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# GLUING DELAUNAY ENDS TO MINIMAL $n$ -NOIDS USING THE DPW METHOD

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*Abstract: we construct constant mean curvature surfaces in euclidean space by gluing  $n$  half Delaunay surfaces to a non-degenerate minimal  $n$ -noid, using the DPW method.*

## 1. INTRODUCTION

In [3], Dorfmeister, Pedit and Wu have shown that surfaces with non-zero constant mean curvature (CMC for short) in euclidean space admit a Weierstrass-type representation, which means that they can be represented in terms of holomorphic data. This representation is now called the DPW method. In [18], we used the DPW method to construct CMC  $n$ -noids: genus zero, CMC surfaces with  $n$  Delaunay type ends. These  $n$ -noids can be described as a unit sphere with  $n$  half Delaunay surfaces with small necksizes attached at prescribed points. They had already been constructed by Kapouleas in [11] using PDE methods.

In the case  $n = 3$ , Alexandrov-embedded CMC trinoids have been classified by Große Brauckman, Kusner and Sullivan in [9]. In particular, equilateral CMC trinoids form a 1-parameter family, parametrized on an open interval. On one end, equilateral trinoids degenerate like the examples described above: they look like a sphere with 3 half Delaunay surfaces with small necksizes attached at the vertices of a spherical equilateral triangle. On the other end, equilateral trinoids limit, after suitable blow-up, to a minimal 3-noid: a genus zero minimal surface with 3 catenoidal ends (see Figure 1).

It seems natural to ask if one can generalize this observation and construct CMC  $n$ -noids by gluing half Delaunay surfaces with small necksizes to a minimal  $n$ -noid. This is indeed the case, and has been done by Mazzeo and Pacard in [14] using PDE methods. In this paper, we propose a quite simple and natural DPW potential to construct these examples. We prove:

**Theorem 1.** *Let  $n \geq 3$  and let  $M_0$  be a non-degenerate minimal  $n$ -noid. There exists a smooth family of CMC surfaces  $(M_t)_{0 < |t| < \epsilon}$  with the following properties:*

- (1)  $M_t$  has genus zero and  $n$  Delaunay ends.
- (2)  $\frac{1}{t}M_t$  converges to  $M_0$  as  $t \rightarrow 0$ .
- (3) If  $M_0$  is Alexandrov-embedded, all ends of  $M_t$  are of unduloid type if  $t > 0$  and of nodoid type if  $t < 0$ . Moreover,  $M_t$  is Alexandrov-embedded if  $t > 0$ .

Non-degeneracy of a minimal  $n$ -noid will be defined in Section 2. The two surfaces  $M_t$  and  $M_{-t}$  are geometrically different: if  $M_t$  has an end of unduloid type then the corresponding end of  $M_{-t}$  is of nodoid type. See Proposition 6 for more details. Of course, a minimal  $n$ -noid is never embedded if  $n \geq 3$  so the surfaces  $M_t$  are not embedded. Alexandrov-embedded

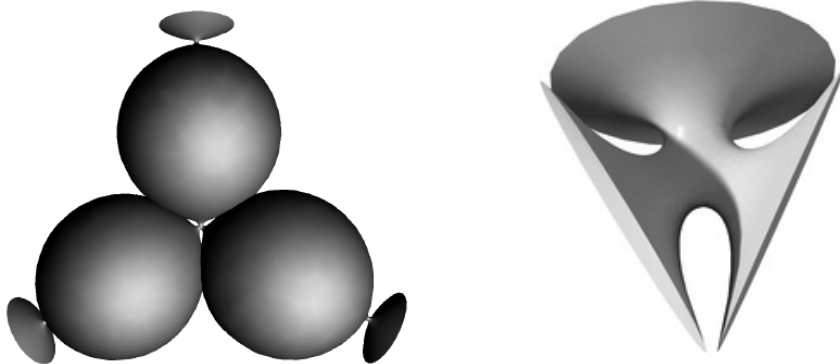


FIGURE 1. A CMC 3-noid (left, image by N. Schmitt [16]) and a minimal 3-noid (right). There is a tiny “copy” of the minimal 3-noid at the center of the CMC 3-noid.

minimal  $n$ -noids whose ends have coplanar axes have been classified by Cosin and Ros in [2], and Alexandrov-embedded CMC  $n$ -noids whose ends have coplanar axes have been classified by Große-Brauckmann, Kusner and Sullivan in [10].

As already said, these surfaces have already been constructed in [14]. Our motivation to construct them with the DPW method is to answer the following questions:

- (1) How can we produce a DPW potential from the Weierstrass data  $(g, \omega)$  of the minimal  $n$ -noid  $M_0$  ?
- (2) How can we prove, with the DPW method, that  $\frac{1}{t}M_t$  converges to  $M_0$  ?

The answer to Question 2 is Theorem 4 in Section 4, a general blow-up result in the context of the DPW method. In [19], we use the DPW method to construct higher genus CMC surfaces with small necks. Theorem 4 is used to ensure that the necks have asymptotically catenoidal shape.

**Remark 1.** *As the referee pointed out, the relation between minimal surfaces and CMC-1 surfaces in the DPW framework has already been investigated by Brander and Dorfmeister [1]. In that paper, the authors propose a DPW potential from the Weierstrass data of a minimal surface  $M$ . The Monodromy Problem is not addressed, however, so the resulting CMC-1 surfaces do not close, unless  $M$  is simply connected.*

I would like to thank the referee for his helpful comments and for providing the references [7] and [8].

## 2. NON-DEGENERATE MINIMAL $n$ -NOIDS

A minimal  $n$ -noid is a complete, immersed minimal surface in  $\mathbb{R}^3$  with genus zero and  $n$  catenoidal ends. Let  $M_0$  be a minimal  $n$ -noid and  $(\Sigma, g, \omega)$  its Weierstrass data. This means that  $M_0$  is parametrized on  $\Sigma$  by the Weierstrass Representation formula:

$$(1) \quad \psi(z) = \operatorname{Re} \int_{z_0}^z \left( \frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega \right)$$

Without loss of generality, we can assume that  $\Sigma = \mathbb{C} \cup \{\infty\} \setminus \{p_1, \dots, p_n\}$ , where  $p_1, \dots, p_n$  are complex numbers and  $g \neq 0, \infty$  at  $p_1, \dots, p_n$  (by rotating  $M_0$  if necessary). Then  $\omega$  needs a double pole at  $p_1, \dots, p_n$  so has  $2n - 2$  zeros, counting multiplicity. Since  $\omega$  needs a zero at each pole of  $g$ , with twice the multiplicity, it follows that  $g$  has  $n - 1$  poles so has degree  $n - 1$ . Hence we may write

$$(2) \quad g = \frac{A(z)}{B(z)} \quad \text{and} \quad \omega = \frac{B(z)^2 dz}{\prod_{i=1}^n (z - p_i)^2}$$

where

$$A(z) = \sum_{i=1}^n a_i z^{n-i} \quad \text{and} \quad B(z) = \sum_{i=1}^n b_i z^{n-i}.$$

We are going to deform this Weierstrass data, so we see  $a_i, b_i$  and  $p_i$  for  $1 \leq i \leq n$  as complex parameters. We denote by  $\mathbf{x} \in \mathbb{C}^{3n}$  the vector of these parameters, and by  $\mathbf{x}_0$  the value of the parameters corresponding to the minimal  $n$ -noid  $M_0$ .

Let  $\gamma_i$  be the homology class of a small circle centered at  $p_i$  and define the following periods for  $1 \leq i \leq n$  and  $0 \leq k \leq 2$ , depending on the parameter vector  $\mathbf{x} \in \mathbb{C}^{3n}$ :

$$P_{i,k}(\mathbf{x}) = \int_{\gamma_i} g^k \omega$$

$$P_i(\mathbf{x}) = (P_{i,0}(\mathbf{x}), P_{i,1}(\mathbf{x}), P_{i,2}(\mathbf{x})) \in \mathbb{C}^3$$

$$Q_i(\mathbf{x}) = \int_{\gamma_i} \left( \frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega \right) \in \mathbb{C}^3.$$

Then

$$Q_i(\mathbf{x}) = \left( \frac{1}{2}(P_{i,0}(\mathbf{x}) - P_{i,2}(\mathbf{x})), \frac{i}{2}(P_{i,0}(\mathbf{x}) + P_{i,2}(\mathbf{x})), P_{i,1}(\mathbf{x}) \right).$$

The components of  $Q_i(\mathbf{x}_0)$  are imaginary because the Period Problem is solved for  $M_0$ . This gives

$$(3) \quad P_{i,2}(\mathbf{x}_0) = \overline{P_{i,0}(\mathbf{x}_0)} \quad \text{and} \quad P_{i,1}(\mathbf{x}_0) \in i\mathbb{R}$$

Moreover,  $\text{Im}(Q_i(\mathbf{x}_0)) = -\phi_i$  where  $\phi_i$  is the flux vector of  $M_0$  at the end  $p_i$ . By the Residue Theorem, we have for all  $\mathbf{x}$  in a neighborhood of  $\mathbf{x}_0$ :

$$\sum_{i=1}^n P_i(\mathbf{x}) = 0$$

Let  $P = (P_1, \dots, P_{n-1})$  and  $Q = (Q_1, \dots, Q_{n-1})$ .

**Definition 1.**  $M_0$  is non-degenerate if the differential of  $P$  (or equivalently,  $Q$ ) at  $\mathbf{x}_0$  has complex rank  $3n - 3$ .

**Remark 2.** If  $n \geq 3$ , we may (using Möbius transformations of the sphere) fix the value of three points, say  $p_1, p_2, p_3$ . Then "non-degenerate" means that the differential of  $P$  with respect to the remaining parameters is an isomorphism of  $\mathbb{C}^{3n-3}$ .

This notion is related to another standard notion of non-degeneracy:

**Definition 2.**  $M_0$  is non-degenerate if its space of bounded Jacobi fields has (real) dimension 3.

**Theorem 2.** *If  $M_0$  is non-degenerate in the sense of Definition 2, then  $M_0$  is non-degenerate in the sense of Definition 1.*

Proof. Assume  $M_0$  is non-degenerate in the sense of Definition 2. Then in a neighborhood of  $M_0$ , the space  $\mathcal{M}$  of minimal  $n$ -noids (up to translation) is a smooth manifold of dimension  $3n - 3$  by a standard application of the Implicit Function Theorem. Moreover, if we write  $\phi_i \in \mathbb{R}^3$  for the flux vector at the  $i$ -th end, then the map  $\phi = (\phi_1, \dots, \phi_n)$  provides a local diffeomorphism between  $\mathcal{M}$  and the space  $V$  of vectors  $v = (v_1, \dots, v_n) \in (\mathbb{R}^3)^n$  such that  $\sum_{i=1}^n v_i = 0$ . (All this is proved in Section 4 of [2] in the case where all ends are coplanar. The argument goes through in the general case.) Hence given a vector  $v \in V$ , there exists a deformation  $M_t$  of  $M_0$  such that  $M_t \in \mathcal{M}$  and  $\frac{d}{dt}\phi(M_t)|_{t=0} = v$ . We may write the Weierstrass data of  $M_t$  as above and obtain a set of parameters  $\mathbf{x}(t)$ , depending smoothly on  $t$ , such that  $\mathbf{x}(0) = \mathbf{x}_0$ . Then  $dQ(\mathbf{x}_0) \cdot \mathbf{x}'(0) = -iv$ . Since  $Q$  is holomorphic, its differential is complex-linear so  $dQ(\mathbf{x}_0)$  has complex rank equal to  $\dim V = 3n - 3$ .  $\square$

If all ends of  $M_0$  have coplanar axes, then  $M_0$  is non-degenerate in the sense of Definition 2 by Proposition 2 in [2]. In particular, the (most symmetric)  $n$ -noids of Jorge-Meeks are non-degenerate. This implies that generic  $n$ -noids in the component of the Jorge-Meeks  $n$ -noid are non-degenerate.

### 3. BACKGROUND

In this section, we recall standard notations and results used in the DPW method. We work in the “untwisted” setting.

**3.1. Loop groups.** A loop is a smooth map from the unit circle  $\mathbb{S}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$  to a matrix group. The circle variable is denoted  $\lambda$  and called the spectral parameter. The unit disk is denoted  $\mathbb{D}$ . For  $\rho > 1$ , we denote  $\mathbb{D}_\rho$  the disk  $|\lambda| < \rho$ ,  $\mathbb{D}_\rho^* = \mathbb{D}_\rho \setminus \{0\}$  and  $\mathbb{A}_\rho$  the annulus  $\frac{1}{\rho} < |\lambda| < \rho$ .

- If  $G$  is a matrix Lie group (or Lie algebra),  $\Lambda G$  denotes the group (or algebra) of smooth maps  $\Phi : \mathbb{S}^1 \rightarrow G$ .
- $\Lambda_+ SL(2, \mathbb{C}) \subset \Lambda SL(2, \mathbb{C})$  is the subgroup of maps  $B$  which extend holomorphically to  $\mathbb{D}$  with  $B(0)$  upper triangular.
- $\Lambda_+^{\mathbb{R}} SL(2, \mathbb{C}) \subset \Lambda_+ SL(2, \mathbb{C})$  is the subgroup of maps  $B$  such that  $B(0)$  has positive entries on the diagonal.

**Theorem 3** (Iwasawa decomposition). *The multiplication  $\Lambda SU(2) \times \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C}) \rightarrow \Lambda SL(2, \mathbb{C})$  is a diffeomorphism. The unique splitting of an element  $\Phi \in \Lambda SL(2, \mathbb{C})$  as  $\Phi = FB$  with  $F \in \Lambda SU(2)$  and  $B \in \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})$  is called Iwasawa decomposition.  $F$  is called the unitary factor of  $\Phi$  and denoted  $\text{Uni}(\Phi)$ .  $B$  is called the positive factor and denoted  $\text{Pos}(\Phi)$ .*

**3.2. The matrix model of  $\mathbb{R}^3$ .** In the DPW method, one identifies  $\mathbb{R}^3$  with the Lie algebra  $\mathfrak{su}(2)$  by

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3 \longleftrightarrow X = -i \begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \in \mathfrak{su}(2).$$

We have  $\det(X) = \|x\|^2$ . The group  $SU(2)$  acts as linear isometries on  $\mathfrak{su}(2)$  by conjugation:  $H \cdot X = HXH^{-1}$ .

**3.3. The DPW method.** The input data for the DPW method is a quadruple  $(\Sigma, \xi, z_0, \phi_0)$  where:

- $\Sigma$  is a Riemann surface.
- $\xi = \xi(z, \lambda)$  is a  $\Lambda\mathfrak{sl}(2, \mathbb{C})$ -valued holomorphic 1-form on  $\Sigma$  called the DPW potential. More precisely,

$$(4) \quad \xi = \begin{pmatrix} \alpha & \lambda^{-1}\beta \\ \gamma & -\alpha \end{pmatrix}$$

where  $\alpha(z, \lambda)$ ,  $\beta(z, \lambda)$ ,  $\gamma(z, \lambda)$  are holomorphic 1-forms on  $\Sigma$  with respect to the  $z$  variable, and are holomorphic with respect to  $\lambda$  in the disk  $\mathbb{D}_\rho$  for some  $\rho > 1$ .

- $z_0 \in \Sigma$  is a base point.
- $\phi_0 \in \Lambda SL(2, \mathbb{C})$  is an initial condition.

Given this data, the DPW method is the following procedure.

- Let  $\tilde{\Sigma}$  be the universal cover of  $\Sigma$  and  $\tilde{z}_0 \in \tilde{\Sigma}$  be an arbitrary element in the fiber of  $z_0$ . Solve the Cauchy Problem on  $\tilde{\Sigma}$ :

$$(5) \quad \begin{cases} d\Phi(z, \lambda) = \Phi(z, \lambda)\xi(z, \lambda) \\ \Phi(\tilde{z}_0, \lambda) = \phi_0(\lambda) \end{cases}$$

to obtain a solution  $\Phi : \tilde{\Sigma} \rightarrow \Lambda SL(2, \mathbb{C})$ .

- Compute the Iwasawa decomposition  $(F(z, \cdot), B(z, \cdot))$  of  $\Phi(z, \cdot)$ .
- Define  $f : \tilde{\Sigma} \rightarrow \mathfrak{su}(2) \sim \mathbb{R}^3$  by the Sym-Bobenko formula:

$$(6) \quad f(z) = -2i \frac{\partial F}{\partial \lambda}(z, 1) F(z, 1)^{-1} =: \text{Sym}(F(z, \cdot)).$$

Then  $f$  is a CMC-1 (branched) conformal immersion.  $f$  is regular at  $z$  (meaning unbranched) if and only if  $\beta(z, 0) \neq 0$ . Its Gauss map is given by

$$(7) \quad N(z) = -iF(z, 1) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} F(z, 1)^{-1} =: \text{Nor}(F(z, \cdot)).$$

The DPW method actually constructs a moving frame for  $f$ . The differential of  $f$  is given by

$$(8) \quad df(z) = 2i B_{11}(z, 0)^2 F(z, 1) \begin{pmatrix} 0 & \beta(z, 0) \\ \bar{\beta}(z, 0) & 0 \end{pmatrix} F(z, 1)^{-1}.$$

Equation (8) can also be obtained by differentiation of the Sym-Bobenko formula (6).

**Remark 3.** In [18], I have opposite signs in Equations (6) and (7). This is unfortunate because it makes the basis  $(f_x, f_y, N)$  negatively oriented. Equation (6) is the right formula, which one obtains by untwisting the standard Sym-Bobenko formula in the twisted case. See [13] or [17].

**3.4. The Monodromy Problem.** Assume that  $\Sigma$  is not simply connected so its universal cover  $\tilde{\Sigma}$  is not trivial. Let  $\text{Deck}(\tilde{\Sigma}/\Sigma)$  be the group of fiber-preserving diffeomorphisms of  $\tilde{\Sigma}$ . Let  $\Phi$  be the solution of the Cauchy Problem (5). For  $\gamma \in \text{Deck}(\tilde{\Sigma}/\Sigma)$ , let

$$\mathcal{M}_\gamma(\Phi)(\lambda) = \Phi(\gamma(z), \lambda)\Phi(z, \lambda)^{-1}$$

be the monodromy of  $\Phi$  with respect to  $\gamma$  (which is independent of  $z \in \tilde{\Sigma}$ ). The standard condition which ensures that the immersion  $f$  descends to a well defined immersion on  $\Sigma$  is the following system of equations, called the Monodromy Problem.

$$(9) \quad \forall \gamma \in \text{Deck}(\tilde{\Sigma}/\Sigma) \quad \begin{cases} \mathcal{M}_\gamma(\Phi) \in \Lambda SU(2) & (i) \\ \mathcal{M}_\gamma(\Phi)(1) = \pm I_2 & (ii) \\ \frac{\partial \mathcal{M}_\gamma(\Phi)}{\partial \lambda}(1) = 0 & (iii) \end{cases}$$

One can identify  $\text{Deck}(\tilde{\Sigma}/\Sigma)$  with the fundamental group  $\pi_1(\Sigma, z_0)$  (see for example Theorem 5.6 in [5]), so we will in general see  $\gamma$  as an element of  $\pi_1(\Sigma, z_0)$ . Under this identification, the monodromy of  $\Phi$  with respect to  $\gamma \in \pi_1(\Sigma, z_0)$  is given by

$$\mathcal{M}_\gamma(\Phi)(\lambda) = \Phi(\tilde{\gamma}(1), \lambda) \Phi(\tilde{\gamma}(0), \lambda)^{-1}$$

where  $\tilde{\gamma} : [0, 1] \rightarrow \tilde{\Sigma}$  is the lift of  $\gamma$  such that  $\tilde{\gamma}(0) = \tilde{z}_0$ .

### 3.5. Gauging.

**Definition 3.** *A gauge on  $\Sigma$  is a map  $G : \Sigma \rightarrow \Lambda_+ SL(2, \mathbb{C})$  such that  $G(z, \lambda)$  depends holomorphically on  $z \in \Sigma$  and  $\lambda \in \mathbb{D}_\rho$  for some  $\rho > 1$ .*

Let  $\Phi$  be a solution of  $d\Phi = \Phi\xi$  and  $G$  be a gauge. Let  $\hat{\Phi} = \Phi \times G$ . Then  $\hat{\Phi}$  and  $\Phi$  define the same immersion  $f$ . This is called ‘‘gauging’’. The gauged potential is

$$\hat{\xi} = \hat{\Phi}^{-1} d\hat{\Phi} = G^{-1} \xi G + G^{-1} dG$$

and will be denoted  $\xi \cdot G$ , the dot denoting the action of the gauge group on the potential.

**3.6. Functional spaces.** We need to introduce a functional space for functions on the unit circle. We need that space to be a Banach algebra, and functions in that space should extend holomorphically to a neighborhood of the unit circle. The following choice is natural. We decompose a smooth function  $f : \mathbb{S}^1 \rightarrow \mathbb{C}$  in Fourier series

$$f(\lambda) = \sum_{i \in \mathbb{Z}} f_i \lambda^i$$

Fix some  $\rho > 1$  and define

$$\|f\| = \sum_{i \in \mathbb{Z}} |f_i| \rho^{|i|}$$

Let  $\mathcal{W}_\rho$  be the space of functions  $f$  with finite norm. This is a Banach algebra, owing to the fact that the weight  $\rho^{|i|}$  is submultiplicative (see Section 4 in [8]). Functions in  $\mathcal{W}_\rho$  extend holomorphically to the annulus  $\mathbb{A}_\rho$ .

We define  $\mathcal{W}_\rho^{>0}$ ,  $\mathcal{W}_\rho^{>0}$ ,  $\mathcal{W}_\rho^{<0}$  and  $\mathcal{W}_\rho^{<0}$  as the subspaces of functions  $f$  such that  $f_i = 0$  for  $i < 0$ ,  $i \leq 0$ ,  $i > 0$  and  $i \geq 0$ , respectively. Functions in  $\mathcal{W}_\rho^{>0}$  extend holomorphically to the disk  $\mathbb{D}_\rho$  and satisfy  $|f(\lambda)| \leq \|f\|$  for all  $\lambda \in \mathbb{D}_\rho$ . We write  $\mathcal{W}^0 \sim \mathbb{C}$  for the subspace of constant functions, so we have a direct sum  $\mathcal{W}_\rho = \mathcal{W}_\rho^{<0} \oplus \mathcal{W}^0 \oplus \mathcal{W}_\rho^{>0}$ . (The Banach algebra  $\mathcal{W}_\rho$  is said to be decomposable, see [7] page 70.) A function  $f$  will be decomposed as  $f = f^- + f^0 + f^+$  with  $(f^-, f^0, f^+) \in \mathcal{W}_\rho^{<0} \times \mathcal{W}^0 \times \mathcal{W}_\rho^{>0}$ .

We define the star operator by

$$f^*(\lambda) = \overline{f(1/\bar{\lambda})} = \sum_{i \in \mathbb{Z}} \overline{f_{-i}} \lambda^i$$

The involution  $f \mapsto f^*$  exchanges  $\mathcal{W}_\rho^{\geq 0}$  and  $\mathcal{W}_\rho^{\leq 0}$ . We have  $\lambda^* = \lambda^{-1}$  and  $c^* = \bar{c}$  if  $c$  is a constant. A function  $f$  is real on the unit circle if and only if  $f = f^*$ . We extend the star involution to loops by  $M^*(\lambda) = \overline{M(1/\bar{\lambda})}^T$ , so a loop  $F$  is unitary if and only if  $F^*F = I_2$ .

If  $L$  is a loop group, we denote  $L_\rho \subset L$  the subgroup of loops whose entries are in  $\mathcal{W}_\rho$ . The loop groups  $\Lambda SL(2, \mathbb{C})_\rho$ ,  $\Lambda SU(2)_\rho$  and  $\Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$  are Banach Lie groups, moreover:

**Proposition 1.** *Iwasawa decomposition restricts to an analytic diffeomorphism between the Banach Lie groups  $\Lambda SL(2, \mathbb{C})_\rho$  and  $\Lambda SU(2)_\rho \times \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$ .*

Proof: let  $\Phi \in \Lambda SL(2, \mathbb{C})_\rho$  and let  $(F, B)$  be its Iwasawa decomposition. We want to prove that  $F$  and  $B$  have entries in  $\mathcal{W}_\rho$ . Since  $F$  is unitary, we have  $\Phi^*\Phi = B^*B$ . Now  $B^*$  has only non-positive powers of  $\lambda$ , so  $(B^*, B)$  is a Birkhoff-type decomposition of  $M = \Phi^*\Phi$ . Since  $M \in \Lambda SL(2, \mathbb{C})_\rho$ , it is known that both factors of its Birkhoff decomposition have entries in  $\mathcal{W}_\rho$ , owing to the fact that  $\mathcal{W}_\rho$  is decomposable (see Theorem 1.4 in [7]). So  $B \in \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$  and  $F \in \Lambda SU(2)_\rho$  follows.

Let  $\Lambda \mathfrak{sl}(2, \mathbb{C})_\rho$ ,  $\Lambda \mathfrak{su}(2)_\rho$  and  $\Lambda_+^{\mathbb{R}} \mathfrak{sl}(2, \mathbb{C})_\rho$  be the Banach Lie algebras of respectively  $\Lambda SL(2, \mathbb{C})_\rho$ ,  $\Lambda SU(2)_\rho$  and  $\Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$ . The following decomposition is standard (the factors can be written explicitly in term of Fourier coefficients):

$$\Lambda \mathfrak{sl}(2, \mathbb{C})_\rho = \Lambda \mathfrak{su}(2)_\rho \oplus \Lambda_+^{\mathbb{R}} \mathfrak{sl}(2, \mathbb{C})_\rho.$$

By the inverse mapping theorem, the multiplication  $\Lambda SU(2)_\rho \times \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho \rightarrow \Lambda SL(2, \mathbb{C})_\rho$  is an analytic local diffeomorphism in a neighborhood of  $(I_2, I_2)$ , and in a neighborhood of any element  $(F, B)$  using left multiplication by  $F$  and right multiplication by  $B$ . Since we already know the multiplication is bijective, it is a global diffeomorphism.  $\square$

#### 4. A BLOW-UP RESULT

In this section, we consider a one-parameter family of DPW potential  $\xi_t$  with solution  $\Phi_t$  and assume that  $\Phi_0(z, \lambda)$  is independent of  $\lambda$ . Then its unitary part  $F_0(z, \lambda)$  is independent of  $\lambda$ . The Sym Bobenko formula yields that  $f_0 \equiv 0$ , so the family  $f_t$  collapses to the origin as  $t = 0$ . The following theorem says that the blow-up  $\frac{1}{t}f_t$  converges to a minimal surface whose Weierstrass data is explicitly computed.

**Theorem 4.** *Let  $\Sigma$  be a Riemann surface,  $(\xi_t)_{t \in I}$  a family of DPW potentials on  $\Sigma$  and  $(\Phi_t)_{t \in I}$  a family of solutions of  $d\Phi_t = \Phi_t \xi_t$  on the universal cover  $\tilde{\Sigma}$  of  $\Sigma$ , where  $I \subset \mathbb{R}$  is a neighborhood of 0. Fix a base point  $z_0 \in \tilde{\Sigma}$ . Assume that*

- (1)  $(t, z) \mapsto \xi_t(z, \cdot)$  and  $t \mapsto \Phi_t(z_0, \cdot)$  are  $C^1$  maps into  $\Lambda \mathfrak{sl}(2, \mathbb{C})_\rho$  and  $\Lambda SL(2, \mathbb{C})_\rho$ , respectively.
- (2) For all  $t \in I$ ,  $\Phi_t$  solves the Monodromy Problem (9).
- (3)  $\Phi_0(z, \lambda)$  is independent of  $\lambda$ :

$$\Phi_0(z, \lambda) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$



Let  $f_t = \text{Sym}(\text{Uni}(\Phi_t)) : \Sigma \rightarrow \mathbb{R}^3$  be the CMC-1 immersion given by the DPW method. Then

$$\lim_{t \rightarrow 0} \frac{1}{t} f_t(z) = \psi(z)$$

where  $\psi : \Sigma \rightarrow \mathbb{R}^3$  is a (possibly branched) minimal immersion with the following Weierstrass data:

$$g(z) = \frac{-a(z)}{c(z)} \quad \text{and} \quad \omega = 4c(z)^2 \frac{\partial \xi_{t;12}^{(-1)}}{\partial t} \Big|_{t=0}.$$

The limit is for the uniform  $C^1$  convergence on compact subsets of  $\Sigma$ .

Here  $\xi_{t;12}^{(-1)}$  denotes the coefficient of  $\lambda^{-1}$  in the upper right entry of  $\xi_t$ . In case  $\omega = 0$ , the minimal immersion degenerates into a point and  $\psi$  is constant.

Proof: by standard ODE theory,  $(t, z) \mapsto \Phi_t(z, \cdot)$  is a  $C^1$  map into  $\Lambda SL(2, \mathbb{C})_\rho$ . Let  $(F_t, B_t)$  be the Iwasawa decomposition of  $\Phi_t$ . By Proposition 1,  $(t, z) \mapsto F_t(z, \cdot)$  and  $(t, z) \mapsto B_t(z, \cdot)$  are real analytic maps into  $\Lambda SU(2)_\rho$  and  $\Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$ , respectively. At  $t = 0$ ,  $\Phi_0$  is constant with respect to  $\lambda$ , so its Iwasawa decomposition is the standard  $QR$  decomposition:

$$F_0 = \frac{1}{\sqrt{|a|^2 + |c|^2}} \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \quad B_0 = \frac{1}{\sqrt{|a|^2 + |c|^2}} \begin{pmatrix} |a|^2 + |c|^2 & \bar{a}b + \bar{c}d \\ 0 & 1 \end{pmatrix}.$$

The Sym-Bobenko formula (6) yields  $f_0 = 0$ . Let  $\mu_t = B_{t;11}^0$  and  $\beta_t = \xi_{t;12}^{(-1)}$ . By Equation (8), we have

$$df_t(z) = 2i \mu_t(z)^2 F_t(z, 1) \begin{pmatrix} 0 & \beta_t(z) \\ \bar{\beta}_t(z) & 0 \end{pmatrix} F_t(z, 1)^{-1}.$$

Hence  $(t, z) \mapsto df_t(z)$  is a  $C^1$  map. At  $t = 0$ ,  $\xi_0$  is constant with respect to  $\lambda$ , so  $\beta_0 = 0$ . Define  $\tilde{f}_t(z) = \frac{1}{t} f_t(z)$  for  $t \neq 0$ . Then  $d\tilde{f}_t(z)$  extends at  $t = 0$ , as a continuous function of  $(t, z)$ , by

$$\begin{aligned} d\tilde{f}_0 &= \frac{d}{dt} df_t \Big|_{t=0} = 2i \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \begin{pmatrix} 0 & \beta' \\ \bar{\beta}' & 0 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ -c & a \end{pmatrix} \\ &= 2i \begin{pmatrix} -ac\beta' - \bar{a}c\bar{\beta}' & a^2\beta' - \bar{c}^2\bar{\beta}' \\ \bar{a}^2\bar{\beta}' - c^2\beta' & ac\beta' + \bar{a}c\bar{\beta}' \end{pmatrix} \end{aligned}$$

where  $\beta' = \frac{d}{dt} \beta_t \Big|_{t=0}$ . In euclidean coordinates, this gives

$$d\tilde{f}_0 = 4 \text{Re} \left[ \frac{1}{2}(c^2 - a^2)\beta', \frac{i}{2}(c^2 + a^2)\beta', -ac\beta' \right].$$

Writing  $g = \frac{-a}{c}$  and  $\omega = 4c^2\beta'$ , we obtain

$$\tilde{f}_0(z) = \tilde{f}_0(z_0) + \text{Re} \int_{z_0}^z \left[ \frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega \right]$$

and we see that  $\tilde{f}_0$  is a minimal surface with Weierstrass data  $(g, \omega)$ . The last statement of Theorem 4 comes from the fact that  $d\tilde{f}_t$  converges uniformly to  $d\tilde{f}_0$  on compact subsets of  $\Sigma$ .  $\square$

4.1. **Example.** As an example, we consider the family of Delaunay surfaces given by the following DPW potential in  $\mathbb{C}^*$ :

$$\xi_t(z, \lambda) = \begin{pmatrix} 0 & \lambda^{-1}r + s \\ \lambda r + s & 0 \end{pmatrix} \frac{dz}{z} \quad \text{with} \quad \begin{cases} r + s = \frac{1}{2} \\ rs = t \\ r < s \end{cases}$$

with initial condition  $\Phi_t(1) = I_2$ . As  $t \rightarrow 0$ , we have  $(r, s) \rightarrow (0, \frac{1}{2})$ . We have

$$\begin{aligned} \Phi_0(z, \lambda) &= \exp \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \log z = \frac{1}{2\sqrt{z}} \begin{pmatrix} z+1 & z-1 \\ z-1 & z+1 \end{pmatrix} \\ \frac{\partial \xi_t}{\partial t} \Big|_{t=0} &= \begin{pmatrix} 0 & 2\lambda^{-1} \\ 2\lambda & 0 \end{pmatrix} \frac{dz}{z}. \end{aligned}$$

Theorem 4 applies and gives

$$g(z) = \frac{1+z}{1-z} \quad \text{and} \quad \omega(z) = 4 \left( \frac{z-1}{2\sqrt{z}} \right)^2 \frac{2dz}{z} = 2 \left( \frac{z-1}{z} \right)^2 dz.$$

This is the Weierstrass data of a horizontal catenoid of waist-radius 4 and axis  $Ox_1$ , with  $x_1 \rightarrow +\infty$  at the end  $z = 0$ .

## 5. THE DPW POTENTIAL

We now start the proof of Theorem 1. Let  $(g, \omega)$  be the Weierstrass data of the given minimal  $n$ -noid  $M_0$ , written as in Section 2. We introduce  $3n$   $\lambda$ -dependent parameters  $a_i, b_i$  and  $p_i$  for  $1 \leq i \leq n$  in the functional space  $\mathcal{W}_\rho^{\geq 0}$ . The vector of these parameters is denoted  $\mathbf{x} \in (\mathcal{W}_\rho^{\geq 0})^{3n}$ . The parameter  $\mathbf{x}$  is in a neighborhood of a (constant) central value  $\mathbf{x}_0 \in (\mathcal{W}^0)^{3n}$  which correspond to the Weierstrass data of  $M_0$ , written as in Section 2. We define

$$\begin{aligned} A_{\mathbf{x}}(z, \lambda) &= \sum_{i=1}^n a_i(\lambda) z^{n-i} \\ B_{\mathbf{x}}(z, \lambda) &= \sum_{i=1}^n b_i(\lambda) z^{n-i} \\ (10) \quad g_{\mathbf{x}}(z, \lambda) &= \frac{A_{\mathbf{x}}(z, \lambda)}{B_{\mathbf{x}}(z, \lambda)} \end{aligned}$$

$$(11) \quad \omega_{\mathbf{x}}(z, \lambda) = \frac{B_{\mathbf{x}}(z, \lambda)^2 dz}{\prod_{i=1}^n (z - p_i(\lambda))^2}.$$

For  $t$  in a neighborhood of 0 in  $\mathbb{R}$ , we consider the following DPW potential:

$$\xi_{t, \mathbf{x}}(z, \lambda) = \begin{pmatrix} 0 & \frac{1}{4}t(\lambda-1)^2\lambda^{-1}\omega_{\mathbf{x}}(z, \lambda) \\ dg_{\mathbf{x}}(z, \lambda) & 0 \end{pmatrix}.$$

We fix a base point  $z_0$ , away from the poles of  $g$  and  $\omega$ , and we take the initial condition

$$\phi_0(\lambda) = \begin{pmatrix} g_{\mathbf{x}}(z_0, \lambda) & 1 \\ -1 & 0 \end{pmatrix}.$$

These choices are motivated by the following observations:

(1) At  $t = 0$ , we have

$$\xi_{0,\mathbf{x}}(z, \lambda) = \begin{pmatrix} 0 & 0 \\ dg_{\mathbf{x}}(z, \lambda) & 0 \end{pmatrix}.$$

The solution of the Cauchy Problem (5) is given by

$$(12) \quad \Phi_{0,\mathbf{x}}(z, \lambda) = \begin{pmatrix} g_{\mathbf{x}}(z, \lambda) & 1 \\ -1 & 0 \end{pmatrix}$$

which is well-defined, so the Monodromy Problem (9) is solved at  $t = 0$ .

(2) The same conclusion holds if  $\lambda = 1$  instead of  $t = 0$ . In particular, Items (ii) and (iii) of the Monodromy Problem (9) are automatically solved.

(3) At  $\mathbf{x} = \mathbf{x}_0$ , we have  $g_{\mathbf{x}_0} = g$  so  $\Phi_{0,\mathbf{x}_0}(z, \lambda)$  is independent of  $\lambda$ . Moreover,

$$\frac{\partial \xi_{t,\mathbf{x}_0;12}^{(-1)}}{\partial t} \Big|_{t=0} = \frac{\omega}{4}.$$

Provided the Monodromy Problem is solved for all  $t$  in a neighborhood of 0, Theorem 4 applies and the limit minimal surface has Weierstrass data  $(g, \omega)$  so is the minimal  $n$ -noid  $M_0$ , up to translation (see details in Section 7.1).

**Remark 4.** *The potential  $\xi_{t,\mathbf{x}}$  is inspired from the potential used in [18] to construct CMC  $n$ -noids by perturbation of a sphere. In fact, in the case  $dg_{\mathbf{x}} = dz$ , the two potentials are dual to each other. (See Section 3.2.8 of [19] for the definition of duality in the DPW method.)*

**5.1. Regularity.** Our potential  $\xi_{t,\mathbf{x}}$  has poles at the zeros of  $B_{\mathbf{x}}$  and the points  $p_1, \dots, p_n$ . (At  $\infty$ , we have  $\omega_{\mathbf{x}} \sim b_1^2 z^{-2} dz$  which is holomorphic.) We want the zeros of  $B_{\mathbf{x}}$  to be apparent singularities, so we require the potential to be gauge-equivalent to a regular potential in a neighborhood of these points. Consider the gauge

$$G_{\mathbf{x}}(z, \lambda) = \begin{pmatrix} g_{\mathbf{x}}(z, \lambda)^{-1} & -1 \\ 0 & g_{\mathbf{x}}(z, \lambda) \end{pmatrix}$$

The gauged potential is

$$\widehat{\xi}_{t,\mathbf{x}} := \xi_{t,\mathbf{x}} \cdot G_{\mathbf{x}} = \begin{pmatrix} 0 & \frac{1}{4}t(\lambda - 1)^2 \lambda^{-1} g_{\mathbf{x}}^2 \omega_{\mathbf{x}} \\ g_{\mathbf{x}}^{-2} dg_{\mathbf{x}} & 0 \end{pmatrix}.$$

We have

$$g_{\mathbf{x}}^{-2} dg_{\mathbf{x}} = \frac{A'_{\mathbf{x}} B_{\mathbf{x}} - A_{\mathbf{x}} B'_{\mathbf{x}}}{A_{\mathbf{x}}^2} \quad \text{and} \quad g_{\mathbf{x}}^2 \omega_{\mathbf{x}} = \frac{A_{\mathbf{x}}^2 dz}{\prod_{i=1}^n (z - p_i)^2}.$$

Let  $\zeta$  be a zero of  $B_{\mathbf{x}_0}$  (recall that  $B_{\mathbf{x}_0}$  does not depend on  $\lambda$ ). Then  $A_{\mathbf{x}_0}(\zeta) \neq 0$ . By continuity, there exists a neighborhood  $U$  of  $\zeta$  such that for  $z \in U$ ,  $\lambda \in \mathbb{D}_\rho$  and  $\mathbf{x}$  close enough to  $\mathbf{x}_0$ ,  $A_{\mathbf{x}}(z, \lambda) \neq 0$ . So  $\widehat{\xi}_{t,\mathbf{x}}$  is holomorphic in  $U \times \mathbb{D}_\rho^*$  and moreover,  $\widehat{\xi}_{t,\mathbf{x};12}^{(-1)} \neq 0$ . This ensures that the immersion extends analytically to  $U$  and is unbranched in  $U$ .

## 6. THE MONODROMY PROBLEM

**6.1. Formulation of the problem.** For  $i \in [1, n]$ , we denote  $p_{i,0}$  the central value of the parameter  $p_i$  (so  $p_{1,0}, \dots, p_{n,0}$  are the ends of the minimal  $n$ -noid  $M_0$ ). We consider the following  $\lambda$ -independent domain on the Riemann sphere:

$$(13) \quad \Omega = \{z \in \mathbb{C} : \forall i \in [1, n], |z - p_{i,0}| > \varepsilon\} \cup \{\infty\}$$

where  $\varepsilon > 0$  is a fixed, small enough number such that the disks  $D(p_{i,0}, 8\varepsilon)$  for  $1 \leq i \leq n$  are disjoint. As in [18], we first construct a family of immersions  $f_t$  on  $\Omega$ . Then we extend  $f_t$  to an  $n$ -punctured sphere in Proposition 4.

Let  $\tilde{\Omega}$  be the universal cover of  $\Omega$  and  $\Phi_{t,\mathbf{x}}(z, \lambda)$  be the solution of the following Cauchy Problem on  $\tilde{\Omega}$ :

$$(14) \quad \begin{cases} d\Phi_{t,\mathbf{x}}(z, \lambda) = \Phi_{t,\mathbf{x}}(z, \lambda)\xi_{t,\mathbf{x}}(z, \lambda) \\ \Phi_{t,\mathbf{x}}(\tilde{z}_0, \lambda) = \phi_0 \end{cases}$$

We denote  $\gamma_1, \dots, \gamma_{n-1}$  a set of generators of the fundamental group  $\pi_1(\Omega, z_0)$ , with  $\gamma_i$  encircling the point  $p_{i,0}$ . We may assume that each  $\gamma_i$  is represented by a fixed curve avoiding the poles of  $\xi_{t,\mathbf{x}}$ . Let

$$M_i(t, \mathbf{x}) = \mathcal{M}_{\gamma_i}(\Phi_{t,\mathbf{x}})$$

be the monodromy of  $\Phi_{t,\mathbf{x}}$  along  $\gamma_i$ . By Equation (12), we have  $M_i(0, \mathbf{x}) = I_2$ . Recall that the matrix exponential is a local diffeomorphism from a neighborhood of 0 in the Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$  (respectively  $\mathfrak{su}(2)$ ) to a neighborhood of  $I_2$  in  $SL(2, \mathbb{C})$  (respectively  $SU(2)$ ). The inverse diffeomorphism is denoted  $\log$ . For  $t \neq 0$  small enough and  $\lambda \in \mathbb{D}_\rho \setminus \{1\}$ , we define as in [18]

$$\tilde{M}_i(t, \mathbf{x})(\lambda) = \frac{4\lambda}{t(\lambda - 1)^2} \log M_i(t, \mathbf{x})(\lambda).$$

**Proposition 2.** (1)  $\tilde{M}_i(t, \mathbf{x})(\lambda)$  extends smoothly at  $t = 0$  and  $\lambda = 1$ , and each entry  $\tilde{M}_{i,kl}$  is a smooth map from a neighborhood of  $(0, \mathbf{x}_0)$  in  $\mathbb{R} \times (\mathcal{W}_\rho^{\geq 0})^3$  to  $\mathcal{W}_\rho$ .

(2) At  $t = 0$ , we have

$$(15) \quad \tilde{M}_i(0, \mathbf{x})(\lambda) = \begin{pmatrix} \mathcal{P}_{i,1}(\mathbf{x}) & \mathcal{P}_{i,2}(\mathbf{x}) \\ -\mathcal{P}_{i,0}(\mathbf{x}) & -\mathcal{P}_{i,1}(\mathbf{x}) \end{pmatrix} \quad \text{where } \mathcal{P}_{i,k}(\mathbf{x}) = \int_{\gamma_i} g_{\mathbf{x}}^k \omega_{\mathbf{x}}.$$

(3) The Monodromy Problem (9) is equivalent to

$$(16) \quad \tilde{M}_i(t, \mathbf{x}) \in \Lambda \mathfrak{su}(2) \quad \text{for } 1 \leq i \leq n - 1.$$

Proof: we follow the proof of Proposition 1 in [18]. We first consider the case where the parameter  $\mathbf{x} = (a_i, b_i, p_i)_{1 \leq i \leq n}$  is constant with respect to  $\lambda$ , so  $\mathbf{x} \in \mathbb{C}^{3n}$ . For  $(\mu, \mathbf{x})$  in a neighborhood of  $(0, \mathbf{x}_0)$  in  $\mathbb{C} \times \mathbb{C}^{3n}$ , we define

$$\hat{\xi}_{\mu,\mathbf{x}}(z) = \begin{pmatrix} 0 & \mu \omega_{\mathbf{x}}(z) \\ dg_{\mathbf{x}}(z) & 0 \end{pmatrix}$$

where  $\omega_{\mathbf{x}}$  and  $g_{\mathbf{x}}$  are defined by Equations (10) and (11), except that  $a_i, b_i, p_i$  are constant complex numbers. Let  $\hat{\Phi}_{\mu,\mathbf{x}}$  be the solution of the Cauchy Problem  $d\hat{\Phi}_{\mu,\mathbf{x}} = \hat{\Phi}_{\mu,\mathbf{x}}\hat{\xi}_{\mu,\mathbf{x}}$  in  $\tilde{\Omega}$  with initial condition  $\hat{\Phi}_{\mu,\mathbf{x}}(\tilde{z}_0) = \phi_0$ . Let  $N_i(\mu, \mathbf{x}) = \mathcal{M}_{\gamma_i}(\hat{\Phi}_{\mu,\mathbf{x}})$ . By standard ODE theory, each entry of  $N_i$  is a holomorphic function of  $(\mu, \mathbf{x})$ . At  $\mu = 0$ ,  $\hat{\Phi}_{0,\mathbf{x}}$  is given by Equation (12), so in particular  $N_i(0, \mathbf{x}) = I_2$ . Hence

$$\tilde{N}_i(\mu, \mathbf{x}) := \frac{1}{\mu} \log N_i(\mu, \mathbf{x})$$

extends holomorphically at  $\mu = 0$  with value  $\widetilde{N}_i(0, \mathbf{x}) = \frac{\partial N_i}{\partial \mu}(0, \mathbf{x})$ . By Proposition 8 in Appendix A of [18] (the same formula appeared before on page 39 of [12]):

$$\frac{\partial N_i}{\partial \mu}(0, \mathbf{x}) = \int_{\gamma_i} \widehat{\Phi}_{0, \mathbf{x}} \frac{\partial \widehat{\xi}_{\mu, \mathbf{x}}}{\partial \mu} \Big|_{\mu=0} \widehat{\Phi}_{0, \mathbf{x}}^{-1}.$$

Hence

$$(17) \quad \widetilde{N}_i(0, \mathbf{x}) = \int_{\gamma_i} \begin{pmatrix} g_{\mathbf{x}} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega_{\mathbf{x}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & g_{\mathbf{x}} \end{pmatrix} = \int_{\gamma_i} \begin{pmatrix} g_{\mathbf{x}} \omega_{\mathbf{x}} & g_{\mathbf{x}}^2 \omega_{\mathbf{x}} \\ -\omega_{\mathbf{x}} & -g_{\mathbf{x}} \omega_{\mathbf{x}} \end{pmatrix}.$$

For  $(t, \mathbf{x})$  in a neighborhood of  $(0, \mathbf{x}_0)$  in  $\mathbb{R} \times (\mathcal{W}_{\rho}^{\geq 0})^{3n}$ , we have

$$\xi_{t, \mathbf{x}}(z, \lambda) = \widehat{\xi}_{\mu(t, \lambda), \mathbf{x}(\lambda)}(z) \quad \text{with} \quad \mu(t, \lambda) = \frac{t(\lambda - 1)^2}{4\lambda}.$$

Hence

$$M_i(t, \mathbf{x})(\lambda) = N_i(\mu(t, \lambda), \mathbf{x}(\lambda)) \quad \text{and} \quad \widetilde{M}_i(t, \mathbf{x})(\lambda) = \widetilde{N}_i(\mu(t, \lambda), \mathbf{x}(\lambda)).$$

By substitution (see Proposition 9 in Appendix B of [18]), each entry of  $\widetilde{M}_i$  is a smooth map from a neighborhood of  $(0, \mathbf{x}_0)$  in  $\mathbb{R} \times (\mathcal{W}_{\rho}^{\geq 0})^3$  to  $\mathcal{W}_{\rho}$ . Moreover,  $\widetilde{M}_i(0, \mathbf{x})$  is given by Equation (17). The fact that  $\widetilde{M}_i$  extends holomorphically at  $\lambda = 1$  implies that Points (ii) and (iii) of Problem (9) are automatically satisfied. Since  $\lambda^{-1}(\lambda - 1)^2 \in \mathbb{R}$  for  $\lambda \in \mathbb{S}^1$ , Equation (i) of Problem (9) is equivalent to Equation (16).  $\square$

**6.2. Solution of the monodromy problem.** Without loss of generality, we may (using a Möbius transformation of the sphere) fix the value of  $p_1$ ,  $p_2$  and  $p_3$ . We still denote  $\mathbf{x} \in (\mathcal{W}_{\rho}^{\geq 0})^{3n-3}$  the vector of the remaining parameters.

**Proposition 3.** *Assume that the given minimal  $n$ -noid is non-degenerate. For  $t$  in a neighborhood of 0, there exists a smooth function  $\mathbf{x}(t) \in (\mathcal{W}_{\rho}^{\geq 0})^{3n-3}$  such that  $\widetilde{M}_i(t, \mathbf{x}(t), \cdot) \in \Lambda \mathfrak{su}(2)$  for  $1 \leq i \leq n - 1$ . Moreover,  $\mathbf{x}(0) = \mathbf{x}_0$ .*

Proof: recalling the definition of  $P_{i,k}$  in Section 2 and  $\mathcal{P}_{i,k}$  in Equation (15), we have

$$\mathcal{P}_{i,k}(\mathbf{x})(\lambda) = P_{i,k}(\mathbf{x}(\lambda)).$$

Hence  $\mathcal{P}_{i,k}$  is a smooth map from a neighborhood of  $\mathbf{x}_0$  in  $(\mathcal{W}_{\rho}^{\geq 0})^{3n-3}$  to  $\mathcal{W}_{\rho}^{\geq 0}$ . Moreover, since  $\mathbf{x}_0$  is constant, we have for  $X \in (\mathcal{W}_{\rho}^{\geq 0})^{3n-3}$ :

$$(18) \quad (d\mathcal{P}_{i,k}(\mathbf{x}_0)X)(\lambda) = dP_{i,k}(\mathbf{x}_0)X(\lambda).$$

Let  $\mathcal{P}_i = (\mathcal{P}_{i,0}, \mathcal{P}_{i,1}, \mathcal{P}_{i,2})$  and  $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{n-1})$ . By the non-degeneracy hypothesis and Remark 2,  $d\mathcal{P}(\mathbf{x}_0)$  is an automorphism of  $\mathbb{C}^{3n-3}$ , so  $d\mathcal{P}(\mathbf{x}_0)$  is an automorphism of  $(\mathcal{W}_{\rho}^{\geq 0})^{3n-3}$  and restricts to an automorphism of  $(\mathcal{W}_{\rho}^{\geq 0})^{3n-3}$ .

We define the following smooth maps with value in  $\mathcal{W}_{\rho}$  (the star operator is defined in Section 3.6)

$$\begin{aligned} \mathcal{F}_i(t, \mathbf{x}) &= \widetilde{M}_{i,11}(t, \mathbf{x}) + \widetilde{M}_{i,11}(t, \mathbf{x})^* \\ \mathcal{G}_i(t, \mathbf{x}) &= \widetilde{M}_{i,12}(t, \mathbf{x}) + \widetilde{M}_{i,21}(t, \mathbf{x})^* \end{aligned}$$

Problem (16) is equivalent to  $\mathcal{F}_i = \mathcal{G}_i = 0$ . Actually, by definition,  $\mathcal{F}_i = \mathcal{F}_i^*$ , so Problem (16) is equivalent to

$$\mathcal{F}_i(t, \mathbf{x})^+ = 0, \quad \operatorname{Re}(\mathcal{F}_i(t, \mathbf{x})^0) = 0 \quad \text{and} \quad \mathcal{G}_i(t, \mathbf{x}) = 0 \quad \text{for } 1 \leq i \leq n-1.$$

At  $t = 0$ , we have by Equation (15):

$$\mathcal{F}_i(0, \mathbf{x}) = \mathcal{P}_{i,1}(\mathbf{x}) + \mathcal{P}_{i,1}(\mathbf{x})^*$$

$$\mathcal{G}_i(0, \mathbf{x}) = \mathcal{P}_{i,2}(\mathbf{x}) - \mathcal{P}_{i,0}(\mathbf{x})^*$$

Equation (3) tells us precisely that that at the central value, we have  $\mathcal{F}_i(0, \mathbf{x}_0) = 0$  and  $\mathcal{G}_i(0, \mathbf{x}_0) = 0$ . We have for  $X \in (\mathcal{W}_\rho^{\geq 0})^{3n-3}$ :

$$d\mathcal{F}_i(0, \mathbf{x}_0)X = d\mathcal{P}_{i,1}(x_0)X + (d\mathcal{P}_{i,1}(\mathbf{x}_0)X)^*$$

$$d\mathcal{G}_i(0, \mathbf{x}_0)X = d\mathcal{P}_{i,2}(x_0)X - (d\mathcal{P}_{i,0}(\mathbf{x}_0)X)^*$$

Projecting on  $\mathcal{W}_\rho^{>0}$  and  $\mathcal{W}_\rho^{<0}$  we obtain:

$$(d\mathcal{F}_i(0, \mathbf{x}_0)X)^+ = d\mathcal{P}_{i,1}(\mathbf{x}_0)X^+$$

$$(d\mathcal{G}_i(0, \mathbf{x}_0)X)^+ = d\mathcal{P}_{i,2}(\mathbf{x}_0)X^+$$

$$(d\mathcal{G}_i(0, \mathbf{x}_0)X)^- = -(d\mathcal{P}_{i,0}(\mathbf{x}_0)X^+)^*$$

$$(d\mathcal{G}_i(0, \mathbf{x}_0)X)^{-*} = -d\mathcal{P}_{i,0}(\mathbf{x}_0)X^+.$$

Hence the operator

$$[d\mathcal{F}_i(0, \mathbf{x}_0)^+, d\mathcal{G}_i(0, \mathbf{x}_0)^+, d\mathcal{G}_i(0, \mathbf{x}_0)^{-*}]_{1 \leq i \leq n-1}$$

only depends on  $X^+$  and is an automorphism of  $(\mathcal{W}_\rho^{>0})^{3n-3}$  because  $d\mathcal{P}(\mathbf{x}_0)$  is. Projecting on  $\mathcal{W}^0$  we obtain:

$$(d\mathcal{F}_i(0, \mathbf{x}_0)X)^0 = 2 \operatorname{Re} (d\mathcal{P}_{i,1}(\mathbf{x}_0)X^0)$$

$$(d\mathcal{G}_i(0, \mathbf{x}_0)X)^0 = d\mathcal{P}_{i,2}(\mathbf{x}_0)X^0 - \overline{d\mathcal{P}_{i,0}(\mathbf{x}_0)X^0}.$$

Hence the  $\mathbb{R}$ -linear operator

$$[\operatorname{Re}(d\mathcal{F}_i(0, \mathbf{x}_0)^0), d\mathcal{G}_i(0, \mathbf{x}_0)^0]_{1 \leq i \leq n-1}$$

only depends on  $X^0$  and is surjective from  $\mathbb{C}^{3n-3}$  to  $(\mathbb{R} \times \mathbb{C})^{3n-3}$ . This implies that the differential of the map  $(\mathcal{F}_i^+, \mathcal{G}_i^+, \mathcal{G}_i^{-*}, \operatorname{Re}(\mathcal{F}_i^0), \mathcal{G}_i^0)_{1 \leq i \leq n-1}$  is surjective from  $(\mathcal{W}_\rho^{\geq 0})^{3n-3}$  to  $((\mathcal{W}_\rho^{>0})^3 \times \mathbb{R} \times \mathbb{C})^{n-1}$ . Proposition 3 follows from the Implicit Function Theorem.  $\square$

**Remark 5.** *The kernel of the differential has real dimension  $3n - 3$  so we have  $3n - 3$  free real parameters. These parameters correspond to deformations of the flux vectors of the minimal  $n$ -noid.*

## 7. GEOMETRY OF THE IMMERSION

From now on, we assume that  $\mathbf{x}(t)$  is given by Proposition 3. We write  $a_{i,t}$ ,  $b_{i,t}$  and  $p_{i,t}$  for the value of the corresponding parameters. (These parameters are in the space  $\mathcal{W}_\rho^{\geq 0}$  so are functions of  $\lambda$ .) For ease of notation, we write  $g_t$ ,  $\omega_t$ ,  $\xi_t$  and  $\Phi_t$  for  $g_{\mathbf{x}(t)}$ ,  $\omega_{\mathbf{x}(t)}$ ,  $\xi_{t,\mathbf{x}(t)}$  and  $\Phi_{t,\mathbf{x}(t)}$ , respectively. Let  $F_t = \text{Uni}(\Phi_t)$ . Since the Monodromy Problem is solved, the Sym-Bobenko formula (6) defines a CMC-1 immersion  $f_t : \Omega \rightarrow \mathbb{R}^3$ , where  $\Omega$  is the (fixed) domain defined by Equation (13).

**Proposition 4.** *The immersion  $f_t$  extends analytically to*

$$\Sigma_t := \mathbb{C} \cup \{\infty\} \setminus \{p_{1,t}(0), \dots, p_{n,t}(0)\}$$

where  $p_{i,t}(0)$  is the value of  $p_{i,t}$  at  $\lambda = 0$ .

We omit the proof which is exactly the same as the proof of Point 1 of Proposition 4 in [18]. It relies on Theorem 3 in [18] which allows for  $\lambda$ -dependent changes of variables in the DPW method.

7.1. Convergence to the minimal  $n$ -noid.

**Proposition 5.**  $\lim_{t \rightarrow 0} \frac{1}{t} f_t = \psi$  where  $\psi$  is (up to translation) the conformal parametrization of the minimal  $n$ -noid given by Equation (1). The limit is the uniform  $C^1$  convergence on compact subsets of  $\Sigma_0 = \mathbb{C} \cup \{\infty\} \setminus \{p_{1,0}, \dots, p_{n,0}\}$ .

Proof: at  $t = 0$ , we have  $g_0 = g$  and  $\omega_0 = \omega$ . By Equation (12) and definition of the potential, we have

$$\Phi_0(z, \lambda) = \begin{pmatrix} g(z) & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial \xi_{t;12}^{(-1)}}{\partial t} \Big|_{t=0} = \frac{\omega}{4}.$$

By Theorem 4,  $\frac{1}{t} f_t$  converges to a minimal surface with Weierstrass data  $(g, \omega)$  on compact subsets of  $\Sigma_0$  minus the poles of  $g$ . In a neighborhood of the poles of  $g$ , we use the gauge introduced in Section 5.1. With the notations of this section and writing  $\widehat{\Phi}_t = \Phi_t G_{\mathbf{x}(t)}$ , we have

$$\widehat{\Phi}_0(z, \lambda) = \begin{pmatrix} 1 & 0 \\ -g(z)^{-1} & 1 \end{pmatrix} \quad \text{and} \quad \frac{\partial \widehat{\xi}_{t;12}^{(-1)}}{\partial t} \Big|_{t=0} = \frac{g^2 \omega}{4}.$$

By Theorem 4 again,  $\frac{1}{t} f_t$  converges to a minimal surface with Weierstrass data  $(g, \omega)$  in a neighborhood of the poles of  $g$ . The two limit minimal surfaces are of course the same, since they coincide in a neighborhood of  $z_0$ .  $\square$

**7.2. Delaunay ends.** In this section, we prove that the immersion  $f_t$  has Delaunay ends. Delaunay ends in the DPW method have been studied in [4, 13]. Following [17], we gauge our potential to a perturbation of the standard Delaunay potential and we use the results in [13].

We denote  $N_0$  the Gauss map of the minimal  $n$ -noid  $M_0$ . For  $1 \leq i \leq n$ , we denote  $\mathcal{C}_i$  the catenoid to which  $M_0$  is asymptotic at  $p_{i,0}$  and  $\tau_i > 0$  the necksize of  $\mathcal{C}_i$ .

**Definition 4.** *We say that  $N_0$  points to the inside in a neighborhood of  $p_{i,0}$  if it points to the component of  $\mathbb{R}^3 \setminus \mathcal{C}_i$  containing the axis of  $\mathcal{C}_i$ .*

**Proposition 6.** *For  $1 \leq i \leq n$  and  $t \neq 0$ :*

(1) The immersion  $f_t$  has a Delaunay end at  $p_{i,t}$ . If we denote  $w_{i,t}$  its weight then

$$\lim_{t \rightarrow 0} t^{-1} w_{i,t} = \pm 2\pi\tau_i$$

where the sign is  $+$  if  $N_0$  points to the inside in a neighborhood of  $p_{i,0}$  and  $-$  otherwise.

(2) Its axis converges as  $t \rightarrow 0$  to the half-line through the origin directed by the vector  $N_0(p_{i,0})$ .

(3) If  $N_0$  points to the inside in a neighborhood of  $p_{i,0}$ , there exists a uniform  $\varepsilon > 0$  such that for  $t > 0$  small enough,  $f_t(D^*(p_{i,0}, \varepsilon))$  is embedded.

Proof: in a neighborhood of the puncture  $p_{i,t}$ , we may use  $w = g_t(z) - g_t(p_{i,t})$  as a local coordinate. Note that  $p_{i,t} \in \mathcal{W}_\rho^{\geq 0}$  so, as a function of  $\lambda$ , extends holomorphically to  $\mathbb{D}_\rho$ . Thus the coordinate  $w$  depends holomorphically on  $\lambda \in \mathbb{D}_\rho$ . This is not a problem by Theorem 3 in [18]. Consider the gauge

$$G(w) = \begin{pmatrix} \frac{k}{\sqrt{w}} & \frac{-1}{2k\sqrt{w}} \\ 0 & \frac{\sqrt{w}}{k} \end{pmatrix}.$$

Here we can take  $k = 1$ , but later on we will take another value of  $k$  so we do the computation for general values of  $k \neq 0$ . The gauged potential is

$$\widehat{\xi}_t := \xi_t \cdot G = \begin{pmatrix} 0 & \frac{dw}{4k^2w} + \frac{wt(\lambda-1)^2}{4k^2\lambda}\omega_t \\ \frac{k^2dw}{w} & 0 \end{pmatrix}$$

Since  $\omega_t$  has a double pole at  $p_{i,t}$ ,  $\widehat{\xi}_t$  has a simple pole at  $w = 0$  with residue

$$A_{i,t}(\lambda) = \begin{pmatrix} 0 & \frac{1}{4k^2} + \frac{t(\lambda-1)^2}{4k^2\lambda}\alpha_{i,t}(\lambda) \\ k^2 & 0 \end{pmatrix}$$

where

$$(19) \quad \alpha_{i,t} = \text{Res}_{p_{i,t}}(w\omega_t) = \text{Res}_{p_{i,t}}(g_t(z) - g_t(p_{i,t}))\omega_t.$$

**Claim 1.** For  $t$  small enough,  $\alpha_{i,t}$  is a real constant (i.e. independent of  $\lambda$ , possibly depending on  $t$ ).

Proof: the proof is similar to the proof of Point 2 of Proposition 4 in [18]. We use the standard theory of Fuchsian systems. Fix  $t \neq 0$  and  $\lambda \in \mathbb{S}^1 \setminus \{1\}$ . Assume that  $\alpha_{i,t}(\lambda) \neq 0$ . Let  $\widehat{\Phi}_t = \Phi_t G$ . The eigenvalues of  $A_{i,t}$  are  $\pm \Lambda_{i,t}$  with

$$\Lambda_{i,t}(\lambda)^2 = \frac{1}{4} + \frac{t(\lambda-1)^2}{4\lambda}\alpha_{i,t}(\lambda).$$

Provided  $t \neq 0$  is small enough,  $\Lambda_{i,t} \notin \mathbb{Z}/2$  so the system is non resonant and  $\widehat{\Phi}_t$  has the following standard  $z^A P$  form in the universal cover of  $D(0, \varepsilon)^*$ :

$$\widehat{\Phi}_t(w, \lambda) = V(\lambda) \exp(A_{i,t}(\lambda) \log w) P(w, \lambda)$$

where  $P(w, \lambda)$  descends to a well defined holomorphic function of  $w \in D(0, \varepsilon)$  with  $P(0, \lambda) = I_2$ . Consequently, its monodromy is

$$\mathcal{M}_{\gamma_i}(\widehat{\Phi}_t) = V(\lambda) \exp(2\pi i A_{i,t}) V(\lambda)^{-1}$$



with eigenvalues  $\exp(\pm 2\pi i \Lambda_{i,t}(\lambda))$ . Since the Monodromy Problem is solved, the eigenvalues are unitary complex numbers, so  $\Lambda_{i,t}(\lambda) \in \mathbb{R}$  which implies that  $\alpha_{i,t}(\lambda) \in \mathbb{R}$ . This of course remains true if  $\alpha_{i,t}(\lambda) = 0$ . Hence  $\alpha_{i,t}$  is real on  $\mathbb{S}^1 \setminus \{1\}$ . Since all the parameters involved in the definition of  $\omega_t$  are in  $\mathcal{W}_\rho^{\geq 0}$ ,  $\alpha_{i,t}$  is holomorphic in the unit disk. Hence it is constant.  $\square$

Returning to the proof of Proposition 6, let  $(r, s) \in \mathbb{R}^2$  be the solution of

$$(20) \quad \begin{cases} rs = \frac{1}{4}t\alpha_{i,t} \\ r + s = \frac{1}{2} \\ r < s \end{cases}$$

Since  $r < s$ ,  $\sqrt{r\lambda + s}$  is well defined and does not vanish for  $\lambda \in \mathbb{D}$ . We take  $k = \sqrt{r\lambda + s}$  in the definition of the gauge  $G$ . Using Equation (20), we have:

$$(r\lambda^{-1} + s)(r\lambda + s) = \frac{1}{4} + rs(\lambda - 1)^2\lambda^{-1} = \frac{1}{4} + \frac{1}{4}t(\lambda - 1)^2\lambda^{-1}\alpha_{i,t}.$$

So the residue of  $\widehat{\xi}_t$  becomes

$$A_{i,t} = \begin{pmatrix} 0 & \frac{1}{r\lambda + s} \left( \frac{1}{4} + \frac{t(\lambda - 1)^2}{4\lambda} \alpha_{i,t} \right) \\ r\lambda + s & 0 \end{pmatrix} = \begin{pmatrix} 0 & r\lambda^{-1} + s \\ r\lambda + s & 0 \end{pmatrix}$$

which is the residue of the standard Delaunay potential. By [13], the immersion  $f_t$  has a Delaunay end at  $p_{i,t}$  of weight  $w_{i,t} = 8\pi rs = 2\pi t\alpha_{i,t}$ . It remains to relate  $\alpha_{i,0}$  to the logarithmic growth  $\tau_i$ . For ease of notation, let us write  $p_i = p_{i,0}$ . Assume that  $N_0$  points to the inside in a neighborhood of  $p_i$ . The flux of  $M_0$  along  $\gamma_i$  is equal to

$$\phi_i = 2\pi\tau_i N_0(p_i) = 2\pi \frac{\tau_i}{|g(p_i)|^2 + 1} (2\operatorname{Re}(g(p_i)), 2\operatorname{Im}(g(p_i)), |g(p_i)|^2 - 1)$$

On the other hand, we have seen in Section 2 that the flux is equal to

$$\phi_i = -2\pi \operatorname{Res}_{p_i} \left( \frac{1}{2}(1 - g^2)\omega, \frac{1}{2}(1 + g^2)\omega, g\omega \right)$$

Comparing these two expressions for  $\phi_i$ , we obtain

$$\operatorname{Res}_{p_i}(g\omega) = -\tau_i \frac{|g(p_i)|^2 - 1}{|g(p_i)|^2 + 1} \quad \text{and} \quad \operatorname{Res}_{p_i}\omega = -2\tau_i \frac{\overline{g(p_i)}}{|g(p_i)|^2 + 1}$$

Using Equation (19), this gives

$$\alpha_{i,0} = \operatorname{Res}_{p_i}(g\omega) - g(p_i) \operatorname{Res}_{p_i}\omega = \tau_i$$

If  $N_0$  points to the outside in a neighborhood of  $p_i$ , then  $\phi_i = -2\pi\tau_i N_0(p_i)$ , so the same computation gives  $\alpha_{i,0} = -\tau_i$ . This proves Point 1 of Proposition 6.

To prove Point 2, we use Theorem 5 in Appendix A. We need to compute  $\widehat{\Phi}_0$  at  $w = 1$ . At  $t = 0$ , we have  $k = \frac{1}{\sqrt{2}}$  so

$$G(1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}.$$

At  $t = 0$ , we have  $w = g(z) - g(p_i)$ , so  $w = 1 \Leftrightarrow g(z) = g(p_i) + 1$ . Using Equation (12),

$$\widehat{\Phi}_0(1) = \frac{1}{\sqrt{2}} \begin{pmatrix} g(p_i) + 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} g(p_i) + 1 & -g(p_i) + 1 \\ -1 & 1 \end{pmatrix}.$$

Fix  $0 < \alpha < 1$ . By Theorem 5 (using  $t\alpha_{i,t}$  as the time parameter), there exists  $\varepsilon > 0$ ,  $T > 0$  and  $c$  such that for  $0 < |t| < T$ :

$$\|f_t(z) - f_{i,t}^{\mathcal{D}}(z)\| \leq c|t||z - p_{i,t}|^\alpha \quad \text{in } D^*(p_{i,t}, \varepsilon)$$

where  $f_{i,t}^{\mathcal{D}} : \mathbb{C} \setminus \{p_{i,t}\} \rightarrow \mathbb{R}^3$  is a Delaunay immersion. We compute the limit axis of  $f_{i,t}^{\mathcal{D}}$  using Point 3 of Theorem 5:

$$\begin{aligned} \widehat{\Phi}_0(1)H &= \begin{pmatrix} g(p_i) & 1 \\ -1 & 0 \end{pmatrix} = \Phi_0(p_i). \\ Q &= F_0(p_i) \\ Qe_3Q^{-1} &= \text{Nor}(F_0(p_i)) = N_0(p_i). \end{aligned}$$

This proves Point 2 of Proposition 6. If  $N_0$  points to the inside in a neighborhood of  $p_{i,0}$ , then for  $t > 0$ ,  $t\alpha_{i,t} > 0$  so Point 3 follows from Point 2 of Theorem 5.  $\square$

**7.3. Alexandrov-embeddedness.** We recall from [2, 9] the definition of Alexandrov-embeddedness in the non-compact case:

**Definition 5.** *A surface  $M$  of finite topology is Alexandrov-embedded if  $M$  is properly immersed, if each end of  $M$  is embedded, and if there exists a compact 3-manifold  $\overline{W}$  with boundary  $\partial\overline{W} = \overline{S}$ ,  $n$  points  $q_1, \dots, q_n \in \overline{S}$  and a proper immersion  $F : W = \overline{W} \setminus \{q_1, \dots, q_n\} \rightarrow \mathbb{R}^3$  whose restriction to  $S = \overline{S} \setminus \{q_1, \dots, q_n\}$  parametrizes  $M$ .*

**Lemma 1.** *Let  $M$  be an Alexandrov-embedded minimal surface with  $n$  catenoidal ends. With the notations of Definition 5, we equip  $W$  with the flat metric induced by  $F$ , so  $F$  is a local isometry, and we denote  $N$  the inside normal to  $S$ . Then there exists a flat 3-manifold  $W'$  containing  $W$ , a local isometry  $F' : W' \rightarrow \mathbb{R}^3$  extending  $F$  and  $r > 0$  such that the tubular neighborhood  $\text{Tab}_r S$  is embedded in  $W'$ . In other words, the map  $(x, s) \mapsto \exp_x(sN(x))$  from  $S \times (-r, r)$  to  $W'$  is well defined and is a diffeomorphism onto its image.*

Proof: since  $M$  has catenoidal ends, there exists  $r > 0$  such that the inside tubular neighborhood map

$$g : S \times (0, r) \rightarrow W, \quad g(x, s) = \exp_x(sN(x))$$

is a diffeomorphism onto its image. Since  $F$  is a local isometry, we have

$$(21) \quad F(g(x, s)) = F(x) + s dF(x)N(x) \quad \text{for } (x, s) \in S \times (0, r).$$

We define  $W'$  as the disjoint union  $(S \times (-r, r)) \sqcup W$  where we identify  $(x, s) \in S \times (0, r)$  with its image  $g(x, s) \in W$ . We define  $F' : W' \rightarrow \mathbb{R}^3$  by  $F' = F$  in  $W$  and

$$F'(x, s) = F(x) + s dF(x)N(x) \quad \text{for } (x, s) \in S \times (-r, r).$$

The map  $F'$  is well defined by Equation (21). We equip  $S \times (-r, r)$  with the flat metric induced by the local diffeomorphism  $F'$ , which extends the metric already defined on  $S \times (0, r)$  by identification with  $W$ . Since

$$dF'(x, 0)(X, T) = dF(x)X + TdF(x)N(x)$$

the metric restricted to  $S \times \{0\}$  is the product metric, so the normal to  $S \times \{0\}$  in  $S \times (-r, r)$  is  $N(x, 0) = (0, 1)$ . Since  $F'$  is a local isometry, we have for  $(x, s) \in S \times (-r, r)$

$$F' \left( \exp_{(x,0)} sN(x, 0) \right) = F'(x, 0) + sdF'(x, 0)(0, 1) = F(x) + sdF(x)N(x) = F'(x, s)$$

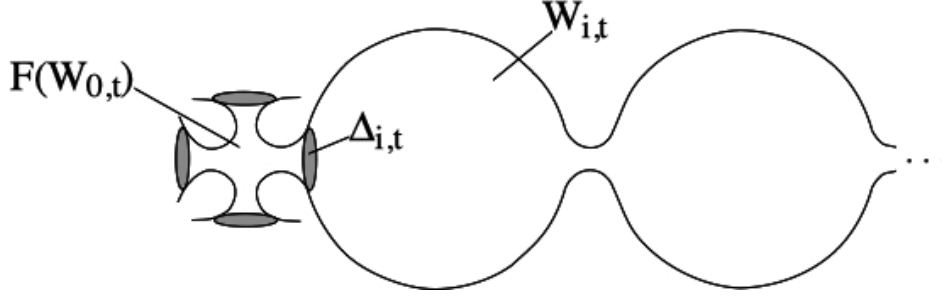


FIGURE 2. Decomposition of a 4-noid into pieces. Only one Delaunay end is represented, and  $F(W_{0,t})$  is represented as an embedded domain for clarity, but in general it will be immersed.

Hence  $\exp_{(x,0)} sN(x,0) = (x,s)$  so  $\text{Tub}_r(S \times \{0\})$  is embedded in  $S \times (-r,r)$ .  $\square$

We now return to the proof of Theorem 1. We orient the minimal  $n$ -noid  $M_0$  so that its Gauss map points to the inside in a neighborhood of  $p_1$ . For  $0 < |t| < \epsilon$ , we denote  $M_t$  the image of the immersion  $f_t$  that we have constructed.

**Proposition 7.** *If  $M_0$  is Alexandrov embedded, then for  $t > 0$  small enough,  $M_t$  is Alexandrov embedded.*

Proof: our strategy is to cut  $M_t$  by suitable planes into pieces which are either close to  $M_0$  or Delaunay surfaces (see Figure 2). Then we prove that each piece, together with flat disks in the cutting planes, is the boundary of a domain, using the Jordan Brouwer Theorem.

Since  $M_0$  is Alexandrov embedded,  $N_0$  points to the inside in a neighborhood of each end, so  $M_t$  has embedded ends by Proposition 6. Let  $\epsilon > 0$  be the number given by our application of Theorem 5 in Section 7.2 and  $f_{i,t}^D : \mathbb{C} \setminus \{p_{i,t}\} \rightarrow \mathbb{R}^3$  be the Delaunay immersion which approximates  $f_t$  in  $D^*(p_{i,t}, \epsilon)$ . Recall that  $f_t(D^*(p_{i,t}, \epsilon))$  is embedded. Let  $\tilde{f}_t = \frac{1}{t} f_t$ . By Proposition 5,  $\tilde{f}_t$  converges to  $\psi$  on compact subsets of  $\Sigma_0$ , where  $\psi : \Sigma_0 \rightarrow \mathbb{R}^3$  is a parametrization of  $M_0$ . Since  $M_0$  has catenoidal ends, we may assume (taking  $\epsilon$  smaller if necessary) that  $\psi(D^*(p_{i,0}, \epsilon))$  is embedded and  $N_0 \neq N_0(p_{i,0})$  in  $D^*(p_{i,0}, \epsilon)$ .

Let  $h_i : \mathbb{R}^3 \rightarrow \mathbb{R}$  be the height function in the direction  $N_0(p_{i,0})$ , defined by

$$h_i(x) = \langle x, N_0(p_{i,0}) \rangle.$$

We shall cut  $M_t$  by the plane  $h_i = \delta$  where  $\delta > 0$  is a fixed, large enough number such that for  $1 \leq i \leq n$ ,

$$\delta > \max_{C(p_{i,0}, \epsilon)} h_i \circ \psi.$$

Since  $\lim_{z \rightarrow p_{i,0}} h_i \circ \psi(z) = +\infty$ , we may fix a positive, small enough  $\epsilon' < \epsilon$  such that

$$\min_{C(p_{i,0}, \epsilon')} h_i \circ \psi > \delta.$$

Let  $\mathcal{A}_{i,t}$  be the annulus defined by  $\epsilon' \leq |z - p_{i,t}| \leq \epsilon$ . Since  $N_0 \neq N_0(p_{i,0})$  in  $\mathcal{A}_{i,0}$ ,

$$\min_{\mathcal{A}_{i,0}} \|N_0(z) - N_0(p_{i,0})\| > 0.$$

For  $t > 0$  small enough:

$$(22) \quad \max_{C(p_{i,t}, \varepsilon)} h_i \circ \tilde{f}_t < \delta$$

$$(23) \quad \min_{C(p_{i,t}, \varepsilon')} h_i \circ \tilde{f}_t > \delta$$

$$(24) \quad \min_{\mathcal{A}_{i,t}} \|N_t(z) - N_0(p_{i,0})\| > 0.$$

Hence the function  $h_i \circ \tilde{f}_t$  has no critical point in the annulus  $\mathcal{A}_{i,t}$ . So  $h_i \circ \tilde{f}_t = \delta$  defines a regular closed curve  $\gamma_{i,t}$  in  $\mathcal{A}_{i,t}$ . At  $t = 0$ ,  $h_i \circ \psi = \delta$  is a single curve around  $p_{i,0}$ , so  $\gamma_{i,t}$  has only one component and is not contractible in  $\mathcal{A}_{i,t}$ . Let  $D_{i,t} \subset \mathbb{C}$  be the topological disk bounded by  $\gamma_{i,t}$  and  $D_{i,t}^* = D_{i,t} \setminus \{p_{i,t}\}$ . Let  $\Delta_{i,t}$  be the closed topological disk bounded by  $\tilde{f}_t(\gamma_{i,t})$  in the plane defined by  $h_i(x) = \delta$ .

**Claim 2.** For  $t > 0$  small enough,  $\tilde{f}_t(D_{i,t}^*) \cap \Delta_{i,t} = \emptyset$ .

Proof: of course,  $h_i \circ \tilde{f}_t > \delta$  in  $D_{i,t}^* \cap \mathcal{A}_{i,t}$ . What we need to prove is that  $\tilde{f}_t(D^*(p_{i,t}, \varepsilon'))$  does not intersect  $\Delta_{i,t}$ . We do this by comparison with the Delaunay surface. Let  $\Pi_i = N_0(p_{i,0})^\perp$  and  $\pi_i = \mathbb{R}^3 \rightarrow \Pi_i$  be the orthogonal projection. Since  $\psi$  has a catenoidal end at  $p_{i,0}$ ,  $\psi(\mathcal{A}_{i,t})$  is a graph over an annulus in the plane  $\Pi_i$ , with inside boundary circle  $\pi_i \circ \psi(C(p_{i,t}, \varepsilon))$  and outside boundary circle  $\pi_i \circ \psi(C(p_{i,t}, \varepsilon'))$ . Moreover,  $N_0$  is close to  $N_0(p_{i,t})$ . Since  $\tilde{f}_t$  is  $C^1$  close to  $\psi$  in  $\mathcal{A}_{i,t}$ , for  $t > 0$  small enough,  $\tilde{f}_t(\mathcal{A}_{i,t})$  is a graph over an annulus in the plane  $\Pi_i$ , with inside boundary circle  $\pi_i \circ \tilde{f}_t(C(p_{i,t}, \varepsilon))$  and outside boundary circle  $\pi_i \circ \tilde{f}_t(C(p_{i,t}, \varepsilon'))$ .

Now we go back to the original scale. Since  $f_t$  is  $C^1$  close to  $f_{i,t}^D$  in  $D^*(p_{i,t}, \varepsilon)$ , we conclude that  $f_{i,t}^D(\mathcal{A}_{i,t})$  is a graph over an annulus in the plane  $\Pi_i$ , with inside boundary circle  $\pi_i \circ f_{i,t}^D(C(p_{i,t}, \varepsilon))$  and outside boundary circle  $\pi_i \circ f_{i,t}^D(C(p_{i,t}, \varepsilon'))$ . Then from the geometry of Delaunay surfaces, there exists a curve  $\gamma_{i,t,0}$  in  $D^*(p_{i,t}, \varepsilon')$  such that  $f_{i,t}^D(\gamma_{i,t,0})$  is a closed curve in the plane  $h_i = \frac{1}{2}$ . Let  $D_{i,t,0}$  be the disk bounded by  $\gamma_{i,t,0}$  and  $\mathcal{A}_{i,t,0}$  be the closed annulus bounded by  $\gamma_{i,t}$  and  $\gamma_{i,t,0}$ . Then  $h_i \circ f_{i,t}^D > \frac{1}{2}$  in  $D_{i,t,0}$  and  $f_{i,t}^D(\mathcal{A}_{i,t,0})$  is a graph over an annulus in the plane  $\Pi_i$ . Since  $f_t$  is  $C^1$  close to  $f_{i,t}^D$  in  $D^*(p_{i,t}, \varepsilon)$ , we conclude that  $h_i \circ f_t > \frac{1}{4}$  in  $D_{i,t,0}^*$  and  $f_t(\mathcal{A}_{i,t,0})$  is a graph over an annulus in the plane  $\Pi_i$ .

Back to the scale  $\frac{1}{t}$ ,  $\tilde{f}_t(D_{i,t} \cap \mathcal{A}_{i,t,0})$  is a graph over an annulus in the plane  $\Pi_i$  whose inside boundary circle is  $\pi_i \circ \tilde{f}_t(\gamma_{i,t}) = \pi_i(\partial\Delta_{i,t})$ , so  $\tilde{f}_t(D_{i,t} \cap \mathcal{A}_{i,t,0}) \cap \Delta_{i,t} = \emptyset$ . Moreover,  $h_i \circ \tilde{f}_t > \frac{1}{4t} \gg \delta$  in  $D_{i,t,0}^*$  so  $\tilde{f}_t(D_{i,t,0}^*) \cap \Delta_{i,t} = \emptyset$ .  $\square$

**Claim 3.** For  $t > 0$  small enough,  $\tilde{f}_t(D_{i,t}^*) \cup \Delta_{i,t}$  is the boundary of a cylindrically bounded domain  $W_{i,t} \subset \mathbb{R}^3$ .

Proof: since  $f_t$  is close to  $f_{i,t}^D$  in  $D_{i,t}^*$ , we can find an increasing diverging sequence  $(R_{t,k})_{k \in \mathbb{N}}$  such that  $f_t(D_{i,t}^*)$  intersects the plane  $h_i = R_{t,k}$  transversally along a closed curve  $f_t(\gamma_{i,t,k})$ . (Explicitly, we can take  $R_{t,k} = \frac{1}{2} + k h_i(T_t)$  where  $T_t \in \mathbb{R}^3$  is the period of the Delaunay surface  $f_{i,t}^D$ .) Let  $\mathcal{A}_{i,t,k}$  be the annulus bounded by  $\gamma_{i,t}$  and  $\gamma_{i,t,k}$ . Let  $\Delta_{i,t,k}$  be the closed disk bounded

by  $\tilde{f}_t(\gamma_{i,t,k})$  in the plane  $h_i = t^{-1}R_{t,k}$ . Then  $\tilde{f}_t(\mathcal{A}_{i,t,k}) \cup \Delta_{i,t} \cup \Delta_{i,t,k}$  is topologically a sphere: the image of  $\mathbb{S}^2$  by an injective continuous map. By the Jordan Brouwer Theorem, it is the boundary of a bounded domain  $W_{i,t,k}$ . Clearly,  $W_{i,t,k} \subset W_{i,t,k+1}$ . We take  $W_{i,t} = \bigcup_{k \in \mathbb{N}} W_{i,t,k}$ .  $\square$

Let  $\Omega_t = \mathbb{C} \cup \{\infty\} \setminus (D_{1,t} \cup \dots \cup D_{n,t})$ . Let  $W'$  be the flat 3-manifold given by Lemma 1 and denote  $F : W' \rightarrow \mathbb{R}^3$  its developing map (instead of  $F'$ ). (Here  $W'$  is an open manifold, meaning not a manifold-with-boundary.)

**Claim 4.** *For  $t > 0$  small enough, there exists a compact domain  $W_{0,t}$  in  $W'$  such that*

$$F(\partial W_{0,t}) = \tilde{f}_t(\Omega_t) \cup \Delta_{1,t} \cup \dots \cup \Delta_{n,t}.$$

Proof: by definition,  $\psi$  lifts to a diffeomorphism  $\widehat{\psi} : \Sigma_0 \rightarrow S \subset W'$  such that  $F \circ \widehat{\psi} = \psi$ . Since  $M_0$  has catenoidal ends, there exists domains  $V_1, \dots, V_n$  in  $W'$  such that for  $1 \leq i \leq n$ :

- $F : V_i \rightarrow F(V_i) \subset \mathbb{R}^3$  is a diffeomorphism,
- $V_i$  is foliated by flat disks on which  $h_i \circ F$  is constant (in particular,  $h_i \circ F$  is constant on  $\partial V_i$ ),
- $\widehat{\psi}(D^*(p_{i,0}, \varepsilon)) \subset V_i$  (which might require taking a smaller  $\varepsilon > 0$ ),
- $h_i < \delta$  on  $V_i \cap \widehat{\psi}(\Sigma_0 \setminus \bigcup_{i=1}^n D(p_{i,0}, \varepsilon))$  (which might require taking a larger  $\delta$ ).

Let  $r > 0$  be the radius of the embedded tubular neighborhood of  $S$  in  $W'$  constructed in Lemma 1. For  $t > 0$  small enough,  $\|\tilde{f}_t - \psi\| < r$  in  $\overline{\Omega}_t$ , so  $\tilde{f}_t$  lifts to  $\widehat{f}_t : \overline{\Omega}_t \rightarrow W'$  such that  $F \circ \widehat{f}_t = \tilde{f}_t$ . (Explicitly,  $\widehat{f}_t(z) = \exp_{\widehat{\psi}(z)}(\tilde{f}_t(z) - \psi(z))$ .) From the properties of  $V_i$  and the convergence of  $\widehat{f}_t$  to  $\widehat{\psi}$  on compact subsets of  $\Sigma_0$ , we have for  $t > 0$  small enough

$$(25) \quad \widehat{f}_t(\overline{\Omega}_t \cap D(p_{i,0}, \varepsilon)) \subset V_i$$

$$(26) \quad h_i < \delta \quad \text{on } V_i \cap \widehat{f}_t(\Sigma_0 \setminus \bigcup_{i=1}^n D(p_{i,0}, \varepsilon)).$$

By Equation (25),  $\widehat{f}_t(\gamma_{i,t}) \subset V_i$  so  $\Delta_{i,t}$  lifts to a closed disk  $\widehat{\Delta}_{i,t} \subset V_i$  such that  $\partial \widehat{\Delta}_{i,t} = \widehat{f}_t(\gamma_{i,t})$  and  $F(\widehat{\Delta}_{i,t}) = \Delta_{i,t}$ . Since  $F$  is a diffeomorphism on  $V_i$ ,  $\widehat{f}_t(\Omega_t \cap D(p_{i,t}, \varepsilon))$  is disjoint from  $\widehat{\Delta}_{i,t}$ . By (26),  $\widehat{f}_t(\Omega_t \setminus \bigcup_{i=1}^n D(p_{i,t}, \varepsilon))$  is disjoint from  $\widehat{\Delta}_{i,t}$ . Hence  $\widehat{f}_t(\Omega_t) \cap \widehat{\Delta}_{i,t} = \emptyset$ . Then  $\widehat{f}_t(\Omega_t) \cup \widehat{\Delta}_{1,t} \cup \dots \cup \widehat{\Delta}_{n,t}$  is a topological sphere in  $W'$ . Since  $M_0$  has genus zero,  $W'$  is homeomorphic to  $\mathbb{R}^3$ . By the Jordan Brouwer Theorem,  $\widehat{f}_t(\Omega_t) \cup \widehat{\Delta}_{1,t} \cup \dots \cup \widehat{\Delta}_{n,t}$  is the boundary of a compact domain  $W_{0,t} \subset W'$ .  $\square$

Returning to the proof of Proposition 7, let  $W_t$  be the abstract 3-manifold with boundary obtained as the disjoint union  $\overline{W}_{0,t} \sqcup \overline{W}_{1,t} \sqcup \dots \sqcup \overline{W}_{n,t}$ , identifying  $\overline{W}_{0,t}$  and  $\overline{W}_{i,t}$  along their boundaries  $\widehat{\Delta}_{i,t}$  and  $\Delta_{i,t}$  via the map  $F$  for  $1 \leq i \leq n$ . Let  $F_t : W_t \rightarrow \mathbb{R}^3$  be the map defined by  $F_t = F$  in  $\overline{W}_{0,t}$  and  $F_t = \text{id}$  in  $\overline{W}_{i,t}$  for  $1 \leq i \leq n$ . Then  $F_t$  is a proper local diffeomorphism whose boundary restriction parametrizes  $M_t$ . Moreover, since each  $\overline{W}_{i,t}$  is homeomorphic to a closed ball minus a boundary point, we may compactify  $W_t$  by adding  $n$  points. This proves that  $M_t$  is Alexandrov-embedded.  $\square$

#### APPENDIX A. APPENDIX: ON DELAUNAY ENDS IN THE DPW METHOD

As already said, Delaunay ends in the DPW method have been studied in [13], where it is proved that if  $\xi$  is a holomorphic perturbation of the standard Delaunay potential, the resulting immersion has a Delaunay end at  $z = 0$  and is close to a Delaunay surface in a disk  $D(0, \varepsilon)$ . In

case the potential  $\xi_t$  depends on a parameter  $t$  and the Delaunay residue has weight  $\sim t$ , their result has been refined by T. Raujouan in [15], yielding a uniform  $\varepsilon$  as  $t \rightarrow 0$ . This is delicate because the corresponding Fuchsian system is resonant at  $t = 0$ . The result of [15] is the key to proving embeddedness of the CMC  $n$ -noids constructed in [18].

We consider the standard Delaunay residue for  $t \leq \frac{1}{16}$ :

$$A_t(\lambda) = \begin{pmatrix} 0 & \lambda^{-1}r + s \\ \lambda r + s & 0 \end{pmatrix} \quad \text{where} \quad \begin{cases} r + s = \frac{1}{2} \\ rs = t \\ r < s \end{cases}$$

In particular, in the limit case  $t = 0$ , we have

$$A_0 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

**Definition 6** ([15]). *A perturbed Delaunay potential is a family of DPW potentials  $\xi_t$  of the form*

$$\xi_t(z, \lambda) = A_t(\lambda) \frac{dz}{z} + R_t(z, \lambda) dz$$

where  $R_t$  is of class  $C^2$  with respect to  $(t, z, \lambda) \in (-T, T) \times D(0, \varepsilon) \times \mathbb{A}_\rho$  for some positive  $\varepsilon$  and  $T$ , and satisfies  $R_0 = 0$ . In particular,  $\xi_0 = A_0 \frac{dz}{z}$ .

Let  $(e_1, e_2, e_3)$  represent the canonical basis of  $\mathbb{R}^3$  in the  $\mathfrak{su}(2)$  model.

**Theorem 5.** *Let  $\xi_t$  be a perturbed Delaunay potential. Let  $\Phi_t(z, \lambda)$  be a family of solutions of  $d\Phi_t = \Phi_t \xi_t$  in the universal cover of the punctured disk  $D^*(0, \varepsilon)$ . Assume that  $\Phi_t(z, \lambda)$  depends continuously on  $(t, z, \lambda)$  and that the Monodromy Problem for  $\Phi_t$  is solved. Let  $f_t = \text{Sym}(\text{Uni}(\Phi_t))$  be the immersion given by the DPW method. Finally, assume that  $\Phi_0(1, \cdot)$  is constant (i.e. independent of  $\lambda$ ).*

Given  $0 < \alpha < 1$ , there exists uniform positive numbers  $\varepsilon' \leq \varepsilon$ ,  $T' \leq T$ ,  $c$  and a family of Delaunay immersions  $f_t^D : \mathbb{C}^* \rightarrow \mathbb{R}^3$  such that:

(1) For  $0 < |t| < T'$  and  $0 < |z| < \varepsilon'$ :

$$\|f_t(z) - f_t^D(z)\| \leq c|t||z|^\alpha.$$

(2) For  $0 < t < T'$ ,  $f_t : D^*(0, \varepsilon') \rightarrow \mathbb{R}^3$  is an embedding.

(3) The end of  $f_t^D$  at  $z = 0$  has weight  $8\pi t$  and its axis converges when  $t \rightarrow 0$  to the half-line spanned by the vector  $Qe_3Q^{-1}$  where

$$Q = \text{Uni}(\Phi_0(1)H) \quad \text{and} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Thomas Raujouan has proved this result in [15], Theorem 3, in the case  $\Phi_0(1, \lambda) = I_2$ . He proves that the limit axis is spanned by  $e_1$ . (In fact, he finds that the limit axis is  $-e_1$ , but this is because he has the opposite sign in the Sym-Bobenko formula. See Remark 3.) Then in Section 2 of [15], he explains, in the case  $r > s$ , how to extend his result to the case where  $\Phi_0(1, \lambda)$  is constant. We adapt his method to the case  $r < s$ .

**Lemma 2.** *Under the assumptions of Theorem 5, there exists a gauge  $G(z)$  and a change of variable  $h(z)$  with  $h(0) = 0$  such that  $\tilde{\xi}_t = (h^*\xi_t) \cdot G$  is a perturbed Delaunay potential (with residue  $A_t$ ) and  $\tilde{\Phi}_t = (h^*\Phi_t) \times G$  satisfies at  $t = 0$*

$$(27) \quad \tilde{\Phi}_0(1, \lambda) = QH^{-1} \in \Lambda SU(2).$$

Proof: we follow the method explained in Section 2 of [15]. We take the change of variable in the form

$$h(z) = \frac{z}{pz + q}$$

where  $p, q$  are complex numbers (independent of  $t$ ) to be determined, with  $q \neq 0$ . We consider the following gauge:

$$G(z) = \frac{1}{\sqrt{q(pz + q)}} \begin{pmatrix} pz + q & pz \\ 0 & q \end{pmatrix}.$$

It is chosen so that

$$(28) \quad G(0) = I_2 \quad \text{and} \quad dG = GA_0 \frac{dz}{z} - A_0 G \frac{dh}{h}.$$

(In fact, the gauge  $G$  is found as the only solution of Problem (28) which is upper triangular.) We have

$$\tilde{\xi}_t = (h^*\xi_t) \cdot G = G^{-1} \left( A_t(\lambda) \frac{dh}{h} + R_t(h, \lambda) dh \right) G + G^{-1} dG.$$

Since  $G(0) = I_2$ ,  $\tilde{\xi}_t$  has a simple pole at  $z = 0$  with residue  $A_t$ . Using Equation (28), we obtain at  $t = 0$ :

$$\tilde{\xi}_0 = G^{-1} A_0 \frac{dh}{h} G + G^{-1} dG = A_0 \frac{dz}{z}.$$

Hence  $\tilde{\xi}_t$  is a perturbed Delaunay potential. It remains to compute  $\tilde{\Phi}_0(1)$ . The matrix  $H$  diagonalises  $A_0$ :

$$A_0 = H \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} H^{-1}$$

Hence

$$\Phi_0(z) = \Phi_0(1) z^{A_0} = \Phi_0(1) H \begin{pmatrix} \frac{1}{\sqrt{z}} & 0 \\ 0 & \sqrt{z} \end{pmatrix} H^{-1}$$

$$\begin{aligned} \tilde{\Phi}_0(1) &= \Phi_0(h(1)) G(1) \\ &= \Phi_0(1) H \begin{pmatrix} \sqrt{p+q} & 0 \\ 0 & \frac{1}{\sqrt{p+q}} \end{pmatrix} H^{-1} \frac{1}{\sqrt{q(p+q)}} \begin{pmatrix} p+q & p \\ 0 & q \end{pmatrix} \\ &= \Phi_0(1) H \begin{pmatrix} \sqrt{q} & \frac{p}{\sqrt{q}} \\ 0 & \frac{1}{\sqrt{q}} \end{pmatrix} H^{-1} \end{aligned}$$

We decompose  $\Phi_0(1)H = QR$  with  $Q \in SU(2)$  and  $R = \begin{pmatrix} \rho & \mu \\ 0 & \frac{1}{\rho} \end{pmatrix}$ . Then

$$\tilde{\Phi}_0(1) = Q \begin{pmatrix} \rho & \mu \\ 0 & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} \sqrt{q} & \frac{p}{\sqrt{q}} \\ 0 & \frac{1}{\sqrt{q}} \end{pmatrix} H^{-1}$$

We take  $q = \frac{1}{\rho^2}$  and  $p = -\frac{\mu}{\rho}$  to cancel the two matrices in the middle and obtain Equation (27).  
 $\square$

We can now prove Theorem 5. Let

$$\widehat{\Phi}_t(z, \lambda) = HQ^{-1}\widetilde{\Phi}_t(z, \lambda) = HQ^{-1}\Phi_t(h(z), \lambda)G(z, \lambda)$$

Since  $\widehat{\Phi}_0(1, \lambda) = I_2$ , we can apply Theorem 3 in [15] which says that the resulting immersion  $\widehat{f}_t$  satisfies Points 1 and 2 of Theorem 5 and its limit axis is spanned by  $e_1$ . We have

$$f_t \circ h = QH^{-1}\widehat{f}_tHQ^{-1}$$

so  $f_t \circ h$  and  $\widehat{f}_t$  differ by a rotation and the limit axis of  $f_t$  is spanned by the vector  $QH^{-1}e_1HQ^{-1}$ .  
 Now

$$H^{-1}e_1H = -\frac{i}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = -i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = e_3.$$

$\square$

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