_j\omega_j \tag{26}$$

where $t \sim 0$ is a real parameter and $\lambda x_1,\ \lambda x_2,\ \lambda x_3 \in \mathcal{W}^{\geq 0}$ are parameters. We aim to determine η_t in dependence of t through the implicit function theorem by imposing the reality condition. We denote the parameter vector then by $x = (x_1, x_2, x_3) \in (\mathcal{W}^{\geq 0})^3$. In particular, η_t is a Fuchsian system for every $\lambda \in \mathbb{D}_a \setminus \{0\}$ and the residues at a puncture is given by a $\Lambda\mathfrak{sl}(2,\mathbb{C})$ element

$$\mathrm{Res}_{z=p_j}\eta_t = tA_j$$

with

$$\begin{aligned} A_1 &= \begin{pmatrix} x_1 & x_2+ix_3 \\ x_2-ix_3 & -x_1\end{pmatrix} \qquad A_2 = \begin{pmatrix} -x_1 & -x_2+ix_3 \\ -x_2-ix_3 & x_1\end{pmatrix}\\ A_3 &= \begin{pmatrix} x_1 & -x_2-ix_3 \\ -x_2+ix_3 & -x_1\end{pmatrix} \quad A_4 = \begin{pmatrix} -x_1 & x_2-ix_3 \\ x_2+ix_3 & x_1\end{pmatrix}\end{aligned} \tag{27}$$

satisfying

$$\det(A_j) = -x_1^2 - x_2^2 - x_3^2 \tag{28}$$

for all $j = 1, ..., 4$.

This ansatz is chosen such that the potential has the following symmetries

$$\begin{aligned}\delta^*\eta_t &= D^{-1}\eta_t D \quad \text{with} \quad D = \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}\\ \tau^*\eta_t &= C^{-1}\eta_t C \quad \text{with} \quad C = \begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}.\end{aligned} \tag{29}$$

In the following section we obtain from the implicit function theorem that these symmetries are automatically satisfied when all parabolic weights at the four punctures of Σ are equal.

Remark 23. *With this ansatz we only study harmonic maps for which the monodromy become trivial for $t \to 0$ and consequently the corresponding Higgs fields $\Psi(t) \to 0$.*

Remark 24. *When comparing to the notations of* [20] *we have $x_1 = ia$, $x_2 = b$ and $x_3 = c$. Moreover, we adjusted the Pauli matrix $\mathfrak{m}_1$ as well. These changes of notations are to facilitate writing compact and symmetric formulas with respect to the parameters x_j.*

3.2. **Potentials and Higgs fields.** A potential of the form (26) with non-vanishing λ^{-1}-part

$$\Psi := \mathrm{res}_{\lambda=0}\eta_t$$

naturally defines a strongly parabolic Higgs bundle where the underlying holomorphic vector bundle is trivial:

Clearly, the functions x_1, x_2, x_3 have first order poles at $\lambda = 0$. Moreover, from (26) we obtain that the non-vanishing λ^{-1}-part Ψ is a meromorphic and $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form on the 4-punctured sphere whose residues at the singular points are nilpotent and non-vanishing. Define the parabolic lines at p_i to be the kernels of the residue at p_i, equipped with parabolic weight t, and equip the full 2-dimensional fibre with parabolic weight $-t$.

By construction, the parabolic Higgs pair consisting of the parabolic Higgs field and the parabolic structure, is uniquely determined by the nilpotent $\mathfrak{sl}(2,\mathbb{C})$-matrix $A_1^{(-1)}$. In fact, we have

$$\Psi = A_1^{(-1)}\frac{dz}{z-p_1} + A_2^{(-1)}\frac{dz}{z-p_2} + A_3^{(-1)}\frac{dz}{z-p_3} + A_4^{(-1)}\frac{dz}{z-p_4}$$

with

$$A_3^{(-1)} = D^{-1}A_1^{(-1)}D, \quad A_2^{(-1)} = (CD)^{-1}A_1^{(-1)}(CD), \quad A_3^{(-1)} = D^{-1}A_1^{(-1)}D, \quad A_4^{(-1)} = C^{-1}A_1^{(-1)}C.$$

Moreover, conjugation of $A_1^{(-1)}$ by C or D leads to a conjugation of the corresponding Higgs field Ψ. To facilitate the notation, we will omit the superscript of $A_1^{(-1)}$ in the following.

Lemma 25. *Let $0 \neq A_1 \in \mathfrak{sl}(2,\mathbb{C})$ be nilpotent and $g \in \mathrm{SL}(2,\mathbb{C})$. The parabolic Higgs fields Ψ and $\tilde{\Psi}$ corresponding to A_1 and $\tilde{A}_1 = g^{-1}A_1 g$ are gauge equivalent if and only if*

$$g \in < C, D > \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Moreover, all these parabolic Higgs fields gives rise to semistable parabolic structures which are generically strictly stable as we require $t \in (0, \frac{1}{4})$. More precisely, we have

Lemma 26. *Let $p_1, \ldots, p_4$ as in (23) and let $(u,v) \in \mathbb{C}^2 \setminus \{0\}$ with*

$$A_1 = \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix}.$$

Then, the corresponding strongly parabolic Higgs pair is semistable. The underlying parabolic structure is strictly stable unless $uv = 0$, $u^2 = v^2$ or $u^2 = -v^2$.

Proof. Since $4t < 1$ we only have to look for degree 0 line subbundles given by complex lines in $\mathbb{C}^2$ as possible destabilising bundles. The kernels $L_1, ..., L_4$ of the residues at $p_1, ..., p_4$ are given by

$$L_1 = \mathbb{C}(u,v), \quad L_2 = \mathbb{C}(-v,u), \quad L_3 = \mathbb{C}(-u,v), \quad L_4 = \mathbb{C}(v,u),$$

respectively. Thus, at most two of these lines can coalesce, and this happens if and only if $uv = 0$, $u^2 = v^2$ or $u^2 = -v^2$.

In these cases, one can compute that the corresponding Higgs field Ψ has non-trivial determinant with simple poles at $p_1, \ldots, p_4$, and therefore must be semistable. In fact an easy computation shows that the Higgs pairs are polystable. Moreover, the Higgs fields with respect to $u = 0$ and $v = 0$ gives rise to gauge equivalent parabolic structures, so does $u = v$ and $u = -v$ as well as $u = iv$ and $u = -iv$. □

We call the parabolic Higgs fields Ψ induced by a single matrix $A_1 \in \mathfrak{sl}(2,\mathbb{C})$ as *symmetric*. The following lemma can then easily be proven.

Lemma 27. *Consider on the holormorphic trivial rank* 2 *bundle a parabolic structure on the 4-punctured sphere with parabolic weights* $\pm t$, *with* $t \in (0, \frac{1}{4})$, *at all four singular points. Then every parabolic structure admit a symmetric parabolic Higgs field. Vice versa, every non-zero polystable strongly parabolic Higgs field is gauge equivalent to a symmetric one.*

Proposition 28. *The moduli space of all polystable strongly parabolic Higgs bundles as in Lemma 27 is given by the cotangent bundle of* $\mathbb{C}P^1$ *modulo a natural* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *action.*

Proof. The gauge class of a polystable strongly parabolic Higgs bundle can be represented with a symmetric parabolic Higgs field induced by a non-zero nilpotent $A_1 \in \mathfrak{sl}(2,\mathbb{C})$. The matrix A_1 is uniquely determined up to sign by the spin covering

$$(u,v) \in \mathbb{C}^2 \setminus \{0\} \mapsto \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix}.$$

Recall that the kernel of A_1 determines the induced parabolic structure. Moreover, by Lemma 25, the gauge class of a strongly parabolic Higgs field (or the induced parabolic structure) determines A_1 unique up to conjugation by the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by C and D. Therefore, up to the action induced by C and D, the moduli space is the blow up of $\mathbb{C}^2 \setminus \{0\}/\mathbb{Z}_2$, which can be identified with the cotangent bundle of $\mathbb{C}P^1$. □

3.3. **The choice of initial data at $t = 0$, and parabolic Higgs fields.** We denote the initial value of the parameters with an underscore. These are chosen such that $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$ at $t = 0$ satisfy

$$\begin{cases} \det(\underline{A}_1) = -1 \\ \underline{A}_1 = \underline{A}_1^*. \end{cases} \tag{30}$$

The $()^*$-operator for matrices is hereby defined as

$$M^*(\lambda) = \overline{M(-1/\overline{\lambda})}^T.$$

The first equation in (30) will later give that the local monodromies of the associated Fuchsian systems lie in the conjugacy class of $\mathrm{diag}(\exp(2\pi i t), \exp(-2\pi i t))$ for every $\lambda \in \mathbb{D}_a$. The second equation in (30) gives an infinitesimal version of $\mathcal{T}$-reality at $t = 0$. The equations (30) on the matrices are equivalent to

$$\begin{cases} \underline{x}_1^2 + \underline{x}_2^2 + \underline{x}_3^2 = 1 \\ \underline{x}_j = \underline{x}_j^*, \quad \forall j, \end{cases}$$

where the induced $()^*$-operator for functions is defined to be

$$f^*(\lambda) = \overline{f(-1/\overline{\lambda})}. \tag{31}$$

Since potentials have at most a first order pole at $\lambda = 0$ each eligible $\underline{x}_j$ must be a degree 1 Laurent polynomial

$$\underline{x}_j = \underline{x}_{j,-1}\lambda^{-1} + \underline{x}_{j,0} + \underline{x}_{j,1}\lambda$$

with

$$\begin{cases} \underline{x}_{j,0} \in \mathbb{R} \\ \underline{x}_{j,1} = -\overline{\underline{x}_{j,-1}}. \end{cases}$$

Then $\underline{x}_1^2 + \underline{x}_2^2 + \underline{x}_3^2 = 1$ is equivalent to

$$\begin{cases} \sum_{j=1}^{3} \underline{x}_{j,-1}^2 = 0 \\ \sum_{j=1}^{3} \underline{x}_{j,-1}\underline{x}_{j,0} = 0 \\ \sum_{j=1}^{3} \underline{x}_{j,0}^2 - 2|\underline{x}_{j,-1}|^2 = 1. \end{cases} \tag{32}$$

For the first equation, consider the standard parametrization of the quadric $\{x^2+y^2+z^2=0\}$ in $\mathbb{C}^3$ given by

$$\underline{x}_{1,-1} = u\,v, \quad \underline{x}_{2,-1} = \tfrac{1}{2}(v^2-u^2) \quad \text{and} \quad \underline{x}_{3,-1} = \tfrac{i}{2}(u^2+v^2) \tag{33}$$

with $(u,v) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Then the second equation of (32) gives

$$u\,v\,\underline{x}_{1,0} + \tfrac{1}{2}(v^2-u^2)\underline{x}_{2,0} + \tfrac{i}{2}(u^2+v^2)\underline{x}_{3,0} = 0 \tag{34}$$

and its (real) solutions are

$$\underline{x}_{1,0} = \rho(|u|^2-|v|^2), \quad \underline{x}_{2,0} = 2\rho\,\mathrm{Re}(u\overline{v}) \quad \text{and} \quad \underline{x}_{3,0} = 2\rho\,\mathrm{Im}(u\overline{v}), \qquad \text{for some } \rho \in \mathbb{R}. \tag{35}$$

Finally, the third equation of (32)

$$\rho^2(|u|^2+|v|^2)^2 - (|u|^2+|v|^2)^2 = 1$$

determines ρ up to sign. The geometric meaning of these equations and its relationship to the classical Weierstrass representation of minimal surfaces in Euclidean 3-space is discussed in Section 5.1. In particular, we obtain from Lemma 39 that the compatible ρ is

$$\rho = -\sqrt{1+(|u|^2+|v|^2)^{-2}}. \tag{36}$$

We have thus fixed the initial conditions of η_t depending on a pair $(u,v) \in \mathbb{C}^2 \setminus \{(0,0)\}$ which parametrizes the space of all eligible non-vanishing λ-residues of the η_t. This gives a 4-fold covering of the open stratum of the moduli space of parabolic Higgs fields. In fact, the symmetries (29) uniquely determine the potentials (27) up to conjugation with C and D for $t > 0$, see also [17, Lemma 17].

Convention 29. *In the following, we slightly abuse the notation and neglect the $\mathbb{Z}_2 \times \mathbb{Z}_2$ issue when referring to completion of the nilpotent orbit as the moduli space of parabolic Higgs fields. This 4-fold covering can be identified with the space of parabolic Higgs fields on the 1-punctured torus (with parabolic weight $\frac{1}{2}-2t$). We will also ignore the complex line of stable parabolic Higgs fields with underlying holomorphic structure $\mathcal{O}(-1)\oplus\mathcal{O}(1)$ at $t=0$, since the $t \to 0$ limit of corresponding representations are far away from the identity.*

Remark 30. *When considering* (26) *for $t > 0$ we have to rephrase the second condition in* (30) *encoding the reality condition at $t = 0$. We require only that η_t^* and η_t lie in the same gauge class. Moreover we fix the λ-residue of η_t at $\lambda = 0$ for all t. This corresponds to constructing families of flat connections that descent to real holomorphic sections of $\mathcal{M}_{DH}(t)$ with prescribed Higgs pair at $\lambda = 0$. Due to this choice of initial conditions, the constructed family of flat connection is gauge equivalent to the associated family of flat connections ∇^λ of an equivariant harmonic map into hyperbolic 3-space, i.e., a twistor line.*

The moduli space of $\mathrm{SL}(2,\mathbb{C})$-Fuchsian systems with prescribed local monodromies over the 4-puncture sphere as well as the space of associated strongly parabolic Higgs fields with fixed weights are both complex 2-dimensional. For the potentials η_t, the underlying holomorphic bundle is trivial, and the weight t is fixed at every puncture. Hence, the parabolic structure is entirely determined by the eigenlines of the residues.

Using the above (u,v)-parametrization the parabolic Higgs field $\Psi = \mathrm{Res}_{\lambda=0}\,\eta_t$ is given by

$$\Psi = t\sum_{j=1}^{3} \underline{x}_{j,-1}\mathfrak{m}_j\omega_j$$

and has determinant

$$\det(\Psi) = \frac{-4t^2(u^4-(p^2+p^{-2})u^2v^2+v^4)}{z^4-(p^2+p^{-2})z^2+1}. \tag{37}$$

Its residues at p_j, $j=1,...,4$ are

$$\begin{aligned} \mathrm{Res}_{p_1}\Psi &= t\begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix} \quad & \mathrm{Res}_{p_2}\Psi &= t\begin{pmatrix} -uv & -v^2 \\ u^2 & uv \end{pmatrix} \\ \mathrm{Res}_{p_3}\Psi &= t\begin{pmatrix} uv & u^2 \\ -v^2 & -uv \end{pmatrix} \quad & \mathrm{Res}_{p_4}\Psi &= t\begin{pmatrix} -uv & v^2 \\ -u^2 & uv \end{pmatrix} \end{aligned} \tag{38}$$

which are all non-zero, nilpotent and their 0-eigenlines viewed as points in $\mathbb{C}P^1$ have homogenous coordinates u/v, $-v/u$, $-u/v$ and v/u, respectively. Their cross-ratio determining the parabolic structure is then

$$\left(\frac{u}{v},\frac{-v}{u};\frac{-u}{v},\frac{v}{u}\right) = \frac{-4u^2v^2}{(u^2-v^2)^2},$$

where the cross-ratio is defined to be

$$(z_1,z_2;z_3,z_4) = \frac{(z_3-z_1)(z_4-z_2)}{(z_3-z_2)(z_4-z_1)}.$$

Using the symmetries a Higgs field is uniquely determined by its residue Ψ_1 at $p\in\Sigma$. Moreover, the cross-ratio is already determined by the ratio u/v (since scaling u and v just scales the Higgs field). Thus an open (and dense) subset of (the 4-fold covering of) the moduli space of strongly parabolic Higgs bundles can identified with the nilpotent orbit in $\mathfrak{sl}(2,\mathbb{C})$ which is given by the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ at the origin.

4. Constructing real holomorphic sections

In order for the potential $d-\eta_t$ to be the lift of a real holomorphic section s we first need to impose the gauge equivalence

$$[d-\eta^\lambda] = \left[d-\overline{\eta^{-\bar\lambda^{-1}}}\right]. \tag{39}$$

To make this condition more explicit, we identify the space of gauge equivalence classes of flat connections with the space of representations of $\pi_1(\Sigma,z_0)$ modulo conjugation via the monodromy representation. Let $p\in\mathbb{C}^+$ and $\Sigma=\Sigma_p$ be the 4-punctured sphere and fix the base point $z_0=0$. Choose the generators γ_1, γ_2, γ_3 and γ_4 of the fundamental group $\pi_1(\Sigma,0)$ as in [20], i.e, let γ_1 be the composition of the real half-line from 0 to $+\infty$ with the imaginary half-line from $+i\infty$ to 0, and more generally γ_k is the product of the half-line from 0 to

$i^{k-1}\infty$ with the half line from $i^k\infty$ to 0, so γ_k encloses p_k and $\gamma_1\gamma_2\gamma_3\gamma_4 = 1$. Let Φ_t be the fundamental solution of the Cauchy Problem

$$d_\Sigma \Phi_t = \Phi_t \eta_t \quad \text{with initial condition} \quad \Phi_t(z=0) = \mathrm{Id} \tag{40}$$

and let $M_k(t) = \mathcal{M}(\Phi_t, \gamma_k)$ be the monodromy of Φ_t along γ_k. The moduli space of representations can be parametrized using so-called Fricke coordinates. Define

$$s_k := \mathrm{trace}(M_k); \qquad s_{k,l} = \mathrm{trace}(M_k M_l).$$

Since $M_k \in \mathrm{SL}(2,\mathbb{C})$, the trace s_k determines the eigenvalues of M_k, i.e., the conjugacy class of the local monodromy M_k. For the symmetric case considered in this paper we restrict to

$$s_1 = s_2 = s_3 = s_4 = 2\cos(2\pi t)$$

for $t \in]0, \frac{1}{4}[$ and the following classical result by Fricke-Voigt holds.

Proposition 31. *Consider a* $\mathrm{SL}(2,\mathbb{C})$*-representation on the* 4*-punctured sphere* Σ*. Let*

$$s = s_1 = \cdots = s_4$$

and let $U = s_{1,2}$*,* $V = s_{2,3}$*,* $W = s_{1,3}$*. Then the following algebraic equation holds*

$$U^2 + V^2 + W^2 + UVW - 2s^2(U+V+W) + 4(s^2-1) + s^4 = 0. \tag{41}$$

When satisfying (41) *the parameters* s *and* U, V, W *together determine a monodromy representation* $\rho\colon \pi_1(\Sigma) \to SL(2,\mathbb{C})$ *from the first fundamental group of* Σ *into* $SL(2,\mathbb{C})$*. By imposing the symmetries* (29)*, this representation is unique up to conjugation.*

Proof. For the first (classical) part of the proposition see for example [11]. Whenever the representations are irreducible, [11] moreover shows that they are uniquely determined by their global traces U, V, W up to conjugation, even without symmetry assumptions. By [11, Lemma 5] a representation is reducible if and only if

$$s_{i,j} \in \{2, -2 + s^2\} = \{2, 2\cos(4\pi t)\}$$

for all i, j. For $t \in]0, \frac{1}{4}[$ (41) then implies that

$$(U, V, W) \in \{(2, 2, 2\cos(4\pi t)), (2, 2\cos(4\pi t), 2), (2\cos(4\pi t), 2, 2)\}.$$

It can be easily checked that for each of the 3 possibilities there is a unique (up to conjugation with C and D) Fuchsian potential satisfying the symmetry assumptions (29). □

Using the above proposition the reality condition (39) on the potentials η_t is equivalent to

$$s_{jk} = s_{jk}^* \quad \text{for } (j,k) \in \{(1,2), (1,3), (2,3)\},$$

with $()^*$ as defined in (31). The goal of this section is thus to solve the following Monodromy Problem

$$\begin{cases} s_{jk} = s_{jk}^* & \text{for } (j,k) \in \{(1,2), (1,3), (2,3)\} \\ \sum_{j=1}^{3} x_j^2 = 1 \end{cases} \tag{42}$$

using a similar implicit function theorem as in [19, 20]. Note that it remains to show that the so-constructed real holomorphic sections are negative (and in the component of twistor lines). This is done in Corollary 36 below.

4.1. **Setup.** As in [20], let $\mathcal{P} = \Phi(z = 1)$ and $\mathcal{Q} = \Phi(z = i)$, where we omitted the index t. Then the traces s_{jk} are given by squares of holomorphic functions in terms of the entries of $\mathcal{P} = (\mathcal{P}_{ij})$ and $\mathcal{Q} = (\mathcal{Q}_{ij})$ as follows:

Proposition 32. *With the notation above we have*

$$s_{12} = 2 - 4\mathfrak{p}^2 \tag{43}$$

$$s_{23} = 2 - 4\mathfrak{q}^2 \tag{44}$$

$$s_{13} = 2 - 4\mathfrak{r}^2 \tag{45}$$

with

$$\mathfrak{p} = \mathcal{P}_{11}\mathcal{P}_{21} - \mathcal{P}_{12}\mathcal{P}_{22}, \quad \mathfrak{q} = i(\mathcal{Q}_{11}\mathcal{Q}_{21} + \mathcal{Q}_{12}\mathcal{Q}_{22})$$

and

$$\begin{aligned}\mathfrak{r} \;=\; & \frac{i}{2}(\mathcal{P}_{22}\mathcal{Q}_{11} + \mathcal{P}_{12}\mathcal{Q}_{21})^2 + \frac{i}{2}(\mathcal{P}_{22}\mathcal{Q}_{12} + \mathcal{P}_{12}\mathcal{Q}_{22})^2 \\ & -\frac{i}{2}(P_{21}Q_{11} + P_{11}Q_{21})^2 - \frac{i}{2}(P_{21}Q_{12} + P_{11}Q_{22})^2.\end{aligned}$$

Proof. The equations (43) and (44) have been proven in [20, Proposition 16]. It remains to show (45). Using the symmetries δ and $\tau \circ \delta$ which fix $z = 1$ and $z = i$ respectively we have as in [20, Proposition 16]

$$\begin{aligned}\Phi(+\infty) &= \mathcal{P}C\mathcal{P}^{-1}C^{-1}, & \Phi(+i\infty) &= \mathcal{Q}DC\mathcal{Q}^{-1}C^{-1}D^{-1} \\ M_1 &= \Phi(+\infty)\Phi(+i\infty)^{-1}, & M_3 &= DM_1D^{-1}\end{aligned} \tag{46}$$

This gives with $C^2 = D^2 = -\mathrm{Id}$ and $CD = -DC$

$$M_1M_3 = -(\mathcal{P}C\mathcal{P}^{-1}C^{-1}DC\mathcal{Q}C^{-1}D^{-1}Q^{-1}D)^2 = -\left(\mathcal{P}C\mathcal{P}^{-1}D\mathcal{Q}CD\mathcal{Q}^{-1}D\right)^2.$$

For $A \in SL(2, \mathbb{C})$ we have

$$\mathrm{trace}(-A^2) = 2 - \mathrm{trace}(A)^2,$$

hence (45) holds with

$$\mathfrak{r} = \frac{1}{2}\mathrm{trace}\left(\mathcal{P}C\mathcal{P}^{-1}D\mathcal{Q}DC\mathcal{Q}^{-1}D\right)$$

which coincides with the formula given in Proposition 32 after a tedious computation. □

Proposition 33. *At $t = 0$, we have*

$$\mathfrak{p}(0) = \mathfrak{q}(0) = \mathfrak{r}(0) = 0$$

with derivatives

$$\mathfrak{p}'(0) = 2\pi x_3, \quad \mathfrak{q}'(0) = 2\pi x_2 \quad \textit{and} \quad \mathfrak{r}'(0) = 2\pi x_1.$$

Proof. At $t = 0$ we have $\eta_t = 0$ thus all monodromies are trivial and $\mathcal{P} = \mathcal{Q} = I_2$ from which the first point follows. For the assertion on the derivatives define as in [20]

$$\Omega_j(z) = \int_0^z \omega_j(z) \tag{47}$$

for $j = 1, 2, 3$, where the integral is computed on the segment from 0 to z. Then

$$\mathcal{P}'(0) = \sum_{j=1}^{3} x_j\Omega_j(1)\mathfrak{m}_j, \quad \text{and} \quad \mathcal{Q}'(0) = \sum_{j=1}^{3} x_j\Omega_j(i)\mathfrak{m}_j,$$

from which we can compute

$$\begin{aligned}\mathfrak{p}'(0) &= \mathcal{P}'_{21}(0) - \mathcal{P}'_{12}(0) = -2ix_3\Omega_3(1)\\ \mathfrak{q}'(0) &= i(\mathcal{Q}'_{21}(0) + \mathcal{Q}'_{12}(0)) = 2ix_2\Omega_2(i)\\ \mathfrak{r}'(0) &= i(\mathcal{P}'_{22}(0) + \mathcal{Q}'_{11}(0) - \mathcal{P}'_{11}(0) - \mathcal{Q}'_{22}(0)) = -2ix_1(\Omega_1(1) - \Omega_1(i)).\end{aligned}$$

By the Residue Theorem, we have for $j = 1, 2, 3$

$$2\pi i = \int_{\gamma_1} \omega_j = \int_0^1 \omega_j - \int_0^1 \tau^*\omega_j - \int_0^i \omega_j + \int_0^i (\tau\delta)^*\omega_j.$$

Using the symmetries (25), this gives

$$\Omega_1(1) - \Omega_1(i) = \pi i, \quad \Omega_2(i) = -\pi i \quad \text{and} \quad \Omega_3(1) = \pi i \tag{48}$$

proving Proposition 33. □

In view of Proposition 32, the monodromy problem (42) can be reformulated to be

$$\begin{cases}\mathfrak{p} = \mathfrak{p}^*\\ \mathfrak{q} = \mathfrak{q}^*\\ \mathfrak{r} = \mathfrak{r}^*\\ \sum_{j=1}^3 x_j^2 = 1\end{cases} \tag{49}$$

and the following proposition implies that it suffices to consider two amongst the three traces.

Proposition 34. *Assume that $(x_1(t), x_2(t), x_3(t))$ are analytic functions of t in a neighborhood of $t = 0$ with $x_j(0) = \underline{x}_j$ and solving the following equations for all t*

$$\begin{cases}\mathfrak{p} = \mathfrak{p}^*\\ \mathfrak{q} = \mathfrak{q}^*\\ \sum_{j=1}^3 x_j^2 = 1.\end{cases} \tag{50}$$

Then also $\mathfrak{r} = \mathfrak{r}^*$ for all t. Analogous statements hold by cyclic permutation of $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$.

Proof. Let

$$U = s_{12}, \quad V = s_{23}, \quad W = s_{13} \quad \text{and} \quad s = \operatorname{trace}(M_k)$$

as in Proposition 31 satisfying the quadratic equation (41)

$$Q := U^2 + V^2 + W^2 + UVW - 2s^2(U + V + W) + 4(s^2 - 1) + s^4 = 0.$$

By substitution of $U = 2 - 4\mathfrak{p}^2$, $V = 2 - 4\mathfrak{q}^2$ and $W = 2 - 4\mathfrak{r}^2$, Q factors as

$$Q = Q_1Q_2 \quad \text{with} \quad Q_j = s^2 + 4(\mathfrak{p}^2 + \mathfrak{q}^2 + \mathfrak{r}^2 - 1) + 8(-1)^j\mathfrak{p}\mathfrak{q}\mathfrak{r}.$$

Since $Q = 0$ for all t, and Q_1, Q_2 are analytic functions of t, one of them must be identically zero. Let

$$\Delta = 64(1 - \mathfrak{p}^2)(1 - \mathfrak{q}^2) - 16s^2$$

be the discriminant of this Q_j, seen as a polynomial in the variable $\mathfrak{r}$. Since $\mathcal{P}$, $\mathcal{Q}$ are well-defined analytic functions of t (with values in $\mathcal{W}_a$), $\mathfrak{r}$ is a well-defined analytic function in t as well and Δ admits a well-defined square root δ such that

$$\mathfrak{r} = (-1)^{j+1}\mathfrak{p}\mathfrak{q} + \frac{\delta}{8}$$

for all t. From the hypotheses of the proposition, we have $\Delta = \Delta^*$ so $\delta^* = \varepsilon\delta$ where the sign $\varepsilon = \pm 1$ does not depend on t because $\mathfrak{r}^*$ is a well-defined analytic function in t. Hence

$$\mathfrak{r}^* = (-1)^{j+1}\mathfrak{p}\mathfrak{q} + \varepsilon\frac{\delta}{8}.$$

The sign ε can be determined using the first order derivatives at $t = 0$. We have

$$\mathfrak{r}' = 2\pi\underline{x}_1 = \frac{\delta'}{8} \quad \text{and} \quad \mathfrak{r}'^* = 2\pi\underline{x}_1^* = 2\pi\underline{x}_1 = \varepsilon\frac{\delta'}{8},$$

so since $\underline{x}_1 \not\equiv 0$, $\varepsilon = 1$. Hence $\mathfrak{r} = \mathfrak{r}^*$ for all t. □

4.2. **Solving the Monodromy Problem.** The moduli space of Higgs fields on $V = \mathcal{O} \oplus \mathcal{O}$ is identified with the completion of the nilpotent orbit in $\mathfrak{sl}(2, \mathbb{C})$ which in turn is given by the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ at the origin. Consider first the regular case of $(u, v) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and consider the quadratic polynomials

$$P_j(\lambda) = \lambda\underline{x}_j(\lambda) = \underline{x}_{j,1}\lambda^2 + \underline{x}_{j,0}\lambda - \overline{\underline{x}_{j,1}}.$$

and denote their discriminants by

$$\Delta_j = \underline{x}_{j,0}^2 + 4|\underline{x}_{j,1}|^2 \in \mathbb{R}.$$

Proposition 35.

(1) *The polynomial P_j has a complex root μ_j with $|\mu_j| < 1$ if and only if $\underline{x}_{j,0} \neq 0$. In this case $\mu_j \in \mathbb{D}_1$ depends real-analytically on (u, v).*
(2) *With the same notation, $P_k(\mu_j)$ and $P_\ell(\mu_j)$ are $\mathbb{R}$-independent complex number if $\{j, k, \ell\} = \{1, 2, 3\}$.*

Proof. If $\underline{x}_{j,1} = 0$, the first point is trivial, because in this case $P_j = \underline{x}_{j,0}\lambda$. Therefore, $\underline{x}_{j,0} \neq 0$ is equivalent to $\mu_j = 0$ which is constant in (u, v). Hence assume in the following $\underline{x}_{j,1} \neq 0$.

(1) If μ is a root of P_j, then $\mu \neq 0$ and $\underline{x}_j = \underline{x}_j^*$ gives that $-\frac{1}{\overline{\mu}} \neq \mu$ is the other root of P_j. Thus

$$|\mu| = 1 \Leftrightarrow \mu - \frac{1}{\overline{\mu}} = 0 \Leftrightarrow \underline{x}_{j,0} = 0.$$

For $x_{j,0} \neq 0$ the root μ_j with $|\mu_j| < 1$ is given by

$$\mu_j = \frac{-\underline{x}_{j,0} + \operatorname{sign}(\underline{x}_{j,0})\sqrt{\Delta_j}}{2\underline{x}_{j,1}}$$

which we may rewrite as

$$\mu_j = \frac{2\,\overline{\underline{x}_{j,1}}}{\underline{x}_{j,0}} f\left(\frac{4|\underline{x}_{j,1}|^2}{\underline{x}_{j,0}^2}\right) \quad \text{with} \quad f(z) = \frac{\sqrt{1+z} - 1}{z}.$$

The function f extends holomorphically to $z = 0$, therefore μ_j depends analytically on (u, v).

(2) To prove the second point, assume for simplicity of notation that $j = 1$. Suppose by contradiction that $P_2(\mu_1)$ and $P_3(\mu_1)$ are linearly dependent over $\mathbb{R}$. First assume that $\mu_1 \neq 0$. Then the complex numbers $\underline{x}_2(\mu_1)$, $\underline{x}_3(\mu_1)$ are linealy dependent. Moreover, $\underline{x}_1(\mu_1) = 0$ so

$$\underline{x}_2(\mu_1)^2 + \underline{x}_3(\mu_1)^2 = 1$$

and this implies that $\underline{x}_2(\mu_1)$ and $\underline{x}_3(\mu_2)$ are real. Since all $\underline{x}_{k,0}$ are real, we obtain

$$\underline{x}_{k,1}\mu_1 - \overline{\underline{x}_{k,1}}\mu_1^{-1} \in \mathbb{R} \quad \text{for } k = 1, 2, 3.$$

Then

$$\sum_{k=1}^{3} \left(\underline{x}_{k,1}\mu_1 - \overline{\underline{x}_{k,1}}\mu_1^{-1}\right)^2 \geq 0.$$

Expanding the squares and using $\sum_{k=1}^{3} \underline{x}_{k,1}^2 = 0$ we obtain

$$\sum_{k=1}^{3} |\underline{x}_{k,1}|^2 \leq 0$$

which implies $u = v = 0$ which is a contradiction.
If $\mu_1 = 0$, we have $P_2(0) = \underline{x}_{2,-1}$ and $P_3(0) = \underline{x}_{3,-1}$. From $P_1(0) = \underline{x}_{1,-1} = 0$ we obtain

$$\underline{x}_{2,-1}^2 + \underline{x}_{3,-1}^2 = 0$$

and since $\underline{x}_{2,-1}$ and $\underline{x}_{3,-1}$ are linearly dependent over $\mathbb{R}$, $\underline{x}_{2,-1} = \underline{x}_{3,-1} = 0$ which again results in $u = v = 0$ leading to a contradiction.

□

Theorem 4. *Fix $(u, v) \neq (0, 0)$, which determines $\underline{x}_{j,-1}$ for $j = 1, 2, 3$. Then there are $\epsilon_0 > 0$ and $a > 1$ such that there exists unique values of the parameters $x = (x_1, x_2, x_3) \in (\mathcal{W}_a^{\geq 0})^3$ in a neighborhood of $\underline{x}$ for all $t \in (-\epsilon_0, \epsilon_0)$ depending analytically on (t, p, u, v) solving (42) with $x(0) = \underline{x}$ and prescribed $x_{j,-1}(t) = \underline{x}_{j,-1}$. Moreover, ϵ_0 and $a > 1$ are uniform with respect to (p, u, v) on compact subsets of $\mathbb{C}^+ \times \mathbb{C}^2 \setminus \{(0, 0)\}$.*

Proof. Fix $(u, v) \neq (0, 0)$. By Proposition 35, at least one of the polynomials P_j has a root μ_j inside the unit λ-disc. By symmetry of the roles played by the parameters $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, we may assume without loss of generality that $j = 1$. By Proposition 34, it suffices to solve Problem (50). We fix $a > 1$ such that $a|\mu_1| < 1$ and consider the corresponding functional space $\mathcal{W}_a^{\geq 0}$. We introduce a parameter $y = (y_1, y_2, y_3)$ in a neighborhood of 0 in $(\mathcal{W}_a^{\geq 0})^3$ and take

$$x_k = \underline{x}_k + y_k, \quad k = 1, 2, 3.$$

Note that the negative part of the potential is fixed to its initial value $x_{k,-1} = \underline{x}_{k,-1}$. Since $\mathfrak{p}$ and $\mathfrak{q}$ are analytic functions of (t, y) which vanish at $t = 0$, the functions

$$\widehat{\mathfrak{p}}(t, y) = \frac{1}{t}\mathfrak{p}(t, y) \quad \text{and} \quad \widehat{\mathfrak{q}}(t, y) = \frac{1}{t}\mathfrak{q}(t, y)$$

extend analytically at $t = 0$ and by Proposition 33 we have at $t = 0$

$$\widehat{\mathfrak{p}}(0, y) = 2\pi(\underline{x}_3 + y_3) \quad \text{and} \quad \widehat{\mathfrak{q}}(0, y) = 2\pi(\underline{x}_2 + y_2).$$

We define

$$\begin{aligned}\mathcal{F}(t,y) &:= \widehat{\mathfrak{p}}(t,y) - \widehat{\mathfrak{p}}(t,y)^* \\ \mathcal{G}(t,y) &:= \widehat{\mathfrak{q}}(t,y) - \widehat{\mathfrak{q}}(t,y)^* \\ \mathcal{K}(y) &:= \sum_{k=1}^{3} x_k^2 = \sum_{k=1}^{3} (\underline{x}_k + y_k)^2.\end{aligned} \tag{51}$$

Then solving Problem(50) is equivalent to solving the equations $\mathcal{F} = \mathcal{G} = 0$ and $\mathcal{K} = 1$. By our choice of the central value, these equations holds at $(t, y) = (0, 0)$. By definition we have $\mathcal{F}^* = -\mathcal{F}$ so $\mathcal{F} = 0$ is equivalent to $\mathcal{F}^+ = 0$ and $\mathrm{Im}(\mathcal{F}^0) = 0$, and the analogous statement holds for $\mathcal{G}$ as well. We have

$$\begin{aligned} d\mathcal{F}(0,0)^+ &= 2\pi dy_3^+ \\ \mathrm{Im}(d\mathcal{F}(0,0)^0) &= 2\pi \mathrm{Im}(dy_{3,0}) \\ d\mathcal{G}(0,0)^+ &= 2\pi dy_2^+ \\ \mathrm{Im}(d\mathcal{G}(0,0)^0) &= 2\pi \mathrm{Im}(dy_{2,0}). \end{aligned} \tag{52}$$

Clearly the partial differential of

$$\big(\mathcal{F}^+, \mathcal{G}^+, \mathrm{Im}(\mathcal{F}^0), \mathrm{Im}(\mathcal{G}^0)\big)$$

with respect to

$$\big(y_2^+, y_3^+, \mathrm{Im}(y_{2,0}), \mathrm{Im}(y_{3,0})\big)$$

is an automorphism of $(\mathcal{W}_a^+)^2 \times \mathbb{R}^2$. The Implicit Function Theorem therefore uniquely determines $(y_2^+, y_3^+, \mathrm{Im}(y_{2,0}), \mathrm{Im}(y_{3,0}))$ as analytic functions of t and the remaining parameters, namely y_1, $\mathrm{Re}(y_{2,0})$ and $\mathrm{Re}(y_{3,0})$. Furthermore, the partial differential of $(y_2^+, y_3^+, \mathrm{Im}(y_{2,0}), \mathrm{Im}(y_{3,0}))$ with respect to these remaining parameters is zero.

It remains to solve the equation $\mathcal{K} = 1$. We write the Euclidean division of the polynomial P_k by $(\lambda - \mu_1)$ as

$$P_k(\lambda) = (\lambda - \mu_1)Q_k + P_k(\mu_1)$$

with $Q_k \in \mathbb{C}[\lambda]$. Note that the Q_k are analytic in (u, v) by Proposition 35. Observe that since

$$\sum_{k=1}^{3} \underline{x}_{k,-1}^2 = 0,$$

$\mathcal{K}$ has no λ^{-2} term so $\lambda\mathcal{K} \in \mathcal{W}^{\geq 0}$. We write the division of $\lambda\mathcal{K}$ by $(\lambda - \mu_1)$ as

$$\lambda\mathcal{K} = (\lambda - \mu_1)\mathcal{S} + \mathcal{R}$$

where $\mathcal{R} \in \mathbb{C}$ and $\mathcal{S} \in \mathcal{W}^{\geq 0}$. Note that since $|\mu_1| < 1$, $\mathcal{R}$ and $\mathcal{S}$ are analytic functions of all parameters by [20, Proposition 4]. We have, since $P_1(\mu_1) = 0$ and $dy_k = \mathrm{Re}(dy_{k,0})$ for $k = 2, 3$

$$\begin{aligned} d(\lambda\mathcal{K})(0,0) &= \sum_{k=1}^{3} 2P_k(\lambda)\, dy_k \\ &= 2(\lambda - \mu_1)Q_1 dy_1 + \sum_{k=2}^{3} 2\big((\lambda - \mu_1)Q_k + P_k(\mu_1)\big)\mathrm{Re}(dy_{k,0}) \end{aligned}$$

so by uniqueness of the division

$$d\mathcal{R}(0,0) = 2P_2(\mu_1)\mathrm{Re}(dy_{2,0}) + 2P_3(\mu_1)\mathrm{Re}(dy_{3,0})$$
$$d\mathcal{S}(0,0) = 2Q_1 dy_1 + 2Q_2\mathrm{Re}(dy_{2,0}) + 2Q_3\mathrm{Re}(dy_{3,0}).$$

If $\underline{x}_{1,1} \neq 0$, we have $\mu_1 \neq 0$ and the other root of P_1 is $-1/\overline{\mu_1}$ so

$$Q_1 = \underline{x}_{1,1}\left(\lambda + \frac{1}{\overline{\mu_1}}\right)$$

is invertible in $\mathcal{W}_a^{\geq 0}$ because $\frac{1}{|\mu_1|} > a$. (It is elementary that $\lambda - c$ is invertible in $\mathcal{W}_a^{\geq 0}$ if and only if $|c| > a$.)

If $\underline{x}_{1,1} = 0$, we have

$$P_1 = \underline{x}_{1,0}\lambda$$

so $Q_1 = \underline{x}_{1,0} \in \mathbb{C}^*$ is invertible in $\mathcal{W}_a^{\geq 0}$ as well. By Proposition 35, $P_2(\mu_1)$ and $P_3(\mu_1)$ are $\mathbb{R}$-independent complex numbers. Hence the partial differential of $(\mathcal{S}, \mathcal{R})$ with respect to $(y_1, \mathrm{Re}(y_{2,0}), \mathrm{Re}(y_{3,0}))$ is an isomorphism from $\mathcal{W}_a^{\geq 0} \times \mathbb{R}^2$ to $\mathcal{W}_a^{\geq 0} \times \mathbb{C}$. The Implicit Function Theorem uniquely determines y_1, $\mathrm{Re}(y_{2,0})$ and $\mathrm{Re}(y_{3,0})$ as analytic functions of t in a neighborhood of 0. □

4.3. **The limit $(u, v) \to (0, 0)$.** Since the limit Higgs bundle moduli space is given by the blow up of $\mathbb{C}^2/\mathbb{Z}^2$ at the origin, rather than expecting the solution $x(t, p, u, v)$ to extend continuously to $(u, v) = (0, 0)$, the limit should depend on the direction in the blow-up. We write

$$u = r\widetilde{u} \quad \text{and} \quad v = r\widetilde{v} \quad \text{with} \quad |\widetilde{u}|^2 + |\widetilde{v}|^2 = 1.$$

Let $0 < |r| \leq \frac{1}{2}$ so that $(u, v) \neq (0, 0)$. With a slight abuse of notation, we write $x = x(t, p, r, \widetilde{u}, \widetilde{v})$. Observe that $(u, v) \to (-u, -v)$ does not change the initial value $\underline{x}$, so the map $x(t, p, r, \widetilde{u}, \widetilde{v})$ is even with respect to r.

Our goal is to prove

Theorem 5.

(1) *There exists $\epsilon_2 > 0$ such that for $|t| < \epsilon_2$, the function $x(t, p, r, \widetilde{u}, \widetilde{v})$ extends analytically at $r = 0$. Moreover, ϵ_2 is uniform with respect to p in compact subsets of $\mathbb{C}^+$ and $(\widetilde{u}, \widetilde{v})$ in $\mathbb{S}^3$.*

(2) *At $r = 0$, $x(t, p, 0, \widetilde{u}, \widetilde{v})$ does not depend on λ and solves the following problem:*

$$\begin{cases} \exists U \in SL(2, \mathbb{C}), \quad \forall k,\ UM_kU^{-1} \in SU(2) \\ x_1^2 + x_2^2 + x_3^2 = 1. \end{cases} \tag{53}$$

(3) *At $(t, r) = (0, 0)$, we have*

$$x_1 = -(|\widetilde{u}|^2 - |\widetilde{v}|^2), \quad x_2 = -2\,\mathrm{Re}\left(\widetilde{u}\,\overline{\widetilde{v}}\right) \quad \textit{and} \quad x_3 = -2\,\mathrm{Im}\left(\widetilde{u}\,\overline{\widetilde{v}}\right).$$

Point (2) of the Theorem shows that the constructed real sections lie in the same connected component as unitary flat connections, solutions with Higgs field $\Psi = 0$, which are negative, showing that our approach is a complex analytic way of constructing twistor lines.

Corollary 36. *The real sections constructed in Theorem 4 and Theorem 5 are twistor lines.*

Proof of Theorem 5. Rewrite the central value $\underline{x}$ in terms of $(r, \widetilde{u}, \widetilde{v})$ as

$$\underline{x}_j = r^2 \underline{\widetilde{x}}_{j,-1} \lambda^{-1} + \widetilde{\rho}(r) \underline{\widetilde{x}}_{j,0} + r^2 \underline{\widetilde{x}}_{j,1} \lambda$$

with

$$\underline{\widetilde{x}}_{1,-1} = \widetilde{u}\,\widetilde{v}, \quad \underline{\widetilde{x}}_{2,-1} = \tfrac{1}{2}(\widetilde{v}^2 - \widetilde{u}^2), \quad \underline{\widetilde{x}}_{3,-1} = \tfrac{i}{2}(\widetilde{u}^2 + \widetilde{v}^2),$$

$$\underline{\widetilde{x}}_{1,0} = |\widetilde{u}|^2 - |\widetilde{v}|^2, \quad \underline{\widetilde{x}}_{2,0} = 2\,\mathrm{Re}\left(\widetilde{u}\,\overline{\widetilde{v}}\right), \quad \underline{\widetilde{x}}_{3,0} = 2\,\mathrm{Im}\left(\widetilde{u}\,\overline{\widetilde{v}}\right),$$

$$\underline{\widetilde{x}}_{j,1} = -\overline{\underline{\widetilde{x}}_{j,-1}}$$

and

$$\widetilde{\rho}(r) = r^2 \rho(r) = -\sqrt{1 + r^4} = -1 + O(r^4).$$

Observe that

$$\sum_{k=1}^{3} \underline{\widetilde{x}}_{k,-1}^2 = 0, \quad \sum_{k=1}^{3} \underline{\widetilde{x}}_{k,0}^2 = 1, \quad \sum_{k=1}^{3} \underline{\widetilde{x}}_{k,-1} \underline{\widetilde{x}}_{k,0} = 0 \tag{54}$$

and

$$\sum_{k=1}^{3} |\underline{\widetilde{x}}_{k,-1}|^2 = \frac{1}{2}. \tag{55}$$

For given $(\widetilde{u}, \widetilde{v}) \in \mathbb{S}^3$, fix $j \in \{1, 2, 3\}$ such that $\underline{\widetilde{x}}_{j,0} \neq 0$. (This is possible by Equation (54)). We take as ansatz that the parameter y_j is of the following form

$$y_j = y_{j,0} + r^2 \widetilde{y}_j^+ \quad \text{with} \quad \widetilde{y}_j^+ \in \mathcal{W}_a^+. \tag{56}$$

We solve the equations $\mathcal{F} = \mathcal{G} = 0$ using the Implicit Function Theorem as in Section 4.2. This determines the parameters y_k^+ and $\mathrm{Im}(y_{k,0})$ with $k \neq j$ as functions of (t, r) and the remaining parameters $y_{j,0}$, $\widetilde{y}_j^+$ and $\mathrm{Re}(y_{k,0})$ with $k \neq j$. Moreover, at $r = 0$, $y_j^+ = 0$ and $y_k^+ = 0$ solves $\mathcal{F}^+ = \mathcal{G}^+ = 0$. Since y_k is an even function of r, this means that we can also write for $k \neq j$

$$y_k = y_{k,0} + r^2 \widetilde{y}_k^+ \quad \text{with} \quad \widetilde{y}_k^+ \in \mathcal{W}^+.$$

We decompose $\mathcal{K} = x_1^2 + x_2^2 + x_3^2$ as

$$\mathcal{K} = \mathcal{K}_{-1} \lambda^{-1} + \mathcal{K}_0 + \mathcal{K}^+ \quad \text{with} \quad \mathcal{K}^+ \in \mathcal{W}^+.$$

When $r = 0$, $\mathcal{K}$ does not depend on λ, i.e., $\mathcal{K}_{-1} = 0$ and $\mathcal{K}^+ = 0$. Being even functions of r, this means that $\widetilde{\mathcal{K}}_{-1} = r^{-2} \mathcal{K}_{-1}$ and $\widetilde{\mathcal{K}}^+ = r^{-2} \mathcal{K}^+$ extend analytically at $r = 0$. More explicitly,

$$\begin{aligned} \mathcal{K} &= \sum_{k=1}^{3} \left(r^2 \underline{\widetilde{x}}_{k,-1} \lambda^{-1} + \widetilde{\rho}\, \underline{\widetilde{x}}_{k,0} + r^2 \underline{\widetilde{x}}_{k,1} \lambda + y_{k,0} + r^2 \widetilde{y}_k^+ \right)^2 \\ &= \sum_{k=1}^{3} \left(\underline{\widetilde{x}}_{k,0} + y_{k,0} \right)^2 + 2r^2 \left(\underline{\widetilde{x}}_{k,0} + y_{k,0} \right) \left(\underline{\widetilde{x}}_{k,-1} \lambda^{-1} + \underline{\widetilde{x}}_{k,1} \lambda + \widetilde{y}_k^+ \right) + O(r^4) \end{aligned}$$

from which we obtain at $r = 0$

$$\widetilde{\mathcal{K}}_{-1}\mid_{r=0} = 2\sum_{k=1}^{3}\left(\underline{\widetilde{x}}_{k,0} + y_{k,0}\right)\underline{\widetilde{x}}_{k,-1}$$

$$\mathcal{K}_0\mid_{r=0} = \sum_{k=1}^{3}\left(\underline{\widetilde{x}}_{k,0} + y_{k,0}\right)^2$$

$$\widetilde{\mathcal{K}}^{+}\mid_{r=0} = 2\sum_{k=1}^{3}\left(\underline{\widetilde{x}}_{k,0} + y_{k,0}\right)\left(\underline{\widetilde{x}}_{k,1}\lambda + \widetilde{y}_k^{+}\right).$$

In particular, at the central value $y = 0$, we have by Equation (54) that $\mathcal{K}_0 = 1$, $\widetilde{\mathcal{K}}_{-1} = 0$ and $\widetilde{\mathcal{K}}^{+} = 0$ and the differentials with respect to y at $(t, r, y) = (0, 0, 0)$ are

$$d\widetilde{\mathcal{K}}_{-1} = 2\sum_{k=1}^{3}\underline{\widetilde{x}}_{k,-1}dy_{k,0}$$

$$d\mathcal{K}_0 = 2\sum_{k=1}^{3}\underline{\widetilde{x}}_{k,0}dy_{k,0}$$

$$d\widetilde{\mathcal{K}}^{+} = 2\sum_{k=1}^{3}\underline{\widetilde{x}}_{k,0}d\widetilde{y}_k^{+} + \lambda\underline{\widetilde{x}}_{k,1}dy_{k,0}.$$

Keep in mind that $\widetilde{y}_k^{+}$ and $\mathrm{Im}(y_k^0)$ for $k \neq j$ have already been determined and their differential at $(t, r) = (0, 0)$ is zero. Consider the polynomials

$$\widetilde{P}_k(\lambda) = \underline{\widetilde{x}}_{k,0}\lambda + \underline{\widetilde{x}}_{k,-1}$$

and let $\widetilde{\mu}_j$ be the root of P_j. Then

$$d\widetilde{\mathcal{K}}_{-1} + \widetilde{\mu}_j d\mathcal{K}_0 = 2\sum_{k\neq j}\widetilde{P}_k(\widetilde{\mu}_j)\mathrm{Re}(dy_{k,0}).$$

By Claim 37 below, this is an isomorphism from $\mathbb{R}^2$ to $\mathbb{C}$. Hence the partial derivative of $(\widetilde{\mathcal{K}}_{-1}, \mathcal{K}_0, \widetilde{\mathcal{K}}^{+})$ with respect to $\left((\mathrm{Re}(dy_{k,0}))_{k\neq j}, y_j^0, \widetilde{y}_j^{+}\right)$ is an isomorphism from $\mathbb{R}^2 \times \mathbb{C} \times \mathcal{W}^{+}$ to $\mathbb{C}^2 \times \mathcal{W}^{+}$. By the Implicit Function Theorem, there thus exists $\epsilon_1 > 0$ such that for $|t| < \epsilon_1$ and $|r| < \epsilon_1$, there exists unique values of the parameters which solve Problem (42). When $\epsilon_1/2 \leq |r| \leq 1/2$, Theorem 4 gives us a uniform $\epsilon_0 > 0$ such that for $|t| < \epsilon_0$, there exists unique values of the parameters which solve Problem (42). They certainly satisfy the ansatz (56) since $|r| \geq \epsilon_1/2$. Take $\epsilon_2 = \min(\epsilon_1, \epsilon_0)$. In the overlap $\epsilon_1/2 \leq |r| < \epsilon_1$, the solutions agree by uniqueness of the implicit function theorem. □

Claim 37. *For $k \neq j$ the two complex numbers $\widetilde{P}_k(\widetilde{\mu}_j)$ are linearly independent over $\mathbb{R}$.*

Proof. Assume for the simplicity of notation that $j = 1$ and that $\widetilde{P}_2(\widetilde{\mu}_1)$ and $\widetilde{P}_3(\widetilde{\mu}_1)$ are linearly dependent. If $\mu_1 \neq 0$, let

$$\alpha_k = \widetilde{\mu}_1^{-1}\widetilde{P}_k(\widetilde{\mu}_1) = \underline{\widetilde{x}}_{k,0} + \underline{\widetilde{x}}_{k,-1}\widetilde{\mu}_1^{-1}.$$

Using Equation (54) we have

$$\sum_{k=1}^{3}\alpha_k^2 = 1.$$

Since $\alpha_1 = 0$ and α_2, α_3 are linearly dependent and all α_k must be real numbers. Since $\underline{\widetilde{x}}_{k,0}$ are real, each $\underline{\widetilde{x}}_{k,-1}\widetilde{\mu}_1^{-1}$ must be real. Then using Equation (54) again, we have

$$\sum_{k=1}^{3}(\underline{\widetilde{x}}_{k,-1}\widetilde{\mu}_1^{-1})^2 = 0$$

implying that $\underline{\widetilde{x}}_{k,-1} = 0$ for all k contradicting Equation (55).
If $\mu_1 = 0$, then $\underline{\widetilde{x}}_{1,-1} = 0$. Then if $\underline{\widetilde{x}}_{2,-1}$, $\underline{\widetilde{x}}_{3,-1}$ are linearly dependent, we would obtain

$$\sum_{k=1}^{3}(\underline{\widetilde{x}}_{k,-1})^2 = 0,$$

which again gives $\underline{\widetilde{x}}_{k,-1} = 0$ for all k contradicting Equation (55). □

Remark 38. *The constructed twistor lines are uniquely determined by the residue $A_1(t,\lambda)$ at $z = p$ of the potential η_t. The deformation of $A = A_1$ in the parameter t can in fact be expressed by a Lax pair type equation*

$$A' = [A, X],$$

for some $X \in \Lambda^+\mathfrak{sl}(2,\mathbb{C})$. Moreover, X unique up to adding $g \cdot A$, where $g\colon \lambda \mapsto g(\lambda)$ is holomorphic around $\lambda = 0$.

To see this recall

$$A = \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix}$$

satisfies $\det A = -1$ *and thus* $A^2 = \mathrm{Id}$. *Therefore,* $A'A + AA' = 0$ *which gives*

$$x_1'x_1 + x_2'x_2 + x_3'x_3 = 0.$$

Let

$$\dot{A} = \begin{pmatrix} x_1' & x_2' + ix_3' \\ x_2' - ix_3' & -x_1' \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}.$$

Then we can choose X to be given by

$$\begin{aligned} \alpha &= -\tfrac{1}{2}(x_2 - ix_3)(x_2' + ix_3') + x_1 R \\ \beta &= \tfrac{1}{2}x_1(x_2' + ix_3') + (x_2 + ix_3)R \\ \gamma &= -\tfrac{1}{2}x_1(x_2' - ix_3') + (x_2 - ix_3)(R + a). \end{aligned} \tag{57}$$

with $R = -\frac{1}{2}(x_{2,0}' + ix_{3,0}')\frac{x_{1,-1}}{x_{2,-1}+ix_{3,-1}}$ *chosen to remove the negative powers of λ.*

5. The hyper-Kähler structure and the non-abelian Hodge correspondence at $t = 0$

In this section, we derive first consequences from our complex analytic approach to the non-abelian Hodge correspondence by varying the parabolic weight t. In particular, we can explicitly describe the (rescaled) metric and the non-abelian Hodge correspondence at the limit $t = 0$. We then compute the first order derivatives with respect to t at $t = 0$.

5.1. **The non-abelian Hodge correspondence at $t = 0$.** The non-abelian Hodge correspondence on the rank 2 hermitian bundle V is a diffeomorphism that associates to each stable Higgs pair $(\bar\partial_V, \Phi)$ the flat connection $\nabla^{\lambda=1}$ of the associated family. In the case of Fuchsian potentials on a 4−punctured sphere with parabolic weight t, the underlying holomorphic structure is trivial and the Higgs pair is specified by the Higgs field only. Due to the symmetries, the residue at $p \in \mathbb{C}^+$ already determines the Higgs field and since $\sum_{j=1}^3 x_{j,-1}^2 = 0$, this residue is nilpotent. On the other hand, the connection 1−form for the flat $\mathrm{SL}(2,\mathbb{C})$-connection $\nabla^{\lambda=1}$ is determined by $A_1(\lambda = 1)$ satisfying $\det A_1 = -1$ and $\mathrm{trace} A_1 = 0$. Hence A_1 lies in the $\mathrm{SL}(2,\mathbb{C})$ adjoint orbit of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Consider the complex 3-dimensional vector space $\mathfrak{sl}(2,\mathbb{C})$ with its complex bilinear inner product

$$< \xi, \eta >= -\frac{1}{2}\mathrm{tr}(\xi\eta).$$

Then its associated quadratic form is the determinant det. Decompose $\mathfrak{sl}(2,\mathbb{C})$ into real subspaces

$$\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$$

consisting of the subspace of skew-hermitian ($A = -\bar A^T$) and the subspace of hermitian symmetric ($A = \bar A^T$) and trace-free matrices. Note that $< .,. >$ is positive-definite on the 3-dimensional real subspace $\mathfrak{su}(2)$ and negative definite on $i\mathfrak{su}(2)$.

Lemma 39. *There is a diffeomorphism between the nilpotent orbit in* $\mathfrak{sl}(2,\mathbb{C})$ *and the* $\mathrm{SL}(2,\mathbb{C})$*-orbit through* $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ *away from hermitian symmetric matrices.*

Proof. Let Ψ be an element in the nilpotent orbit. Then, $\Phi = \Psi - \bar\Psi^T$ is skew-hermitian and there is a unique skew-hermitian $N \in \mathfrak{su}(2)$ of length 1 such that

$$< N, \Psi - \bar\Psi^T >= 0, \quad < N, i\Psi + i\bar\Psi^T >= 0$$

and

$$N \times (\Psi - \bar\Psi^T) = \tfrac{1}{2}[N, \Psi - \bar\Psi^T] = i\Psi + i\Psi^T.$$

Moreover,

$$A_\Psi := \sqrt{1+ < \Phi, \Phi >}\, iN + \Phi \in \mathfrak{sl}(2,\mathbb{C})$$

has determinant -1 and the map

$$\Psi \longmapsto A_\Psi$$

is smooth. Note that

$$A_1(\lambda) = \lambda^{-1}\Psi + \sqrt{1+ < \Phi, \Phi >}\, iN - \lambda\bar\Psi^T$$

is of the form (30). Plugging in the formulas for the Higgs filed Ψ in terms of the (u, v) coordinates, we obtain that $\sqrt{1+ < \Phi, \Phi >}\, iN$ is the constant term of the initial values and this fixes the sign of ρ to be negative.

For the reverse, let $A \in \mathfrak{sl}(2,\mathbb{C})$ be of determinant -1, such that A is not hermitian symmetric, i.e.,

$$\Phi := \frac{1}{2}(A - \bar A^T) \neq 0.$$

Since $\det(A) = -1$ and $< .,. >$ is positive definite on $\mathfrak{su}(2)$, also

$$\xi := \tfrac{1}{2}(A + \bar A^T)$$

does not vanish. Define

$$N := \frac{-i}{\sqrt{- < \xi, \xi >}} \xi \in \mathfrak{su}(2).$$

Note that $< N, \Phi >= 0$ since $\det(A)$ is real. Then,

$$\Psi_A := \tfrac{1}{2}(\Phi - iN \times \Phi)$$

is nilpotent, and

$$A_{\Psi_A} = A.$$

Moreover,

$$A \longmapsto \Psi_A$$

is also smooth. □

We identify the nilpotent orbit

$$\mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z}_2 \cong \{\Psi \in \mathfrak{sl}(2,\mathbb{C}) \mid \Psi \neq 0;\ \det(\Psi) = 0\}$$

via

$$(v,u) \mapsto \Psi := \begin{pmatrix} u\,v & -u^2 \\ v^2 & -u\,v \end{pmatrix},$$

which is $2:1$ as $\pm(u,v)$ maps onto the same Ψ. Note that

$$\Psi \begin{pmatrix} u \\ v \end{pmatrix} = 0.$$

And we obtain

Theorem 6. *The diffeomorphism in Lemma 39 extends to a diffeomorphism of the blow-up of* $\mathbb{C}^2/\mathbb{Z}_2$ *at* $(0,0)$ *which is* $T^*\mathbb{C}P^1$*, to the full adjoint* $\mathrm{SL}(2,\mathbb{C})$*-orbit through* $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$*. This is the limit the non-abelian Hodge correspondence for* $t \to 0$*.*

Proof. Take $u = r\widetilde{u}$ and $v = r\widetilde{v}$ with $|\widetilde{u}|^2 = |\widetilde{v}|^2 = 1$ and $r \in \mathbb{R}_{>0}$ and consider $r \to 0$. Let $\Psi(u,v) = r^2\widetilde{\Psi}(\widetilde{u},\widetilde{v})$ be the associated nilpotent matrix. Then the map $\Psi \mapsto A_\Psi$ extends to $r = 0$ with

$$\lim_{r\to 0} A_{\Psi(u,v)} = i\widetilde{N}$$

with $\widetilde{N} \in \mathfrak{su}(2,\mathbb{C})$ of length 1 satisfying

$$< \widetilde{N}, \widetilde{\Psi} - \overline{\widetilde{\Psi}}^T >= 0 \quad \text{and} \quad < \widetilde{N}, i\widetilde{\Psi} + i\overline{\widetilde{\Psi}}^T >= 0.$$

Moreover, all hermitian symmetric matrices of determinant -1 can be realized as a limit. □

5.2. **The rescaled metric at** $t = 0$**.** We have seen that the Higgs bundle moduli space at $t = 0$ is the completion of the nilpotent $\mathrm{SL}(2,\mathbb{C})$-orbit, and that the deRham moduli space at $t = 0$ is the $\mathrm{SL}(2,\mathbb{C})$-orbit through $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Next, we derive that the rescaled limit metric is the Eguchi-Hanson metric. Note that the scaling factor is chosen so that the central sphere $\mathbb{C}P^1$ in $\mathcal{M}_{Higgs}$, which is the moduli space of semi-stable parabolic bundles (i.e., Higgs pairs with vanishing Higgs fields), has constant area with respect to the weight t.

To identify the limit hyper-Kähler metric at $t = 0$ we first compute the twisted symplectic form. Consider the moduli space $\mathcal{M}^{\frac{l}{k}}$ of logarithmic connections ∇ on the 4-punctured sphere Σ such that its residue $\mathrm{Res}_{p_j}(\nabla)$ at the puncture p_j lies in the conjugacy class of

$$\frac{l}{k}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

for some coprime integers l, k with $0 < \frac{l}{k} < \frac{1}{2}$ and every $j = 1, ..., 4$. Consider the k-fold covering Σ_k of $\mathbb{C}P^1$ defined by the equation

$$y^k = \frac{(z-p_1)(z-p_3)}{(z-p_2)(z-p_4)}$$

totally branched over $p_1, \ldots, p_4$, and let $\pi\colon \Sigma_k \to \mathbb{C}P^1$ denote the covering map. Then $\pi^*\nabla$ is gauge equivalent to a smooth (flat) connection on Σ_k equivariant with respect to the natural $\mathbb{Z}_k$ action on Σ_k. In this sense, the moduli space of logarithmic connections on the 4-punctured sphere Σ with rational weights $\frac{l}{k}$ can be identified with a complex subspace of the moduli space of all flat connections on Σ_k (away from reducible connections). By uniqueness of the non-abelian Hodge correspondence and due to the equivariance of the pull-back connections, $\mathcal{M}^{\frac{l}{k}}$ is a holomorphic submanifold with respect to both complex structures I, J on $\mathcal{M}_{SD}(\Sigma_k)$.

Theorem 7. *Consider the hyper-Kähler subspace $\mathcal{M}^{\frac{l}{k}}$ of $\mathcal{M}_{SD}(\Sigma_k)$, and the open subset $\mathcal{U} \subset \mathcal{M}^{\frac{l}{k}}$ whose associated families of flat connections are given by pull-backs of Fuchsian potentials η on some (punctured) a-disc in λ. Let $\eta \in \mathcal{U}$ and consider tangent vectors*

$$[X_q] \in T_{[\eta]}\mathcal{M}^{\frac{l}{k}}$$

represented by meromorphic 1-forms

$$X_q \in \mathbb{H}^0(\Sigma, K \otimes (\lambda^{-2}\Lambda^+\mathfrak{sl}(2,\mathbb{C})))$$

$q = 1, 2$ with first order poles on the 4-punctured sphere. Then,

$$\varpi_{[\eta]}([X_1],[X_2]) = 2\pi i k^3 \sum_{j=1}^{4} \mathrm{trace}(\, Res_{p_j}(\eta)[Res_{p_j}(X_1), Res_{p_j}(X_2)]\,), \tag{58}$$

where ϖ is as in (4).

Proof. The factor k^3 comes from the fact that we have to take pull-backs to a k-fold covering totally branched over $p_1 \ldots, p_4$, and that we pull-back three different 1-forms. The theorem is therefore a direct consequence of Corollary 21 with Proposition 18, after identifying twistor lines in the Deligne-Hitchin moduli space with the associated $\mathbb{C}^*$-families of flat connections as in (18), as the tangent vectors are meromorphic and only the residue terms remain. □

Remark 40. *On the 4-punctured sphere with rational weights $\frac{l}{k}$ we consider $\mathcal{M}^{\frac{l}{k}}$ as the complex submanifold of equivariant solutions upstairs on Σ_k. The moduli space of self-duality solutions $\mathcal{M}_{SD}(\Sigma_k)$ is hyper-Kähler with respect to Hitchin's metric. This therefore induces a hyper-Kähler structure downstairs on $\mathcal{M}^{\frac{l}{k}}$. This hyper-Kähler structure extends to irrational weights t using the real analytic dependence of harmonic metrics on the parabolic weights* [23]. *The hyper-Kähler structure on $\mathcal{M}^t$ should coincide with the one defined by Konno* [24] *directly for the moduli space of strongly parabolic Higgs bundles on the 4-punctured sphere (up to a*

normalizing factor), but we have not found a proper reference. In the following, we therefore use the (rescaled) twisted holomorphic symplectic form

$$\varpi = \tfrac{2\pi i}{t^3}\sum_{j=1}^{4} trace(B_j[dB_j \wedge dB_j])$$

on the moduli space of Fuchsian systems

$$d + \sum_{j=1}^{4} B_j \frac{dz}{z - z_j}$$

with $det(B_j) = -t^2$. *Note that this (rescaled) twisted holomorphic symplectic form coincides with the natural twisted holomorphic symplectic form upstairs when* $t = \frac{1}{k}$, *and it coincides at each puncture with* $\frac{1}{t}$ *(instead of* $\frac{1}{t^3}$*) times the Kirillov symplectic form on the adjoint orbit through* $\left(\begin{smallmatrix} t & 0 \\ 0 & -t \end{smallmatrix}\right)$. *By continuity (in* t*) the twistor construction gives rise to a positive definite complete hyper-Kähler metric such that the (rescaled) volume of (semi-stable) parabolic bundles (i.e., Higgs pairs with vanishing parabolic Higgs fields) is constant in* t.

Corollary 41. *Let* $t \sim 0$ *and consider the space of solutions (parametrized by* $(u, v) \in \mathbb{C}^2 \setminus \{(0, 0)\}$*) provided by Theorem 4 and Theorem 5. In terms of the parameters* $x_1(\lambda), x_2(\lambda), x_3(\lambda)$ *in* (27)*, the rescaled twisted holomorphic symplectic form is given by*

$$\varpi = 32\pi \frac{dx_2 \wedge dx_3}{x_1}.$$

Proof. By construction all 4 residues give the same contribution and residues are given by $B_j = tA_j$. Therefore, a direct computation shows that

$$\text{trace}[(B_1[dB_1 \wedge dB_1]) = -4it^3(x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2). \tag{59}$$

On the other hand,

$$0 = x_1 dx_1 + x_2 dx_2 + x_3 dx_3$$

which combines with (59) to

$$\tfrac{2\pi i}{t^3}\text{trace}[(B_1[dB_1 \wedge dB_1]) = 2\pi i \text{trace}[(A_1[dA_1 \wedge dA_1]) = 8\pi \frac{dx_2 \wedge dx_3}{x_1}$$

proving the corollary. □

To explicitly compute the rescaled twisted symplectic form in Corollary 41 at $t = 0$ let $A = A_1(\lambda)$ be the residue of the rescaled potential $\frac{1}{t}\eta_{t=0}$ at $t = 0$. Recall that

$$A = \lambda^{-1}\begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix} + \rho\begin{pmatrix} |u|^2 - |v|^2 & 2u\overline{v} \\ 2\overline{u}v & |v|^2 - |u|^2 \end{pmatrix} + \lambda\begin{pmatrix} -\overline{uv} & -\overline{v}^2 \\ \overline{u}^2 & \overline{uv} \end{pmatrix},$$

i.e., we compute the symplectic form using the Higgs field coordinates (u, v). Let $r^2 = |u|^2 + |v|^2$ and $\rho = -\sqrt{1 + r^{-4}}$. A direct computation then gives at $t = 0$

$$\begin{aligned} \varpi = 64\pi i \Big(&-\frac{r^6 + |v|^2}{\rho r^6} du \wedge d\overline{u} - \lambda^{-1} du \wedge dv + \frac{\overline{u}v}{\rho r^6} du \wedge d\overline{v} \\ &- \frac{u\overline{v}}{\rho r^6} d\overline{u} \wedge dv - \lambda d\overline{u} \wedge d\overline{v} - \frac{r^6 + |u|^2}{\rho r^6} dv \wedge d\overline{v} \Big). \end{aligned} \tag{60}$$

On the other hand,

$$\varpi = \lambda^{-1}(\omega_J + i\omega_K) - 2\omega_I - \lambda(\omega_J - i\omega_K)$$

from which we obtain

$$\begin{aligned}\omega_I &= \frac{32\pi i}{\rho r^6}\left((r^6+|v|^2)du\wedge d\overline{u} - \overline{u}v\, du\wedge d\overline{v} + u\overline{v}\, d\overline{u}\wedge dv + (r^6+|u|^2)dv\wedge d\overline{v}\right)\\ \omega_J &= 32\pi i\left(-du\wedge dv + d\overline{u}\wedge d\overline{v}\right)\\ \omega_K &= -32\pi\left(du\wedge dv + d\overline{u}\wedge d\overline{v}\right).\end{aligned}$$

Proof of Theorem 1. Since (u,v) corresponds to Higgs bundle coordinates, we use the complex structure I to compute $g=\omega_I(.,I.)$. Consider the tangent space basis

$$\mathcal{B} = \left(\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial \overline{u}}, \tfrac{\partial}{\partial v}, \tfrac{\partial}{\partial \overline{v}}\right)$$

Then the complex structure I can be represented by the matrix

$$i\left(\begin{array}{c|c} I_2 & 0\\ \hline 0 & -I_2\end{array}\right)$$

and we obtain

$$\begin{aligned} g \;=\; & 32\pi\sqrt{1+r^{-4}}\Big[du\otimes d\overline{u} + d\overline{u}\otimes du + dv\otimes d\overline{v} + d\overline{v}\otimes dv - \frac{1}{r^2(1+r^4)}\Big(u\overline{u}(du\otimes d\overline{u} + d\overline{u}\otimes du)\\ & + \overline{u}v(du\otimes d\overline{v} + d\overline{v}\otimes du) + u\overline{v}(dv\otimes d\overline{u} + d\overline{u}\otimes dv) + v\overline{v}(dv\otimes d\overline{v} + d\overline{v}\otimes dv)\Big)\Big]\end{aligned}$$

which by [25, Equation 2] identifies with the Eguchi-Hanson metric with $n=2$ and $a=1$ scaled by 32π. □

5.3. **The complex structure J and K in terms of Higgs field coordinates at $t=0$.** Next we compute the complex structure J. With respect to the basis $\mathcal{B}$ the symplectic form ω_J is represented by the matrix

$$\mathrm{Mat}_{\mathcal{B}}(\omega_J) = 32\pi i\left(\begin{array}{c|c} -N & 0\\ \hline 0 & N\end{array}\right) \quad \text{with} \quad N = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$$

Thus the matrix representing the complex structure J is given by

$$\mathrm{Mat}_{\mathcal{B}}(J) = \mathrm{Mat}_{\mathcal{B}}(\omega_J)^{-1}\,\mathrm{Mat}_{\mathcal{B}}(g) = \frac{-i}{\rho r^6}\left(\begin{array}{c|c} N & 0\\ \hline 0 & -N\end{array}\right)\left(\begin{array}{c|c} 0 & B\\ \hline \overline{B} & 0\end{array}\right) = \frac{-i}{\rho r^6}\left(\begin{array}{c|c} 0 & NB\\ \hline -\overline{NB} & 0\end{array}\right)$$

with

$$NB = \begin{pmatrix} -u\overline{v} & r^6+|u|^2\\ -r^6-|v|^2 & \overline{u}v\end{pmatrix}.$$

Similarly the complex structure K is then given by

$$\mathrm{Mat}_{\mathcal{B}}(\omega_K) = -32\pi\left(\begin{array}{c|c} N & 0\\ \hline 0 & N\end{array}\right)$$

$$\mathrm{Mat}_{\mathcal{B}}(K) = \mathrm{Mat}_{\mathcal{B}}(\omega_K)^{-1}\,\mathrm{Mat}_{\mathcal{B}}(g) = \frac{-1}{\rho r^6}\left(\begin{array}{c|c} -N & 0\\ \hline 0 & -N\end{array}\right)\left(\begin{array}{c|c} 0 & B\\ \hline \overline{B} & 0\end{array}\right) = \frac{1}{\rho r^6}\left(\begin{array}{c|c} 0 & NB\\ \hline \overline{NB} & 0\end{array}\right)$$

One can then check that $J^2=K^2=-\mathrm{Id}$ and $IJ=K$ as expected and the complex structure J is that of the adjoint $\mathrm{SL}(2,\mathbb{C})$ orbit through $\left(\begin{smallmatrix} 1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

5.4. **First order approximations.** The main advantage of the implicit function theorem approach is that we obtain, analogously to [20], an iterative way of computing the power series expansion of the twistor lines leading to an approach towards explicitly computing the non-abelian Hodge correspondence and all involved geometric quantities in this symmetric setup. In this section we first compute the first order derivatives of the parameters from which in particular first order derivatives of the relative twisted holomorphic symplectic form is computed, which yield derivatives of the non-abelian Hodge correspondence as well as derivatives of the hyper-Kähler metric.

5.4.1. *First order derivatives of the parameters.* In order to write down closed form formulas for the to be computed derivatives we define for $1 \leq j, k \leq 3$ and ω_k, Ω_j given in (24) and (47), respectively,

$$\Omega_{jk}(z) = \int_0^z \Omega_j \omega_k.$$

The shuffle relation (see e.g. [20, Appendix]) then gives

$$\Omega_j \Omega_k = \Omega_{jk} + \Omega_{kj}. \tag{61}$$

Let $()'$ and $()''$ denote the first and second order derivatives of a quantity with respect to t at $t = 0$. Then the following Proposition holds.

Proposition 42. *The first order derivatives x_j' for $j = 1, 2, 3$ are polynomials of degree at most 2 in λ:*

$$x_j' = x_{j,0}' + x_{j,1}'\lambda + x_{j,2}'\lambda^2 = x_{j,0}' + (x_j')^+$$

with the positive parts given by

$$\begin{cases} (x_1')^+ = \dfrac{4i}{\pi} \operatorname{Im}(\Omega_{21}(1) + \Omega_{31}(i)) \, (\underline{x}_2 \underline{x}_3)^+ \\ (x_2')^+ = \dfrac{-4i}{\pi} \operatorname{Im}(\Omega_{31}(i)) \, (\underline{x}_1 \underline{x}_3)^+ \\ (x_3')^+ = \dfrac{-4i}{\pi} \operatorname{Im}(\Omega_{21}(1)) \, (\underline{x}_1 \underline{x}_2)^+, \end{cases} \tag{62}$$

and the constant terms given by

$$\begin{cases} x_{1,0}' = \dfrac{-1}{\rho r^4} \left((|u|^2 - |v|^2) X - 2\rho u v Y \right) \\ x_{2,0}' = \dfrac{-1}{\rho r^4} \left(2 \operatorname{Re}(u \, \overline{v}) X - \rho (v^2 - u^2) Y \right) \\ x_{3,0}' = \dfrac{-1}{\rho r^4} \left(2 \operatorname{Im}(u \, \overline{v}) X - \rho i (u^2 + v^2) Y \right), \end{cases} \tag{63}$$

where $r^2 = |u|^2 + |v|^2$ and

$$\begin{aligned} X &= \sum_{j=1}^{3} \underline{x}_{j,-1} x_{j,1}' \\ Y &= \sum_{j=1}^{3} \left(\underline{x}_{j,-1} x_{j,2}' + \underline{x}_{j,0} x_{j,1}' \right) \end{aligned} \tag{64}$$

and the $x_{j,1}'$ and $x_{j,2}'$ are determined from (62).

Remark 43. *The required Ω integrals are computed in Proposition 45 which gives*

$$\mathrm{Im}(\Omega_{21}(1)) = 2\pi \log\left|\frac{p^2-1}{2p}\right| \quad \textit{and} \quad \mathrm{Im}(\Omega_{31}(i)) = -2\pi\log\left|\frac{p^2+1}{2p}\right|.$$

Proof. Let Φ be the fundamental solution of the equation $d\Phi_t = \Phi_t\eta_t$ with $\Phi_t(0) = \mathrm{Id}$. Recall that $\eta_{t=0} = 0$ and therefore $\Phi_{t=0} = \mathrm{Id}$. Differentiating the equation $d\Phi_t = \Phi_t\eta_t$ twice at $t = 0$ we thus obtain

$$d\Phi'' = \eta'' + 2\Phi'\eta',$$

hence

$$\Phi''(z) = \int_0^z (\eta'' + 2\Phi'\eta').$$

Then we have using the proof of Proposition 33

$$\eta' = \sum_{j=1}^{3} \underline{x}_j\omega_j\mathfrak{m}_j, \quad \Phi' = 2\sum_{j=1}^{3}\underline{x}_j\Omega_j\mathfrak{m}_j, \quad \eta'' = 2\sum_{j=1}^{3} x'_j\omega_j\mathfrak{m}_j.$$

This gives

$$\int_0^z \eta'' = 2\begin{pmatrix} x'_1\Omega_1 & x'_2\Omega_2 + ix'_3\Omega_3 \\ x'_2\Omega_2 - ix'_3\Omega_3 & -x'_1\Omega_1 \end{pmatrix}$$

and

$$\int_0^z \Phi'\eta' = \begin{pmatrix} \sum_{j=1}^3 \underline{x}_j^2\Omega_{jj} + i\underline{x}_2\underline{x}_3(\Omega_{32}-\Omega_{23}) & \underline{x}_1\underline{x}_2(\Omega_{12}-\Omega_{21}) + i\underline{x}_1\underline{x}_3(\Omega_{13}-\Omega_{31}) \\ \underline{x}_1\underline{x}_2(\Omega_{21}-\Omega_{12}) + i\underline{x}_1\underline{x}_3(\Omega_{13}-\Omega_{31}) & \sum_{j=1}^3 \underline{x}_j^2\Omega_{jj} + i\underline{x}_2\underline{x}_3(\Omega_{23}-\Omega_{32}) \end{pmatrix}.$$

Using Leibniz rule, the shuffle relation (61) and that we solved Equation 48 we obtain

$$\begin{aligned} \mathfrak{p}'' &= \mathcal{P}''_{12} - \mathcal{P}''_{21} + 2(\mathcal{P}'_{11}\mathcal{P}'_{21} - \mathcal{P}'_{12}\mathcal{P}'_{22}) \\ &= -4ix'_3\Omega_3(1) + 4\underline{x}_1\underline{x}_2\left(\Omega_{21}(1) - \Omega_{12}(1)\right) + 4\underline{x}_1\underline{x}_2\Omega_1(1)\Omega_2(1) \\ &= 4\pi x'_3 + 8\underline{x}_1\underline{x}_2\Omega_{21}(1) \end{aligned} \tag{65}$$

$$\begin{aligned} \mathfrak{q}'' &= i(\mathcal{Q}''_{21} + \mathcal{Q}''_{12}) + 2i(\mathcal{Q}'_{11}\mathcal{Q}'_{21} + \mathcal{Q}'_{12}\mathcal{Q}'_{22}) \\ &= 4ix'_2\Omega_2(i) - 4\underline{x}_1\underline{x}_3(\Omega_{13}(i) - \Omega_{31}(i)) + 4\underline{x}_1\underline{x}_3\Omega_1(i)\Omega_3(i) \\ &= 4\pi x'_2 + 8\underline{x}_1\underline{x}_3\Omega_{31}(i). \end{aligned} \tag{66}$$

Since we have solved $\mathfrak{p}(t) = \mathfrak{p}(t)^*$ for all t, we have $\mathfrak{p}'' = (\mathfrak{p}'')^*$. Moreover, $\underline{x}_j = \underline{x}_j^*$, hence

$$4\pi x'_3 - 4\pi(x'_3)^* = -16i\,\underline{x}_1\underline{x}_2\mathrm{Im}(\Omega_{21}(1)). \tag{67}$$

Projecting onto $\mathcal{W}^+$ and remembering that $x' \in \mathcal{W}^{\geq 0}$, we obtain the formula for $(x'_3)^+$ stated in Equation (62), by keeping in mind that $(\underline{x}_1\underline{x}_2)^+$ is a degree-2 polynomial. In the same vein $\mathfrak{q}'' = (\mathfrak{q}'')^*$ gives

$$4\pi x'_2 - 4\pi(x'_2)^* = -16i\,\underline{x}_1\underline{x}_3\mathrm{Im}(\Omega_{31}(i)) \tag{68}$$

which determines $(x'_2)^+$.

For $(x'_1)^+$ consider the equation $\mathcal{K} = 1$ which holds for all t. Therefore,

$$\mathcal{K}' = 0 = 2\sum_{j=1}^{3}\underline{x}_j x'_j. \tag{69}$$

Then $\mathcal{K}' = \mathcal{K}'^*$ and Equations (67), (68) give the equation

$$4\pi \underline{x}_1 x_1' - 4\pi \underline{x}_1 (x_1')^* = 16i\, \underline{x}_1 \underline{x}_2 \underline{x}_3 \mathrm{Im}(\Omega_{21}(1) + \Omega_{31}(i)). \tag{70}$$

Dividing by $\underline{x}_1 \not\equiv 0$ and taking the positive part determines $(x_1')^+$.

To compute the constant terms, we consider the coefficients of λ^{-1}, λ^0 and λ in $\mathcal{K}$:

$$\mathcal{K}'_{-1} = 2\sum_{j=1}^{3} \underline{x}_{j,1} x'_{j,0} = 0$$

$$\mathcal{K}'_{0} = 2\sum_{j=1}^{3} (\underline{x}_{j,0} x'_{j,0} + \underline{x}_{j,-1} x'_{j,1}) = 0$$

$$\mathcal{K}'_{1} = 2\sum_{j=1}^{3} (\underline{x}_{j,1} x'_{j,0} + \underline{x}_{j,0} x'_{j,1} + \underline{x}_{j,-1} x'_{j,2})$$

which yield the system of equations

$$\begin{cases} \sum_{j=1}^{3} \underline{x}_{j,-1} x'_{j,0} = 0 \\ \sum_{j=1}^{3} \underline{x}_{j,0} x'_{j,0} = -X \\ \sum_{j=1}^{3} \underline{x}_{j,1} x'_{j,0} = -Y \end{cases}$$

with X, Y as in Proposition 42. Its determinant simplifies to

$$\det(\underline{x}_{j,k})_{1\le j\le 3, -1\le k\le 1} = \frac{i}{2}\rho r^6.$$

Using the Cramer rule, we obtain (63) after simplification. □

Remark 44. *The second derivative of $\mathfrak{r}$ is computed in the following. Through the character variety equation and the formulas for $\mathfrak{p}''$ and $\mathfrak{q}''$ it will give rise to an Ω-identity. Using Proposition 32 we have*

$$\begin{aligned} \mathfrak{r}'' &= i\big[\mathcal{P}''_{22} + 2\mathcal{P}'_{22}\mathcal{Q}'_{11} + \mathcal{Q}''_{11} + 2\mathcal{P}'_{12}\mathcal{Q}'_{21} - 2\mathcal{P}'_{21}\mathcal{Q}'_{12} - \mathcal{P}''_{11} - 2\mathcal{P}'_{11}\mathcal{Q}'_{22} - \mathcal{Q}''_{22} \\ &\quad + (\mathcal{P}'_{22} + \mathcal{Q}'_{11})^2 + (\mathcal{P}'_{12} + \mathcal{Q}'_{12})^2 - (\mathcal{P}'_{21} + \mathcal{Q}'_{21})^2 - (\mathcal{P}'_{11} + \mathcal{Q}'_{22})^2\big] \\ &= i\big[\mathcal{P}''_{22} - \mathcal{P}''_{11} + \mathcal{Q}''_{11} - \mathcal{Q}''_{22} + 2(\mathcal{P}'_{12} - \mathcal{P}'_{21})(\mathcal{Q}'_{12} + \mathcal{Q}'_{21}) + (\mathcal{P}'_{12})^2 - (\mathcal{P}'_{21})^2 + (\mathcal{Q}'_{12})^2 - (\mathcal{Q}'_{21})^2\big] \end{aligned}$$

where we used that $\mathcal{P}'_{11} + \mathcal{P}'_{22} = \mathcal{Q}'_{11} + \mathcal{Q}'_{22} = 0$. This gives

$$\begin{aligned} \mathfrak{r}'' &= i\big[-4x_1'\Omega_1(1) + 4i\underline{x}_2\underline{x}_3(\Omega_{23}(1) - \Omega_{32}(1)) + 4x_1'\Omega_1(i) - 4i\underline{x}_2\underline{x}_3(\Omega_{23}(i) - \Omega_{32}(i)) \\ &\quad + 8i\underline{x}_2\underline{x}_3\Omega_3(1)\Omega_2(i) + 4i\underline{x}_2\underline{x}_3\Omega_3(1)\Omega_2(1) + 4i\underline{x}_2\underline{x}_3\Omega_2(i)\Omega_3(i)\big] \\ &= 4\pi x_1' - 8\underline{x}_2\underline{x}_3\left(\Omega_{23}(1) + \Omega_{32}(i) + \pi^2\right) \end{aligned} \tag{71}$$

using that we solved Equation 48 and the shuffle relation (61)*. Observe the similarity with* (65) *and* (66)*. Remember from the proof of Proposition 34 that $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{r}$ satisfy the character variety equation for all t*

$$4\cos(2\pi t)^2 + 4(\mathfrak{p}^2 + \mathfrak{q}^2 + \mathfrak{r}^2 - 1) + 8(-1)^j \mathfrak{p}\mathfrak{q}\mathfrak{r} = 0 \tag{72}$$

with either $j = 1$ or $j = 2$. Taking the third order derivative of the above equation yields (since $\mathfrak{p}(0) = \mathfrak{q}(0) = \mathfrak{r}(0) = 0$)

$$\mathfrak{p}'\mathfrak{p}'' + \mathfrak{q}'\mathfrak{q}'' + \mathfrak{r}'\mathfrak{r}'' + 2(-1)^j\mathfrak{p}'\mathfrak{q}'\mathfrak{r}' = 0$$

Using Proposition 33 and Equations (65), (66), (71) *we then obtain*

$$8\pi^2(\underline{x}_1x_1' + \underline{x}_2x_2' + \underline{x}_3x_3') + 16\pi\underline{x}_1\underline{x}_2\underline{x}_3\big(\Omega_{21}(1) + \Omega_{31}(i) - \Omega_{23}(1) - \Omega_{32}(i) - \pi^2 + (-1)^j\pi^2\big) = 0.$$

The first summand vanishes, as $x_1^2 + x_2^2 + x_3^2 = 1$ for all t, thus

$$\Omega_{21}(1) + \Omega_{31}(i) - \Omega_{23}(1) - \Omega_{32}(i) - \pi^2 + (-1)^j\pi^2 = 0.$$

In the most symmetric case of the 4-punctured sphere where $p = e^{i\pi/4}$ we have by symmetry

$$\Omega_{31}(i) = -\Omega_{21}(1) \quad \textit{and} \quad \Omega_{32}(i) = -\Omega_{23}(i).$$

Hence $j = 2$ and we have proved the following identity that holds for all $p \in \mathbb{C}^+$:

$$\Omega_{23}(1) + \Omega_{32}(i) = \Omega_{21}(1) + \Omega_{31}(i). \tag{73}$$

Proposition 45. *For $p \in \mathbb{C}^+$ we have*

$$\Omega_{21}(1) = 2\pi i \log\left(\frac{p^2-1}{2ip}\right)$$

$$\Omega_{31}(i) = -2\pi i \log\left(\frac{p^2+1}{2p}\right)$$

where log *denote the principal valuation of the logarithm on $\mathbb{C} \setminus \mathbb{R}^-$.*

Proof. We first prove the Proposition for $p = e^{i\varphi}$ with $0 < \varphi < \pi/2$. In that case, we have by [20, Proposition 35]

$$\begin{aligned} \Omega_{21}(1) - \Omega_{12}(1) &= 4\pi i \log(\sin\varphi)) - i(\pi - 2\varphi)\log\left(\frac{1-\cos(\varphi)}{1+\cos(\varphi)}\right) \\ \Omega_{31}(i) - \Omega_{13}(i) &= -4\pi i \log(\cos(\varphi)) + 2i\varphi\log\left(\frac{1-\sin(\varphi)}{1+\sin(\varphi)}\right). \end{aligned} \tag{74}$$

On the other hand, by the shuffle product formula and using (48)

$$\Omega_{21}(1) + \Omega_{12}(1) = \Omega_1(1)\Omega_2(1) = i(\pi - 2\varphi)\log\left(\frac{1-\cos(\varphi)}{1+\cos(\varphi)}\right)$$

$$\Omega_{31}(i) + \Omega_{13}(i) = \Omega_1(i)\Omega_3(i) = -2i\varphi\log\left(\frac{1-\sin(\varphi)}{1+\sin(\varphi)}\right).$$

Hence

$$\Omega_{21}(1) = 2\pi i \log(\sin\varphi)$$

$$\Omega_{31}(i) = -2\pi i \log(\cos(\varphi)$$

proving the result for $p = e^{i\varphi}$.

For $p \in \mathbb{C}^+$, both $\frac{p^2-1}{2ip}$ and $\frac{p^2+1}{2p}$ are in $\mathbb{C}\setminus\mathbb{R}^-$ and both sides of the formulas of Proposition 45 are well-defined holomorphic functions in $p \in \mathbb{C}^+$ which coincide when $p \in \mathbb{C}^+ \cap \mathbb{S}^1$, therefore they are equal. □

5.5. **First order derivative of the metric.** By Corollary 41 we have for all t

$$\varpi = 32\pi \frac{dx_2 \wedge dx_3}{x_1}$$

Since we prescribe the Higgs field in our construction, we have that the complex structure I is independent of t and therefore also the holomorphic symplectic forms $\omega_J \pm i\omega_K$ are independent of t. This gives that the first order derivatives of the twisted holomorphic symplectic form is given by the derivative of its constant term

$$\varpi_0 = -2\omega_I = \frac{32\pi}{\underline{x}_{1,-1}} \left(\frac{-x_{1,0}}{\underline{x}_{1,-1}} d\underline{x}_{2,-1} \wedge d\underline{x}_{3,-1} + dx_{2,0} \wedge d\underline{x}_{3,-1} + d\underline{x}_{2,-1} \wedge x_{3,0} \right). \tag{75}$$

Using the formulas for $x'_{1,0}$, $x'_{2,0}$ and $x'_{3,0}$, we obtain

$$\begin{aligned}
\varpi'_0 \;=\; & \frac{128\, i\, \mathrm{Im}(\Omega_{21}(1))}{r^8} \left[\left(-r^4 + 3(\overline{u}v + u\overline{v})^2\right) |v|^2 + r^8 \left(|u|^2 - |v|^2\right) \right] du \wedge d\overline{u} \\
+ \; & \frac{128\, i\, \mathrm{Im}(\Omega_{31}(i))}{r^8} \left[\left(r^4 + 3(\overline{u}v - u\overline{v})^2\right) |v|^2 + r^8 \left(|v|^2 - |u|^2\right) \right] du \wedge d\overline{u} \\
+ \; & \frac{128\, i\, \mathrm{Im}(\Omega_{21}(1))}{r^8} \left[\left(r^4 - 3(\overline{u}v + u\overline{v})^2\right) \overline{u}v + r^8 \left(-\overline{u}v - 3u\overline{v}\right) \right] du \wedge d\overline{v} \\
+ \; & \frac{128\, i\, \mathrm{Im}(\Omega_{31}(i))}{r^8} \left[\left(-r^4 - 3(\overline{u}v - u\overline{v})^2\right) \overline{u}v + r^8 \left(\overline{u}v - 3u\overline{v}\right) \right] du \wedge d\overline{v} \\
+ \; & \frac{128\, i\, \mathrm{Im}(\Omega_{21}(1))}{r^8} \left[\left(r^4 - 3(\overline{u}v + u\overline{v})^2\right) u\overline{v} + r^8 \left(-u\overline{v} - 3\overline{u}v\right) \right] dv \wedge d\overline{u} \\
+ \; & \frac{128\, i\, \mathrm{Im}(\Omega_{31}(i))}{r^8} \left[\left(-r^4 - 3(\overline{u}v - u\overline{v})^2\right) u\overline{v} + r^8 \left(u\overline{v} - 3\overline{u}v\right) \right] dv \wedge d\overline{u} \\
+ \; & \frac{128\, i\, \mathrm{Im}(\Omega_{21}(1))}{r^8} \left[\left(-r^4 + 3(\overline{u}v + u\overline{v})^2\right) |u|^2 + r^8 \left(|v|^2 - |u|^2\right) \right] dv \wedge d\overline{v} \\
+ \; & \frac{128\, i\, \mathrm{Im}(\Omega_{31}(i))}{r^8} \left[\left(r^4 + 3(\overline{u}v - u\overline{v})^2\right) |u|^2 + r^8 \left(|u|^2 - |v|^2\right) \right] dv \wedge d\overline{v}.
\end{aligned} \tag{76}$$

Note that the terms $du \wedge dv$ and $d\overline{u} \wedge d\overline{v}$ vanish as expected, and the required Ω-integrals are given by Proposition 45.

5.6. **Energy.** Analogously to [20, equation (34)] and similar to the derivation of Corollary 41, the general formula for the (rescaled) L^2-energy of the real section η is given by

$$\mathcal{E} = 4\pi \sum_j \mathrm{Res}_{q_j} \mathrm{trace} \left(\eta_{-1} G_{j,1} G_{j,0}^{-1} \right)$$

on the covering surface Σ_k, where $G_j = G_{j,0} + G_{j,1}\lambda + \dots$ is a desingularizing gauge.

This energy is the harmonic map energy of the (smooth) equivariant harmonic map for rational $t = \frac{l}{k}$ on the compact surface Σ_k, and it is a Kähler potential for the metric with respect to the complex structure J, see [21].

Assume that $t = \frac{1}{2g+2}$ and let $\pi : \widetilde{\Sigma} \to \Sigma$ be the $g+1$ covering. A desingularizing gauge in a neighborhood of $\widetilde{p}_1$ is

$$G_1 = \begin{pmatrix} 1 & 0 \\ \frac{1-x_1}{x_2+ix_3} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{w}} & 0 \\ 0 & \sqrt{w} \end{pmatrix}$$

with w a local coordinate such that $w^{g+1} = z - p_1$. Then

$$\mathrm{Res}_{\widetilde{p}_1}\mathrm{trace}\left(\widetilde{\eta}_{-1}G_{1,1}G_{1,0}^{-1}\right) = 1-x_{1,0}+\frac{x_{1,-1}(x_{2,0}+ix_{3,0})}{x_{2,-1}+ix_{3,-1}} = 1-x_{1,0}-\frac{(x_{2,-1}-ix_{3,-1})(x_{2,0}+ix_{3,0})}{x_{1,-1}}.$$

By the substitution $(x_1, x_2, x_3) \to (-x_1, -x_2, x_3)$

$$\mathrm{Res}_{\widetilde{p}_2}\mathrm{trace}\left(\widetilde{\eta}_{-1}G_{2,1}G_{2,0}^{-1}\right) = 1 + x_{1,0} + \frac{(x_{2,-1}+ix_{3,-1})(x_{2,0}-ix_{3,0})}{x_{1,-1}}$$

and substituting $(x_1, x_2, x_3) \to (x_1, -x_2, -x_3)$ we obtain for $k = 1, 2$

$$\mathrm{Res}_{\widetilde{p}_{k+2}}\mathrm{trace}\left(\widetilde{\eta}_{-1}G_{k+2,1}G_{k+2,0}^{-1}\right) = \mathrm{Res}_{\widetilde{p}_k}\mathrm{trace}\left(\widetilde{\eta}_{-1}G_{k,1}G_{k,0}^{-1}\right)$$

This gives

$$\mathcal{E} = 8\pi\left(1+\frac{i}{x_{1,-1}}(-x_{2,-1}x_{3,0}+x_{2,0}x_{3,-1})\right).$$

Using the central value of the parameters we find at $t = 0$ that

$$\mathcal{E} = 8\pi(1-\rho r^2) = 8\pi(1+\sqrt{1+r^4}), \tag{77}$$

with $r^2 = |u|^2 + |v|^2$. The first order derivatives of the parameters then gives

$$\begin{aligned}\mathcal{E}' &= 8\,\mathrm{Im}(\Omega_{21}(1))\left(|u|^4+|v|^4-3\overline{u}^2v^2-3u^2\overline{v}^2-4|u|^2|v|^2\right)\\ &\quad -8\,\mathrm{Im}(\Omega_{31}(i))\left(|u|^4+|v|^4+3\overline{u}^2v^2+3u^2\overline{v}^2-4|u|^2|v|^2\right).\end{aligned} \tag{78}$$

For the most symmetric case of $p = e^{i\pi/4}$, using $\Omega_{21}(1) = -\pi i\log(2)$ from Appendix A of [20] and $\Omega_{31}(i) = -\Omega_{21}(1)$, this simplifies to

$$\mathcal{E}' = -16\pi\log(2)\left(|u|^4+|v|^4-4|u|^2|v|^2\right). \tag{79}$$

6. Higher order derivatives

6.1. **The algorithm.** Just as in [20, Section 5] we give an iterative algorithm for computing higher order derivatives of the parameters in terms of the multiple polylogarithm (MPL) function. The difference to the minimal surface case [20] is that we have no extrinsic closing conditions but complex parameters here.

The computation of the higher order derivatives of the parameters involves the iterated integral $\Omega_{i_1,\cdots,i_n}$ defined recursively by

$$\Omega_{i_1,\cdots,i_n}(z) = \int_0^z \Omega_{i_1,\cdots,i_{n-1}}\omega_{i_n}, \tag{80}$$

where ω_i is as in (24) for $i = 1, 2, 3$. It is shown in [20, A.3] that (and how) these iterated integrals can be expressed in terms of multiple polylogarithm $\mathrm{Li}_{n_1,\ldots,n_d}$. For positive integers $n_1, \ldots, n_d \in \mathbb{Z}_{>0}$, and $z_i \in \mathbb{C}^d$ in the region given by $|z_i \ldots z_d| < 1$, the multiple polylogarithm $\mathrm{Li}_{n_1,\ldots,n_d}$ is defined by

$$\mathrm{Li}_{n_1,\ldots,n_d}(z_1,\ldots,z_d) = \sum_{0<k_1<k_2<\cdots<k_d}\frac{z_1^{k_1}\cdots z_d^{k_d}}{k_1^{n_1}\cdots k_d^{n_d}}.$$

This function is extended to a multivalued function by analytic continuation. Here the *depth* d counts the number of indices $n_1, \ldots, n_d$, and the *weight* is given by the sum $n_1 + \cdots + n_d$ of the indices. The functions $\Omega_{i_1,\cdots,i_n}(z)$ can be expressed in terms of multiple polylogarithms of depth n and weight n.

In the following we denote by $x_i^{(n)}$, $\mathcal{P}^{(n)}$, $\mathcal{Q}^{(n)}$ the n-th order derivatives of x_i, $\mathcal{P}$, $\mathcal{Q}$ with respect to t at $t=0$ and we suppress the dependence of (u,v) and p.

Proposition 46. *For $n \geq 1$ and $1 \leq i \leq 3$, $x_i^{(n)}$ are polynomials (with respect to λ) of degree at most $n+1$ and the coefficients of $\mathcal{P}^{(n+1)}$, $\mathcal{Q}^{(n+1)}$ are Laurent polynomials of degree at most $n+1$, which can be expressed explicitly in terms of multiple-polylogarithms of depth $n+1$ and weight $n+1$.*

Proof. The proof is by induction on n. Let H_n be the statement of the proposition. We have already proved H_1 in Section 5.4. Fix $n \geq 1$ and assume that H_k is true for all $1 \leq k < n$ and let the index 'lower' denote all terms of a quantity that depends only on derivatives of lower order. As in [20, Proposition 37] we have:

$$\mathcal{P}^{(n+1)} = \sum_{\ell=1}^{n+1} \frac{(n+1)!}{(n+1-\ell)!} \sum_{i_1,\cdots,i_\ell} x_{i_1,\cdots,i_\ell}^{(n+1-\ell)} \mathfrak{m}_{i_1,\cdots,i_\ell} \Omega_{i_1,\cdots,i_\ell}(1)$$

with

$$x_{i_1,\cdots,i_\ell} = \prod_{j=1}^{\ell} x_{i_j} \quad \text{and} \quad \mathfrak{m}_{i_1,\cdots,i_\ell} = \prod_{j=1}^{\ell} \mathfrak{m}_{i_j}.$$

We rewrite this as

$$\mathcal{P}^{(n+1)} = (n+1) \sum_{i=1}^{3} x_i^{(n)} \mathfrak{m}_i \Omega_i(1) + \mathcal{P}_{\text{lower}}^{(n+1)}$$

with

$$\mathcal{P}_{\text{lower}}^{(n+1)} = \sum_{\ell=2}^{n+1} \frac{(n+1)!}{(n+1-\ell)!} \sum_{i_1,\cdots,i_\ell} x_{i_1,\cdots,i_\ell}^{(n+1-\ell)} \mathfrak{m}_{i_1,\cdots,i_\ell} \Omega_{i_1,\cdots,i_\ell}(1).$$

Similar formula holds for $\mathcal{Q}^{(n+1)}$ with $\Omega_{i_1,\cdots,i_\ell}(1)$ replaced by $\Omega_{i_1,\cdots,i_\ell}(i)$. Using Leibniz rule we have

$$\mathfrak{p}^{(n+1)} = 2\pi(n+1) x_3^{(n)} + \mathfrak{p}_{\text{lower}}^{(n+1)}$$

with

$$\mathfrak{p}_{\text{lower}}^{(n+1)} = \sum_{k=1}^{n} \binom{n+1}{k} \left(\mathcal{P}_{11}^{(k)} \mathcal{P}_{21}^{(n+1-k)} - \mathcal{P}_{12}^{(k)} \mathcal{P}_{22}^{(n+1-k)} \right) + \mathcal{P}_{\text{lower},21}^{(n+1)} - \mathcal{P}_{\text{lower},12}^{(n+1)}$$

and from $\mathfrak{p}^{(n+1)} = (\mathfrak{p}^{(n+1)})^*$ we obtain by taking the positive part:

$$(x_3^{(n)})^+ = \frac{1}{2\pi(n+1)} \left((\mathfrak{p}_{\text{lower}}^{(n+1)})^{-*} - (\mathfrak{p}_{\text{lower}}^{(n+1)})^+ \right).$$

In the same way

$$\mathfrak{q}^{(n+1)} = 2\pi(n+1) x_2^{(n)} + \mathfrak{q}_{\text{lower}}^{(n+1)}$$

with

$$\mathfrak{q}_{\text{lower}}^{(n+1)} = i \sum_{k=1}^{n} \binom{n+1}{k} \left(\mathcal{Q}_{11}^{(k)} \mathcal{Q}_{21}^{(n+1-k)} + \mathcal{Q}_{12}^{(k)} \mathcal{Q}_{22}^{(n+1-k)} \right) + i\mathcal{Q}_{\text{lower},21}^{(n+1)} + i\mathcal{Q}_{\text{lower},12}^{(n+1)}$$

and we obtain

$$(x_2^{(n)})^+ = \frac{1}{2\pi(n+1)} \left((\mathfrak{q}_{\text{lower}}^{(n+1)})^{-*} - (\mathfrak{q}_{\text{lower}}^{(n+1)})^+ \right).$$

By inspection and the induction hypothesis, $(x_2^{(n)})^+$ and $(x_3^{(n)})^+$ are polynomials of degree at most $n+1$.

Remark 47. *We could also determine* $\mathrm{Im}(x_{3,0}^{(n)})$ *and* $\mathrm{Im}(x_{2,0}^{(n)})$ *from the zero part of* $\mathfrak{p}^{(n)} = (\mathfrak{p}^{(n)})^*$ *and* $\mathfrak{q}^{(n)} = (\mathfrak{q}^{(n)})^*$*, but it is simpler to determine the three complex parameters* $x_{i,0}^{(n)}$ *by solving a complex linear system, see below.*

Using Leibnitz rule:

$$0 = \mathcal{K}^{(n)} = 2\sum_{i=1}^{3} \underline{x}_i x_i^{(n)} + \mathcal{K}^{(n)}_{\text{lower}}$$

with

$$\mathcal{K}^{(n)}_{\text{lower}} = \sum_{i=1}^{3}\sum_{k=1}^{n-1} \binom{n}{k} x_i^{(k)} x_i^{(n-k)}.$$

We multiply by λ and obtain (recall that $P_j = \lambda \underline{x}_j$):

$$P_1 x_1^{(n)} + P_2 x_{2,0}^{(n)} + P_3 x_{3,0}^{(n)} = -\frac{\lambda}{2}\mathcal{K}^{(n)}_{\text{lower}} - P_2 (x_2^{(n)})^+ - P_3 (x_3^{(n)})^+. \tag{81}$$

The right side of (81) is already known and is a polynomial in λ of degree at most $n+3$. Hence $P_1 x_1^{(n)}$ is a polynomial of degree at most $n+3$. When $u \sim v$, P_1 has two roots in $\mathbb{D}_a$, so since $x_1^{(n)}$ cannot have poles in $\mathbb{D}_a$, it must be a polynomial of degree at most $n+1$. This remains true for all (u,v) by analyticity. Let Q, R be the quotient and remainder of the division of the right side of (81) with respect to P_1. Then

$$(x_1^{(n)})^+ = Q^+$$

and looking at the coefficients of λ^0, λ^1 and λ^2 in (81), we obtain a system of three complex equations with unknowns $x_{1,0}^{(n)}$, $x_{2,0}^{(n)}$ and $x_{3,0}^{(n)}$, whose determinant is

$$\det\left(\underline{x}_{i,j}\right)_{\substack{-1\le i\le 1\\ 1\le j\le 3}} = \tfrac{1}{2} i\rho r^6 \neq 0$$

. □

As a direct corollary we can express the hyper-Kähler metric of the moduli space $\mathcal{M}(t)$ in terms of multiple polylogarithms.

Proof of Theorem 2. By twistor theory, we can compute the hyper-Kähler metric explicitly in terms of the relative holomorphic symplectic form $\varpi = 32\pi \frac{dx_2 \wedge dx_3}{x_1}$. Since x_1, x_2, x_3 depend real analytic on t the theorem follows from Proposition 46. □

The algorithm has been implemented and using Mathematica we obtain, for example, for $p = e^{i\pi/4}$

$$\mathcal{E}'' = \frac{-32\,\pi}{\rho r^6}(|u|^2 - |v|^2)^2\big(3r^8 + 2r^4 + 4|u|^2|v|^2\big)\log(2)^2. \tag{82}$$

6.2. **Results for the nilpotent cone of the most symmetric case.** Note that all computations in the following is conducted and simplified using Mathematica. The fully documented Mathematica notebook can be downloaded on the webpage

`http://www.lmpt.univ-tours.fr/~traizet/`

We restrict to the case $p = e^{i\pi/4}$ and assume $v = pu$, i.e., by (37) we are in the nilpotent cone. The computations of higher order derivatives then simplify to:

$$\mathcal{E}'' = 0$$

$$\mathcal{E}''' = 192\,\pi|u|^4\left(127|u|^4 + 20\right)\zeta(3) \tag{83}$$

where ζ is the Riemann ζ-function. If $\widetilde{\varpi}$ denotes the restriction of ϖ to the nilpotent cone, we obtain

$$\widetilde{\varpi}' = 256\,\pi i|u|^2\log(2)\,du\wedge d\overline{u}$$

$$\widetilde{\varpi}'' = 0$$

$$\widetilde{\varpi}''' = 3072\,\pi i\,\zeta(3)(127|u|^6 + 10|u|^2)\,du\wedge d\overline{u}. \tag{84}$$

6.3. **New Ω-identities.** Write

$$\varpi = \sum_{k=0}^{\infty}\varpi_k\lambda^k.$$

We can compute ϖ_k'' for $k = 0, 1, 2...$ from the derivatives of the parameters. On the other hand, we know from section 2.6 and Corollary 41 that $\varpi_k'' = 0$ for $k \geq 1$, and also

$$\varpi''_{0,u,v} = \varpi''_{0,\overline{u},\overline{v}} = 0.$$

Note that $\varpi''_{0,\overline{u},\overline{v}} = 0$ is trivial because of (75) and because $d\underline{x}_{2,-1}$ and $d\underline{x}_{3,-1}$ are holomorphic 1-forms so vanish on $(\frac{\partial}{\partial\overline{u}}, \frac{\partial}{\partial\overline{v}})$. On the other hand, $\varpi''_{0,u,v} = 0$ is not trivial and gives identities involving Ω-integrals. In this section, we use the notation

$$\Omega_{i_1,\cdots,i_n} = \Omega_{i_1,\cdots,i_n}(1) \quad \text{and} \quad \Theta_{i_1,\cdots,i_n} = \Omega_{i_1,\cdots,i_n}(i).$$

From

$$\varpi_0'' = \frac{32\pi}{\underline{x}_{1,-1}}\left(\frac{-x''_{1,0}}{\underline{x}_{1,-1}}d\underline{x}_{2,-1}\wedge d\underline{x}_{3,-1} + dx''_{2,0}\wedge d\underline{x}_{3,-1} + d\underline{x}_{2,-1}\wedge x''_{3,0}\right)$$

we obtain

$$\begin{aligned}\varpi''_{0,u,v} = \frac{32\rho}{\pi}\big[&6(u\overline{u}^2\overline{v} - \overline{u}v\overline{v}^2)(I_1 + \overline{I_1}) + 8(\overline{u}^3v - u\overline{v}^3)(I_2 + \overline{I_2})\\ &+\left(\frac{u\overline{u}^6}{\overline{v}^3} - \frac{v\overline{v}^6}{\overline{u}^3} + \frac{3\overline{u}^5v}{\overline{v}^2} - \frac{3u\overline{v}^5}{\overline{u}^2}\right)(I_3 + \overline{I_3})\big]\end{aligned} \tag{85}$$

with

$$\begin{cases} I_1 = 6\pi(\Omega_{333} - \Theta_{222}) + i((\Omega_{21})^2 + (\Theta_{31})^2) + 2\pi(\Omega_{223} - \Theta_{332}) - 8\pi(\Omega_{311} - \Theta_{211}) + 10i\Omega_{21}\Theta_{31}\\ I_2 = i((\Omega_{21})^2 - (\Theta_{31})^2) + 2\pi(\Omega_{223} + \Theta_{332} + \Omega_{311} + \Theta_{211}) - 4\pi(\Omega_{333} + \Theta_{222})\\ I_3 = -i((\Omega_{21})^2 + (\Theta_{31})^2) + 2\pi(\Omega_{333} - \Theta_{222} - \Omega_{223} + \Theta_{332}) - 2i\Omega_{21}\Theta_{31}.\end{cases}$$

Since $\varpi''_{0,u,v} = 0$, I_1, I_2 and I_3 are pure imaginary. Since they are holomorphic functions of $p \in \mathbb{C}^+$, they must be constant. We find the constants by evaluating at $p = e^{i\pi/4}$, where all integrals are known from [20, Appendix], and find

$$\begin{cases} I_1 = -i\frac{\pi^4}{3} \\ I_2 = 0 \\ I_3 = -i\pi^4. \end{cases}$$

Using the elementary values

$$\Omega_{333} = \frac{1}{6}(\Omega_3)^3 = \frac{-i\pi^3}{6} \quad \text{and} \quad \Theta_{222} = \frac{1}{6}(\Theta_2)^2 = \frac{i\pi^3}{6}$$

we obtain the following identities for all $p \in \mathbb{C}^+$:

$$\begin{cases} \Omega_{223} + \Omega_{311} = -\frac{i}{2\pi}(\Omega_{21})^2 \\ \Theta_{211} + \Theta_{332} = \frac{i}{2\pi}(\Theta_{31})^2 \\ \Omega_{223} - \Theta_{332} = -\frac{i}{2\pi}(\Omega_{21} + \Theta_{31})^2 + \frac{i\pi^3}{6}. \end{cases} \tag{86}$$

From higher order derivatives in t and higher order terms ϖ_k we expect a hierarchy of identities for Ω-values. Another source of identities is the character variety of the 4-punctured sphere. Analogously to Remark (44), taking the 4th-order derivative of (72) gives rise to the following three identities, which express linear combinations of Ω-integrals of depth 3 as a function of Ω-integrals of depth at most 2, and are non-trivial in the sense that they do not follow from shuffle product relations alone:

$$\begin{aligned} &\Omega_{212} - \Theta_{121}\Theta_{212} = \\ &\frac{1}{2}\Omega_1\Theta_{12} - \frac{1}{2}\Omega_1\Theta_{21} - \frac{1}{2}\Omega_2\Omega_{12} + \frac{1}{2}\Omega_2\Omega_{21} + \frac{1}{2}\Omega_1\Omega_2^2 - \frac{1}{4}\Theta_1\Theta_2^2 - \frac{i\Omega_{12}^2}{4\pi} \\ &+ \frac{3i\Omega_{21}^2}{4\pi} - \frac{i\Omega_{12}\Omega_{21}}{2\pi} + \frac{i\Omega_1^2\Omega_2^2}{4\pi} + \frac{i\pi\Omega_1^2}{4} + \frac{i\pi\Theta_1^2}{4} - \frac{\pi^2\Omega_1}{4} + \frac{i\pi^3}{6} \end{aligned} \tag{87}$$

$$\begin{aligned} &\Omega_{131} + \Omega_{313} - \Theta_{313} = \\ &\frac{1}{2}\Omega_1\Omega_{31} - \frac{1}{2}\Omega_1\Omega_{13} + \frac{1}{2}\Theta_3\Theta_{13} - \frac{1}{2}\Theta_3\Theta_{31} + \frac{1}{4}\Omega_1\Theta_3^2 + \frac{1}{4}\Omega_1\Omega_3^2 - \frac{1}{4}\Theta_1\Theta_3^2 - \frac{i\Theta_{13}^2}{4\pi} \\ &+ \frac{3i\Theta_{31}^2}{4\pi} - \frac{i\Theta_{13}\Theta_{31}}{2\pi} + \frac{i\Omega_1^2\Theta_3^2}{4\pi} + \frac{i\pi\Omega_{13}}{2} - \frac{i\pi\Omega_{31}}{2} + \frac{i\pi\Omega_1^2}{2} + \frac{3\pi^2\Omega_1}{4} - \frac{i\pi^3}{3} \end{aligned} \tag{88}$$

$$\begin{aligned} &\Omega_{232} - \Theta_{323} = \\ &\frac{1}{2}\Omega_2\Omega_{23} - \frac{1}{2}\Omega_2\Omega_{32} + \frac{1}{2}\Theta_3\Omega_{32} - \frac{1}{2}\Theta_3\Omega_{21} + \Theta_3\Omega_{23} + \frac{1}{2}\Theta_3\Theta_{23} - \frac{1}{2}\Theta_3\Theta_{31} \\ &- \frac{i\Omega_{21}\Theta_{23}}{2\pi} - \frac{i\Omega_{32}\Theta_{23}}{2\pi} + \frac{3i\Omega_{21}\Theta_{31}}{2\pi} - \frac{i\Omega_{32}\Theta_{31}}{2\pi} - \frac{i\Omega_{21}\Omega_{32}}{2\pi} - \frac{i\Theta_{23}\Theta_{31}}{2\pi} \\ &+ \frac{3i\Omega_{21}^2}{4\pi} - \frac{i\Omega_{32}^2}{4\pi} - \frac{i\Theta_{23}^2}{4\pi} + \frac{3i\Theta_{31}^2}{4\pi} + \frac{i\pi\Omega_2^2}{4} + \frac{i\pi\Theta_3^2}{4} - i\pi\Omega_{21} - i\pi\Omega_{32} \\ &- i\pi\Theta_{23} - i\pi\Theta_{31} - \pi^2\Omega_2 + \pi^2\Theta_3 - \frac{i\pi^3}{3} \end{aligned} \tag{89}$$

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