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UNIQUENESS OF THE CAUCHY DATUM FOR THE TEMPERED-IN-TIME RESPONSE AND CONDUCTIVITY OPERATOR OF A PLASMA

OLIVIER LAFITTE^{1,2} AND OMAR MAJ^{3,4}

ABSTRACT. We study the linear Vlasov equation with a given electric field $E \in \mathcal{S}$, where \mathcal{S} is the space of Schwartz functions. The associated damped partial differential equation has a unique tempered solution, which fixes the needed Cauchy datum. This tempered solution then converges to the causal solution of the linear Vlasov equation when the damping parameter goes to zero. This result allows us to define the plasma conductivity operator σ , which gives the current density $j = \sigma(E)$ induced by the electric field E . We prove that σ is continuous from \mathcal{S} to its dual \mathcal{S}' . We can treat rigorously the case of uniform non-magnetized non-relativistic plasma (linear Landau damping) and the case of uniform magnetized relativistic plasma (cyclotron damping). In both cases, we demonstrate that the main part of the conductivity operator is a pseudo-differential operator and we give its expression rigorously. This matches the formal results widely used in the theoretical physics community.

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1. INTRODUCTION

A plasma is a collection of a sufficiently large number of electrically charged particles of various species (electrons, protons, and ions of different elements), subject to electromagnetic fields. In kinetic theory, the configuration of a plasma is specified by a family of functions $f_s : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}_+$, labeled by the index of particle species s and defined so that $f_s(t, x, p)$ gives the density of particles of the species s at the time t , position x and relativistic momentum p .

The equations governing the evolution of the distribution functions $\{f_s\}_s$, together with the electric field $\mathbf{E} : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and the magnetic field $\mathbf{B} : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, are given by the relativistic Vlasov-Maxwell-Landau system, which writes

$$(1) \quad \begin{cases} \partial_t f_s + v_s \cdot \nabla_x f_s + q_s (\mathbf{E} + v_s \times \mathbf{B}/c) \cdot \nabla_p f_s = C_s(\{f_{s'}\}_{s'}), \\ \partial_t \mathbf{E} - c \nabla \times \mathbf{B} = -4\pi \sum_s q_s \int_{\mathbb{R}^3} v_s(p) f_s(t, x, p) dp, \\ \partial_t \mathbf{B} = -c \nabla \times \mathbf{E}, \\ \nabla \cdot \mathbf{B} = 0, \\ \nabla \cdot \mathbf{E} = 4\pi \sum_s q_s \int_{\mathbb{R}^3} f_s(t, x, p) dp, \end{cases}$$

where the relativistic velocity v_s is defined by $v_s(p) = p/[m_s \gamma_s(p)]$, with $\gamma_s(p) = (1 + p^2/m_s^2 c^2)^{1/2}$, C_s is the relativistic Landau collision operator [6, 13] and depends on $\{f_{s'}\}_{s'}$, c is the speed of light, m_s is the mass of particles of the species s , and q_s is their electric charge (c.g.s. units are used throughout the paper).

However, a variety of reduced models are also in use for modeling plasmas and gases in special cases. For instance, at moderate energies the non-relativistic version is used, which follows from (1) by setting $\gamma_s(p) = 1$, so that $v_s(p) = p/m_s$, and by replacing C_s with the non-relativistic Landau collision operator. When the collision operator C_s can be neglected, one recovers the Vlasov-Maxwell system both in the relativistic and non-relativistic versions. The Vlasov-Maxwell system can be further reduced, when all effects of the magnetic field can be neglected; then Maxwell's equations are replaced by the Poisson equation for the electrostatic potential ϕ , $\mathbf{E} = -\nabla \phi$, and the Vlasov-Maxwell system reduces to the Vlasov-Poisson system. For electrically neutral particles (a gas), $q_s = 0$, the electromagnetic part of the system can be dropped and the collision operator C_s is given by the Boltzmann operator. This gives the Boltzmann equation [16]. One can also replace the collision operator by simpler models, such as the BGK (Bhatnagar, Gross and Krook [8]) operator, leading to the BGK kinetic model [16].

The cases mentioned above are just some of the most common kinetic models for plasmas and gases but other “combinations” of self-consistent forces and collision

operators are also considered. The literature on kinetic models is vast and has many applications. We shall not attempt to give a review here.

In this paper we are particularly interested in applications to the study of high-frequency electromagnetic waves in high-temperature plasmas. For such problems, relativistic effects have to be accounted for (at least for the electrons), but collisions can be neglected, i.e. $C_s = 0$, since the time scale of interest is much shorter than the collision time. In addition, the wave is a small perturbation of the electromagnetic fields of the plasma so that a formal linearization of the Vlasov equation can be physically justified. As a result, the linearized relativistic Vlasov-Maxwell system is the physically appropriate model for such applications. Simpler models such as the linearized non-relativistic Vlasov-Maxwell or the linearized Vlasov-Poisson system, even in reduced dimension $d < 3$, can be interesting as well, for fundamental plasma theory.

It is however worth starting from an overview of the mathematical results for the non-linear models, in order to understand the expected regularity of the distribution functions and the electromagnetic field. The mathematical works on such a large class of models focus in particular on the associated Cauchy problem [25, 21, 41, and references therein].

1.1. Known results on existence and uniqueness. For the non-relativistic Vlasov-Maxwell system, i.e. equation (1) with $\gamma = 1$ and $C_s = 0$, Wollman [56] obtained a local existence and uniqueness result: for a single-species non-relativistic plasma and with initial data in H^s , $s \geq 5$, and with compactly supported initial particle distribution, there is a time $T > 0$ depending on the initial conditions such that a unique solution $\mathbf{f} \in C([0, T], H^s(\mathbb{R}^6)) \cap C^1([0, T], H^{s-1}(\mathbb{R}^6))$ exists. Later Asano and Ukai [3], Degond [19], and Schaeffer [53] independently proved similar results on the local existence and uniqueness of a solution $(\mathbf{f}_s, \mathbf{E}, \mathbf{B})$ with lifespan independent of the speed of light c . The fact that the lifespan is independent of c allowed them to take the limit $c \rightarrow \infty$ and to show convergence to a solution of the Vlasov-Poisson system. E.g., Degond showed the local existence and uniqueness of the solution in Sobolev spaces H^s with $s \geq 3$; specifically, if the initial data are in H^3 , and the initial distribution function is non-negative, $\mathbf{f}_s^0 \geq 0$, and compactly supported in velocity, Degond has shown that there exist a time $T > 0$, depending on the initial data but not on the speed of light c , and a solution $(\mathbf{f}_s, \mathbf{E}, \mathbf{B})$ in $L^\infty(0, T; H^3(\mathbb{R}^6) \times H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3))$. Wollman [57] improved his earlier result: again for a single-species non-relativistic plasma, he showed local existence of a C^1 solution with initial data $\mathbf{f}^0 \in C_0^1(\mathbb{R}^6)$ and $\mathbf{E}^0, \mathbf{B}^0 \in H^3(\mathbb{R}^3)$; specifically there is $T > 0$ depending on the initial condition such that there exists a solution $\mathbf{f} \in C^1([0, T] \times \mathbb{R}^6)$ which is unique as an element of $L^1(0, T; H^3(\mathbb{R}^6)) \cap C([0, T], H^2(\mathbb{R}^6))$.

For the relativistic Vlasov-Maxwell system with multiple species Glassey and Strauss [27] proved existence of a unique global solution in $C^1(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$ with initial data $\mathbf{f}_s^0 \in C_0^1(\mathbb{R}^3 \times \mathbb{R}^3)$, $\mathbf{E}^0, \mathbf{B}^0 \in C^2(\mathbb{R}^3)$ under the *a priori* assumption that a solution \mathbf{f}_s is compactly supported in momenta and the radius of the support is bounded by a continuous function of time. A simpler proof was given by Bouchut, Golse and Pallard [12], and a variant of this result has been proposed by Klainerman and Staffilani [36]. Existence and uniqueness of a global C^1 solution for $t \in [0, +\infty)$, $(x, p) \in \mathbb{R}^6$ has been shown by Glassey and Schaeffer [26] with compactly supported initial data satisfying appropriate conditions that require, in particular, the initial distribution function and electromagnetic fields to be small

in C^1 and C^2 norms respectively. We note also the results on global existence with small data for the relativistic Vlasov-Maxwell system obtained by Horst [32]. The key observation is that the decay of $\|\mathbf{E}(t)\|_{L^2}^2 + \|\mathbf{B}(t)\|_{L^2}^2$ for $t \rightarrow +\infty$ completely determine the electromagnetic field, without initial conditions. Horst makes use of a fixed-point argument to show global existence of solutions: given the electromagnetic fields (\mathbf{E}, \mathbf{B}) in a suitable class of functions, he constructs the characteristic flow for the kinetic equation, from which he computes the charge and electric current densities that generate new electromagnetic fields $(\mathbf{E}', \mathbf{B}')$. This defines an operator $Q : (\mathbf{E}, \mathbf{B}) \mapsto (\mathbf{E}', \mathbf{B}')$ which is a contraction if the initial distribution \mathbf{f}^0 and its derivatives are small enough. In this construction, the field $(\mathbf{E}', \mathbf{B}')$ is obtained as the solution of Maxwell's equations with the condition $\|\mathbf{E}(t)\|_{L^2}^2 + \|\mathbf{B}(t)\|_{L^2}^2 \rightarrow 0$ for $t \rightarrow +\infty$, [32, definition 2.6 and 3.5].

Global existence of weak solution without small data assumption is due to Di Perna and Lions [22]: for the non-relativistic case and with one particle species, given initial data $\mathbf{f}^0 \in L^1 \cap L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ and $\mathbf{E}^0, \mathbf{B}^0 \in L^2(\mathbb{R}^3)$ with the conditions $\mathbf{f}^0 \geq 0$ and

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 \mathbf{f}^0 dx dv < +\infty,$$

they prove existence of

$$\mathbf{f} \in L^\infty(0, +\infty; L^1(\mathbb{R}^3 \times \mathbb{R}^3)), \quad \mathbf{E}, \mathbf{B} \in L^\infty(0, +\infty; L^2(\mathbb{R}^3)),$$

that satisfy the non-relativistic Vlasov-Maxwell system in the sense of distributions. The conditions on the data are the natural ones since $\mathbf{f}(t, \cdot, \cdot)$ is a phase-space density of particles (and thus must be non-negative and in L^1) and the quantity

$$\mathcal{E} = \frac{1}{2} m \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 \mathbf{f} dx dv + \frac{1}{8\pi} (\|\mathbf{E}\|_{L^2}^2 + \|\mathbf{B}\|_{L^2}^2),$$

is the total energy of the system. Di Perna and Lions have shown that for such global weak solutions, one has $\mathcal{E}(t) \leq \mathcal{E}(0)$, that is, they are finite-energy solutions. The key idea of this proof is the use of renormalized solutions [21, 41], and this idea has been applied to the Boltzmann equation [23] as well. More recently, the relativistic version has been addressed by Rein [51], while time-periodic weak solutions in bounded (in space) domains have been considered by Bostan [11].

As for the Vlasov-Poisson system the study of the Cauchy problem developed along the same lines, moving from local-in-time classical solutions up to global weak solutions [26, 21, and references therein]. Particularly, the works by Asano and Ukai, Degond, and Schaeffer cited above give results on the existence and uniqueness of local solutions the Vlasov-Poisson system. There are however earlier results on local solutions [30, 31] and on global weak solutions [2, 33]. Existence of a global C^1 solutions with small initial data has been established by Bardos and Degond [5]. The first results on global classical solutions is due to Pfaffelmoser [47], where ‘‘classical’’ here means that the characteristics system for the Vlasov equation has a unique classical solution and \mathbf{f} is constant along the characteristics.

All these results are for the fully nonlinear problem. In this work, however, we address the linearized problem and we focus specifically on the associated linear kinetic equation. Wollman [57, section 3] reports the classical results on the existence of C^1 solutions for such linear problems, the proof of which is based on the standard method of characteristics. Specifically, if the electric and magnetic fields \mathbf{E}, \mathbf{B} are in $C([0, T]; H^3(\mathbb{R}^3))$ with H^3 -norm bounded uniformly in time, the linear

non-relativistic equation with initial datum $f^0 \in C_0^1(\mathbb{R}^3 \times \mathbb{R}^3)$ has a unique classical solution $f \in C^1([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)$. Here T is the life-span of the fields and does not depend on the initial distribution.

1.2. Framework of this paper. In this paper, we consider a given stationary configuration $\{F_{s,0}(x, p)\}_s$ of the particle distribution functions, with zero electric field $E_0 = 0$, and a constant magnetic field B_0 . Then we address the linearized system around the stationary solution $(\{F_{s,0}\}_s, E_0, B_0)$. The associated unknowns are the linear perturbations $(\{f_s\}_s, E, B)$, where f_s is the perturbed distribution function for the particle species s , while E and B are the perturbations of electric and magnetic field, respectively.

Since we consider a linearized problem, the solution for f_s is not necessarily non-negative, but $F_{s,0} + f_s \geq 0$ (essentially, if $F_{s,0} + f_s$ fails to be non-negative, we no longer are in the linear regime). We need to keep the assumption on the existence of the velocity moments of f_s and the L^1 -in- x behavior (or L_{loc}^1 in the idealized case of a plasma with an infinite number of particles, e.g., a uniform plasma over the whole space).

We could expect to need and to be able to consider given arbitrary initial data. Our aim however is characterizing and determining what is referred to as the *dielectric response of the plasma* in the physics literature [14, 54, 38, 10]. The underlying physical idea is that the application of a small-amplitude electromagnetic perturbation determines a small change in the distribution functions f_s , which represents the response of the plasma to the imposed electromagnetic disturbance. It is similar to the construction of the operator Q in Horst's fixed-point argument mentioned above. In addition however, physical reasoning suggests that the response of the plasma should be uniquely determined by and depend continuously on this imposed perturbation, hence there should be no need to prescribe a Cauchy datum.

The evolution of a small perturbation f_s induced by an externally imposed small electric field disturbance E is governed by the linear relativistic Vlasov equation [9, 14, 35, 54],

$$(2) \quad \partial_t f_s + v_s \cdot \nabla_x f_s + q_s(v_s \times B_0/c) \cdot \nabla_p f_s = -q_s(E + v_s \times B/c) \cdot \nabla_p F_{s,0},$$

where the electric field E of the disturbance is given, e.g., $E \in [\mathcal{S}(\mathbb{R}^4)]^3$, and the magnetic field depends linearly on the electric field via the Faraday law,

$$(3) \quad \partial_t B + c \nabla \times E = 0.$$

Since E is given, this is a system of partial differential equations for (f_s, B) which reads

$$\begin{cases} \partial_t f_s + v_s \cdot \nabla_x f_s + q_s(v_s \times B_0/c) \cdot \nabla_p f_s = -q_s(E + v_s \times B/c) \cdot \nabla_p F_{s,0}, \\ \partial_t B = -c \nabla \times E. \end{cases}$$

From this system, by introducing the new unknown $g_s = \partial_t f_s$, one deduces the decoupled system

$$(4) \quad \begin{cases} \partial_t g_s + v_s \cdot \nabla_x g_s + q_s(v_s \times B_0/c) \cdot \nabla_p g_s = -q_s(\partial_t E - v_s \times \nabla \times E) \cdot \nabla_p F_{s,0}, \\ \partial_t B = -c \nabla \times E. \end{cases}$$

The associated homogeneous equation for g_s is

$$\mathcal{H}(g_s) := \partial_t g_s + v_s \cdot \nabla_x g_s + q_s(v_s \times B_0/c) \cdot \nabla_p g_s = 0.$$

We shall see that if we impose a control of the growth at $t \rightarrow \pm\infty$, the modified equation $\mathcal{H}(g_{s,\nu}) + \nu g_{s,\nu} = 0$ for any $\nu > 0$ has the unique solution $g_{s,\nu} = 0$. This is similar to scattering theory [39], in which the scattered field is determined by conditions at infinity. The introduction of the damping term $\nu g_{s,\nu}$ is analogous to the limiting absorption principle [39, 24, 52] for elliptic equations of the form $Au = (\lambda + i\varepsilon)u + f$ for $\varepsilon \rightarrow 0^+$ (resp. $\varepsilon \rightarrow 0^-$), where one constructs two resolvents for the elliptic operator A . A classical application of the limiting absorption principle consists in selecting the unique outgoing-wave solution of the Helmholtz equation. Other examples include the solution of elliptic equations, as well as the extension of the resolvent of the operator $-\Delta + V$ [24, 45, 15, and references therein]. Recently the idea of using scattering theory for the linearized Vlasov-Maxwell system has been developed by Després in order to prove linear Landau damping for inhomogeneous equilibrium distributions [20]. In this paper, however, we shall not take advantage of scattering theory, but rather focus on the solution selected by the growth conditions at infinity in time and the limiting absorption principle.

The main idea here is to apply the limiting absorption principle to either the inhomogeneous problem (2) or (4). We shall see that for $\nu > 0$, there is a unique tempered solution, which has a limit for $\nu \rightarrow 0^+$, and the limit itself is a tempered solution of either (2) or (4) without damping. We will find that the limit amounts exactly to the solution which is referred to as the *causal solution* in the physics literature and which describes the response of the plasma to the imposed E .

Finding the response of a system is classical. For example, in the modeling of electric circuits with capacitance C inductance L and resistance R , the electric charge $q : t \mapsto q(t)$ satisfies the ordinary differential equation

$$LC\ddot{q} + RC\dot{q} + q = CU_0 \cos(\omega t),$$

which leads to the response

$$q(t) = \operatorname{Re} \left[\frac{CU_0 e^{i\omega t}}{1 - LC\omega^2 + i\omega RC} \right],$$

even though one has infinitely many solutions (depending on Cauchy data).

Returning to the linearized Vlasov equation, one can also envisage the use of other dissipation mechanisms such as the Fokker-Planck operator $\nu \nabla_v \cdot (\nabla_v f - vf)$ or the collision operators mentioned above, but this is not addressed here.

Having identified the unique solution of the linearized Vlasov equation which describes the response of the plasma to the imposed electric field disturbance, the corresponding unique perturbation of the electric current density is defined by

$$(5) \quad j(t, x) := \sum_s q_s \int_{\mathbb{R}^3} v_s(p) f_s(t, x, p) dp,$$

which requires that the solution f_s is at least in L^1 (the relativistic velocity is bounded by the speed of light, $|v_s(p)| < c$). We shall generalize such integrals to the case of distributional solutions and see that, since f_s depends linearly on E , the induced current density j is given by the action of a linear operator on the electric field E , namely,

$$(6) \quad j = \sigma(E).$$

This is referred to as the linear constitutive relation of the plasma and the operator σ is the conductivity operator. Equation (6) is also referred to as (generalized) Ohm's law.

A precise mathematical analysis of the response of a plasma is important because it is the basis for the construction of constitutive relations for linear plasma waves, the simplest of which being the Ohm's law. Together with Maxwell's equations, it determines the linear wave equation describing plasma waves [10]. The same problem has been considered by Omnes for a bounded plasma [46]. More recently, Chevry and Fontaine [18, 17] have addressed the characteristic variety (or dispersion relation) for the linearized Maxwell-Vlasov system using asymptotic methods, but here we focus on the properties of the plasma constitutive relation as an operator.

We carry out this ideas for the one-dimensional non-relativistic linearized Vlasov equation without background magnetic field (non-magnetized) and for the three-dimensional relativistic linearized Vlasov equation with uniform background and with constant magnetic field.

1.3. Main results 1: the non-relativistic, one-dimensional case. We consider first the case of a non-magnetized, non-relativistic plasma in one dimension in space and velocity. We also restrict ourselves to the case of a single particle species (and thus drop the index s). This last simplification does not imply any further loss of generality as the current density is the sum of the currents carried by the individual species. With background distribution $F_0 \in \mathcal{S}(\mathbb{R})$ depending on $v \in \mathbb{R}$ only, and $E \in \mathcal{S}(\mathbb{R}^2)$, we consider the linear kinetic equation

$$(7) \quad \mathcal{L}f := \partial_t f + v \partial_x f = -\frac{q}{m} E F'_{s,0},$$

which is a reduced version of equation (2). The damped form is

$$(8) \quad \mathcal{L}_\nu f_\nu := \partial_t f_\nu + \nu f_\nu + v \partial_x f_\nu = -\frac{q}{m} E F'_0.$$

If f is a generic distribution (not necessarily a solution of (7)) with finite first velocity moment, i.e., with $v \mapsto v f(t, x, v)$ in L^1 for all (t, x) , we define the operator

$$(9) \quad \mathbb{J}(f)(t, x) := q \int_{\mathbb{R}} v f(t, x, v) dv,$$

which gives the associated current density, cf. equation (5), in the one-dimensional, non-relativistic case.

Theorem 1.1. *Let $F_0 \in \mathcal{S}(\mathbb{R})$ be given.*

- (i) *If $\nu > 0$, for any $E \in \mathcal{S}(\mathbb{R}^2)$ equation (8) has a solution $f_\nu \in \mathcal{S}(\mathbb{R}^3)$ which is unique as an element of $\mathcal{S}'(\mathbb{R}^3)$, and $j_\nu = \mathbb{J}(f_\nu) \in \mathcal{S}'(\mathbb{R}^2)$.*
- (ii) *For $\nu \rightarrow 0^+$, f_ν and j_ν have pointwise limits $f \in C_b^\infty(\mathbb{R}^3)$ and $j \in C_b^\infty(\mathbb{R}^2)$, respectively; in addition, $f_\nu \rightarrow f$, and $j_\nu \rightarrow j$ also in the topology of \mathcal{S}' .*
- (iii) *The limit f is a solution of equation (7) and $j = \mathbb{J}(f)$.*

By the limiting absorption principle of theorem 1.1, to each $E \in \mathcal{S}(\mathbb{R}^2)$ we thus can associate a unique f , and thus a unique current density j . The conductivity operator is then defined as the map

$$(10) \quad \sigma : \mathcal{S}(\mathbb{R}^2) \ni E \mapsto j \in C_b^\infty(\mathbb{R}^2) \subset \mathcal{S}'(\mathbb{R}^2).$$

For any $\chi \in C_0^\infty(\mathbb{R})$ with $\chi = 1$ in a neighborhood of zero, we also introduce the following Fourier multiplier (cf. appendix A for definitions and notations)

$$(11) \quad \mathcal{F}(\sigma_{1-\chi}(E))(\omega, k) = (1 - \chi(k)) \hat{\sigma}_{\text{ph}}(\omega, k) \hat{E}(\omega, k),$$

where \mathcal{F} denotes the Fourier transform and $\hat{\sigma}_{\text{ph}}$ is the physical conductivity tensor (explicit formula given in equation (33)).

Theorem 1.2. *The map σ defined by equation (10) is linear and continuous from $\mathcal{S}(\mathbb{R}^2) \rightarrow \mathcal{S}'(\mathbb{R}^2)$ and for every $\chi \in C_0^\infty(\mathbb{R})$ with $\chi = 1$ near zero, $\sigma(E) = \sigma_{1-\chi}(E)$ for all E satisfying $\hat{E}(\omega, k) = 0$ if $k \in \text{supp } \chi$.*

Remark 1.

- (1) The limit f established in theorem 1.1 coincides with the causal solution of (7), which is reviewed in appendix C.
- (2) Expressions for the solutions f_ν , f , their Fourier transforms, the associated currents, and the operators σ_ν and σ are given in section 3.
- (3) The operator $\sigma_\chi := \sigma - \sigma_{1-\chi}$ is well defined and $\sigma_\chi(E) = 0$ when $\hat{E}(\omega, k) = 0$ for k small. Expressions of both σ and σ_χ are available (see proposition 3.5).
- (4) Theorem 1.1 can be straightforwardly generalized to the case of a non-magnetized non-relativistic plasma with a spatially non-homogeneous equilibrium distribution of the form $F_0(x, v) = n_0(x)\tilde{F}_0(v)$, for which the velocity distribution is the same at any point in space. For such equilibria, $\sigma_{1-\chi}$ is a pseudo-differential operator. The expression of the symbol is obtained in section 3.2, remark 7.

1.4. Main results 2: relativistic, three-dimensional case. The second case under consideration is the relativistic Vlasov equation with a uniform background magnetic field, that is, B_0 in equation (2) is taken constant and non-zero. We choose $B_0 = |B_0|e_\parallel$ directed along the third axis of a Cartesian frame $\{e_1, e_2, e_3 = e_\parallel\}$. It is natural to normalize the relativistic momentum p to $m_s c$, and thus to introduce normalized momentum variables

$$(12) \quad u := p/(m_s c) \quad u_\perp := (u_1^2 + u_2^2)^{1/2}, \quad u_\parallel := u_3.$$

We define the relativistic cyclotron frequency for the considered species,

$$(13) \quad \Omega_s(u) := \frac{1}{\gamma(u)} \frac{q_s |B_0|}{m_s c} = \text{sgn}(q_s) \frac{\omega_{c,s}}{\gamma(u)} > 0,$$

which has the sign of the charge q_s and depends on u^2 through $\gamma(u) = (1 + u^2)^{1/2}$, whereas the classical cyclotron frequency $\omega_{c,s} := |q_s B_0|/(m_s c) > 0$ is a positive constant.

The background distribution functions are taken uniform and *gyrotropic*, i.e., $F_{s,0}$ is constant in time t and space x and depends only on u_\parallel , u_\perp , namely,

$$(14) \quad F_{s,0}(t, x, p) = \frac{n_{s,0}}{(m_s c)^3} G_s(u_\parallel, u_\perp),$$

where $n_{s,0} > 0$ is the constant background particle density and G_s is such that $u \mapsto G_s(u_3, (u_1^2 + u_2^2)^{1/2})$ belongs to $\mathcal{S}(\mathbb{R}^3)$. Such property is usually satisfied by the background distribution functions of practical interest. The momentum distribution G_s has unit norm in $L^1(\mathbb{R} \times \mathbb{R}_+, 2\pi u_\perp du_\parallel du_\perp)$.

Instead of addressing equation (2) directly, we consider the kinetic equation in (4) for $g_s = \partial_t f_s$, which amounts to

$$(15) \quad \mathcal{V}_s g_s = -q_s (\partial_t E - v_s \times \nabla \times E) \cdot \nabla_p F_{s,0},$$

where

$$(16) \quad \mathcal{V}_s = \partial_t + v_s \cdot \nabla_x + \operatorname{sgn}(q_s) \frac{\omega_{c,s}}{\gamma} (u \times e_{\parallel}) \cdot \nabla_u,$$

with $v_s = p/(m_s \gamma_s) = cu/\gamma(u)$ being the relativistic velocity as a function of the normalized momentum.

We add to equation (15) a damping term, with the idea of applying the limiting absorption principle. Because of the u -dependent relativistic factor $\omega_{c,s}/\gamma$ the damping coefficient will be multiplied by γ after Fourier transform in time and space of equation (15). It is therefore convenient to allow the damping coefficient ν_s to depend on the species s and on momentum u from the beginning, subject to the conditions

$$(17) \quad \begin{cases} \nu_s \in C^\infty(\mathbb{R}^3), \\ \text{there exists } \nu_0 > 0 : \gamma(u)\nu_s(u) \geq \nu_0, \\ (u_1 \partial_{u_2} - u_2 \partial_{u_1})\nu_s(u) = 0, \\ \text{and there exists } m \in \mathbb{R} : |\partial_u^\alpha \nu_s(u)| \leq C_\alpha (1 + u^2)^m, \forall u \in \mathbb{R}^3, \forall \alpha \in \mathbb{N}_0^3, \end{cases}$$

where $C_\alpha \in \mathbb{R}$ are constants depending only on the order α of the derivatives. For example, if $\nu_s \in C_b^\infty(\mathbb{R}^3)$ is a function of $u_1^2 + u_2^2$ and u_3 only, then conditions (17) are fulfilled with $m = 0$. We shall see that the dissipation-less limit is independent of the choice of this damping function.

For $\varepsilon > 0$ and for any function ν_s , satisfying conditions (17), we consider the regularized equation

$$(18) \quad \mathcal{V}_{s,\varepsilon} g_{s,\varepsilon} = -q_s (\partial_t E - v \times \nabla \times E) \cdot \nabla_p F_{s,0},$$

where $\mathcal{V}_{s,\varepsilon} = \mathcal{V}_s + \varepsilon \nu_s$.

If g_s are generic distributions with finite first velocity moment, i.e., $g_s(t, x, \cdot) \in L^1(\mathbb{R}^3)$ for every (t, x) , we define

$$(19) \quad \mathbb{K}(\{g_s\})(t, x) := \sum_s q_s (m_s c)^3 \int_{\mathbb{R}^3} v_s(u) g_s(t, x, u) du,$$

which gives the time-derivative of the current (5) when $g_s = \partial_t f_s$. For this model, the limiting absorption principle parallels theorem 1.2.

Theorem 1.3. *Let $F_{s,0} \in \mathcal{S}(\mathbb{R}^3)$ be given uniform gyrotropic distribution functions and let ν_s be any function satisfying conditions (17).*

- (i) *If $\varepsilon > 0$, for any $E \in [\mathcal{S}(\mathbb{R}^4)]^3$ equation (18) has a solution $g_{s,\varepsilon} \in \mathcal{S}(\mathbb{R}^7)$ which is unique as an element of $\mathcal{S}'(\mathbb{R}^7)$, $\partial_t j_\varepsilon = \mathbb{K}(\{g_{s,\varepsilon}\}) \in [\mathcal{S}(\mathbb{R}^4)]^3$.*
- (ii) *For $\varepsilon \rightarrow 0^+$, $g_{s,\varepsilon}$ and $\partial_t j_\varepsilon$ have limits g_s and $\partial_t j$ in \mathcal{S}' , independent of ν_s .*
- (iii) *The limit g_s belongs to $C_b^\infty(\mathbb{R}^7)$ and is a classical solution of equation (15); in addition g_s belongs to the domain of \mathbb{K} and $\partial_t j = \mathbb{K}(\{g_s\}) \in C_b^\infty(\mathbb{R}^4)$.*

Therefore we can define the map

$$(20) \quad \varsigma : [\mathcal{S}(\mathbb{R}^4)]^3 \ni E \mapsto \partial_t j \in [\mathcal{S}'(\mathbb{R}^4)]^3.$$

We shall see that it can be represented by a Fourier multiplier if we exclude the hyperplane $\omega = 0$ in Fourier space. For any cut-off function $\chi \in C_0^\infty(\mathbb{R})$ with $\chi = 1$ in a neighborhood of zero, we define the Fourier multiplier $\varsigma_{1-\chi}$ by

$$(21) \quad \mathcal{F}(\varsigma_{1-\chi}(E))(\omega, k) = (1 - \chi(\omega)) \hat{\varsigma}_0(\omega, k) \hat{E}(\omega, k),$$

where $\hat{\varsigma}_0(\omega, k)$ is the limit established in proposition 5.10. An expression for $\hat{\varsigma}_0$ is given in equation (81), and proposition 5.10 establishes that $\hat{\varsigma}_0$ is continuous for $\omega \neq 0$ and C^∞ where $\omega^2 \neq c^2 k^2 + n^2 \omega_{c,s}^2$ for all $n \in \mathbb{Z}$ and all species s .

Theorem 1.4. *The map ς defined in (20) is continuous and for any $\chi \in C_0^\infty(\mathbb{R})$ with $\chi = 1$ in a neighborhood of zero, $\varsigma(E) = \varsigma_{1-\chi}(E)$ if $\hat{E}(\omega, k) = 0$ for $\omega \in \text{supp } \chi$.*

Remark 2.

- (1) The hypothesis that ν_s is in the kernel of the operator $u_1 \partial_{u_2} - u_2 \partial_{u_1}$ expresses the fact that ν_s must be gyrotropic.
- (2) The solution g_s is the unique causal solution of (15) (defined in appendix C).
- (3) We also have pointwise convergence of $g_{s,\varepsilon} \rightarrow g_s$.
- (4) Existence and uniqueness of the solution $g_{s,\varepsilon}$ is established via Fourier transform, while the causal solution is obtained by integration along the characteristics, cf. appendix C. Hence the proof of theorem 1.3 (iii) establishes a link between the formulations in Fourier and physical variables.
- (5) An explicit expression of the linear operator ς valid without restriction of the support of E is given below in proposition 5.9.

1.5. Concluding remarks and structure of the paper. Theorems 1.2 and 1.4 in particular show that the response of a uniform plasma to oscillatory electromagnetic disturbances can be expressed by a Fourier multiplier. Although limited to a simple plasma equilibrium, these results support the physics theories that rely on the pseudo-differential form of the conductivity operator [37, 7, 44, 42, 43, 10]. More precisely, even though the response of a plasma is rigorously not a pseudo-differential operator, it can be written as the sum of a pseudo-differential operator plus a remainder which vanishes if the spectrum of the electric-field disturbance is supported away from $\omega = 0$ (or $k = 0$ in the simpler case of theorem 1.1); this is typically the case in the envisaged applications, since the frequency of the perturbation is set by an external source and it is tuned to resonate with the cyclotron motion of a particle species.

In order to illustrate these specific applications at least qualitatively, we return to the linearized Vlasov-Maxwell system, that is,

$$(22) \quad \begin{cases} \partial_t f_s + v_s(p) \cdot \nabla_x f_s + q_s(v_s \times B_0/c) \cdot \nabla_p f_s = -q_s(E + v_s \times B/c) \cdot \nabla_p F_{s,0}, \\ \partial_t E - c \nabla \times B = -4\pi \sum_s q_s \int_{\mathbb{R}^3} v_s f_s dp, \\ \partial_t B + c \nabla \times E = 0, \end{cases}$$

for f_s , E , and B . The two equations for the electromagnetic field (E, B) imply (formally at least, by taking the time-derivative of the Ampère-Maxwell law)

$$\partial_t^2 E + c^2 \nabla \times (\nabla \times E) + 4\pi \partial_t j = 0,$$

where $\partial_t j = \mathbb{K}(\{\partial_t f_s\})$. This equation depends on the time-derivative of the induced current $\partial_t j$ rather than on j alone, and the map (20) give $\partial_t j = \varsigma(E)$ when $E \in \mathcal{S}(\mathbb{R}^4)$. However, if the solution is highly oscillatory (high-frequency waves), $\hat{E}(\omega, k) = 0$ near $\omega = 0$ and we can replace ς by the Fourier multiplier (21) with the low-frequency cut-off, obtaining

$$(23) \quad D(i\partial_t, -i\partial_x)E := \partial_t^2 E + c^2 \nabla \times (\nabla \times E) + 4\pi \varsigma_{1-\chi}(E) = 0,$$

which is a constant-coefficients pseudo-differential equation for the electric field only. Theorem 1.4 implies that the operator $D(i\partial_t, -i\partial_x) : [\mathcal{S}(\mathbb{R}^4)]^3 \rightarrow [\mathcal{S}'(\mathbb{R}^4)]^3$ is continuous and we have established regularity results for its symbol in proposition 5.10. The symbol, in particular, is polynomially bounded and thus the operator extends to

$$D(i\partial_t, -i\partial_x) : [H_{(s)}(\mathbb{R}^4)]^3 \rightarrow [H_{(s-m)}(\mathbb{R}^4)]^3, \quad s \in \mathbb{R},$$

where $H_{(s)}(\mathbb{R}^4)$ is the space of $w \in \mathcal{S}'(\mathbb{R}^4)$ such that $(1 + (\frac{\omega}{\hat{\omega}})^2 + (\frac{ck}{\hat{\omega}})^2)^{\frac{s}{2}} \hat{w} \in L^2(\mathbb{R}^4)$, with the normalization frequency chosen by $\hat{\omega} := \max_s |\omega_{c,s}|$ and with $m = \max\{2M, 2\}$ where the integer M being the degree of the polynomial bound for $\hat{\zeta}_0$ established in proposition 5.10 in section 5.4.

The semi-classical methods commonly used to find approximate solutions of the wave equation (23) (cf. the work of Prater et al. [48] for an overview of computational tools) are valid under strong assumptions on the symbol of the operator D . These assumptions are smoothness of the symbol and the fact that its anti-Hermitian part is small in a certain sense (weakly non-Hermitian operators) [37, 7, 44, 42, 43, 10]. In this paper we analyze the construction of the operator in detail. Theorem 1.4 shows that the full operator ς defined in (21) has an additional contribution that accounts for the low-frequency response of the plasma. For the high-frequency part (operator (21)), smoothness of the symbol is established almost everywhere in Fourier space, cf. proposition 5.10. As for the assumption of weak anti-Hermitian part, this is always violated near cyclotron resonances and the application of standard computational methods is justified by heuristic arguments only. Propagation near a resonance has been addressed in the physics literature [55, 4] but a satisfactory theory is not available. Then the only rigorous approach to the problem would be the direct numerical computation of the solution of the linearized Vlasov-Maxwell system, which is computationally too expensive in realistic cases. The precise characterization of response operators may help to improve the available methods toward including resonances.

In section 2, a simple case study is presented in order to illustrate the ideas. The rest of the paper is dedicated to the proofs. In section 3 the case of a non-relativistic isotropic plasma in one dimension is addressed, while section 5 is dedicated to relativistic uniform magnetized plasmas. An overview of notations and standard definitions together with technical results can be found in the appendices. In appendix C in particular a precise definition of causal solution is given for advection equations associated to global-in-time flows.

2. CHARACTERIZATION OF THE RESPONSE OPERATOR: A SIMPLE CASE STUDY

In this section, we study a simple model which however contains the essential elements of the full problem. The aim of these simple considerations is showing how the limiting absorption principle determines the causal solution of a hyperbolic equation. All proofs are straightforward and reported in appendix B for the reader's convenience.

Given $v \in \mathcal{S}(\mathbb{R}^{1+d})$, we consider the equation

$$\partial_t u(t, x) = v(t, x), \quad u(0, \cdot) = u_0 \in C_b^\infty(\mathbb{R}^d),$$

where C_b^∞ is the space of smooth bounded functions with bounded derivatives, cf. appendix A for the precise definition.

Proposition 2.1. *For $v \in \mathcal{S}(\mathbb{R}^{1+d})$, there exists a unique solution u in $C_b^\infty(\mathbb{R}^{1+d})$ of the equation $\partial_t u = v$ such that $\lim_{t \rightarrow -\infty} u(t, x) = 0$ pointwise in x , and*

$$u(t, x) = \int_{-\infty}^t v(s, x) ds.$$

The map $v \mapsto u$ is a continuous linear operator both from $\mathcal{S}(\mathbb{R}^{1+d}) \rightarrow \mathcal{S}'(\mathbb{R}^{1+d})$ and from $\mathcal{S}(\mathbb{R}^{1+d}) \rightarrow L^\infty(\mathbb{R}^{1+d})$.

If we think of u as the response to a localized perturbation v , causality requires that $u \rightarrow 0$ for $t \rightarrow -\infty$ since the perturbation decreases faster than any polynomial when $t \rightarrow -\infty$. Hence, the solution given in proposition 2.1 is referred to as the causal solution. Since $v \in \mathcal{S}(\mathbb{R}^{1+d})$,

$$\int_{-\infty}^0 v(s, \cdot) ds$$

is finite. The causality principle selects a unique initial condition u_0 given by

$$u_0(x) = \int_{-\infty}^0 v(s, x) ds,$$

thereby allowing us to define the linear continuous operator $v \mapsto u$, which we view as the response of the operator ∂_t for a perturbation v . We also note that the limit for $t \rightarrow +\infty$ of the solution gives the time integral of the perturbation $\int_{\mathbb{R}} v(s, x) ds$.

The following simple result provides us with a characterization of the causal solution, that is used in this paper for more general problems.

Proposition 2.2. *The damped problem $\partial_t u^\nu + \nu u^\nu = v$ in $\mathcal{S}'(\mathbb{R}^{1+d})$ with $\nu > 0$ and $v \in \mathcal{S}(\mathbb{R}^{1+d})$ has a unique solution $u^\nu \in \mathcal{S}'(\mathbb{R}^{1+d})$. It belongs to $\mathcal{S}(\mathbb{R}^{1+d})$ and it is given by*

$$u^\nu(t, x) = \int_{-\infty}^t e^{-\nu(t-t')} v(t', x) dt'.$$

Furthermore, $u^\nu \rightarrow u$ in $\mathcal{S}'(\mathbb{R}^{1+d})$ as $\nu \rightarrow 0^+$, where u is the causal solution obtained in proposition 2.1.

Remark 3. The integral in the definition of u^ν is absolutely convergent as $t - t' \geq 0$ on the domain of integration and $v(\cdot, x) \in L^1(\mathbb{R})$. The fact that $u^\nu \in \mathcal{S}(\mathbb{R}^{1+d})$ is proven by showing that the Fourier transform \hat{u}^ν is in $\mathcal{S}(\mathbb{R}^{1+d})$.

Proposition 2.2 establishes that the admissible solution u of the model without dissipation (in the sense that $u \rightarrow 0$ when $t \rightarrow -\infty$) is the limit of the unique solution u^ν in \mathcal{S}' of the model with dissipation.

3. UNIFORM ISOTROPIC PLASMAS IN ONE SPATIAL DIMENSION: THE STANDARD LINEAR LANDAU DAMPING

Here we consider in detail the case of a non-magnetized non-relativistic plasma in one spatial dimension and for a single particle species. Coupled to the Poisson equation this is the textbook example for linear Landau damping. Equation (2) reduces to

$$(24) \quad \partial_t f(t, x, v) + v \partial_x f(t, x, v) = -(q/m) E(t, x) F'_0(v),$$

where $F_0 \in \mathcal{S}(\mathbb{R})$ is the equilibrium distribution function, and $E \in \mathcal{S}(\mathbb{R}^2)$ is the electric field perturbation. We drop the species index s since we consider one species only. In this case the linearized Vlasov operator is the free-transport operator

$$\mathcal{L} = \partial_t + v\partial_x.$$

Viewed as an operator from $\mathcal{S}'(\mathbb{R}^3)$ into itself, \mathcal{L} has a non-trivial null space given by all tempered distributions with partial Fourier transform in (t, x) of the form $\hat{f} = 2\pi\kappa^*(\delta \otimes h)$ for $h \in \mathcal{S}'(\mathbb{R}^2)$ and with $\kappa : (\omega, k, v) \mapsto (\omega - kv, k, v)$ being a volume-preserving diffeomorphism of \mathbb{R}^3 , and κ^* denotes the pull-back of distributions. Usually, this is formally written as $\hat{f}(\omega, k, v) = 2\pi\delta(\omega - kv)\hat{h}(k, v)$, in the physics literature. In particular, (24) has infinitely many solutions in \mathcal{S}' .

We consider a damped version of the advection operator and then pass to the limit to recover a solution of the original problem (limiting absorption principle). More precisely, for a given $F_0 \in \mathcal{S}(\mathbb{R})$, we first prove the existence and uniqueness of the solution f_ν in $\mathcal{S}'(\mathbb{R}^3)$ of (8). We find that the unique solution $f_\nu \in \mathcal{S}'(\mathbb{R}^3)$ is an element of $\mathcal{S}(\mathbb{R}^3)$ and we thus define a map from $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}(\mathbb{R}^3)$. We first calculate the Fourier transform of f_ν in (t, x) (proposition 3.1), from which we deduce the function itself. We obtain from this unique solution the damped current $j_\nu(t, x) := \int_{\mathbb{R}} v f_\nu(t, x, v) dv$ through its Fourier transform, and the limits when $\nu \rightarrow 0^+$, respectively in $\mathcal{S}'(\mathbb{R}^3)$ and in $\mathcal{S}'(\mathbb{R}^2)$ of f_ν (proposition 3.2) and of j_ν (proposition 3.3). This shows that $\lim_{\nu \rightarrow 0^+} j_\nu = \sigma(E)$ where σ is an operator (called the conductivity operator), for which we give expressions. Indeed, equality (34) below gives its pointwise limit as a Fourier multiplier and its global expression is described in proposition 3.5, which rewrites using the plasma physics language as proposition 3.6 on the object called conductivity.

3.1. Solution of the linearized Vlasov equation. In this section we prove existence and uniqueness of the solution of the damped problem (8), which reads

$$\mathcal{L}_\nu f_\nu = -(q/m)EF'_0,$$

where $\mathcal{L}_\nu = \mathcal{L} + \nu = \partial_t + v\partial_x + \nu$. Let us also define the function

$$f_{\nu,*}(x, v) := -(q/m)F'_0(v) \int_{-\infty}^0 e^{\nu s} E(s, x + vs) ds,$$

the integral being absolutely convergent.

Proposition 3.1. *For any $\nu > 0$, $E \in \mathcal{S}(\mathbb{R}^2)$ and $F_0 \in \mathcal{S}(\mathbb{R})$, equation (8) has a solution $f_\nu \in \mathcal{S}(\mathbb{R}^3)$, given by its Fourier transform*

$$(25) \quad \hat{f}_\nu = -i(q/m) \frac{\hat{E}F'_0}{\omega - kv + i\nu}.$$

This is the unique solution in $\mathcal{S}'(\mathbb{R}^3)$ and the unique (classical) solution of the Cauchy problem

$$\mathcal{L}_\nu f_\nu(t, x, v) = -(q/m)E(t, x)F'_0(v), \quad f_\nu(0, x, v) = f_{\nu,*}(x, v),$$

with initial condition given at $t = 0$.

Remark 4. In proposition 3.1, we distinguish between the equation in \mathcal{S}' and the equation stated for $C^1(\mathbb{R}^2)$ functions.

Proof. For $F_0 \in \mathcal{S}(\mathbb{R})$, $E \in \mathcal{S}(\mathbb{R}^2)$ and $\nu > 0$, after partial Fourier transform in (t, x) we observe that necessarily a solution in \mathcal{S}' of the considered equation is given by (25). Since the function $(\omega, k, v) \mapsto (\omega - kv + i\nu)^{-1}$ belongs to C^∞ and has polynomially bounded derivatives, from equation (25) we check that $\hat{f}_\nu \in \mathcal{S}(\mathbb{R}^3)$, and thus $f_\nu \in \mathcal{S}(\mathbb{R}^3)$. However the sequence $\{f_\nu\}_\nu$ is not uniformly bounded in \mathcal{S} .

The inverse partial Fourier transform gives

$$(26) \quad f_\nu(t, x, v) = -(q/m)F_0'(v) \int_{-\infty}^t e^{-\nu \cdot (t-s)} E(s, x - v \cdot (t-s)) ds,$$

and we note that

$$X(s, t, x, v) = x - v \cdot (t-s), \quad V(s, t, x, v) = v,$$

is the solution of the equations for the characteristic curves of \mathcal{L}_ν integrated backward in time from (t, x, v) . Therefore (26) is the classical solution as claimed. \square

Remark 5. This result shows that the requirement $f_\nu \in \mathcal{S}'(\mathbb{R}^3)$ leads to the selection of a specific initial condition $f_{\nu,*}$ for the Cauchy problem, thus uniquely determining the solution. Conversely, the solution of the Cauchy problem with initial condition $f_{\nu,*}$ is an element of \mathcal{S} and thus of \mathcal{S}' .

As an alternative way to illustrate how the condition $f_\nu \in \mathcal{S}'(\mathbb{R}^3)$ leads to the selection of a specific Cauchy datum, one can consider for $f_{\nu,0} \in L^2(\mathbb{R}^2)$ the initial value problem in $C^1(\mathbb{R}, L^2(\mathbb{R}^2))$,

$$\begin{cases} \partial_t f_\nu(t, x, v) + v \partial_x f_\nu(t, x, v) + \nu f_\nu(t, x, v) = -(q/m)E(t, x)F_0'(v), \\ f_\nu(0, x, v) = f_{\nu,0}(x, v). \end{cases}$$

Performing the Fourier transform in space, we obtain an ordinary differential equation almost everywhere in the (k, v) -space,

$$\begin{cases} \partial_t \tilde{f}_\nu(t, k, v) + ikv \tilde{f}_\nu(t, k, v) + \nu \tilde{f}_\nu(t, k, v) = -(q/m)\tilde{E}(t, k)F_0'(v), \\ \tilde{f}_\nu(0, k, v) = \tilde{f}_{\nu,0}(k, v), \end{cases}$$

the solution of which is

$$\tilde{f}_\nu(t, k, v) = e^{-(\nu+ikv)t} \left[\tilde{f}_{\nu,0}(k, v) - (q/m)F_0'(v) \int_0^t e^{(\nu+ikv)s} \tilde{E}(s, k) ds \right].$$

We see that $f_\nu, \tilde{f}_\nu \in C^\infty(\mathbb{R}, L^2(\mathbb{R}^2))$. The integral factor has a finite limit for $t \rightarrow -\infty$,

$$\begin{aligned} \tilde{f}_{\nu,*}(k, v) &= (q/m)F_0'(v) \lim_{t \rightarrow -\infty} \int_0^t e^{(\nu+ikv)s} \tilde{E}(s, k) ds \\ &= -(q/m)F_0'(v) \int_{-\infty}^0 e^{(\nu+ikv)s} \tilde{E}(s, k) ds, \end{aligned}$$

and $\tilde{f}_{\nu,*} \in L^2(\mathbb{R}^2)$. We show that if $\tilde{f}_{\nu,0} \neq \tilde{f}_{\nu,*}$, then the solution \tilde{f}_ν , or equivalently f_ν , is not tempered in time for any k, v fixed. With this aim we write

$$\tilde{f}_\nu(t, k, v) = e^{-(\nu+ikv)t} [\tilde{f}_\nu^{(0)}(k, v) - \tilde{f}_{\nu,*}(k, v)] - \frac{qF_0'(v)}{m} \int_{-\infty}^t e^{-(\nu+ikv)(t-s)} \tilde{E}(s, k) ds,$$

and one observes that, for every k, v , the second term on the right-hand side belongs to $C_b^\infty(\mathbb{R})$ and thus to $\mathcal{S}'(\mathbb{R})$. As for the first term on the right-hand side, for any k, v , and $\nu > 0$ the function $t \mapsto e^{-\nu t + ikvt}$ is not tempered, since there exists

a test function $\varphi(t) = e^{-\nu\sqrt{1+t^2}-ikvt}$ in $\mathcal{S}(\mathbb{R})$ such that $\int e^{-\nu t+ikvt}\varphi(t)dt = +\infty$. Hence the first term cannot be the partial Fourier transform of a distribution, which was the hypotheses that allowed us to write the equation in Fourier space.

It follows that we have $\tilde{f}_\nu \in \mathcal{S}'$ if and only if the initial condition satisfies $\tilde{f}_\nu^{(0)}(k, v) = \tilde{f}_{\nu,*}(k, v)$ almost everywhere, that is,

$$\tilde{f}_{\nu,0}(k, v) = \tilde{f}_{\nu,*}(k, v) = -(q/m)F'_0(v) \int_{-\infty}^0 e^{(\nu+ikv)s} \tilde{E}(s, k) ds.$$

The corresponding solution amounts to

$$\tilde{f}_\nu(t, k, v) = -(q/m)F'_0(v) \int_{-\infty}^t e^{-(\nu+ikv)(t-s)} \tilde{E}(s, k) ds,$$

and this is the unique solution in $\mathcal{S}'(\mathbb{R}^3)$. Upon inserting the full Fourier transform of $E(t, x)$, one can check that this gives (26).

We apply now the limiting absorption principle, that is we consider the limit of the distribution function f_ν for $\nu \rightarrow 0^+$.

Proposition 3.2. *For any $E \in \mathcal{S}(\mathbb{R}^2)$ and $F_0 \in \mathcal{S}(\mathbb{R})$, the solution f_ν defined in (26) has a pointwise limit for $\nu \rightarrow 0^+$ given by*

$$f(t, x, v) = -(q/m)F'_0(v) \int_{-\infty}^t E(s, x - v \cdot (t - s)) ds,$$

which is in $C_b^\infty(\mathbb{R}^3)$, with $f(t, x, \cdot) \in \mathcal{S}(\mathbb{R})$, and solves $\mathcal{L}_0 f = -(q/m)EF'_0$.

Proof. We observe that for $\nu > 0$ and $s < t$ the function

$$s \mapsto e^{-\nu \cdot (t-s)} E(s, x - v \cdot (t - s)),$$

is bounded by

$$|E(s, x - v \cdot (t - s))| \leq \frac{1}{(1 + s^2)^m} \sup_{t,x} |(1 + t^2 + x^2)^m E(t, x)|,$$

for all $m \geq 0$. If we choose $m > 1/2$, then $1/(1 + s^2)^m$ is integrable and by the dominated convergence theorem, for any $(t, x, v) \in \mathbb{R}^3$,

$$f_\nu(t, x, v) \xrightarrow{\nu \rightarrow 0^+} f(t, x, v) := -(q/m)F'_0(v) \int_{-\infty}^t E(s, x - v \cdot (t - s)) ds.$$

We observe that the pointwise limit is the causal solution of linear advection equation $\mathcal{L}_0 f = -(q/m)F'_0 E$ in the sense of appendix C and the characteristic flow satisfies the hypothesis of proposition C.1. Hence proposition C.2 gives $f \in C_b^\infty(\mathbb{R}^3)$. Since $f(t, x, v)$ is proportional to $F'_0(v)$, we have $f(t, x, \cdot) \in \mathcal{S}(\mathbb{R})$. \square

Remark 6. We can deduce other properties of the solution f . Since $f(t, x, \cdot)$ is rapidly decreasing, we also have $f(t, x, \cdot) \in L^1(\mathbb{R})$ as it should be (in view of its meaning as particle density). Continuity implies that $f(t, \cdot, \cdot)$ is in $L^1(K \times \mathbb{R})$ for every compact $K \subset \mathbb{R}$, and this is physically appropriate for such an idealized model, which, being spatially uniform, has an infinite number of particles: only the number of particles $\|f(t, \cdot, \cdot)\|_{L^1(K \times \mathbb{R})}$ in a compact spatial domain K has to be finite.

The pointwise limit obtained in proposition 3.2 is referred to as the response of the plasma to the perturbation E .

3.2. Current density and conductivity operator. We can now compute the electric current density via equation (5), namely,

$$(27) \quad j_\nu(t, x) = q \int_{\mathbb{R}} v f_\nu(t, x, v) dv,$$

and $j_\nu \in \mathcal{S}(\mathbb{R}^2)$. The map $E \mapsto j_\nu = \sigma_\nu(E)$ defines a linear continuous operator $\sigma_\nu : \mathcal{S}(\mathbb{R}^2) \rightarrow \mathcal{S}(\mathbb{R}^2)$ which is given by the Fourier multiplier

$$(28) \quad \hat{j}_\nu(\omega, k) = \hat{\sigma}_\nu(\omega, k) \hat{E}(\omega, k), \quad \hat{\sigma}_\nu(\omega, k) = -i \frac{q^2}{m} \int_{\mathbb{R}} \frac{v F_0'(v)}{\omega - kv + i\nu} dv.$$

The continuity of σ_ν in particular follows from the estimate

$$|\partial_\omega^\alpha \partial_k^\beta \hat{\sigma}_\nu(\omega, k)| \leq \frac{C}{\nu^{\alpha+\beta+1}} \int_{\mathbb{R}} |v^{\beta+1} F_0'(v)| dv,$$

for any non-negative integers α, β , where the constant C depends only on q^2/m , α , and β . We observe that this estimate is not uniform in ν as expected, since the sequence f_ν is not uniformly bounded in \mathcal{S} .

For the slightly more general case of non-homogeneous equilibria of the form $F_0(x, v) = n_0(x) \tilde{F}_0(v)$ with $n_0 \in C_b^\infty$, one can define $\tilde{\sigma}_\nu : \mathcal{S}(\mathbb{R}^2) \rightarrow \mathcal{S}(\mathbb{R}^2)$ as the Fourier multiplier with symbol

$$(29) \quad \hat{\tilde{\sigma}}_\nu(\omega, k) = -i \frac{q^2}{m} \int_{\mathbb{R}} \frac{v \tilde{F}_0'(v)}{\omega - kv + i\nu} dv,$$

and obtain the induced current

$$(30) \quad \hat{j}_\nu(\omega, k) = \hat{\tilde{\sigma}}_\nu(\omega, k) \widehat{n_0 E}(\omega, k).$$

In this case the conductivity operator is

$$j_\nu = \sigma_\nu(E) := \tilde{\sigma}_\nu(n_0 E),$$

and it amounts to the pseudo-differential operator

$$(31) \quad j_\nu(t, x) = \frac{1}{(2\pi)^2} \int e^{-i\omega(t-t') + ik(x-x')} n_0(x') \hat{\tilde{\sigma}}_\nu(\omega, k) E(t', x') dt' dx' d\omega dk,$$

where the integral is in the sense of oscillatory integrals and the symbol of the operator is $\hat{\sigma}_\nu(x', \omega, k) = n_0(x') \hat{\tilde{\sigma}}_\nu(\omega, k)$.

By using the dominated convergence theorem we have that the limit of the current density j_ν is equal to the current carried by the limit distribution function f . Specifically we have

Proposition 3.3. *With $E \in \mathcal{S}(\mathbb{R}^2)$ and $F_0 \in \mathcal{S}(\mathbb{R})$, the function defined by*

$$j(t, x) = -\frac{q^2}{m} \int_{D_t} v F_0'(v) E(s, x - v \cdot (t - s)) ds dv,$$

with $D_t = (-\infty, t] \times \mathbb{R}$, belongs to $C_b^\infty(\mathbb{R}^2)$, hence to \mathcal{S}' . The map $\sigma : E \mapsto j = \sigma(E)$ is a linear continuous operator from $\mathcal{S}(\mathbb{R}^2) \rightarrow \mathcal{S}'(\mathbb{R}^2)$.

Proof. In the domain $D_t = (-\infty, t] \times \mathbb{R}$, we change variables to $s' = t - s$, $v' = v$ for $(s, v) \in D_t$, and thus $(s', v') \in [0, +\infty) \times \mathbb{R}$. Then

$$j(t, x) = -\frac{q^2}{m} \int_0^{+\infty} \int_{\mathbb{R}} v F_0'(v) E(t - s, x - vs) dv ds.$$

The integrand $\mathcal{E}(t, x, s, v) = vF'_0(v)E(t-s, x-vs)$ is such that

$$\partial_t^\alpha \partial_x^\beta \mathcal{E}(t, x, s, v) = vF'_0(v) \partial_t^\alpha \partial_x^\beta E(t-s, x-vs) \leq \frac{|vF'_0(v)|}{(1+(t-s)^2)^m} \|E\|_{\alpha+\beta+2m}.$$

For any $t \in \mathbb{R}$, this upper bound belongs to $L^1(\mathbb{R}^2)$. By the dominated convergence theorem, we deduce that $j \in C_b^\infty(\mathbb{R}^2)$ and $|\partial_t^\alpha \partial_x^\beta j(t, x)| \leq C_m \|E\|_{\alpha+\beta+2m}$.

As $C_b^\infty(\mathbb{R}^2) \subset \mathcal{S}'(\mathbb{R}^2)$ and $\|j\|_{L^\infty(\mathbb{R}^2)} \leq C \|E\|_{2m}$, one has that the map $E \mapsto j$ from $\mathcal{S} \rightarrow \mathcal{S}'$ is continuous. \square

Proposition 3.4. *When $\nu \rightarrow 0^+$, the sequence j_ν defined in equation (28) converges to j both pointwise in \mathbb{R}^2 and in $\mathcal{S}'(\mathbb{R}^2)$.*

Proof. With $\mathcal{E}(t, x, s, v) = vF'_0(v)E(s, x-v \cdot (t-s))$ as in the proof of proposition 3.3, we have

$$j_\nu(t, x) - j(t, x) = \frac{q^2}{m} \int_{D_t} (1 - e^{-\nu(t-s)}) \mathcal{E}(t, x, s, v) ds dv.$$

Since $e^{-\nu(t-s)} \leq 1$ for $(s, v) \in D_t$, we have

$$(1 - e^{-\nu(t-s)}) |\mathcal{E}(t, x, s, v)| \leq \frac{\|F_0\|_{2m_2+2} \|E\|_{2m_1}}{(1+s^2)^{m_1} (1+v^2)^{m_2}}$$

and for $m_1, m_2 > 1/2$ the bound is in L^1 . The dominated convergence theorem then yields pointwise convergence: $\lim (j_\nu(t, x) - j(t, x)) = 0$ for all $(t, x) \in \mathbb{R}^2$. The same estimate also gives convergence in $\mathcal{S}'(\mathbb{R}^2)$: for any test function $\phi \in \mathcal{S}(\mathbb{R}^2)$,

$$\langle j_\nu - j, \phi \rangle = \frac{q^2}{m} \int_{\mathbb{R}^2} \int_{D_t} (1 - e^{-\nu(t-s)}) \mathcal{E}(t, x, s, v) ds dv \phi(t, x) dt dx,$$

and the integrand is bounded by $\|F_0\|_{2m_2+2} \|E\|_{2m_1} (1+s^2)^{-m_1} (1+v^2)^{-m_2} |\phi(t, x)|$ which is integrable. Again the dominated convergence theorem allows us to pass to the limit in the integral and obtain $\langle j_\nu - j, \phi \rangle \rightarrow 0$. \square

For an explicit calculation of the conductivity operator we consider the limit $\nu \rightarrow 0^+$ in Fourier space. As a tempered distribution, \hat{j}_ν acts on $\psi \in \mathcal{S}(\mathbb{R}^2)$ by

$$(32) \quad \langle \hat{j}_\nu, \psi \rangle = -i(q^2/m) \int_{\mathbb{R}^2 \times \mathbb{R}} vF'_0(v) \frac{\hat{E}(\omega, k) \psi(\omega, k)}{\omega - kv + i\nu} d\omega dk dv.$$

We now want to pass to the limit for $\nu \rightarrow 0^+$. We use the Hilbert transform (appendix D).

Let $G(v) = vF'_0(v)/n_0$ where $n_0 > 0$ is the uniform background plasma density and let $\omega_p = \sqrt{4\pi q^2 n_0/m}$ be the plasma frequency of the considered species. For $(\omega, k) \in \mathbb{R}^2, k \neq 0$, let

$$(33) \quad \hat{\sigma}_{\text{ph}}(\omega, k) := -i \frac{\omega_p^2}{4\pi} \frac{1}{k} \left[\pi \mathcal{H}(G)(\omega/k) - i\pi G(\omega/k) \right],$$

where $\mathcal{H}(G)$ is the Hilbert transform of G . The tensor $\hat{\sigma}_{\text{ph}}$ is the same as the one obtained formally in the physics literature. We have $4\pi i \omega \hat{\sigma}_{\text{ph}}(\omega, k) = \omega_p^2 H(\omega/k)$ for $k \neq 0$, where the function $H \in C^\infty(\mathbb{R})$ is given by $H(z) = z[\pi \mathcal{H}(G)(z) - i\pi G(z)]$. Then we have

$$(34) \quad \lim_{\nu \rightarrow 0^+} \hat{\sigma}_\nu = \hat{\sigma}_{\text{ph}} \text{ pointwise in } (\omega, k) \in \mathbb{R}^2, k \neq 0,$$

since $\hat{\sigma}_\nu(\omega, k) = -i(\omega_p^2/4\pi)A_\nu(\omega, k)$, where A_ν is defined in appendix D, and equation (34) follows from lemma D.2.

Let us introduce $\lambda > 0$, a cut-off function $\chi \in C_0^\infty(\mathbb{R})$, $\chi(z) = 1$ for $|z| \leq 1/2$ and $\text{supp } \chi \subset (-1, 1)$, and let $\chi_\lambda(k) = \chi(\lambda k)$. The real number λ can be interpreted as a scale-length in the frequency domain. Then we define two operators $\sigma_{\lambda, 1-\chi}, \sigma_{\lambda, \chi} : \mathcal{S}(\mathbb{R}^2) \rightarrow \mathcal{S}'(\mathbb{R}^2)$ given by

$$(35) \quad \sigma_{\lambda, 1-\chi}(E) := \mathcal{F}^{-1}((1 - \chi_\lambda)\hat{\sigma}_{\text{ph}}\hat{E}), \quad \sigma_{\lambda, \chi}(E) := \sigma - \sigma_{\lambda, 1-\chi}.$$

We note that $\sigma_{\lambda, 1-\chi}$ is continuous since it is the composition of continuous operations, and we have shown in proposition 3.3 that σ is continuous, therefore $\sigma_{\lambda, \chi}$ is continuous.

Proposition 3.5. *The operators σ and $\sigma_{\lambda, \chi}$ defined in proposition 3.3 and equation (35), respectively, are such that*

$$\begin{aligned} \langle \mathcal{F}(\sigma_{\lambda, \chi}(E)), \hat{\psi} \rangle &= i \frac{\omega_p^2}{4\pi} \int_{\mathbb{R}^2} \chi_\lambda(k) G(v) \left[\pi \mathcal{H}(\hat{E}\hat{\psi}(\cdot, k))(kv) + i\pi \hat{E}\hat{\psi}(kv, k) \right] dkdv, \\ \langle \mathcal{F}(\sigma(E)), \hat{\psi} \rangle &= i \frac{\omega_p^2}{4\pi} \int_{\mathbb{R}^2} G(v) \left[\pi \mathcal{H}(\hat{E}\hat{\psi}(\cdot, k))(kv) + i\pi \hat{E}\hat{\psi}(kv, k) \right] dkdv, \end{aligned}$$

for all $E, \psi \in \mathcal{S}(\mathbb{R}^2)$.

Proof. Let $G(v) = vF_0'(v)/n_0$, $\omega_p^2 = 4\pi q^2 n_0/m$, and for any $\phi \in \mathcal{S}(\mathbb{R}^2)$, let A_ν and B_ν be the integrals defined in appendix D. Let

$$\begin{aligned} I_{\lambda, 1-\chi}^\nu(\phi) &:= -i \frac{\omega_p^2}{4\pi} \int_{\mathbb{R}^3} (1 - \chi_\lambda(k)) G(v) \frac{\phi(\omega, k)}{\omega - kv + i\nu} d\omega dkdv \\ &= -i \frac{\omega_p^2}{4\pi} \int_{\mathbb{R}^2} (1 - \chi_\lambda(k)) \phi(\omega, k) A_\nu(\omega, k) d\omega dk, \\ I_{\lambda, \chi}^\nu(\phi) &:= -i \frac{\omega_p^2}{4\pi} \int_{\mathbb{R}^3} \chi_\lambda(k) G(v) \frac{\phi(\omega, k)}{\omega - kv + i\nu} d\omega dkdv \\ &= -i \frac{\omega_p^2}{4\pi} \int_{\mathbb{R}^2} \chi_\lambda(k) G(v) B_\nu(v, k) dkdv. \end{aligned}$$

Since $k \neq 0$ in the support of $1 - \chi_\lambda$, the integrands have a pointwise limit as $\nu \rightarrow 0^+$ computed in lemma D.2. In addition, lemma D.2 shows that the integrands are bounded by a function in L^1 uniformly in ν . The dominated convergence theorem applies and we can pass to the limit $\nu \rightarrow 0^+$ in the integrals, obtaining

$$\begin{aligned} I_{\lambda, 1-\chi}^\nu(\phi) &\rightarrow -i \frac{\omega_p^2}{4\pi} \int_{\mathbb{R}^2} (1 - \chi_\lambda(k)) \phi(\omega, k) \frac{1}{k} \left[\pi \mathcal{H}(G)(\omega/k) - i\pi G(\omega/k) \right] d\omega dk, \\ I_{\lambda, \chi}^\nu(\phi) &\rightarrow -i \frac{\omega_p^2}{4\pi} \int_{\mathbb{R}^2} \chi_\lambda(k) G(v) \left[-\pi \mathcal{H}(\phi(\cdot, k))(kv) - i\pi \phi(kv, k) \right] dkdv. \end{aligned}$$

Particularly, $I_{\lambda, 1-\chi}^\nu(\phi) \rightarrow \int (1 - \chi_\lambda(k)) \hat{\sigma}_{\text{ph}}(\omega, k) \phi(\omega, k) d\omega dk$. Then we have

$$\begin{aligned} \langle \hat{j}_\nu, \hat{\psi} \rangle &= I_{\lambda, 1-\chi}^\nu(\hat{E}\hat{\psi}) + I_{\lambda, \chi}^\nu(\hat{E}\hat{\psi}) \xrightarrow{\nu \rightarrow 0^+} \langle \mathcal{F}(\sigma_{\lambda, 1-\chi}(E)), \hat{\psi} \rangle \\ &\quad + i \frac{\omega_p^2}{4\pi} \int_{\mathbb{R}^2} \chi_\lambda(k) G(v) \left[\pi \mathcal{H}(\hat{E}\hat{\psi}(\cdot, k))(kv) + i\pi \hat{E}\hat{\psi}(kv, k) \right] dkdv. \end{aligned}$$

On the other hand we know from proposition 3.4 that as $\nu \rightarrow 0^+$, $j_\nu \rightarrow j = \sigma(E)$ in \mathcal{S}' , hence

$$\langle \hat{j}_\nu, \hat{\psi} \rangle \xrightarrow{\nu \rightarrow 0^+} \langle \mathcal{F}(\sigma(E)), \hat{\psi} \rangle,$$

and by definition $\sigma(E) = \sigma_{\lambda, 1-\chi}(E) + \sigma_{\lambda, \chi}(E)$. Uniqueness of the limit gives the claimed expression for $\sigma_{\lambda, \chi}$.

The claimed expression for $\sigma(E)$ follows analogously on noting that

$$\langle \hat{j}_\nu, \hat{\psi} \rangle = I^\nu(\hat{E}\hat{\psi}) := -i \frac{\omega_p^2}{4\pi} \int_{\mathbb{R}^2} G(v) B_\nu(v, k) dk dv,$$

where the function B_ν is now computed with $\phi = \hat{E}\hat{\psi}$, i.e.,

$$B_\nu(\omega, k) = \int_{\mathbb{R}} \frac{\hat{E}(\omega, k) \hat{\psi}(\omega, k)}{\omega - kv + i\nu} d\omega.$$

As $\nu \rightarrow 0^+$ the right-hand side converges to $\langle \mathcal{F}(\sigma(E)), \hat{\psi} \rangle$, while the limit of the left-hand side is dealt with as in the case of $I'_{\lambda, \chi}$. \square

The operator $\sigma_{\lambda, \chi}$ does not play any role when the electric field perturbation is supported away from $k = 0$, i.e., for non-constant fields. More precisely we have the following result, which expresses the usual Ohm's law for a uniform plasma.

Corollary 3.6. *If $E \in \mathcal{S}(\mathbb{R}^2)$ is such that $\hat{E}(\omega, k) = 0$ for $|k| \leq 1/\lambda$, then $\hat{j} \in C^\infty(\mathbb{R}^2)$ and $\hat{j}(\omega, k) = \hat{\sigma}_{\text{ph}}(\omega, k) \hat{E}(\omega, k)$.*

Proof. By hypothesis $\chi_\lambda(k) \hat{E}(\omega, k) = 0$ for all $(\omega, k) \in \mathbb{R}^2$, hence, $\sigma_{\lambda, \chi}(E) = 0$; this follows directly from the expression given in proposition 3.5 since, in particular, $\chi_\lambda(k) \mathcal{H}(\hat{E}\hat{\psi}(\cdot, k))(kv) = \mathcal{H}(\chi_\lambda \hat{E}\hat{\psi}(\cdot, k))(kv)$. Then

$$\mathcal{F}(\sigma(E)) = \mathcal{F}(\sigma_{\lambda, 1-\chi}(E)) = (1 - \chi_\lambda) \hat{\sigma}_{\text{ph}} \hat{E} = \hat{\sigma}_{\text{ph}} \hat{E},$$

since $(1 - \chi_\lambda) \hat{E} = \hat{E}$. The fact that \hat{j} is in C^∞ follows from the properties of the Hilbert transform summarized in proposition D.1, that imply in particular, $\mathcal{H}(G) \in H^\infty(\mathbb{R})$. \square

Remark 7. In the case of non-homogeneous equilibria of the form $F_0(x, v) = n_0(x) \tilde{F}_0(v)$, the statement of corollary 3.6 remains true with G and \hat{E} replaced by $v \tilde{F}_0(v)$ and $\widehat{n_0 E}$, respectively. Particularly, one has

$$\mathcal{F}(\sigma_{\lambda, 1-\chi}(E))(\omega, k) = (1 - \chi_\lambda(k)) \widehat{\hat{\sigma}_{\text{ph}}}(\omega, k) \widehat{n_0 E}(\omega, k),$$

where $\widehat{\hat{\sigma}_{\text{ph}}}$ is obtained from (29) in analogy with $\hat{\sigma}_{\text{ph}}$. Then, the operator

$$\begin{aligned} \sigma_{\lambda, 1-\chi}(E)(t, x) &= \frac{1}{(2\pi)^2} \int e^{-i\omega(t-t') + ik(x-x')} \\ &\quad \times [n_0(x') (1 - \chi_\lambda(k)) \widehat{\hat{\sigma}_{\text{ph}}}(\omega, k)] E(t', x') dt' dx' d\omega dk \end{aligned}$$

is pseudo-differential with symbol $n_0(x') (1 - \chi_\lambda(k)) \widehat{\hat{\sigma}_{\text{ph}}}(\omega, k)$, recovering an expression similar to (31).

3.3. Proof of the main results for the non-magnetized non-relativistic one-dimensional case (section 1.3). We collect at last the partial results of this section and give the proofs of the two theorems stated in section 1.3.

Proof of theorem 1.1. (i) The fact that f_ν belongs to $\mathcal{S}(\mathbb{R}^3)$ and is the unique solution of equation (8) in $\mathcal{S}'(\mathbb{R}^3)$ is proven in proposition 3.1. The current density j_ν and the operator σ_ν are given in equation (27) and (28) and related comments.

(ii) and (iii) Pointwise convergence $f_\nu \rightarrow f$ is established in proposition 3.2 and an expression for the solution f is given there. In proposition 3.2, it is also proven that $f(t, x, \cdot) \in \mathcal{S}(\mathbb{R})$ for every $(t, x) \in \mathbb{R}^2$. As for the convergence of $f_\nu \rightarrow f$ in the topology of \mathcal{S}' , proposition 3.2 establishes pointwise (but not uniform) convergence $f_\nu \rightarrow f$ with limit $f \in C_b^\infty$; in addition, for any integer $m \geq 1/2$, we have

$$|f_\nu(t, x, v)| \leq \|qF'_0/m\|_0 \cdot \|E\|_{2m} \int_{-\infty}^{+\infty} \frac{ds}{(1+s^2)^m},$$

uniformly in $\nu \in [0, +\infty)$. Therefore for every $\varphi \in \mathcal{S}(\mathbb{R}^3)$, the function $(f_\nu - f)\varphi$ satisfies

$$|f_\nu(t, x, v) - f(t, x, v)| |\varphi(t, x, v)| \leq C |\varphi(t, x, v)|.$$

As $|\varphi| \in L^1$, $(f_\nu - f)\varphi$ satisfies the hypothesis of the dominated convergence theorem and

$$\langle f_\nu - f, \phi \rangle = \int_{\mathbb{R}^3} (f_\nu - f) \phi dt dx dv \rightarrow 0, \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^3).$$

In proposition 3.3, it is shown that $j = \mathbb{J}(f) \in C_b^\infty(\mathbb{R}^2)$, and proposition 3.4 establishes the limit $j_\nu \rightarrow j$ both pointwise and in $\mathcal{S}'(\mathbb{R}^2)$. \square

Proof of theorem 1.2. Proposition 3.3 also establishes the continuity of the operator $\sigma : E \mapsto j$. The relation to the physical conductivity operator is proven in corollary 3.6. \square

4. STUDY OF A PDE WITH PARAMETERS OF THE FORM

$$-(u_1 \partial_{u_2} - u_2 \partial_{u_1}) \varphi(\theta, u) - ia(\theta, u) \varphi(\theta, u) = \psi(\theta, u)$$

In this section we establish existence and uniqueness results for a partial differential equation with parameters that arises in the study of the relativistic, linear Vlasov equation with uniform magnetic field B_0 , addressed below in section 5. Specifically the equation is

$$(36) \quad -(u_1 \partial_{u_2} - u_2 \partial_{u_1}) \varphi(\theta, u) - ia(\theta, u) \varphi(\theta, u) = \psi(\theta, u),$$

where $a, \psi \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3)$ are given complex-valued functions of $u \in \mathbb{R}^3$ and depend on parameters $\theta \in \mathbb{R}^l$. The operator $-(u_1 \partial_{u_2} - u_2 \partial_{u_1})$ originates from the Lorentz force term $q(v \times B_0) \cdot \nabla_p$ with $q > 0$ and B_0 constant and directed along the third axis. Eventually, the parameters θ will be related to the Fourier variables (τ, ξ) , and a to a_ε defined in equation (56) of section 5 below. Therefore, we assume that a satisfies a condition similar to (17), that is,

$$(37a) \quad a \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3, \mathbb{C}), \quad (u_1 \partial_{u_2} - u_2 \partial_{u_1}) a(\theta, u) = 0, \quad |\operatorname{Im} a(\theta, u)| \geq \eta > 0,$$

for a given constant $\eta > 0$. As we need to control the growth of derivatives of the solution at infinity, we shall also assume that

$$(37b) \quad |\partial_\theta^\alpha \partial_u^\beta a(\theta, u)| \leq C_{\alpha, \beta} (1 + \theta^2 + u^2)^m, \quad \forall \alpha \in \mathbb{N}_0^l, \quad \forall \beta \in \mathbb{N}_0^3,$$

uniformly in $(\theta, u) \in \mathbb{R}^l \times \mathbb{R}^3$ for a given $m \in \mathbb{R}$ and with constants $C_{\alpha, \beta} > 0$ depending on the multi-indices.

First we establish the uniqueness of the solution under rather general conditions.

Lemma 4.1. *Let $a \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3)$ satisfy condition (37a), $\Theta \subseteq \mathbb{R}^l$ be an open set, and let $\varphi \in L^2_{\text{loc}}(\Theta \times \mathbb{R}^3)$ be a function with weak derivatives $\partial_{u_1}\varphi, \partial_{u_2}\varphi \in L^2_{\text{loc}}(\Theta \times \mathbb{R}^3)$ and such that*

$$-(u_1\partial_{u_2} - u_2\partial_{u_1})\varphi - ia\varphi = 0, \quad \text{a.e. in } \Theta \times \mathbb{R}^3.$$

Then, $\varphi = 0$ a.e. in $\Theta \times \mathbb{R}^3$.

Proof. For almost all $(\theta, u_3) \in \Theta \times \mathbb{R}$, the function $\tilde{\varphi}(u_1, u_2) := \varphi(\theta, u_1, u_2, u_3)$ belongs to $H^1(B_r(0))$ for every $r > 0$, where $B_r(0) \subset \mathbb{R}^2$ is the open ball of radius r and centered in zero in \mathbb{R}^2 . From the equation we deduce

$$-(u_1\partial_{u_2} - u_2\partial_{u_1})|\tilde{\varphi}|^2 + 2\text{Im}(a)|\tilde{\varphi}|^2 = 0.$$

The first term amounts to the divergence of the vector field $(-u_2, u_1)|\tilde{\varphi}|^2$ which is tangent to $\partial B_r(0)$, hence Gauss theorem for the divergence, which holds for H^1 functions, gives

$$0 = \int_{B_r(0)} (u_1\partial_{u_2} - u_2\partial_{u_1})|\tilde{\varphi}|^2 du_1 du_2 = 2 \int_{B_r(0)} \text{Im } a |\tilde{\varphi}|^2 du_1 du_2,$$

for every radius $r > 0$. We can now conclude upon accounting for hypotheses (37a). If $\text{Im } a \geq \eta > 0$, we have

$$0 \leq \eta \int_{B_r(0)} |\tilde{\varphi}|^2 du_1 du_2 \leq \int_{B_r(0)} \text{Im } a |\tilde{\varphi}|^2 du_1 du_2 = 0.$$

If instead $-\text{Im } a \geq \eta > 0$,

$$0 \leq \eta \int_{B_r(0)} |\tilde{\varphi}|^2 du_1 du_2 \leq - \int_{B_r(0)} \text{Im } a |\tilde{\varphi}|^2 du_1 du_2 = 0.$$

In both cases we deduce

$$\int_{B_r(0)} |\tilde{\varphi}|^2 du_1 du_2 = 0,$$

and thus $\tilde{\varphi} = 0$ a.e. in $B_r(0)$ for all r and for almost all $(\theta, u_3) \in \Theta \times \mathbb{R}$. It follows that $\varphi = 0$ a.e. in $\Theta \times \mathbb{R}^3$. \square

In the remaining part of this section, we first give an existence result for the case in which the source term ψ is a polynomial in (u_1, u_2) ; this is based on an algebraic argument. Then, we prove the existence of a smooth solution $\varphi \in C^\infty$ when $\psi \in C^\infty$ and of a solution $\varphi \in \mathcal{S}$ when $\psi \in \mathcal{S}$. The latter implies uniqueness of the solution in \mathcal{S}' .

4.1. Equation with a polynomial source term. Let the source term in equation (36) be a polynomial of the form

$$(38) \quad \psi(\theta, u) = \sum_{0 \leq m+n \leq L} \mathcal{Y}_{m,n}(\theta, u_3) u_1^m u_2^n, \quad \mathcal{Y}_{m,n} \in C^\infty,$$

and let us consider for $z \in \mathbb{C} \setminus \mathbb{Z}$ the equation

$$(39) \quad -(u_1\partial_{u_2} - u_2\partial_{u_1})\tilde{\varphi}(z; \theta, u) - iz\tilde{\varphi}(z; \theta, u) = \psi(\theta, u).$$

We can search for solutions of the form

$$(40) \quad \tilde{\varphi}(z; \theta, u) = \sum_{0 \leq m+n \leq L} \mathcal{X}_{m,n}(z; \theta, u_3) u_1^m u_2^n,$$

that is, a polynomial with at most the same degree as the source term. Substitution into (39) yields

$$\begin{aligned} & - \sum_{m=1} \sum_{n=0} (n+1) \mathcal{X}_{m-1, n+1} u_1^m u_2^n + \sum_{m=0} \sum_{n=1} (m+1) \mathcal{X}_{m+1, n-1} u_1^m u_2^n \\ & - iz \sum_{m=0} \sum_{n=0} \mathcal{X}_{m,n} u_1^m u_2^n = \sum_{m=0} \sum_{n=0} \mathcal{Y}_{m,n} u_1^m u_2^n, \end{aligned}$$

where the sums are all finite since $m+n \leq L$. We observe that only the coefficients $\mathcal{X}_{m,n}$ with $m+n = \ell$ for $\ell = 0, 1, 2, \dots$ are coupled. For every integer $0 \leq \ell \leq L$, we define

$$x_\ell = (\mathcal{X}_{\ell-j, j})_{j=0}^\ell, \quad y_\ell = (-i \mathcal{Y}_{\ell-j, j})_{j=0}^\ell,$$

then the linear equation for the coefficients splits into $(\ell+1)$ -dimensional blocks of the form

$$(41) \quad (A_\ell - z)x_\ell = y_\ell, \quad 0 \leq \ell \leq L,$$

where the matrix $A_\ell \in \mathbb{C}^{(\ell+1) \times (\ell+1)}$ is defined and given in appendix E. For each ℓ , equation (41) has a unique solution when z is not an eigenvalue of the matrix A_ℓ . Lemma E.1 shows that the spectrum of A_ℓ is given by $\{2s - \ell : s = 0, 1, \dots, \ell\}$ and it is contained in the set of relative integers \mathbb{Z} for any ℓ . Hence, if $z \in \mathbb{C} \setminus \mathbb{Z}$ equation (41) has a unique solution for all ℓ .

Lemma 4.2. *If $z \in \mathbb{C} \setminus \mathbb{Z}$ and ψ is given by (38), equation (39) has a solution which is of the form (40) with $\mathcal{X}_{m,n}(\cdot; \theta, u_3)$ analytic in $\mathbb{C} \setminus \mathbb{Z}$, and $\mathcal{X}_{m,n}(z; \cdot) \in C^\infty(\mathbb{R}^l \times \mathbb{R})$.*

Proof. Lemma E.1 establishes that the matrix A_ℓ is diagonalizable with eigenvalues $2s - \ell$, $s = 0, \dots, \ell$. We denote by S and $T = S^{-1}$ the matrices (explicitly given in the proof of lemma E.1) such that $TA_\ell S$ is diagonal. Then, for $z \in \mathbb{C} \setminus \mathbb{Z}$, $A_\ell - z$ is invertible and equation (41) has a unique solution $x_\ell = (\mathcal{X}_{m,n})_{m+n=\ell}$ given by

$$\mathcal{X}_{\ell-j, j}(z; \theta, u_3) = \sum_{r,s=0}^\ell \frac{S_{jr} T_{rs}}{2r - \ell - z} \mathcal{Y}_{\ell-s, s}(\theta, u_3),$$

which is analytic in $z \in \mathbb{C} \setminus \mathbb{Z}$, and C^∞ in (θ, u_3) . \square

We can now use $\tilde{\varphi}$ to construct the unique solution of (36).

Proposition 4.3. *Let $a \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3, \mathbb{C})$ satisfy the condition (37a), ψ be given in the form (38), $\tilde{\varphi}$ be the solution established in lemma 4.2, and let*

$$\varphi(\theta, u) := \tilde{\varphi}(a(\theta, u); \theta, u).$$

Then $\varphi \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3)$ is the unique solution of (36).

Proof. The fact that $\varphi \in C^\infty$ is a solution follows by equation (39) and assumption (37a) which in particular implies

$$-(u_1 \partial_{u_2} - u_2 \partial_{u_1}) \varphi(\theta, u) = -(u_1 \partial_{u_2} - u_2 \partial_{u_1}) \tilde{\varphi}(z; \theta, u)|_{z=a(\theta, u)}.$$

Uniqueness has been proven in lemma 4.1. \square

4.2. **Equation with source term in \mathcal{S} and uniqueness in \mathcal{S}' .** If $\varphi \in C^1$ is a solution of (36) and $(u_\perp, \phi) \in \mathbb{R}_+ \times [0, 2\pi]$ are polar coordinates defined by $u_1 = u_\perp \cos \phi$, $u_2 = -u_\perp \sin \phi$, then the functions

$$\begin{aligned} U(r, \phi) &= \varphi(\theta, u_\perp \cos \phi, -u_\perp \sin \phi, u_3), \\ V(r, \phi) &= \psi(\theta, u_\perp \cos \phi, -u_\perp \sin \phi, u_3), \end{aligned}$$

with parameter $r = (\theta, u_\perp, u_3)$ satisfy

$$(42) \quad \partial_\phi U(r, \phi) - i\tilde{a}(r)U(r, \phi) = V(r, \phi), \quad U(r, 0) = U(r, 2\pi),$$

with

$$\tilde{a}(r) = a(\theta, u_\perp \cos \phi, -u_\perp \sin \phi, u_3),$$

which is independent of ϕ because of condition (37a).

Remark 8. The choice of the angle ϕ , in the clockwise direction, is unusual for polar coordinates. This is motivated by the fact that, with this definition, ϕ increases in the direction of gyration of a positively charged particle under the Lorentz force.

For smooth solutions $\varphi \in C^\infty$ of (36), we find that the derivatives $\partial_\theta^\alpha \partial_u^\beta \varphi$ with the same order of differentiation in (u_1, u_2) are related to the solution of an ordinary differential equation analogous to (42). In fact, differentiating equation (36) yields

$$\begin{aligned} \partial_\theta^\alpha \partial_u^\beta [-(u_1 \partial_{u_2} - u_2 \partial_{u_1})\varphi] &= -(u_1 \partial_{u_2} - u_2 \partial_{u_1})(\partial_\theta^\alpha \partial_u^\beta \varphi) \\ &\quad - \beta_1 \partial_\theta^\alpha \partial_{u_1}^{\beta_1-1} \partial_{u_2}^{\beta_2+1} \partial_{u_3}^{\beta_3} \varphi + \beta_2 \partial_\theta^\alpha \partial_{u_1}^{\beta_1+1} \partial_{u_2}^{\beta_2-1} \partial_{u_3}^{\beta_3} \varphi, \end{aligned}$$

for any multi-index $\alpha \in \mathbb{N}_0^1$ and $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}_0^3$. This can be shown either directly using the identities

$$\begin{aligned} \partial_{u_1}^{\beta_1} \partial_{u_2}^{\beta_2} (u_1 \partial_{u_2} \varphi) &= u_1 \partial_{u_1}^{\beta_1} \partial_{u_2}^{\beta_2+1} \varphi + \beta_1 \partial_{u_1}^{\beta_1-1} \partial_{u_2}^{\beta_2+1} \varphi, \\ \partial_{u_1}^{\beta_1} \partial_{u_2}^{\beta_2} (u_2 \partial_{u_1} \varphi) &= u_2 \partial_{u_1}^{\beta_1+1} \partial_{u_2}^{\beta_2} \varphi + \beta_2 \partial_{u_1}^{\beta_1+1} \partial_{u_2}^{\beta_2-1} \varphi, \end{aligned}$$

or by induction over β_1 and β_2 . Therefore, the $(\ell + 1)$ -dimensional complex-vector-valued function defined by

$$(43) \quad \varphi_{\alpha, \beta, \ell}(\theta, u) = (\partial_\theta^\alpha \partial_u^\beta \varphi(\theta, u))_{\beta_1 + \beta_2 = \ell} = (\partial_\theta^\alpha \partial_{u_1}^{\ell-j} \partial_{u_2}^j \partial_{u_3}^{\beta_3} \varphi(\theta, u))_{j=0}^\ell,$$

satisfies the system of partial differential equations

$$(44) \quad -(u_1 \partial_{u_2} - u_2 \partial_{u_1})\varphi_{\alpha, \beta, \ell} - i(a + {}^t A_\ell)\varphi_{\alpha, \beta, \ell} = \psi_{\alpha, \beta, \ell},$$

where A_ℓ are the same matrices introduced in equation (41) and studied in appendix E, and the right-hand side is the $(\ell + 1)$ -dimensional-vector-valued function

$$(45) \quad \psi_{\alpha, \beta, \ell} = \left(\partial_\theta^\alpha \partial_u^\beta \psi + i \sum_{\alpha' < \alpha} \sum_{\beta' < \beta} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} (\partial_\theta^{\alpha-\alpha'} \partial_u^{\beta-\beta'} a) (\partial_\theta^{\alpha'} \partial_u^{\beta'} \varphi) \right)_{j=0}^\ell,$$

with $\beta = (\beta_1, \beta_2, \beta_3)$, $\beta_1 = \ell - j$ and $\beta_2 = j$, $j = 0, \dots, \ell$. For $(u_1, u_2) \neq (0, 0)$, let

$$\begin{aligned} U_{\alpha, \beta, \ell}(r, \phi) &= \varphi_{\alpha, \beta, \ell}(\theta, u_\perp \cos \phi, -u_\perp \sin \phi, u_3), \\ V_{\alpha, \beta, \ell}(r, \phi) &= \psi_{\alpha, \beta, \ell}(\theta, u_\perp \cos \phi, -u_\perp \sin \phi, u_3). \end{aligned}$$

Then, if $\varphi \in C^\infty$ is a solution of (36), necessarily it must hold that

$$(46) \quad \begin{cases} \partial_\phi U_{\alpha, \beta, \ell}(r, \phi) - i(\tilde{a}(r) + {}^t A_\ell)U_{\alpha, \beta, \ell}(r, \phi) = V_{\alpha, \beta, \ell}(r, \phi), \\ U_{\alpha, \beta, \ell}(r, 0) = U_{\alpha, \beta, \ell}(r, 2\pi). \end{cases}$$

For $\ell = 0$ the matrix A_ℓ reduces to $A_0 = 0$, hence equation (42) is a special case of (46) obtained for $\alpha = \beta = 0$ and $\ell = 0$. In general, equation (46) is a system of $\ell + 1$ first-order ordinary differential equations on $[0, 2\pi]$ with periodic boundary conditions, for which we have the following result.

Proposition 4.4. *For $\ell \in \mathbb{N}_0$, let M_ℓ be a $(\ell + 1) \times (\ell + 1)$ diagonalizable, complex matrix with eigenvalues $\lambda_{\ell,j} \in \mathbb{C} \setminus \mathbb{Z}$, $j = 0, 1, \dots, \ell$. Then, for any function $V_\ell \in C^\infty([0, 2\pi], \mathbb{C}^{\ell+1})$ satisfying $V_\ell(0) = V_\ell(2\pi)$, there exists a unique solution $U_\ell \in C^\infty([0, 2\pi], \mathbb{C}^{\ell+1})$ of*

$$U_\ell'(\phi) - iM_\ell U_\ell(\phi) = V_\ell(\phi), \quad U_\ell(0) = U_\ell(2\pi),$$

given in Fourier series by

$$U_\ell(\phi) = \sum_{n \in \mathbb{Z}} [i(M_\ell - n)^{-1} \hat{V}_{\ell,n}] e^{+in\phi}, \quad \hat{V}_{\ell,n} = \frac{1}{2\pi} \int_0^{2\pi} V_\ell(\phi) e^{-in\phi} d\phi,$$

and if $\text{Im } \lambda_{\ell,j} \neq 0$, the solution satisfies, for all $\phi \in [0, 2\pi]$,

$$|U_\ell(\phi)|_\infty \leq \frac{\kappa_\ell}{\lambda_{\ell,m}} \max_{\phi' \in [0, 2\pi]} |V_\ell(\phi')|_\infty,$$

where, for $z = (z_0, z_1, \dots, z_\ell) \in \mathbb{C}^{\ell+1}$, $|z|_\infty := \max_j |z_j|$ is the L^∞ norm in $\mathbb{C}^{\ell+1}$, κ_ℓ is a constant depending only on M_ℓ , and $\lambda_{\ell,m} = \min_j |\text{Im } \lambda_{\ell,j}|$.

Proof. Since $V_\ell \in C^\infty([0, 2\pi], \mathbb{C}^{\ell+1})$, for any $\mu \in \mathbb{N}_0$, the Fourier coefficients satisfy $|n|^\mu |\hat{V}_{\ell,n}|_\infty \leq \max_{\phi'} |\partial_\phi^\mu V_\ell(\phi')|_\infty$, hence the corresponding Fourier series converges in $C^k([0, 2\pi], \mathbb{C}^{\ell+1})$ for every $k \in \mathbb{N}_0$. Dini's test implies that the sum of the Fourier series is equal to V_ℓ , that is, V_ℓ can be represented by a Fourier series.

Analogously a function $U_\ell \in C^1([0, 2\pi], \mathbb{C}^{\ell+1})$ can be represented by a Fourier series, with convergence in $C^1([0, 2\pi], \mathbb{C}^{\ell+1})$ and it is a solution if and only if the Fourier coefficients $\hat{U}_{\ell,n}$ satisfy

$$(M_\ell - n)\hat{U}_{\ell,n} = i\hat{V}_{\ell,n}.$$

As it was assumed that M_ℓ is diagonalizable, that is, there exists a non-singular complex matrix S_ℓ such that $S_\ell^{-1} M_\ell S_\ell = \Lambda_\ell$ where $\Lambda_\ell = \text{diag}(\lambda_{\ell,0}, \lambda_{\ell,1}, \dots, \lambda_{\ell,\ell})$ is the diagonal matrix of eigenvalues. Then $M_\ell - n = S_\ell(\Lambda_\ell - n)S_\ell^{-1}$ is non-singular for all $n \in \mathbb{Z}$, if and only if $\lambda_{\ell,j} \notin \mathbb{Z}$. Since this is the case, the Fourier coefficients of the solution are uniquely determined and given by $\hat{U}_{\ell,n} = i(M_\ell - n)^{-1} \hat{V}_{\ell,n}$. The norm of the Fourier coefficients can be readily estimated by

$$|\hat{U}_{\ell,n}|_\infty \leq |S_\ell|_\infty \cdot |(\Lambda_\ell - n)^{-1}|_\infty \cdot |S_\ell^{-1}|_\infty \cdot |\hat{V}_{\ell,n}|_\infty \leq \frac{\kappa_\ell}{\delta_\ell} |\hat{V}_{\ell,n}|_\infty,$$

where $\kappa_\ell = |S_\ell|_\infty |S_\ell^{-1}|_\infty$ is the condition number of the matrix S_ℓ and thus depends only on M_ℓ , while $\delta_\ell = \min_{j,n} |\lambda_{\ell,j} - n| > 0$ measures the distance of the eigenvalues from \mathbb{Z} . Since for $n \neq 0$, $|\hat{V}_{\ell,n}|_\infty = O(|n|^{-\mu})$ for all $\mu \in \mathbb{N}_0$, the Fourier series of U_ℓ converges in $C^k([0, 2\pi], \mathbb{C}^{\ell+1})$ for every $k \in \mathbb{N}_0$, hence the sum U_ℓ belongs to $C^\infty([0, 2\pi], \mathbb{C}^{\ell+1})$ and it is the unique classical solution of the problem.

We obtain an equivalent representation of the classical solution. In fact U_ℓ must necessarily satisfy

$$(e^{-iM_\ell \phi} U_\ell(\phi))' = e^{-iM_\ell \phi} V_\ell(\phi),$$

and the general solution of this equation, with arbitrary initial condition $U_\ell(0)$, is

$$e^{-iM_\ell\phi}U_\ell(\phi) = U_\ell(0) + \int_0^\phi e^{-iM_\ell\phi'}V_\ell(\phi')d\phi'.$$

Then the periodic boundary condition $U_\ell(0) = U_\ell(2\pi)$ amounts to

$$(e^{-2\pi iM_\ell} - 1)U_\ell(0) = \int_0^{2\pi} e^{-iM_\ell\phi'}V_\ell(\phi')d\phi'.$$

The matrix on the left-hand side is diagonalizable with eigenvalues $e^{-2\pi i\lambda_{\ell,j}} - 1$; for $\lambda_{\ell,j} \in \mathbb{C} \setminus \mathbb{Z}$ the eigenvalues are all non-zero, the matrix is invertible, and the integration constant $U_\ell(0)$ is uniquely determined. At last one finds that there is a unique periodic solution given by

$$(47) \quad U_\ell(\phi) = [e^{-2\pi iM_\ell} - 1]^{-1} \int_0^{2\pi} e^{+iM_\ell(\phi-\phi')}V_\ell(\phi')d\phi' + \int_0^\phi e^{+iM_\ell(\phi-\phi')}V_\ell(\phi')d\phi'.$$

Equation (47) shows that $U_\ell \in C^\infty([0, 2\pi], \mathbb{C}^{\ell+1})$. The components of the vector $S_\ell^{-1}U_\ell$ are given by

$$(48) \quad (S_\ell^{-1}U_\ell(\phi))_j = \frac{1}{e^{-2\pi i\lambda_{\ell,j}} - 1} \int_0^{2\pi} e^{+i\lambda_{\ell,j}(\phi-\phi')} (S_\ell^{-1}V_\ell(\phi'))_j d\phi' \\ + \int_0^\phi e^{+i\lambda_{\ell,j}(\phi-\phi')} (S_\ell^{-1}V_\ell(\phi'))_j d\phi'.$$

Therefore,

$$|(S_\ell^{-1}U_\ell(\phi))_j| \leq \max_{\phi'} |(S_\ell^{-1}V_\ell(\phi'))_j| \left[\left| \frac{1}{e^{-2\pi i\lambda_{\ell,j}} - 1} \right| \int_0^{2\pi} e^{-\text{Im} \lambda_{\ell,j}(\phi-\phi')} d\phi' \right. \\ \left. + \int_0^\phi e^{-\text{Im} \lambda_{\ell,j}(\phi-\phi')} d\phi' \right].$$

For the factor in square brackets, we use

$$|1 - e^{-2\pi i\lambda_{\ell,j}}| \geq |1 - |e^{-2\pi i\lambda_{\ell,j}}|| = |1 - e^{2\pi \text{Im} \lambda_{\ell,j}}|,$$

so that

$$\left| \frac{1}{e^{-2\pi i\lambda_{\ell,j}} - 1} \right| \leq \frac{1}{|e^{2\pi \text{Im} \lambda_{\ell,j}} - 1|}.$$

For any $\phi_1 > 0$ and $y \neq 0$ we have

$$\int_0^{\phi_1} e^{-y(\phi-\phi')} d\phi' = \frac{e^{-y\phi}}{y} (e^{y\phi_1} - 1) = \frac{e^{-y\phi}}{|y|} |e^{y\phi_1} - 1|,$$

and the two needed integrals are obtained for $\phi_1 = 2\pi$ and $\phi_1 = \phi$. Hence,

$$|(S_\ell^{-1}U_\ell(\phi))_j| \leq \frac{1}{|\text{Im} \lambda_{\ell,j}|} \left[e^{-\text{Im} \lambda_{\ell,j}\phi} + |1 - e^{-\text{Im} \lambda_{\ell,j}\phi}| \right] \max_{\phi'} |(S_\ell^{-1}V_\ell(\phi'))_j|.$$

If $\text{Im} \lambda_{\ell,j} > 0$, the term in square brackets is equal to one and we obtain

$$(49) \quad |(S_\ell^{-1}U_\ell(\phi))_j| \leq \frac{1}{|\text{Im} \lambda_{\ell,j}|} \max_{\phi'} |(S_\ell^{-1}V_\ell(\phi'))_j|.$$

If $\text{Im } \lambda_{\ell,j} < 0$, we consider $(S_\ell^{-1}U_\ell(2\pi - \phi))_j$ and upon changing integration variable in (48) we obtain

$$(S_\ell^{-1}U_\ell(2\pi - \phi))_j = - \left[\frac{1}{e^{2\pi i \lambda_{\ell,j}} - 1} \int_0^{2\pi} e^{-i\lambda_{\ell,j}(\phi - \phi')} (S_\ell^{-1}V_\ell(2\pi - \phi'))_j d\phi' \right. \\ \left. + \int_0^\phi e^{-i\lambda_{\ell,j}(\phi - \phi')} (S_\ell^{-1}V_\ell(2\pi - \phi'))_j d\phi' \right].$$

The factor in square brackets has the same form as the right-hand side of (48) with $\lambda_{\ell,j}$ replaced by $-\lambda_{\ell,j}$ and now $\text{Im}(-\lambda_{\ell,j}) > 0$. Hence we obtain

$$|(S_\ell^{-1}U_\ell(2\pi - \phi))_j| \leq \frac{1}{|\text{Im } \lambda_{\ell,j}|} \max_{\phi'} |(S_\ell^{-1}V_\ell(\phi'))_j|.$$

Since ϕ is arbitrary, this is equivalent to (49) for $\text{Im } \lambda_{\ell,j} < 0$. Since $|\text{Im } \lambda_{\ell,j}| \geq \lambda_{\ell,m} > 0$, taking the maximum over j in (49) yields

$$|S_\ell^{-1}U_\ell(\phi)|_\infty \leq \frac{1}{\lambda_{\ell,m}} \max_j \max_{\phi'} |(S_\ell^{-1}V_\ell(\phi'))_j| = \frac{1}{\lambda_{\ell,m}} \max_{\phi'} |S_\ell^{-1}V_\ell(\phi')|_\infty.$$

Then

$$|U_\ell(\phi)|_\infty \leq |S_\ell|_\infty |S_\ell^{-1}U_\ell(\phi)|_\infty \leq \frac{\kappa_\ell}{\lambda_{\ell,m}} \max_{\phi'} |V_\ell(\phi')|_\infty,$$

which is the claimed estimate. \square

As a corollary we obtain the solution of the problem stated in proposition 4.4 with generic parameters $r \in \mathcal{O} \subseteq \mathbb{R}^m$, where \mathcal{O} is an open set.

Corollary 4.5. *For $m \in \mathbb{N}$, $\mathcal{O} \subseteq \mathbb{R}^m$, and $\ell \in \mathbb{N}_0$, let $M_\ell \in C^\infty(\mathcal{O}, \mathbb{R}^{(\ell+1) \times (\ell+1)})$ and $V_\ell \in C^\infty(\mathcal{O} \times [0, 2\pi], \mathbb{C}^{\ell+1})$ be such that*

- 1) *for any $r \in \mathcal{O}$, there is a non-singular matrix $S_\ell(r)$ for which $\Lambda_\ell(r) = S_\ell(r)^{-1}M_\ell(r)S_\ell(r)$ is diagonal with eigenvalues $\lambda_{\ell,j}(r) \in \mathbb{C} \setminus \mathbb{R}$ satisfying $|\text{Im } \lambda_{\ell,j}(r)| \geq \eta > 0$ for $j = 0, \dots, \ell$, $r \in \mathcal{O}$ and for a given $\eta > 0$,*
- 2) *$V(r, 0) = V(r, 2\pi)$ for $r \in \mathcal{O}$, and*
- 3) *S_ℓ , S_ℓ^{-1} , and $\lambda_{\ell,j}$ are of class $C^\infty(\mathcal{O})$.*

Then, the problem

$$\partial_\phi U_\ell(r, \phi) - iM_\ell(r)U_\ell(r, \phi) = V_\ell(r, \phi), \quad U_\ell(r, 0) = U_\ell(r, 2\pi),$$

has a unique solution $U_\ell \in C^\infty(\mathcal{O} \times [0, 2\pi], \mathbb{C}^{\ell+1})$, and, for all $\phi \in [0, 2\pi]$,

$$|U_\ell(r, \phi)| \leq \frac{\kappa_\ell}{\eta} \max_{\phi' \in [0, 2\pi]} |V_\ell(r, \phi')|.$$

Proof. A function U_ℓ is a solution if and only if, for any $r \in \mathcal{O}$, $U_\ell(r, \cdot)$ solves the ordinary differential equation in proposition 4.4. Assumptions 1) and 2) imply that all the hypotheses of proposition 4.4 are verified and we note that $\lambda_{\ell,m}(r) \geq \eta$ uniformly for $r \in \mathcal{O}$. Therefore, for any $r \in \mathcal{O}$, there is a unique solution $U_\ell(r, \cdot)$ to the problem of corollary 4.5. The fact that U_ℓ is in $C^\infty(\mathcal{O} \times [0, 2\pi])$ follows from the explicit formula (47) and assumption 3) by using the classical results of differentiation in the integral. The estimates follow from the ones proven in proposition 4.4. \square

We can now give the general result for equation (36) with right-hand side in C^∞ and then in the Schwartz space, which will immediately imply uniqueness in \mathcal{S}' . Uniqueness of the C^∞ solution, in particular, is a special case of lemma 4.1, but here we give a different more explicit argument.

Proposition 4.6. *Let $a \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3)$ satisfy condition (37a). Then for any $\psi \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3)$, equation (36) has a unique solution $\varphi \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3)$.*

Proof. Uniqueness of a C^1 solution. If $\varphi \in C^1(\mathbb{R}^l \times \mathbb{R}^3)$ is a solution of (36), evaluating the equations at $(u_1, u_2) = (0, 0)$ yields

$$\varphi(\theta, 0, 0, u_3) = i\psi(\theta, 0, 0, u_3)/a(\theta, 0, 0, u_3),$$

while for $(u_1, u_2) \neq (0, 0)$,

$$\varphi(\theta, u) = U(r, \phi),$$

where U is the unique solution of (42) constructed in using corollary 4.5. These conditions completely define the value of a C^1 solution everywhere in $\mathbb{R}^l \times \mathbb{R}^3$.

Existence of a C^∞ solution. First we address the special case

$$(50) \quad \psi(\theta, u) = \sum_{m+n=k} u_1^m u_2^n \tilde{\psi}_{m,n}(\theta, u),$$

where $\tilde{\psi}_{m,n} \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3)$ and $k \geq 2$ is a given integer. Let us consider the ordinary differential equation (42) with source term V determined by the ψ given in (50). This equation is a special cases of the problem addressed in corollary 4.5 with $\ell = 0$; particularly, because of assumption (37a), $M_0(r) = \tilde{a}(r)$ satisfies the hypotheses of the corollary. Therefore, equation (42) has a unique solution $U \in C^\infty(\mathcal{O} \times [0, 2\pi])$. Due to the special choice of ψ and the estimate in corollary 4.5, one deduces that for any given point (θ, u_3) and $\delta > 0$, there are constants $c_{U,\theta,u_3,\delta}$ and $c_{V,\theta,u_3,\delta} > 0$ for which

$$|U(r, \phi)| \leq u_\perp^k c_{U,\theta,u_3,\delta}, \quad |V(r, \phi)| \leq u_\perp^k c_{V,\theta,u_3,\delta},$$

uniformly in $u_\perp \in (0, \delta]$ and $\phi \in [0, 2\pi]$. Let us construct the function

$$\varphi(\theta, u) := \begin{cases} U(r, \phi), & \text{for } (u_1, u_2) \neq (0, 0), \\ 0, & \text{for } (u_1, u_2) = (0, 0). \end{cases}$$

Since polar coordinates in the region $(u_1, u_2) \neq (0, 0)$ define a diffeomorphism which maps the partial differential equation (36) into the ordinary differential equation (42), the function φ is of class C^∞ and solves equation (36) in the open set $\{(\theta, u) \in \mathbb{R}^l \times \mathbb{R}^3 : (u_1, u_2) \neq (0, 0)\}$. We also have $|\varphi(\theta, u)| \leq c_{U,\theta,u_3,\delta} |(u_1, u_2)|^k$, and thus φ is continuous on whole domain $\mathbb{R}^l \times \mathbb{R}^3$.

We now show that $\varphi \in C^{k-1}(\mathbb{R}^l \times \mathbb{R}^3)$. We need to check the existence of derivatives at $(u_1, u_2) = (0, 0)$ and their continuity. With this aim we collect the derivatives of φ with the same order ℓ of differentiation in (u_1, u_2) into the vector-valued functions $\varphi_{\alpha,\beta,\ell}$, as defined in (43); analogously let $\psi_{\alpha,\beta,\ell}$ be given by (45). For $(u_1, u_2) \neq (0, 0)$, φ is of class C^∞ and solves equation (36), so that $\varphi_{\alpha,\beta,\ell}$ satisfies equation (44) with source $\psi_{\alpha,\beta,\ell}$. In polar coordinates those equations amount to ordinary differential equations (46) for the vector-valued functions given by $U_{\alpha,\beta,\ell}(r, \phi) = \varphi_{\alpha,\beta,\ell}(\theta, u_\perp \cos \phi, -u_\perp \sin \phi, u_3)$ and with source $V_{\alpha,\beta,\ell}(r, \phi) = \psi_{\alpha,\beta,\ell}(\theta, u_\perp \cos \phi, -u_\perp \sin \phi, u_3)$. From lemma E.1 we know that the matrices A_ℓ , and thus ${}^t A_\ell$, are diagonalizable with integer eigenvalues. It follows that the matrices $M_\ell(r) = \tilde{a}(r) + {}^t A_\ell$ in equation (46) are diagonalizable and the

imaginary part of the eigenvalues coincides with $\text{Im } \tilde{a}$. In view of assumption (37a), we have $|\text{Im } \tilde{a}| \geq \eta > 0$, and the hypotheses of corollary 4.5 are therefore satisfied. We can conclude that

$$(51) \quad |U_{\alpha,\beta,\ell}(r, \phi)|_\infty \leq \frac{\kappa_\ell}{\eta} \max_{\phi'} |V_{\alpha,\beta,\ell}(r, \phi')|_\infty.$$

We can use this estimate to show that, for $(u_1, u_2) \neq (0, 0)$ we have

$$(52) \quad |\partial_\theta^\alpha \partial_u^\beta \varphi(\theta, u)| \leq K_{\alpha,\beta}(\theta, u_3) u_\perp^{k-\ell} \text{ with } \ell = \beta_1 + \beta_2, \text{ and } 0 < |(u_1, u_2)| \leq \delta.$$

We prove this by induction over α , β_3 and $\ell = \beta_1 + \beta_2$. For $\alpha = 0$, $\beta_3 = 0$, and $\ell = 0$ the claim follows directly from the estimate in corollary 4.5 and $V = u_\perp^k V_R$. For the induction step, let us assume that the claim holds for all $\alpha' < \alpha$, $\beta'_3 < \beta_3$, and $\ell' = \beta'_1 + \beta'_2 < \ell = \beta_1 + \beta_2$. Then from (45) we deduce

$$|V_{\alpha,\beta,\ell}(r, \phi)|_\infty = |\psi_{\alpha,\beta,\ell}(\theta, u)|_\infty \leq c_{\alpha,\beta}^\psi(\theta, u_3) u_\perp^{k-\ell},$$

and from estimate (51) we deduce that, if $\beta_1 + \beta_2 = \ell$,

$$|\partial_\theta^\alpha \partial_u^\beta \varphi(\theta, u)| \leq |\varphi_{\alpha,\beta,\ell}(\theta, u)|_\infty = |U_{\alpha,\beta,\ell}(r, \phi)|_\infty \leq \frac{\kappa_\ell}{\eta} c_{\alpha,\beta}^\psi(\theta, u_3) u_\perp^{k-\ell},$$

which is (52) as claimed. Therefore,

$$\frac{|\partial_\theta^\alpha \partial_u^\beta \varphi(\theta, u)|}{|(u_1, u_2)|} \leq K_{\alpha,\beta}(\theta, u_3) |(u_1, u_2)|^{k-\ell-1}, \text{ for } \beta_1 + \beta_2 = \ell \leq k-2,$$

which implies that the derivatives $\partial_\theta^\alpha \partial_u^\beta \varphi(\theta, 0, 0, u_3)$ exist and are zero for $\beta_1 + \beta_2 \leq k-1$. Continuity of $\partial_\theta^\alpha \partial_u^\beta \varphi$ follows from inequality (52). Hence $\varphi \in C^{k-1}(\mathbb{R}^l \times \mathbb{R}^3)$ as claimed. Since $k \geq 2$, $\varphi \in C^1(\mathbb{R}^l \times \mathbb{R}^3)$ and equation (36) is satisfied also at $(u_1, u_2) = (0, 0)$ since all terms vanish if $u_1 = u_2 = 0$.

For the general case $\psi \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3)$, for any integer $k \geq 2$ we write

$$\psi = \psi_{k-1} + \psi_{r,k},$$

where ψ_{k-1} is the Taylor polynomial of degree $k-1$ in (u_1, u_2) centered at $(u_1, u_2) = (0, 0)$ and $\psi_{r,k}$ is the remainder, which is of the form (50). Hence the above argument applies to $\psi_{r,k}$. Let $\varphi_{r,k} \in C^{k-1}(\mathbb{R}^l \times \mathbb{R}^3)$ be the unique solution obtained with $\psi_{r,k}$ as a source term. On the other hand proposition 4.3 established the existence of a unique solution $\varphi_{k-1} \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3)$ for the case with source term ψ_{k-1} . The sum $\varphi = \varphi_{k-1} + \varphi_{r,k}$ is of class C^{k-1} and it is the unique solution of (36). Since k is arbitrary we conclude that $\varphi \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3)$. \square

Corollary 4.7. *Let $a \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3)$ satisfy both conditions (37) and let $m \in \mathbb{R}$ be the constant in (37b). Then the unique solution $\varphi \in C^\infty(\mathbb{R}^l \times \mathbb{R}^3)$ of equation (36) obtained in proposition 4.6 is such that:*

(i) *If there are $m_0 \in \mathbb{R}$ and $n_0 \in \mathbb{N}_0$ such that*

$$|\partial_\theta^\alpha \partial_u^\beta \psi(\theta, u)| \leq C_{\alpha,\beta}^\psi (1 + \theta^2 + u^2)^{m_0}, \quad |\alpha| + |\beta| \leq n_0,$$

then

$$|\partial_\theta^\alpha \partial_u^\beta \varphi(\theta, u)| \leq C_{\eta,\alpha,\beta}^\varphi (1 + \theta^2 + u^2)^{m_{\alpha,\beta}}, \quad |\alpha| + |\beta| \leq n_0,$$

where $m_{\alpha,\beta} = m_0$ if $m \leq 0$, and $m_{\alpha,\beta} = m_0 + (|\alpha| + |\beta|)m$ if $m > 0$.

(ii) *If $\psi \in \mathcal{S}(\mathbb{R}^l \times \mathbb{R}^3)$, then $\varphi \in \mathcal{S}(\mathbb{R}^l \times \mathbb{R}^3)$.*

Proof. (i) We proceed by induction as in the proof of (52). For $\alpha = 0$ and $\beta = 0$, the claim follows directly from the estimate in corollary 4.5 since for $u_1^2 + u_2^2 \geq \rho^2 > 0$,

$$|\varphi(\theta, u)| = |U(r, \phi)| \leq \frac{1}{\eta} \max_{\phi} |V(r, \phi)| \leq \frac{1}{\eta} C_{0,0}^{\psi} (1 + \theta^2 + u^2)^{m_0}.$$

The constant is independent of the radius ρ , hence the claim. For the induction step, if the claim is true for all multi-indices $\alpha' < \alpha$ and $\beta' < \beta$, where α and β are any multi-indices satisfying $|\alpha| + |\beta| \leq n_0$, then, again for $u_1^2 + u_2^2 \geq \rho^2 > 0$, we estimate $|V_{\alpha,\beta,\ell}(r, \phi)|_{\infty} = |\psi_{\alpha,\beta,\ell}(\theta, u)|_{\infty}$ by

$$\begin{aligned} |V_{\alpha,\beta,\ell}|_{\infty} &\leq \max \left\{ |\partial_{\theta}^{\alpha} \partial_u^{\beta} \psi| + \sum_{\alpha' < \alpha} \sum_{\beta' < \beta} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} |\partial_{\theta}^{\alpha-\alpha'} \partial_u^{\beta-\beta'} a| |\partial_{\theta}^{\alpha'} \partial_u^{\beta'} \varphi| \right\} \\ &\leq \max \left\{ C_{\alpha,\beta}^{\psi} (1 + \theta^2 + u^2)^{m_0} + \sum_{\alpha' < \alpha} \sum_{\beta' < \beta} C_{\eta,\alpha',\beta'}^{\alpha,\beta} (1 + \theta^2 + u^2)^{m+m_{\alpha',\beta'}} \right\}, \end{aligned}$$

where the maximum is computed over all (β_1, β_2) such that $\beta_1 + \beta_2 = \ell$, holding α and β_3 fixed. We observe that, if $m \leq 0$, then $m + m_{\alpha',\beta'} = m + m_0 \leq m_0 = m_{\alpha,\beta}$, while if $m > 0$, $m + m_{\alpha',\beta'} = m_0 + (|\alpha'| + |\beta'| + 1)m \leq m_0 + (|\alpha| + |\beta|)m = m_{\alpha,\beta}$. In both cases we have

$$\max_{\phi} |V_{\alpha,\beta,\ell}(r, \phi)|_{\infty} \leq \tilde{C}_{\eta,\alpha,\beta_3,\ell}^{\psi} (1 + \theta^2 + u^2)^{m_{\alpha,\beta}}.$$

We can now apply inequality (51) and deduce

$$|\partial_{\theta}^{\alpha} \partial_u^{\beta} \varphi(\theta, u)| \leq \frac{\kappa_{\ell}}{\eta} \tilde{C}_{\eta,\alpha,\beta_3,\ell}^{\psi} (1 + \theta^2 + u^2)^{m_{\alpha,\beta}},$$

where $\ell = \beta_1 + \beta_2$; this proves the claim for α and β satisfying $|\alpha| + |\beta| \leq n_0$.

(ii) If $\psi \in \mathcal{S}(\mathbb{R}^l \times \mathbb{R}^3)$, then it satisfies the assumption of item (i) for all $m_0 \in \mathbb{R}$ and for all α, β . For any $\mu \in \mathbb{R}$, $\alpha \in \mathbb{N}_0^l$, and $\beta \in \mathbb{N}_0^3$, let us apply the estimate proven in item (i) with $m_0 = \mu - (|\alpha| + |\beta|)m$; we obtain $|\partial_{\theta}^{\alpha} \partial_u^{\beta} \varphi(\theta, u)| \leq C_{\eta,\alpha,\beta}^{\varphi} (1 + \theta^2 + u^2)^{\mu}$. Hence $\varphi \in \mathcal{S}(\mathbb{R}^l \times \mathbb{R}^3)$ as claimed. \square

The existence of a solution $\varphi \in \mathcal{S}(\mathbb{R}^l \times \mathbb{R}^3)$ of equation (36) implies (by duality) uniqueness in $\mathcal{S}'(\mathbb{R}^l \times \mathbb{R}^3)$ for the linear equation

$$(53) \quad -(u_1 \partial_{u_2} - u_2 \partial_{u_1})h - ia h = s,$$

for $s \in \mathcal{S}'(\mathbb{R}^l \times \mathbb{R}^3)$ and a satisfying conditions (37).

Proposition 4.8. *If $a \in C^{\infty}(\mathbb{R}^l \times \mathbb{R}^3)$ satisfies all conditions (37), equation (53) has at most one solution in \mathcal{S}' .*

Proof. We show that the only solution of the associate homogeneous equation is the trivial solution $h = 0$. Explicitly, this means that if

$$\langle h, +(u_1 \partial_{u_2} - u_2 \partial_{u_1})\chi - ia\chi \rangle = 0,$$

for all $\chi \in \mathcal{S}(\mathbb{R}^l \times \mathbb{R}^3)$, then $h = 0$.

Given an arbitrary test function $\psi \in \mathcal{S}(\mathbb{R}^l \times \mathbb{R}^3)$ let us consider the equation

$$(u_1 \partial_{u_2} - u_2 \partial_{u_1})\varphi - ia\varphi = \psi.$$

Since $-a$ satisfies conditions (37), we have established in corollary 4.7 that this equation has a unique solution $\varphi \in \mathcal{S}$. Then for any $\psi \in \mathcal{S}$,

$$\langle h, \psi \rangle = \langle h, +(u_1 \partial_{u_2} - u_2 \partial_{u_1})\varphi - ia\varphi \rangle = 0,$$

and thus $h = 0$ as a tempered distribution. \square

5. RESPONSE OF A UNIFORM MAGNETIZED PLASMA

In this section we address the case of a uniform magnetized plasma and prove the results stated in section 1.4. With this aim we shall rely heavily on the preparatory results of section 4 and on a stationary-phase argument postponed to section 6.

5.1. Notation. We shall make use of normalized momentum (12) and for $(u_1, u_2) \neq (0, 0)$ we define the two additional systems of cylindrical coordinates

$$(54) \quad u_1 = u_\perp \cos \phi, \quad u_2 = \mp u_\perp \sin \phi, \quad u_3 = u_\parallel,$$

with $u_\perp \in \mathbb{R}_+$ and $\phi \in [0, 2\pi]$. We re-write the linearized Vlasov equation in one of these two cylindrical coordinate systems depending on the electric charge of the considered particle species: we choose the sign $-$ (resp., $+$) for a positively (resp., negatively) charged particle species.

With normalized Fourier variables

$$(55) \quad \tau := \omega/\omega_{c,s}, \quad \xi := ck/\omega_{c,s}, \quad \text{and with } \kappa_s(u) := \gamma(u)\nu_s(u)/\omega_{c,s},$$

we define the quantities

$$(56) \quad a_\varepsilon(\tau, \xi, u) := \frac{\omega + i\varepsilon\nu - k_3v_3}{\omega_{c,s}/\gamma} = \gamma(u)\tau - \xi_3u_3 + i\varepsilon\kappa_s(u),$$

$$b_i(\xi, u) := \frac{k_iv_\perp}{\omega_{c,s}/\gamma} = u_\perp\xi_i, \quad i = 1, 2.$$

Written in terms of normalized variables, the functions $a_0 := a_\varepsilon|_{\varepsilon=0}$, b_i , and γ are independent of the particle species.

With G_s defined in (14), the functions of $u \in \mathbb{R}^3$ defined by

$$(57) \quad \mathcal{F}_s(u) := \frac{1}{u_\perp} \frac{\partial G_s}{\partial u_\perp}(u_\parallel, u_\perp), \quad \mathcal{G}_s(u) := \frac{\partial G_s}{\partial u_\parallel}(u_\parallel, u_\perp) - \frac{u_\parallel}{u_\perp} \frac{\partial G_s}{\partial u_\perp}(u_\parallel, u_\perp),$$

belong $\mathcal{S}(\mathbb{R}^3)$ because of the assumptions on G_s .

Next we define the first-order partial differential operators

$$(58) \quad Q_{s,j}(\tau, \xi, u, \partial_\xi) := \mathcal{F}_s(u)\Phi_j(\tau, \xi, u, \partial_\xi) + \mathcal{G}_s(u)\Psi_j(\tau, \xi, u, \partial_\xi),$$

where Φ_j, Ψ_j , for $j = 1, 2, 3$, are given in terms of the coefficients

$$(59a) \quad \Gamma_0(\tau, \xi, u) := \tau\gamma(u)e^{\pm i(\xi_1u_2 - \xi_2u_1)},$$

$$(59b) \quad \Gamma_j(\tau, \xi, u) := \xi_j e^{\pm i(\xi_1u_2 - \xi_2u_1)}, \quad j = 1, 2, 3,$$

by

$$\begin{aligned} \Phi_1 &:= \pm\Gamma_0\partial_{\xi_2}, & \Phi_2 &:= \mp\Gamma_0\partial_{\xi_1}, & \Phi_3 &:= iu_\parallel\Gamma_0, \\ \Psi_1 &:= \pm\Gamma_3\partial_{\xi_2}, & \Psi_2 &:= \mp\Gamma_3\partial_{\xi_1}, & \Psi_3 &:= i\Gamma_0\mp(\Gamma_1\partial_{\xi_2} - \Gamma_2\partial_{\xi_1}). \end{aligned}$$

The coefficients Γ_j all satisfy

$$|\partial_{\tau, \xi, u}^\alpha \Gamma_j(\tau, \xi, u)| \leq C_\alpha(1 + \tau^2 + \xi^2 + u^2)^{1+|\alpha|},$$

for all $\alpha \in \mathbb{N}_0^7$, and thus multiplication by Γ_j is closed in \mathcal{S} .

We also introduce the functions of $b = (b_1, b_1)$ given by

$$(60) \quad A_n^\pm(b) := \sum_{k, \ell \in \mathbb{Z}: k \pm \ell = n} (\pm i)^\ell J_k(b_1)J_\ell(b_2),$$

where J_k are the Bessel's functions of the first kind. If $b = (b_1, b_2) = (0, 0)$, the only non-zero term in the series is the one for $k = 0$ and $\ell = 0$, hence $A_n^\pm(0) = 1$ for $n = 0$, and $A_n^\pm(0) = 0$ for $n \neq 0$; if, on the other hand, $b = (b_1, b_2) \neq (0, 0)$ we can write $b_1 = |b| \cos \theta$, $b_2 = \mp |b| \sin \theta$ and

$$(61) \quad A_n^\pm(b) = J_n(|b|)e^{-in\theta}, \quad b \neq 0,$$

This identity follows from Jacobi-Anger expansions [1, p.361]

$$e^{iz \cos \phi} = \sum_{n \in \mathbb{Z}} i^n J_n(z) e^{in\phi}, \quad e^{iz \sin \phi} = \sum_{n \in \mathbb{Z}} J_n(z) e^{in\phi},$$

for $z \in \mathbb{C}$ and $\phi \in \mathbb{R}$, that imply

$$(62) \quad e^{\pm i(b_2 \cos \phi \pm b_1 \sin \phi)} = \sum_{n \in \mathbb{Z}} A_n^\pm(b) e^{in\phi}.$$

With $b_1 = |b| \cos \theta$ and $b_2 = \mp |b| \sin \theta$, one computes

$$\sum_{n \in \mathbb{Z}} A_n^\pm(b) e^{in\phi} = e^{i|b| \sin(\phi - \theta)} = \sum_{n \in \mathbb{Z}} J_n(|b|) e^{in(\phi - \theta)},$$

which yields identity (61). At last let

$$(63) \quad \begin{aligned} r_\varepsilon^\pm(\tau, \xi, u) &:= \int_0^{+\infty} e^{ia_\varepsilon(\tau, \xi, u)\lambda - i(\xi_1 u_1 + \xi_2 u_2) \sin \lambda \pm i(\xi_2 u_1 - \xi_1 u_2) \cos \lambda} d\lambda \\ &= \frac{i}{2} \frac{e^{-i\pi a_\varepsilon(\tau, \xi, u)}}{\sin(\pi a_\varepsilon(\tau, \xi, u))} P_\varepsilon^\pm(\tau, \xi, u), \end{aligned}$$

with

$$(64) \quad P_\varepsilon^\pm(\tau, \xi, u) := \int_0^{2\pi} e^{ia_\varepsilon(\tau, \xi, u)\lambda - i(\xi_1 u_1 + \xi_2 u_2) \sin \lambda \pm i(\xi_2 u_1 - \xi_1 u_2) \cos \lambda} d\lambda.$$

The quantities defined in equations (59)-(64) are independent on the particle species.

Lemma 5.1. *For $\varepsilon > 0$ and for each choice of the sign, the equation*

$$\mp(u_1 \partial_{u_2} - u_2 \partial_{u_1}) r_\varepsilon^\pm - ia_\varepsilon r_\varepsilon^\pm = e^{\pm i(\xi_2 u_1 - \xi_1 u_2)},$$

has a unique solution in $C^\infty(\mathbb{R}^7)$ given by (63) and for every $\alpha \in \mathbb{N}_0^4$, $\beta \in \mathbb{N}_0^3$ there are constants $C_{\alpha, \beta, \varepsilon, \nu} > 0$ and $m_{\alpha, \beta} \in \mathbb{R}$ such that

$$|\partial_{\tau, \xi}^\alpha \partial_u^\beta r_\varepsilon^\pm(\tau, \xi, u)| \leq C_{\alpha, \beta, \varepsilon, \nu} (1 + \tau^2 + \xi^2 + u^2)^{m_{\alpha, \beta}},$$

uniformly in (τ, ξ, u) .

Proof. For both choices of the sign, the equation for r_ε^\pm is of the form (36) with right-hand side in C^∞ and with $\theta = (\tau, \xi)$. (For r_ε^- in particular, one can multiply the equation by -1 and set $a = -a_\varepsilon$ and $\psi = -e^{\pm i(\xi_2 u_1 - \xi_1 u_2)}$.) For $\varepsilon > 0$, the function a_ε defined in (56) is such that all conditions (37) are true. Therefore proposition 4.6 ensures the existence a unique solution $r_\varepsilon^\pm \in C^\infty(\mathbb{R}^7)$ for each choice of the sign. For the claimed estimates it is enough to show that for any $n_0 \in \mathbb{N}_0$ the right-hand side of the equation satisfies the hypothesis of corollary 4.7 (i), and this is straightforward.

The integral expressions (63) can be checked by direct substitution into the equation. In fact if $I_\varepsilon^\pm(\tau, \xi, u; \lambda)$ denotes the integrand in (63), we have the identity

$$[\mp(u_1 \partial_{u_2} - u_2 \partial_{u_1}) - ia_\varepsilon(\tau, \xi, u)] I_\varepsilon^\pm(\tau, \xi, u; \lambda) = -\frac{\partial}{\partial \lambda} I_\varepsilon^\pm(\tau, \xi, u; \lambda),$$

and $I_\varepsilon^\pm(\tau, \xi, u; 0) = e^{\pm i(\xi_2 u_1 - \xi_1 u_2)}$. For the second form of r_ε^\pm , we notice that, following Qin et al. [49],

$$r_\varepsilon^\pm(\tau, \xi, u) = \int_0^{+\infty} e^{ia_\varepsilon(\tau, \xi, u)\lambda - i(\xi_1 u_1 + \xi_2 u_2) \sin(\lambda + 2\pi) \pm i(\xi_2 u_1 - \xi_1 u_2) \cos(\lambda + 2\pi)} d\lambda,$$

and changing integration variable we have

$$\begin{aligned} r_\varepsilon^\pm(\tau, \xi, u) &= e^{-2\pi ia_\varepsilon(\tau, \xi, u)} \int_{2\pi}^{+\infty} e^{ia_\varepsilon(\tau, \xi, u)\lambda - i(\xi_1 u_1 + \xi_2 u_2) \sin \lambda \pm i(\xi_2 u_1 - \xi_1 u_2) \cos \lambda} d\lambda \\ &= e^{-2\pi ia_\varepsilon(\tau, \xi, u)} \left[r_\varepsilon^\pm(\tau, \xi, u) \right. \\ &\quad \left. - \int_0^{2\pi} e^{ia_\varepsilon(\tau, \xi, u)\lambda - i(\xi_1 u_1 + \xi_2 u_2) \sin \lambda \pm i(\xi_2 u_1 - \xi_1 u_2) \cos \lambda} d\lambda \right], \end{aligned}$$

hence,

$$r_\varepsilon^\pm(\tau, \xi, u) = \frac{-e^{-2\pi ia_\varepsilon(\tau, \xi, u)}}{1 - e^{-2\pi ia_\varepsilon(\tau, \xi, u)}} \int_0^{2\pi} e^{ia_\varepsilon(\tau, \xi, u)\lambda - i(\xi_1 u_1 + \xi_2 u_2) \sin \lambda \pm i(\xi_2 u_1 - \xi_1 u_2) \cos \lambda} d\lambda,$$

from which the second expression for r_ε^\pm follows. \square

5.2. The roots of $a_0 - n$ and the distribution $\lim_{\varepsilon \rightarrow 0^+} (1/\sin \pi a_\varepsilon)$. From expression (63), one can see that the main issue in computing the limit for $\varepsilon \rightarrow 0^+$ of r_ε^\pm consists in the sets of points $(\tau, \xi, u) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ for which

$$a_0(\tau, \xi, u) = \gamma(u)\tau - \xi_3 u_3 \in \mathbb{Z}.$$

We note a few preliminary facts about such points.

Remark 9. The condition $a_0(\tau, \xi, u) = n \in \mathbb{Z}$ is equivalent to

$$\omega - k_3 v_3 = n\Omega_s(u), \quad n \in \mathbb{Z},$$

which defines the cyclotron resonances: particles of the species s that satisfy this condition along their orbit, for some integer n , resonate to a plane wave with frequency and wave vector (ω, k) . For resonant particles, the Doppler-shifted wave frequency $\omega - k_3 v_3$ matches a multiple (also referred to as a harmonic) of the gyration frequency Ω_s of the particle's orbit around the magnetic field.

For any given $n \in \mathbb{Z}$ and $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^3$, $\tau \neq 0$, let

$$(65) \quad \mathcal{R}_n(\tau, \xi) = \{u \in \mathbb{R}^3 : a_0(\tau, \xi, u) = n\}.$$

Physically $\mathcal{R}_n(\tau, \xi)$ is the set of normalized particle momenta u that resonate with the n -th harmonic of the cyclotron frequency when the wave field is a plane wave with frequency and wave vector are given by (τ, ξ) . Since a_0 is constant in (ξ_1, ξ_2) , the sets $\mathcal{R}_n(\tau, \xi)$ for a fixed harmonic number n depend only on τ , and ξ_3 . In addition it is enough to study them for $\tau > 0$ and $\xi_3 \geq 0$, in view of the symmetries of the function a_0 . (The hyperplane $\tau = 0$ will be excluded in our main results.) A necessary condition for $u \in \mathcal{R}_n(\tau, \xi)$ is

$$(\tau^2 - \xi_3^2)u_3^2 + \tau^2(1 + u_1^2 + u_2^2) - 2n\xi_3 u_3 - n^2 = 0.$$

which defines a family of surfaces of revolution obtained by the rotation of conics around the u_3 -axis. Specifically we find ellipsoids for $\tau^2 > \xi_3^2$, a paraboloid for $\tau^2 = \xi_3^2$, and one branch of a hyperboloid for $\tau^2 < \xi_3^2$. We shall speak of elliptic,

parabolic, and hyperbolic resonances, with reference these three conditions, respectively. In the elliptic case, $\mathcal{R}_n(\tau, \xi)$ is non-empty if and only if $n\tau > 0$, that is when, τ and n are both non-zero and have the same sign; ellipsoids degenerate to a point when $\tau^2 - \xi_3^2 = n^2 > 0$. In the hyperbolic case one can check that $\mathcal{R}_n(\tau, \xi)$ is non-empty for all integers n . The special case $n = 0$ is a particular hyperbolic resonance, to be referred to as Landau resonance, for which $\mathcal{R}_0(\tau, \xi)$ is non empty only if $\tau^2 < \xi_3^2$.

We shall study the limit in $\mathcal{S}'(\mathbb{R}^3)$ of functions of the form

$$(66) \quad \tilde{r}_\varepsilon(\tau, \xi, u) = \frac{\tilde{P}_\varepsilon(\tau, \xi, u)}{\sin(\pi a_\varepsilon(\tau, \xi, u))},$$

where \tilde{P}_ε is a family of functions parameterized by $\varepsilon \in [0, \varepsilon_0]$ for a fixed $\varepsilon_0 > 0$. The choices of \tilde{P}_ε relevant to our analysis are $\tilde{P}_\varepsilon = ie^{-i\pi a_\varepsilon} \Phi_j P_\varepsilon^\pm / 2$ and $\tilde{P}_\varepsilon = ie^{-i\pi a_\varepsilon} \Psi_j P_\varepsilon^\pm / 2$, with notation of section 5.1. With this aim, it will be sufficient to consider a family of functions \tilde{P}_ε that satisfy the following conditions:

$$(67) \quad \begin{cases} \tilde{P}_\varepsilon \in C^\infty(\mathbb{R}^7), & \forall \varepsilon \in [0, \varepsilon_0], \\ |\partial^\mu \tilde{P}_\varepsilon(\tau, \xi, u)| \leq \hat{c}_\mu (1 + \tau^2 + \xi^2)^{\hat{m}_\mu} (1 + u^2)^{\hat{n}_\mu}, & \forall \varepsilon \in [0, \varepsilon_0], \forall \mu \in \mathbb{N}_0^7, \\ \tilde{P}_\varepsilon(\tau, \xi, u) \rightarrow \tilde{P}_0(\tau, \xi, u) \text{ for } \varepsilon \rightarrow 0^+, & \forall (\tau, \xi, u) \in \mathbb{R}^7, \end{cases}$$

where the constants $\hat{c}_\mu, \hat{m}_\mu, \hat{n}_\mu \in \mathbb{R}$ are independent of $\varepsilon \in [0, \varepsilon_0]$ and $(\tau, \xi, u) \in \mathbb{R}^7$.

We state the main result for the limit of (66) as $\varepsilon \rightarrow 0^+$. We shall see that it is sufficient to consider the case $\kappa_s = 1$, or $\nu_s = \omega_{c,s}/\gamma$; then a_ε is independent of the particle species. This is a valid choice of the damping coefficient ν_s , since it satisfies conditions (17).

Proposition 5.2. *Let $\varepsilon_0, \tau_0 > 0$, and let \tilde{r}_ε be the family of functions defined for $\varepsilon \in (0, \varepsilon_0]$ in equation (66) with \tilde{P}_ε satisfying condition (67), and with $\kappa_s = 1$. Then, for every (τ, ξ) , $\tau \neq 0$, there is $\tilde{r}_0(\tau, \xi, \cdot) \in \mathcal{S}'(\mathbb{R}^3)$, such that*

- (i) *in the limit $\varepsilon \rightarrow 0^+$, $\tilde{r}_\varepsilon(\tau, \xi, \cdot) \rightarrow \tilde{r}_0(\tau, \xi, \cdot)$ in $\mathcal{S}'(\mathbb{R}^3)$;*
- (ii) *for every $\phi \in \mathcal{S}(\mathbb{R}^3)$ the function $(\tau, \xi) \mapsto \langle \tilde{r}_0(\tau, \xi, \cdot), \phi \rangle$ is continuous on $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^3$, and C^∞ near points (τ, ξ) such that $\tau^2 \neq \xi_3^2 + n^2$ for all integers $n \geq 0$;*
- (iii) *for any $\phi \in \mathcal{S}(\mathbb{R}^3)$ there are reals $K_0, M > 0$, such that*

$$|\langle \tilde{r}_\varepsilon(\tau, \xi, \cdot), \phi \rangle| \leq K_0 (1 + \tau^2 + \xi^2)^M,$$

for all (τ, ξ) with $|\tau| \geq \tau_0$ and $\varepsilon \in [0, \varepsilon_0]$.

Remark 10 (Degenerate resonances). The varieties $\tau^2 = \xi^2 + n^2$ for $n \geq 1$ in the Fourier space correspond to plane waves for which the elliptic resonance $\mathcal{R}_n(\tau, \xi)$ degenerates to a point. The special case $n = 0$, that is $\tau^2 - \xi^2 = 0$, corresponds to the parabolic resonance which separates elliptic and hyperbolic resonances. The function $(\tau, \xi) \mapsto \langle \tilde{r}_0(\tau, \xi, \cdot), \phi \rangle$ is smooth away from such topological transitions, where we can show continuity only.

For the proof of proposition 5.2, we need a few preparatory results. Let us choose a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ such that

$$0 \leq \chi(z) \leq 1, \quad \chi(z) = 1, \text{ for } |z| < \frac{1}{4}, \quad \chi(z) = 0, \text{ for } |z| \geq \frac{1}{3},$$

and for any $n \in \mathbb{Z}$, $\delta \in (0, 1)$ let

$$\chi_{\delta,n}(\tau, \xi, u) = \chi((a_0(\tau, \xi, u) - n)/\delta).$$

One can see that $\chi_{\delta,n} \in C^\infty(\mathbb{R}^7)$ but it is not necessarily compactly supported; in fact, $\chi_{\delta,n}(\tau, \xi, \cdot)$ is localized around $\mathcal{R}_n(\tau, \xi)$ which is unbounded when $\tau^2 \leq \xi_3^2$.

Lemma 5.3. *If $(\tau, \xi, u) \in \text{supp}(\chi_{\delta,n})$ for a certain $n \in \mathbb{Z}$ then $\chi_{\delta,m}(\tau, \xi, u) = 0$ for all $m \neq n$.*

Proof. The set $\text{supp} \chi_{\delta,n}$ is given by the condition $|a_0 - n| \leq \delta/3$, so that

$$|a_0(\tau, \xi, u) - m| \geq |m - n| - |a_0(\tau, \xi, u) - n| > 1 - \delta/3 > \delta/3,$$

and thus $\chi_{\delta,m}(\tau, \xi, u) = 0$. \square

It follows from lemma 5.3 that the sum

$$(68) \quad \chi_\delta = \sum_{n \in \mathbb{Z}} \chi_{\delta,n},$$

is locally finite and thus it defines a family of smooth functions χ_δ for $\delta \in (0, 1)$.

Lemma 5.4. *For every $\delta \in (0, 1)$, the cut-off function χ_δ in (68) is such that*

$$\begin{aligned} |\cos(\pi a_0(\tau, \xi, u))| &\geq \cos(\pi\delta/3) > 0, & \text{for } (\tau, \xi, u) \in \text{supp}(\chi_\delta), \\ |\sin(\pi a_0(\tau, \xi, u))| &\geq \sin(\pi\delta/4) > 0, & \text{for } (\tau, \xi, u) \in \text{supp}(1 - \chi_\delta). \end{aligned}$$

Proof. For every point (τ, ξ, u) in the support of χ_δ , there is an integer n such that

$$|a_0 - n| \leq \delta/3, \quad |\cos(\pi a_0)| = |\cos(\pi(a_0 - n))| \geq \cos(\pi\delta/3) > 0.$$

Analogously, on the support of $1 - \chi_\delta$,

$$|a_0 - n| \geq \delta/4, \quad |\sin(\pi a_0)| = |\sin(\pi(a_0 - n))| \geq \sin(\pi\delta/4) > 0,$$

for all integers n . \square

The cut-off function χ_δ allows us to isolate the singularities of (66) when $\varepsilon \rightarrow 0^+$. If ν_s in equation (55) is chosen so that $\kappa_s = 1$, for $(\tau, \xi, u) \in \text{supp}(\chi_\delta)$ we can write

$$\sin(\pi a_\varepsilon) = \cos(\pi a_0) [\tan(\pi a_0) \cosh(\pi\varepsilon) + i \sinh(\pi\varepsilon)],$$

and

$$(69) \quad \frac{1}{\sin(\pi a_\varepsilon)} = \frac{-i}{\cos(\pi a_0)} \int_0^{+\infty} e^{i\lambda[\tan(\pi a_0) \cosh(\pi\varepsilon) + i \sinh(\pi\varepsilon)]} d\lambda.$$

Therefore, for any $\psi \in C_0^\infty(\mathbb{R}^3)$,

$$(70) \quad \langle \tilde{r}_\varepsilon(\tau, \xi, \cdot), \psi \rangle = -i \int_0^{+\infty} e^{-\lambda \sinh(\pi\varepsilon)} \mathcal{I}_{\delta,\varepsilon}^c(\psi)(\tau, \xi, \lambda) d\lambda + \mathcal{I}_{\delta,\varepsilon}^s(\psi)(\tau, \xi),$$

where we have defined the functions

$$(71a) \quad \vartheta_\varepsilon(\tau, \xi, u) := \tan(\pi a_0(\tau, \xi, u)) \cosh(\pi\varepsilon),$$

$$(71b) \quad \mathcal{I}_{\delta,\varepsilon}^c(\psi)(\tau, \xi, \lambda) := \int_{\mathbb{R}^3} e^{i\lambda\vartheta_\varepsilon(\tau, \xi, u)} \frac{\tilde{P}_\varepsilon(\tau, \xi, u)\psi(u)}{\cos(\pi a_0(\tau, \xi, u))} \chi_\delta(\tau, \xi, u) du,$$

$$(71c) \quad \mathcal{I}_{\delta,\varepsilon}^s(\psi)(\tau, \xi) := \int_{\mathbb{R}^3} \frac{\tilde{P}_\varepsilon(\tau, \xi, u)\psi(u)}{\sin(\pi a_\varepsilon(\tau, \xi, u))} (1 - \chi_\delta(\tau, \xi, u)) du.$$

We also define

$$(71d) \quad \mathcal{I}_{\delta,0}^c(\psi)(\tau, \xi, \lambda) := \int_{\mathbb{R}^3} e^{i\lambda \tan(\pi a_0(\tau, \xi, u))} \frac{\tilde{P}_0(\tau, \xi, u)\psi(u)}{\cos(\pi a_0(\tau, \xi, u))} \chi_\delta(\tau, \xi, u) du,$$

$$(71e) \quad \mathcal{I}_{\delta,0}^s(\psi)(\tau, \xi) := \int_{\mathbb{R}^3} \frac{\tilde{P}_0(\tau, \xi, u)\psi(u)}{\sin(\pi a_0(\tau, \xi, u))} (1 - \chi_\delta(\tau, \xi, u)) du.$$

The main step in the proof of proposition 5.2 consists of an application of the stationary phase formula [29, Chapter 7] in order to prove that $\mathcal{I}_{\delta,\varepsilon}^c(\psi)(\tau, \xi, \cdot)$ is bounded in L^1 uniformly in ε . The real-valued phase is given by ϑ_ε and the parameter is $\lambda \geq 1$. All technical results needed in the proof of proposition 5.2 are collected in section 6, below.

Proof of proposition 5.2. We start from identity (70),

$$\langle \tilde{r}_\varepsilon(\tau, \xi, \cdot), \psi \rangle = -i \int_0^{+\infty} e^{-\lambda \sinh(\pi\varepsilon)} \mathcal{I}_{\delta,\varepsilon}^c(\psi)(\tau, \xi, \lambda) d\lambda + \mathcal{I}_{\delta,\varepsilon}^s(\psi)(\tau, \xi),$$

for $\psi \in C_0^\infty(\mathbb{R}^3)$ and lemma 6.4 (i) shows that this is extended to $\phi \in \mathcal{S}(\mathbb{R}^3)$.

For every $\delta \in (0, 1)$, lemma 6.4 (iii) and (v) allows us to define the functional

$$(72) \quad \langle \tilde{r}_{\delta,0}(\tau, \xi, \cdot), \phi \rangle = -i \int_0^{+\infty} \mathcal{I}_{\delta,0}^c(\phi)(\tau, \xi, \lambda) d\lambda + \mathcal{I}_{\delta,0}^s(\phi)(\tau, \xi),$$

over $\mathcal{S}(\mathbb{R}^3)$ for every (τ, ξ) , $\tau \neq 0$.

In view of lemma 6.4 (ii), as $\varepsilon \rightarrow 0^+$, we have $\mathcal{I}_{\delta,\varepsilon}^c(\phi)(\tau, \xi) \rightarrow \mathcal{I}_{\delta,0}^c(\phi)(\tau, \xi)$ and $\mathcal{I}_{\delta,\varepsilon}^s(\phi)(\tau, \xi) \rightarrow \mathcal{I}_{\delta,0}^s(\phi)(\tau, \xi)$ for all $\phi \in \mathcal{S}(\mathbb{R}^3)$. By the dominated convergence theorem and lemma 6.4 (iii) we deduce

$$\int_0^{+\infty} e^{-\lambda \sinh(\pi\varepsilon)} \mathcal{I}_{\delta,\varepsilon}^c(\phi)(\tau, \xi, \lambda) d\lambda \rightarrow \int_0^{+\infty} \mathcal{I}_{\delta,0}^c(\phi)(\tau, \xi, \lambda) d\lambda,$$

for all $\phi \in \mathcal{S}(\mathbb{R}^3)$, hence $\tilde{r}_\varepsilon(\tau, \xi, \cdot) \rightarrow \tilde{r}_{\delta,0}(\tau, \xi, \cdot)$ in $\mathcal{S}'(\mathbb{R}^3)$. By uniqueness of the limit, we have that $\tilde{r}_{\delta,0}(\tau, \xi, \cdot)$ is the same tempered distribution for all $\delta \in (0, 1)$ which we denote by $\tilde{r}_0(\tau, \xi, \cdot)$. This proves (i).

As for the regularity of $\langle \tilde{r}_0(\tau, \xi, \cdot), \phi \rangle$ with respect to (τ, ξ) , lemma 6.4 (i) show in particular that $\mathcal{I}_{\delta,0}^s(\phi) \in C^\infty$, hence it is enough to address

$$\langle \tilde{r}_{\delta,0}^c(\tau, \xi, \cdot), \phi \rangle := \langle \tilde{r}_{\delta,0}(\tau, \xi, \cdot), \phi \rangle - \mathcal{I}_{\delta,0}^s(\phi)(\tau, \xi) = -i \int_0^{+\infty} \mathcal{I}_{\delta,0}^c(\phi)(\tau, \xi, \lambda) d\lambda.$$

In lemma 6.4 (i) we have established that $\mathcal{I}_{\delta,0}^c(\phi)$ is C^∞ , and the inequality in item (iii) of the same lemma 6.4 gives a function $\tilde{\mathcal{B}}_{\varepsilon_0,\delta} \in L^1(\mathbb{R}_+)$ such that

$$|\mathcal{I}_{\delta,0}^c(\phi)(\tau, \xi, \lambda)| \leq \tilde{\mathcal{B}}_{\varepsilon_0,\delta}(\lambda),$$

uniformly in $(\tau, \xi) \in K$ where K is any compact where $|\tau| > 0$. Then the dominated convergence theorem can be applied to show that $\langle \tilde{r}_{\delta,0}^c(\tau, \xi, \cdot), \phi \rangle$ is continuous at any point (τ, ξ) , where $|\tau| > 0$.

Concerning the derivatives of $\mathcal{I}_{\delta,0}^c(\phi)$ with respect to (τ, ξ) , let us fix a point $(\bar{\tau}, \bar{\xi})$ such that $|\bar{\tau}| \neq 0$ and $\bar{\tau}^2 \neq \bar{\xi}_3^2 + n^2$ for all $n \in \mathbb{N}_0$. For a sufficiently small radius $\rho > 0$ the closed ball $K = \{(\tau, \xi) : (\tau - \bar{\tau})^2 + (\xi - \bar{\xi})^2 \leq \rho^2\}$ satisfies the assumptions of lemma 6.4 item (iv). Therefore, there is a value of δ depending only on K and an upper bound $|\partial_{\tau,\xi}^\alpha \mathcal{I}_{\delta,\varepsilon}^c(\phi)(\tau, \xi, \lambda)| \leq \mathcal{B}_{\varepsilon_0,\delta}^{(\alpha)}(\lambda, \tau_*)$ with $\mathcal{B}_{\varepsilon_0,\delta}^{(\alpha)}(\cdot, \tau_*) \in L^1$,

where $\tau_* = \min |\tau|$ in K . Hence, $\langle \tilde{r}_{\delta,0}^c(\tau, \xi, \cdot), \phi \rangle$ is of class C^∞ near any point where $|\tau| > 0$ and $\tau^2 \neq \xi_3^2 + n^2$, as claimed in (ii).

Again lemma 6.4, items (iii) and (v), imply

$$|\langle \tilde{r}_\varepsilon(\tau, \xi, \cdot), \phi \rangle| \leq \|\mathcal{B}_{\varepsilon_0, \delta}(\cdot, \tau_0)\|_{L^1(\mathbb{R}_+)} (1 + \tau^2 + \xi^2)^{m_4 + \frac{5}{2}} \|\phi\|_{2\tilde{m}_4 + 10} \\ + K_0^s (1 + \tau^2 + \xi^2)^{\ell_{0,0}} \|\phi\|_{2\tilde{\ell}_{0,0} + 4},$$

uniformly in (τ, ξ) where $|\tau| \geq \tau_0 > 0$, where $m_j, \tilde{m}_j, \ell_{0,0}$ and $\tilde{\ell}_{0,0}$ have been defined in lemma 6.3 and 6.4, respectively. This estimate is uniform in $\varepsilon \in [0, \varepsilon_0]$. Since $m_4 \geq \ell_{0,0}$, we obtain claim (iii) with constant K_0 depending in particular on τ_0, ε_0 and δ and with $M = m_4 + 5/2$. \square

Remark 11. The estimate in proposition 5.2 item (iii) could be replaced by

$$|\langle \tilde{r}_\varepsilon(\tau, \xi, \cdot), \phi \rangle| \leq K_0(\tau)(1 + \tau^2 + \xi^2)^M, \quad \tau \neq 0,$$

but $K_0(\tau)$ is not bounded near $\tau = 0$. Hence our argument does not allow any conclusion for $\tau = 0$ and we have excluded all frequencies in $|\tau| < \tau_0$ with τ_0 arbitrarily small and fixed.

5.3. Solution of the linear Vlasov equation for the magnetized case. We address equation (15) for a given species s , and thus drop the subscript s for simplicity. Particularly, we shall denote by Q_j the differential operator $Q_{s,j}$ defined in equation (58), which depends on the equilibrium distribution function and the sign of the electric charge of the consider particle species.

We shall first address the existence of solutions of the linear Vlasov equation (18) for the time-derivative of the distribution function including a damping term and address its dissipation-less limit. The result will then be used to compute the time-derivative of the induced current (19).

Theorem 5.5. *Let $\varepsilon > 0$ and let ν be any function satisfying conditions (17). Then equation (18) has a solution $g_\varepsilon \in \mathcal{S}(\mathbb{R}^7)$ which is unique as an element of $\mathcal{S}'(\mathbb{R}^7)$ and:*

(i) *The Fourier transform of the unique tempered solution g_ε is*

$$\hat{g}_\varepsilon(\omega, k, u) = \frac{qn_0}{(mc)^4} \sum_{j=1}^3 \hat{E}_j(\omega, k) Q_j(\tau, \xi, u, \partial_\xi) r_\varepsilon^\pm(\tau, \xi, u),$$

where the sign $+$ (resp. $-$) is chosen for $q > 0$ (resp. $q < 0$).

(ii) *For $(u_1, u_2) \neq (0, 0)$,*

$$\hat{g}_\varepsilon(\omega, k, u) = i \frac{qn_0}{(mc)^4} \sum_{j=1}^3 \hat{E}_j(\omega, k) \sum_{n \in \mathbb{Z}} \frac{Q_j(\tau, \xi, u, \partial_\xi) A_n^\pm(b)}{a_\varepsilon - n} e^{+in\phi},$$

where ϕ is defined in (54) and $b = (b_1, b_2)$ in (56).

Remark 12. The expression of the solution in item (i) in terms of the integral r_ε^\pm defined in (63) was first proposed by Qin et al. [49], cf. also the subsequent discussion in the literature [40, 50].

Remark 13. The solution \hat{g}_ε given in item (ii) is in agreement with the standard expression obtained in the physics literature [9, 14, 54]. For $\varepsilon \rightarrow 0^+$ it exposes a countable number of poles for $a_0 \in \mathbb{Z}$ where $\sin(\pi a_0) = 0$. These poles correspond to cyclotron resonances briefly discussed in section 5.2.

Proof of theorem 5.5. We look for tempered solutions of equation (18) and thus we can equivalently consider the Fourier transform \hat{g}_ε of the distribution function g_ε . If we define

$$(73) \quad \hat{h}_\varepsilon = e^{\pm i(\xi_2 u_1 - \xi_1 u_2)} \hat{g}_\varepsilon,$$

where $\tau = \omega/\omega_{c,s}$, $\xi = ck/\omega_{c,s}$, and with sign + (resp. -) for $q > 0$ (resp. $q < 0$), the Fourier transform of equation (18) is equivalent to equation (53) with source

$$\hat{s}^\pm = \frac{|q|n_0}{(mc)^4} e^{\pm i(\xi_2 u_1 - \xi_1 u_2)} \sum_{j=1}^3 \hat{E}_j Q_j(\tau, \xi, u, \partial_\xi) e^{\pm i(\xi_2 u_1 - \xi_1 u_2)},$$

and with the sign chosen according to the electric charge q . (As in the proof of proposition 5.1, in the case $q < 0$ we can multiply by -1 the equation for \hat{h}_ε in order to recast it into the form of (53); hence $a = -a_\varepsilon$.) Under the hypotheses, s^\pm belongs to $\mathcal{S}(\mathbb{R}^7)$. Proposition 4.6 gives a solution $\hat{h}_\varepsilon \in \mathcal{S}(\mathbb{R}^7)$, and thus $\hat{g}_\varepsilon \in \mathcal{S}(\mathbb{R}^7)$. This is the unique solution in C^∞ , as proven in proposition 4.6, as well as in $\mathcal{S}'(\mathbb{R}^7)$, as proven in proposition 4.8.

(i) Substitution of the claimed expression \hat{g}_ε into (73) gives

$$\hat{h}_\varepsilon(\omega, k, u) = \frac{qn_0}{(mc)^4} \sum_{j=1}^3 \hat{E}_j(\omega, k) e^{\pm i(\xi_2 u_1 - \xi_1 u_2)} Q_j(\tau, \xi, u, \partial_\xi) r_\varepsilon^\pm(\tau, \xi, u),$$

where $\tau = \omega/\omega_{c,s}$, $\xi = ck/\omega_{c,s}$, and we observe that the operator $e^{\pm i(\xi_2 u_1 - \xi_1 u_2)} Q_j$ commutes with $u_2 \partial_{u_1} - u_1 \partial_{u_2}$. Then upon using lemma 5.1, we have that \hat{h}_ε solves (53) and thus \hat{g}_ε is the solution of equation (15).

(ii) Upon using coordinates (54), for either choice of the sign of the particle charge the equation for \hat{h}_ε reduces to (42) with $\tilde{a} = a_\varepsilon$ and with source

$$V(\phi) = \frac{qn_0}{(mc)^4} e^{\pm i(\xi_2 u_1 - \xi_1 u_2)} \sum_{j=1}^3 \hat{E}_j Q_j e^{\pm i(\xi_2 u_1 - \xi_1 u_2)}.$$

We evaluate the Fourier coefficients of the source, that is,

$$\begin{aligned} \hat{V}_n &= \frac{qn_0}{(mc)^4} \frac{1}{2\pi} \int_0^{2\pi} e^{-in\phi} e^{\pm i(\xi_2 u_1 - \xi_1 u_2)} \sum_{j=1}^3 \hat{E}_j Q_j e^{\pm i(\xi_2 u_1 - \xi_1 u_2)} d\phi \\ &= \frac{qn_0}{(mc)^4} e^{\pm i(\xi_2 u_1 - \xi_1 u_2)} \sum_{j=1}^3 \hat{E}_j Q_j \frac{1}{2\pi} \int_0^{2\pi} e^{-in\phi} e^{\pm i(\xi_2 u_1 - \xi_1 u_2)} d\phi, \end{aligned}$$

and, in the second identity, we have used the fact that the coefficients of the operator $e^{\pm i(\xi_2 u_1 - \xi_1 u_2)} Q_j$ are independent of ϕ , i.e., the exponential factor is canceled by the corresponding factor in the definition of the coefficients in equations (59). It is now sufficient to compute the Fourier coefficients of

$$e^{\pm i(\xi_2 u_1 - \xi_1 u_2)} = e^{\pm i(b_2 \cos \phi \pm b_1 \sin \phi)},$$

and those are equal to $A_n^\pm(b)$ as we have shown in equation (62). Then

$$\hat{V}_n = \frac{qn_0}{(mc)^4} e^{\pm i(\xi_2 u_1 - \xi_1 u_2)} \sum_{j=1}^3 \hat{E}_j Q_j A_n^\pm.$$

The Fourier expansion in lemma 4.4 then yields the claimed identity. \square

We shall now show that the solution g_ε for $\varepsilon \rightarrow 0^+$ approaches the causal solution of (15) as computed by integration along the characteristics curves. This result in addition clarifies the relation between the analytical expressions for the solution in Fourier and physical spaces. In the position-momentum variables, the characteristic curve $t' \mapsto (X(t'; t, x, p), P(t'; t, x, p))$ with terminal condition (x, p) at $t' = t$ is [9]

$$(74) \quad \begin{cases} X_1(t'; t, x, p) = x_1 - \frac{p_1}{m\gamma\Omega} \sin(\Omega \cdot (t - t')) - \frac{p_2}{m\gamma\Omega} [\cos(\Omega \cdot (t - t')) - 1], \\ X_2(t'; t, x, p) = x_2 + \frac{p_1}{m\gamma\Omega} [\cos(\Omega \cdot (t - t')) - 1] - \frac{p_2}{m\gamma\Omega} \sin(\Omega \cdot (t - t')), \\ X_3(t'; t, x, p) = x_3 - v_3(t - t'), \\ P_1(t'; t, x, p) = p_1 \cos(\Omega \cdot (t - t')) - p_2 \sin(\Omega \cdot (t - t')), \\ P_2(t'; t, x, p) = p_1 \sin(\Omega \cdot (t - t')) + p_2 \cos(\Omega \cdot (t - t')), \\ P_3(t'; t, x, p) = p_3, \end{cases}$$

where γ and thus Ω are constants of motion. Under assumption (17) ν is constant along the characteristics as well. The characteristic flow has the semi-group property: for every t_1, t_2, t_3 with $t_1 \leq t_2 \leq t_3$ and (x, p) ,

$$(75) \quad \begin{aligned} X(t_1; t_2, X(t_2; t_3, x, p), P(t_2; t_3, x, p)) &= X(t_1; t_3, x, p), \\ P(t_1; t_2, X(t_2; t_3, x, p), P(t_2; t_3, x, p)) &= P(t_1; t_3, x, p), \end{aligned}$$

which can be verified directly from (74). For E and F_0 at least in C^1 , let $\mathfrak{s} = -q(\partial_t E - v \times \nabla \times E) \cdot \nabla_p F_0$ for brevity, and $F_0 = n_0 G$ as in (14). We shall always imply the relation $u = p/(mc)$ between normalized and physical momentum variables.

Proposition 5.6. *For every $\varepsilon > 0$, $E \in [\mathcal{S}(\mathbb{R}^4)]^3$, $F_0 = n_0 G \in \mathcal{S}(\mathbb{R}^3)$, and $g_{0,\varepsilon} \in \mathcal{S}(\mathbb{R}^6)$, the Cauchy problem for equation (15) with initial condition $g_{0,\varepsilon}$ at time $t = t_0$ has a unique classical solution $g_\varepsilon \in C^\infty(\mathbb{R}^7)$ and it holds that:*

- (i) *There is a unique Cauchy datum for which the solution $g_\varepsilon \in \mathcal{S}'$ and that is given by*

$$g_{*,\varepsilon}(x, u) := \int_{-\infty}^{t_0} e^{-\varepsilon\nu(t_0-t')} \mathfrak{s}(t', X(t'; t_0, x, p), P(t'; t_0, x, p)) dt',$$

which belongs to $\mathcal{S}(\mathbb{R}^6)$.

- (ii) *For $g_{0,\varepsilon} = g_{*,\varepsilon}$ the solution is*

$$g_\varepsilon(t, x, u) = \int_{-\infty}^t e^{-\varepsilon\nu(t-t')} \mathfrak{s}(t', X(t'; t, x, p), P(t'; t, x, p)) dt',$$

and \hat{g}_ε is the same function defined in theorem 5.5 (i).

- (iii) *For $\varepsilon \rightarrow 0^+$, the function g_ε has a pointwise limit*

$$g(t, x, u) = \int_{-\infty}^t \mathfrak{s}(t', X(t'; t, x, p), P(t'; t, x, p)) dt',$$

for every $(t, x, u) \in \mathbb{R}^7$. The limit g belongs to $C_b^\infty(\mathbb{R}^7)$, it is a classical solution of equation (15), and it is independent of the choice of ν .

Remark 14. The first part of this statement is essentially Wollman's result on the linear Vlasov equation [57, theorem 3.1], but for the case of a uniform plasma

equilibrium and with somewhat relaxed hypotheses on the support of the initial datum.

Proof. By hypothesis the damping function ν is constant along the characteristics. The application of the standard method of characteristics [34] gives the unique classical solution of the Cauchy problem,

$$g_\varepsilon(t, x, u) = g_{0,\varepsilon}(X(t_0; t, x, p), P(t_0; t, x, p))e^{-\varepsilon\nu(t-t_0)} + \int_{t_0}^t e^{-\varepsilon\nu(t-t')} \mathfrak{s}(t', X(t'; t, x, p), P(t'; t, x, p)) dt',$$

where X and P are given in equation (74), and $g_\varepsilon \in C^\infty$ follows by Leibniz rule for differentiation of integrals. This completes the first part of the proposition.

Since $e^{-\varepsilon\nu(t-t')}$ is integrable for $t' \in (-\infty, t]$, the integral

$$H_\varepsilon(t, x, p) = \int_{-\infty}^t e^{-\varepsilon\nu(t-t')} \mathfrak{s}(t', X(t'; t, x, p), P(t'; t, x, p)) dt'$$

is finite. If the electric field is given in Fourier transform, using

$$E(t, x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{-i\omega t + ik \cdot x} \hat{E}(\omega, k) d\omega dk,$$

and applying Fubini's theorem, we get

$$H_\varepsilon(t, x, p) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{-i\omega t + ik \cdot x} \tilde{H}_\varepsilon(t, \omega, k, p) d\omega dk,$$

with, using $F_0 = n_0 G$,

$$\tilde{H}_\varepsilon(t, \omega, k, p) = \frac{qn_0}{(mc)^4} \sum_{j=1}^3 \hat{E}_j(\omega, k) Q_j(\tau, \xi, u, \partial_\xi) R_\varepsilon^\pm(t, \omega, k, p),$$

where Q_j are the operators defined in (58) for the considered particle species, and

$$R_\varepsilon^\pm(t, \omega, k, p) = \frac{\omega c}{\gamma} \int_{-\infty}^t e^{-\varepsilon\nu(t-t') + i\omega(t-t') - ik_\parallel v_\parallel(t-t')} e^{\pm i(\xi_2 u_1 - \xi_1 u_2) \cos(\frac{\omega c}{\gamma}(t-t')) - i(\xi_1 u_1 + \xi_2 u_2) \sin(\frac{\omega c}{\gamma}(t-t'))} dt',$$

with sign chosen according to the particle charge q . The change of variable $\lambda = \frac{\omega c}{\gamma}(t-t')$ shows that R_ε^\pm is actually independent of time t and $R_\varepsilon^\pm(t, \omega, k, p) = r_\varepsilon^\pm(\tau, \xi, u)$ where r_ε^\pm has been defined in (63). Then, \tilde{H}_ε is also independent of time and $\tilde{H}_\varepsilon(t, \omega, k, p) = \hat{g}_\varepsilon(\omega, k, p)$ with \hat{g}_ε the unique tempered solution established in theorem 5.5 (i). We have $\hat{g}_\varepsilon \in \mathcal{S}$ and thus H_ε is the inverse Fourier transform of a Schwartz function, so that $H_\varepsilon \in \mathcal{S}(\mathbb{R}^7)$. By definition $g_{*,\varepsilon}(x, p) = H_\varepsilon(t_0, x, p) = g_\varepsilon(t_0, x, p)$, hence $g_{*,\varepsilon} \in \mathcal{S}$ and it is the Cauchy datum of the unique tempered solution $g_\varepsilon = H_\varepsilon$. This proves (i) and item (ii) follows from the expression for H_ε .

As for the pointwise limit of the solution, item (iii), we observe that the characteristic flow and the source term \mathfrak{s} satisfy the hypothesis of proposition C.2; in particular, the flow is of the form (88), while \mathfrak{s} is defined as the sum of the products of a function in $\mathcal{S}(\mathbb{R}^4)$ in frequency and wave-vector and a function in $\mathcal{S}(\mathbb{R}^3)$ in momentum, hence $\mathfrak{s} \in \mathcal{S}(\mathbb{R}^7)$. The function g is then the causal solution of (16) in

the sense of appendix C and proposition C.2 gives $g \in C_b^\infty(\mathbb{R}^7)$. For every $m \in \mathbb{N}_0$ one has the estimate

$$\begin{aligned} & (1 + s^2)^m |\mathfrak{s}(s, X(s; t, x, p), P(s; t, x, p))| \\ & \leq (1 + s^2 + X(s; t, x, p)^2 + P(s; t, x, p)^2)^m |\mathfrak{s}(s, X(s; t, x, p), P(s; t, x, p))|, \end{aligned}$$

which gives

$$|\mathfrak{s}(s, X(s; t, x, p), P(s; t, x, p))| \leq \frac{\|\mathfrak{s}\|_{2m}}{(1 + s^2)^m},$$

where $\|\cdot\|_m$ are the semi-norms defined in appendix A. Therefore, for every (t, x, p) and for m large enough, dominated convergence allows us to pass to the limit $\varepsilon \rightarrow 0^+$ in the integrand. \square

5.4. Current density and conductivity operator. First we show that the functions r_ε^\pm in lemma 5.1 have a limit in \mathcal{S}' . For simplicity let

$$(76) \quad \zeta^\pm(\xi_1, \xi_2, u_1, u_2, \lambda) = (\xi_1 u_1 + \xi_2 u_2) \sin \lambda \mp (\xi_2 u_1 - \xi_1 u_2) \cos \lambda,$$

and we recall the definition of a_0 in equation (56) and normalized Fourier variables (55).

Proposition 5.7. *For every integer $m \geq 2$,*

(i) *the linear map*

$$\begin{aligned} \psi \mapsto & \int \int_0^1 e^{ia_0(\tau, \xi, u)\lambda - i\zeta^\pm(\xi_1, \xi_2, u_1, u_2, \lambda)} d\lambda \psi(\tau, \xi, u) d\tau d\xi du \\ & + i^m \int \int_1^{+\infty} \lambda^{-m} e^{ia_0(\tau, \xi, u)\lambda - i\zeta^\pm(\xi_1, \xi_2, u_1, u_2, \lambda)} d\lambda \frac{\partial_\tau^m \psi(\tau, \xi, u)}{\gamma(u)^m} d\tau d\xi du, \end{aligned}$$

is continuous on $\mathcal{S}(\mathbb{R}^7)$ and thus defines a tempered distribution $r_0^{\pm, m}$;

(ii) *for $\varepsilon \rightarrow 0^+$, r_ε^\pm has a limit r_0^\pm in \mathcal{S}' , and $r_0^\pm = r_0^{\pm, m}$ for all $m \geq 2$. The limit is independent of the choice of the function ν_ε .*

Proof. The two terms defining the linear map on ψ are bounded by the norms $\|\psi\|_8$ and $\|\psi\|_{m+8}$, respectively, and this shows continuity in the topology of $\mathcal{S}(\mathbb{R}^7)$. Then the map is a tempered distribution which is denoted by $r_0^{\pm, m}$. Since

$$\frac{(-i)^m}{\gamma^m \lambda^m} \partial_\tau^m e^{ia_\varepsilon \lambda - i\zeta^\pm} = e^{ia_\varepsilon \lambda - i\zeta^\pm},$$

after integration by parts in τ ,

$$\begin{aligned} \langle r_\varepsilon^\pm, \psi \rangle = & \int \int_0^1 e^{ia_\varepsilon \lambda - i\zeta^\pm} d\lambda \psi d\tau d\xi du \\ & + i^m \int \left[\int_1^{+\infty} \frac{e^{ia_\varepsilon \lambda - i\zeta^\pm}}{\lambda^m} d\lambda \right] \frac{\partial_\tau^m \psi}{\gamma^m} d\tau d\xi du, \end{aligned}$$

and if $m \geq 2$ the integrand is uniformly bounded by an integrable function. We can then pass to the limit in the integral and obtain

$$\langle r_\varepsilon^\pm, \psi \rangle \xrightarrow{\varepsilon \rightarrow 0^+} \langle r_0^{\pm, m}, \psi \rangle,$$

for every $\psi \in \mathcal{S}(\mathbb{R}^7)$ and every $m \geq 2$. Uniqueness of the limit implies that all distributions $r_0^{\pm, m}$ for $m \geq 2$ are equal to the limit $r_0^\pm = \lim_{\varepsilon \rightarrow 0^+} r_\varepsilon^\pm$. \square

This result is already sufficient to compute the limit of the time derivative of the induced current density as a tempered distribution under fairly general assumptions. In the following we define, for any \mathbf{s} labeling a particle species,

$$r_{\mathbf{s},\varepsilon}(\tau, \xi, u) := r_{\varepsilon}^{\pm}(\tau, \xi, \nu), \quad r_{\mathbf{s}} := r_0^{\pm},$$

with sign $+$ (resp. $-$) for $q_{\mathbf{s}} > 0$ (resp. $q_{\mathbf{s}} < 0$), with r_{ε}^{\pm} defined in equation (63), and with r_0^{\pm} being the distributional limit in lemma 5.7.

By definition, cf. equation (19), the current density associated to $\{g_{\mathbf{s},\varepsilon}\}$ is

$$\partial_t j_{\varepsilon} = \mathbb{K}(\{g_{\mathbf{s},\varepsilon}\}).$$

We recall that $\omega_{p,\mathbf{s}} = \sqrt{4\pi q_{\mathbf{s}}^2 n_{\mathbf{s},0}/m_{\mathbf{s}}}$ is the plasma frequency of the \mathbf{s} -th species. We also need the operator $Q_{\mathbf{s},j}$ defined in equation (58). Relations (55) are also implied.

Proposition 5.8. *The function $\partial_t j_{\varepsilon}$ belongs to $[\mathcal{S}(\mathbb{R}^4)]^3$ and*

$$(77a) \quad \widehat{\partial_t j_{\varepsilon}}(\omega, k) = \widehat{\zeta}_{\varepsilon}(\omega, k) \widehat{E}(\omega, k),$$

where the tensor $\widehat{\zeta}_{\varepsilon}$ is given component-wise by

$$(77b) \quad \widehat{\zeta}_{\varepsilon,ij}(\omega, k) = \sum_{\mathbf{s}} \frac{\omega_{p,\mathbf{s}}^2}{4\pi} \int_{\mathbb{R}^3} \frac{1}{\gamma(u)} u_i Q_{\mathbf{s},j}(\tau, \xi, u, \partial_{\xi}) r_{\mathbf{s},\varepsilon}(\tau, \xi, u) du.$$

Proof. In theorem 5.5 we have shown that $g_{\mathbf{s},\varepsilon} \in \mathcal{S}(\mathbb{R}^7)$, hence for every $l, m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^4$,

$$(1 + t^2 + x^2)^m (1 + u^2)^n |\partial_t^l \partial_x^{\alpha} g_{\mathbf{s},\varepsilon}(t, x, u)| \leq \|g_{\mathbf{s},\varepsilon}\|_{|\alpha|+l+2m+2n}.$$

Since for $n > 3/2$, $(1 + u^2)^{-n}$ is integrable, from equation (19) we deduce that $\partial_t j_{\varepsilon} = \mathbb{K}(\{g_{\mathbf{s},\varepsilon}\})$ is C^{∞} and the same inequality implies that all derivatives are rapidly decaying at infinity, that is, $\partial_t j_{\varepsilon} \in [\mathcal{S}(\mathbb{R}^4)]^3$.

Then the Fourier transform of $\partial_t j_{\varepsilon}$ exists in the classical sense and, by Fubini's theorem, we have

$$\widehat{\partial_t j_{\varepsilon}}(\omega, k) = \sum_{\mathbf{s}} q_{\mathbf{s}} (m_{\mathbf{s}} c)^3 \int v_{\mathbf{s}}(u) \widehat{g}_{\mathbf{s},\varepsilon}(\omega, k, u) du.$$

Upon using the expression for $\widehat{g}_{\mathbf{s},\varepsilon}$ in theorem 5.5 (i), we arrive at equation (77). \square

Remark 15. Convergence of the integral in (77b) is ensured by the rapid decay in u at infinity of the coefficients (57) in the operators $Q_{\mathbf{s},j}$.

We recall that $Q_{\mathbf{s},j}$ are first-order partial differential operators and we can define the formal adjoint $Q'_{\mathbf{s},j}$ by

$$\int_{\mathbb{R}^7} \chi_1(\tau, \xi, u) Q_{\mathbf{s},j} \chi_2(\tau, \xi, u) d\tau d\xi du = \int_{\mathbb{R}^7} \chi_2(\tau, \xi, u) Q'_{\mathbf{s},j} \chi_1(\tau, \xi, u) d\tau d\xi du,$$

for every $\chi_1, \chi_2 \in \mathcal{S}(\mathbb{R}^7)$. The adjoint $Q'_{\mathbf{s},j}$ is again a first-order partial differential operator and it is continuous from $\mathcal{S} \rightarrow \mathcal{S}$. Therefore we can define the distribution $\partial_t j \in [\mathcal{S}'(\mathbb{R}^4)]^3$ by

$$\langle \widehat{\partial_t j}, \widehat{\varphi} \rangle := \sum_{j=1}^3 \sum_{\mathbf{s}} \frac{\omega_{c,\mathbf{s}}^4 \omega_{p,\mathbf{s}}^2}{4\pi c^3} \langle r_{\mathbf{s}}, Q'_{\mathbf{s},j}(\frac{\widehat{\varphi} \cdot u}{\gamma} \widehat{E}_j) \rangle,$$

for every vector test-function $\widehat{\varphi} \in [\mathcal{S}(\mathbb{R}^4)]^3$.

Proposition 5.9. *As $\varepsilon \rightarrow 0^+$, $\partial_t j_\varepsilon \rightarrow \partial_t j$ in $[\mathcal{S}'(\mathbb{R}^4)]^3$ and the limit satisfies $\partial_t j = \mathbb{K}(\{g_s\}) \in [C_b^\infty(\mathbb{R}^4)]^3$.*

Proof. Step 1: $\partial_t j_\varepsilon \rightarrow \partial_t j$. For $\varepsilon > 0$, one has

$$\langle \widehat{\partial_t j_\varepsilon}, \widehat{\varphi} \rangle = \sum_s \frac{\omega_{c,s}^4 \omega_{p,s}^2}{4\pi c^3} \sum_{j=1}^3 \langle r_{s,\varepsilon}, Q'_{s,j} \left(\frac{\widehat{\varphi} \cdot u}{\gamma} \widehat{E}_j \right) \rangle,$$

where the factor $\omega_{c,s}^4 c^{-3}$ comes for the chance of variable $d\omega dk = (\omega_{c,s}^4 / c^3) d\tau d\xi$. The result then follows from the convergence of $r_{s,\varepsilon}$ in \mathcal{S}' proven in proposition 5.7.

Step 2: g_s belongs to the domain of the operator \mathbb{K} . We observe that the flow (74) is polynomially bounded and thus satisfies the hypothesis of lemma C.3. Then, since $|v(u)| = c|u|/\gamma(u) \leq c$, for every $n, l \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$,

$$(78) \quad (1+u^2)^n |v(u) \partial_t^l \partial_x^\alpha g_s(t, x, u)| \leq c(1+u^2)^n |\partial_t^l \partial_x^\alpha g_s(t, x, u)| \leq C,$$

and upon choosing $n > 3/2$ we deduce that $u \mapsto v(u) \partial_t^l \partial_x^\alpha g_s(t, x, u)$ is bounded by an L^1 -function. For $l = 0$ and $\alpha = 0$, this shows that g_s has finite first velocity moment. Iterated application of the dominated convergence theorem also shows that $\mathbb{K}(\{g_s\}) \in C_b^\infty(\mathbb{R}^4)$.

Step 3: $\partial_t j = \mathbb{K}(\{g_s\})$. Let $\widetilde{\partial_t j} = \mathbb{K}(\{g_s\})$. For every $\varphi \in [\mathcal{S}(\mathbb{R}^4)]^3$,

$$\langle \partial_t j - \widetilde{\partial_t j}, \varphi \rangle = -\langle \partial_t j_\varepsilon - \partial_t j, \varphi \rangle + \langle \mathbb{K}(\{g_{s,\varepsilon} - g_s\}), \varphi \rangle.$$

The first terms on the right-hand side converges to zero for $\varepsilon \rightarrow 0^+$. The second term is estimated by

$$\langle \mathbb{K}(\{g_{s,\varepsilon} - g_s\}), \varphi \rangle = \sum_s q_s (m_s c)^3 \int_{\mathbb{R}^4} \int_{\mathbb{R}^3} (g_{s,\varepsilon} - g_s)(t, x, u) \varphi(t, x) \cdot v(u) du dt dx.$$

We know that $g_{s,\varepsilon} \in \mathcal{S}$ and upon using estimate (78) with $l = 0$ and $\alpha = 0$, we can bound the integrand by an L^1 -function independent on ε so thus pass to the limit in the integrand. At last from proposition 5.6, we have $g_{s,\varepsilon} - g_s \rightarrow 0$. \square

Proposition 5.9 is sufficient to prove the continuity of $\varsigma : [\mathcal{S}(\mathbb{R}^4)]^3 \rightarrow [\mathcal{S}'(\mathbb{R}^4)]^3$, cf. equation (20). However, we wish to understand under which conditions ς reduces to a Fourier multiplier and, when this is the case, we wish to address the regularity of its symbol.

Since the limit $\partial_t j$ is independent of the choice of the damping function, we choose ν_s such that $\kappa_s = 1$; one can check that this satisfies condition (17).

We start from equation (77) which can be rewritten as

$$(79) \quad \widehat{\varsigma}_{\varepsilon,ij}(\omega, k) = \sum_s \frac{\omega_{p,s}^2}{4\pi} \left[\int_{\mathbb{R}^3} \frac{u_i}{\gamma} \Phi_j r_{s,\varepsilon}(\tau, \xi, u) \mathcal{F}_s(u) du + \int_{\mathbb{R}^3} \frac{u_i}{\gamma} \Psi_j r_{s,\varepsilon}(\tau, \xi, u) \mathcal{G}_s(u) du \right].$$

Since a_ε depends on ξ through ξ_3 only, we have

$$(80) \quad \Phi_j r_{s,\varepsilon} = \frac{ie^{-i\pi a_\varepsilon} \Phi_j P_\varepsilon^\pm}{2 \sin(\pi a_\varepsilon)}, \quad \Psi_j r_{s,\varepsilon} = \frac{ie^{-i\pi a_\varepsilon} \Psi_j P_\varepsilon^\pm}{2 \sin(\pi a_\varepsilon)},$$

with sign + (resp., -) for $q_s > 0$ (resp., $q_s < 0$).

We shall now apply proposition 5.2, which holds for generic functions of the form (66) with \tilde{P}_ε subject to condition (67), to the case of interest, which is given

in equations (79) and (80). We choose a normalization time-scale common to all particle species, that is, $\hat{\omega} := \max_{\mathfrak{s}} \omega_{c,\mathfrak{s}}$.

Proposition 5.10. *Under the same hypotheses as in theorem 5.5, for $\omega \neq 0$, the tensor $\hat{\zeta}_\varepsilon(\omega, k)$ defined in (77), has a pointwise limit $\hat{\zeta}_0(\omega, k)$, which can be computed, cf. equation (81), and such that:*

- (i) *The limit $\hat{\zeta}_0$ is continuous on $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^3$ and it is C^∞ where $\omega \neq 0$ and $\omega^2 \neq (ck_3)^2 + n^2\omega_{c,\mathfrak{s}}^2$ for all $n \in \mathbb{Z}$ and species index \mathfrak{s} .*
- (ii) *For any $\omega_0 > 0$, there is a constant $K(\omega_0)$ for which*

$$|\hat{\zeta}_0(\omega, k)| \leq K \left(1 + \left(\frac{\omega}{\hat{\omega}}\right)^2 + \left(\frac{ck}{\hat{\omega}}\right)^2\right)^M,$$

uniformly for $(\omega, k) \in \mathbb{R}^4$, $|\omega| \geq \omega_0$, with exponent M independent of ω_0 .

- (iii) *For all $E \in [\mathcal{S}(\mathbb{R}^4)]^3$ such that $\omega \neq 0$ in $\text{supp } \hat{E}$ we have $\partial_t j = \mathcal{F}^{-1}(\hat{\zeta}_0 \hat{E})$.*

Proof. We start from equation (79) which reads

$$\hat{\zeta}_{\varepsilon,ij}(\omega, k) = \sum_{\mathfrak{s}} \frac{\omega_{p,\mathfrak{s}}^2}{4\pi} \left[\left\langle \Phi_j r_{\mathfrak{s},\varepsilon}(\tau, \xi, \cdot), \frac{1}{\gamma} u_i \mathcal{F}_{\mathfrak{s}} \right\rangle + \left\langle \Psi_j r_{\mathfrak{s},\varepsilon}(\tau, \xi, \cdot), \frac{1}{\gamma} u_i \mathcal{G}_{\mathfrak{s}} \right\rangle \right],$$

where $\tau = \omega/\omega_{c,\mathfrak{s}}$ and $\xi = ck/\omega_{c,\mathfrak{s}}$, cf. equation (55).

As observed in equations (80), both $\Phi_j r_{\mathfrak{s},\varepsilon}$ and $\Psi_j r_{\mathfrak{s},\varepsilon}$ are special cases of (66) corresponding to $\hat{P}_\varepsilon = (i/2)e^{-i\pi a_\varepsilon} \Phi_j P_\varepsilon^\pm$ and $\hat{P}_\varepsilon = (i/2)e^{-i\pi a_\varepsilon} \Psi_j P_\varepsilon^\pm$, respectively. Lemma 6.5 shows that in both cases \hat{P}_ε satisfies condition (67). Proposition 5.2 gives, for any (τ, ξ) with $\tau \neq 0$, distributions $r_{\mathfrak{s},j}^\Phi(\tau, \xi, \cdot), r_{\mathfrak{s},j}^\Psi(\tau, \xi, \cdot) \in \mathcal{S}'(\mathbb{R}^3)$ such that

$$\Phi_j r_{\mathfrak{s},\varepsilon}(\tau, \xi, \cdot) \rightarrow r_{\mathfrak{s},j}^\Phi(\tau, \xi, \cdot), \quad \Psi_j r_{\mathfrak{s},\varepsilon}(\tau, \xi, \cdot) \rightarrow r_{\mathfrak{s},j}^\Psi(\tau, \xi, \cdot),$$

in the topology of $\mathcal{S}'(\mathbb{R}^3)$. In this case, let

$$(81) \quad \hat{\zeta}_{0,ij}(\omega, k) = \sum_{\mathfrak{s}} \frac{\omega_{p,\mathfrak{s}}^2}{4\pi} \left[\left\langle r_{\mathfrak{s},j}^\Phi(\tau, \xi, \cdot), \frac{1}{\gamma} u_i \mathcal{F}_{\mathfrak{s}} \right\rangle + \left\langle r_{\mathfrak{s},j}^\Psi(\tau, \xi, \cdot), \frac{1}{\gamma} u_i \mathcal{G}_{\mathfrak{s}} \right\rangle \right],$$

and we have $\hat{\zeta}_{\varepsilon,ij}(\omega, k) \rightarrow \hat{\zeta}_{0,ij}(\omega, k)$ pointwise in Fourier space where $\omega \neq 0$.

(i) Proposition 5.2 implies that $\hat{\zeta}_{0,ij}$ is continuous in $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^3$ and C^∞ where $\tau \neq 0$ and $\tau^2 \neq \xi_3^2 + n^2$ for all integers n , and for all particle species.

(ii) We observe that for any \mathfrak{s} , $\hat{\omega}/\omega_{c,\mathfrak{s}} \geq 1$, hence,

$$1 + \tau^2 + \xi^2 \leq \frac{\hat{\omega}^2}{\omega_{c,\mathfrak{s}}^2} \left(1 + \left(\frac{\omega}{\hat{\omega}}\right)^2 + \left(\frac{ck}{\hat{\omega}}\right)^2\right).$$

Then proposition 5.2 item (iii) implies that there are constants $K, M > 0$ independent of ε such that

$$|\hat{\zeta}_\varepsilon(\omega, k)| \leq K \left(1 + \left(\frac{\omega}{\hat{\omega}}\right)^2 + \left(\frac{ck}{\hat{\omega}}\right)^2\right)^M,$$

uniformly in $\varepsilon \in [0, \varepsilon_0]$.

(iii) We use the estimate proven in (ii), which implies

$$|\hat{\zeta}_\varepsilon(\omega, k) - \hat{\zeta}_0(\omega, k)| \leq 2K \left(1 + \left(\frac{\omega}{\hat{\omega}}\right)^2 + \left(\frac{ck}{\hat{\omega}}\right)^2\right)^M,$$

uniformly in $\varepsilon \in [0, \varepsilon_0]$ and for $|\omega| \geq \omega_0$. Since by hypothesis $\omega \neq 0$ and thus $\tau \neq 0$ in $\text{supp } \hat{E}$, we can choose ω_0 small enough that, for any $\hat{\varphi} \in [\mathcal{S}(\mathbb{R}^4)]^3$,

$$\left| \left[(\hat{\zeta}_\varepsilon(\omega, \tau) - \hat{\zeta}_0(\omega, k)) \hat{E}(\omega, k) \right] \cdot \hat{\varphi}(\omega, k) \right| \leq 2K \left(1 + \left(\frac{\omega}{\hat{\omega}}\right)^2 + \left(\frac{ck}{\hat{\omega}}\right)^2\right)^M |\hat{E}(\omega, k)| |\hat{\varphi}(\omega, k)|$$

uniformly in $\varepsilon \in [0, \varepsilon_0]$ and $(\omega, k) \in \mathbb{R}^4$, $|\omega| \geq \omega_0$. Since $E, \varphi \in [\mathcal{S}(\mathbb{R}^4)]^3$, the right-hand side of the above inequality belongs to $L^1(\mathbb{R}^4)$ and we have already

established that $\hat{\zeta}_\varepsilon(\omega, \tau) - \hat{\zeta}_0(\omega, k) \rightarrow 0$ pointwise. Therefore the hypothesis of the dominated convergence theorem are satisfied and we can conclude

$$\int_{\mathbb{R}^4} [(\hat{\zeta}_\varepsilon(\omega, \tau) - \hat{\zeta}_0(\omega, k))\hat{E}(\omega, k)] \cdot \varphi(\omega, k) d\omega dk \rightarrow 0,$$

for all $\hat{\varphi} \in [\mathcal{S}(\mathbb{R}^4)]^3$ which means that $\hat{\zeta}_\varepsilon \hat{E} \rightarrow \hat{\zeta}_0 \hat{E}$ in $\mathcal{S}'(\mathbb{R}^4)$. At last proposition 5.9 establishes that the \mathcal{S}' -limit of $\hat{\zeta}_\varepsilon \hat{E}$ is exactly $\widehat{\partial_t j}$. \square

5.5. Proof of the results on the relativistic, three-dimensional case with uniform magnetic field (section 1.4). We can now collect the partial results obtained in this section and give a proof of the main theorems stated in section 1.4.

Proof of theorem 1.3. (i) For each \mathfrak{s} and ε , existence of a solution $g_{\mathfrak{s}, \varepsilon} \in \mathcal{S}(\mathbb{R}^7)$ of the Vlasov equation with damping and its uniqueness is established in theorem 5.5. We have shown that $\widehat{\partial_t j_\varepsilon} \in [\mathcal{S}(\mathbb{R}^4)]^3$ in proposition 5.9.

(ii) Proposition 5.6 proves that $g_{\mathfrak{s}, \varepsilon}$ converges pointwise to $g_{\mathfrak{s}} \in C_b^\infty$ and the limit is independent of the damping function ν . Explicit expressions for $g_{\mathfrak{s}, \varepsilon}$ and $g_{\mathfrak{s}}$ are given. Using the inequality, cf. the proof of proposition 5.6,

$$(1 + s^2)^m |\mathfrak{s}(s, X(s; t, x, p), P(s; t, x, p))| \leq \|\mathfrak{s}\|_{2m},$$

yields the bound

$$|g_{\mathfrak{s}, \varepsilon}(t, x, u)| \leq C,$$

uniformly in ε and thus for any $\varphi \in \mathcal{S}(\mathbb{R}^7)$,

$$|(g_{\mathfrak{s}, \varepsilon}(t, x, u) - g_{\mathfrak{s}}(t, x, u))\varphi(t, x, u)| \leq 2C|\varphi(t, x, u)|,$$

and $|\varphi|$ is in L^1 . Hence the dominated convergence theorem yields

$$\langle g_{\mathfrak{s}, \varepsilon} - g_{\mathfrak{s}}, \varphi \rangle = \int (g_{\mathfrak{s}, \varepsilon}(t, x, u) - g_{\mathfrak{s}}(t, x, u))\varphi(t, x, u) dt dx du \rightarrow 0,$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^7)$. This is equivalent to $g_{\mathfrak{s}, \varepsilon} \rightarrow g_{\mathfrak{s}}$ in \mathcal{S}' . Existence of the \mathcal{S}' -limit of $\widehat{\partial_t j_\varepsilon}$ is proven in proposition 5.9, in which we also show that the limit $\widehat{\partial_t j}$ is determined by $g_{\mathfrak{s}}$ only, hence it is independent of ν .

(iii) The fact that the limit $g_{\mathfrak{s}}$ is a classical solution of the linearized Vlasov equation in C_b^∞ is proven in proposition 5.6. At last, the identity $\widehat{\partial_t j} = \mathbb{K}(\{g_{\mathfrak{s}}\})$ is shown in proposition 5.9. \square

Proof of theorem 1.4. After Fourier transform, the action of $\zeta(E)$ on a test function is given in proposition 5.9. Since $r_{\mathfrak{s}}$ is a tempered distribution and $Q'_{\mathfrak{s}, j}$ is continuous from $\mathcal{S} \rightarrow \mathcal{S}$, we have

$$|\langle \widehat{\partial_t j}, \hat{\varphi} \rangle| \leq C \|u \cdot \hat{\varphi} \hat{E} / \gamma\|_\ell,$$

and Leibniz rule for the derivative yields

$$|\langle \widehat{\partial_t j}, \hat{\varphi} \rangle| \leq \tilde{C} \|\hat{\varphi}\|_\ell \|\hat{E}\|_\ell.$$

If $\hat{E} \in [\mathcal{S}(\mathbb{R}^4)]^3$, then $(1 - \chi(\omega))\hat{E}(\omega, k)$ satisfies the hypothesis of proposition 5.10 which gives $\zeta(E) = \mathcal{F}^{-1}((1 - \chi)\hat{\zeta}_0 \hat{E})$. \square

6. STATIONARY AND NON-STATIONARY PHASE RESULTS FOR SOME INTEGRALS

This section is devoted to the proof of four lemmas, in which we study the phase $\vartheta_\varepsilon = \tan(\pi a_0) \cosh(\pi\varepsilon)$ with the aim of evaluating integrals in u via stationary phase results. It replaces the study of $\frac{1}{\sin \pi a_\varepsilon}$ by the study of $-\frac{i}{\cos \pi a_0} \int_0^{+\infty} e^{i\lambda\theta_\varepsilon} d\lambda$ and exchanging integration in λ and u in integrals defining the current, cf. equations (70) and (71) in section 5. In addition, in this section, as the distribution $\frac{1}{\sin \pi a_\varepsilon}$ acts on the functions \tilde{P}_ε , we give results on the functions \tilde{P}_ε relevant to our applications (lemma 6.5 on functions satisfying (67) below). In this section, we let $\kappa_s = 1$, cf. comments before proposition 5.2.

We note basic facts about the stationary phase points, that are the same as the critical points of a_0 , since $\tan(\pi a_0)$ behaves as $\pi(a_0 - n)$. Because of the symmetries of the function a_0 , it is sufficient to consider the case $\tau > 0$ and $\xi_3 \geq 0$.

Lemma 6.1. *For $\varepsilon > 0$, let ϑ_ε be defined in equation (71a) and let $\tau > 0$, $\xi \in \mathbb{R}^3$ with $\xi_3 \geq 0$.*

(i) *For $\tau^2 - \xi_3^2 \leq 0$, $\vartheta_\varepsilon(\tau, \xi, \cdot)$ has no critical points and*

$$|\nabla_u \vartheta_\varepsilon(\tau, \xi, u)| > (\pi\tau/2)(1 + u^2)^{-1}, \quad \text{for all } u \in \mathbb{R}^3.$$

(ii) *For $\tau^2 - \xi_3^2 > 0$, $\vartheta_\varepsilon(\tau, \xi, \cdot)$ has an isolated non-degenerate critical point at*

$$u = u_c(\tau, \xi) = (0, 0, \xi_3/(\tau^2 - \xi_3^2)^{1/2}),$$

and at the critical point $u = u_c$ one has

$$|\det \vartheta_\varepsilon''(\tau, \xi, u_c)| \geq (\pi\tau)^3 (1 + u_c^2)^{-5/2},$$

$$|\langle \vartheta_\varepsilon''(\tau, \xi, u_c)^{-1} \nabla_u, \nabla_u \rangle \Upsilon| \leq \frac{3}{\pi\tau} (1 + u_c^2)^{3/2} \max_{|\beta|=2} |\partial_u^\beta \Upsilon|,$$

for all $\Upsilon(\tau, \xi, u)$ of class C^2 , where ϑ_ε'' denotes the Hessian matrix of ϑ_ε with respect to u , and $\langle \cdot, \cdot \rangle$ is the Euclidean product in \mathbb{R}^3 .

Proof. A direct computation gives

$$\nabla_u \vartheta_\varepsilon = \pi(1 + \tan^2(\pi a_0)) \cosh(\pi\varepsilon) \nabla_u a_0,$$

and thus $\nabla_u \vartheta_\varepsilon = 0$ if and only if $\nabla_u a_0 = 0$. On the other hand the critical points of a_0 for a given (τ, ξ) are solution to

$$u_1 = u_2 = 0, \quad \frac{\tau u_3}{\sqrt{1 + u_3^2}} = \xi_3.$$

For $\tau \neq 0$, this equation is satisfied only if $\xi_3^2/\tau^2 \leq 1$. In that case the solution is given by u_c as claimed. The Hessian matrix

$$\begin{aligned} \vartheta_\varepsilon'' &= 2\pi^2 \tan(\pi a_0) (1 + \tan^2(\pi a_0)) \cosh(\pi\varepsilon) \nabla a_0 \otimes \nabla a_0 \\ &\quad + \pi(1 + \tan^2(\pi a_0)) \cosh(\pi\varepsilon) a_0'', \end{aligned}$$

at the critical point reduces to

$$\vartheta_\varepsilon''(\tau, \xi, u_c) = \pi(1 + \tan^2(\pi a_0(\tau, \xi, u_c))) \cosh(\pi\varepsilon) a_0''(\tau, \xi, u_c),$$

with

$$a_0''(\tau, \xi, u_c) = a_0(\tau, \xi, u_c) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - \xi_3^2/\tau^2 \end{pmatrix}$$

being the Hessian matrix of $a_0(\tau, \xi, \cdot)$ evaluated at $u = u_c$ and we have accounted for the identity $a_0(\tau, \xi, u_c) = (\tau^2 - \xi_3^2)^{1/2}$.

Proof of the inequality in (i). We write

$$|\nabla_u a_0(\tau, \xi, u)|^2 = \frac{\tau^2}{1+u^2} \left[u_1^2 + u_2^2 + \left(u_3 - \frac{\xi_3}{\tau} \sqrt{1+u^2} \right)^2 \right],$$

and observe that for $\tau > 0$, $\xi_3 \geq 0$, $\tau^2 - \xi_3^2 \leq 0$ and all $u \in \mathbb{R}^3$,

$$\left| u_3 - \frac{\xi_3}{\tau} \sqrt{1+u^2} \right| \geq \sqrt{1+u_3^2} - |u_3|,$$

therefore $(1+u^2)|\nabla_u a_0(\tau, \xi, u)|^2 \geq \tau^2 \psi(u_3)$, with $\psi(u_3) = (\sqrt{1+u_3^2} - |u_3|)^2$. The function $u_3 \mapsto (1+u_3^2)\psi(u_3)$ is even and for $u_3 \geq 0$, it decreases monotonically starting from the value $\psi(0) = 1$, and approaching $1/4$ as $u_3 \rightarrow +\infty$. Hence $(1+u_3^2)\psi(u_3) > 1/4$ for all $u_3 \in \mathbb{R}$. This yields the claimed inequality.

Estimates in (ii). We compute

$$\det a_0''(\tau, \xi, u_c) = (\tau^2 - \xi_3^2)^{5/2} / \tau^2,$$

and, for every function $\Upsilon \in C^2$,

$$\begin{aligned} |\langle (\vartheta_\varepsilon''(\tau, \xi, u_c))^{-1} \nabla_u, \nabla_u \rangle \Upsilon| &= \frac{|\partial_{u_1}^2 \Upsilon + \partial_{u_2}^2 \Upsilon + \frac{\tau^2}{\tau^2 - \xi_3^2} \partial_{u_3}^2 \Upsilon|}{\pi(1 + \tan^2(\pi a_0)) \cosh(\pi \varepsilon) (\tau^2 - \xi_3^2)^{1/2}} \\ &\leq \frac{3\tau^2}{\pi(\tau^2 - \xi_3^2)^{3/2}} \max_{|\alpha|=2} |\partial_u^\alpha \Upsilon|, \end{aligned}$$

since $\tau^2 / (\tau^2 - \xi_3^2) \geq 1$. At last it is enough noting that $\sqrt{\tau^2 - \xi_3^2} = \tau / \sqrt{1+u_c^2}$. \square

Remark 16 (Non-relativistic limits). In the non-relativistic and weakly relativistic limits, one has

$$a_0(\tau, \xi, u) \approx a_{\text{nr},0}(\tau, \xi, u) := \tau - \xi_3 u_3,$$

and

$$a_0(\tau, \xi, u) \approx a_{\text{wr},0}(\tau, \xi, u) := \tau(1 + u^2/2) - \xi_3 u_3,$$

respectively. In the first case (the non-relativistic limit), there is no stationary phase point if $\xi_3 \neq 0$, but the phase reduces to a constant (in u) when $\xi_3 = 0$; particularly, when $\xi_3 = 0$, all values of $u \in \mathbb{R}^3$ are either in resonance if $\tau \in \mathbb{Z}$ or not in resonance if $\tau \notin \mathbb{Z}$, that is, the set $\mathcal{R}(\tau, \xi)|_{\xi_3=0}$ is not a closed surface. On the other hand, in the weakly non-relativistic limit there is a non-degenerate stationary phase point $u_c = (0, 0, \xi_3/\tau)$ for $\tau \neq 0$. Then, $u_c \in \text{supp } \chi_\delta(\tau, \xi, \cdot)$, where χ_δ is the cut-off function introduced in equation (68) of section 5, only if there is an integer n such that $\tau - \xi_3^2/(2\tau) \in [n - \delta/3, n + \delta/3]$. The Hessian matrix of the phase at the critical point in the weakly non-relativistic case amounts to

$$\vartheta_{\text{wr},\varepsilon}''(\tau, \xi, u_c) = \pi\tau(1 + \tan^2(\pi a_{\text{wr},0})) \cosh(\pi\varepsilon) I$$

where I is the identity matrix. In a sense, the relativistic Lorentz factor removes the degeneracy of the case $\xi_3 = 0$. The stationary phase argument developed here applies to both the relativistic and weakly relativistic cases.

Lemma 6.2. For any $\alpha \in \mathbb{N}_0^4$, $\alpha \neq 0$, and any integer $n \geq 0$, there are polynomials π_α , $\pi_{\alpha,j}$, $j = 1, \dots, |\alpha|$, and $\pi_\alpha^{(n)}$ of one real variable such that

$$(82) \quad \partial_{\tau,\xi}^\alpha \vartheta_\varepsilon = \pi_\alpha (\tan(\pi a_0)) \cosh(\pi\varepsilon) (\nabla_{\tau,\xi} a_0)^\alpha,$$

$$(83) \quad e^{-i\lambda\vartheta_\varepsilon} \partial_{\tau,\xi}^\alpha e^{i\lambda\vartheta_\varepsilon} = \sum_{j=1}^{|\alpha|} (i\lambda \cosh(\pi\varepsilon))^j \pi_{\alpha,j} (\tan(\pi a_0)) (\nabla_{\tau,\xi} a_0)^\alpha, \quad \lambda \in \mathbb{R},$$

$$(84) \quad |\partial_{\tau,\xi}^\alpha [\vartheta_\varepsilon^n e^{i\lambda\vartheta_\varepsilon}]| \leq \pi_\alpha^{(n)}(\lambda) (1 + u^2)^{\frac{|\alpha|}{2}}, \quad u \in \text{supp } \chi_\delta(\tau, \xi, \cdot), \quad \lambda \geq 0,$$

where χ_δ is the cut-off function defined in equation (68).

Proof. The key observation is that a_0 is a linear function of (τ, ξ) , hence $\partial_{\tau,\xi}^\beta a_0 = 0$ for $|\beta| \geq 2$. Identities (82) and (83) are true for $|\alpha| = 1$ and can be extended to all α with $|\alpha| \geq 1$ by induction. As for inequality (84), Leibniz formula applied to the function $G_n(\lambda, z) = z^n e^{i\lambda z}$ with $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$, gives for any positive integer ℓ ,

$$\partial_z^\ell G_n(\lambda, z) = e^{i\lambda z} \sum_{m=0}^{\ell} \binom{\ell}{m} (i\lambda)^{\ell-m} \partial_z^m (z^n).$$

Because of lemma 5.4, for $u \in \text{supp } \chi_\delta(\tau, \xi, \cdot)$ and $\varepsilon \leq \varepsilon_0$,

$$|\vartheta_\varepsilon| \leq |\tan(\pi a_0) \cosh(\pi\varepsilon)| + |\sinh(\pi\varepsilon)| \leq \cosh(\pi\varepsilon_0) / \cos(\pi\delta/3) + \sinh(\pi\varepsilon_0),$$

therefore, for $\lambda \geq 0$,

$$|\partial_z^\ell G_n(\lambda, \vartheta_\varepsilon)| \leq \tilde{\pi}_\ell^{(n)}(\lambda),$$

where $\tilde{\pi}_\ell^{(n)}(\lambda)$ is a polynomial of degree ℓ in λ and with positive real coefficients. Since $|\nabla_{\tau,\xi} a_0(\tau, \xi, u)| \leq (1 + u^2)^{1/2}$, lemma 5.4 and identity (82) imply

$$|\partial_{\tau,\xi}^\alpha \vartheta_\varepsilon(\tau, \xi, u)| \leq c_\alpha (1 + u^2)^{|\alpha|/2}, \quad \text{for } \varepsilon \in [0, \varepsilon_0], \quad |\alpha| \geq 1 \text{ and } u \in \text{supp } \chi_\delta(\tau, \xi, \cdot).$$

The Faà di Bruno's formula for $\partial_{\tau,\xi}^\alpha [\vartheta_\varepsilon^n e^{i\lambda\vartheta_\varepsilon}] = \partial_{\tau,\xi}^\alpha [G_n(\lambda, \vartheta_\varepsilon)]$ amounts to the sum of terms of the form

$$\partial_z^\ell G_n(\lambda, \vartheta_\varepsilon) \partial_{\tau,\xi}^{\alpha_1} \vartheta_\varepsilon \cdots \partial_{\tau,\xi}^{\alpha_\ell} \vartheta_\varepsilon,$$

where the multi-indices $\alpha_1, \dots, \alpha_\ell$ in the ℓ factors are such that $\partial_{\tau,\xi}^\alpha = \partial_{\tau,\xi}^{\alpha_1} \cdots \partial_{\tau,\xi}^{\alpha_\ell}$, and in particular, $|\alpha_1| + \cdots + |\alpha_\ell| = |\alpha|$. It follows that

$$|\partial_{\tau,\xi}^\alpha [\vartheta_\varepsilon^n e^{i\lambda\vartheta_\varepsilon}]| \leq \pi_\alpha^{(n)}(\lambda) (1 + u^2)^{|\alpha|/2},$$

where $\pi_\alpha^{(n)}$ is a polynomial of degree $|\alpha|$ in λ , with positive real coefficients. \square

Next, we establish estimates for the integrands in equations (71). (Let us recall that $\|\phi\|_j$ denotes the Schwartz semi-norms of $\phi \in \mathcal{S}$ as defined in appendix A.)

Lemma 6.3. Let $\varepsilon_0 > 0$ and $\delta \in (0, 1)$ be fixed and let a_ε and χ_δ be defined in equation (56) and (68). For every $\phi \in \mathcal{S}(\mathbb{R}^3)$, \tilde{P}_ε satisfying condition (67), $\alpha \in \mathbb{N}_0^4$, $\rho \in \mathbb{N}_0^3$, and integer $m \geq 0$, there are constants $C_{\alpha,\rho}^c$ and $C_{\alpha,\rho}^s$, dependent on ε_0 , and δ but independent of m , such that

$$(1 + u^2)^m \left| \partial_{\tau,\xi}^\alpha \partial_u^\rho \left[\frac{\chi_\delta \tilde{P}_\varepsilon \phi}{\cos(\pi a_0)} \right] \right| \leq C_{\alpha,\rho}^c (1 + \tau^2 + \xi^2)^{\ell_{\alpha,\rho} + \frac{|\rho|}{2}} \|\phi\|_{2(m + \bar{\ell}_{\alpha,\rho}) + |\alpha| + |\rho|}$$

$$(1 + u^2)^m \left| \partial_{\tau,\xi}^\alpha \partial_u^\rho \left[\frac{(1 - \chi_\delta) \tilde{P}_\varepsilon \phi}{\sin(\pi a_\varepsilon)} \right] \right| \leq C_{\alpha,\rho}^s (1 + \tau^2 + \xi^2)^{\ell_{\alpha,\rho} + \frac{|\rho|}{2}} \|\phi\|_{2(m + \bar{\ell}_{\alpha,\rho}) + |\alpha| + |\rho|},$$

uniformly in (τ, ξ, u) and $\varepsilon \in [0, \varepsilon_0]$, with $\ell_{\alpha, \rho} = \min\{\ell \in \mathbb{N}_0 : \ell \geq \hat{m}_\mu, \forall \mu : |\mu| \leq |\alpha| + |\rho|\}$ and $\tilde{\ell}_{\alpha, \rho} = \min\{\ell \in \mathbb{N}_0 : \ell \geq \hat{n}_\mu, \forall \mu : |\mu| \leq |\alpha| + |\rho|\}$.

Proof. For $\alpha = 0$ and $\rho = 0$, both inequalities follows immediately from (67) and lemma 5.4. For the first inequality with $|\alpha| + |\rho| \geq 1$, the Leibniz rule yields

$$\left| \partial_{\tau, \xi}^\alpha \partial_u^\rho \left[\frac{\tilde{P}_\varepsilon \phi}{\cos(\pi a_0)} \chi_\delta \right] \right| \leq \hat{C}_{\alpha, \rho} \max_{\beta, \rho'} \left| \partial_{\tau, \xi}^\beta \partial_u^{\rho'} \left(\frac{\chi_\delta}{\cos(\pi a_0)} \right) \right| \max_{\beta, \rho'} |\partial_{\tau, \xi}^\beta \partial_u^{\rho'} \tilde{P}^\varepsilon| \max_{\rho'} |\partial_u^{\rho'} \phi|,$$

where the max are on multi-indices $\beta \in \mathbb{N}_0^4$ and $\rho' \in \mathbb{N}_0^3$ such that $\beta_i \leq \alpha_i$ for all $i = 0, 1, 2, 3$ and $\rho'_i \leq \rho_i$ for all $i = 1, 2, 3$, and with constant depending only of α, ρ .

The function a_0 is linear in (τ, ξ) , hence $\partial_{\tau, \xi}^\beta a_0 = 0$ for all multi-indices β with $|\beta| \geq 2$, while $|\partial_{\tau, \xi} a_0| \leq (1 + u^2)^{1/2}$, and

$$|\partial_u^\rho a_0| \leq (1 + \tau^2 + \xi^2)^{1/2}, \quad |\partial_{\tau, \xi} \partial_u^\rho a_0| \leq \tilde{c}_\rho, \quad \text{for all } \rho \in \mathbb{N}_0^3 : |\rho| \geq 1,$$

where $\tilde{c}_\rho = \max\{1, \sup_u |\partial_u^\rho \sqrt{1 + u^2}|\}$.

Because of lemma 5.3, near any point $(\tau, \xi, u) \in \text{supp } \chi_\delta$, there is a unique integer n such that $\chi_\delta = \chi((a_0 - n)/\delta)$. One can estimate the derivatives of $\chi((a_0 - n)/\delta)/\cos(\pi a_0)$ by means of a multivariate version of the standard Faà di Bruno's formula [28]. However, we observe that each factor $\partial_{\tau, \xi}^\beta \partial_u^{\rho'} a_0$ grows at most like either $(1 + \tau^2 + \xi^2)^{1/2}$ or $(1 + u^2)^{1/2}$ regardless of the order of the derivative. Therefore it is sufficient to estimate the term in the Faà di Bruno's formula with the highest number of factors, that is,

$$\partial_{\tau, \xi}^\beta \partial_u^{\rho'} \left[\frac{\chi_\delta}{\cos(\pi a_0)} \right] = (\nabla_{\tau, \xi, u} a_0)^\mu \frac{d^N}{dz^N} \left[\frac{\chi((z - n)/\delta)}{\cos(\pi z)} \right]_{z=a_0} + \dots$$

where $\mu = (\beta, \rho') \in \mathbb{N}_0^7$, and $N = |\mu| = |\beta| + |\rho'|$. All the derivative with respect to z are bounded because of the support of χ and its derivatives as in lemma 5.4. Hence, for all $\beta \leq \alpha$ and $\rho' \leq \rho$,

$$\left| \partial_{\tau, \xi}^\beta \partial_u^{\rho'} \left[\frac{\chi_\delta}{\cos(\pi a_0)} \right] \right| \leq c_{\alpha, \rho}(\delta) (1 + u^2)^{|\alpha|/2} (1 + \tau^2 + \xi^2)^{|\rho|/2}.$$

Assumption (67) then gives

$$\left| \partial_{\tau, \xi}^\alpha \partial_u^\rho \left[\frac{\tilde{P}_\varepsilon \phi}{\cos(\pi a_0)} \chi_\delta \right] \right| \leq C_{\alpha, \rho}^c (1 + \tau^2 + \xi^2)^{\ell_{\alpha, \rho} + \frac{|\rho|}{2}} (1 + u^2)^{\tilde{\ell}_{\alpha, \rho} + \frac{|\alpha|}{2}} \max_{|\rho'| \leq |\rho|} |\partial_u^{\rho'} \phi|,$$

where $\ell_{\alpha, \rho}$ and $\tilde{\ell}_{\alpha, \rho}$ are smallest integers larger than all the exponents \hat{m}_μ and \hat{n}_μ for $|\mu| \leq |\alpha| + |\rho|$, respectively. Upon multiplying by $(1 + u^2)^m$, one obtains the claimed inequality. The second inequality follows analogously. \square

The functions defined in (71) have the following properties.

Lemma 6.4. *Let $\varepsilon_0 > 0$ be fixed, a_ε and χ_δ be defined in equation (56) and (68), $\ell_{\alpha, \rho}$ and $\tilde{\ell}_{\alpha, \rho}$ be the integers given in lemma 6.3, and $m_j = \min\{\ell \in \mathbb{N}_0 : \ell \geq \ell_{0, \rho}, |\rho| \leq j\}$, $\tilde{m}_j = \min\{\ell \in \mathbb{N}_0 : \ell \geq \tilde{\ell}_{0, \rho}, |\rho| \leq j\}$. If \tilde{P}_ε , $\varepsilon \in [0, \varepsilon_0]$, satisfies condition (67),*

- (i) for any $\delta \in (0, 1)$, $\mathcal{I}_{\delta, \varepsilon}^c(\psi)$, $\mathcal{I}_{\delta, \varepsilon}^s(\psi)$, $\mathcal{I}_{\delta, 0}^c(\psi)$, and $\mathcal{I}_{\delta, 0}^s(\psi)$ defined in (71) for $\psi \in C_0^\infty$ are defined and of class C^∞ for any $\psi \in \mathcal{S}(\mathbb{R}^3)$;
- (ii) for any $\delta \in (0, 1)$ and $\phi \in \mathcal{S}(\mathbb{R}^3)$, $\mathcal{I}_{\delta, \varepsilon}^c(\phi) \rightarrow \mathcal{I}_{\delta, 0}^c(\phi)$ and $\mathcal{I}_{\delta, \varepsilon}^s(\phi) \rightarrow \mathcal{I}_{\delta, 0}^s(\phi)$, pointwise as $\varepsilon \rightarrow 0^+$;

- (iii) for any $\delta \in (0, 1)$, there exists a function $\mathcal{B}_{\varepsilon_0, \delta} : \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\}) \rightarrow \mathbb{R}_+$ such that $\mathcal{B}_{\varepsilon_0, \delta}(\cdot, \tau) \in L^1(\mathbb{R}_+)$, $\mathcal{B}_{\varepsilon_0, \delta}(\lambda, \tau_1) \leq \mathcal{B}_{\varepsilon_0, \delta}(\lambda, \tau_2)$ for $|\tau_1| \geq |\tau_2|$, and

$$|\mathcal{I}_{\delta, \varepsilon}^c(\phi)(\tau, \xi, \lambda)| \leq \mathcal{B}_{\varepsilon_0, \delta}(\lambda, \tau)(1 + \tau^2 + \xi^2)^{m_4 + \frac{5}{2}} \|\phi\|_{2\bar{m}_4 + 10},$$

for all $\phi \in \mathcal{S}(\mathbb{R}^3)$, $(\tau, \xi, \lambda) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^3 \times \mathbb{R}_+$, and $\varepsilon \in [0, \varepsilon_0]$;

- (iv) for $\alpha \in \mathbb{N}_0^4$, $|\alpha| \geq 1$, $\phi \in \mathcal{S}(\mathbb{R}^3)$, and for any connected, compact set $K \subset \mathbb{R}^4$ with non-empty interior, such that $\tau \neq 0$ and $|\tau^2 - \xi_3^2 - n^2| \geq \delta_K > 0$ for all $n \in \mathbb{N}_0$, $(\tau, \xi) \in K$, there are $\delta \in (0, 1)$ depending on K and functions $\mathcal{B}_{\varepsilon_0, \delta}^{(\alpha)} : \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\}) \rightarrow \mathbb{R}_+$, such that $\mathcal{B}_{\varepsilon_0, \delta}^{(\alpha)}(\cdot, \tau) \in L^1(\mathbb{R}_+)$, $\mathcal{B}_{\varepsilon_0, \delta}^{(\alpha)}(\lambda, \tau_1) \leq \mathcal{B}_{\varepsilon_0, \delta}^{(\alpha)}(\lambda, \tau_2)$ for $|\tau_1| \geq |\tau_2|$, and $|\partial_{\tau, \xi}^\alpha \mathcal{I}_{\delta, \varepsilon}^c(\phi)(\tau, \xi, \lambda)| \leq \mathcal{B}_{\varepsilon_0, \delta}^{(\alpha)}(\lambda, \tau)$;
- (v) for any multi-index $\alpha \in \mathbb{N}_0^4$, there is $K_\alpha^s > 0$ dependent on ε_0 and δ such that

$$|\partial_{\tau, \xi}^\alpha \mathcal{I}_{\delta, \varepsilon}^s(\phi)(\tau, \xi)| \leq K_\alpha^s (1 + \tau^2 + \xi^2)^{\ell_{\alpha, 0}} \|\phi\|_{2\bar{\ell}_{\alpha, 0} + |\alpha| + 4},$$

uniformly for $\varepsilon \in [0, \varepsilon_0]$, for all $\phi \in \mathcal{S}(\mathbb{R}^3)$.

Proof. (i) We shall show that, for any $\phi \in \mathcal{S}(\mathbb{R}^3)$, the integrands in equations (71b)-(71e) and their derivatives with respect to (τ, ξ, λ) are uniformly bounded by an integrable function of $u \in \mathbb{R}^3$ for (τ, ξ) and λ in a bounded set and for $\varepsilon \in [0, \varepsilon_0]$. Then it follows that all integrals in equations (71) are finite also when $\psi \in C_0^\infty(\mathbb{R}^3)$ is replaced by $\phi \in \mathcal{S}(\mathbb{R}^3)$, and the dominated convergence theorem allows us to differentiate in the integral.

For the case of $\mathcal{I}_{\delta, \varepsilon}^c$ and $\mathcal{I}_{\delta, 0}^s$, the needed uniform upper bound follows directly from the second inequality of lemma 6.3 with $m > 3/2$ and $\rho = 0$.

As for $\mathcal{I}_{\delta, \varepsilon}^c$, if $\phi \in \mathcal{S}(\mathbb{R}^3)$,

$$\begin{aligned} \partial_{\tau, \xi}^\alpha \partial_\lambda^n \left[e^{i\lambda\vartheta_\varepsilon} \frac{\chi_\delta \tilde{P}_\varepsilon \phi}{\cos(\pi a_0)} \right] &= \partial_{\tau, \xi}^\alpha \left[(i\vartheta_\varepsilon)^n e^{i\lambda\vartheta_\varepsilon} \frac{\chi_\delta \tilde{P}_\varepsilon \phi}{\cos(\pi a_0)} \right] \\ &= i^n \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial_{\tau, \xi}^{\alpha-\beta} [\vartheta_\varepsilon^n e^{i\lambda\vartheta_\varepsilon}] \partial_{\tau, \xi}^\beta \left[\frac{\chi_\delta \tilde{P}_\varepsilon \phi}{\cos(\pi a_0)} \right]. \end{aligned}$$

For any $R > 0$, inequality (84) in lemma 6.2 gives

$$|\partial_{\tau, \xi}^{\alpha-\beta} [\vartheta_\varepsilon^n e^{i\lambda\vartheta_\varepsilon}]| \leq M_{R, \alpha, n} (1 + u^2)^{\frac{|\alpha-\beta|}{2}},$$

for $u \in \text{supp } \chi_\delta((\tau, \xi, \cdot))$ and $|\lambda| \leq R$. Then the first inequality proven in lemma 6.3 with $\rho = 0$ gives

$$(85) \quad \left| \partial_{\tau, \xi}^\alpha \partial_\lambda^n \left[e^{i\lambda\vartheta_\varepsilon} \frac{\chi_\delta \tilde{P}_\varepsilon \phi}{\cos(\pi a_0)} \right] \right| \leq \frac{\widetilde{M}_{R, \alpha, n}}{(1 + u^2)^m} \|\phi\|_{2(m + \bar{\ell}_\alpha) + |\alpha|}$$

uniformly for $\varepsilon \in [0, \varepsilon_0]$, (τ, ξ, λ) in the ball $\tau^2 + \xi^2 + \lambda^2 \leq R^2$ for all R , and with $\bar{\ell}_\alpha = \max_{\beta \leq \alpha} \bar{\ell}_{\beta, 0}$. For $m > 3/2$ the right-hand side is integrable over \mathbb{R}^3 .

(ii) The integrands in equations (71b)-(71c) are continuous for $\varepsilon \rightarrow 0^+$ and the inequalities proven in lemma 6.3 with $m > 3/2$, $\alpha = 0$, and $\rho = 0$ imply upper bounds by L^1 functions uniformly in $\varepsilon \in [0, \varepsilon_0]$. Then, the hypothesis of the dominated convergence theorem are satisfied and one can pass to the limit in the integral.

(iii) Let us first address the case $\mathcal{I}_{\delta,\varepsilon}^c(\psi)$ for $\psi \in C_0^\infty(\mathbb{R}^3)$. Since $\mathcal{I}_{\delta,\varepsilon}^c(\psi)(\tau, \xi, \cdot)$ is continuous, it is measurable and integrable on compact intervals. We write

$$\int_0^{+\infty} \mathcal{I}_{\delta,\varepsilon}^c(\psi)(\tau, \xi, \lambda) d\lambda = \int_0^1 \mathcal{I}_{\delta,\varepsilon}^c(\psi)(\tau, \xi, \lambda) d\lambda + \int_1^{+\infty} \mathcal{I}_{\delta,\varepsilon}^c(\psi)(\tau, \xi, \lambda) d\lambda.$$

The first integral on the right-hand side is bounded by

$$\int_{\mathbb{R}^3} \left| \frac{\tilde{P}_\varepsilon \psi}{\cos(\pi a_0)} \chi_\delta \right| du \leq C_{0,0}^c (1 + \tau^2 + \xi^2)^{\ell_{0,0}} \|\psi\|_{2(m+\tilde{\ell}_{0,0})} \int_{\mathbb{R}^3} \frac{du}{(1+u^2)^m},$$

in view of the first inequality of lemma 6.3 with $\alpha = 0$, $\rho = 0$, and $m > 3/2$.

For the second integral, we estimate the decay in λ of $\mathcal{I}_{\delta,\varepsilon}^c(\psi)(\tau, \xi, \cdot)$ by means of the stationary phase formula.

For $\tau \neq 0$ and $\tau^2 - \xi_3^2 \leq 0$, lemma 6.1 shows that there are no stationary phase points, and we have a lower bound for the gradient of the phase. The standard stationary phase lemma [29, Theorem 7.7.1] gives

$$\left| \int_{\mathbb{R}^3} e^{i\lambda\vartheta_\varepsilon} \frac{\chi_\delta \tilde{P}_\varepsilon \psi}{\cos(\pi a_0)} du \right| \leq \frac{c_{\text{sp},\ell}}{\lambda^\ell} \sum_{|\rho| \leq \ell} \sup \left(\frac{1}{|\nabla_u \vartheta_\varepsilon|^{2\ell-|\rho|}} \left| \partial_u^\rho \left[\frac{\chi_\delta \tilde{P}_\varepsilon \psi}{\cos(\pi a_0)} \right] \right| \right),$$

for any integer $\ell \geq 0$. The right-hand side is integrable in $\lambda \in [1, +\infty)$ if $\ell \geq 2$. We choose $\ell = 2$ and lemma 6.1 (i) gives, for $|\rho| \leq \ell = 2$,

$$\frac{1}{|\nabla_u \vartheta_\varepsilon|^{2\ell-|\rho|}} \leq \frac{(1+u^2)^{4-|\rho|}}{(\pi\tau/2)^{4-|\rho|}} \leq \frac{16}{\pi^2 \tau^{4-|\rho|}} (1+u^2)^4,$$

and lemma 6.3 with $m = 4$ and $\alpha = 0$ implies that, for $|\rho| \leq \ell = 2$,

$$\frac{1}{|\nabla_u \vartheta_\varepsilon|^{2\ell-|\rho|}} \left| \partial_u^\rho \left[\frac{\chi_\delta \tilde{P}_\varepsilon \psi}{\cos(\pi a_0)} \right] \right| \leq \frac{16C_{0,\rho}^c}{\pi^2 \tau^{4-|\rho|}} (1 + \tau^2 + \xi^2)^{\ell_{0,\rho} + |\rho|/2} \|\psi\|_{2\tilde{\ell}_{0,\rho} + |\rho| + 8},$$

which in turns yields

$$\left| \int_{\mathbb{R}^3} e^{i\lambda\vartheta_\varepsilon} \frac{\chi_\delta \tilde{P}_\varepsilon \psi}{\cos(\pi a_0)} du \right| \leq \mathcal{B}'_{\varepsilon_0,\delta}(\lambda, \tau) (1 + \tau^2 + \xi^2)^{m_2+1} \|\psi\|_{2\tilde{m}_2+10}.$$

where

$$\mathcal{B}'_{\varepsilon_0,\delta}(\lambda, \tau) = \frac{16c_{\text{sp},2}}{\pi^2 \lambda^2} \sum_{|\rho| \leq 2} \frac{C_{0,\rho}^c}{\tau^{4-|\rho|}}, \quad \lambda \geq 1.$$

Since by definition $m_2 \leq m_4$ this proves the claim for $\tau^2 - \xi_3^2 \leq 0$.

For $\tau^2 - \xi_3^2 > 0$, there is an isolated non-degenerate stationary phase point $u = u_c(\tau, \xi)$ as shown in lemma 6.1. In this case, the stationary phase lemma [29, Theorem 7.7.5] gives

$$\begin{aligned} \left| \int_{\mathbb{R}^3} e^{i\lambda\vartheta_\varepsilon} \frac{\chi_\delta \tilde{P}_\varepsilon \psi}{\cos(\pi a_0)} du \right| &\leq \left| \det(\lambda \vartheta_\varepsilon'' / 2\pi) \right|^{-\frac{1}{2}} \sum_{j=0}^{\ell-1} \lambda^{-j} |L_{j,\tau,\xi}(\psi)| \\ &\quad + \frac{c'_{\text{sp},\ell}}{\lambda^\ell} \sum_{|\rho| \leq 2\ell} \sup \left| \partial_u^\rho \left[\frac{\chi_\delta \tilde{P}_\varepsilon \psi}{\cos(\pi a_0)} \right] \right|, \end{aligned}$$

where the Hessian matrix ϑ_ε'' and $L_{j,\tau,\xi}(\psi)$ are differential operators acting on ψ and evaluated at the stationary phase point $u = u_c(\tau, \xi)$. We seek an upper bound

in L^1 , hence it is sufficient to choose $\ell = 2$, and we have

$$L_{0,\tau,\xi}(\psi) = \frac{\chi_\delta \tilde{P}_\varepsilon \psi}{\cos(\pi a_0)},$$

$$L_{1,\tau,\xi}(\psi) = -\frac{i}{2} \langle (\vartheta''_\varepsilon)^{-1} \nabla_u, \nabla_u \rangle \left[\frac{\chi_\delta \tilde{P}_\varepsilon \psi}{\cos(\pi a_0)} \right],$$

evaluated at $(\tau, \xi, u) = (\tau, \xi, u_c)$. Using the results of lemma 6.1 (ii) together with the first inequality of lemma 6.3 yields

$$\begin{aligned} \frac{|L_{0,\tau,\xi}(\psi)|}{|\det(\vartheta''_\varepsilon)|^{1/2}} &\leq \frac{1}{(\pi\tau)^{3/2}} (1 + u_c^2)^{5/4} \left| \frac{\chi_\delta \tilde{P}_\varepsilon \psi}{\cos(\pi a_0)} \right| \\ &\leq \frac{1}{(\pi\tau)^{3/2}} C_0 (1 + \tau^2 + \xi^2)^{m_0} \|\psi\|_{2\tilde{m}_0+4}, \\ \frac{|L_{1,\tau,\xi}(\psi)|}{|\det(\vartheta''_\varepsilon)|^{1/2}} &\leq \frac{3/2}{(\pi\tau)^{5/2}} \max_{|\beta|=2} \left| (1 + u_c^2)^{11/4} \partial_u^\beta \left[\frac{\chi_\delta \tilde{P}_\varepsilon \psi}{\cos(\pi a_0)} \right] \right| \\ &\leq \frac{3/2}{(\pi\tau)^{5/2}} C_2 (1 + \tau^2 + \xi^2)^{m_2+1} \|\psi\|_{2\tilde{m}_2+8}, \end{aligned}$$

and

$$\left| \partial_u^\rho \left[\frac{\chi_\delta \tilde{P}_\varepsilon \psi}{\cos(\pi a_0)} \right] \right| \leq C_4 (1 + \tau^2 + \xi^2)^{m_4+2} \|\psi\|_{2\tilde{m}_4+4},$$

where $C_j = \max_{|\rho| \leq j} C_{0,\rho}^c$, $j \in \mathbb{N}_0$. Therefore,

$$\left| \int_{\mathbb{R}^3} e^{i\lambda\vartheta_\varepsilon} \frac{\chi_\delta \tilde{P}_\varepsilon \psi}{\cos(\pi a_0)} du \right| \leq \mathcal{B}_{\varepsilon_0,\delta}''(\lambda, \tau) (1 + \tau^2 + \xi^2)^{m_4 + \frac{5}{2}} \|\psi\|_{2\tilde{m}_4+8},$$

with

$$\mathcal{B}_{\varepsilon_0,\delta}''(\lambda, \tau) = \left(\frac{2\pi}{\lambda} \right)^{3/2} \left[\frac{C_0}{(\pi\tau)^{3/2}} + \frac{3}{2} \frac{C_2}{(\pi\tau)^{5/2}} \frac{1}{\lambda} \right] + \frac{C_4'}{\lambda^2}, \quad \lambda \geq 1,$$

with the constant C_4' depending on δ and ε_0 . At last we can combine the estimates obtained in the three cases by defining $\mathcal{B}_{\varepsilon_0,\delta} = \mathcal{B}'_{\varepsilon_0,\delta} + \mathcal{B}''_{\varepsilon_0,\delta}$ for $\lambda \geq 1$ and extending it to a constant for $\lambda \in [0, 1]$. This gives the claimed estimate for $\psi \in C_0^\infty(\mathbb{R}^3)$. Then we observe that for any (τ, ξ, λ) with $\tau \neq 0$, the map $\psi \mapsto \mathcal{I}_{\delta,\varepsilon}^c(\psi)(\tau, \xi, \lambda)$ defines a linear functional on $C_0^\infty(\mathbb{R}^3)$, bounded by a Schwartz semi-norm. Since $C_0^\infty(\mathbb{R}^3)$ is dense in $\mathcal{S}(\mathbb{R}^3)$ the inequality remains true for $\psi \in \mathcal{S}(\mathbb{R}^3)$: given a sequence $\psi_i \in C_0^\infty$, $i \in \mathbb{N}$, converging to $\phi \in \mathcal{S}(\mathbb{R}^3)$ in the topology of \mathcal{S} , inequality (85) with $\alpha = 0$ and $n = 0$ shows that $|\mathcal{I}_{\delta,\varepsilon}^c(\psi_i)| \rightarrow |\mathcal{I}_{\delta,\varepsilon}^c(\phi)|$, while by definition of convergence we have $\|\psi_i\|_j \rightarrow \|\phi\|_j$; then we can pass to the limit $i \rightarrow +\infty$ on both sides of the inequality.

(iv) First, let $\psi \in C_0^\infty(\mathbb{R}^3)$. From (i), we have that

$$\partial_{\tau,\xi}^\alpha \mathcal{I}_{\delta,\varepsilon}^c(\psi)(\tau, \xi, \lambda) = i^n \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^3} \partial_{\tau,\xi}^{\alpha-\beta} [e^{i\lambda\vartheta_\varepsilon}] \partial_{\tau,\xi}^\beta \left[\frac{\chi_\delta \tilde{P}_\varepsilon \psi}{\cos(\pi a_0)} \right] du.$$

We exploit the second identity in lemma 6.2 for the factor $\partial_{\tau,\xi}^{\alpha-\beta} e^{i\lambda\vartheta_\varepsilon}$ with the result that

$$\partial_{\tau,\xi}^\alpha \mathcal{I}_{\delta,\varepsilon}^c(\psi)(\tau, \xi, \lambda) = \sum_{j=0}^{|\alpha|} \lambda^j \cosh(\pi\varepsilon)^j \int_{\mathbb{R}^3} e^{i\lambda\vartheta_\varepsilon} \psi_{\delta,\varepsilon,j}(\tau, \xi, u) du,$$

where $\psi_{\delta,\varepsilon,j}$ are combinations of polynomials in $\tan(\pi a_0)$ and (τ, ξ) -derivatives of the function $\chi_\delta \tilde{P}_\varepsilon \psi / \cos(\pi a_0)$. The assumption states that (τ, ξ) varies in a connected, compact set K with non-empty interior, therefore the continuous function $(\tau, \xi) \mapsto \tau^2 - \xi_3^2$ maps K into a closed interval $[c_1, c_2] \subset \mathbb{R}$. The assumption with $n = 0$ also implies that $|\tau^2 - \xi_3^2| \geq \delta_K$, hence zero is not in the interval, that is, either $c_1 < c_2 < 0$, or $0 < c_1 < c_2$. In the first case, $\tau^2 - \xi_3^2 < 0$ and lemma 6.1 established that the phase ϑ_ε has no critical points. In the second case, let $n_0 \in \mathbb{N}_0$ be the unique non-negative integer for which

$$n_0^2 + \delta_K \leq \tau^2 - \xi_3^2 \leq (n_0 + 1)^2 - \delta_K, \quad (\tau, \xi) \in K.$$

We have shown in the proof of lemma 6.1 that $a_0(\tau, \xi, u_c) = \sqrt{\tau^2 - \xi_3^2}$ is the value of a_0 at the stationary phase point $u_c(\tau, \xi)$. If $n_c \in \{n_0, n_0 + 1\}$ is the closest integer to $a_0(\tau, \xi, u_c)$, we find $|a_0(\tau, \xi, u_c) - n_c| \geq c(\delta_K)$, and for any $n \in \mathbb{Z}$,

$$|a_0(\tau, \xi, u_c) - n| \geq |a_0(\tau, \xi, u_c) - n_c| \geq c(\delta_K).$$

We can now choose δ sufficiently small that $\delta/3 < c(\delta_K)$, which depends only on the set K , and we obtain $\chi_\delta(\tau, \xi, u_c) = 0$: there are no critical points of the phase in the support of χ_δ for $(\tau, \xi) \in K$. We can then apply the same argument as in (iii) in order to show that each of the integrals on the right-hand side decreases like $1/\lambda^\ell$ for all positive integers ℓ and since (τ, ξ) varies in a compact set K we obtain the claimed inequality with functions $\mathcal{B}_{\varepsilon_0, \delta}^{(\alpha)}$ depending on $\sup\{(1 + \tau^2 + \xi^2) : (\tau, \xi) \in K\}$ and the Schwartz semi-norms of ψ . Then the inequality can be extended to $\phi \in \mathcal{S}(\mathbb{R}^3)$ by choosing a sequence $\psi_n \in C_0^\infty(\mathbb{R}^3)$ converging to ϕ in $\mathcal{S}(\mathbb{R}^3)$.

(v). The derivatives of $\mathcal{I}_{\delta, \varepsilon}^s(\phi)$ for $\phi \in \mathcal{S}(\mathbb{R}^3)$ are estimated by the second inequality in lemma 6.3 with $\rho = 0$, and $m = 2$. \square

We conclude this section with the proof that the functions defined in equation (80) are of the form (66). The function in the following lemma are defined in section 5.1.

Lemma 6.5. *The functions $e^{-i\pi a_\varepsilon} \Phi_j P_\varepsilon^\pm$ and $e^{-i\pi a_\varepsilon} \Psi_j P_\varepsilon^\pm$ satisfy condition (67).*

Proof. The considered functions are all defined as integrals in λ over a compact interval and with smooth integrands, hence they are of class C^∞ . As for the polynomial-growth estimate, the derivatives of any

$$F_\varepsilon \in \{e^{-i\pi a_\varepsilon} \Phi_j P_\varepsilon^\pm : j = 1, 2, 3\} \cup \{e^{-i\pi a_\varepsilon} \Psi_j P_\varepsilon^\pm : j = 1, 2, 3\}$$

are given by

$$\partial^\mu F_\varepsilon(\tau, \xi, u) = \int_0^{2\pi} \Pi_\mu(\tau, \xi, u, \lambda) e^{i(\lambda - \pi)a_\varepsilon - i\tilde{\zeta}} d\lambda,$$

where $\tilde{\zeta}(\tau, \xi, u, \lambda) = (\xi_1 u_1 + \xi_2 u_2) \sin \lambda \mp (\xi_2 u_1 - \xi_1 u_2)(\cos \lambda - 1)$ and Π_μ is a linear combination of monomials of the form

$$\gamma^m \tau^{\ell_0} \xi^\alpha u^\beta \lambda^{\ell_1} (\cos \lambda)^{\ell_2} (\sin \lambda)^{\ell_3},$$

for $m \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{N}_0^3$, $\ell_j \in \mathbb{N}_0$. This can be checked directly for $\mu = 0$ and $|\mu| = 1$, then extended to all $\mu \in \mathbb{N}_0^7$ by induction. We also have that Π_μ is independent of ε . For $\lambda \in [0, 2\pi]$ we readily have

$$|\gamma^m \tau^{\ell_0} \xi^\alpha u^\beta \lambda^{\ell_1} (\cos \lambda)^{\ell_2} (\sin \lambda)^{\ell_3}| \leq (2\pi)^{\ell_1} (1 + \tau^2 + \xi^2)^{(\ell_0 + |\alpha|)/2} (1 + u^2)^{(m + |\beta|)/2},$$

and (since we have chosen $\kappa = 1$)

$$|e^{i(\lambda-\pi)a_\varepsilon - i\tilde{\zeta}}| \leq e^{\pi\varepsilon}, \quad \text{for } \lambda \in [0, 2\pi],$$

hence

$$|\partial^\mu F_\varepsilon(\tau, \xi, u)| \leq C_{F,\mu} e^{\pi\varepsilon} (1 + \tau^2 + \xi^2)^{\hat{m}_{F,\mu}} (1 + u^2)^{\hat{n}_{F,\mu}},$$

with constant $C_{F,\mu}$ and exponents $\hat{m}_{F,\mu}, \hat{n}_{F,\mu}$ depending on the specific function F . This is the claimed inequality if $\varepsilon \in [0, \varepsilon_0]$. By means of the same identity one can also check continuity of all derivatives for $\varepsilon \rightarrow 0$, since

$$\begin{aligned} |\partial^\mu F_\varepsilon(\tau, \xi, u) - \partial^\mu F_0(\tau, \xi, u)| &\leq \int_0^{2\pi} |\Pi_\mu(\tau, \xi, u, \lambda)| \cdot |e^{-(\lambda-\pi)\varepsilon} - 1| d\lambda \\ &\leq \tilde{C}_{F,\mu}(\tau, \xi, u) |e^{\pi\varepsilon} - 1|, \end{aligned}$$

and thus, as $\varepsilon \rightarrow 0^+$, $\partial^\mu F_\varepsilon(\tau, \xi, u) \rightarrow \partial^\mu F_0(\tau, \xi, u)$ pointwise. \square

Remark 17. This induction argument has the advantage of avoiding lengthy calculations, but does not allow us to compute the exponents $\hat{m}_{F,\mu}$ and $\hat{n}_{F,\mu}$ explicitly.

APPENDIX A. NOTATION AND BASIC DEFINITIONS

Positive integers are denoted by $\mathbb{N} = \{1, 2, \dots\}$ and natural numbers by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ including zero. Integers including zero, reals and complex numbers are denoted by \mathbb{Z} , \mathbb{R} , and \mathbb{C} , respectively. Strictly positive real numbers are denoted by $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. We use the standard multi-index notation, a multi-index of dimension $n \in \mathbb{N}$ being an element $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$. The length of a multi-index is $|\alpha| = \sum_i \alpha_i$, and for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.

As usual $C^k(\mathbb{R}^n)$ denotes the space of k -times continuously differentiable functions of n variables, $C^\infty(\mathbb{R}^n) = \bigcap_k C^k(\mathbb{R}^n)$.

Definition 1. We denote by $C_b^\infty(\mathbb{R}^n)$ the space of functions $u \in C^\infty(\mathbb{R}^n)$ that are bounded with all derivatives bounded, that is, for every $\alpha \in \mathbb{N}_0^n$ there are real constants $C_\alpha > 0$ such that $|\partial_x^\alpha u(x)| \leq C_\alpha$ uniformly.

We work with tempered distributions on \mathbb{R}^n . As usually we denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions on \mathbb{R}^n . Those are functions $\varphi \in C^\infty(\mathbb{R}^n)$ for which

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta \varphi(x)| < \infty.$$

Semi-norms in \mathcal{S} are defined by

$$\|\varphi\|_j = \max_{|\alpha|+|\beta| \leq j} \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta \varphi(x)|, \quad j \in \mathbb{N}_0.$$

(With the condition $|\alpha| + |\beta| \leq j$, instead of equality, these are actually norms.) The countable set of semi-norms gives the Schwartz space the topology of a Fréchet space and its topological dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions, namely, the space of continuous linear functionals $u : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{C}$; the action of $u \in \mathcal{S}'$ on a test-function $\varphi \in \mathcal{S}$ is equivalently denoted by $\langle u, \varphi \rangle = u(\varphi)$. In this case continuity of a linear functional u means that there exists an integer $j \geq 0$ and a constant $C_j > 0$ such that $|\langle u, \varphi \rangle| = |u(\varphi)| \leq C_j \|\varphi\|_j$.

We also note that for any $\varphi \in \mathcal{S}$, for every $m \in \mathbb{R}$, and $\beta \in \mathbb{N}_0^d$, there is a constant $C_{m,\beta} > 0$ such that

$$(86) \quad (1+x^2)^m |\partial_x^\beta \varphi(x)| \leq C_{m,\beta},$$

uniformly in x . For $m \leq 0$, this inequality follows from $(1+x^2)^m \leq 1$, while for $m > 0$, one can observe that $(1+x^2)^m \leq (1+x^2)^l$ for any integer $l \geq m$, and use the multi-nomial formula.

The Fourier transform of a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{-i\xi \cdot x} dx.$$

The Fourier transform $\mathcal{F} : \varphi \mapsto \hat{\varphi}$ is continuous from \mathcal{S} into itself and extends to \mathcal{S}' by duality. Specifically, this means

$$\hat{u}(\varphi) = u(\hat{\varphi}).$$

That \hat{u} is a continuous linear functional follows from the continuity of u and of the Fourier transform $\varphi \rightarrow \hat{\varphi}$. For a function $\varphi(t, x)$ in $\mathcal{S}(\mathbb{R}^{1+d})$ of physical time and space ($d = 3$), we write

$$\hat{\varphi}(\omega, k) = \int_{\mathbb{R}^{1+d}} \varphi(t, x) e^{i\omega t - ik \cdot x} dt dx,$$

where, $\omega \in \mathbb{R}$ is the angular frequency and $k \in \mathbb{R}^d$ is the wave vector.

Definition 2 (Fourier multipliers). Let $a \in C(\mathbb{R}^n)$ and $m, C_0 \in \mathbb{R}$, be such that $|a(\xi)| \leq C_0(1+|\xi|)^m$. The Fourier multiplier with symbol a is the continuous linear operator on $\mathcal{S}(\mathbb{R}^n)$ defined by

$$A\varphi(x) = \mathcal{F}^{-1}(a\mathcal{F}(\varphi))(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(\xi) \hat{\varphi}(\xi) d\xi,$$

where the inverse Fourier transform is an absolutely convergent integral, and the integrand has partial derivatives in x of order k bounded by the L^1 -function $\xi \mapsto |\xi|^k |a(\xi)| |\hat{\varphi}(\xi)|$; hence, $A : \mathcal{S}(\mathbb{R}^n) \rightarrow C_b^\infty(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.

If in addition, $a \in C^\infty(\mathbb{R}^n)$ with all derivatives bounded by

$$|\partial_\xi^\alpha a(\xi)| \leq C_\alpha (1+|\xi|)^{m_\alpha},$$

for all $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$, with constants $C_\alpha, m_\alpha \in \mathbb{R}$ depending only on α , then for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $a\hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ and the inverse Fourier transform gives $\mathcal{F}^{-1}(a\hat{\varphi}) \in \mathcal{S}(\mathbb{R}^n)$. Such operations are continuous on \mathcal{S} . Hence, the corresponding Fourier multiplier defined by a is a continuous linear operator from $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$.

Fourier multipliers are relevant to the case of a uniform plasma equilibrium. Specifically, the high-frequency component of the current density induced in a uniform plasma by an electromagnetic disturbance is related to the electric field of the disturbance by a Fourier multiplier.

APPENDIX B. PROOFS FOR THE CASE STUDY OF SECTION 2

Proof of proposition 2.1. With an initial condition u_0 in C_b^∞ , the solution of this problem is

$$u(t, x) = u_0(x) + \int_0^t v(s, x) ds,$$

and we have $u \in C^\infty(\mathbb{R}^{1+d})$. In addition, $u \in C_b^\infty(\mathbb{R}^{1+d})$, thanks to $v \in \mathcal{S}$. This follows upon considering the derivatives $\partial_t^\ell \partial_x^\alpha u(t, x)$. For $\ell > 1$ one has $\partial_t^\ell \partial_x^\alpha u(t, x) = \partial_t^{\ell-1} \partial_x^\alpha v(t, x)$ with $v \in \mathcal{S}(\mathbb{R}^{1+d})$, while for $\ell = 0$ and for every $m > 1/2$ we have

$$|\partial_x^\alpha u(t, x)| \leq |\partial_x^\alpha u_0(x)| + \sup_{s \in \mathbb{R}} |(1+s^2)^m \partial_x^\alpha v(s, x)| \int_{-\infty}^{+\infty} \frac{ds}{(1+s^2)^m},$$

with $\partial_x^\alpha u_0$ bounded by hypothesis.

The initial condition

$$u_0(x) = \int_{-\infty}^0 v(s, x) ds,$$

belongs to $\mathcal{S}(\mathbb{R}^d)$ and thus to $C_b^\infty(\mathbb{R}^d)$. The corresponding unique solution in $C_b^\infty(\mathbb{R}^{1+d})$ is

$$u(t, x) = \left(\int_{-\infty}^0 + \int_0^t \right) v(s, x) ds = \int_{-\infty}^t v(s, x) ds.$$

For every integers $\ell \geq 0$ and $t \in \mathbb{R}$, $\partial_t^\ell u(t, \cdot) \in \mathcal{S}(\mathbb{R}^d)$ and u satisfies the condition $\lim_{t \rightarrow -\infty} u(t, x) = 0$. As for the uniqueness, if $u_* \in C_b^\infty(\mathbb{R}^d)$ is another initial condition such that the limit for $t \rightarrow -\infty$ of the corresponding solution vanishes, then

$$0 = u_*(x) + \int_0^{-\infty} v(s, x) ds = u_*(x) - u_0(x),$$

which shows that $u_* = u_0$. Since, in particular, $u \in L^\infty(\mathbb{R}^{1+d})$, it defines a tempered distribution by integration,

$$\langle u, \varphi \rangle = \int_{\mathbb{R}^{1+d}} u(t, x) \varphi(t, x) dt dx, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^{1+d}).$$

The continuity of the map $\varphi \mapsto \langle u, \varphi \rangle$ follows for $m > (1+d)/2$ from

$$|\langle u, \varphi \rangle| \leq \|u\|_{L^\infty(\mathbb{R}^{1+d})} \sup_{y \in \mathbb{R}^{1+d}} |(1+y^2)^m \varphi(y)| \int_{\mathbb{R}^{1+d}} \frac{dy}{(1+y^2)^m},$$

and, if $m > 1/2$ is an integer,

$$\sup_{y \in \mathbb{R}^{1+d}} |(1+y^2)^m \varphi(y)| \leq C_m \|\varphi\|_{2m},$$

On the other hand,

$$|u(t, x)| \leq \sup_{t \in \mathbb{R}} |(1+t^2)^\mu v(t, x)| \int_{-\infty}^{+\infty} \frac{ds}{(1+s^2)^\mu}, \quad \mu > 1/2,$$

and thus

$$\|u\|_{L^\infty(\mathbb{R}^{1+d})} \leq \left(\int_{-\infty}^{+\infty} \frac{ds}{(1+s^2)^\mu} \right) \sup_{(t,x) \in \mathbb{R}^{1+d}} |(1+t^2)^\mu v(t, x)|.$$

Since, if we choose $\mu > 1/2$ in \mathbb{N} ,

$$\sup_{(t,x) \in \mathbb{R}^{1+d}} |(1+t^2)^\mu v(t, x)| \leq \sup_{(t,x) \in \mathbb{R}^{1+d}} |(1+t^2+x^2)^\mu v(t, x)| \leq C_\mu \|v\|_{2\mu},$$

we have

$$\|u\|_{L^\infty(\mathbb{R}^{1+d})} \leq \tilde{C}_\mu \|v\|_{2\mu}, \quad \text{and} \quad |\langle u, \varphi \rangle| \leq K_{m,\mu} \|v\|_{2\mu} \|\varphi\|_{2m},$$

for $m > (1+d)/2$ and $\mu > 1/2$, both integers. \square

Proof of proposition 2.2. If u is a solution in \mathcal{S}' of the damped equation, its Fourier transform satisfies

$$-i(\omega + i\nu)\hat{u}^\nu = \hat{v}.$$

For $\nu > 0$, this has one and only one solution

$$\hat{u}^\nu(\omega, k) = i \frac{\hat{v}(\omega, k)}{\omega + i\nu},$$

and we have $\hat{u}^\nu \in \mathcal{S}(\mathbb{R}^{1+d})$ since $(\omega + i\nu)^{-n}$ is smooth and polynomially bounded for $\omega \in \mathbb{R}$ and for all integers $n > 0$. Hence, its inverse Fourier transform belongs to $\mathcal{S}(\mathbb{R}^{1+d})$. We recall that

$$u^\nu(-t, -x) = (2\pi)^{-(1+d)} \hat{u}^\nu(t, x),$$

so that, if $\check{\varphi}(t, x) = \varphi(-t, -x)$,

$$\langle u^\nu, \varphi \rangle = \langle (2\pi)^{-(1+d)} \hat{u}^\nu, \check{\varphi} \rangle = \langle \hat{u}^\nu, (2\pi)^{-(1+d)} \hat{\check{\varphi}} \rangle.$$

Let us introduce, for any $\varphi \in \mathcal{S}(\mathbb{R}^{1+d})$, the function $\hat{\psi} \in \mathcal{S}(\mathbb{R})$ given by

$$\hat{\psi}(\omega) = (2\pi)^{-(1+d)} \int_{\mathbb{R}^d} \hat{v}(\omega, k) \hat{\varphi}(-\omega, -k) dk.$$

We deduce

$$\langle u^\nu, \varphi \rangle = \int_{\mathbb{R}} \frac{i\hat{\psi}(\omega)}{\omega + i\nu} d\omega.$$

However, the sequence $\{u^\nu\}_{\nu \in \mathbb{R}_+}$ is not bounded in \mathcal{S} . In order to take the limit, we use the identity

$$\frac{i}{\omega + i\nu} = \int_0^{+\infty} e^{i(\omega + i\nu)t} dt,$$

and note that the function $(t, \omega) \mapsto e^{i(\omega + i\nu)t} \hat{\psi}(\omega)$ belongs to $L^1(\mathbb{R}_+ \times \mathbb{R})$ so that, by Fubini's theorem,

$$\begin{aligned} \langle u^\nu, \varphi \rangle &= \int_0^{+\infty} e^{-\nu t} \left(\int_{-\infty}^{+\infty} e^{i\omega t} \hat{\psi}(\omega) d\omega \right) dt \\ &= 2\pi \int_0^{+\infty} e^{-\nu t} \psi(-t) dt. \end{aligned}$$

Also the Fourier inversion theorem gives

$$\begin{aligned} \psi(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega t} \hat{\psi}(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega t} \int_{\mathbb{R}^d} \hat{v}(\omega, k) (2\pi)^{-(1+d)} \hat{\varphi}(-\omega, -k) dk d\omega \\ &= \frac{1}{(2\pi)^{d+2}} \int_{-\infty}^{+\infty} e^{-i\omega t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{1+d}} e^{-i(k \cdot x_1 - \omega t_1)} v(t_1, x_1) dt_1 dx_1 \\ &\quad \times \int_{\mathbb{R}^{1+d}} e^{+i(k \cdot x_2 - \omega t_2)} \varphi(t_2, x_2) dt_2 dx_2 dk d\omega \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} e^{-i\omega t} \int_{\mathbb{R}^{1+d}} e^{+i\omega t_1} v(t_1, x_1) dt_1 \int_{\mathbb{R}} e^{-i\omega t_2} \varphi(t_2, x_1) dt_2 dx_1 d\omega \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^{1+d}} v(t', x_1) \varphi(t' - t, x_1) dt' dx_1, \end{aligned}$$

and this yields

$$\langle u^\nu, \varphi \rangle = \int_0^{+\infty} \int_{\mathbb{R}^{1+d}} e^{-\nu t''} v(t', x) \varphi(t' + t'', x) dt' dx dt''.$$

By the change of variables $t'' = t - s$, $t' = s$, one has

$$\langle u^\nu, \varphi \rangle = \int_{\mathbb{R}^{1+d}} \int_{-\infty}^t e^{-\nu(t-s)} v(s, x) \varphi(t, x) ds dt dx,$$

which shows that the distribution u^ν is regular and equal to the C^∞ function

$$u^\nu(t, x) = \int_{-\infty}^t e^{-\nu(t-s)} v(s, x) ds,$$

as claimed. In addition, $e^{-\nu(t-s)} \leq 1$ for $s \in (-\infty, t]$, and with u defined in proposition 2.1,

$$\lim_{\nu \rightarrow 0^+} \langle u^\nu, \varphi \rangle = \int_{\mathbb{R}^{1+d}} \int_{-\infty}^t v(s, x) \varphi(t, x) ds dt dx = \int u(t, x) \varphi(t, x) dt dx,$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^{1+d})$, that is,

$$u^\nu \rightarrow u, \quad \text{in } \mathcal{S}'(\mathbb{R}^{1+d}),$$

and the limit is the causal solution of proposition 2.1. \square

APPENDIX C. CAUSAL SOLUTIONS OF LINEAR KINETIC EQUATIONS

Let Ω be a domain in \mathbb{R}^d and $\varphi \in C^\infty(\mathbb{R} \times \Omega)$. We denote $\varphi_t = \varphi(t, \cdot)$ and assume that

- (i) $\varphi_t : \Omega \rightarrow \Omega$ for every $t \in \mathbb{R}$,
- (ii) $\varphi_0 = \text{Id}$ is the identity map on Ω ,
- (iii) $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for every $t, s \in \mathbb{R}$.
- (iv) For any multi-index $\alpha \in \mathbb{N}^d$ there are constants $C, m \in \mathbb{R}$ such that

$$|\partial_x^\alpha \varphi_t(x)| \leq C(1 + t^2 + |\varphi_t(x)|^2)^m.$$

Remark 18. In condition (iv) the case $\alpha = 0$ is excluded because, for $\alpha = 0$, $|\varphi_t(x)| \leq (1 + t^2 + |\varphi_t(x)|^2)^{1/2}$ for any map φ , and the inequality in (iv) is trivially verified.

As a consequence of properties (i)-(iii), $\{\varphi_t : t \in \mathbb{R}\}$ is a one-parameter Abelian group of diffeomorphisms of Ω . Particularly, $\varphi_t^{-1} = \varphi_{-t}$. We can associate to φ_t the autonomous vector field $X \in C^\infty(\Omega, \mathbb{R}^d)$ defined by

$$(87) \quad X(x) = \left. \frac{d\varphi_t(x)}{dt} \right|_{t=0},$$

and, with $x = \varphi_t(x_0)$ for every $x_0 \in \Omega$, we have

$$X(\varphi_t(x_0)) = \left. \frac{d\varphi_s}{ds}(\varphi_t(x_0)) \right|_{s=0} = \left. \frac{d}{ds} \varphi_{t+s}(x_0) \right|_{s=0} = \left. \frac{d\varphi_t(x_0)}{dt} \right|_{t=0},$$

which shows that the orbit $x(t) = \varphi_t(x_0)$ of the group solves the Cauchy problem

$$\frac{dx}{dt} = X(x), \quad x(0) = x_0,$$

globally in time and for every initial point $x_0 \in \Omega$. While conditions (i)-(iii) are standard properties of flows of autonomous vector fields, condition (iv) is specific to the application considered here. An example of flow satisfying condition (iv) is

$$(88) \quad \varphi_t(x) = A(t)x,$$

where $A(t)$ is a $d \times d$ matrix satisfying suitable conditions.

Proposition C.1. *If the matrix-valued function $A \in C^\infty(\mathbb{R}, \mathbb{R}^{d \times d})$ is such that, $A(0) = I$, $A(t+s) = A(t)A(s)$ and, in the norm induced by the standard Euclidean norm in \mathbb{R}^d ,*

$$\|A(t)\| \leq C(1+t^2)^m,$$

for given constants $C, m > 0$, then the map (88) satisfies conditions (i)-(iv) above with $\Omega = \mathbb{R}^d$.

Proof. Condition (i) is true because of definition (88), while (ii) and (iii) follow directly from the assumptions. As for condition (iv), since the flow is linear in x the spatial derivative vanish for $|\alpha| \geq 2$. For $|\alpha| = 1$, condition (iv) is implied by the polynomial growth of A . \square

Remark 19. Both the characteristics flow of the kinetic equation considered in section 3 and 5 are of this form.

In this section we consider the linear advection equation

$$(89) \quad \partial_t f + X \cdot \nabla f = g, \quad \text{in } \mathbb{R} \times \Omega,$$

with given source $g \in \mathcal{S}(\mathbb{R}^{1+d})$. By construction the orbits of the group are the characteristics curves of (89) and exist globally in time.

We define the function

$$(90) \quad f(t, x) := \int_{-\infty}^t g(s, \varphi_{t-s}^{-1}(x)) ds, \quad (t, x) \in \mathbb{R} \times \Omega,$$

for which we prove the following.

Proposition C.2. *Let φ be a map satisfying properties (i)-(iv) above. Then, for any $g \in \mathcal{S}(\mathbb{R}^{1+d})$ the function f defined in (90) belongs to $C_b^\infty(\mathbb{R} \times \Omega)$ and is a classical solution of the linear advection equation (89).*

Proof. First we observe that, for every $t \in \mathbb{R}$ the function $s \mapsto (1+s^2)/(1+(t-s)^2)$ is in $C^\infty(\mathbb{R})$, strictly positive, tends to 1 for $s \rightarrow \pm\infty$, and for $t \neq 0$ has two critical points at $s = s_\pm = (t \pm (t^2 + 4)^{1/2})/2$ that correspond to a local minimum and a local maximum depending of the sign of t . The local maximum, in particular, is also the global maximum and thus

$$(1 + s^2) \leq C_t(1 + (t - s)^2),$$

uniformly in s . The constant C_t is the value of the function at the maximum which is $C_t = [4 + t^2 + |t|(4 + t^2)^{1/2}] / [4 + t^2 - |t|(4 + t^2)^{1/2}]$. The trivial case $t = 0$ is included with $C_t = 1$. The substitutions $s \rightarrow s/a$ and $t \rightarrow t/a$ yield $(a^2 + s^2) \leq C_{t/a}(a^2 + (t - s)^2)$, for all $a \neq 0$. Explicitly, $C_{t/a} = (\xi + |t|)/(\xi - |t|)$ where $\xi = \sqrt{4a^2 + t^2}$; this is a monotonically decreasing function of ξ in the interval $[\sqrt{4 + t^2}, +\infty)$ corresponding to $a \geq 1$, and the maximum is exactly C_t . Therefore, for every $a \geq 1$,

$$(91) \quad (a^2 + s^2) \leq C_t(a^2 + (t - s)^2),$$

uniformly in $s \in \mathbb{R}$ and the constant is independent of a .

We now apply this inequality to the function (90). After the change of variable $s' = t - s$, we have

$$f(t, x) = \int_0^{+\infty} g(t - s', \varphi_{s'}^{-1}(x)) ds'.$$

We shall show that all derivatives $\partial_t^n \partial_x^\alpha [g(t - s', \varphi_{s'}^{-1}(x))]$ are uniformly bounded by L^1 -functions of s . If this is the case, repeated use of the dominated convergence theorem gives $f \in C^\infty(\mathbb{R} \times \Omega)$ with all derivative bounded as claimed.

In the case $\alpha = 0$, for any real $k \geq 0$ and for any integer $n \geq 0$, we have

$$\begin{aligned} |\partial_t^n g(t - s, \varphi_s^{-1}(x))| &= \frac{(1 + (t - s)^2 + |\varphi_s^{-1}(x)|^2)^k |\partial_t^n g(t - s, \varphi_s^{-1}(x))|}{(1 + (t - s)^2 + |\varphi_s^{-1}(x)|^2)^k} \\ &\leq \frac{(1 + (t - s)^2 + |\varphi_s^{-1}(x)|^2)^k |\partial_t^n g(t - s, \varphi_s^{-1}(x))|}{(1 + (t - s)^2)^k} \end{aligned}$$

and since $g \in \mathcal{S}$ inequality (86) gives

$$|\partial_t^n g(t - s, \varphi_s^{-1}(x))| \leq \frac{C}{(1 + (t - s)^2)^k},$$

with constant C independent on s and $s \mapsto (1 + (t - s)^2)^{-k}$ is in L^1 for $k > 1/2$. With $n = 0$, this also shows that $f(t, x)$ is bounded.

Including spatial derivative requires condition (iv). In the case $|\alpha| = 1$,

$$\begin{aligned} \partial_t^n \partial_x^\alpha [g(t - s, \varphi_s^{-1}(x))] &= \partial_x^\alpha \varphi_s^{-1}(x) \cdot (\partial_t^n \nabla g)(t - s, \varphi_s^{-1}(x)) \\ &\leq d \cdot |\partial_x^\alpha \varphi_{-s}(x)| \cdot \max_j |(\partial_t^n \partial_{x_j} g)(t - s, \varphi_s^{-1}(x))|. \end{aligned}$$

Hypothesis (iv) and the estimate (91) give, for every $k > 0$,

$$\begin{aligned} |\partial_x^\alpha \varphi_{-s}(x)| &\leq C(1 + s^2 + |\varphi_s^{-1}(x)|^2)^m \\ &\leq \tilde{C}_t(1 + (t - s)^2 + |\varphi_s^{-1}(x)|^2)^m \\ &\leq \tilde{C}_t \frac{(1 + (t - s)^2 + |\varphi_s^{-1}(x)|^2)^{m+k}}{(1 + (t - s)^2)^k}. \end{aligned}$$

Proceeding as before, we choose $k > 1/2$ and obtain that the first-order derivatives are uniformly bounded by an L^1 -function.

The case of general α is complicated by the form of the chain rule. However, the same argument can be applied to the explicit formula for the multi-variate chain rule (Faà di Bruno formula) which is linear in the derivatives of g and polynomial in the derivatives of φ .

The fact that f is a classical solution of (89) can be checked by substitution. \square

If the flow is polynomially bounded, we can deduce that f , viewed as a distribution on Ω , has finite moments at all orders. This is a consequence of the following result.

Lemma C.3. *Let φ satisfy assumptions (i)-(iv) and $g \in \mathcal{S}(\mathbb{R}^{1+d})$. In addition let us assume there are $n \in \mathbb{N}_0$ and $C > 0$ such that $|\varphi_t(x)| \leq C(1 + x^2)^n$, uniformly in $\mathbb{R} \times \Omega$. Then, $p \cdot \partial_t^l \partial_x^\alpha f(t, \cdot) \in L^\infty(\Omega)$ for every polynomial $p = p(x)$, $l \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$.*

Proof. It is enough to show that $|x|^k \partial_t^l \partial_x^\alpha f(t, x)$ is in $L^\infty(\Omega)$. Since $x = \varphi_s(\varphi_s^{-1}(x))$,

$$|x|^k \leq C^k (1 + |\varphi_s^{-1}(x)|^2)^{kn} \leq C^k (1 + (t-s)^2 + |\varphi_s^{-1}(x)|^2)^{kn},$$

and

$$\begin{aligned} |x|^k |f(t, x)| &\leq C^k \int_0^{+\infty} (1 + (t-s)^2 + |\varphi_s^{-1}(x)|^2)^{kn} \partial_t^l \partial_x^\alpha [g(t-s, \varphi_s^{-1}(x))] ds \\ &\leq C_{k,l,m,\alpha} \int_0^{+\infty} \frac{ds}{(1 + (t-s)^2)^m}, \end{aligned}$$

where the derivatives of g are estimates as in proposition C.2. \square

The function (90) is referred to as the causal solution of equation (89).

APPENDIX D. THE HILBERT TRANSFORM AND ITS ACTION ON SYMBOLS

The Hilbert transform is defined by

$$\mathcal{H}(\phi)(x) = \frac{1}{\pi} \text{p.v.} \int \frac{\phi(y)}{x-y} dy = \frac{1}{\pi} (\text{p.v.} \frac{1}{x}) * \phi(x),$$

for a function $\phi \in C_0^\infty$. It has the following properties, which we state without proof.

Proposition D.1. *The Hilbert transform \mathcal{H} defined above is extended to functions in $\mathcal{S}(\mathbb{R})$ through the equality*

$$\mathcal{H}(\phi)(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{2u} [\phi(x-u) - \phi(x+u)] du.$$

Moreover, it extends to an isometry of Sobolev spaces $H^k(\mathbb{R})$ through the equality

$$\widehat{\mathcal{H}(u)} = -i \text{sgn}(\xi) \hat{u}, \quad \forall u \in H^k(\mathbb{R}),$$

for every non-negative integer k . Particularly if $u \in \mathcal{S}(\mathbb{R})$, then $\mathcal{H}(u) \in H^\infty(\mathbb{R})$.

With $\nu > 0$, $G \in \mathcal{S}(\mathbb{R})$, and $\phi \in \mathcal{S}(\mathbb{R}^2)$ let us consider the integrals

$$A_\nu(\omega, k) := \int_{\mathbb{R}} \frac{G(v)}{\omega - kv + i\nu} dv, \quad B_\nu(v, k) := \int_{\mathbb{R}} \frac{\phi(\omega, k)}{\omega - kv + i\nu} d\omega,$$

that involve the same denominator as in (28).

Lemma D.2. *Let $G \in \mathcal{S}(\mathbb{R})$, $\phi \in \mathcal{S}(\mathbb{R}^2)$, and $\nu > 0$.*

(i) *There are constants $C_{A,j}$, $j = 0, 1$, depending on G , such that for $k \neq 0$,*

$$\begin{aligned} |A_\nu(\omega, k)| &\leq C_{A,0} + C_{A,1}/|k|, \\ A_\nu(\omega, k) &\xrightarrow{\nu \rightarrow 0^+} \frac{1}{k} \left[\pi \mathcal{H}(G)(\omega/k) - i\pi G(\omega/k) \right]. \end{aligned}$$

(ii) *For any integer $m \geq 0$ there exists a constant C_B , depending on ϕ and m , such that*

$$\begin{aligned} |B_\nu(v, k)| &\leq C_B (1 + k^2)^{-m}, \\ B_\nu(v, k) &\xrightarrow{\nu \rightarrow 0^+} -\pi \mathcal{H}(\phi(\cdot, k))(kv) - i\pi \phi(kv, k). \end{aligned}$$

Proof. Let $G \in \mathcal{S}(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R}^2)$. For $k \neq 0$, define the two functions G_0 and G_1 by

$$\begin{aligned} G_0\left(\frac{\omega}{k}, u\right) &:= \frac{1}{2} \left[G\left(\frac{\omega}{k} - u\right) + G\left(\frac{\omega}{k} + u\right) \right], \\ uG_1\left(\frac{\omega}{k}, u\right) &:= \frac{1}{2} \left[G\left(\frac{\omega}{k} - u\right) - G\left(\frac{\omega}{k} + u\right) \right]. \end{aligned}$$

Similarly, for all $(\omega, k, v) \in \mathbb{R}^3$ we define

$$\begin{aligned} \phi_0(kv, \varpi, k) &:= \frac{1}{2} (\phi(kv + \varpi, k) + \phi(kv - \varpi, k)), \\ \varpi\phi_1(kv, \varpi, k) &:= \frac{1}{2} [\phi(kv + \varpi, k) - \phi(kv - \varpi, k)]. \end{aligned}$$

We observe that,

$$(92) \quad |G_0\left(\frac{\omega}{k}, u\right)| \leq \|G\|_0, \quad (1 + k^2)^m |\phi_0(kv, \varpi, k)| \leq \|\phi\|_{2m}.$$

The first inequality follows directly from the definition of G_0 , and we have

$$\begin{aligned} (1 + k^2)^m |\phi_0(kv, \varpi, k)| &\leq \frac{1}{2} [(1 + (kv + \varpi)^2 + k^2)^m |\phi(kv + \varpi, k)| \\ &\quad + (1 + (kv - \varpi)^2 + k^2)^m |\phi(kv - \varpi, k)|] \leq \|\phi\|_{2m}. \end{aligned}$$

Moreover,

$$|G_1\left(\frac{\omega}{k}, u\right)| \leq \|G\|_1, \quad (1 + k^2)^m |\phi_1(kv, \varpi, k)| \leq \|\phi\|_{1+2m},$$

and this can be proven by Taylor's formula and

$$\begin{aligned} |G_1\left(\frac{\omega}{k}, u\right)| &\leq \frac{1}{2} \int_{-1}^{+1} |G'\left(\frac{\omega}{k} + \lambda u\right)| d\lambda \leq \sup |G'(v)|, \\ (1 + k^2)^m |\phi_1(kv, \varpi, k)| &\leq \frac{1}{2} \int_{-1}^{+1} (1 + k^2)^m |\partial_\omega \phi(kv + \lambda \varpi, k)| d\lambda \\ &\leq \sup [(1 + (kv + \lambda \varpi)^2 + k^2)^m |\partial_\omega \phi(kv + \lambda \varpi, k)|]. \end{aligned}$$

When $|u| > 1$ we also have, for any $M \in \mathbb{N}_0$,

$$\begin{aligned} |G_1\left(\frac{\omega}{k}, u\right)| &\leq \frac{1}{2} \left[|G\left(\frac{\omega}{k} - u\right)| + |G\left(\frac{\omega}{k} + u\right)| \right] \\ &\leq \frac{1}{2} \left[\frac{1}{(1 + (\frac{\omega}{k} - u)^2)^M} + \frac{1}{(1 + (\frac{\omega}{k} + u)^2)^M} \right] \sup [(1 + v^2)^M |G(v)|], \end{aligned}$$

and, analogously, when $|\varpi| > 1$,

$$\begin{aligned} (1 + k^2)^m |\phi_1(kv, \varpi, k)| &\leq \frac{1}{2} \left[\frac{1}{(1 + (kv + \varpi)^2)^M} + \frac{1}{(1 + (kv - \varpi)^2)^M} \right] \\ &\quad \times \sup [(1 + \omega^2 + k^2)^{M+m} |\phi(\omega, k)|]. \end{aligned}$$

Therefore,

$$(93) \quad |G_1\left(\frac{\omega}{k}, u\right)| \leq G_1^*\left(\frac{\omega}{k}, u\right), \quad (1 + k^2)^m |\phi_1(kv, \varpi, k)| \leq \phi_1^*(kv, \varpi),$$

where

$$G_1^*\left(\frac{\omega}{k}, u\right) := \begin{cases} \|G\|_1, & |u| \leq 1, \\ \frac{1}{2} \left[\frac{1}{(1 + (\frac{\omega}{k} - u)^2)^M} + \frac{1}{(1 + (\frac{\omega}{k} + u)^2)^M} \right] \|G\|_{2M}, & |u| > 1. \end{cases}$$

and

$$\phi_1^*(kv, \varpi) := \begin{cases} \|\phi\|_{1+2m}, & |\varpi| \leq 1, \\ \frac{1}{2} \left[\frac{1}{(1+(kv-\varpi)^2)^M} + \frac{1}{(1+(kv+\varpi)^2)^M} \right] \|\phi\|_{2(M+m)}, & |\varpi| > 1. \end{cases}$$

For $M \geq 1$, one has $G_1^*(\frac{\omega}{k}, \cdot), \phi_1^*(kv, \cdot) \in L^1(\mathbb{R})$ with

$$(94) \quad \begin{aligned} \|G_1^*(\frac{\omega}{k}, \cdot)\|_{L^1} &\leq 2\|G\|_1 + \|G\|_{2M} \int_{\mathbb{R}} \frac{dt}{(1+t^2)^M} =: C_{M,G}, \\ \|\phi_1^*(kv, \cdot)\|_{L^1} &\leq 2\|\phi\|_{1+2m} + \|\phi\|_{2(M+m)} \int_{\mathbb{R}} \frac{dt}{(1+t^2)^M} =: C_{M,m,\phi}, \end{aligned}$$

where $C_{M,G}$ is a constant depending only on the Schwartz semi-norms of G and the integer M and $C_{M,m,\phi}$ depends only on M, m , and the Schwartz semi-norms of ϕ .

Then, one has the identities

$$\begin{aligned} A_\nu(\omega, k) &= \int_{\mathbb{R}} \frac{G_0(\frac{\omega}{k}, u) + uG_1(\frac{\omega}{k}, u)}{ku + i\nu} du, \\ B_\nu(v, k) &= \int_{\mathbb{R}} \frac{\phi_0(kv, \varpi, k) + \varpi\phi_1(kv, \varpi, k)}{\varpi + i\nu} d\varpi. \end{aligned}$$

As G_0, G_1 are even functions in u , and ϕ_0, ϕ_1 are even functions in ϖ , one deduces

$$\begin{aligned} kA_\nu(\omega, k) &= \int_{\mathbb{R}} \frac{k^2u^2G_1(\frac{\omega}{k}, u) - ik\nu G_0(\frac{\omega}{k}, u)}{k^2u^2 + \nu^2} du \\ &= \int_{\mathbb{R}} \frac{k^2u^2G_1(\frac{\omega}{k}, u)}{k^2u^2 + \nu^2} du - ik \int_{\mathbb{R}} \frac{G_0(\frac{\omega}{k}, \nu t)}{k^2t^2 + 1} dt, \\ B_\nu(v, k) &= \int_{\mathbb{R}} \frac{\varpi^2\phi_1(kv, \varpi, k) - i\nu\phi_0(kv, \varpi, k)}{\varpi^2 + \nu^2} d\varpi \\ &= \int_{\mathbb{R}} \frac{\varpi^2\phi_1(kv, \varpi, k)}{\varpi^2 + \nu^2} d\varpi - i \int_{\mathbb{R}} \frac{\phi_0(kv, \nu t, k)}{t^2 + 1} dt. \end{aligned}$$

We observe that $\frac{k^2u^2}{k^2u^2 + \nu^2}$ and $\frac{\varpi^2}{\varpi^2 + \nu^2}$ are uniformly bounded by 1. Moreover, $t \mapsto \frac{1}{t^2+1}$ is in $L^1(\mathbb{R})$, and, for $k \neq 0$, $t \mapsto \frac{1}{k^2t^2+1}$ belongs to $L^1(\mathbb{R})$, with the values $\int_{\mathbb{R}} \frac{dt}{t^2+1} = \pi$ and $\int_{\mathbb{R}} \frac{dt}{k^2t^2+1} = \frac{\pi}{k}$.

Then estimates (92), (93), and (94) give

$$\begin{aligned} |kA_\nu(\omega, k)| &\leq \|G_1^*(\frac{\omega}{k}, \cdot)\|_{L^1} + \pi|k|\|G\|_0 \leq C_{M,G} + \pi|k|\|G\|_0, \\ (1+k^2)^m |B_\nu(v, k)| &\leq \|\phi_1^*(kv, \cdot)\|_{L^1} + \pi\|\phi\|_{2m} \leq C_{M,m,\phi} + \pi\|\phi\|_{2m}. \end{aligned}$$

As for the limits of A_ν and B_ν as $\nu \rightarrow 0^+$, estimate (92) and (93) with $M \geq 1$ imply that the integrands are bounded by an L^1 function uniformly in ν . Then the dominated convergence theorem allows us to conclude that, as $\nu \rightarrow 0^+$,

$$\begin{aligned} kA_\nu(\omega, k) &\rightarrow \int_{\mathbb{R}} G_1(\frac{\omega}{k}, u) du - i\pi G_0(\frac{\omega}{k}, 0), \\ B_\nu(v, k) &\rightarrow \int_{\mathbb{R}} \phi_1(kv, \varpi, k) d\varpi - i\pi\phi_0(kv, 0, k). \end{aligned}$$

Upon accounting for the definitions of G_0, G_1 and ϕ_0, ϕ_1 that reads

$$\begin{aligned} kA_\nu(\omega, k) &\rightarrow \pi\mathcal{H}(G)(\omega/k) - i\pi G_0(\omega/k, 0), \\ B_\nu(v, k) &\rightarrow -\pi\mathcal{H}(\phi(\cdot, k))(kv) - i\pi\phi(kv, k), \end{aligned}$$

as claimed. \square

APPENDIX E. A USEFUL LINEAR ALGEBRA RESULT

In this appendix we establish a linear algebra result which constitutes a key step in the proof of the results in section 4.

Lemma E.1. *For any integer $\ell \geq 0$, the matrices*

$$A_\ell = \begin{pmatrix} 0 & i & 0 & 0 & \dots \\ -i\ell & 0 & 2i & 0 & \dots \\ 0 & -i(\ell-1) & 0 & 3i & \dots \\ 0 & 0 & -i(\ell-2) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{C}^{(\ell+1) \times (\ell+1)}$$

are diagonalizable with integer eigenvalues $\{(2s-\ell) : s = 0, 1, \dots, \ell\}$. For $z \in \mathbb{C} \setminus \mathbb{Z}$, $A_\ell - z$ is invertible and

$$|(A_\ell - z)^{-1}|_1 \leq (\ell+1)^2 2^\ell / \delta_z,$$

where the norm $|\cdot|_1$ is induced by the L^1 -norm on $\mathbb{C}^{\ell+1}$ and $\delta_z = \min\{|z - m| : m \in \mathbb{Z}\}$.

Proof. The map $i_1 : \mathbb{C}^{\ell+1} \rightarrow C^\infty(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{N})$, defined by

$$x \mapsto U(\phi) = \sum_{r=0}^{\ell} x_r (\cos \phi)^{\ell-r} (\sin \phi)^r,$$

is a linear embedding of $\mathbb{C}^{\ell+1}$ and becomes an isomorphism when restricted to its range. Injectivity in particular holds since, if $x \in \mathbb{C}^{\ell+1}$ is such that $U = i_1(x) = 0$, then for any $R \geq 0$

$$R^\ell U(\phi) = \sum_{r=0}^{\ell} x_r u_1^{\ell-r} u_2^r = 0,$$

uniformly for $(u_1, u_2) = (R \cos \phi, R \sin \phi) \in \mathbb{R}^2$. Since monomials $u_1^{\ell-r} u_2^r$ are linearly independent, we deduce $x = 0$ and thus i_1 is injective.

A second embedding is $i_2 : \mathbb{C}^{\ell+1} \rightarrow C^\infty(\mathbb{T})$ with

$$v \mapsto V(\phi) = \sum_{s=0}^{\ell} v_s e^{i(\ell-2s)\phi},$$

which is injective since the exponential are linearly independent.

We claim that i_1, i_2 have the same range i.e., $V_\ell := i_1(\mathbb{C}^{\ell+1}) = i_2(\mathbb{C}^{\ell+1})$. In fact, $i_1(\mathbb{C}^{\ell+1})$ is spanned by functions $f_r(\phi) = (\cos \phi)^{\ell-r} (\sin \phi)^r$ while $i_2(\mathbb{C}^{\ell+1})$ is

spanned by $g_s(\phi) = e^{i(\ell-2s)\phi}$. On the one hand, the binomial formula gives

$$\begin{aligned} f_r(\phi) &= \frac{(-i)^r}{2^\ell} (e^{i\phi} + e^{-i\phi})^{\ell-r} (e^{i\phi} - e^{-i\phi})^r \\ &= \frac{(-i)^r}{2^\ell} \sum_{m=0}^{\ell-r} \sum_{n=0}^r (-1)^n \binom{\ell-r}{m} \binom{r}{n} e^{i(\ell-2(m+n))\phi}, \end{aligned}$$

and thus

$$f_r(\phi) = \sum_{s=0}^{\ell} T_{sr} g_s(\phi), \quad T_{sr} = \frac{(-i)^r}{2^\ell} \sum_{(m,n) \in \Sigma(r,s)} (-1)^n \binom{\ell-r}{m} \binom{r}{n},$$

where the last sum is over the set of indices

$$\Sigma(r, s) = \{(m, n) : m = 0, \dots, \ell - r, n = 0, \dots, r, m + n = s\}.$$

On the other hand,

$$\begin{aligned} g_s(\phi) &= e^{i(\ell-2s)\phi} = (\cos \phi + i \sin \phi)^{\ell-s} (\cos \phi - i \sin \phi)^s \\ &= \sum_{m=0}^{\ell-s} \sum_{n=0}^s i^{m-n} \binom{\ell-s}{m} \binom{s}{n} (\cos \phi)^{\ell-(m+n)} (\sin \phi)^{m+n}, \end{aligned}$$

or

$$g_s(\phi) = \sum_{r=0}^{\ell} S_{rs} f_r(\phi), \quad S_{rs} = \sum_{(m,n) \in \Sigma(s,r)} i^{m-n} \binom{\ell-r}{m} \binom{r}{n}.$$

In summary, we have obtained that

$$f_r = \sum_{s=0}^{\ell} T_{sr} g_s, \quad g_s = \sum_{r=0}^{\ell} S_{rs} f_r,$$

hence for any ℓ , $\{f_r\}_{r=0}^{\ell}$ and $\{g_s\}_{s=0}^{\ell}$ span the same linear space V_ℓ . The matrices (T_{sr}) and (S_{sr}) correspond to the linear operators

$$T = i_2^{-1} \circ i_1 : \mathbb{C}^{\ell+1} \rightarrow \mathbb{C}^{\ell+1}, \quad S = i_1^{-1} \circ i_2 : \mathbb{C}^{\ell+1} \rightarrow \mathbb{C}^{\ell+1}.$$

At last, let us introduce the differential operator

$$B = i \frac{d}{d\phi} : C^\infty(\mathbb{T}) \rightarrow C^\infty(\mathbb{T}),$$

for which a direct calculation shows that

$$B \circ i_1(x) = \sum_{r=0}^{\ell} (A_\ell x)_r f_r(\phi).$$

This means that B can be restricted to operators from $V_\ell \rightarrow V_\ell$ and

$$(95) \quad A_\ell = i_1^{-1} \circ B \circ i_1.$$

On the other hand

$$(96) \quad B \circ i_2(v) = \sum_{s=0}^{\ell} (2s - \ell) v_s g_s(\phi),$$

that is, the operator $i_2^{-1} \circ B \circ i_2$ is diagonal with eigenvalues $(2s - \ell)$.

For equation (95) it follows that

$$A_\ell = i_1^{-1} \circ B \circ i_1 = S \circ (i_2^{-1} \circ B \circ i_2) \circ T,$$

which shows that A_ℓ is diagonalizable with eigenvalues $a = (2s - \ell)$ for $s = 0, \dots, \ell$.

Therefore if $z \in \mathbb{C} \setminus \mathbb{Z}$, then $A_\ell - z$ is invertible and

$$(A_\ell - z)^{-1} = S \circ (i_2^{-1} \circ (B - z)^{-1} \circ i_2) \circ T,$$

hence

$$(97) \quad |(A_\ell - z)^{-1}|_1 \leq |S|_1 \cdot |i_2^{-1} \circ (B - z)^{-1} \circ i_2|_1 \cdot |T|_1.$$

We claim that

$$(98) \quad |i_2^{-1} \circ (B - z)^{-1} \circ i_2|_1 \leq \delta_z^{-1}, \quad |T|_1 \leq (\ell + 1), \quad |S|_1 \leq (\ell + 1)2^\ell.$$

If this is true, then inequality (97) becomes $|(A_\ell - z)^{-1}|_1 \leq 2^\ell(\ell + 1)^2/\delta_z$, which is the claimed estimate. Therefore, in order to complete the proof it is sufficient to show that (98) holds.

From (96), one deduces the expression for the inverse,

$$(B - z)^{-1} \circ i_2(v) = \sum_{s=0}^{\ell} \frac{v_s}{2s - \ell - z} e^{i(\ell - 2s)\phi},$$

and thus, in the L^1 -norm,

$$|i_2^{-1} \circ (B - z)^{-1} \circ i_2(v)|_1 = \sum_{s=0}^{\ell} \frac{|v_s|}{|2s - \ell - z|}.$$

The definition of δ_z implies

$$|2s - \ell - z| \geq \delta_z = \min_{m \in \mathbb{Z}} |z - m|,$$

so that

$$(99) \quad |i_2^{-1} \circ (B - z)^{-1} \circ i_2(v)|_1 \leq \frac{1}{\delta_z} \sum_{s=0}^{\ell} |v_s| = \frac{|v|_1}{\delta_z},$$

or $|i_2^{-1} \circ (B - z)^{-1} \circ i_2|_1 \leq \delta_z^{-1}$. As for the linear operator T ,

$$|T_{sr}| \leq \frac{1}{2^\ell} \sum_{m=0}^{\ell-r} \sum_{n=0}^r \binom{\ell-r}{m} \binom{r}{n},$$

and for the last sum we can use the identity,

$$(a + b)^\ell = (a + b)^{\ell-r} (a + b)^r = \sum_{m=0}^{\ell-r} \sum_{n=0}^r \binom{\ell-r}{m} \binom{r}{n} a^{\ell-m-n} b^{m+n},$$

which for $a = b = 1$ and together with the previous inequality implies $|T_{sr}| \leq 1$; then

$$|Tx|_1 = \sum_{s=0}^{\ell} \left| \sum_{r=0}^{\ell} T_{sr} x_s \right| \leq \sum_{s=0}^{\ell} \sum_{r=0}^{\ell} |T_{sr}| |x_s| \leq (\ell + 1) |x|_1,$$

or $|T|_1 \leq (\ell + 1)$. Analogously for S ,

$$|S_{rs}| \leq \sum_{m=0}^{\ell-s} \sum_{n=0}^s \binom{\ell-s}{m} \binom{s}{n} = 2^\ell,$$

and $|S|_1 \leq (\ell + 1)2^\ell$. □

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REFERENCES

1. M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, ninth dover printing, tenth gpo printing ed., Dover, New York, 1964.
2. A.A. Arsen'ev, *Global existence of a weak solution of Vlasov's system of equations*, USSR Computational Mathematics and Mathematical Physics **15** (1975), no. 1, 131 – 143.
3. K. Asano and S. Ukai, *On the Vlasov-Poisson limit of the Vlasov-Maxwell equation*, Patterns and Waves (Takaaki Nishida, Masayasu Mimura, and Hiroshi Fujii, eds.), Studies in Mathematics and Its Applications, vol. 18, Elsevier, 1986, pp. 369 – 383.
4. A. A. Balakin, M. A. Balakina, G. V. Permitin, and A. I. Smirnov, *Quasi-optical description of wave beams in smoothly inhomogeneous anisotropic media*, Journal of Physics D: Applied Physics **40** (2007), no. 14, 4285.
5. C. Bardos and P. Degond, *Global existence for the Vlasov-Poisson equation in 3 space variables with small initial data*, Annales de l'Institut Henri Poincaré (C) Non Linear Analysis **2** (1985), no. 2, 101 – 118.
6. S. T. Beliaev and G. I. Budker, *The relativistic kinetic equation*, Sov. Phys. Dokl. **1** (1956), 218.
7. I. B. Bernstein, *Geometric optics in space- and time-varying plasmas*, Physics of Fluids **18** (1975), no. 3, 320–324.
8. P. L. Bhatnagar, E. P. Gross, and M. Krook, *A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems*, Phys. Rev. **94** (1954), 511–525.
9. M. Bornatici, R. Cano, O. De Barbieri, and F. Engelmann, *Electron cyclotron emission and absorption in fusion plasmas*, Nuclear Fusion **23** (1983), no. 9, 1153.
10. M. Bornatici and O. Maj, *Geometrical optics response tensors and the transport of the wave energy density*, Plasma Physics and Controlled Fusion **45** (2003), no. 8, 1511.
11. M. Bostan, *Asymptotic behavior of weak solutions for the relativistic Vlasov-Maxwell equations with large light speed*, Journal of Differential Equations **227** (2006), no. 2, 444 – 498.
12. F. Bouchut, F. Golse, and C. Pallard, *Classical solutions and the Glassey-Strauss theorem for the 3d Vlasov-Maxwell system.*, Archive for Rational Mechanics & Analysis **170** (2003), no. 1, 1 – 15.
13. B. J. Braams and C. F. Karney, *Conductivity of a relativistic plasma*, Phys. Fluids B **1** (1989), 1355–1368.
14. M. Brambilla, *Kinetic theory of plasma waves: Homogeneous plasmas*, Oxford University Press, Oxford, 1998.
15. F. Cacciafesta, P. D'Ancona, and Lucá R., *A limiting absorption principle for the Helmholtz equation with variable coefficients*, J. Spectr. Theory **8** (2018), no. 4, 1349–1392.
16. C. Cercignani, *The Boltzmann equation and its applications*, Springer, 1975.
17. C. Cheverry and A. Fontaine, *Dispersion relation in hot magnetized plasmas*, Journal of Mathematical Analysis and Applications **466** (2017), no. 2, 1238–1280.
18. ———, *Dispersion relations in cold magnetized plasmas*, Kinetic & Related Models **10** (2017), 373.
19. P. Degond and H. Neunzert, *Local existence of solutions of the Vlasov-Maxwell equations and convergence to the Vlasov-Poisson equations for infinite light velocity*, Mathematical Methods in the Applied Sciences **8** (1986), no. 1, 533–558.
20. B. Després, *Scattering structure and Landau damping for linearized Vlasov equations with inhomogeneous Boltzmannian states*, Annales Henri Poincaré **20** (2019), no. 8, 2767–2818.
21. J. R. Di Perna and P.-L. Lions, *Global weak solutions of kinetic equations*, Rend. Semin. Mat., Torino **46** (1988), no. 3, 259–288.
22. ———, *Global weak solutions of Vlasov-Maxwell systems*, Comm. Pure Appl. Math **42** (1989), 729–757.
23. ———, *On the Cauchy problem for the Boltzmann equation: Global existence and weak stability*, Annals of Mathematics **130** (1989), 321–366.

24. D. Eidus, *The limiting absorption and amplitude principles for the diffraction problem with two unbounded media*, Comm. Math. Phys. **107** (1986), no. 1, 29–38.
25. R. T. Glassey, *The Cauchy problem in kinetic theory*, SIAM, 1996.
26. R. T. Glassey and J. Schaeffer, *Global existence for the relativistic Vlasov-Maxwell system with nearly neutral initial data*, Comm. Math. Phys. **119** (1988), 353–384.
27. R. T. Glassey and W. A. Strauss, *Singularity formation in a collisionless plasma could occur only at high velocities*, Archive for Rational Mechanics and Analysis **92** (1986), no. 1, 59–90.
28. H. Gzyl, *Multidimensional extension of faa di bruno’s formula*, Journal of Mathematical Analysis and Applications **116** (1986), no. 2, 450–455.
29. L. Hörmander, *The analysis of linear partial differential operators i: Distribution theory and fourier analysis*, second edition ed., Springer Berlin Heidelberg, 1990.
30. E. Horst, *On the classical solutions of the initial value problem for the unmodified non-linear vlasov equation i general theory*, Math. Meth. Appl. Sci. **3** (1981), no. 1, 229–248.
31. ———, *On the classical solutions of the initial value problem for the unmodified non-linear vlasov equation ii special cases*, Math. Meth. Appl. Sci. **4** (1982), no. 1, 19–32.
32. ———, *Global solutions of the relativistic Vlasov-Maxwell system of plasma physics*, Dissertationes Mathematicae (1990), 1–63.
33. E. Horst, R. Hunze, and H. Neunzert, *Weak solutions of the initial value problem for the unmodified non-linear vlasov equation*, Mathematical Methods in the Applied Sciences **6** (1984), no. 1, 262–279.
34. F. John, *Partial differential equations.*, Springer Berlin Heidelberg, 1980.
35. C. F. Kennel and F. Engelmann, *Velocity space diffusion from weak plasma turbulence in a magnetic field*, Physics of Fluids (1958-1988) **9** (1966), no. 12, 2377–2388.
36. S. Klainerman and G. Staffilani, *A new approach to study the vlasov-maxwell system*, Communications on Pure and Applied Analysis **1** (2002), no. 1, 103–125.
37. Yu. A. Kravtsov, *The geometric optics approximation in the general case of inhomogeneous and nonstationary media with frequency and spatial dispersion*, Sov. Phys. - JETP **28** (1969), no. 4, 769.
38. J. Larsson, *Current responses of first and second order in a collisionless plasma. I. Stationary plasma*, Journal of Mathematical Physics **23** (1982), no. 1, 176–182.
39. P. D. Lax and R. S. Phillips, *Scattering theory*, Academic Press, 1967.
40. I. Lerche, R. Schlickeiser, and R. C. Tautz, *Comment on "a new derivation of the plasma susceptibility tensor for a hot magnetized plasma without infinite sums of products of Bessel functions" [phys. plasmas 14, 092103 (2007)]*, Physics of Plasmas **15** (2008), no. 2, 024701.
41. P.-L. Lions, *On kinetic equations*, Proc. of the International Congress of Mathematicians (1991), 1173–1185.
42. S. W. McDonald, *Phase-space representations of wave equations with applications to the eikonal approximation for short-wavelength waves*, Physics Reports **158** (1988), 337 – 416.
43. S. W. McDonald, *Wave kinetic equation in a fluctuating medium*, Phys. Rev. A **43** (1991), 4484–4499.
44. S. W. McDonald and A. N. Kaufman, *Weyl representation for electromagnetic waves: The wave kinetic equation*, Phys. Rev. A **32** (1985), 1708–1713.
45. G. Nakamura and J.-N. Wang, *The limiting absorption principle for the two-dimensional inhomogeneous anisotropic elasticity system*, Transactions of the American Mathematical Society **358** (2006), no. 1, 147–165.
46. P. Omnes, *Dielectric conductivity of a bounded plasma and its rate of convergence towards its infinite-geometry value*, Journal of Plasma Physics **69** (2003), no. 5, 449–463.
47. K. Pfaffelmoser, *Global classical solutions of the vlasov-poisson system in three dimensions for general initial data*, J. Differential Equations **95** (1992), 281–303.
48. R. Prater, D. Farina, Yu. Gribov, R.W. Harvey, A.K. Ram, Y.-R. Lin-Liu, E. Poli, A.P. Smirnov, F. Volpe, E. Westerhof, A. Zvonkov, and the ITPA Steady State Operation Topical Group, *Benchmarking of codes for electron cyclotron heating and electron cyclotron current drive under iter conditions*, Nuclear Fusion **48** (2008), no. 3, 035006.
49. H. Qin, C. K. Phillips, and R. C. Davidson, *A new derivation of the plasma susceptibility tensor for a hot magnetized plasma without infinite sums of products of Bessel functions*, Physics of Plasmas **14** (2007), no. 9, 092103.

50. ———, *Response to "comment on 'a new derivation of the plasma susceptibility tensor for a hot magnetized plasma without infinite sums of products of bessel functions'" [phys. plasmas 15, 024701 (2008)]*, *Physics of Plasmas* **15** (2008), no. 2, 024702.
51. G. Rein, *Global weak solutions to the relativistic Vlasov-Maxwell system revisited*, *Commun. Math. Sci.* **2** (2004), no. 2, 145–158.
52. E. Sanchez-Palencia, *Non-homogeneous media and vibration theory*, Springer, 1980.
53. J. Schaeffer, *The classical limit of the relativistic Vlasov-Maxwell system*, *Comm. Math. Phys.* **104** (1986), no. 3, 403–421.
54. T. H. Stix, *Waves in plasmas*, Springer-Verlag, New York, 1992.
55. E. Westerhof, *Wave propagation through an electron cyclotron resonance layer*, *Plasma Physics and Controlled Fusion* **39** (1997), no. 6, 1015.
56. S. Wollman, *An existence and uniqueness theorem for the Vlasov-Maxwell system*, *Communications on Pure and Applied Mathematics* **37** (1984), no. 4, 457–462.
57. ———, *Local existence and uniqueness theory of the Vlasov-Maxwell system*, *Journal of Mathematical Analysis and Applications* **127** (1987), no. 1, 103 – 121.

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