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GLUING DELAUNAY ENDS TO MINIMAL n -NOIDS USING THE DPW METHOD

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Abstract: we construct constant mean curvature surfaces in euclidean space by gluing n half Delaunay surfaces to a non-degenerate minimal n -noid, using the DPW method.

1. INTRODUCTION

In [3], Dorfmeister, Pedit and Wu have shown that surfaces with non-zero constant mean curvature (CMC for short) in euclidean space admit a Weierstrass-type representation, which means that they can be represented in terms of holomorphic data. This representation is now called the DPW method. In [18], we used the DPW method to construct CMC n -noids: genus zero, CMC surfaces with n Delaunay type ends. These n -noids can be described as a unit sphere with n half Delaunay surfaces with small necksizes attached at prescribed points. They had already been constructed by Kapouleas in [11] using PDE methods.

In the case $n = 3$, Alexandrov-embedded CMC trinoids have been classified by Große Brauckman, Kusner and Sullivan in [9]. In particular, equilateral CMC trinoids form a 1-parameter family, parametrized on an open interval. On one end, equilateral trinoids degenerate like the examples described above: they look like a sphere with 3 half Delaunay surfaces with small necksizes attached at the vertices of a spherical equilateral triangle. On the other end, equilateral trinoids limit, after suitable blow-up, to a minimal 3-noid: a genus zero minimal surface with 3 catenoidal ends (see Figure 1).

It seems natural to ask if one can generalize this observation and construct CMC n -noids by gluing half Delaunay surfaces with small necksizes to a minimal n -noid. This is indeed the case, and has been done by Mazzeo and Pacard in [14] using PDE methods. In this paper, we propose a quite simple and natural DPW potential to construct these examples. We prove:

Theorem 1. *Let $n \geq 3$ and let M_0 be a non-degenerate minimal n -noid. There exists a smooth family of CMC surfaces $(M_t)_{0 < |t| < \epsilon}$ with the following properties:*

- (1) M_t has genus zero and n Delaunay ends.
- (2) $\frac{1}{t}M_t$ converges to M_0 as $t \rightarrow 0$.
- (3) If M_0 is Alexandrov-embedded, all ends of M_t are of unduloid type if $t > 0$ and of nodoid type if $t < 0$. Moreover, M_t is Alexandrov-embedded if $t > 0$.

Non-degeneracy of a minimal n -noid will be defined in Section 2. The two surfaces M_t and M_{-t} are geometrically different: if M_t has an end of unduloid type then the corresponding end of M_{-t} is of nodoid type. See Proposition 6 for more details. Of course, a minimal n -noid is never embedded if $n \geq 3$ so the surfaces M_t are not embedded. Alexandrov-embedded

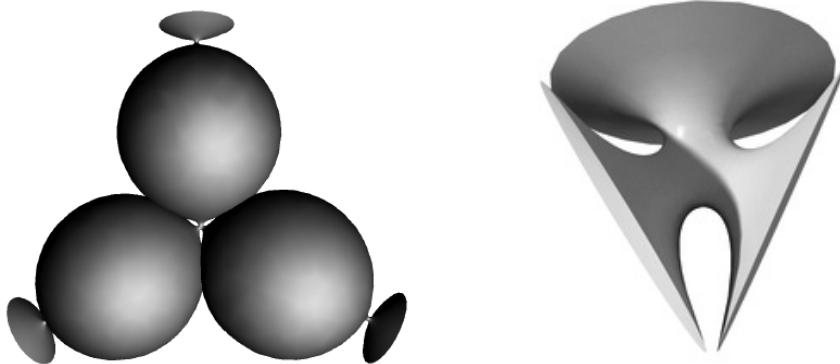


FIGURE 1. A CMC 3-noid (left, image by N. Schmitt [16]) and a minimal 3-noid (right). There is a tiny “copy” of the minimal 3-noid at the center of the CMC 3-noid.

minimal n -noids whose ends have coplanar axes have been classified by Cosin and Ros in [2], and Alexandrov-embedded CMC n -noids whose ends have coplanar axes have been classified by Große-Brauckmann, Kusner and Sullivan in [10].

As already said, these surfaces have already been constructed in [14]. Our motivation to construct them with the DPW method is to answer the following questions:

- (1) How can we produce a DPW potential from the Weierstrass data (g, ω) of the minimal n -noid M_0 ?
- (2) How can we prove, with the DPW method, that $\frac{1}{t}M_t$ converges to M_0 ?

The answer to Question 2 is Theorem 4 in Section 4, a general blow-up result in the context of the DPW method. In [19], we use the DPW method to construct higher genus CMC surfaces with small necks. Theorem 4 is used to ensure that the necks have asymptotically catenoidal shape.

Remark 1. *As the referee pointed out, the relation between minimal surfaces and CMC-1 surfaces in the DPW framework has already been investigated by Brander and Dorfmeister [1]. In that paper, the authors propose a DPW potential from the Weierstrass data of a minimal surface M . The Monodromy Problem is not addressed, however, so the resulting CMC-1 surfaces do not close, unless M is simply connected.*

I would like to thank the referee for his helpful comments and for providing the references [7] and [8].

2. NON-DEGENERATE MINIMAL n -NOIDS

A minimal n -noid is a complete, immersed minimal surface in \mathbb{R}^3 with genus zero and n catenoidal ends. Let M_0 be a minimal n -noid and (Σ, g, ω) its Weierstrass data. This means that M_0 is parametrized on Σ by the Weierstrass Representation formula:

$$(1) \quad \psi(z) = \operatorname{Re} \int_{z_0}^z \left(\frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega \right)$$

Without loss of generality, we can assume that $\Sigma = \mathbb{C} \cup \{\infty\} \setminus \{p_1, \dots, p_n\}$, where p_1, \dots, p_n are complex numbers and $g \neq 0, \infty$ at p_1, \dots, p_n (by rotating M_0 if necessary). Then ω needs a double pole at p_1, \dots, p_n so has $2n - 2$ zeros, counting multiplicity. Since ω needs a zero at each pole of g , with twice the multiplicity, it follows that g has $n - 1$ poles so has degree $n - 1$. Hence we may write

$$(2) \quad g = \frac{A(z)}{B(z)} \quad \text{and} \quad \omega = \frac{B(z)^2 dz}{\prod_{i=1}^n (z - p_i)^2}$$

where

$$A(z) = \sum_{i=1}^n a_i z^{n-i} \quad \text{and} \quad B(z) = \sum_{i=1}^n b_i z^{n-i}.$$

We are going to deform this Weierstrass data, so we see a_i, b_i and p_i for $1 \leq i \leq n$ as complex parameters. We denote by $\mathbf{x} \in \mathbb{C}^{3n}$ the vector of these parameters, and by \mathbf{x}_0 the value of the parameters corresponding to the minimal n -noid M_0 .

Let γ_i be the homology class of a small circle centered at p_i and define the following periods for $1 \leq i \leq n$ and $0 \leq k \leq 2$, depending on the parameter vector $\mathbf{x} \in \mathbb{C}^{3n}$:

$$P_{i,k}(\mathbf{x}) = \int_{\gamma_i} g^k \omega$$

$$P_i(\mathbf{x}) = (P_{i,0}(\mathbf{x}), P_{i,1}(\mathbf{x}), P_{i,2}(\mathbf{x})) \in \mathbb{C}^3$$

$$Q_i(\mathbf{x}) = \int_{\gamma_i} \left(\frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega \right) \in \mathbb{C}^3.$$

Then

$$Q_i(\mathbf{x}) = \left(\frac{1}{2}(P_{i,0}(\mathbf{x}) - P_{i,2}(\mathbf{x})), \frac{i}{2}(P_{i,0}(\mathbf{x}) + P_{i,2}(\mathbf{x})), P_{i,1}(\mathbf{x}) \right).$$

The components of $Q_i(\mathbf{x}_0)$ are imaginary because the Period Problem is solved for M_0 . This gives

$$(3) \quad P_{i,2}(\mathbf{x}_0) = \overline{P_{i,0}(\mathbf{x}_0)} \quad \text{and} \quad P_{i,1}(\mathbf{x}_0) \in i\mathbb{R}$$

Moreover, $\text{Im}(Q_i(\mathbf{x}_0)) = -\phi_i$ where ϕ_i is the flux vector of M_0 at the end p_i . By the Residue Theorem, we have for all \mathbf{x} in a neighborhood of \mathbf{x}_0 :

$$\sum_{i=1}^n P_i(\mathbf{x}) = 0$$

Let $P = (P_1, \dots, P_{n-1})$ and $Q = (Q_1, \dots, Q_{n-1})$.

Definition 1. M_0 is non-degenerate if the differential of P (or equivalently, Q) at \mathbf{x}_0 has complex rank $3n - 3$.

Remark 2. If $n \geq 3$, we may (using Möbius transformations of the sphere) fix the value of three points, say p_1, p_2, p_3 . Then "non-degenerate" means that the differential of P with respect to the remaining parameters is an isomorphism of \mathbb{C}^{3n-3} .

This notion is related to another standard notion of non-degeneracy:

Definition 2. M_0 is non-degenerate if its space of bounded Jacobi fields has (real) dimension 3.

Theorem 2. *If M_0 is non-degenerate in the sense of Definition 2, then M_0 is non-degenerate in the sense of Definition 1.*

Proof. Assume M_0 is non-degenerate in the sense of Definition 2. Then in a neighborhood of M_0 , the space \mathcal{M} of minimal n -noids (up to translation) is a smooth manifold of dimension $3n - 3$ by a standard application of the Implicit Function Theorem. Moreover, if we write $\phi_i \in \mathbb{R}^3$ for the flux vector at the i -th end, then the map $\phi = (\phi_1, \dots, \phi_n)$ provides a local diffeomorphism between \mathcal{M} and the space V of vectors $v = (v_1, \dots, v_n) \in (\mathbb{R}^3)^n$ such that $\sum_{i=1}^n v_i = 0$. (All this is proved in Section 4 of [2] in the case where all ends are coplanar. The argument goes through in the general case.) Hence given a vector $v \in V$, there exists a deformation M_t of M_0 such that $M_t \in \mathcal{M}$ and $\frac{d}{dt}\phi(M_t)|_{t=0} = v$. We may write the Weierstrass data of M_t as above and obtain a set of parameters $\mathbf{x}(t)$, depending smoothly on t , such that $\mathbf{x}(0) = \mathbf{x}_0$. Then $dQ(\mathbf{x}_0) \cdot \mathbf{x}'(0) = -iv$. Since Q is holomorphic, its differential is complex-linear so $dQ(\mathbf{x}_0)$ has complex rank equal to $\dim V = 3n - 3$. \square

If all ends of M_0 have coplanar axes, then M_0 is non-degenerate in the sense of Definition 2 by Proposition 2 in [2]. In particular, the (most symmetric) n -noids of Jorge-Meeks are non-degenerate. This implies that generic n -noids in the component of the Jorge-Meeks n -noid are non-degenerate.

3. BACKGROUND

In this section, we recall standard notations and results used in the DPW method. We work in the “untwisted” setting.

3.1. Loop groups. A loop is a smooth map from the unit circle $\mathbb{S}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ to a matrix group. The circle variable is denoted λ and called the spectral parameter. The unit disk is denoted \mathbb{D} . For $\rho > 1$, we denote \mathbb{D}_ρ the disk $|\lambda| < \rho$, $\mathbb{D}_\rho^* = \mathbb{D}_\rho \setminus \{0\}$ and \mathbb{A}_ρ the annulus $\frac{1}{\rho} < |\lambda| < \rho$.

- If G is a matrix Lie group (or Lie algebra), ΛG denotes the group (or algebra) of smooth maps $\Phi : \mathbb{S}^1 \rightarrow G$.
- $\Lambda_+ SL(2, \mathbb{C}) \subset \Lambda SL(2, \mathbb{C})$ is the subgroup of maps B which extend holomorphically to \mathbb{D} with $B(0)$ upper triangular.
- $\Lambda_+^{\mathbb{R}} SL(2, \mathbb{C}) \subset \Lambda_+ SL(2, \mathbb{C})$ is the subgroup of maps B such that $B(0)$ has positive entries on the diagonal.

Theorem 3 (Iwasawa decomposition). *The multiplication $\Lambda SU(2) \times \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C}) \rightarrow \Lambda SL(2, \mathbb{C})$ is a diffeomorphism. The unique splitting of an element $\Phi \in \Lambda SL(2, \mathbb{C})$ as $\Phi = FB$ with $F \in \Lambda SU(2)$ and $B \in \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})$ is called Iwasawa decomposition. F is called the unitary factor of Φ and denoted $\text{Uni}(\Phi)$. B is called the positive factor and denoted $\text{Pos}(\Phi)$.*

3.2. The matrix model of \mathbb{R}^3 . In the DPW method, one identifies \mathbb{R}^3 with the Lie algebra $\mathfrak{su}(2)$ by

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3 \longleftrightarrow X = -i \begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \in \mathfrak{su}(2).$$

We have $\det(X) = \|x\|^2$. The group $SU(2)$ acts as linear isometries on $\mathfrak{su}(2)$ by conjugation: $H \cdot X = HXH^{-1}$.

3.3. The DPW method. The input data for the DPW method is a quadruple $(\Sigma, \xi, z_0, \phi_0)$ where:

- Σ is a Riemann surface.
- $\xi = \xi(z, \lambda)$ is a $\Lambda\mathfrak{sl}(2, \mathbb{C})$ -valued holomorphic 1-form on Σ called the DPW potential. More precisely,

$$(4) \quad \xi = \begin{pmatrix} \alpha & \lambda^{-1}\beta \\ \gamma & -\alpha \end{pmatrix}$$

where $\alpha(z, \lambda)$, $\beta(z, \lambda)$, $\gamma(z, \lambda)$ are holomorphic 1-forms on Σ with respect to the z variable, and are holomorphic with respect to λ in the disk \mathbb{D}_ρ for some $\rho > 1$.

- $z_0 \in \Sigma$ is a base point.
- $\phi_0 \in \Lambda SL(2, \mathbb{C})$ is an initial condition.

Given this data, the DPW method is the following procedure.

- Let $\tilde{\Sigma}$ be the universal cover of Σ and $\tilde{z}_0 \in \tilde{\Sigma}$ be an arbitrary element in the fiber of z_0 . Solve the Cauchy Problem on $\tilde{\Sigma}$:

$$(5) \quad \begin{cases} d\Phi(z, \lambda) = \Phi(z, \lambda)\xi(z, \lambda) \\ \Phi(\tilde{z}_0, \lambda) = \phi_0(\lambda) \end{cases}$$

to obtain a solution $\Phi : \tilde{\Sigma} \rightarrow \Lambda SL(2, \mathbb{C})$.

- Compute the Iwasawa decomposition $(F(z, \cdot), B(z, \cdot))$ of $\Phi(z, \cdot)$.
- Define $f : \tilde{\Sigma} \rightarrow \mathfrak{su}(2) \sim \mathbb{R}^3$ by the Sym-Bobenko formula:

$$(6) \quad f(z) = -2i \frac{\partial F}{\partial \lambda}(z, 1) F(z, 1)^{-1} =: \text{Sym}(F(z, \cdot)).$$

Then f is a CMC-1 (branched) conformal immersion. f is regular at z (meaning unbranched) if and only if $\beta(z, 0) \neq 0$. Its Gauss map is given by

$$(7) \quad N(z) = -iF(z, 1) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} F(z, 1)^{-1} =: \text{Nor}(F(z, \cdot)).$$

The DPW method actually constructs a moving frame for f . The differential of f is given by

$$(8) \quad df(z) = 2i B_{11}(z, 0)^2 F(z, 1) \begin{pmatrix} 0 & \beta(z, 0) \\ \bar{\beta}(z, 0) & 0 \end{pmatrix} F(z, 1)^{-1}.$$

Equation (8) can also be obtained by differentiation of the Sym-Bobenko formula (6).

Remark 3. In [18], I have opposite signs in Equations (6) and (7). This is unfortunate because it makes the basis (f_x, f_y, N) negatively oriented. Equation (6) is the right formula, which one obtains by untwisting the standard Sym-Bobenko formula in the twisted case. See [13] or [17].

3.4. The Monodromy Problem. Assume that Σ is not simply connected so its universal cover $\tilde{\Sigma}$ is not trivial. Let $\text{Deck}(\tilde{\Sigma}/\Sigma)$ be the group of fiber-preserving diffeomorphisms of $\tilde{\Sigma}$. Let Φ be the solution of the Cauchy Problem (5). For $\gamma \in \text{Deck}(\tilde{\Sigma}/\Sigma)$, let

$$\mathcal{M}_\gamma(\Phi)(\lambda) = \Phi(\gamma(z), \lambda)\Phi(z, \lambda)^{-1}$$

be the monodromy of Φ with respect to γ (which is independent of $z \in \tilde{\Sigma}$). The standard condition which ensures that the immersion f descends to a well defined immersion on Σ is the following system of equations, called the Monodromy Problem.

$$(9) \quad \forall \gamma \in \text{Deck}(\tilde{\Sigma}/\Sigma) \quad \begin{cases} \mathcal{M}_\gamma(\Phi) \in \Lambda SU(2) & (i) \\ \mathcal{M}_\gamma(\Phi)(1) = \pm I_2 & (ii) \\ \frac{\partial \mathcal{M}_\gamma(\Phi)}{\partial \lambda}(1) = 0 & (iii) \end{cases}$$

One can identify $\text{Deck}(\tilde{\Sigma}/\Sigma)$ with the fundamental group $\pi_1(\Sigma, z_0)$ (see for example Theorem 5.6 in [5]), so we will in general see γ as an element of $\pi_1(\Sigma, z_0)$. Under this identification, the monodromy of Φ with respect to $\gamma \in \pi_1(\Sigma, z_0)$ is given by

$$\mathcal{M}_\gamma(\Phi)(\lambda) = \Phi(\tilde{\gamma}(1), \lambda) \Phi(\tilde{\gamma}(0), \lambda)^{-1}$$

where $\tilde{\gamma} : [0, 1] \rightarrow \tilde{\Sigma}$ is the lift of γ such that $\tilde{\gamma}(0) = \tilde{z}_0$.

3.5. Gauging.

Definition 3. *A gauge on Σ is a map $G : \Sigma \rightarrow \Lambda_+ SL(2, \mathbb{C})$ such that $G(z, \lambda)$ depends holomorphically on $z \in \Sigma$ and $\lambda \in \mathbb{D}_\rho$ for some $\rho > 1$.*

Let Φ be a solution of $d\Phi = \Phi\xi$ and G be a gauge. Let $\hat{\Phi} = \Phi \times G$. Then $\hat{\Phi}$ and Φ define the same immersion f . This is called ‘‘gauging’’. The gauged potential is

$$\hat{\xi} = \hat{\Phi}^{-1} d\hat{\Phi} = G^{-1} \xi G + G^{-1} dG$$

and will be denoted $\xi \cdot G$, the dot denoting the action of the gauge group on the potential.

3.6. Functional spaces. We need to introduce a functional space for functions on the unit circle. We need that space to be a Banach algebra, and functions in that space should extend holomorphically to a neighborhood of the unit circle. The following choice is natural. We decompose a smooth function $f : \mathbb{S}^1 \rightarrow \mathbb{C}$ in Fourier series

$$f(\lambda) = \sum_{i \in \mathbb{Z}} f_i \lambda^i$$

Fix some $\rho > 1$ and define

$$\|f\| = \sum_{i \in \mathbb{Z}} |f_i| \rho^{|i|}$$

Let \mathcal{W}_ρ be the space of functions f with finite norm. This is a Banach algebra, owing to the fact that the weight $\rho^{|i|}$ is submultiplicative (see Section 4 in [8]). Functions in \mathcal{W}_ρ extend holomorphically to the annulus \mathbb{A}_ρ .

We define $\mathcal{W}_\rho^{>0}$, $\mathcal{W}_\rho^{>0}$, $\mathcal{W}_\rho^{<0}$ and $\mathcal{W}_\rho^{<0}$ as the subspaces of functions f such that $f_i = 0$ for $i < 0$, $i \leq 0$, $i > 0$ and $i \geq 0$, respectively. Functions in $\mathcal{W}_\rho^{>0}$ extend holomorphically to the disk \mathbb{D}_ρ and satisfy $|f(\lambda)| \leq \|f\|$ for all $\lambda \in \mathbb{D}_\rho$. We write $\mathcal{W}^0 \sim \mathbb{C}$ for the subspace of constant functions, so we have a direct sum $\mathcal{W}_\rho = \mathcal{W}_\rho^{<0} \oplus \mathcal{W}^0 \oplus \mathcal{W}_\rho^{>0}$. (The Banach algebra \mathcal{W}_ρ is said to be decomposable, see [7] page 70.) A function f will be decomposed as $f = f^- + f^0 + f^+$ with $(f^-, f^0, f^+) \in \mathcal{W}_\rho^{<0} \times \mathcal{W}^0 \times \mathcal{W}_\rho^{>0}$.

We define the star operator by

$$f^*(\lambda) = \overline{f(1/\bar{\lambda})} = \sum_{i \in \mathbb{Z}} \overline{f_{-i}} \lambda^i$$

The involution $f \mapsto f^*$ exchanges $\mathcal{W}_\rho^{\geq 0}$ and $\mathcal{W}_\rho^{\leq 0}$. We have $\lambda^* = \lambda^{-1}$ and $c^* = \bar{c}$ if c is a constant. A function f is real on the unit circle if and only if $f = f^*$. We extend the star involution to loops by $M^*(\lambda) = \overline{M(1/\bar{\lambda})}^T$, so a loop F is unitary if and only if $F^*F = I_2$.

If L is a loop group, we denote $L_\rho \subset L$ the subgroup of loops whose entries are in \mathcal{W}_ρ . The loop groups $\Lambda SL(2, \mathbb{C})_\rho$, $\Lambda SU(2)_\rho$ and $\Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$ are Banach Lie groups, moreover:

Proposition 1. *Iwasawa decomposition restricts to an analytic diffeomorphism between the Banach Lie groups $\Lambda SL(2, \mathbb{C})_\rho$ and $\Lambda SU(2)_\rho \times \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$.*

Proof: let $\Phi \in \Lambda SL(2, \mathbb{C})_\rho$ and let (F, B) be its Iwasawa decomposition. We want to prove that F and B have entries in \mathcal{W}_ρ . Since F is unitary, we have $\Phi^*\Phi = B^*B$. Now B^* has only non-positive powers of λ , so (B^*, B) is a Birkhoff-type decomposition of $M = \Phi^*\Phi$. Since $M \in \Lambda SL(2, \mathbb{C})_\rho$, it is known that both factors of its Birkhoff decomposition have entries in \mathcal{W}_ρ , owing to the fact that \mathcal{W}_ρ is decomposable (see Theorem 1.4 in [7]). So $B \in \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$ and $F \in \Lambda SU(2)_\rho$ follows.

Let $\Lambda \mathfrak{sl}(2, \mathbb{C})_\rho$, $\Lambda \mathfrak{su}(2)_\rho$ and $\Lambda_+^{\mathbb{R}} \mathfrak{sl}(2, \mathbb{C})_\rho$ be the Banach Lie algebras of respectively $\Lambda SL(2, \mathbb{C})_\rho$, $\Lambda SU(2)_\rho$ and $\Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$. The following decomposition is standard (the factors can be written explicitly in term of Fourier coefficients):

$$\Lambda \mathfrak{sl}(2, \mathbb{C})_\rho = \Lambda \mathfrak{su}(2)_\rho \oplus \Lambda_+^{\mathbb{R}} \mathfrak{sl}(2, \mathbb{C})_\rho.$$

By the inverse mapping theorem, the multiplication $\Lambda SU(2)_\rho \times \Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho \rightarrow \Lambda SL(2, \mathbb{C})_\rho$ is an analytic local diffeomorphism in a neighborhood of (I_2, I_2) , and in a neighborhood of any element (F, B) using left multiplication by F and right multiplication by B . Since we already know the multiplication is bijective, it is a global diffeomorphism. \square

4. A BLOW-UP RESULT

In this section, we consider a one-parameter family of DPW potential ξ_t with solution Φ_t and assume that $\Phi_0(z, \lambda)$ is independent of λ . Then its unitary part $F_0(z, \lambda)$ is independent of λ . The Sym Bobenko formula yields that $f_0 \equiv 0$, so the family f_t collapses to the origin as $t = 0$. The following theorem says that the blow-up $\frac{1}{t}f_t$ converges to a minimal surface whose Weierstrass data is explicitly computed.

Theorem 4. *Let Σ be a Riemann surface, $(\xi_t)_{t \in I}$ a family of DPW potentials on Σ and $(\Phi_t)_{t \in I}$ a family of solutions of $d\Phi_t = \Phi_t \xi_t$ on the universal cover $\tilde{\Sigma}$ of Σ , where $I \subset \mathbb{R}$ is a neighborhood of 0. Fix a base point $z_0 \in \tilde{\Sigma}$. Assume that*

- (1) $(t, z) \mapsto \xi_t(z, \cdot)$ and $t \mapsto \Phi_t(z_0, \cdot)$ are C^1 maps into $\Lambda \mathfrak{sl}(2, \mathbb{C})_\rho$ and $\Lambda SL(2, \mathbb{C})_\rho$, respectively.
- (2) For all $t \in I$, Φ_t solves the Monodromy Problem (9).
- (3) $\Phi_0(z, \lambda)$ is independent of λ :

$$\Phi_0(z, \lambda) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

Let $f_t = \text{Sym}(\text{Uni}(\Phi_t)) : \Sigma \rightarrow \mathbb{R}^3$ be the CMC-1 immersion given by the DPW method. Then

$$\lim_{t \rightarrow 0} \frac{1}{t} f_t(z) = \psi(z)$$

where $\psi : \Sigma \rightarrow \mathbb{R}^3$ is a (possibly branched) minimal immersion with the following Weierstrass data:

$$g(z) = \frac{-a(z)}{c(z)} \quad \text{and} \quad \omega = 4c(z)^2 \frac{\partial \xi_{t;12}^{(-1)}}{\partial t} \Big|_{t=0}.$$

The limit is for the uniform C^1 convergence on compact subsets of Σ .

Here $\xi_{t;12}^{(-1)}$ denotes the coefficient of λ^{-1} in the upper right entry of ξ_t . In case $\omega = 0$, the minimal immersion degenerates into a point and ψ is constant.

Proof: by standard ODE theory, $(t, z) \mapsto \Phi_t(z, \cdot)$ is a C^1 map into $\Lambda SL(2, \mathbb{C})_\rho$. Let (F_t, B_t) be the Iwasawa decomposition of Φ_t . By Proposition 1, $(t, z) \mapsto F_t(z, \cdot)$ and $(t, z) \mapsto B_t(z, \cdot)$ are real analytic maps into $\Lambda SU(2)_\rho$ and $\Lambda_+^{\mathbb{R}} SL(2, \mathbb{C})_\rho$, respectively. At $t = 0$, Φ_0 is constant with respect to λ , so its Iwasawa decomposition is the standard QR decomposition:

$$F_0 = \frac{1}{\sqrt{|a|^2 + |c|^2}} \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \quad B_0 = \frac{1}{\sqrt{|a|^2 + |c|^2}} \begin{pmatrix} |a|^2 + |c|^2 & \bar{a}b + \bar{c}d \\ 0 & 1 \end{pmatrix}.$$

The Sym-Bobenko formula (6) yields $f_0 = 0$. Let $\mu_t = B_{t;11}^0$ and $\beta_t = \xi_{t;12}^{(-1)}$. By Equation (8), we have

$$df_t(z) = 2i \mu_t(z)^2 F_t(z, 1) \begin{pmatrix} 0 & \beta_t(z) \\ \bar{\beta}_t(z) & 0 \end{pmatrix} F_t(z, 1)^{-1}.$$

Hence $(t, z) \mapsto df_t(z)$ is a C^1 map. At $t = 0$, ξ_0 is constant with respect to λ , so $\beta_0 = 0$. Define $\tilde{f}_t(z) = \frac{1}{t} f_t(z)$ for $t \neq 0$. Then $d\tilde{f}_t(z)$ extends at $t = 0$, as a continuous function of (t, z) , by

$$\begin{aligned} d\tilde{f}_0 &= \frac{d}{dt} df_t \Big|_{t=0} = 2i \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \begin{pmatrix} 0 & \beta' \\ \bar{\beta}' & 0 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ -c & a \end{pmatrix} \\ &= 2i \begin{pmatrix} -ac\beta' - \bar{a}c\bar{\beta}' & a^2\beta' - \bar{c}^2\bar{\beta}' \\ \bar{a}^2\bar{\beta}' - c^2\beta' & ac\beta' + \bar{a}c\bar{\beta}' \end{pmatrix} \end{aligned}$$

where $\beta' = \frac{d}{dt} \beta_t \Big|_{t=0}$. In euclidean coordinates, this gives

$$d\tilde{f}_0 = 4 \text{Re} \left[\frac{1}{2}(c^2 - a^2)\beta', \frac{i}{2}(c^2 + a^2)\beta', -ac\beta' \right].$$

Writing $g = \frac{-a}{c}$ and $\omega = 4c^2\beta'$, we obtain

$$\tilde{f}_0(z) = \tilde{f}_0(z_0) + \text{Re} \int_{z_0}^z \left[\frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega \right]$$

and we see that \tilde{f}_0 is a minimal surface with Weierstrass data (g, ω) . The last statement of Theorem 4 comes from the fact that $d\tilde{f}_t$ converges uniformly to $d\tilde{f}_0$ on compact subsets of Σ . \square

4.1. **Example.** As an example, we consider the family of Delaunay surfaces given by the following DPW potential in \mathbb{C}^* :

$$\xi_t(z, \lambda) = \begin{pmatrix} 0 & \lambda^{-1}r + s \\ \lambda r + s & 0 \end{pmatrix} \frac{dz}{z} \quad \text{with} \quad \begin{cases} r + s = \frac{1}{2} \\ rs = t \\ r < s \end{cases}$$

with initial condition $\Phi_t(1) = I_2$. As $t \rightarrow 0$, we have $(r, s) \rightarrow (0, \frac{1}{2})$. We have

$$\begin{aligned} \Phi_0(z, \lambda) &= \exp \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \log z = \frac{1}{2\sqrt{z}} \begin{pmatrix} z+1 & z-1 \\ z-1 & z+1 \end{pmatrix} \\ \frac{\partial \xi_t}{\partial t} \Big|_{t=0} &= \begin{pmatrix} 0 & 2\lambda^{-1} \\ 2\lambda & 0 \end{pmatrix} \frac{dz}{z}. \end{aligned}$$

Theorem 4 applies and gives

$$g(z) = \frac{1+z}{1-z} \quad \text{and} \quad \omega(z) = 4 \left(\frac{z-1}{2\sqrt{z}} \right)^2 \frac{2dz}{z} = 2 \left(\frac{z-1}{z} \right)^2 dz.$$

This is the Weierstrass data of a horizontal catenoid of waist-radius 4 and axis Ox_1 , with $x_1 \rightarrow +\infty$ at the end $z = 0$.

5. THE DPW POTENTIAL

We now start the proof of Theorem 1. Let (g, ω) be the Weierstrass data of the given minimal n -noid M_0 , written as in Section 2. We introduce $3n$ λ -dependent parameters a_i, b_i and p_i for $1 \leq i \leq n$ in the functional space $\mathcal{W}_\rho^{\geq 0}$. The vector of these parameters is denoted $\mathbf{x} \in (\mathcal{W}_\rho^{\geq 0})^{3n}$. The parameter \mathbf{x} is in a neighborhood of a (constant) central value $\mathbf{x}_0 \in (\mathcal{W}^0)^{3n}$ which correspond to the Weierstrass data of M_0 , written as in Section 2. We define

$$\begin{aligned} A_{\mathbf{x}}(z, \lambda) &= \sum_{i=1}^n a_i(\lambda) z^{n-i} \\ B_{\mathbf{x}}(z, \lambda) &= \sum_{i=1}^n b_i(\lambda) z^{n-i} \\ (10) \quad g_{\mathbf{x}}(z, \lambda) &= \frac{A_{\mathbf{x}}(z, \lambda)}{B_{\mathbf{x}}(z, \lambda)} \end{aligned}$$

$$(11) \quad \omega_{\mathbf{x}}(z, \lambda) = \frac{B_{\mathbf{x}}(z, \lambda)^2 dz}{\prod_{i=1}^n (z - p_i(\lambda))^2}.$$

For t in a neighborhood of 0 in \mathbb{R} , we consider the following DPW potential:

$$\xi_{t, \mathbf{x}}(z, \lambda) = \begin{pmatrix} 0 & \frac{1}{4}t(\lambda-1)^2\lambda^{-1}\omega_{\mathbf{x}}(z, \lambda) \\ dg_{\mathbf{x}}(z, \lambda) & 0 \end{pmatrix}.$$

We fix a base point z_0 , away from the poles of g and ω , and we take the initial condition

$$\phi_0(\lambda) = \begin{pmatrix} g_{\mathbf{x}}(z_0, \lambda) & 1 \\ -1 & 0 \end{pmatrix}.$$

These choices are motivated by the following observations:

(1) At $t = 0$, we have

$$\xi_{0,\mathbf{x}}(z, \lambda) = \begin{pmatrix} 0 & 0 \\ dg_{\mathbf{x}}(z, \lambda) & 0 \end{pmatrix}.$$

The solution of the Cauchy Problem (5) is given by

$$(12) \quad \Phi_{0,\mathbf{x}}(z, \lambda) = \begin{pmatrix} g_{\mathbf{x}}(z, \lambda) & 1 \\ -1 & 0 \end{pmatrix}$$

which is well-defined, so the Monodromy Problem (9) is solved at $t = 0$.

(2) The same conclusion holds if $\lambda = 1$ instead of $t = 0$. In particular, Items (ii) and (iii) of the Monodromy Problem (9) are automatically solved.

(3) At $\mathbf{x} = \mathbf{x}_0$, we have $g_{\mathbf{x}_0} = g$ so $\Phi_{0,\mathbf{x}_0}(z, \lambda)$ is independent of λ . Moreover,

$$\frac{\partial \xi_{t,\mathbf{x}_0;12}^{(-1)}}{\partial t} \Big|_{t=0} = \frac{\omega}{4}.$$

Provided the Monodromy Problem is solved for all t in a neighborhood of 0, Theorem 4 applies and the limit minimal surface has Weierstrass data (g, ω) so is the minimal n -noid M_0 , up to translation (see details in Section 7.1).

Remark 4. *The potential $\xi_{t,\mathbf{x}}$ is inspired from the potential used in [18] to construct CMC n -noids by perturbation of a sphere. In fact, in the case $dg_{\mathbf{x}} = dz$, the two potentials are dual to each other. (See Section 3.2.8 of [19] for the definition of duality in the DPW method.)*

5.1. Regularity. Our potential $\xi_{t,\mathbf{x}}$ has poles at the zeros of $B_{\mathbf{x}}$ and the points p_1, \dots, p_n . (At ∞ , we have $\omega_{\mathbf{x}} \sim b_1^2 z^{-2} dz$ which is holomorphic.) We want the zeros of $B_{\mathbf{x}}$ to be apparent singularities, so we require the potential to be gauge-equivalent to a regular potential in a neighborhood of these points. Consider the gauge

$$G_{\mathbf{x}}(z, \lambda) = \begin{pmatrix} g_{\mathbf{x}}(z, \lambda)^{-1} & -1 \\ 0 & g_{\mathbf{x}}(z, \lambda) \end{pmatrix}$$

The gauged potential is

$$\widehat{\xi}_{t,\mathbf{x}} := \xi_{t,\mathbf{x}} \cdot G_{\mathbf{x}} = \begin{pmatrix} 0 & \frac{1}{4}t(\lambda - 1)^2 \lambda^{-1} g_{\mathbf{x}}^2 \omega_{\mathbf{x}} \\ g_{\mathbf{x}}^{-2} dg_{\mathbf{x}} & 0 \end{pmatrix}.$$

We have

$$g_{\mathbf{x}}^{-2} dg_{\mathbf{x}} = \frac{A'_{\mathbf{x}} B_{\mathbf{x}} - A_{\mathbf{x}} B'_{\mathbf{x}}}{A_{\mathbf{x}}^2} \quad \text{and} \quad g_{\mathbf{x}}^2 \omega_{\mathbf{x}} = \frac{A_{\mathbf{x}}^2 dz}{\prod_{i=1}^n (z - p_i)^2}.$$

Let ζ be a zero of $B_{\mathbf{x}_0}$ (recall that $B_{\mathbf{x}_0}$ does not depend on λ). Then $A_{\mathbf{x}_0}(\zeta) \neq 0$. By continuity, there exists a neighborhood U of ζ such that for $z \in U$, $\lambda \in \mathbb{D}_\rho$ and \mathbf{x} close enough to \mathbf{x}_0 , $A_{\mathbf{x}}(z, \lambda) \neq 0$. So $\widehat{\xi}_{t,\mathbf{x}}$ is holomorphic in $U \times \mathbb{D}_\rho^*$ and moreover, $\widehat{\xi}_{t,\mathbf{x};12}^{(-1)} \neq 0$. This ensures that the immersion extends analytically to U and is unbranched in U .

6. THE MONODROMY PROBLEM

6.1. Formulation of the problem. For $i \in [1, n]$, we denote $p_{i,0}$ the central value of the parameter p_i (so $p_{1,0}, \dots, p_{n,0}$ are the ends of the minimal n -noid M_0). We consider the following λ -independent domain on the Riemann sphere:

$$(13) \quad \Omega = \{z \in \mathbb{C} : \forall i \in [1, n], |z - p_{i,0}| > \varepsilon\} \cup \{\infty\}$$

where $\varepsilon > 0$ is a fixed, small enough number such that the disks $D(p_{i,0}, 8\varepsilon)$ for $1 \leq i \leq n$ are disjoint. As in [18], we first construct a family of immersions f_t on Ω . Then we extend f_t to an n -punctured sphere in Proposition 4.

Let $\tilde{\Omega}$ be the universal cover of Ω and $\Phi_{t,\mathbf{x}}(z, \lambda)$ be the solution of the following Cauchy Problem on $\tilde{\Omega}$:

$$(14) \quad \begin{cases} d\Phi_{t,\mathbf{x}}(z, \lambda) = \Phi_{t,\mathbf{x}}(z, \lambda)\xi_{t,\mathbf{x}}(z, \lambda) \\ \Phi_{t,\mathbf{x}}(\tilde{z}_0, \lambda) = \phi_0 \end{cases}$$

We denote $\gamma_1, \dots, \gamma_{n-1}$ a set of generators of the fundamental group $\pi_1(\Omega, z_0)$, with γ_i encircling the point $p_{i,0}$. We may assume that each γ_i is represented by a fixed curve avoiding the poles of $\xi_{t,\mathbf{x}}$. Let

$$M_i(t, \mathbf{x}) = \mathcal{M}_{\gamma_i}(\Phi_{t,\mathbf{x}})$$

be the monodromy of $\Phi_{t,\mathbf{x}}$ along γ_i . By Equation (12), we have $M_i(0, \mathbf{x}) = I_2$. Recall that the matrix exponential is a local diffeomorphism from a neighborhood of 0 in the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ (respectively $\mathfrak{su}(2)$) to a neighborhood of I_2 in $SL(2, \mathbb{C})$ (respectively $SU(2)$). The inverse diffeomorphism is denoted \log . For $t \neq 0$ small enough and $\lambda \in \mathbb{D}_\rho \setminus \{1\}$, we define as in [18]

$$\tilde{M}_i(t, \mathbf{x})(\lambda) = \frac{4\lambda}{t(\lambda - 1)^2} \log M_i(t, \mathbf{x})(\lambda).$$

Proposition 2. (1) $\tilde{M}_i(t, \mathbf{x})(\lambda)$ extends smoothly at $t = 0$ and $\lambda = 1$, and each entry $\tilde{M}_{i,kl}$ is a smooth map from a neighborhood of $(0, \mathbf{x}_0)$ in $\mathbb{R} \times (\mathcal{W}_\rho^{\geq 0})^3$ to \mathcal{W}_ρ .

(2) At $t = 0$, we have

$$(15) \quad \tilde{M}_i(0, \mathbf{x})(\lambda) = \begin{pmatrix} \mathcal{P}_{i,1}(\mathbf{x}) & \mathcal{P}_{i,2}(\mathbf{x}) \\ -\mathcal{P}_{i,0}(\mathbf{x}) & -\mathcal{P}_{i,1}(\mathbf{x}) \end{pmatrix} \quad \text{where } \mathcal{P}_{i,k}(\mathbf{x}) = \int_{\gamma_i} g_{\mathbf{x}}^k \omega_{\mathbf{x}}.$$

(3) The Monodromy Problem (9) is equivalent to

$$(16) \quad \tilde{M}_i(t, \mathbf{x}) \in \Lambda \mathfrak{su}(2) \quad \text{for } 1 \leq i \leq n - 1.$$

Proof: we follow the proof of Proposition 1 in [18]. We first consider the case where the parameter $\mathbf{x} = (a_i, b_i, p_i)_{1 \leq i \leq n}$ is constant with respect to λ , so $\mathbf{x} \in \mathbb{C}^{3n}$. For (μ, \mathbf{x}) in a neighborhood of $(0, \mathbf{x}_0)$ in $\mathbb{C} \times \mathbb{C}^{3n}$, we define

$$\hat{\xi}_{\mu,\mathbf{x}}(z) = \begin{pmatrix} 0 & \mu \omega_{\mathbf{x}}(z) \\ dg_{\mathbf{x}}(z) & 0 \end{pmatrix}$$

where $\omega_{\mathbf{x}}$ and $g_{\mathbf{x}}$ are defined by Equations (10) and (11), except that a_i, b_i, p_i are constant complex numbers. Let $\hat{\Phi}_{\mu,\mathbf{x}}$ be the solution of the Cauchy Problem $d\hat{\Phi}_{\mu,\mathbf{x}} = \hat{\Phi}_{\mu,\mathbf{x}}\hat{\xi}_{\mu,\mathbf{x}}$ in $\tilde{\Omega}$ with initial condition $\hat{\Phi}_{\mu,\mathbf{x}}(\tilde{z}_0) = \phi_0$. Let $N_i(\mu, \mathbf{x}) = \mathcal{M}_{\gamma_i}(\hat{\Phi}_{\mu,\mathbf{x}})$. By standard ODE theory, each entry of N_i is a holomorphic function of (μ, \mathbf{x}) . At $\mu = 0$, $\hat{\Phi}_{0,\mathbf{x}}$ is given by Equation (12), so in particular $N_i(0, \mathbf{x}) = I_2$. Hence

$$\tilde{N}_i(\mu, \mathbf{x}) := \frac{1}{\mu} \log N_i(\mu, \mathbf{x})$$

extends holomorphically at $\mu = 0$ with value $\widetilde{N}_i(0, \mathbf{x}) = \frac{\partial N_i}{\partial \mu}(0, \mathbf{x})$. By Proposition 8 in Appendix A of [18] (the same formula appeared before on page 39 of [12]):

$$\frac{\partial N_i}{\partial \mu}(0, \mathbf{x}) = \int_{\gamma_i} \widehat{\Phi}_{0, \mathbf{x}} \frac{\partial \widehat{\xi}_{\mu, \mathbf{x}}}{\partial \mu} \Big|_{\mu=0} \widehat{\Phi}_{0, \mathbf{x}}^{-1}.$$

Hence

$$(17) \quad \widetilde{N}_i(0, \mathbf{x}) = \int_{\gamma_i} \begin{pmatrix} g_{\mathbf{x}} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega_{\mathbf{x}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & g_{\mathbf{x}} \end{pmatrix} = \int_{\gamma_i} \begin{pmatrix} g_{\mathbf{x}} \omega_{\mathbf{x}} & g_{\mathbf{x}}^2 \omega_{\mathbf{x}} \\ -\omega_{\mathbf{x}} & -g_{\mathbf{x}} \omega_{\mathbf{x}} \end{pmatrix}.$$

For (t, \mathbf{x}) in a neighborhood of $(0, \mathbf{x}_0)$ in $\mathbb{R} \times (\mathcal{W}_{\rho}^{\geq 0})^{3n}$, we have

$$\xi_{t, \mathbf{x}}(z, \lambda) = \widehat{\xi}_{\mu(t, \lambda), \mathbf{x}(\lambda)}(z) \quad \text{with} \quad \mu(t, \lambda) = \frac{t(\lambda - 1)^2}{4\lambda}.$$

Hence

$$M_i(t, \mathbf{x})(\lambda) = N_i(\mu(t, \lambda), \mathbf{x}(\lambda)) \quad \text{and} \quad \widetilde{M}_i(t, \mathbf{x})(\lambda) = \widetilde{N}_i(\mu(t, \lambda), \mathbf{x}(\lambda)).$$

By substitution (see Proposition 9 in Appendix B of [18]), each entry of \widetilde{M}_i is a smooth map from a neighborhood of $(0, \mathbf{x}_0)$ in $\mathbb{R} \times (\mathcal{W}_{\rho}^{\geq 0})^3$ to \mathcal{W}_{ρ} . Moreover, $\widetilde{M}_i(0, \mathbf{x})$ is given by Equation (17). The fact that \widetilde{M}_i extends holomorphically at $\lambda = 1$ implies that Points (ii) and (iii) of Problem (9) are automatically satisfied. Since $\lambda^{-1}(\lambda - 1)^2 \in \mathbb{R}$ for $\lambda \in \mathbb{S}^1$, Equation (i) of Problem (9) is equivalent to Equation (16). \square

6.2. Solution of the monodromy problem. Without loss of generality, we may (using a Möbius transformation of the sphere) fix the value of p_1 , p_2 and p_3 . We still denote $\mathbf{x} \in (\mathcal{W}_{\rho}^{\geq 0})^{3n-3}$ the vector of the remaining parameters.

Proposition 3. *Assume that the given minimal n -noid is non-degenerate. For t in a neighborhood of 0, there exists a smooth function $\mathbf{x}(t) \in (\mathcal{W}_{\rho}^{\geq 0})^{3n-3}$ such that $\widetilde{M}_i(t, \mathbf{x}(t), \cdot) \in \Lambda \mathfrak{su}(2)$ for $1 \leq i \leq n - 1$. Moreover, $\mathbf{x}(0) = \mathbf{x}_0$.*

Proof: recalling the definition of $P_{i,k}$ in Section 2 and $\mathcal{P}_{i,k}$ in Equation (15), we have

$$\mathcal{P}_{i,k}(\mathbf{x})(\lambda) = P_{i,k}(\mathbf{x}(\lambda)).$$

Hence $\mathcal{P}_{i,k}$ is a smooth map from a neighborhood of \mathbf{x}_0 in $(\mathcal{W}_{\rho}^{\geq 0})^{3n-3}$ to $\mathcal{W}_{\rho}^{\geq 0}$. Moreover, since \mathbf{x}_0 is constant, we have for $X \in (\mathcal{W}_{\rho}^{\geq 0})^{3n-3}$:

$$(18) \quad (d\mathcal{P}_{i,k}(\mathbf{x}_0)X)(\lambda) = dP_{i,k}(\mathbf{x}_0)X(\lambda).$$

Let $\mathcal{P}_i = (\mathcal{P}_{i,0}, \mathcal{P}_{i,1}, \mathcal{P}_{i,2})$ and $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{n-1})$. By the non-degeneracy hypothesis and Remark 2, $d\mathcal{P}(\mathbf{x}_0)$ is an automorphism of \mathbb{C}^{3n-3} , so $d\mathcal{P}(\mathbf{x}_0)$ is an automorphism of $(\mathcal{W}_{\rho}^{\geq 0})^{3n-3}$ and restricts to an automorphism of $(\mathcal{W}_{\rho}^{\geq 0})^{3n-3}$.

We define the following smooth maps with value in \mathcal{W}_{ρ} (the star operator is defined in Section 3.6)

$$\begin{aligned} \mathcal{F}_i(t, \mathbf{x}) &= \widetilde{M}_{i,11}(t, \mathbf{x}) + \widetilde{M}_{i,11}(t, \mathbf{x})^* \\ \mathcal{G}_i(t, \mathbf{x}) &= \widetilde{M}_{i,12}(t, \mathbf{x}) + \widetilde{M}_{i,21}(t, \mathbf{x})^* \end{aligned}$$

Problem (16) is equivalent to $\mathcal{F}_i = \mathcal{G}_i = 0$. Actually, by definition, $\mathcal{F}_i = \mathcal{F}_i^*$, so Problem (16) is equivalent to

$$\mathcal{F}_i(t, \mathbf{x})^+ = 0, \quad \operatorname{Re}(\mathcal{F}_i(t, \mathbf{x})^0) = 0 \quad \text{and} \quad \mathcal{G}_i(t, \mathbf{x}) = 0 \quad \text{for } 1 \leq i \leq n-1.$$

At $t = 0$, we have by Equation (15):

$$\mathcal{F}_i(0, \mathbf{x}) = \mathcal{P}_{i,1}(\mathbf{x}) + \mathcal{P}_{i,1}(\mathbf{x})^*$$

$$\mathcal{G}_i(0, \mathbf{x}) = \mathcal{P}_{i,2}(\mathbf{x}) - \mathcal{P}_{i,0}(\mathbf{x})^*$$

Equation (3) tells us precisely that at the central value, we have $\mathcal{F}_i(0, \mathbf{x}_0) = 0$ and $\mathcal{G}_i(0, \mathbf{x}_0) = 0$. We have for $X \in (\mathcal{W}_\rho^{\geq 0})^{3n-3}$:

$$d\mathcal{F}_i(0, \mathbf{x}_0)X = d\mathcal{P}_{i,1}(x_0)X + (d\mathcal{P}_{i,1}(\mathbf{x}_0)X)^*$$

$$d\mathcal{G}_i(0, \mathbf{x}_0)X = d\mathcal{P}_{i,2}(x_0)X - (d\mathcal{P}_{i,0}(\mathbf{x}_0)X)^*$$

Projecting on $\mathcal{W}_\rho^{>0}$ and $\mathcal{W}_\rho^{<0}$ we obtain:

$$(d\mathcal{F}_i(0, \mathbf{x}_0)X)^+ = d\mathcal{P}_{i,1}(\mathbf{x}_0)X^+$$

$$(d\mathcal{G}_i(0, \mathbf{x}_0)X)^+ = d\mathcal{P}_{i,2}(\mathbf{x}_0)X^+$$

$$(d\mathcal{G}_i(0, \mathbf{x}_0)X)^- = -(d\mathcal{P}_{i,0}(\mathbf{x}_0)X^+)^*$$

$$(d\mathcal{G}_i(0, \mathbf{x}_0)X)^{-*} = -d\mathcal{P}_{i,0}(\mathbf{x}_0)X^+.$$

Hence the operator

$$[d\mathcal{F}_i(0, \mathbf{x}_0)^+, d\mathcal{G}_i(0, \mathbf{x}_0)^+, d\mathcal{G}_i(0, \mathbf{x}_0)^{-*}]_{1 \leq i \leq n-1}$$

only depends on X^+ and is an automorphism of $(\mathcal{W}_\rho^{>0})^{3n-3}$ because $d\mathcal{P}(\mathbf{x}_0)$ is. Projecting on \mathcal{W}^0 we obtain:

$$(d\mathcal{F}_i(0, \mathbf{x}_0)X)^0 = 2 \operatorname{Re} (d\mathcal{P}_{i,1}(\mathbf{x}_0)X^0)$$

$$(d\mathcal{G}_i(0, \mathbf{x}_0)X)^0 = d\mathcal{P}_{i,2}(\mathbf{x}_0)X^0 - \overline{d\mathcal{P}_{i,0}(\mathbf{x}_0)X^0}.$$

Hence the \mathbb{R} -linear operator

$$[\operatorname{Re}(d\mathcal{F}_i(0, \mathbf{x}_0)^0), d\mathcal{G}_i(0, \mathbf{x}_0)^0]_{1 \leq i \leq n-1}$$

only depends on X^0 and is surjective from \mathbb{C}^{3n-3} to $(\mathbb{R} \times \mathbb{C})^{3n-3}$. This implies that the differential of the map $(\mathcal{F}_i^+, \mathcal{G}_i^+, \mathcal{G}_i^{-*}, \operatorname{Re}(\mathcal{F}_i^0), \mathcal{G}_i^0)_{1 \leq i \leq n-1}$ is surjective from $(\mathcal{W}_\rho^{\geq 0})^{3n-3}$ to $((\mathcal{W}_\rho^{>0})^3 \times \mathbb{R} \times \mathbb{C})^{n-1}$. Proposition 3 follows from the Implicit Function Theorem. \square

Remark 5. *The kernel of the differential has real dimension $3n - 3$ so we have $3n - 3$ free real parameters. These parameters correspond to deformations of the flux vectors of the minimal n -noid.*

7. GEOMETRY OF THE IMMERSION

From now on, we assume that $\mathbf{x}(t)$ is given by Proposition 3. We write $a_{i,t}$, $b_{i,t}$ and $p_{i,t}$ for the value of the corresponding parameters. (These parameters are in the space $\mathcal{W}_\rho^{\geq 0}$ so are functions of λ .) For ease of notation, we write g_t , ω_t , ξ_t and Φ_t for $g_{\mathbf{x}(t)}$, $\omega_{\mathbf{x}(t)}$, $\xi_{t,\mathbf{x}(t)}$ and $\Phi_{t,\mathbf{x}(t)}$, respectively. Let $F_t = \text{Uni}(\Phi_t)$. Since the Monodromy Problem is solved, the Sym-Bobenko formula (6) defines a CMC-1 immersion $f_t : \Omega \rightarrow \mathbb{R}^3$, where Ω is the (fixed) domain defined by Equation (13).

Proposition 4. *The immersion f_t extends analytically to*

$$\Sigma_t := \mathbb{C} \cup \{\infty\} \setminus \{p_{1,t}(0), \dots, p_{n,t}(0)\}$$

where $p_{i,t}(0)$ is the value of $p_{i,t}$ at $\lambda = 0$.

We omit the proof which is exactly the same as the proof of Point 1 of Proposition 4 in [18]. It relies on Theorem 3 in [18] which allows for λ -dependent changes of variables in the DPW method.

7.1. Convergence to the minimal n -noid.

Proposition 5. $\lim_{t \rightarrow 0} \frac{1}{t} f_t = \psi$ where ψ is (up to translation) the conformal parametrization of the minimal n -noid given by Equation (1). The limit is the uniform C^1 convergence on compact subsets of $\Sigma_0 = \mathbb{C} \cup \{\infty\} \setminus \{p_{1,0}, \dots, p_{n,0}\}$.

Proof: at $t = 0$, we have $g_0 = g$ and $\omega_0 = \omega$. By Equation (12) and definition of the potential, we have

$$\Phi_0(z, \lambda) = \begin{pmatrix} g(z) & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial \xi_{t;12}^{(-1)}}{\partial t} \Big|_{t=0} = \frac{\omega}{4}.$$

By Theorem 4, $\frac{1}{t} f_t$ converges to a minimal surface with Weierstrass data (g, ω) on compact subsets of Σ_0 minus the poles of g . In a neighborhood of the poles of g , we use the gauge introduced in Section 5.1. With the notations of this section and writing $\widehat{\Phi}_t = \Phi_t G_{\mathbf{x}(t)}$, we have

$$\widehat{\Phi}_0(z, \lambda) = \begin{pmatrix} 1 & 0 \\ -g(z)^{-1} & 1 \end{pmatrix} \quad \text{and} \quad \frac{\partial \widehat{\xi}_{t;12}^{(-1)}}{\partial t} \Big|_{t=0} = \frac{g^2 \omega}{4}.$$

By Theorem 4 again, $\frac{1}{t} f_t$ converges to a minimal surface with Weierstrass data (g, ω) in a neighborhood of the poles of g . The two limit minimal surfaces are of course the same, since they coincide in a neighborhood of z_0 . \square

7.2. Delaunay ends. In this section, we prove that the immersion f_t has Delaunay ends. Delaunay ends in the DPW method have been studied in [4, 13]. Following [17], we gauge our potential to a perturbation of the standard Delaunay potential and we use the results in [13].

We denote N_0 the Gauss map of the minimal n -noid M_0 . For $1 \leq i \leq n$, we denote \mathcal{C}_i the catenoid to which M_0 is asymptotic at $p_{i,0}$ and $\tau_i > 0$ the necksize of \mathcal{C}_i .

Definition 4. *We say that N_0 points to the inside in a neighborhood of $p_{i,0}$ if it points to the component of $\mathbb{R}^3 \setminus \mathcal{C}_i$ containing the axis of \mathcal{C}_i .*

Proposition 6. *For $1 \leq i \leq n$ and $t \neq 0$:*

(1) The immersion f_t has a Delaunay end at $p_{i,t}$. If we denote $w_{i,t}$ its weight then

$$\lim_{t \rightarrow 0} t^{-1} w_{i,t} = \pm 2\pi\tau_i$$

where the sign is $+$ if N_0 points to the inside in a neighborhood of $p_{i,0}$ and $-$ otherwise.

(2) Its axis converges as $t \rightarrow 0$ to the half-line through the origin directed by the vector $N_0(p_{i,0})$.

(3) If N_0 points to the inside in a neighborhood of $p_{i,0}$, there exists a uniform $\varepsilon > 0$ such that for $t > 0$ small enough, $f_t(D^*(p_{i,0}, \varepsilon))$ is embedded.

Proof: in a neighborhood of the puncture $p_{i,t}$, we may use $w = g_t(z) - g_t(p_{i,t})$ as a local coordinate. Note that $p_{i,t} \in \mathcal{W}_\rho^{\geq 0}$ so, as a function of λ , extends holomorphically to \mathbb{D}_ρ . Thus the coordinate w depends holomorphically on $\lambda \in \mathbb{D}_\rho$. This is not a problem by Theorem 3 in [18]. Consider the gauge

$$G(w) = \begin{pmatrix} \frac{k}{\sqrt{w}} & \frac{-1}{2k\sqrt{w}} \\ 0 & \frac{\sqrt{w}}{k} \end{pmatrix}.$$

Here we can take $k = 1$, but later on we will take another value of k so we do the computation for general values of $k \neq 0$. The gauged potential is

$$\widehat{\xi}_t := \xi_t \cdot G = \begin{pmatrix} 0 & \frac{dw}{4k^2 w} + \frac{wt(\lambda-1)^2}{4k^2 \lambda} \omega_t \\ \frac{k^2 dw}{w} & 0 \end{pmatrix}$$

Since ω_t has a double pole at $p_{i,t}$, $\widehat{\xi}_t$ has a simple pole at $w = 0$ with residue

$$A_{i,t}(\lambda) = \begin{pmatrix} 0 & \frac{1}{4k^2} + \frac{t(\lambda-1)^2}{4k^2 \lambda} \alpha_{i,t}(\lambda) \\ k^2 & 0 \end{pmatrix}$$

where

$$(19) \quad \alpha_{i,t} = \text{Res}_{p_{i,t}}(w\omega_t) = \text{Res}_{p_{i,t}}(g_t(z) - g_t(p_{i,t}))\omega_t.$$

Claim 1. For t small enough, $\alpha_{i,t}$ is a real constant (i.e. independent of λ , possibly depending on t).

Proof: the proof is similar to the proof of Point 2 of Proposition 4 in [18]. We use the standard theory of Fuchsian systems. Fix $t \neq 0$ and $\lambda \in \mathbb{S}^1 \setminus \{1\}$. Assume that $\alpha_{i,t}(\lambda) \neq 0$. Let $\widehat{\Phi}_t = \Phi_t G$. The eigenvalues of $A_{i,t}$ are $\pm \Lambda_{i,t}$ with

$$\Lambda_{i,t}(\lambda)^2 = \frac{1}{4} + \frac{t(\lambda-1)^2}{4\lambda} \alpha_{i,t}(\lambda).$$

Provided $t \neq 0$ is small enough, $\Lambda_{i,t} \notin \mathbb{Z}/2$ so the system is non resonant and $\widehat{\Phi}_t$ has the following standard $z^A P$ form in the universal cover of $D(0, \varepsilon)^*$:

$$\widehat{\Phi}_t(w, \lambda) = V(\lambda) \exp(A_{i,t}(\lambda) \log w) P(w, \lambda)$$

where $P(w, \lambda)$ descends to a well defined holomorphic function of $w \in D(0, \varepsilon)$ with $P(0, \lambda) = I_2$. Consequently, its monodromy is

$$\mathcal{M}_{\gamma_i}(\widehat{\Phi}_t) = V(\lambda) \exp(2\pi i A_{i,t}) V(\lambda)^{-1}$$

with eigenvalues $\exp(\pm 2\pi i \Lambda_{i,t}(\lambda))$. Since the Monodromy Problem is solved, the eigenvalues are unitary complex numbers, so $\Lambda_{i,t}(\lambda) \in \mathbb{R}$ which implies that $\alpha_{i,t}(\lambda) \in \mathbb{R}$. This of course remains true if $\alpha_{i,t}(\lambda) = 0$. Hence $\alpha_{i,t}$ is real on $\mathbb{S}^1 \setminus \{1\}$. Since all the parameters involved in the definition of ω_t are in $\mathcal{W}_\rho^{\geq 0}$, $\alpha_{i,t}$ is holomorphic in the unit disk. Hence it is constant. \square

Returning to the proof of Proposition 6, let $(r, s) \in \mathbb{R}^2$ be the solution of

$$(20) \quad \begin{cases} rs = \frac{1}{4}t\alpha_{i,t} \\ r + s = \frac{1}{2} \\ r < s \end{cases}$$

Since $r < s$, $\sqrt{r\lambda + s}$ is well defined and does not vanish for $\lambda \in \mathbb{D}$. We take $k = \sqrt{r\lambda + s}$ in the definition of the gauge G . Using Equation (20), we have:

$$(r\lambda^{-1} + s)(r\lambda + s) = \frac{1}{4} + rs(\lambda - 1)^2\lambda^{-1} = \frac{1}{4} + \frac{1}{4}t(\lambda - 1)^2\lambda^{-1}\alpha_{i,t}.$$

So the residue of $\widehat{\xi}_t$ becomes

$$A_{i,t} = \begin{pmatrix} 0 & \frac{1}{r\lambda + s} \left(\frac{1}{4} + \frac{t(\lambda - 1)^2}{4\lambda} \alpha_{i,t} \right) \\ r\lambda + s & 0 \end{pmatrix} = \begin{pmatrix} 0 & r\lambda^{-1} + s \\ r\lambda + s & 0 \end{pmatrix}$$

which is the residue of the standard Delaunay potential. By [13], the immersion f_t has a Delaunay end at $p_{i,t}$ of weight $w_{i,t} = 8\pi rs = 2\pi t\alpha_{i,t}$. It remains to relate $\alpha_{i,0}$ to the logarithmic growth τ_i . For ease of notation, let us write $p_i = p_{i,0}$. Assume that N_0 points to the inside in a neighborhood of p_i . The flux of M_0 along γ_i is equal to

$$\phi_i = 2\pi\tau_i N_0(p_i) = 2\pi \frac{\tau_i}{|g(p_i)|^2 + 1} (2\operatorname{Re}(g(p_i)), 2\operatorname{Im}(g(p_i)), |g(p_i)|^2 - 1)$$

On the other hand, we have seen in Section 2 that the flux is equal to

$$\phi_i = -2\pi \operatorname{Res}_{p_i} \left(\frac{1}{2}(1 - g^2)\omega, \frac{1}{2}(1 + g^2)\omega, g\omega \right)$$

Comparing these two expressions for ϕ_i , we obtain

$$\operatorname{Res}_{p_i}(g\omega) = -\tau_i \frac{|g(p_i)|^2 - 1}{|g(p_i)|^2 + 1} \quad \text{and} \quad \operatorname{Res}_{p_i}\omega = -2\tau_i \frac{\overline{g(p_i)}}{|g(p_i)|^2 + 1}$$

Using Equation (19), this gives

$$\alpha_{i,0} = \operatorname{Res}_{p_i}(g\omega) - g(p_i) \operatorname{Res}_{p_i}\omega = \tau_i$$

If N_0 points to the outside in a neighborhood of p_i , then $\phi_i = -2\pi\tau_i N_0(p_i)$, so the same computation gives $\alpha_{i,0} = -\tau_i$. This proves Point 1 of Proposition 6.

To prove Point 2, we use Theorem 5 in Appendix A. We need to compute $\widehat{\Phi}_0$ at $w = 1$. At $t = 0$, we have $k = \frac{1}{\sqrt{2}}$ so

$$G(1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}.$$

At $t = 0$, we have $w = g(z) - g(p_i)$, so $w = 1 \Leftrightarrow g(z) = g(p_i) + 1$. Using Equation (12),

$$\widehat{\Phi}_0(1) = \frac{1}{\sqrt{2}} \begin{pmatrix} g(p_i) + 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} g(p_i) + 1 & -g(p_i) + 1 \\ -1 & 1 \end{pmatrix}.$$

Fix $0 < \alpha < 1$. By Theorem 5 (using $t\alpha_{i,t}$ as the time parameter), there exists $\varepsilon > 0$, $T > 0$ and c such that for $0 < |t| < T$:

$$\|f_t(z) - f_{i,t}^{\mathcal{D}}(z)\| \leq c|t||z - p_{i,t}|^\alpha \quad \text{in } D^*(p_{i,t}, \varepsilon)$$

where $f_{i,t}^{\mathcal{D}} : \mathbb{C} \setminus \{p_{i,t}\} \rightarrow \mathbb{R}^3$ is a Delaunay immersion. We compute the limit axis of $f_{i,t}^{\mathcal{D}}$ using Point 3 of Theorem 5:

$$\begin{aligned} \widehat{\Phi}_0(1)H &= \begin{pmatrix} g(p_i) & 1 \\ -1 & 0 \end{pmatrix} = \Phi_0(p_i). \\ Q &= F_0(p_i) \\ Qe_3Q^{-1} &= \text{Nor}(F_0(p_i)) = N_0(p_i). \end{aligned}$$

This proves Point 2 of Proposition 6. If N_0 points to the inside in a neighborhood of $p_{i,0}$, then for $t > 0$, $t\alpha_{i,t} > 0$ so Point 3 follows from Point 2 of Theorem 5. \square

7.3. Alexandrov-embeddedness. We recall from [2, 9] the definition of Alexandrov-embeddedness in the non-compact case:

Definition 5. *A surface M of finite topology is Alexandrov-embedded if M is properly immersed, if each end of M is embedded, and if there exists a compact 3-manifold \overline{W} with boundary $\partial\overline{W} = \overline{S}$, n points $q_1, \dots, q_n \in \overline{S}$ and a proper immersion $F : W = \overline{W} \setminus \{q_1, \dots, q_n\} \rightarrow \mathbb{R}^3$ whose restriction to $S = \overline{S} \setminus \{q_1, \dots, q_n\}$ parametrizes M .*

Lemma 1. *Let M be an Alexandrov-embedded minimal surface with n catenoidal ends. With the notations of Definition 5, we equip W with the flat metric induced by F , so F is a local isometry, and we denote N the inside normal to S . Then there exists a flat 3-manifold W' containing W , a local isometry $F' : W' \rightarrow \mathbb{R}^3$ extending F and $r > 0$ such that the tubular neighborhood $\text{Tab}_r S$ is embedded in W' . In other words, the map $(x, s) \mapsto \exp_x(sN(x))$ from $S \times (-r, r)$ to W' is well defined and is a diffeomorphism onto its image.*

Proof: since M has catenoidal ends, there exists $r > 0$ such that the inside tubular neighborhood map

$$g : S \times (0, r) \rightarrow W, \quad g(x, s) = \exp_x(sN(x))$$

is a diffeomorphism onto its image. Since F is a local isometry, we have

$$(21) \quad F(g(x, s)) = F(x) + s dF(x)N(x) \quad \text{for } (x, s) \in S \times (0, r).$$

We define W' as the disjoint union $(S \times (-r, r)) \sqcup W$ where we identify $(x, s) \in S \times (0, r)$ with its image $g(x, s) \in W$. We define $F' : W' \rightarrow \mathbb{R}^3$ by $F' = F$ in W and

$$F'(x, s) = F(x) + s dF(x)N(x) \quad \text{for } (x, s) \in S \times (-r, r).$$

The map F' is well defined by Equation (21). We equip $S \times (-r, r)$ with the flat metric induced by the local diffeomorphism F' , which extends the metric already defined on $S \times (0, r)$ by identification with W . Since

$$dF'(x, 0)(X, T) = dF(x)X + TdF(x)N(x)$$

the metric restricted to $S \times \{0\}$ is the product metric, so the normal to $S \times \{0\}$ in $S \times (-r, r)$ is $N(x, 0) = (0, 1)$. Since F' is a local isometry, we have for $(x, s) \in S \times (-r, r)$

$$F' \left(\exp_{(x,0)} sN(x, 0) \right) = F'(x, 0) + s dF'(x, 0)(0, 1) = F(x) + s dF(x)N(x) = F'(x, s)$$

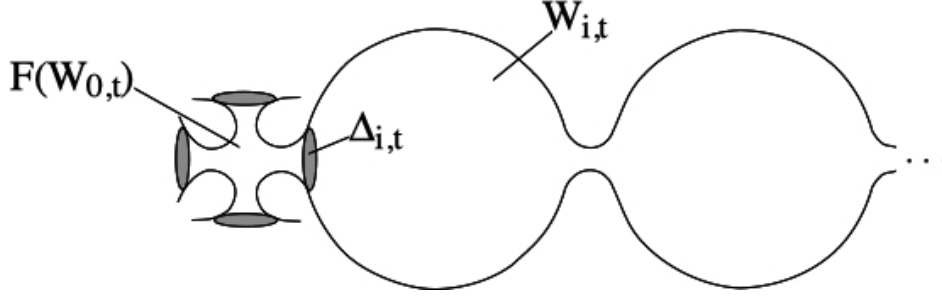


FIGURE 2. Decomposition of a 4-noid into pieces. Only one Delaunay end is represented, and $F(W_{0,t})$ is represented as an embedded domain for clarity, but in general it will be immersed.

Hence $\exp_{(x,0)} sN(x,0) = (x,s)$ so $\text{Tub}_r(S \times \{0\})$ is embedded in $S \times (-r,r)$. \square

We now return to the proof of Theorem 1. We orient the minimal n -noid M_0 so that its Gauss map points to the inside in a neighborhood of p_1 . For $0 < |t| < \epsilon$, we denote M_t the image of the immersion f_t that we have constructed.

Proposition 7. *If M_0 is Alexandrov embedded, then for $t > 0$ small enough, M_t is Alexandrov embedded.*

Proof: our strategy is to cut M_t by suitable planes into pieces which are either close to M_0 or Delaunay surfaces (see Figure 2). Then we prove that each piece, together with flat disks in the cutting planes, is the boundary of a domain, using the Jordan Brouwer Theorem.

Since M_0 is Alexandrov embedded, N_0 points to the inside in a neighborhood of each end, so M_t has embedded ends by Proposition 6. Let $\epsilon > 0$ be the number given by our application of Theorem 5 in Section 7.2 and $f_{i,t}^D : \mathbb{C} \setminus \{p_{i,t}\} \rightarrow \mathbb{R}^3$ be the Delaunay immersion which approximates f_t in $D^*(p_{i,t}, \epsilon)$. Recall that $f_t(D^*(p_{i,t}, \epsilon))$ is embedded. Let $\tilde{f}_t = \frac{1}{t} f_t$. By Proposition 5, \tilde{f}_t converges to ψ on compact subsets of Σ_0 , where $\psi : \Sigma_0 \rightarrow \mathbb{R}^3$ is a parametrization of M_0 . Since M_0 has catenoidal ends, we may assume (taking ϵ smaller if necessary) that $\psi(D^*(p_{i,0}, \epsilon))$ is embedded and $N_0 \neq N_0(p_{i,0})$ in $D^*(p_{i,0}, \epsilon)$.

Let $h_i : \mathbb{R}^3 \rightarrow \mathbb{R}$ be the height function in the direction $N_0(p_{i,0})$, defined by

$$h_i(x) = \langle x, N_0(p_{i,0}) \rangle.$$

We shall cut M_t by the plane $h_i = \delta$ where $\delta > 0$ is a fixed, large enough number such that for $1 \leq i \leq n$,

$$\delta > \max_{C(p_{i,0}, \epsilon)} h_i \circ \psi.$$

Since $\lim_{z \rightarrow p_{i,0}} h_i \circ \psi(z) = +\infty$, we may fix a positive, small enough $\epsilon' < \epsilon$ such that

$$\min_{C(p_{i,0}, \epsilon')} h_i \circ \psi > \delta.$$

Let $\mathcal{A}_{i,t}$ be the annulus defined by $\epsilon' \leq |z - p_{i,t}| \leq \epsilon$. Since $N_0 \neq N_0(p_{i,0})$ in $\mathcal{A}_{i,0}$,

$$\min_{\mathcal{A}_{i,0}} \|N_0(z) - N_0(p_{i,0})\| > 0.$$

For $t > 0$ small enough:

$$(22) \quad \max_{C(p_{i,t}, \varepsilon)} h_i \circ \tilde{f}_t < \delta$$

$$(23) \quad \min_{C(p_{i,t}, \varepsilon')} h_i \circ \tilde{f}_t > \delta$$

$$(24) \quad \min_{\mathcal{A}_{i,t}} \|N_t(z) - N_0(p_{i,0})\| > 0.$$

Hence the function $h_i \circ \tilde{f}_t$ has no critical point in the annulus $\mathcal{A}_{i,t}$. So $h_i \circ \tilde{f}_t = \delta$ defines a regular closed curve $\gamma_{i,t}$ in $\mathcal{A}_{i,t}$. At $t = 0$, $h_i \circ \psi = \delta$ is a single curve around $p_{i,0}$, so $\gamma_{i,t}$ has only one component and is not contractible in $\mathcal{A}_{i,t}$. Let $D_{i,t} \subset \mathbb{C}$ be the topological disk bounded by $\gamma_{i,t}$ and $D_{i,t}^* = D_{i,t} \setminus \{p_{i,t}\}$. Let $\Delta_{i,t}$ be the closed topological disk bounded by $\tilde{f}_t(\gamma_{i,t})$ in the plane defined by $h_i(x) = \delta$.

Claim 2. For $t > 0$ small enough, $\tilde{f}_t(D_{i,t}^*) \cap \Delta_{i,t} = \emptyset$.

Proof: of course, $h_i \circ \tilde{f}_t > \delta$ in $D_{i,t}^* \cap \mathcal{A}_{i,t}$. What we need to prove is that $\tilde{f}_t(D^*(p_{i,t}, \varepsilon'))$ does not intersect $\Delta_{i,t}$. We do this by comparison with the Delaunay surface. Let $\Pi_i = N_0(p_{i,0})^\perp$ and $\pi_i = \mathbb{R}^3 \rightarrow \Pi_i$ be the orthogonal projection. Since ψ has a catenoidal end at $p_{i,0}$, $\psi(\mathcal{A}_{i,t})$ is a graph over an annulus in the plane Π_i , with inside boundary circle $\pi_i \circ \psi(C(p_{i,t}, \varepsilon))$ and outside boundary circle $\pi_i \circ \psi(C(p_{i,t}, \varepsilon'))$. Moreover, N_0 is close to $N_0(p_{i,t})$. Since \tilde{f}_t is C^1 close to ψ in $\mathcal{A}_{i,t}$, for $t > 0$ small enough, $\tilde{f}_t(\mathcal{A}_{i,t})$ is a graph over an annulus in the plane Π_i , with inside boundary circle $\pi_i \circ \tilde{f}_t(C(p_{i,t}, \varepsilon))$ and outside boundary circle $\pi_i \circ \tilde{f}_t(C(p_{i,t}, \varepsilon'))$.

Now we go back to the original scale. Since f_t is C^1 close to $f_{i,t}^D$ in $D^*(p_{i,t}, \varepsilon)$, we conclude that $f_{i,t}^D(\mathcal{A}_{i,t})$ is a graph over an annulus in the plane Π_i , with inside boundary circle $\pi_i \circ f_{i,t}^D(C(p_{i,t}, \varepsilon))$ and outside boundary circle $\pi_i \circ f_{i,t}^D(C(p_{i,t}, \varepsilon'))$. Then from the geometry of Delaunay surfaces, there exists a curve $\gamma_{i,t,0}$ in $D^*(p_{i,t}, \varepsilon')$ such that $f_{i,t}^D(\gamma_{i,t,0})$ is a closed curve in the plane $h_i = \frac{1}{2}$. Let $D_{i,t,0}$ be the disk bounded by $\gamma_{i,t,0}$ and $\mathcal{A}_{i,t,0}$ be the closed annulus bounded by $\gamma_{i,t}$ and $\gamma_{i,t,0}$. Then $h_i \circ f_{i,t}^D > \frac{1}{2}$ in $D_{i,t,0}$ and $f_{i,t}^D(\mathcal{A}_{i,t,0})$ is a graph over an annulus in the plane Π_i . Since f_t is C^1 close to $f_{i,t}^D$ in $D^*(p_{i,t}, \varepsilon)$, we conclude that $h_i \circ f_t > \frac{1}{4}$ in $D_{i,t,0}^*$ and $f_t(\mathcal{A}_{i,t,0})$ is a graph over an annulus in the plane Π_i .

Back to the scale $\frac{1}{t}$, $\tilde{f}_t(D_{i,t} \cap \mathcal{A}_{i,t,0})$ is a graph over an annulus in the plane Π_i whose inside boundary circle is $\pi_i \circ \tilde{f}_t(\gamma_{i,t}) = \pi_i(\partial\Delta_{i,t})$, so $\tilde{f}_t(D_{i,t} \cap \mathcal{A}_{i,t,0}) \cap \Delta_{i,t} = \emptyset$. Moreover, $h_i \circ \tilde{f}_t > \frac{1}{4t} \gg \delta$ in $D_{i,t,0}^*$ so $\tilde{f}_t(D_{i,t,0}^*) \cap \Delta_{i,t} = \emptyset$. \square

Claim 3. For $t > 0$ small enough, $\tilde{f}_t(D_{i,t}^*) \cup \Delta_{i,t}$ is the boundary of a cylindrically bounded domain $W_{i,t} \subset \mathbb{R}^3$.

Proof: since f_t is close to $f_{i,t}^D$ in $D_{i,t}^*$, we can find an increasing diverging sequence $(R_{t,k})_{k \in \mathbb{N}}$ such that $f_t(D_{i,t}^*)$ intersects the plane $h_i = R_{t,k}$ transversally along a closed curve $f_t(\gamma_{i,t,k})$. (Explicitly, we can take $R_{t,k} = \frac{1}{2} + k h_i(T_t)$ where $T_t \in \mathbb{R}^3$ is the period of the Delaunay surface $f_{i,t}^D$.) Let $\mathcal{A}_{i,t,k}$ be the annulus bounded by $\gamma_{i,t}$ and $\gamma_{i,t,k}$. Let $\Delta_{i,t,k}$ be the closed disk bounded

by $\tilde{f}_t(\gamma_{i,t,k})$ in the plane $h_i = t^{-1}R_{t,k}$. Then $\tilde{f}_t(\mathcal{A}_{i,t,k}) \cup \Delta_{i,t} \cup \Delta_{i,t,k}$ is topologically a sphere: the image of \mathbb{S}^2 by an injective continuous map. By the Jordan Brouwer Theorem, it is the boundary of a bounded domain $W_{i,t,k}$. Clearly, $W_{i,t,k} \subset W_{i,t,k+1}$. We take $W_{i,t} = \bigcup_{k \in \mathbb{N}} W_{i,t,k}$. \square

Let $\Omega_t = \mathbb{C} \cup \{\infty\} \setminus (D_{1,t} \cup \dots \cup D_{n,t})$. Let W' be the flat 3-manifold given by Lemma 1 and denote $F : W' \rightarrow \mathbb{R}^3$ its developing map (instead of F'). (Here W' is an open manifold, meaning not a manifold-with-boundary.)

Claim 4. *For $t > 0$ small enough, there exists a compact domain $W_{0,t}$ in W' such that*

$$F(\partial W_{0,t}) = \tilde{f}_t(\Omega_t) \cup \Delta_{1,t} \cup \dots \cup \Delta_{n,t}.$$

Proof: by definition, ψ lifts to a diffeomorphism $\widehat{\psi} : \Sigma_0 \rightarrow S \subset W'$ such that $F \circ \widehat{\psi} = \psi$. Since M_0 has catenoidal ends, there exists domains V_1, \dots, V_n in W' such that for $1 \leq i \leq n$:

- $F : V_i \rightarrow F(V_i) \subset \mathbb{R}^3$ is a diffeomorphism,
- V_i is foliated by flat disks on which $h_i \circ F$ is constant (in particular, $h_i \circ F$ is constant on ∂V_i),
- $\widehat{\psi}(D^*(p_{i,0}, \varepsilon)) \subset V_i$ (which might require taking a smaller $\varepsilon > 0$),
- $h_i < \delta$ on $V_i \cap \widehat{\psi}(\Sigma_0 \setminus \bigcup_{i=1}^n D(p_{i,0}, \varepsilon))$ (which might require taking a larger δ).

Let $r > 0$ be the radius of the embedded tubular neighborhood of S in W' constructed in Lemma 1. For $t > 0$ small enough, $\|\tilde{f}_t - \psi\| < r$ in $\overline{\Omega}_t$, so \tilde{f}_t lifts to $\widehat{f}_t : \overline{\Omega}_t \rightarrow W'$ such that $F \circ \widehat{f}_t = \tilde{f}_t$. (Explicitly, $\widehat{f}_t(z) = \exp_{\widehat{\psi}(z)}(\tilde{f}_t(z) - \psi(z))$.) From the properties of V_i and the convergence of \widehat{f}_t to $\widehat{\psi}$ on compact subsets of Σ_0 , we have for $t > 0$ small enough

$$(25) \quad \widehat{f}_t(\overline{\Omega}_t \cap D(p_{i,0}, \varepsilon)) \subset V_i$$

$$(26) \quad h_i < \delta \quad \text{on } V_i \cap \widehat{f}_t(\Sigma_0 \setminus \bigcup_{i=1}^n D(p_{i,0}, \varepsilon)).$$

By Equation (25), $\widehat{f}_t(\gamma_{i,t}) \subset V_i$ so $\Delta_{i,t}$ lifts to a closed disk $\widehat{\Delta}_{i,t} \subset V_i$ such that $\partial \widehat{\Delta}_{i,t} = \widehat{f}_t(\gamma_{i,t})$ and $F(\widehat{\Delta}_{i,t}) = \Delta_{i,t}$. Since F is a diffeomorphism on V_i , $\widehat{f}_t(\Omega_t \cap D(p_{i,t}, \varepsilon))$ is disjoint from $\widehat{\Delta}_{i,t}$. By (26), $\widehat{f}_t(\Omega_t \setminus \bigcup_{i=1}^n D(p_{i,t}, \varepsilon))$ is disjoint from $\widehat{\Delta}_{i,t}$. Hence $\widehat{f}_t(\Omega_t) \cap \widehat{\Delta}_{i,t} = \emptyset$. Then $\widehat{f}_t(\Omega_t) \cup \widehat{\Delta}_{1,t} \cup \dots \cup \widehat{\Delta}_{n,t}$ is a topological sphere in W' . Since M_0 has genus zero, W' is homeomorphic to \mathbb{R}^3 . By the Jordan Brouwer Theorem, $\widehat{f}_t(\Omega_t) \cup \widehat{\Delta}_{1,t} \cup \dots \cup \widehat{\Delta}_{n,t}$ is the boundary of a compact domain $W_{0,t} \subset W'$. \square

Returning to the proof of Proposition 7, let W_t be the abstract 3-manifold with boundary obtained as the disjoint union $\overline{W}_{0,t} \sqcup \overline{W}_{1,t} \sqcup \dots \sqcup \overline{W}_{n,t}$, identifying $\overline{W}_{0,t}$ and $\overline{W}_{i,t}$ along their boundaries $\widehat{\Delta}_{i,t}$ and $\Delta_{i,t}$ via the map F for $1 \leq i \leq n$. Let $F_t : W_t \rightarrow \mathbb{R}^3$ be the map defined by $F_t = F$ in $\overline{W}_{0,t}$ and $F_t = \text{id}$ in $\overline{W}_{i,t}$ for $1 \leq i \leq n$. Then F_t is a proper local diffeomorphism whose boundary restriction parametrizes M_t . Moreover, since each $\overline{W}_{i,t}$ is homeomorphic to a closed ball minus a boundary point, we may compactify W_t by adding n points. This proves that M_t is Alexandrov-embedded. \square

APPENDIX A. APPENDIX: ON DELAUNAY ENDS IN THE DPW METHOD

As already said, Delaunay ends in the DPW method have been studied in [13], where it is proved that if ξ is a holomorphic perturbation of the standard Delaunay potential, the resulting immersion has a Delaunay end at $z = 0$ and is close to a Delaunay surface in a disk $D(0, \varepsilon)$. In

case the potential ξ_t depends on a parameter t and the Delaunay residue has weight $\sim t$, their result has been refined by T. Raujouan in [15], yielding a uniform ε as $t \rightarrow 0$. This is delicate because the corresponding Fuchsian system is resonant at $t = 0$. The result of [15] is the key to proving embeddedness of the CMC n -noids constructed in [18].

We consider the standard Delaunay residue for $t \leq \frac{1}{16}$:

$$A_t(\lambda) = \begin{pmatrix} 0 & \lambda^{-1}r + s \\ \lambda r + s & 0 \end{pmatrix} \quad \text{where} \quad \begin{cases} r + s = \frac{1}{2} \\ rs = t \\ r < s \end{cases}$$

In particular, in the limit case $t = 0$, we have

$$A_0 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Definition 6 ([15]). *A perturbed Delaunay potential is a family of DPW potentials ξ_t of the form*

$$\xi_t(z, \lambda) = A_t(\lambda) \frac{dz}{z} + R_t(z, \lambda) dz$$

where R_t is of class C^2 with respect to $(t, z, \lambda) \in (-T, T) \times D(0, \varepsilon) \times \mathbb{A}_\rho$ for some positive ε and T , and satisfies $R_0 = 0$. In particular, $\xi_0 = A_0 \frac{dz}{z}$.

Let (e_1, e_2, e_3) represent the canonical basis of \mathbb{R}^3 in the $\mathfrak{su}(2)$ model.

Theorem 5. *Let ξ_t be a perturbed Delaunay potential. Let $\Phi_t(z, \lambda)$ be a family of solutions of $d\Phi_t = \Phi_t \xi_t$ in the universal cover of the punctured disk $D^*(0, \varepsilon)$. Assume that $\Phi_t(z, \lambda)$ depends continuously on (t, z, λ) and that the Monodromy Problem for Φ_t is solved. Let $f_t = \text{Sym}(\text{Uni}(\Phi_t))$ be the immersion given by the DPW method. Finally, assume that $\Phi_0(1, \cdot)$ is constant (i.e. independent of λ).*

Given $0 < \alpha < 1$, there exists uniform positive numbers $\varepsilon' \leq \varepsilon$, $T' \leq T$, c and a family of Delaunay immersions $f_t^{\mathcal{D}} : \mathbb{C}^* \rightarrow \mathbb{R}^3$ such that:

(1) For $0 < |t| < T'$ and $0 < |z| < \varepsilon'$:

$$\|f_t(z) - f_t^{\mathcal{D}}(z)\| \leq c|t||z|^\alpha.$$

(2) For $0 < t < T'$, $f_t : D^*(0, \varepsilon') \rightarrow \mathbb{R}^3$ is an embedding.

(3) The end of $f_t^{\mathcal{D}}$ at $z = 0$ has weight $8\pi t$ and its axis converges when $t \rightarrow 0$ to the half-line spanned by the vector Qe_3Q^{-1} where

$$Q = \text{Uni}(\Phi_0(1)H) \quad \text{and} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Thomas Raujouan has proved this result in [15], Theorem 3, in the case $\Phi_0(1, \lambda) = I_2$. He proves that the limit axis is spanned by e_1 . (In fact, he finds that the limit axis is $-e_1$, but this is because he has the opposite sign in the Sym-Bobenko formula. See Remark 3.) Then in Section 2 of [15], he explains, in the case $r > s$, how to extend his result to the case where $\Phi_0(1, \lambda)$ is constant. We adapt his method to the case $r < s$.

Lemma 2. *Under the assumptions of Theorem 5, there exists a gauge $G(z)$ and a change of variable $h(z)$ with $h(0) = 0$ such that $\tilde{\xi}_t = (h^*\xi_t) \cdot G$ is a perturbed Delaunay potential (with residue A_t) and $\tilde{\Phi}_t = (h^*\Phi_t) \times G$ satisfies at $t = 0$*

$$(27) \quad \tilde{\Phi}_0(1, \lambda) = QH^{-1} \in \Lambda SU(2).$$

Proof: we follow the method explained in Section 2 of [15]. We take the change of variable in the form

$$h(z) = \frac{z}{pz + q}$$

where p, q are complex numbers (independent of t) to be determined, with $q \neq 0$. We consider the following gauge:

$$G(z) = \frac{1}{\sqrt{q(pz + q)}} \begin{pmatrix} pz + q & pz \\ 0 & q \end{pmatrix}.$$

It is chosen so that

$$(28) \quad G(0) = I_2 \quad \text{and} \quad dG = GA_0 \frac{dz}{z} - A_0 G \frac{dh}{h}.$$

(In fact, the gauge G is found as the only solution of Problem (28) which is upper triangular.) We have

$$\tilde{\xi}_t = (h^*\xi_t) \cdot G = G^{-1} \left(A_t(\lambda) \frac{dh}{h} + R_t(h, \lambda) dh \right) G + G^{-1} dG.$$

Since $G(0) = I_2$, $\tilde{\xi}_t$ has a simple pole at $z = 0$ with residue A_t . Using Equation (28), we obtain at $t = 0$:

$$\tilde{\xi}_0 = G^{-1} A_0 \frac{dh}{h} G + G^{-1} dG = A_0 \frac{dz}{z}.$$

Hence $\tilde{\xi}_t$ is a perturbed Delaunay potential. It remains to compute $\tilde{\Phi}_0(1)$. The matrix H diagonalises A_0 :

$$A_0 = H \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} H^{-1}$$

Hence

$$\Phi_0(z) = \Phi_0(1) z^{A_0} = \Phi_0(1) H \begin{pmatrix} \frac{1}{\sqrt{z}} & 0 \\ 0 & \sqrt{z} \end{pmatrix} H^{-1}$$

$$\begin{aligned} \tilde{\Phi}_0(1) &= \Phi_0(h(1)) G(1) \\ &= \Phi_0(1) H \begin{pmatrix} \sqrt{p+q} & 0 \\ 0 & \frac{1}{\sqrt{p+q}} \end{pmatrix} H^{-1} \frac{1}{\sqrt{q(p+q)}} \begin{pmatrix} p+q & p \\ 0 & q \end{pmatrix} \\ &= \Phi_0(1) H \begin{pmatrix} \sqrt{q} & \frac{p}{\sqrt{q}} \\ 0 & \frac{1}{\sqrt{q}} \end{pmatrix} H^{-1} \end{aligned}$$

We decompose $\Phi_0(1)H = QR$ with $Q \in SU(2)$ and $R = \begin{pmatrix} \rho & \mu \\ 0 & \frac{1}{\rho} \end{pmatrix}$. Then

$$\tilde{\Phi}_0(1) = Q \begin{pmatrix} \rho & \mu \\ 0 & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} \sqrt{q} & \frac{p}{\sqrt{q}} \\ 0 & \frac{1}{\sqrt{q}} \end{pmatrix} H^{-1}$$

We take $q = \frac{1}{\rho^2}$ and $p = -\frac{\mu}{\rho}$ to cancel the two matrices in the middle and obtain Equation (27).
 \square

We can now prove Theorem 5. Let

$$\widehat{\Phi}_t(z, \lambda) = HQ^{-1}\widetilde{\Phi}_t(z, \lambda) = HQ^{-1}\Phi_t(h(z), \lambda)G(z, \lambda)$$

Since $\widehat{\Phi}_0(1, \lambda) = I_2$, we can apply Theorem 3 in [15] which says that the resulting immersion \widehat{f}_t satisfies Points 1 and 2 of Theorem 5 and its limit axis is spanned by e_1 . We have

$$f_t \circ h = QH^{-1}\widehat{f}_tHQ^{-1}$$

so $f_t \circ h$ and \widehat{f}_t differ by a rotation and the limit axis of f_t is spanned by the vector $QH^{-1}e_1HQ^{-1}$.
 Now

$$H^{-1}e_1H = -\frac{i}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = -i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = e_3.$$

\square

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