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Hydrodynamic and hydrostatic limit for a contact process with random slowdowns in finite volume and with slow boundaries

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Abstract

Abstract We consider an interacting particle system which models the sterile insect technique. It is the superposition of a generalized contact process with exchanges of particles on a finite cylinder with open boundaries (see *Kuoch et al., 2017*). We show that when the system is in contact with reservoirs at different slow-down rates, the hydrodynamic limit is a set of coupled non linear reaction-diffusion equations with mixed boundary conditions. We also prove the hydrostatic limit when the macroscopic equations exhibit a unique attractor.

Keywords: *Hydrodynamic limit; Hydrostatic limit; Random reservoirs; System of reaction-diffusion equations; Generalized contact process; Stirring process; Mixed boundary conditions;*

MSC: *60K35; 82C22*

1 Introduction

In this paper, we consider the interacting particle system which was introduced in [17] to model the sterile insect technique. This technique was developed, among others, by E. Knipling (see [16]) to eradicate New World screw worms in the 1950's, a serious pest for warm blooded animals. The method is still used today, for instance in France, to protect crops from the very invasive Mediterranean flies, and it is also being tested to fight mosquitoes which transmit dengue in countries like Panama or Brazil. The sterile insect technique works as follows: male insects are sterilized in captivity using gamma rays. They are then released in the wild population, where females mate only once, giving rise to no off springs if they mate with a sterile male. When enough sterile individuals are released, the wild population eventually becomes extinct. From a mathematical perspective, the sterile insect technique has mainly been modeled in a deterministic way through the study of partial differential equations (see [1]).

The sterile insect technique was studied from a probabilistic perspective in [17] and [18] using interacting particle systems. In [17], a phase transition result is proved at the microscopic level. Recently, another probabilistic model was studied in [14], also at the microscopic level. In [18], the study is carried at the macroscopic

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level (hydrodynamic limit) in finite volume with reservoirs, in order to account for the migration/immigration mechanism.

Here, we aim at studying the hydrodynamic limit and hydrostatic limit of that interacting particle system under the effect of slow reservoirs. The slow-down mechanism models the fact that beyond the boundary through which insects arrive into the system or leave it, there are very few insects (the exterior of the system might be a territory which is much less favorable to the development of these insects).

In the perspective of interacting particle systems, the sterile insect technique is modeled as follows: insects evolve on a d -dimensional finite set $B_N = \{-N, \dots, N\} \times \mathbb{T}_N^{d-1}$, where $N \geq 1$, and the evolution of the population is described by a continuous time Markov process $(\eta_t^N)_{t \geq 0}$ with state space E^{B_N} where E is a countable set. The quantity of interest here is not the number of insects per site but the types of insects present at a given site. Precisely, $E = \{0, 1, 2, 3\}$ and for x in B_N ,

$$\eta(x) = \begin{cases} 0 & \text{if there are no insects in } x, \\ 1 & \text{if there are only wild insects in } x, \\ 2 & \text{if there are only sterile insects in } x, \\ 3 & \text{if there is a combination of wild and sterile insects in } x. \end{cases}$$

The dynamics of the Markov process is the superposition of three Markovian jump processes:

- (i) A process which models the fact that insects move in an isotropic way within B_N and which is parameterized by a diffusivity constant $D > 0$. Precisely, for a configuration η and x, y two sites in B_N , the states of sites x and y in η are exchanged at rate D .
- (ii) A birth and death dynamics which models births of individuals due to the mating of a wild females with wild or sterile insects, as well as deaths of individuals. This is parameterized by a release rate $r > 0$ and growth rates $\lambda_1, \lambda_2 > 0$. Sterile males are injected on a site at rate r independently of everything else. The rate at which wild males give birth (to wild males) on neighbouring sites is λ_1 at sites in state 1, and λ_2 at sites in state 3. Sterile males do not give birth. We take $\lambda_2 < \lambda_1$ to reflect the fact that fertility is reduced at sites in state 3. Deaths for each type of male insects occur independently and at rate 1.
- (iii) A boundary dynamics which models the slow migration/immigration mechanism. This is parameterized by a function $\hat{b} = (b_1, b_2, b_3) : \{-1, 1\} \times \mathbb{T}^{d-1} \rightarrow [0, 1]^3$ and two constants θ_ℓ and θ_r in \mathbb{R}^+ . For $i \in \{0, 1, 2, 3\}$, a particle of type $i \in \{0, 1, 2, 3\}$ is injected in the system through $x \in \{-N\} \times \mathbb{T}_N^{d-1}$, resp. $x \in \{N\} \times \mathbb{T}_N^{d-1}$ at rate $N^{-\theta_\ell} b_i(x/N)$, resp. $N^{-\theta_r} b_i(x/N)$ and is expelled from the system through x at rate $N^{-\theta_\ell} (1 - b_i(x/N))$, resp. $N^{-\theta_r} (1 - b_i(x/N))$, with $b_0 = 1 - b_1 - b_2 - b_3$.

The birth and death mechanism is referred to as a contact process with random slowdowns (or CPRS). Indeed, without the presence of sterile insects, it would be a basic contact process (as defined for instance in [20]) with parameter λ_1 , and the presence of sterile insects can be interpreted as a random decrease of the fertility rate due to the presence of sites containing sterile and wild individuals. In [17], the microscopic study of the birth and death dynamics alone leads to the following phase transition result: for certain values of λ_1 and λ_2 , when r is large enough, the healthy population almost surely becomes extinct and survives otherwise. In [18], the hydrodynamic limit of the superposition of the three dynamics above, where the first and the third one are accelerated in the diffusive scaling N^2 , and where $\theta_\ell = \theta_r = 0$, is proven to be a system of non linear reaction-diffusion equations with Dirichlet boundary conditions.

In this paper, we prove the finite volume hydrodynamic limit of this interacting particle system for any values of $\theta_\ell, \theta_r \geq 0$. The hydrodynamic equation obtained has mixed boundary conditions which depend on the values of θ_ℓ , resp. θ_r . Precisely, for $\theta_\ell \in [0, 1)$, resp. $\theta_r \in [0, 1)$, we get a Dirichlet type boundary condition at the left hand side, resp. right hand side of the system. For $\theta_\ell = 1$, resp. $\theta_r = 1$, we get a Robin type boundary condition at the left hand side, resp. right hand side of the system. For $\theta_\ell > 1$, resp. $\theta_r > 1$, we get a Neumann type boundary condition at the left hand side, resp. right hand side of the system.

We then prove the finite volume hydrostatic limit of the interacting particle system for a specific class of parameters regarding the dynamics. Within that class of parameters, the sequence of invariant measures of

the interacting particle system is associated to a profile which is the stationary solution of the hydrodynamic equation with corresponding mixed boundary conditions.

Our paper is, up to our knowledge, the first one regarding the effect of mixed reservoirs in and out of equilibrium (hydrodynamic and hydrostatic limit) for a multi species process in finite volume. The effect of reservoirs on a one dimensional conservative system has been widely studied in finite volume (see for instance [5], [8]). Much is now known both at the microscopic and macroscopic level. Recently, the effect of slow reservoirs has aroused much interest for the symmetric simple exclusion process in one dimension (see for instance [2], [10], [11], [12] and references therein). In [9], authors proved a hydrostatic principle for a boundary driven gradient symmetric exclusion process using the fact that the stationary profile is a global attractor for the hydrodynamic equation. This method inspired our proof for the hydrostatic limit. However, the coupled equations obtained for the hydrodynamic limit, and the fact that we work in any dimension make the analysis more subtle.

The proof of the hydrodynamic limit for each of these regime is established in Section 3 via the Entropy Method. Among other things, as we work in arbitrary dimension, some care must be taken to define and characterize the solution of the hydrodynamic limit at the boundary, through the use of the Trace Operator (see subsection 3.4). The proof of the hydrostatic limit, established in Section 4 and inspired by [9] relies on the use of a change of coordinates for the coupled equations. Under this change of coordinates (inspired by some simulations see Appendix B), a comparison principle holds. It allows us to find a unique attractor when some conditions on the parameters are satisfied. Outside that class of parameters, although uniqueness of the invariant measure holds, we do not even know whether there is uniqueness of the stationary solution of the hydrodynamic equation and simulations show (see Appendix B) that for Neumann type boundary conditions there are several stationary profiles. However, we believe that a more general hydrostatic principle in the spirit of the one proved in [19] is valid.

2 Notations and results

2.1 The microscopic model

The dynamics of our interacting particle system is given by three generators, one for the diffusive dynamics, one for the contact dynamics and one for the boundary dynamics. In order to explicit each one of those generators, let us give a few notations. Let $N \in \mathbb{N}$. Denote $B_N = \{-N, \dots, N\} \times \mathbb{T}_N^{d-1}$ the bulk and $\Gamma_N = \{-N, N\} \times \mathbb{T}_N^{d-1}$, resp. $\Gamma_N^+ = \{N\} \times \mathbb{T}_N^{d-1}$, resp. $\Gamma_N^- = \{-N\} \times \mathbb{T}_N^{d-1}$ the boundary, resp. left hand side boundary, resp. right hand side boundary of the bulk. Denote $B = (-1, 1) \times \mathbb{T}^{d-1}$ the continuous counter part of the bulk, $\bar{B} = [-1, 1] \times \mathbb{T}^{d-1}$ its closure, $\Gamma = \{-1, 1\} \times \mathbb{T}^{d-1}$, $\Gamma^- = \{-1\} \times \mathbb{T}^{d-1}$ and $\Gamma^+ = \{1\} \times \mathbb{T}^{d-1}$.

The microscopic state space is denoted $\Omega_N := \{0, 1, 2, 3\}^{B_N}$ and its elements, also called configurations, are denoted η . Therefore, for $x \in B_N$, $\eta(x) \in \{0, 1, 2, 3\}$. To describe the dynamics of our model, we will use the correspondence introduced in [18] between the state space Ω_N and $\widehat{\Sigma}_N := (\{0, 1\} \times \{0, 1\})^{B_N}$ where the correspondence between an element $(\xi, \omega) \in \widehat{\Sigma}_N$ and $\eta \in \Omega_N$ is given as follows: for $x \in B_N$,

$$\begin{aligned}
 \eta(x) = 0 &\iff (1 - \xi(x))(1 - \omega(x)) = 1, \\
 \eta(x) = 1 &\iff \xi(x)(1 - \omega(x)) = 1, \\
 \eta(x) = 2 &\iff (1 - \xi(x))\omega(x) = 1, \\
 \eta(x) = 3 &\iff \xi(x)\omega(x) = 1.
 \end{aligned}
 \tag{2.1}$$

In other words, $(\xi(x), \omega(x)) = (0, 0)$ if x is in state 0, $(1, 0)$ if it is in state 1, $(0, 1)$ if it is in state 2 and $(1, 1)$ if it is in state 3. Also, in order to describe the evolution of the density of sites in state 1, resp. 2, resp 3,

resp 0, we define for x in B_N and a configuration $\eta \in \Omega_N$ with associated configuration $(\xi, \omega) \in \widehat{\Sigma}_N$,

$$\begin{cases} \eta_1(x) := \mathbb{1}_{\eta(x)=1} = \xi(x)(1 - \omega(x)), \\ \eta_2(x) := \mathbb{1}_{\eta(x)=2} = (1 - \xi(x))\omega(x), \\ \eta_3(x) := \mathbb{1}_{\eta(x)=3} = \xi(x)\omega(x), \\ \eta_0(x) := \mathbb{1}_{\eta(x)=0} = (1 - \xi(x))(1 - \omega(x)). \end{cases} \quad (2.2)$$

Finally, we also express the correspondence (2.1) by the following application from $\widehat{\Sigma}_N$ to Ω_N :

$$\eta = \eta(\xi, \omega), \quad \text{where, for any } x \in B_N, \quad \eta(x) = 2\omega(x) + \xi(x). \quad (2.3)$$

- **Generator for the diffusion mechanism:** it corresponds to the usual stirring mechanism where each site has an exponential clock with rate D and independent from all the other clocks. When the clock rings, a neighbouring site is chosen uniformly at random and the states of both sites are exchanged. The action of the generator on functions $f : \widehat{\Sigma}_N \rightarrow \mathbb{R}$ is therefore given by:

$$\mathcal{L}_N f(\xi, \omega) := \sum_{k=1}^d \sum_{(x, x+e_k) \in B_N} D \left(f(\xi^{x, x+e_k}, \omega^{x, x+e_k}) - f(\xi, \omega) \right) \quad (2.4)$$

where (e_1, \dots, e_d) is the canonical basis of \mathbb{Z}^d and for $\zeta \in \{0, 1\}^{B_N}$ and $x, y \in B_N$, $\zeta^{x, y}$ is the configuration obtained from ζ by exchanging the occupation variables $\zeta(x)$ and $\zeta(y)$, i.e.,

$$\zeta^{x, y}(z) = \begin{cases} \zeta(x) & \text{if } z = y, \\ \zeta(y) & \text{if } z = x, \\ \zeta(z) & \text{otherwise.} \end{cases}$$

- **Generator for the contact process in the bulk:** following the description of the CPRS in the introduction, the birth and death mechanism in the bulk has the following rates: for $\eta \in \Omega_N$ and $x \in B_N$,

$$\begin{aligned} 0 \rightarrow 1 & \text{ at rate } \lambda_1 n_1(x, \eta) + \lambda_2 n_3(x, \eta), & 1 \rightarrow 0 & \text{ at rate } 1, & 0 \rightarrow 2 & \text{ at rate } r, \\ 2 \rightarrow 0 & \text{ at rate } 1, & 1 \rightarrow 3 & \text{ at rate } r, & 3 \rightarrow 1 & \text{ at rate } 1, \\ 2 \rightarrow 3 & \text{ at rate } \lambda_1 n_1(x, \eta) + \lambda_2 n_3(x, \eta), & 3 \rightarrow 2 & \text{ at rate } 1. \end{aligned} \quad (2.5)$$

Therefore, using the correspondence (2.1), the generator $\mathbb{L}_N = \mathbb{L}_{N, \lambda_1, \lambda_2, r}$ of the CPRS acts as follows on functions $f : \widehat{\Sigma}_N \rightarrow \mathbb{R}$:

$$\mathbb{L}_N f(\xi, \omega) = \sum_{x \in B_N} \mathbb{L}_{B_N}^x f(\xi, \omega), \quad (2.6)$$

where for $x \in B_N$,

$$\begin{aligned} \mathbb{L}_{B_N}^x f(\xi, \omega) & := \left(r(1 - \omega(x)) + \omega(x) \right) \left[f(\xi, \sigma^x \omega) - f(\xi, \omega) \right] \\ & + \left(\beta_{B_N}(x, \xi, \omega)(1 - \xi(x)) + \xi(x) \right) \left[f(\sigma^x \xi, \omega) - f(\xi, \omega) \right], \end{aligned} \quad (2.7)$$

$$\beta_{B_N}(x, \eta) := \lambda_1 \sum_{\substack{y \sim x \\ y \in B_N}} \eta_1(y) + \lambda_2 \sum_{\substack{y \sim x \\ y \in B_N}} \eta_3(y)$$

where $x \sim y$ means that x and y are neighbouring sites in B_N , and where for $\zeta \in \{0, 1\}^{B_N}$, $\sigma^x \zeta$ is the configuration obtained from ζ by flipping the configuration at x , i.e.

$$\sigma^x \zeta(z) = \begin{cases} 1 - \zeta(x) & \text{if } z = x, \\ \zeta(z) & \text{otherwise.} \end{cases}$$

- **Generator for the boundary dynamics:** the generator of the dynamics at the boundary is parametrized by $\widehat{\theta} = (\theta_\ell, \theta_r)$ with $\theta_\ell, \theta_r \geq 0$ and a positive function $\widehat{b} = (b_1, b_2, b_3) : \Gamma \rightarrow \mathbb{R}_+^3$ satisfying the following conditions: there exists a neighbourhood V of \overline{B} in $\mathbb{R} \times \mathbb{T}^{d-1}$ and a smooth function $\widehat{g} = (g_1, g_2, g_3) : V \rightarrow (0, 1)^3$ in $\mathcal{C}^2(V, \mathbb{R})$ such that

$$0 < c^* < \min_{1 \leq i \leq 3} |g_i| \leq \max_{1 \leq i \leq 3} |g_i| \leq C^* < 1 \quad (2.8)$$

for two positive constants c^* and C^* , and such that the restriction of \widehat{g} to Γ is equal to \widehat{b} . The dynamics at the boundary can then be described as follows: a site $x \in \Gamma_N^-$, resp. $x \in \Gamma_N^+$ goes from state $i \in \{0, 1, 2, 3\}$ to state $j \in \{0, 1, 2, 3\} \setminus \{i\}$ at rate $N^{-\theta_\ell} b_j(x/N)$, resp. $N^{-\theta_r} b_j(x/N)$. In order to express the generator of the boundary dynamics, we make use of $\eta_i = \eta_i(\xi, \omega)$ for $i \in \{0, 1, 2, 3\}$ which is the configuration in $\{0, 1\}^{B_N}$ obtained from $(\xi, \omega) \in \widehat{\Sigma}_N$ according to (2.2). For $f : \widehat{\Sigma}_N \rightarrow \mathbb{R}$, the boundary generator acts on f as follows:

$$L_{\widehat{b}, \widehat{\theta}, N} f(\xi, \omega) := \sum_{x \in \Gamma_N} L_{\widehat{b}, \widehat{\theta}, N}^x f(\xi, \omega),$$

where

$$\begin{aligned} L_{\widehat{b}, \widehat{\theta}, N}^x f(\xi, \omega) &:= c_x(\widehat{b}(x/N), \xi, \sigma^x \omega) \left[f(\xi, \sigma^x \omega) - f(\xi, \omega) \right] + c_x(\widehat{b}(x/N), \sigma^x \xi, \omega) \left[f(\sigma^x \xi, \omega) - f(\xi, \omega) \right] \\ &\quad + c_x(\widehat{b}(x/N), \sigma^x \xi, \sigma^x \omega) \left[f(\sigma^x \xi, \sigma^x \omega) - f(\xi, \omega) \right], \end{aligned}$$

and with

$$c_x(\widehat{b}(x/N), \xi, \omega) := \begin{cases} N^{-\theta_\ell} \sum_{i=0}^3 b_i(x/N) \eta_i(x) & \text{if } x \in \Gamma_N^-, \\ N^{-\theta_r} \sum_{i=0}^3 b_i(x/N) \eta_i(x) & \text{if } x \in \Gamma_N^+ \end{cases}$$

and $b_0(x/N) := 1 - \sum_{i=1}^3 b_i(x/N)$. Note that the following simpler expression holds for $L_{\widehat{b}, \widehat{\theta}, N}$: for $f : \widehat{\Sigma}_N \rightarrow \mathbb{R}$ and $(\xi, \omega) \in \widehat{\Sigma}_N$,

$$\begin{aligned} L_{\widehat{b}, \widehat{\theta}, N} f(\xi, \omega) &= N^{-\theta_\ell} \sum_{i=0}^3 \sum_{x \in \Gamma_N^-} b_i(x/N) \left(f(\sigma_{i,x}(\xi, \omega)) - f(\xi, \omega) \right) \\ &\quad + N^{-\theta_r} \sum_{i=0}^3 \sum_{x \in \Gamma_N^+} b_i(x/N) \left(f(\sigma_{i,x}(\xi, \omega)) - f(\xi, \omega) \right) \end{aligned} \quad (2.9)$$

with $\sigma_{i,x}(\xi, \omega) := \sigma_{i,x} \eta(\xi, \omega)$, the configuration in $\widehat{\Sigma}_N$ associated to $\sigma_{i,x} \eta$, where

$$\sigma_{i,x} \eta(z) := \begin{cases} i & \text{if } z = x, \\ \eta(\xi, \omega)(z) & \text{otherwise} \end{cases}$$

with $\eta(\xi, \omega)$ as defined in (2.3).

Fix a time horizon $T > 0$ and denote $\{(\xi_t^N, \omega_t^N), t \in [0, T]\}$ the Markov process associated to the generator

$$L_N := N^2 \mathcal{L}_N + N^2 L_{\widehat{b}, \widehat{\theta}, N} + \mathbb{L}_N. \quad (2.10)$$

Let $D_{\widehat{\Sigma}_N}([0, T])$ be the path space of càdlàg trajectories with values in $\widehat{\Sigma}_N$. Given a measure μ_N on $\widehat{\Sigma}_N$, denote by \mathbb{P}_{μ_N} the probability measure on $D_{\widehat{\Sigma}_N}([0, T])$ induced by μ_N and $(\xi_t, \omega_t)_{t \geq 0}$ and denote \mathbb{E}_{μ_N} the expectation with respect to \mathbb{P}_{μ_N} .

Invariant measures for the diffusive and boundary dynamics:

Consider $\hat{\alpha} = (\alpha_1, \alpha_2, \alpha_3) : \overline{B} \rightarrow (0, 1)^3$ a smooth function satisfying the following conditions:

$$\exists c^*, C^* > 0, \quad 0 < c^* < \min_{1 \leq i \leq 3} |\alpha_i| \leq \max_{1 \leq i \leq 3} |\alpha_i| \leq C^* < 1 \quad (2.11)$$

and

$$\forall x \in \Gamma, \quad \hat{\alpha}(x) = \hat{b}(x). \quad (2.12)$$

Denote $\nu_{\hat{\alpha}}^N$ the Bernoulli product measure on B_N with parameter $\hat{\alpha}$: for $(\xi, \omega) \in \widehat{\Sigma}_N$,

$$\nu_{\hat{\alpha}}^N(\xi, \omega) := \frac{1}{Z_{\hat{\alpha}, N}} \exp \left(\sum_{i=1}^3 \sum_{x \in B_N} \left(\log \frac{\alpha_i(x/N)}{\alpha_0(x/N)} \right) \eta_i(x) \right),$$

where $Z_{\hat{\alpha}, N}$ is the normalizing constant

$$Z_{\hat{\alpha}, N} = \prod_{x \in B_N} \left(1 + \sum_{i=1}^3 \exp(\alpha_i(x/N)) \right)$$

and with $\alpha_0 = 1 - \alpha_1 - \alpha_2 - \alpha_3$. Note that $\nu_{\hat{\alpha}}^N$ is such that for every $1 \leq i \leq 3$ and $x \in B_N$,

$$\mathbb{E}_{\nu_{\hat{\alpha}}^N}[\eta_i(x)] = \alpha_i(x/N).$$

The following statements are easy to verify:

- Consider $\hat{\alpha}$ a constant profile. Then $\nu_{\hat{\alpha}}^N$ is an invariant measure for the exchange dynamics so for any $f : \widehat{\Sigma}_N \rightarrow \mathbb{R}$,

$$\int_{\widehat{\Sigma}_N} \mathcal{L}_N f(\xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) = 0. \quad (2.13)$$

- Consider $\hat{\alpha}$ a smooth profile satisfying (2.11) and (2.12). Then $\nu_{\hat{\alpha}}^N$ is an invariant measure for the boundary dynamics so for any $f : \widehat{\Sigma}_N \rightarrow \mathbb{R}$,

$$\int_{\widehat{\Sigma}_N} L_{\hat{b}, \hat{\theta}, N} f(\xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) = 0. \quad (2.14)$$

For any $\hat{\theta} \in (\mathbb{R}^+)^3$, at fixed N , the dynamics defined by (2.10) is irreducible and the state space is finite. Therefore, there is a unique invariant measure that in the sequel we denote $\mu_N^{ss}(\hat{\theta})$.

Useful (in)equalities: For any $A, B > 0$,

$$A(B - A) = -\frac{1}{2}(B - A)^2 + \frac{1}{2}(B^2 - A^2). \quad (2.15)$$

For any a, b, A and $N \in \mathbb{N}$,

$$2ab \leq \frac{N}{A}a^2 + \frac{A}{N}b^2. \quad (2.16)$$

2.2 The macroscopic equations

Let us first introduce a few notations. We will write functions with values in \mathbb{R} with normal letters (for instance G) and the ones with values in \mathbb{R}^3 with letters with a hat (for instance \hat{G}). For $n, m \in \mathbb{N}$, denote $\mathcal{C}^{n, m}([0, T] \times \overline{B})$ the space of functions that are n times differentiable in time and m times differentiable in space, $\mathcal{C}_0^{n, m}$, resp. $\mathcal{C}_{0, -}^{n, m}$, resp. $\mathcal{C}_{0, +}^{n, m}$ the ones in $\mathcal{C}^{n, m}([0, T] \times \overline{B})$ which are zero on Γ , resp. Γ^- , resp. Γ^+ . Denote $\mathcal{C}_k^\infty(B)$ the space of smooth functions with compact support in B , $\mathcal{C}^m(\overline{B})$ the space of functions that

are m times differentiable in space and $\mathcal{C}(\overline{B})$ when $m = 0$. For $\widehat{\theta} = (\theta_\ell, \theta_r)$ in $(\mathbb{R}^+)^2$, we will use the following notations to denote these functional spaces:

$$\mathcal{C}_{\widehat{\theta}} := \begin{cases} \mathcal{C}_0^{1,2} & \text{if } \widehat{\theta} \in [0, 1]^2, \\ \mathcal{C}_{0,-}^{1,2} & \text{if } \theta_\ell \in [0, 1), \theta_r \geq 1, \\ \mathcal{C}_{0,+}^{1,2} & \text{if } \theta_r \in [0, 1), \theta_\ell \geq 1, \\ \mathcal{C}^{1,2} & \text{if } \theta_r, \theta_\ell \geq 1. \end{cases} \quad (2.17)$$

Let $\langle \cdot, \cdot \rangle$ be the $L^2(\overline{B})$ inner product and $\langle \cdot, \cdot \rangle_\mu$ the inner product with respect to a measure μ . For $\widehat{f} = (f_1, f_2, f_3)$ and $\widehat{g} = (g_1, g_2, g_3)$ in $(L^2(\overline{B}))^3$, $\langle \widehat{f}, \widehat{g} \rangle = \sum_{i=1}^3 \langle f_i, g_i \rangle$. Introduce $\mathcal{H}^1(B)$ the set of functions in $L^2(\overline{B})$ such that for any $1 \leq k \leq d$, there exists an element $\partial_{e_k} g$ in $L^2(B)$ such that for any φ in $\mathcal{C}_k^\infty(B)$,

$$\langle \partial_{e_k} \varphi, g \rangle = - \langle \varphi, \partial_{e_k} g \rangle.$$

The $\mathcal{H}^1(B)$ norm is then defined as follows:

$$\|g\|_{\mathcal{H}^1(B)} = \left(\|g\|_{L^2(B)}^2 + \sum_{k=1}^d \|\partial_{e_k} g\|_{L^2(B)}^2 \right)^{1/2}.$$

Denote $L^2([0, T], \mathcal{H}^1(B))$ the space of functions $f : [0, T] \rightarrow \mathcal{H}^1(B)$ such that

$$\int_0^T \|f(t, \cdot)\|_{\mathcal{H}^1(B)}^2 dt < \infty.$$

In order to define the value of an element G in $\mathcal{H}^1(B)$ at the boundary, we need to introduce the notion of trace of functions on such Sobolev spaces. The trace operator in the Sobolev space $\mathcal{H}^1(B)$ can be defined as a bounded linear operator, $\text{Tr} : \mathcal{H}^1(B) \rightarrow L^2(\Gamma)$ such that Tr extends the classical trace, that is $\text{Tr}(G) = G|_\Gamma$, for any $G \in \mathcal{H}^1(B) \cap \mathcal{C}(\overline{B})$. We refer to [6, Part II Section 5] for a detailed survey of the trace operator. In the sequel, for $s, u \in \mathbb{R}^+ \times \Gamma$ and for any $f \in L^2((0, T); \mathcal{H}^1(B))$, $f(s, u)$ stands for $\text{Tr}(f(s, \cdot))(u)$.

To lighten notations, for a function \widehat{G} depending on time and space we will often write \widehat{G}_s instead of $\widehat{G}(s, \cdot)$. Finally, for $\widehat{\theta} \in (\mathbb{R}^+)^2$, introduce the following linear functional on $L^2([0, T], \mathcal{H}^1(B))$ parametrized by a test function \widehat{G} in $\mathcal{C}_{\widehat{\theta}}$: for $t \in [0, T]$,

$$\begin{aligned} I_{\widehat{G}}(\widehat{\rho})(t) &:= \langle \widehat{\rho}_t, \widehat{G}_t \rangle - \langle \widehat{\rho}_0, \widehat{G}_0 \rangle - \int_0^t \langle \widehat{\rho}_s, \partial_s \widehat{G}_s \rangle ds \\ &\quad - D \int_0^t \langle \widehat{\rho}_s, \Delta \widehat{G}_s \rangle ds - \int_0^t \langle \widehat{F}(\widehat{\rho}_s), \widehat{G}_s \rangle ds \end{aligned} \quad (2.18)$$

where $\widehat{F} = (F_1(\widehat{\rho}), F_2(\widehat{\rho}), F_3(\widehat{\rho})) : [0, 1]^3 \rightarrow \mathbb{R}^3$ is defined by

$$\begin{cases} F_1(\rho_1, \rho_2, \rho_3) = 2d(\lambda_1 \rho_1 + \lambda_2 \rho_3) \rho_0 + \rho_3 - (r+1) \rho_1 \\ F_2(\rho_1, \rho_2, \rho_3) = r \rho_0 + \rho_3 - 2d(\lambda_1 \rho_1 + \lambda_2 \rho_3) \rho_2 - \rho_2 \\ F_3(\rho_1, \rho_2, \rho_3) = 2d(\lambda_1 \rho_1 + \lambda_2 \rho_3) \rho_2 + r \rho_1 - 2 \rho_3, \end{cases} \quad (2.19)$$

with $\rho_0 = 1 - \rho_1 - \rho_2 - \rho_3$.

The hydrodynamic equation is a reaction diffusion coupled equation with mixed boundary conditions depending on $\widehat{\theta}$. If θ_ℓ , resp. θ_r in $[0, 1)$, the boundary conditions are of Dirichlet type on Γ^- , resp. Γ^+ . If $\theta_\ell = 1$, resp. $\theta_r = 1$, they are of Robin type on Γ^- , resp. Γ^+ . If $\theta_\ell > 1$, resp. $\theta_r > 1$, they are of Neumann type on Γ^- , resp. Γ^+ . We will focus on the cases where $\theta_\ell \in [0, 1), \theta_r = 1$ resp. $\theta_\ell > 1, \theta_r = 1$ corresponding to a Dirichlet boundary condition on Γ^- and a Robin boundary condition on Γ^+ , resp. a Neumann boundary condition on Γ^- and a Robin boundary condition on Γ^+ . All the other cases can easily be adapted (see the Table 1).

Definition 1. Let $\hat{\gamma} : B \rightarrow \mathbb{R}^3$ be a continuous function.

- Hydrodynamic equation for $\theta_\ell \in [0, 1)$ and $\theta_r = 1$. We say that a bounded function $\hat{\rho} = (\rho_1, \rho_2, \rho_3) : [0, T] \times B \rightarrow \mathbb{R}^3$ is a weak solution of the Dirichlet + Robin mixed boundary problem

$$\begin{cases} \partial_t \hat{\rho} = D\Delta \hat{\rho} + \hat{F}(\hat{\rho}) \text{ in } B \times (0, T), \\ \hat{\rho}(0, \cdot) = \hat{\gamma} \text{ in } B, \\ \hat{\rho}(t, \cdot)|_{\Gamma^-} = \hat{b} \text{ for } 0 < t \leq T, \\ \partial_{e_1} \hat{\rho}(t, \cdot)|_{\Gamma^+} = \frac{1}{D}(\hat{b} - \hat{\rho})|_{\Gamma^+} \text{ for } 0 < t \leq T \end{cases} \quad (2.20)$$

if, for any $1 \leq i \leq 3$,

$$\rho_i \in L^2((0, T), \mathcal{H}^1(B)), \quad (2.21)$$

for any function $\hat{G} \in \mathcal{C}_{\hat{\rho}}$, for any $t \in [0, T]$,

$$\begin{aligned} I_{\hat{G}}(\hat{\rho})(t) + D \sum_{i=1}^3 \int_0^t \int_{\Gamma^-} b_i(r) (\partial_{e_1} G_{i,s})(r) n_1(r) \cdot dS(r) ds \\ + D \sum_{i=1}^3 \int_0^t \int_{\Gamma^+} \rho_i(s, r) (\partial_{e_1} G_{i,s})(r) n_1(r) \cdot dS(r) ds - \sum_{i=1}^3 \int_0^t \int_{\Gamma^+} G_i(r) (b_i(r) - \rho_i(s, r)) n_1(r) \cdot dS(r) ds = 0, \end{aligned} \quad (2.22)$$

where $n_1(r)$ is the outward unit normal vector to the boundary surface Γ and $dS(r)$ is an element of surface on Γ . And,

$$\hat{\rho}(0, \cdot) = \hat{\gamma}(\cdot) \text{ almost surely.} \quad (2.23)$$

- Hydrodynamic equation for $\theta_\ell > 1$ and $\theta_r = 1$. We say that a bounded function $\hat{\rho} = (\rho_1, \rho_2, \rho_3) : [0, T] \times B \rightarrow \mathbb{R}^3$ is a weak solution of the Neumann + Robin mixed boundary problem

$$\begin{cases} \partial_t \hat{\rho} = D\Delta \hat{\rho} + \hat{F}(\hat{\rho}) \text{ in } B \times (0, T), \\ \hat{\rho}(0, \cdot) = \hat{\gamma} \text{ in } B, \\ \partial_{e_1} \hat{\rho}(t, \cdot)|_{\Gamma^-} = 0 \text{ for } 0 < t \leq T \\ \partial_{e_1} \hat{\rho}(t, \cdot)|_{\Gamma^+} = \frac{1}{D}(\hat{b} - \hat{\rho})|_{\Gamma^+} \text{ for } 0 < t \leq T \end{cases} \quad (2.24)$$

if $\hat{\rho}$ satisfies conditions (2.21) and (2.23) as well as the following: for any $\hat{G} \in \mathcal{C}_{\hat{\rho}}$, for any $t \in [0, T]$,

$$\begin{aligned} I_{\hat{G}}(\hat{\rho})(t) + D \sum_{i=1}^3 \int_0^t \int_{\Gamma^-} \rho_i(s, r) (\partial_{e_1} G_{i,s})(r) n_1(r) \cdot dS(r) ds \\ + D \sum_{i=1}^3 \int_0^t \int_{\Gamma^+} \rho_i(s, r) (\partial_{e_1} G_{i,s})(r) n_1(r) \cdot dS(r) ds - \int_{\Gamma^+} G_i(r) (b_i(r) - \rho_i(s, r)) n_1(r) \cdot dS(r) ds = 0. \end{aligned} \quad (2.25)$$

Remark 1. In (2.22), the integral over Γ^- corresponds to the Dirichlet boundary condition. In (2.25) the integral over Γ^- comes from an integration by part of the terms involved in the bulk. Both in (2.22) and (2.25) the first integral over Γ^+ comes from an integration by part of the terms involved in the bulk and the second integral over Γ^+ corresponds to the Robin boundary condition.

Definition 2. Stationary solution of the hydrodynamic equation.

(θ_ℓ, θ_r)	$\theta_r \in [0, 1)$	$\theta_r = 1$	$\theta_r > 1$
$\theta_\ell \in [0, 1)$	(D ; D)	(D ; R)	(D ; N)
$\theta_\ell = 1$	(R ; D)	(R ; R)	(R ; N)
$\theta_\ell > 1$	(N ; D)	(N ; R)	(N ; N)

Figure 1: Mixed boundary conditions depending on the values of θ_ℓ and θ_r . The letters D, resp. R, resp. N denote a Dirichlet, resp. Robin, resp. Neumann boundary condition. For instance (D ; N) denotes a left hand side Dirichlet boundary condition and a right hand side Neumann boundary condition.

- A function $\bar{\rho} = (\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3)$ in $(\mathcal{H}^1(B))^3$ is said to be a stationary solution of (2.20) if for every function $\hat{G} = (G_1, G_2, G_3) \in C_0^2(B)^3$, for all $1 \leq i \leq 3$,

$$\begin{aligned}
D \langle \rho_i, \Delta G_i \rangle + \langle F_i(\hat{\rho}), G_i \rangle &= D \int_{\Gamma^-} b_i(r) (\partial_{e_1} G_i)(r) n_1(r) . dS(r) \\
&+ D \int_{\Gamma^+} \rho_i(r) (\partial_{e_1} G_i)(r) n_1(r) . dS(r) - \int_{\Gamma^+} G_i(r) (b_i(r) - \rho_i(r)) n_1(r) . dS(r).
\end{aligned} \tag{2.26}$$

- A function $\bar{\rho} = (\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3)$ in $(\mathcal{H}^1(B))^3$ is said to be a stationary solution of (2.24) if for every function $\hat{G} = (G_1, G_2, G_3) \in C_0^2(B)^3$, for all $1 \leq i \leq 3$,

$$\begin{aligned}
D \langle \rho_i, \Delta G_i \rangle + \langle F_i(\hat{\rho}), G_i \rangle &= D \int_{\Gamma^-} \rho_i(r) (\partial_{e_1} G_i)(r) n_1(r) . dS(r) \\
&+ D \int_{\Gamma^+} \rho_i(r) (\partial_{e_1} G_i)(r) n_1(r) . dS(r) - \int_{\Gamma^+} G_i(r) (b_i(r) - \rho_i(r)) n_1(r) . dS(r).
\end{aligned} \tag{2.27}$$

2.3 Hydrodynamic and hydrostatic results

Let us state the main results proved in this paper. The first one (Theorem 1) establishes the hydrodynamic limit of the dynamics defined above and the second one (Theorem 2) establishes its hydrostatic limit. Before stating Theorem 1, let us first define the empirical measure $(\hat{\pi}^N(\xi_t, \omega_t))_{t \geq 0} = (\hat{\pi}_t^N)_{t \geq 0}$ associated to $(\xi_t, \omega_t)_{t \geq 0}$. Recall how in (2.2), we built $\eta_i \in \{0, 1\}^{B_N}$ from $(\xi, \omega) \in \hat{\Sigma}_N$ for $0 \leq i \leq 3$. For any $t \geq 0$,

$$\begin{aligned}
\hat{\pi}^N(\xi_t, \omega_t) &:= \left(\frac{1}{N^d} \sum_{x \in B_N} \eta_{1,t}(x) \delta_{x/N}, \frac{1}{N^d} \sum_{x \in B_N} \eta_{2,t}(x) \delta_{x/N}, \frac{1}{N^d} \sum_{x \in B_N} \eta_{3,t}(x) \delta_{x/N} \right) \\
&=: (\pi_{1,t}^N(\xi_t, \omega_t), \pi_{2,t}^N(\xi_t, \omega_t), \pi_{3,t}^N(\xi_t, \omega_t))
\end{aligned}$$

where $\delta_{x/N}$ is the point mass at x/N . For \hat{G} in $\mathcal{C}^{1,2}([0, T] \times B)$ and $t \geq 0$, write

$$\langle \hat{\pi}_t^N, \hat{G}_t \rangle := \sum_{i=1}^3 \langle \pi_{i,t}^N, G_i(t, \cdot) \rangle = \sum_{i=1}^3 \frac{1}{N^d} \sum_{x \in B_N} \eta_{i,t}(x) G_i(t, \frac{x}{N}).$$

The empirical measure is therefore the triplet of empirical measures associated to the density of sites in state 1, resp. 2, resp. 3. Denote \mathcal{M} the set of positive measures on B with total mass bounded by $\pm 2^d$. The process $(\widehat{\pi}_t^N)_{t \geq 0}$, is a Markov process with state space \mathcal{M}^3 and its trajectories are in $D([0, T], \mathcal{M}^3)$, the path space of càdlàg time trajectories with values in \mathcal{M}^3 . We endow the path space with the Skorohod topology (we refer to [3] for a detailed survey on the Skorohod topology). For $\widehat{\theta} \in (\mathbb{R}^+)^2$ and μ_N a measure on $\widehat{\Sigma}_N$, denote $Q_N^{\widehat{\theta}} = \mathbb{P}_{\mu_N}(\widehat{\pi}^N)^{-1}$ the law of the process $(\widehat{\pi}^N(\xi_t, \omega_t))_{t \geq 0}$ when $(\xi_0, \omega_0) \sim \mu_N$ and where $(\xi_t, \omega_t)_{t \geq 0}$ evolves according to the dynamics given by (2.10), with parameter $\widehat{\theta}$ for the boundary reservoirs. The hydrodynamic result states as follows:

Theorem 1. (*Hydrodynamic limit*). *For any sequence of initial probability measure $(\mu_N)_{N \geq 1}$ on $\widehat{\Sigma}_N$, the sequence of probability measures $(Q_N^{\widehat{\theta}})_{N \geq 1}$ is weakly relatively compact and all its converging subsequences converge to some limit $Q^{\widehat{\theta},*}$ that is concentrated on the set of weak solutions of hydrodynamic equation that are in $L^2(0, T; \mathcal{H}^1(B))$. Furthermore, if there is an initial continuous profile $\widehat{\gamma} : B \rightarrow [0, 1]^3$ such that for any $\delta > 0$ and any $\widehat{G} \in C_k^\infty(B)$,*

$$\limsup_{N \rightarrow \infty} \mathbb{P}_{\mu_N} \left[\left| \langle \widehat{\pi}_N, \widehat{G} \rangle - \langle \widehat{\gamma}, \widehat{G} \rangle \right| > \delta \right] = 0,$$

then, $(Q_N^{\widehat{\theta}})_{N \geq 1}$ converges to the Dirac mass $Q^{\widehat{\theta}}$ concentrated on the unique weak solution $\widehat{\rho}$ of the boundary value problem associated to $\widehat{\theta}$ and with initial condition $\widehat{\gamma}$. Therefore, for any $t \in [0, T]$, $\delta > 0$ and any function $\widehat{G} \in C_c^{1,2}([0, T] \times B)$,

$$\limsup_{N \rightarrow \infty} \mathbb{P}_{\mu_N} \left[\left| \langle \widehat{\pi}_t^N, \widehat{G}_t \rangle - \langle \widehat{\rho}_t, \widehat{G}_t \rangle \right| > \delta \right] = 0.$$

We prove Theorem 1 in Section 3.

For the hydrostatic limit, we need to introduce the following sets of conditions:

$$(H_1) : \begin{cases} D \geq 1 \\ r + 1 > 2d(\lambda_1 - \lambda_2) \\ 1 > 2d\lambda_2 \end{cases}$$

$$(H_2) : \begin{cases} D\delta_1 + r + 2 > 2d(\lambda_1 - \lambda_2) \\ D\delta_1 + 1 > 2d\lambda_2 \end{cases}$$

$$(H_3) : \begin{cases} r + 2 > 2d(\lambda_1 - \lambda_2) \\ 1 > 2d\lambda_2 \end{cases}$$

where δ_1 is the smallest eigenvalue of the Laplacian with Dirichlet boundary conditions (see (3.70)). Recall that $\mu_N^{ss}(\widehat{\theta})$ denotes the sequence of unique invariant measures for the irreducible dynamics defined by (2.10). The hydrostatic result states as follows.

Theorem 2. (*Hydrostatic limit*). *Suppose that conditions (H_1) hold. There exists a unique stationary solution of (2.20) that we denote $\bar{\rho}^{D,R}$, and a unique stationary solution of (2.24) that we denote $\bar{\rho}^{N,R}$. Furthermore, the following statements hold.*

- Consider $\widehat{\theta} = (\theta_\ell, \theta_r)$ with $\theta_\ell \in [0, 1)$ and $\theta_r = 1$. For any continuous function $\widehat{G} : B \rightarrow [0, 1]^3$,

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N^{ss}(\widehat{\theta})} \left(\left| \sum_{i=1}^3 \frac{1}{N^d} \sum_{x \in B_N} \eta_i(x) G_i(x/N) - \sum_{i=1}^3 \int_B G_i(u) \bar{\rho}_i^{D,R}(u) du \right| \right) = 0. \quad (2.28)$$

In other words, the sequence $(\mu_N^{ss}(\widehat{\theta}))_{N \geq 1}$ is associated to the unique stationary profile $\bar{\rho}^{D,R}$.

- Consider $\widehat{\theta} = (\theta_\ell, \theta_r)$ with $\theta_\ell > 1$ and $\theta_r = 1$. For any continuous function $\widehat{G} : B \rightarrow [0, 1]^3$,

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N^{ss}(\widehat{\theta})} \left(\left| \sum_{i=1}^3 \frac{1}{N^d} \sum_{x \in B_N} \eta_i(x) G_i(x/N) - \sum_{i=1}^3 \int_{\overline{B}} G_i(u) \overline{\rho}_i^{N,R}(u) du \right| \right) = 0. \quad (2.29)$$

In other words, the sequence $(\mu_N^{ss}(\widehat{\theta}))_{N \geq 1}$ is associated to the unique stationary profile $\overline{\rho}^{N,R}$.

Remark 2. For all the other mixed boundary regimes corresponding to other values of $\widehat{\theta}$, the hydrostatic principle states in the same way, replacing $\overline{\rho}_i^{D,R}$ or $\overline{\rho}_i^{N,R}$ by the stationary solution of the associated hydrodynamic equation. In the cases where only Dirichlet and Robin boundary conditions are involved, one can slightly weaken the conditions (H_1) by using conditions (H_2) and (H_3) instead. Precisely: in the $(D ; D)$, $(D ; R)$, $(R ; D)$ regimes, the hydrostatic principle holds under conditions (H_2) and in the $(N ; N)$ regime, it holds under conditions (H_3) .

The proof of Theorem 2 is done in Section 4. It essentially relies on an intermediate result stated in Theorem 6 regarding the convergence of solutions of the hydrodynamic equation towards the unique stationary state. This result is non standard as it involves a system of coupled equations and we prove it in the second section of Section 4.

3 Proof of the hydrodynamic limit

As said before, we focus on the cases where $\theta_\ell \in [0, 1], \theta_r = 1$ and $\theta_\ell > 1, \theta_r = 1$. We follow the entropy method introduced by Guo, Papanicolaou and Varadhan in [13] to prove the hydrodynamic limit. First, we prove tightness of the sequence of measures $(Q_N^{\widehat{\theta}})_{N \geq 1}$. Then, we show that any limit point of $(Q_N^{\widehat{\theta}})_{N \geq 1}$ is a Dirac mass concentrated on a weak solution of (2.20) if $\theta_\ell \in [0, 1], \theta_r = 1$, or (2.24) if $\theta_\ell > 1, \theta_r = 1$. Finally, we prove uniqueness of the solution of the hydrodynamic equations at fixed initial data.

3.1 The martingale property and tightness

By Dynkin's formula (see [15, Appendix A.1]), for $1 \leq i \leq 3$, $t \in [0, T]$ and $\widehat{G} \in \mathcal{C}_c^{1,2}([0, T] \times \overline{B})$,

$$\begin{aligned} M_{i,t}^N(\widehat{G}) := & \langle \pi_{i,t}^N, G_{i,t} \rangle - \langle \pi_{i,0}^N, G_{i,0} \rangle - \int_0^t \langle \pi_{i,s}^N, \partial_s G_{i,s} \rangle ds \\ & - N^2 \int_0^t \mathcal{L}_N \langle \pi_{i,s}^N, G_{i,s} \rangle ds - \int_0^t \mathbb{L}_N \langle \pi_{i,s}^N, G_{i,s} \rangle ds - N^2 \int_0^t L_{\widehat{b}, \widehat{\theta}, N} \langle \pi_{i,s}^N, G_{i,s} \rangle ds \end{aligned} \quad (3.1)$$

is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(\eta_s, s \leq t)$ and with quadratic variation given by:

$$\begin{aligned} & N^2 \int_0^t \mathcal{L}_N \langle \pi_{i,s}^N, G_{i,s} \rangle^2 ds + \int_0^t \mathbb{L}_N \langle \pi_{i,s}^N, G_{i,s} \rangle^2 ds + N^2 \int_0^t L_{\widehat{b}, \widehat{\theta}, N} \langle \pi_{i,s}^N, G_{i,s} \rangle^2 ds \\ & - 2N^2 \int_0^t \langle \pi_{i,s}^N, G_{i,s} \rangle \mathcal{L}_N \langle \pi_{i,s}^N, G_{i,s} \rangle ds - 2 \int_0^t \langle \pi_{i,s}^N, G_{i,s} \rangle \mathbb{L}_N \langle \pi_{i,s}^N, G_{i,s} \rangle ds \\ & - 2N^2 \int_0^t \langle \pi_{i,s}^N, G_{i,s} \rangle L_{\widehat{b}, \widehat{\theta}, N} \langle \pi_{i,s}^N, G_{i,s} \rangle ds. \end{aligned} \quad (3.2)$$

We then have that

$$\widehat{M}_t^N(\widehat{G}) := \sum_{i=1}^3 M_{i,t}^N(\widehat{G})$$

is also a martingale whose quadratic variation is known.

In order to develop the integral terms in (3.1), introduce the discrete second derivative in the direction e_k (for $1 \leq k \leq d$) in the bulk, the discrete laplacian, and the discrete gradient in the direction e_1 at the boundary: for $x \in B_N \setminus \Gamma_N$,

$$(\partial_{e_k}^N)^2 G(x/N) := \sum_{k=1}^d N^2 \left(G\left(\frac{x+e_k}{N}\right) + G\left(\frac{x-e_k}{N}\right) - 2G\left(\frac{x}{N}\right) \right),$$

$$\Delta_N G(x/N) := \sum_{k=1}^d (\partial_{e_k}^N)^2 G(x/N), \quad (\partial_{e_1}^N)^- H(x/N) := N \left(H\left(\frac{x}{N}\right) - H\left(\frac{x-e_1}{N}\right) \right)$$

and

$$(\partial_{e_1}^N)^+ H(x/N) := N \left(H\left(\frac{x+e_1}{N}\right) - H\left(\frac{x}{N}\right) \right).$$

Computations yield

$$\begin{aligned} M_{i,t}^N(\widehat{G}) &= \langle \pi_{i,t}^N, G_{i,t} \rangle - \langle \pi_{i,0}^N, G_{i,0} \rangle - \int_0^t \langle \pi_{i,s}^N, \partial_s G_{i,s} \rangle ds \\ &- \int_0^t \frac{D}{N^d} \sum_{x \in B_N \setminus \Gamma_N} \Delta_N G_{i,s}(x/N) \eta_{i,s}(x) ds - \int_0^t \frac{D}{N^d} \sum_{x \in \Gamma_N} \sum_{k=2}^d (\partial_{e_k}^N)^2 G_{i,s}(x/N) \eta_{i,s}(x) ds \\ &- \int_0^t \left[\frac{D}{N^{d-1}} \sum_{x \in \Gamma_N^+} (\partial_{e_1}^N)^- G_{i,s}(x/N) \eta_{i,s}(x) - \frac{D}{N^{d-1}} \sum_{x \in \Gamma_N^-} (\partial_{e_1}^N)^+ G_{i,s}(x/N) \eta_{i,s}(x) \right] ds \\ &- \int_0^t \frac{1}{N^d} \sum_{x \in B_N} G_{i,s}(x/N) \tau_x f_i(\eta_s) ds \\ &+ \frac{N^2}{N^{d+\theta_\ell}} \int_0^t \sum_{x \in \Gamma_N^-} G_{i,s}(x/N) (\eta_{i,s}(x) - b_i(x/N)) ds + \frac{N^2}{N^{d+\theta_r}} \int_0^t \sum_{x \in \Gamma_N^+} G_{i,s}(x/N) (\eta_{i,s}(x) - b_i(x/N)) ds, \end{aligned} \quad (3.3)$$

where we used that

$$\mathbb{L}_N \langle \pi_{i,s}^N, G_{i,s} \rangle = \frac{1}{N^d} \sum_{x \in B_N} G_{i,s}(x/N) \tau_x f_i(\eta_s), \quad (3.4)$$

with

$$\begin{aligned} \mathbb{L}_N \eta_1(0) &= \beta_{B_N}(0, \eta) \eta_0(0) + \eta_3(0) - (r+1) \eta_1(0) =: f_1(\xi, \omega), \\ \mathbb{L}_N \eta_2(0) &= r \eta_0(0) + \eta_3(0) - \beta_{B_N}(0, \eta) \eta_2(0) - \eta_2(0) =: f_2(\xi, \omega), \\ \mathbb{L}_N \eta_3(0) &= \beta_{B_N}(0, \eta) \eta_2(0) + r \eta_1(0) - 2 \eta_3(0) =: f_3(\xi, \omega). \end{aligned}$$

The second and third line in (3.3) correspond to the computation of the time integral associated to $N^2 \mathcal{L}_N$, the fourth line in (3.3) corresponds to the time integral associated to \mathbb{L}_N and the last term, to the integral associated to $N^2 L_{\widehat{b}, \widehat{\theta}, N}$.

Proposition 1. *The sequence of probability measures $(Q_N^{\widehat{\theta}})_{N \geq 1}$ is tight in the Skorohod topology of $D([0, T], \mathcal{M}^3)$.*

We refer to [15, Section 4] for details regarding the proof of tightness of a sequence of probability measures. Recall that it is enough to show that for every H in a dense subset of $\mathcal{C}(\overline{B})$ for the L^2 norm, for every $1 \leq i \leq 3$,

$$\limsup_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\sup_{|t-s| \leq \delta} \left| \langle \pi_{i,t}^N, H \rangle - \langle \pi_{i,s}^N, H \rangle \right| \right] = 0. \quad (3.5)$$

By density of $\mathcal{C}_0^2(\overline{B})$ in $\mathcal{C}^2(\overline{B})$ for the L^1 norm, it is enough to show (3.5) with H in $\mathcal{C}_0^2(\overline{B})$. To prove that, we use the martingale and its quadratic variation introduced in (3.1) and (3.2), and show that

$$\limsup_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\sup_{|t-s| \leq \delta} \left| M_{i,t}^N(H) - M_{i,s}^N(H) \right| \right] = 0, \quad (3.6)$$

and

$$\limsup_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\sup_{|t-s| \leq \delta} \left| \int_s^t L_N < \pi_{i,r}^N, H > dr \right| \right] = 0. \quad (3.7)$$

We get (3.6) using the triangular inequality, Doob's inequality and the fact that the quadratic variation converges to zero. To prove (3.7), one shows that there is a constant C depending only on H such that for every $r \in [0, T]$,

$$\left| L_N < \pi_{i,r}^N, H > \right| \leq C. \quad (3.8)$$

For that, use the decomposition of L_N and the fact that H vanishes at the boundary as well as explicit computations and the fact that the f_i 's are uniformly bounded in N .

3.2 Replacement Lemmas

In order to characterize the limit points of a sequence $(Q_N^{\hat{\theta}})_{N \geq 1}$, we need to close the equation (3.3). That means that we want to show that each term of the martingale converges to a term that appears in the weak formulation of the solution of the hydrodynamic equation, and that the martingale converges to zero. For that, we perform a replacement lemma in the bulk and one at the boundary. The replacement lemma in the bulk (Proposition 2) is exactly the same as in [18, Lemma 4.2] and we refer to that article for a detailed proof. Here we focus on the replacement lemmas at the boundary and more specifically on the left hand side boundary (the same statements hold on the right hand side). There are two replacement lemmas: one for $\theta_\ell \in [0, 1)$ whose formulation coincides with the replacement lemma at the boundary in [18, Proposition 4.3] (corresponding to a Dirichlet condition), and one for $\theta_r \geq 1$, whose formulation involves particle densities over small macroscopic boxes.

3.2.1 Dirichlet forms

Let us recall the expressions introduced in [18, Section 5] of the Dirichlet forms associated to each dynamics: for $f : \hat{\Sigma}_N \rightarrow \mathbb{R}$ and μ a measure on $\hat{\Sigma}_N$,

$$\begin{aligned} \mathcal{D}_N(f, \mu) &:= \sum_{k=1}^d \sum_{(x, x+e_k) \in B_N^2} \int_{\hat{\Sigma}_N} D \left(\sqrt{f(\xi^{x, x+e_k}, \omega^{x, x+e_k})} - \sqrt{f(\xi, \omega)} \right)^2 d\mu(\xi, \omega), \\ \mathcal{D}_{\hat{b}, \hat{\theta}, N}(f, \mu) &:= N^{-\theta_\ell} \sum_{i=0}^3 \sum_{x \in \Gamma_N^-} \int_{\hat{\Sigma}_N} b_i(x/N)(1 - \eta_i(x)) \left(\sqrt{f(\sigma_{i,x}(\xi, \omega))} - \sqrt{f(\xi, \omega)} \right)^2 d\mu(\xi, \omega) \\ &\quad + N^{-\theta_r} \sum_{i=0}^3 \sum_{x \in \Gamma_N^+} \int_{\hat{\Sigma}_N} b_i(x/N)(1 - \eta_i(x)) \left(\sqrt{f(\sigma_{i,x}(\xi, \omega))} - \sqrt{f(\xi, \omega)} \right)^2 d\mu(\xi, \omega) \end{aligned}$$

and

$$\begin{aligned} \mathbb{D}_N(f, \mu) &:= \sum_{x \in B_N} \int_{\hat{\Sigma}_N} \left[r(1 - \omega(x)) + \omega(x) \right] \left(\sqrt{f(\xi, \sigma^x \omega)} - \sqrt{f(\xi, \omega)} \right)^2 d\mu(\xi, \omega) \\ &\quad + \sum_{x \in B_N} \int_{\hat{\Sigma}_N} \left[\beta_{B_N}(x, \xi, \omega)(1 - \xi(x)) + \xi(x) \right] \left(\sqrt{f(\sigma^x \xi, \omega)} - \sqrt{f(\xi, \omega)} \right)^2 d\mu(\xi, \omega). \end{aligned}$$

In the proofs of the Replacement lemmas, we will widely make use of the following inequalities.

Lemma 1. *(i) Consider $\hat{\alpha}$ a smooth profile which satisfies (2.11) and (2.12). There is a constant $C_1 > 0$ such that for any density function $f : \hat{\Sigma}_N \rightarrow \mathbb{R}$ with respect to the measure $\nu_{\hat{\alpha}}^N$,*

$$\langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\hat{\alpha}}^N} \leq -\frac{1}{4} \mathcal{D}_N(f, \nu_{\hat{\alpha}}^N) + C_1 N^{d-2}. \quad (3.9)$$

(ii) Consider $\hat{\alpha}$ a smooth profile which either satisfies (2.11) and (2.12) or is constant. There is a constant $C_2 > 0$ such that for any density function $f : \hat{\Sigma}_N \rightarrow \mathbb{R}$ with respect to the measure $\nu_{\hat{\alpha}}^N$,

$$\langle \mathbb{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\hat{\alpha}}^N} \leq C_2 N^d. \quad (3.10)$$

(iii) Consider $\hat{\alpha}$ a smooth profile which satisfies (2.11) and (2.12), then for any density function $f : \hat{\Sigma}_N \rightarrow \mathbb{R}$ with respect to the measure $\nu_{\hat{\alpha}}^N$,

$$\langle L_{\hat{b}, \hat{\theta}, N} \sqrt{f}, \sqrt{f} \rangle_{\nu_{\hat{\alpha}}^N} = -\frac{1}{2} D_{\hat{b}, \hat{\theta}, N}(f, \nu_{\hat{\alpha}}^N). \quad (3.11)$$

Proof. Both (i) and (ii) are stated and proved in [18, Section 6, Lemma 6.1] and (iii) follows from the fact that $\nu_{\hat{\alpha}}^N$ is invariant under the dynamics generated by $L_{\hat{b}, \hat{\theta}, N}$ when $\hat{\alpha}$ satisfies (2.11) and (2.12). We recall the proof of (ii). Using the correspondence (2.1) and the expression of the generator for the contact process in the bulk (2.6) we have

$$\begin{aligned} \langle \mathbb{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\hat{\alpha}}^N} &= \sum_{x \in B_N} \int (\beta_{B_N}(x, \xi, \omega)(1 - \xi(x)) + \xi(x)) \sqrt{f(\xi, \omega)} (\sqrt{f(\sigma^x \xi, \omega)} - \sqrt{f(\xi, \omega)}) d\nu_{\hat{\alpha}}^N(\xi, \omega) \\ &\quad + \sum_{x \in B_N} \int (r(1 - \omega(x)) + \omega(x)) \sqrt{f(\xi, \omega)} (\sqrt{f(\xi, \sigma^x \omega)} - \sqrt{f(\xi, \omega)}) d\nu_{\hat{\alpha}}^N(\xi, \omega). \end{aligned}$$

As the terms $\beta_{B_N}(x, \xi, \omega)(1 - \xi(x)) + \xi(x)$ and $r(1 - \omega(x)) + \omega(x)$ are uniformly bounded in N by a constant $C(\lambda_1, \lambda_2, r, d)$, the above is less than

$$\begin{aligned} I_1 + I_2 &:= \sum_{x \in B_N} \int C(\lambda_1, \lambda_2, r, d) \sqrt{f(\xi, \omega)} (\sqrt{f(\sigma^x \xi, \omega)} + \sqrt{f(\xi, \omega)}) d\nu_{\hat{\alpha}}^N(\xi, \omega) \\ &\quad + \sum_{x \in B_N} \int C(\lambda_1, \lambda_2, r, d) \sqrt{f(\xi, \omega)} (\sqrt{f(\xi, \sigma^x \omega)} + \sqrt{f(\xi, \omega)}) d\nu_{\hat{\alpha}}^N(\xi, \omega). \end{aligned}$$

To deal with I_1 (the same method holds for I_2). Use (2.16) with $A = 2N$ to get that

$$I_1 \leq 2C(\lambda_1, \lambda_2, r, d) \sum_{x \in B_N} \int f(\xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) + C(\lambda_1, \lambda_2, r, d) \sum_{x \in B_N} \int f(\sigma^x \xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega).$$

Since f is a density with respect to $\nu_{\hat{\alpha}}^N$, the first term is less than CN^d where $C > 0$ is a uniform constant in N . Now note that for $x \in B_N$, using

$$\begin{aligned} \int f(\sigma^x \xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) &= \sum_{0 \leq i \leq 3} \int \eta_i(x) f(\sigma^x \xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) \\ &\leq \sum_{0 \leq i \leq 3} \sum_{\substack{j \neq i \\ \alpha_j(x/N) \neq 0}} \frac{\alpha_i(x/N)}{\alpha_j(x/N)} \int \eta_j(x) f(\xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) \end{aligned}$$

where we used the change of variable $(\xi, \omega) \rightarrow (\sigma^x \xi, \omega)$ and formulas (A.5) in the second inequality. Now as $\hat{\alpha}$ satisfies (2.11) and (2.12), the set

$$\left\{ \frac{\alpha_i(x/N)}{\alpha_j(x/N)}, \alpha_j(x/N) \neq 0, 0 \leq i, j \leq 3 \right\}$$

is uniformly bounded in N . Using again that f is a density we get the desired result. \square

3.2.2 Replacement lemma in the bulk.

Let us first introduce a few notations. Given a smooth profile $\hat{\alpha}$, and a function $\phi : \widehat{\Sigma}_N \rightarrow \mathbb{R}$, denote $\tilde{\phi}(\hat{\alpha})$ the expectation of ϕ under $\nu_{\hat{\alpha}}^N$. For $\ell \in \mathbb{N}$, introduce

$$\Lambda_x^\ell = \{y \in B_N, \|y - x\| \leq \ell\}$$

where $\|y - x\| = \max\{|y_i - x_i|, 1 \leq i \leq d\}$, and denote $\eta_i^\ell(x)$ the average of η in Λ_x^ℓ , that is,

$$\eta_i^\ell(x) = \frac{1}{|\Lambda_x^\ell|} \sum_{y \in \Lambda_x^\ell} \eta_i(y), \quad \text{for } 1 \leq i \leq 3.$$

Introduce the vector

$$\hat{\eta}^\ell(x) = (\hat{\eta}_1^\ell(x), \hat{\eta}_2^\ell(x), \hat{\eta}_3^\ell(x))$$

and for $\varepsilon > 0$,

$$V_{\varepsilon N}(\xi, \omega) = \left| \frac{1}{|\Lambda_x^{\lfloor \varepsilon N \rfloor}|} \sum_{y \in \Lambda_x^{\lfloor \varepsilon N \rfloor}} \tau_y \phi(\xi, \omega) - \tilde{\phi}(\hat{\eta}^{\lfloor \varepsilon N \rfloor}(0)) \right|.$$

In the sequel, we will write εN instead of $\lfloor \varepsilon N \rfloor$. The replacement lemma in the bulk stated and proved in [18, Lemma 4.2] is the following:

Proposition 2. *For any $G \in C_0^{1,2}$ and for any function $\phi : \widehat{\Sigma}_N \rightarrow \mathbb{R}$,*

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\frac{1}{N^d} \sum_{x \in B_N} \int_0^T |G_s(x/N)| \tau_x V_{\varepsilon N}(\xi_s, \omega_s) ds \right] = 0.$$

3.2.3 Replacement lemma at the left hand side boundary for $\theta_\ell \in [0, 1)$.

Here we fix θ_ℓ in $[0, 1)$ and prove the replacement lemma at the left hand side boundary. It essentially states that when performing the macroscopic limit $N \rightarrow \infty$, one can replace $\eta_i(x)$ by $b_i(x/N)$. For $\theta_r \in [0, 1)$, the replacement lemma at the right hand side boundary is exactly the same. Recall that this result has been proved for $\theta_\ell = \theta_r = 0$ in [18, Section 6] and we generalize it here to the case where the left hand side (or right hand side) parameter θ_ℓ is allowed to vary in $[0, 1)$.

Proposition 3. *For any sequence of measures $(\mu_N)_{N \geq 0}$ on $\widehat{\Sigma}_N$, for any $G \in C^{1,2}([0, T] \times \overline{B})$ and any $i \in \{1, 2, 3\}$, for any $t \in [0, T]$, for all $\delta > 0$,*

$$\limsup_{N \rightarrow \infty} \mathbb{P}_{\mu_N} \left[\left| \int_0^t \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) (\eta_{i,s}(x) - b_i(x/N)) ds \right| > \delta \right] = 0. \quad (3.12)$$

Note that the replacement lemma at the right hand side boundary for $\theta_r \in [0, 1)$ states as above, with the sum in x carrying over Γ_N^+ rather than Γ_N^- .

Proof. Fix an $i \in \{1, 2, 3\}$. It is enough to show that

$$\limsup_{N \rightarrow \infty} \frac{1}{N^d} \log \left(\mathbb{P}_{\mu_N} \left[\left| \int_0^t \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) (\eta_{i,s}(x) - b_i(x/N)) ds \right| > \delta \right] \right) = -\infty.$$

Consider $\hat{\alpha}$ a smooth profile satisfying conditions (2.11) and (2.12). For $a > 0$,

$$\begin{aligned}
& \mathbb{P}_{\mu_N} \left[\left| \int_0^t \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) (\eta_{i,s}(x) - b_i(x/N)) ds \right| > \delta \right] \\
& \leq \sup_{(\xi, \omega) \in \tilde{\Sigma}_N} \frac{d\mu_N}{d\nu_{\hat{\alpha}}^N}(\xi, \omega) \times \mathbb{P}_{\nu_{\hat{\alpha}}^N} \left[\left| \int_0^t \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) (\eta_{i,s}(x) - b_i(x/N)) ds \right| > \delta \right] \\
& \leq \exp(K_0 N^d) \mathbb{E}_{\mu_N} \left[\exp \left(a N^d \left| \int_0^t \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) (\eta_{i,s}(x) - b_i(x/N)) ds \right| \right) \right] \exp(-a \delta N^d).
\end{aligned}$$

We used, in the first inequality, that the Radon-Nikodym derivative of μ_N with respect to $\nu_{\hat{\alpha}}^N$ is bounded by $\exp(K_0 N^d)$ with K_0 a constant, and Tchebychev's inequality in the second line. Therefore,

$$\begin{aligned}
& \frac{1}{N^d} \log \left(\mathbb{P}_{\mu_N} \left[\left| \int_0^t \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) (\eta_{i,s}(x) - b_i(x/N)) ds \right| > \delta \right] \right) \\
& \leq -a\delta + K_0 + \frac{1}{N^d} \log \left(\mathbb{E}_{\nu_{\hat{\alpha}}^N} \left[\exp \left(a N^d \left| \int_0^t \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) (\eta_{i,s}(x) - b_i(x/N)) ds \right| \right) \right] \right).
\end{aligned} \tag{3.13}$$

It is enough to show that the last term is uniformly bounded in a and N and then, take $a \rightarrow \infty$ with N . Since $e^{|x|} \leq e^x + e^{-x}$ and

$$\limsup_N \frac{1}{N} \log(a_N + b_N) = \max \left[\limsup_N \frac{1}{N} \log a_N, \limsup_N \frac{1}{N} \log b_N \right], \tag{3.14}$$

we show that the last term in (3.13) without the absolute values, is uniformly bounded in a and N . Apply Feynman-Kac's formula with

$$V(s, (\xi_s, \omega_s)) = \frac{aN^d}{N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) (\eta_{i,s}(x) - b_i(x/N)).$$

Since G is bounded,

$$\begin{aligned}
& \frac{1}{N^d} \log \left(\mathbb{E}_{\nu_{\hat{\alpha}}^N} \left[\exp \left(a N^d \int_0^t \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) (\eta_{i,s}(x) - b_i(x/N)) ds \right) \right] \right) \\
& \leq \int_0^t ds \left\{ \sup_f \left[\int_{\tilde{\Sigma}_N} \frac{a}{N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) (b_i(x/N) - \eta_i(x)) f(\xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) + \frac{1}{N^d} < L_N \sqrt{f}, \sqrt{f} >_{\nu_{\hat{\alpha}}^N} \right] \right\}.
\end{aligned}$$

Note that for $x \in \Gamma_N^-$,

$$b_i(x/N) - \eta_i(x) = \sum_{j \neq i} (b_i(x/N) \eta_j(x) - b_j(x/N) \eta_i(x))$$

and for $j \neq i$,

$$\int \eta_i(x) b_j(x/N) f(\xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) = \int \eta_j(x) f(\sigma_{i,x}(\xi, \omega)) b_i(x/N) d\nu_{\hat{\alpha}}^N(\xi, \omega). \tag{3.15}$$

Therefore,

$$\begin{aligned}
& G(s, x/N) \int (b_i(x/N) - \eta_i(x)) f(\xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) \\
&= G(s, x/N) \int b_i(x/N) \sum_{j \neq i} \eta_j(x) f(\xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) - G(s, x/N) \int b_i(x/N) \sum_{j \neq i} \eta_j(x) f(\sigma_{i,x}(\xi, \omega)) d\nu_{\hat{\alpha}}^N(\xi, \omega) \\
&= -G(s, x/N) \int b_i(x/N) (1 - \eta_i(x)) (f(\sigma_{i,x}(\xi, \omega)) - f(\xi, \omega)) d\nu_{\hat{\alpha}}^N(\xi, \omega) \\
&\leq \frac{A}{2} \int b_i(x/N) (1 - \eta_i(x)) \left(\sqrt{f(\sigma_{i,x}(\xi, \omega))} - \sqrt{f(\xi, \omega)} \right)^2 d\nu_{\hat{\alpha}}^N(\xi, \omega) \\
&+ \frac{1}{2A} (G(s, x/N))^2 \int b_i(x/N) (1 - \eta_i(x)) \left(\sqrt{f(\sigma_{i,x}(\xi, \omega))} + \sqrt{f(\xi, \omega)} \right)^2 d\nu_{\hat{\alpha}}^N(\xi, \omega)
\end{aligned} \tag{3.16}$$

where we used (2.16) in the last line replacing A by AN , with $A > 0$. Summing (3.16) over Γ_N^- and multiplying by $\frac{a}{N^{d-1}}$ yields,

$$\int_{\eta} \frac{a}{N^{d-1}} \sum_{x \in \Gamma_N^-} (b_i(x/N) - \eta_i(x)) f(\xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) \leq \frac{aAN^{\theta_\ell}}{2N^{d-1}} D_{\hat{b}, \hat{\theta}, N}(f, \nu_{\hat{\alpha}}^N) + \frac{a\|b\|_\infty \|G\|_\infty^2}{2A}$$

where the second term comes from Cauchy-Schwarz's inequality, the fact that f is a density, the change of variable formula (3.15) and the fact that b is bounded and that there are of order N^{d-1} terms in Γ_N^- . Therefore, taking $a = A = N^{\frac{1-\theta_\ell}{2}}$, using (3.9), (3.10) and (3.11) to bound $\langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\hat{\alpha}}^N}$ and the fact that a Dirichlet form is positive we are left with

$$\begin{aligned}
& \frac{1}{N^d} \log \left(\mathbb{E}_{\nu_{\hat{\alpha}}^N} \left[\exp \left(aN^d \int_0^t \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) (\eta_{i,s}(x) - b_i(x/N)) ds \right) \right] \right) \\
&\leq T \sup_f \left\{ \frac{N^2}{2N^d} D_{\hat{b}, \hat{\theta}, N}(f, \nu_{\hat{\alpha}}^N) + \frac{\|G\|_\infty^2 \|b\|_\infty}{2} + \frac{1}{N^d} \langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\hat{\alpha}}^N} \right\} \\
&\leq T \sup_f \left\{ \frac{\|G\|_\infty^2 \|b\|_\infty}{2} - \frac{N^2}{4N^d} \mathcal{D}_N(f, \nu_{\hat{\alpha}}^N) + C_1 + C_2 \right\} \\
&\leq T \sup_f \left\{ \left(\frac{aAN^{\theta_\ell}}{2N^{d-1}} - \frac{N^2}{2N^d} \right) D_{\hat{b}, \hat{\theta}, N}(f, \nu_{\hat{\alpha}}^N) + \frac{\|G\|_\infty^2 a \|b\|_\infty}{2A} \right\} + TC_1 + TC_2 \\
&\leq T \frac{\|b\|_\infty \|G\|_\infty^2}{2} + TC_1 + TC_2,
\end{aligned}$$

hence the desired result. \square

3.2.4 Replacement lemma at the left hand side boundary for $\theta_\ell \geq 1$.

For $\theta_\ell \geq 1$, the replacement lemma at the boundary involves particle densities over small macroscopic boxes. Again, the same replacement lemma holds at the right hand side boundary for $\theta_r \geq 1$. In fact, we will see in the proof that the lemma holds for any positive value of θ_ℓ , resp. θ_r regardless of whether θ_ℓ resp. $\theta_r \geq 1$.

Proposition 4. *For any sequence of probability measures $(\mu_N)_{N \geq 0}$ on $\hat{\Sigma}_N$, for any $G \in \mathcal{C}^{1,2}([0, T] \times \bar{B})$, for all $i \in \{1, 2, 3\}$ and any $t \in [0, T]$,*

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} \int_0^t G(s, x/N) (\eta_{i,s}^{\varepsilon N}(x) - \eta_{i,s}(x)) ds \right| \right] = 0. \tag{3.17}$$

Proof. Consider $\hat{\alpha}$ a smooth profile satisfying conditions (2.11) and (2.12). By the entropy inequality (see [15, Appendix 1]), for any $A > 0$,

$$\begin{aligned} & \mathbb{E}_{\mu_N} \left[\left| \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} \int_0^t G(s, x/N) (\eta_{i,s}^{\varepsilon N}(x) - \eta_{i,s}(x)) ds \right| \right] \\ & \leq \frac{1}{AN^d} H(\mu_N | \nu_{\hat{\alpha}}^N) + \frac{1}{AN^d} \log \mathbb{E}_{\nu_{\hat{\alpha}}^N} \left[\exp \left(AN^d \left| \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} \int_0^t G(s, x/N) (\eta_{i,s}^{\varepsilon N}(x) - \eta_{i,s}(x)) ds \right| \right) \right]. \end{aligned} \quad (3.18)$$

As B_N is finite, there is a constant $K_0 > 0$ such that $H(\mu_N | \nu_{\hat{\alpha}}^N) \leq K_0 N^d$ so the first term in (3.18) is bounded by K_0/A . Let us show that the second term tends to zero when $N \rightarrow \infty$ and $\varepsilon \rightarrow 0$ and then take A arbitrarily big. Again, by (3.14), it is enough to show that the second term in (3.18) without the absolute values in the exponential, tends to zero. By Feynman-Kac's inequality,

$$\begin{aligned} & \frac{1}{AN^d} \log \mathbb{E}_{\nu_{\hat{\alpha}}^N} \left[\exp \left(AN^d \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} \int_0^t G(s, x/N) (\eta_{i,s}^{\varepsilon N}(x) - \eta_{i,s}(x)) ds \right) \right] \\ & \leq \int_0^t ds \sup_f \left[\int \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) (\eta_i^{\varepsilon N}(x) - \eta_i(x)) f(\xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) + \frac{1}{AN^d} \langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\hat{\alpha}}^N} \right] \end{aligned} \quad (3.19)$$

where the supremum is taken over densities. Write $\eta_i^{\varepsilon N}(x) - \eta_i(x)$ as a d -dimensional telescopic sum. For that, note that for any $y \in \Lambda_{\varepsilon N}$, there is an integer $p_y \leq d\varepsilon N$ and p_y points $u_1 = x, \dots, u_{p_y} = x + y$ such that for every $1 \leq j \leq p_y - 1$, $u_{j+1} - u_j = e_{r_j}$ where $r_j \in \{1, \dots, d\}$. Therefore,

$$\eta_i^{\varepsilon N}(x) - \eta_i(x) = \frac{1}{(\varepsilon N)^d} \sum_{y \in \Lambda_{\varepsilon N}} \sum_{j=1}^{p_y-1} (\eta_i(u_j + e_{r_j}) - \eta_i(u_j)). \quad (3.20)$$

Performing the change of variable $(\xi, \omega) \rightarrow (\xi^{u_j, u_j + e_{r_j}}, \omega^{u_j, u_j + e_{r_j}}) =: (\xi^{u_j}, \omega^{u_j})$ and using (A.2),

$$\frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) \int (\eta_i^{\varepsilon N}(x) - \eta_i(x)) f(\xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) \quad (3.21)$$

$$= \frac{1}{2N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) \int \frac{1}{(\varepsilon N)^d} \sum_{y \in \Lambda_{\varepsilon N}} \sum_{j=1}^{p_y-1} (\eta_i(u_j + e_{r_j}) - \eta_i(u_j)) (f(\xi, \omega) - f(\xi^{u_j}, \omega^{u_j})) d\nu_{\hat{\alpha}}^N(\xi, \omega) \quad (3.22)$$

$$+ \frac{1}{2N^{d-1}} \sum_{x \in \Gamma_N^-} G(s, x/N) \int \frac{1}{(\varepsilon N)^d} \sum_{y \in \Lambda_{\varepsilon N}} \sum_{j=1}^{p_y-1} (\eta_i(u_j + e_{r_j}) - \eta_i(u_j)) f(\xi^{u_j}, \omega^{u_j}) \left(1 - \frac{\nu_{\hat{\alpha}}^N(\xi^{u_j}, \omega^{u_j})}{\nu_{\hat{\alpha}}^N(\xi, \omega)} \right) d\nu_{\hat{\alpha}}^N(\xi, \omega). \quad (3.23)$$

First we deal with (3.22). Using that $|\eta_i(u_j + e_{r_j}) - \eta_i(u_j)| \leq 1$ and inequality (2.16),

$$\begin{aligned} (3.22) & \leq \frac{1}{2N^{d-1}} \sum_{x \in \Gamma_N^-} \frac{B}{2(\varepsilon N)^d} \int \sum_{y \in \Lambda_{\varepsilon N}} \sum_{j=1}^{p_y-1} \left(\sqrt{f(\xi^{u_j}, \omega^{u_j})} - \sqrt{f(\xi, \omega)} \right)^2 d\nu_{\hat{\alpha}}^N(\xi, \omega) \\ & \quad + \frac{1}{2N^{d-1}} \sum_{x \in \Gamma_N^-} \frac{\|G\|_{\infty}^2}{2B(\varepsilon N)^d} \int \sum_{y \in \Lambda_{\varepsilon N}} \sum_{j=1}^{p_y-1} \left(\sqrt{f(\xi^{u_j}, \omega^{u_j})} + \sqrt{f(\xi, \omega)} \right)^2 d\nu_{\hat{\alpha}}^N(\xi, \omega) \end{aligned} \quad (3.24)$$

where $B > 0$ will be chosen later. Now for $x \in \Gamma_N^-$,

$$\sum_{y \in \Lambda_{\varepsilon N}} \sum_{j=1}^{p_y-1} \left(\sqrt{f(\xi^{u_j}, \omega^{u_j})} - \sqrt{f(\xi, \omega)} \right)^2 \leq d\varepsilon N \sum_{k=1}^d \sum_{(z, z+e_k) \in x + \Lambda_{\varepsilon N}} \left(\sqrt{f(\xi^{z, z+e_k}, \omega^{z, z+e_k})} - \sqrt{f(\xi, \omega)} \right)^2$$

and

$$\sum_{y \in \Lambda_{\varepsilon N}} \sum_{j=1}^{p_y-1} \left(\sqrt{f(\xi^{u_j}, \omega^{u_j})} + \sqrt{f(\xi, \omega)} \right)^2 \leq d\varepsilon N \sum_{k=1}^d \sum_{(z, z+e_k) \in x + \Lambda_{\varepsilon N}} \left(\sqrt{f(\xi^{z, z+e_k}, \omega^{z, z+e_k})} + \sqrt{f(\xi, \omega)} \right)^2.$$

Therefore,

$$\begin{aligned} & \int \sum_{x \in \Gamma_N^-} \frac{1}{(\varepsilon N)^d} \sum_{y \in \Lambda_{\varepsilon N}} \sum_{j=1}^{p_y-1} \left(\sqrt{f(\xi^{u_j}, \omega^{u_j})} - \sqrt{f(\xi, \omega)} \right)^2 d\nu_{\hat{\alpha}}^N(\xi, \omega) \\ & \leq d\varepsilon N \int \sum_{x \in \Gamma_N^-} \frac{1}{(\varepsilon N)^d} \sum_{k=1}^d \sum_{(z, z+e_k) \in x + \Lambda_{\varepsilon N}} \left(\sqrt{f(\xi^{z, z+e_k}, \omega^{z, z+e_k})} - \sqrt{f(\xi, \omega)} \right)^2 d\nu_{\hat{\alpha}}^N(\xi, \omega) \quad (3.25) \\ & \leq d \int \sum_{k=1}^d \sum_{(z, z+e_k) \in B_N^2} \left(\sqrt{f(\xi^{z, z+e_k}, \omega^{z, z+e_k})} - \sqrt{f(\xi, \omega)} \right)^2 d\nu_{\hat{\alpha}}^N(\xi, \omega) = \frac{d}{D} \mathcal{D}_N(f, \nu_{\hat{\alpha}}^N) \end{aligned}$$

where the last inequality comes from an integration by part formula and the fact that there are of order N^{d-1} elements in Γ_N^- . Now using that f is a density,

$$\frac{1}{2N^{d-1}} \int \sum_{x \in \Gamma_N^-} \frac{1}{(\varepsilon N)^d} \sum_{y \in \Lambda_{\varepsilon N}} \sum_{j=1}^{p_y-1} \left(\sqrt{f(\xi^{u_j}, \omega^{u_j})} + \sqrt{f(\xi, \omega)} \right)^2 d\nu_{\hat{\alpha}}^N(\xi, \omega) \quad (3.26)$$

$$\leq d\varepsilon N + \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} \frac{d\varepsilon N}{(\varepsilon N)^d} \sum_{(z, z+e_k) \in x + \Lambda_{\varepsilon N}} \int f(\xi^{z, z+e_k}, \omega^{z, z+e_k}) d\nu_{\hat{\alpha}}^N(\xi, \omega). \quad (3.27)$$

Note that $(\xi, \omega) \mapsto f(\xi^{z, z+e_k}, \omega^{z, z+e_k})$ is not a density. Using the explicit expression of the product measure $\nu_{\hat{\alpha}}^N$, one has that for $(x, x + e_k) \in B_N^2$

$$\frac{\nu_{\hat{\alpha}}^N(\xi^{x, x+e_k}, \omega^{x, x+e_k})}{\nu_{\hat{\alpha}}^N(\xi, \omega)} = \prod_{i=1}^3 \left(\frac{\alpha_i(x/N)}{\alpha_0(x/N)} \right)^{\eta_i(x+e_k) - \eta_i(x)} \left(\frac{\alpha_i((x+e_k)/N)}{\alpha_0((x+e_k)/N)} \right).$$

Now using that $\alpha_i(\frac{x+e_k}{N}) = \alpha_i(\frac{x}{N}) + O(\frac{1}{N})$ we have the following inequality: there is a constant $C > 0$ such that

$$\left| 1 - \frac{\nu_{\hat{\alpha}}^N(\xi^{x, x+e_k}, \omega^{x, x+e_k})}{\nu_{\hat{\alpha}}^N(\xi, \omega)} \right| \leq \frac{C}{N}. \quad (3.28)$$

Therefore, for $(x, x + e_k) \in B_N^2$,

$$\int f(\xi^{x, x+e_k}, \omega^{x, x+e_k}) d\nu_{\hat{\alpha}}^N(\xi, \omega) \leq 1 + \frac{C}{N} \quad (3.29)$$

so the last term in (3.27) is smaller than

$$\frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} \frac{d\varepsilon N}{(\varepsilon N)^d} \sum_{z, z+e_k \in x + \Lambda_{\varepsilon N}} \left(1 + \frac{C}{N} \right) = \frac{d\varepsilon N}{B} + \frac{d\varepsilon}{B}. \quad (3.30)$$

Collecting inequalities (3.24), (3.25) and (3.30), we get that

$$(3.22) \leq \frac{dB}{2D} \mathcal{D}_N(f, \nu_{\hat{\alpha}}^N) + \|G\|_\infty^2 \left(\frac{d\varepsilon N}{B} + \frac{d\varepsilon}{B} \right). \quad (3.31)$$

Now we deal with (3.23). By inequalities (3.28) and (3.29) and using that $\eta_i \in \{0, 1\}$,

$$\begin{aligned} (3.23) &\leq \frac{\|G\|_\infty}{N^{d-1}} \sum_{x \in \Gamma_N^-} \int \frac{1}{(\varepsilon N)^d} \sum_{y \in \Lambda_{\varepsilon N}} \sum_{j=1}^{p_y-1} f(\xi^{u_j}, \omega^{u_j}) \left(1 - \frac{\nu_{\hat{\alpha}}^N(\xi^{u_j}, \omega^{u_j})}{\nu_{\hat{\alpha}}^N(\xi, \omega)} \right) d\nu_{\hat{\alpha}}^N(\xi, \omega) \\ &\leq \frac{C\|G\|_\infty}{N^d} \sum_{x \in \Gamma_N^-} \frac{1}{(\varepsilon N)^d} \sum_{y \in \Lambda_{\varepsilon N}} \sum_{j=1}^{p_y-1} \left(1 + \frac{C}{N} \right) \\ &\leq \|G\|_\infty \left(1 + \frac{C}{N} \right) C d \varepsilon. \end{aligned} \quad (3.32)$$

Therefore, collecting inequalities (3.31) and (3.32), for any density f ,

$$\int \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} (\eta_i^{\varepsilon N}(x) - \eta_i(x)) f(\xi, \omega) d\nu_{\hat{\alpha}}^N(\xi, \omega) \leq \left(\frac{dB}{2DN^{d-1}} \mathcal{D}_N(f, \nu_{\hat{\alpha}}^N) + \frac{d\varepsilon N}{B} + \frac{d\varepsilon}{B} + C d \varepsilon + \frac{C^2 d \varepsilon}{N} \right).$$

This, combined with (3.19) as well as Lemma 1 yields:

$$\begin{aligned} &\frac{1}{AN^d} \log \mathbb{E}_{\nu_{\hat{\alpha}}^N} \left[\exp \left(AN^d \left| \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} \int_0^t G(s, x/N) (\eta_{i,s}^{\varepsilon N}(x) - \eta_{i,s}(x)) ds \right| \right) \right] \\ &\leq \sup_f \left[\left(\frac{T\|G\|_\infty dB}{2DN^{d-1}} - \frac{N^2}{4AN^d} \right) \mathcal{D}_N(f, \nu_{\hat{\alpha}}^N) + \frac{T\|G\|_\infty \varepsilon}{B} (N+d) + T\|G\|_\infty C d \varepsilon \left(1 + \frac{C}{N} \right) + \frac{C_4}{A} \right] \end{aligned} \quad (3.33)$$

with $C_4 > 0$, a constant that only depends on d . Taking $B = (ND)/(2dT\|G\|_\infty A)$ and putting together (3.19) and (3.33) yields

$$\mathbb{E}_{\mu_N} \left[\left| \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^-} \int_0^t G(s, x/N) (\eta_{i,s}^{\varepsilon N}(x) - \eta_{i,s}(x)) ds \right| \right] \leq K_0 A \varepsilon \left(1 + \frac{1}{N} \right) + T\|G\|_\infty C d \varepsilon \left(1 + \frac{C}{N} \right) + \frac{C_4}{A}$$

with K_0 a constant depending on T, G, d and taking $\varepsilon \rightarrow 0$ and then $A \rightarrow \infty$ we get the desired result. \square

3.3 Energy estimates

In view of the proof of uniqueness of the limit of the sequence of probability measures $(Q_N^{\hat{\theta}})_{N \geq 1}$, we state that any limiting measure $Q^{\hat{\theta}}$ is concentrated on a trajectory belonging to a specific functional space. This allows to define the hydrodynamic limit at the boundary.

Proposition 5. *Let $\hat{\theta} \in (\mathbb{R}^+)^2$ and $Q^{\hat{\theta}}$ be a limit point of the sequence of probability measures $(Q_N^{\hat{\theta}})_{N \geq 1}$. Then, the probability measure $Q^{\hat{\theta}}$ is concentrated on paths $\hat{\rho}(t, u) du$ such that for every $1 \leq i \leq 3$, ρ_i belongs to $L^2((0, T); \mathcal{H}^1(B))$.*

This follows from the Lemma below and the Riesz Representation Theorem. A similar proof can be found, for instance in [15, section 5].

Lemma 2. *For any $\hat{\theta} \in (\mathbb{R}^+)^2$, there is a constant $K_{\hat{\theta}} > 0$ such that for every $1 \leq i \leq 3$,*

$$\mathbb{E}_{Q^{\hat{\theta}}} \left[\sup_H \left(\int_0^T \int_B \sum_{k=1}^d \partial_{e_k} H(s, u) \rho_i(s, u) duds - K_{\hat{\theta}} \int_0^T \int_B H(s, u)^2 duds \right) \right] < \infty \quad (3.34)$$

where the supremum is carried over functions $H \in \mathcal{C}_c^{0,2}([0, T] \times B)$.

For the proof of Lemma 2, one can follow the arguments in [15, Section 5]. First prove (3.34) for a dense and countable set of elements of $C_c^{0,2}([0, T] \times B)$ thanks to Feynmann-Kac's inequality. Then, use an integration by part to deal with the spatial derivatives in H , as well as a change of variable.

3.4 Characterization of the limit point in the (Dirichlet ; Robin) mixed regime

In order to show that the limit point of the sequence of probability measures $(Q_N^{\hat{\theta}})_{N \geq 1}$ lies on the trajectory with density profile the unique solution of the hydrodynamic equation associated to $\hat{\theta}$ and $\hat{\gamma}$, we give a characterization result (see Proposition 6). We will focus on the (Dirichlet ; Robin) mixed regime since the (Neumann ; Robin) mixed regime can be proved following the same lines. Therefore, take $\theta_\ell \in [0, 1)$ and $\theta_r \geq 1$.

As mentioned in the introduction, in one dimension, the macroscopic trajectories are continuous in space and their values at the boundaries are defined in the classical sense. This is no longer valid in higher dimension. To deal with this difficulty we use the regularity of the trajectories proved in Proposition 5: the trajectories lie in $L^2((0, T); \mathcal{H}^1(B))$ so their values at the boundary are defined via the trace operator (see Lemma 3).

Proposition 6. *If $Q^{\hat{\theta}}$ is a limit point of the sequence of probability measures $(Q_N^{\hat{\theta}})_{N \geq 1}$, then*

$$\begin{aligned} Q^\theta \left[\hat{\pi}, \left| I_{\hat{G}}(\hat{\rho})(t) + D \sum_{i=1}^3 \int_0^t \left[\int_{\Gamma^-} b_i(r)(\partial_{e_1} G_{i,s})(r) n_1(r).dS(r) + \int_{\Gamma^+} \rho_i(s, r)(\partial_{e_1} G_{i,s})(r) n_1(r).dS(r) \right] ds \right. \right. \\ \left. \left. - \sum_{i=1}^3 \int_0^t \int_{\Gamma^+} G_i(r)(b_i(r) - \rho_i(s, r)) n_1(r).dS(r) ds \right| = 0, \forall t \in [0, T], \forall \hat{G} \in \mathcal{C}_{\hat{\theta}} \right] = 1 \end{aligned} \quad (3.35)$$

where recall that $I_{\hat{G}}(\hat{\rho})$ was defined in (2.18).

Proof. The fact that any limit point is concentrated on trajectories which are absolutely continuous with respect to the Lebesgue measure comes from Proposition 5. Let $Q^{\hat{\theta}}$ be a limit point of the sequence of probability measures $(Q_N^{\hat{\theta}})_{N \geq 1}$. To prove (3.35), it is enough to show that for any fixed $\delta > 0$ and $\hat{G} \in \mathcal{C}_{\hat{\theta}}^{1,2}$,

$$\begin{aligned} Q^{\hat{\theta}} \left[\hat{\pi}, \sup_{0 \leq t \leq T} \left| I_{\hat{G}}(\hat{\rho})(t) + D \sum_{i=1}^3 \int_0^t \left[\int_{\Gamma^-} b_i(r)(\partial_{e_1} G_{i,s})(r) n_1(r).dS(r) + \int_{\Gamma^+} \rho_i(s, r)(\partial_{e_1} G_{i,s})(r) n_1(r).dS(r) \right] ds \right. \right. \\ \left. \left. - \sum_{i=1}^3 \int_0^t \int_{\Gamma^+} G_i(r)(b_i(r) - \rho_i(s, r)) n_1(r).dS(r) ds \right| > \delta \right] = 0. \end{aligned} \quad (3.36)$$

Here, note that for $s \in [0, T]$ and $r \in \Gamma$, $\rho_i(s, r)$ stands for $\text{Tr}(\rho)(s, r)$ which is well defined since ρ is in $L^2([0, T], \mathcal{H}^1(B))$. By the triangular inequality, it suffices to prove that for any $1 \leq i \leq 3$,

$$\begin{aligned} Q^{\hat{\theta}} \left[\hat{\pi}, \sup_{0 \leq t \leq T} \left| I_{G_i}(\rho_i)(t) + D \int_0^t \left[\int_{\Gamma^-} b_i(r)(\partial_{e_1} G_{i,s})(r) n_1(r).dS(r) + \int_{\Gamma^+} \rho_i(s, r)(\partial_{e_1} G_{i,s})(r) n_1(r).dS(r) \right] ds \right. \right. \\ \left. \left. - \int_0^t \int_{\Gamma^+} G_i(r)(b_i(r) - \rho_i(s, r)) n_1(r).dS(r) ds \right| > \delta \right] = 0. \end{aligned} \quad (3.37)$$

As usual, we would like to approximate ρ by a convolution of its associated empirical measure with an approximation of the identity. Indeed, that convolution product can then be written in terms of the mean value of the configuration in a microscopic box. This is straightforward in the bulk, however, for the boundary

terms, one needs to justify that such an approximation works (see (3.49)). Without loss of generality, let us deal with $i = 1$. We turn to our martingales (3.1) $M_{1,t}^N(\widehat{G})$ and recall that we have proved that its quadratic variation vanishes as $N \uparrow \infty$. For $\varepsilon > 0$, introduce the set

$$B_{N,\varepsilon} = \{-N(1-\varepsilon), \dots, N(1-\varepsilon)\} \times \mathbb{T}_N^{d-1}.$$

By Proposition 2,

$$\begin{aligned} M_{1,t}^N(\widehat{G}) &= \langle \pi_{1,t}^N, G_{1,t} \rangle - \langle \pi_{1,0}^N, G_{1,0} \rangle - \int_0^t \langle \pi_{1,s}^N, \partial_s G_{1,s} \rangle ds \\ &\quad - \int_0^t \frac{D}{N^d} \sum_{x \in B_N \setminus \Gamma_N} \Delta G_{1,s}(x/N) \eta_{1,s}(x) ds \\ &\quad + \int_0^t \frac{D}{N^{d-1}} \sum_{x \in \Gamma_N^-} b_1(x/N) \partial_{e_1} G_{1,s}(x/N) ds + \int_0^t \frac{D}{N^{d-1}} \sum_{x \in \Gamma_N^-} \partial_{e_1} G_{1,s}(x/N) (\eta_{1,s}(x) - b_1(x/N)) ds \\ &\quad - \int_0^t \frac{D}{N^{d-1}} \sum_{x \in \Gamma_N^+} \partial_{e_1} G_{1,s}(x/N) \eta_{1,s}^{\varepsilon N}(x) ds + \int_0^t \frac{D}{N^{d-1}} \sum_{x \in \Gamma_N^+} \partial_{e_1} G_{1,s}(x/N) (\eta_{1,s}^{\varepsilon N}(x) - \eta_{1,s}(x)) ds \quad (3.38) \\ &\quad + \int_0^t \frac{D}{N^{d-1}} \sum_{x \in \Gamma_N^+} G_{1,s}(x/N) (\eta_{1,s}^{\varepsilon N}(x) - b(x/N)) ds \\ &\quad - \int_0^t \frac{1}{N^d} \sum_{x \in B_{N,\varepsilon}} G_{1,s}\left(\frac{x}{N}\right) \left(2d(\lambda_1 \eta_1^{\varepsilon N}(x) + \lambda_2 \eta_3^{\varepsilon N}(x)) \eta_0^{\varepsilon N}(x) + \eta_3^{\varepsilon N}(x) - (r+1) \eta_1^{\varepsilon N}(x)\right) ds \\ &\quad + R(N, \varepsilon, G_1, (\eta_t)_{t \in [0, T]}), \end{aligned}$$

where $R(N, \varepsilon, G_1, (\eta_t)_{t \in [0, T]})$ is a random variable satisfying $\lim_{\varepsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu^N} \left[R(N, \varepsilon, G_1, (\eta_t)_{t \in [0, T]}) \right] = 0$.

From Proposition 3 and Proposition 4, the martingale $M_{1,t}^N(\widehat{G})$ can be rewritten as

$$\begin{aligned} M_{1,t}^N(\widehat{G}) &= \langle \pi_{1,t}^N, G_{1,t} \rangle - \langle \pi_{1,0}^N, G_{1,0} \rangle - \int_0^t \langle \pi_{1,s}^N, \partial_s G_{1,s} \rangle ds \\ &\quad - \int_0^t \frac{D}{N^d} \sum_{x \in B_N \setminus \Gamma_N} \Delta G_{1,s}(x/N) \eta_{1,s}(x) ds + \int_0^t \frac{D}{N^{d-1}} \sum_{x \in \Gamma_N^-} b_1(x/N) \partial_{e_1} G_{1,s}(x/N) ds \\ &\quad - \int_0^t \frac{D}{N^{d-1}} \sum_{x \in \Gamma_N^+} \partial_{e_1} G_{1,s}(x/N) \eta_{1,s}^{\varepsilon N}(x) ds + \int_0^t \frac{D}{N^{d-1}} \sum_{x \in \Gamma_N^+} G_{1,s}(x/N) (\eta_{1,s}^{\varepsilon N}(x) - b(x/N)) ds \quad (3.39) \\ &\quad - \int_0^t \frac{1}{N^d} \sum_{x \in B_{N,\varepsilon}} G_{1,s}\left(\frac{x}{N}\right) \left(2d(\lambda_1 \eta_1^{\varepsilon N}(x) + \lambda_2 \eta_3^{\varepsilon N}(x)) \eta_0^{\varepsilon N}(x) + \eta_3^{\varepsilon N}(x) - (r+1) \eta_1^{\varepsilon N}(x)\right) ds \\ &\quad + R'(N, \varepsilon, G_1, (\eta_t)_{t \in [0, T]}), \end{aligned}$$

where $R'(N, \varepsilon, G_1, (\eta_t)_{t \in [0, T]})$ is a random variable satisfying $\lim_{\varepsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu^N} \left[R'(N, \varepsilon, G_1, (\eta_t)_{t \in [0, T]}) \right] = 0$.

On the other hand, a computation of the quadratic variation of the martingale $M_{1,t}^N(\widehat{G})$ shows that its expectation vanishes as $N \uparrow \infty$. Therefore, by Doob's inequality, for every $\delta > 0$,

$$\limsup_{N \rightarrow \infty} \mathbb{P}_{\mu^N} \left[\sup_{0 \leq t \leq T} \left| M_{1,t}^N(\widehat{G}) \right| > \delta \right] = 0, \quad (3.40)$$

Now, introduce the following approximations of the identity on B :

$$u_\varepsilon(x) = \frac{1}{(2\varepsilon)^d} \mathbf{1}_{[-\varepsilon, \varepsilon]^d}(x), \quad (3.41)$$

$$u_\varepsilon^{right}(x) = \frac{1}{\varepsilon(2\varepsilon)^{d-1}} \mathbf{1}_{[0,\varepsilon] \times [-\varepsilon,\varepsilon]^{d-1}}(x), \quad \text{and} \quad u_\varepsilon^{left}(x) = \frac{1}{\varepsilon(2\varepsilon)^{d-1}} \mathbf{1}_{[-\varepsilon,0] \times [-\varepsilon,\varepsilon]^{d-1}}(x). \quad (3.42)$$

Note that for $\varepsilon > 0$, $1 \leq i \leq 3$, $x \in B_{N,\varepsilon}$, $y \in \Gamma_N^+$, and $z \in \Gamma_N^-$,

$$\eta_i^{\varepsilon N}(x) = \frac{(2\varepsilon N)^d}{(2\varepsilon N + 1)^d} (\pi_i^N * u_\varepsilon)(x/N), \quad (3.43)$$

$$\eta_i^{\varepsilon N}(y) = \frac{(2\varepsilon N)^{d-1}}{(2\varepsilon N + 1)^{d-1}} (\pi_i^N * u_\varepsilon^{right})(y/N), \quad \text{and} \quad \eta_i^{\varepsilon N}(z) = \frac{(2\varepsilon N)^{d-1}}{(2\varepsilon N + 1)^{d-1}} (\pi_i^N * u_\varepsilon^{left})(z/N). \quad (3.44)$$

Here we will only make use of (3.43) and the first relation in (3.44) since we need to replace elements in the bulk and the right hand side boundary of the system to recover the weak formulation of the equation in the (Dirichlet; Robin) regime. For regimes where a replacement is needed on the left hand side boundary, we use the second relation in (3.44) in the same way.

We may thus replace in (3.39) and (3.40), $\eta_i^{\varepsilon N}$ by $\pi_i^N * u_\varepsilon$ in the bulk and $\eta_i^{\varepsilon N}$ by $\pi_i^N * u_\varepsilon^{right}$ at the right boundary. Therefore, for any $\delta > 0$,

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} Q_N^{\hat{\theta}} \left[\sup_{0 \leq t \leq T} \left| \mathcal{F}_{1,N,\varepsilon}^{\hat{G},t}(\hat{\pi}) \right| \geq \delta \right] = 0,$$

where for any trajectory $\hat{\pi}$ and for any $t \in [0, T]$,

$$\begin{aligned} \mathcal{F}_{1,N,\varepsilon}^{\hat{G},t}(\hat{\pi}) &= \langle \pi_{1,t}, G_{1,t} \rangle - \langle \pi_{1,0}, G_{1,0} \rangle - \int_0^t \langle \pi_{1,s}, \partial_s G_{1,s} \rangle ds \\ &\quad - \int_0^t D \langle \pi_{1,s}, \Delta G_{1,s} \rangle ds + \int_0^t \frac{D}{N^{d-1}} \sum_{x \in \Gamma_N^-} b_1(x/N) \partial_{e_1} G_{1,s}(x/N) ds \\ &\quad - \int_0^t \frac{D}{N^{d-1}} \sum_{x \in \Gamma_N^+} \partial_{e_1} G_{1,s}(x/N) (\pi_{1,s} * u_\varepsilon^{right})(x) ds \\ &\quad + \int_0^t \frac{D}{N^{d-1}} \sum_{x \in \Gamma_N^+} G_{1,s}(x/N) ((\pi_{1,s} * u_\varepsilon^{right})(x) - b(x/N)) ds \\ &\quad - \int_0^t \frac{1}{N^d} \sum_{x \in B_{N,\varepsilon}} G_{1,s}(x/N), F_1(\pi_{1,s} * u_\varepsilon(x/N), \pi_{2,s} * u_\varepsilon(x/N), \pi_{3,s} * u_\varepsilon(x/N)) ds, \end{aligned} \quad (3.45)$$

where functions F_i , $i = 1, 2, 3$ are defined in (2.19). By approximating Lebesgue integrals by Riemann sums, on the bulk and at the boundary, we obtain

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} Q_N^{\hat{\theta}} \left[\sup_{0 \leq t \leq T} \left| \mathcal{F}_{1,\varepsilon}^{\hat{G},t}(\hat{\pi}) \right| \geq \delta \right] = 0,$$

where for any trajectory $\hat{\pi}$ and for any $t \in [0, T]$,

$$\begin{aligned} \mathcal{F}_{1,\varepsilon}^{\hat{G},t}(\hat{\pi}) &= \langle \pi_{1,t}, G_{1,t} \rangle - \langle \pi_{1,0}, G_{1,0} \rangle - \int_0^t \langle \pi_{1,s}, \partial_s G_{1,s} \rangle ds \\ &\quad - D \int_0^t \langle \pi_{1,s}, \Delta G_{1,s} \rangle ds + D \int_0^t \int_{\Gamma^-} b_1(r) \partial_{e_1} G_{1,s}(r) dr ds \\ &\quad - D \int_0^t \int_{\Gamma^+} \partial_{e_1} G_{1,s}(r) (\pi_{1,s} * u_\varepsilon^{right})(r) dr ds + \int_0^t \int_{\Gamma^+} G_{1,s}(r) ((\pi_{1,s} * u_\varepsilon^{right})(r) - b(r)) dr ds \\ &\quad - \int_0^t \int_{B_{1,\varepsilon}} G_{1,s}(r), F_1(\pi_{1,s} * u_\varepsilon(r), \pi_{2,s} * u_\varepsilon(r), \pi_{3,s} * u_\varepsilon(r)) dr ds, \end{aligned} \quad (3.46)$$

with $B_{1,\varepsilon} = [-1 + \varepsilon, 1 + \varepsilon] \times \mathbb{T}^{d-1}$. By the continuity of the function $\hat{\pi} \rightarrow \mathcal{F}_{1,\varepsilon}^{\hat{G},t}(\hat{\pi})$, for each $\varepsilon > 0$, we get for any limit point $Q^{\hat{\theta}}$ of the sequence of probability measures $(Q_N^{\hat{\theta}})_{N \geq 1}$,

$$\limsup_{\varepsilon \rightarrow 0} Q^{\hat{\theta}} \left[\sup_{0 \leq t \leq T} \left| \mathcal{F}_{1,\varepsilon}^{\hat{G},t}(\hat{\pi}) \right| \geq \delta \right] = 0. \quad (3.47)$$

To conclude the proof, it remains to prove that we may replace the convolutions appearing in the functional $\mathcal{F}_{1,\varepsilon}^{\hat{G},t}$ by the associated density of the trajectory. By Proposition 5, $Q^{\hat{\theta}}$ is concentrated on paths $(\hat{\pi}(t, dr))_{t \in [0, T]} = (\hat{\rho}(t, r) dr)_{t \in [0, T]}$ which are absolutely continuous with respect to the Lebesgue measure and such that for every $1 \leq i \leq 3$, ρ_i belongs to $L^2((0, T); \mathcal{H}^1(B))$. For the replacement of the convolution with the density in the bulk, since u_ε is an approximation of the identity in $L^1(B)$ and the functions F_i are Lipschitz, the random variables

$$\int_0^t \int_{B_{1,\varepsilon}} G_{1,s}(r), F_1(\pi_{1,s} * u_\varepsilon(r), \pi_{2,s} * u_\varepsilon(r), \pi_{3,s} * u_\varepsilon(r)) dr ds$$

converge $Q^{\hat{\theta}}$ almost surely to

$$\int_0^t \int_{B_{1,\varepsilon}} G_{1,s}(r), F_1(\rho_{1,s}(r), \rho_{2,s}(r), \rho_{3,s}(r)) dr ds. \quad (3.48)$$

For the replacement of the convolution at the boundary we use the following result which follows from [7, Section 5.3]: for any $H \in \mathcal{H}^1(B)$

$$\lim_{\varepsilon \rightarrow 0} H * u_\varepsilon^{right} = \text{Tr}(H) \quad \text{a.s in } \Gamma^+. \quad (3.49)$$

For the other terms in $\mathcal{F}_{1,\varepsilon}^{\hat{G},t}$, by the dominated convergence Theorem, for almost every trajectory $(\hat{\pi}(t, dr))_{t \in [0, T]} = (\hat{\rho}(t, r) dr)_{t \in [0, T]}$ with $\rho_1 \in L^2((0, T); \mathcal{H}^1(B))$,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} D \int_0^t \int_{\Gamma^+} \partial_{e_1} G_{1,s}(r) (\pi_{1,s} * u_\varepsilon^r)(r) dr ds - \int_0^t \int_{\Gamma^+} G_{1,s}(r) ((\pi_{1,s} * u_\varepsilon^r)(r) - b(r)) dr ds \\ &= D \int_0^t \int_{\Gamma^+} \partial_{e_1} G_{1,s}(r) \text{Tr}(\rho_{1,s})(r) dr ds - \int_0^t \int_{\Gamma^+} G_{1,s}(r) (\text{Tr}(\rho_{1,s})(r) - b(r)) dr ds. \end{aligned} \quad (3.50)$$

Collecting (3.47), (3.48), (3.49) and (3.50), we obtain (3.37) and conclude the proof. \square

3.5 Uniqueness of the limit points

In order to finish the proof of the hydrodynamic limit specific to each regime we are left to show that each boundary valued problem (2.20) and (2.24) with fixed initial data admits a unique solution. For that, we use the standard method which consists in decomposing the difference of two solutions on the orthonormal basis of a well chosen eigenvectors of the Laplacian. The choice of the family of eigenvectors is not necessarily intuitive and depends on the boundary conditions of the mixed regime considered. As we are working in dimension $d \geq 1$, we will need to control integral terms on the boundary. Therefore, we will make use of the following result regarding the continuity of the trace operator. We refer to [6, Part II Section 5] for a detailed survey of the trace operator.

Theorem 3. (see [6]) Fix $1 \leq p < \infty$ and Ω an open subspace of \mathbb{R}^d with smooth boundary $\partial\Omega$. There is a constant $C_{tr} > 0$ depending only on Ω and p such that for any $\varphi \in \mathcal{C}^\infty(\bar{\Omega})$,

$$\|\varphi\|_{L^p(\partial\Omega)} \leq C_{tr} \|\varphi\|_{W^{1,p}}$$

where $\|\cdot\|_{L^p(\partial\Omega)}$ denotes the L^p norm on $\partial\Omega$ and $\|\cdot\|_{W^{1,p}}$ the Sobolev norm on Ω given by

$$\|\varphi\|_{W^{1,p}} = \left(\|\varphi\|_{L^p(\Omega)}^p + \|\nabla\varphi\|_{L^p(\Omega)}^p \right)^{1/p}$$

where

$$\|\nabla\varphi\|_{L^p(\Omega)}^p = \sum_{i=1}^d \|\partial_{e_i}\varphi\|_{L^p(\Omega)}^p.$$

Remark 3. For $p = 2$ and $\Omega = B$,

$$\|\varphi\|_{L^2(\partial\Omega)}^2 \leq \|\varphi\|_{L^2(\Omega)}^2 + \|\nabla\varphi\|_{L^2(\Omega)}^2 \quad (3.51)$$

In particular, $C_{tr} = 1$.

In the sequel we only make use of (3.51) but we stated Theorem 3 by sake of completeness.

3.5.1 Uniqueness of the solution in the (Neumann ; Robin) mixed regime

Theorem 4. *There exists a unique solution to the Neumann + Robin boundary problem (2.24).*

Proof. By Liouville's Theorem stated for instance in [6], there is a countable system $\{V_n, \alpha_n, n \geq 1\}$ of eigensolutions for the problem

$$\begin{cases} -\Delta\phi = \alpha\phi \\ \partial_{e_1}\phi|_{\Gamma} = 0 \end{cases} \quad (3.52)$$

in $\mathcal{H}^1(B)$ and containing all possible eigenvalues. The set $\{V_n, n \geq 1\}$ forms a complete, orthonormal system in the Hilbert space $L^2(B)$ and the eigenvalues $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_n \xrightarrow{n \rightarrow \infty} \infty$ have finite multiplicity. Note that for any $U, W \in \mathcal{H}^1(B)$,

$$\langle U, W \rangle_2 = \lim_{n \rightarrow \infty} \sum_{k=1}^n \langle U, V_k \rangle \langle W, V_k \rangle, \quad (3.53)$$

$$\langle \nabla U, \nabla W \rangle_2 = \lim_{n \rightarrow \infty} \sum_{k=1}^n \alpha_k \langle U, V_k \rangle \langle W, V_k \rangle, \quad (3.54)$$

$$\|U\|_{L^2(\Gamma^+)}^2 = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\int_{\Gamma^+} U(r) \check{V}_k(r) n_1(r) dS(r) \right)^2. \quad (3.55)$$

One can check that since we are working on $(-1, 1) \times \mathbb{T}^{d-1}$, for $k = (k_1, \dots, k_d) \in \mathbb{N} \times (\mathbb{N}^*)^{d-1}$,

$$V_k(x_1, \dots, x_d) = 2^{\frac{d-1}{2}} \cos\left(\frac{k_1\pi x_1}{2} + \frac{\pi}{2}\right) \prod_{i=2}^d \sin(k_i\pi x_i) \quad \text{and} \quad \alpha_k = \frac{(k_1\pi)^2}{4} + \sum_{i=2}^d k_i^2\pi^2,$$

$$\check{V}_k(x_2, \dots, x_d) = 2^{\frac{d-1}{2}} \prod_{i=2}^d \sin(k_i\pi x_i) \quad \text{and} \quad \check{\alpha}_k = \sum_{i=2}^d k_i^2\pi^2. \quad (3.56)$$

Note that by abuse of notations we indexed the family V_k by \mathbb{N}^* instead of $\mathbb{N} \times (\mathbb{N}^*)^{d-1}$ but this is not a problem because we can give an order to elements of $\mathbb{N} \times (\mathbb{N}^*)^{d-1}$.

Consider $\hat{\rho}^1$ and $\hat{\rho}^2$ two solutions of (2.24) associated to the same initial profile and for $n \in \mathbb{N}$ and $t > 0$, introduce

$$G_n(t) = \sum_{i=1}^3 \sum_{k=1}^n | \langle \rho_i^1 - \rho_i^2, V_k \rangle |^2. \quad (3.57)$$

Let us show that $\lim_{n \rightarrow \infty} G_n(t) = \|\widehat{\rho}^1 - \widehat{\rho}^2\|_2^2 =: G(t) = 0$. For that, apply the weak formulation (2.24) with V_k : for any $1 \leq i \leq 3$

$$\begin{aligned} \langle (\rho_i^1 - \rho_i^2)(t, \cdot), V_k \rangle &= -D\alpha_k \int_0^t \langle (\rho_i^1 - \rho_i^2)(s, \cdot), V_k \rangle ds + \int_0^t \langle (F_i(\widehat{\rho}^1) - F_i(\widehat{\rho}^2))(s, \cdot), V_k \rangle ds \\ &\quad - \int_0^t \int_{\Gamma^+} (\rho_i^1 - \rho_i^2)(s, r) V_k(r) n_1(r) .dS(r) ds. \end{aligned} \quad (3.58)$$

Therefore $\langle \rho_i^1(t, \cdot) - \rho_i^2(t, \cdot), V_k \rangle$ is time differentiable with derivative:

$$\begin{aligned} \partial_t \langle \rho_i^1(t, \cdot) - \rho_i^2(t, \cdot), V_k \rangle &= -D\alpha_k \langle \rho_i^1(t, \cdot) - \rho_i^2(t, \cdot), V_k \rangle + \langle F_i(\widehat{\rho}^1(t, \cdot)) - F_i(\widehat{\rho}^2(t, \cdot)), V_k \rangle \\ &\quad - \int_{\Gamma^+} (\rho_i^1 - \rho_i^2)(t, r) V_k(r) n_1(r) .dS(r) \end{aligned} \quad (3.59)$$

and so is G_n , with

$$\begin{aligned} G'_n(t) &= -2D \sum_{i=1}^3 \sum_{k=1}^n \alpha_k |\langle \rho_{i,t}^1 - \rho_{i,t}^2, V_k \rangle|^2 + 2 \sum_{i=1}^3 \sum_{k=1}^n \langle F_i(\widehat{\rho}_t^1) - F_i(\widehat{\rho}_t^2), V_k \rangle \langle \rho_{i,t}^1 - \rho_{i,t}^2, V_k \rangle \\ &\quad - 2 \sum_{i=1}^3 \sum_{k=1}^n \int_{\Gamma^+} (\rho_i^1 - \rho_i^2)(t, r) \check{V}_k(r) n_1(r) .dS(r) \langle \rho_{i,t}^1 - \rho_{i,t}^2, V_k \rangle \\ &\leq -2D \sum_{i=1}^3 \sum_{k=1}^n \alpha_k |\langle \rho_{i,t}^1 - \rho_{i,t}^2, V_k \rangle|^2 + \sum_{i=1}^3 \sum_{k=1}^n \langle F_i(\widehat{\rho}_t^1) - F_i(\widehat{\rho}_t^2), V_k \rangle^2 + G_n(t) \\ &\quad + \frac{1}{A} \sum_{i=1}^3 \sum_{k=1}^n \left(\int_{\Gamma^+} (\rho_i^1 - \rho_i^2)(t, r) \check{V}_k(r) n_1(r) .dS(r) \right)^2 + AG_n(t), \end{aligned} \quad (3.60)$$

for any $A > 0$, where we used both the Cauchy-Schwarz and (2.16) inequalities in the last line. By (3.53), (3.54) and (3.55), the right hand side of (3.60) converges to

$$-2D \|\nabla(\widehat{\rho}^1 - \widehat{\rho}^2)\|_2^2 + \sum_{i=1}^3 \|F_i(\widehat{\rho}^1) - F_i(\widehat{\rho}^2)\|_2^2 + (1+A) \|\widehat{\rho}^1 - \widehat{\rho}^2\|_2^2 + \frac{1}{A} \|\widehat{\rho}^1 - \widehat{\rho}^2\|_{L^2(\Gamma)}^2. \quad (3.61)$$

By the trace inequality (3.51),

$$\|\widehat{\rho}^1 - \widehat{\rho}^2\|_{L^2(\Gamma)}^2 \leq \|\widehat{\rho}^1 - \widehat{\rho}^2\|_{L^2(B)}^2 + \|\nabla(\widehat{\rho}^1 - \widehat{\rho}^2)\|_2^2. \quad (3.62)$$

Furthermore, using that $\widehat{\rho}^1$ and $\widehat{\rho}^2$ take their values in $[0, 1]^3$, there is a constant $C := C(\lambda_1, \lambda_2, r, d) > 0$ such that for any $\widehat{\rho}^a, \widehat{\rho}^b \in [0, 1]^3$ and $1 \leq i \leq 3$,

$$|F_i(\widehat{\rho}^a) - F_i(\widehat{\rho}^b)| \leq C \sum_{j=1}^3 |\rho_j^a - \rho_j^b|.$$

Then, by Cauchy-Schwarz's inequality, there is a constant $C' > 0$ such that for any $1 \leq i \leq 3$,

$$\|F_i(\widehat{\rho}^a) - F_i(\widehat{\rho}^b)\|_2^2 \leq C' \sum_{j=1}^3 \|\rho_j^a - \rho_j^b\|_2^2. \quad (3.63)$$

Putting together (3.61), (3.62), (3.63), taking $A > \frac{1}{D}$ and applying the dominated convergence theorem, we are left with

$$G'(t) \leq (C' + 2 + A)G(t). \quad (3.64)$$

Grönwall's inequality and the fact that $G(0) = 0$ yields $G(t) = 0$ at any time. \square

3.5.2 Uniqueness of the solution in the (Dirichlet ; Robin) mixed regime

Theorem 5. *There exists a unique solution to the Dirichlet + Robin boundary problem (2.20).*

Proof. The proof follows the same lines as the previous one except that we consider another family of eigenfunctions of the Laplacian. Indeed, consider the following boundary-eigenvalue problem for the Laplacian:

$$\begin{cases} -\Delta\phi = \gamma\phi \\ \phi(x) = 0 \text{ for } x \in \Gamma^- \times \mathbb{T}^{d-1} \\ \partial_{e_1}\phi(x) = 0 \text{ for } x \in \Gamma^+ \times \mathbb{T}^{d-1} \\ \phi \in \mathcal{H}^1(B). \end{cases} \quad (3.65)$$

Again, one can check that the countable system of eigensolutions $\{W_n, \gamma_n, n \geq 1\}$ given below (in (3.66)) for the problem (3.65) contains all possible eigenvalues and is a complete, orthonormal system in the Hilbert space $L^2(B)$, that the eigenvalues γ_n have finite multiplicity and that $0 < \gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_n \rightarrow \infty$. Furthermore, (3.53), (3.54) and (3.55) stay valid when one replaces V_k by W_k , where, for $k = (k_1, \dots, k_d) \in \mathbb{N} \times (\mathbb{N}^*)^{d-1}$,

$$W_k(x) = 2^{\frac{d-1}{2}} \left[(-1)^{k_1} \cos\left(\left(\frac{\pi}{4} + \frac{k_1\pi}{2}\right)x\right) + \sin\left(\left(\frac{\pi}{4} + \frac{k_1\pi}{2}\right)x\right) \right] \prod_{i=2}^d \sin(k_i \pi x_i) \quad (3.66)$$

with

$$\gamma_k = \left(\frac{\pi}{4} + \frac{k_1\pi}{2}\right)^2 + \sum_{i=2}^d k_i^2 \pi^2.$$

Again, by abuse of notations we have indexed the W_k 's by \mathbb{N}^* instead of $(\mathbb{N}^*)^d$.

As before, take $\hat{\rho}^1$ and $\hat{\rho}^2$ two solutions of (2.20) with same initial data and introduce

$$H_n(t) = \sum_{i=1}^3 \sum_{k=1}^n \left| \langle \rho_i^1(t, \cdot) - \rho_i^2(t, \cdot), W_k \rangle \right|^2 \quad (3.67)$$

and

$$H(t) = \|(\hat{\rho}^1 - \hat{\rho}^2)(t, \cdot)\|_2^2. \quad (3.68)$$

Using the weak formulation (2.22) with W_k , we get that for any $1 \leq i \leq 3$,

$$\begin{aligned} \langle (\rho_i^1 - \rho_i^2)(t, \cdot), W_k \rangle &= -D\gamma_k \int_0^t \langle (\rho_i^1 - \rho_i^2)(s, \cdot), W_k \rangle ds + \int_0^t \langle (F_i(\hat{\rho}^1) - F_i(\hat{\rho}^2))(s, \cdot), W_k \rangle ds \\ &\quad - \int_0^t \int_{\Gamma^+} (\rho_i^1 - \rho_i^2)(s, r) \check{W}_k(r) n_1(r) . dS(r) ds \end{aligned} \quad (3.69)$$

where the $\check{W}_k = \check{V}_k$ are defined in (3.56). Then, one concludes following exactly the same lines as the proof of Theorem 4. \square

3.5.3 Uniqueness of the solution in the other regimes

In order to prove uniqueness in the other regimes, one can follow the same classic method used above. The orthonormal basis used to decompose the difference of two solutions as in (3.57) or (3.67) then depends on the boundary conditions. For the (Dirichlet ; Dirichlet) regime, the decomposition is carried out on the eigenvectors of the following boundary-eigenvalue problem for the Laplacian:

$$\begin{cases} -\Delta\phi = \delta\phi \\ \phi \in \mathcal{H}_0^1(B). \end{cases} \quad (3.70)$$

for which the associated family of eigenvectors is

$$U_k(x_1, \dots, x_d) = 2^{\frac{d-1}{2}} \prod_{i=1}^d \sin(k_i \pi x_i),$$

with eigenvalues given by

$$\delta_k = \sum_{i=1}^d k_i^2 \pi^2$$

for $k = (k_1, \dots, k_d) \in (\mathbb{N}^*)^d$. As before, for $V, W \in L^2(B)$,

$$\langle V, W \rangle_2 = \lim_{n \rightarrow \infty} \sum_{k=1}^n \langle V, U_k \rangle_2 \langle W, U_k \rangle_2, \quad (3.71)$$

$$\langle \nabla V, \nabla W \rangle_2 = \lim_{n \rightarrow \infty} \sum_{k=1}^n \delta_k \langle V, U_k \rangle \langle W, U_k \rangle, \quad (3.72)$$

$$\|V\|_{L^2(\Gamma)}^2 = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\int_{\Gamma} V(r) \tilde{U}_k(r) n_1(r) dS(r) \right)^2 \quad (3.73)$$

where the $\tilde{U}_k = \tilde{V}_k$ are defined in (3.56).

4 Hydrostatic limit

In this section, we prove Theorem 2 which states that when the parameters $r, \lambda_1, \lambda_2, d, D$ satisfy certain conditions, starting from an invariant measure, the system converges to the stationary profile of the corresponding hydrodynamic equation. Precisely, recall that in Section 2, for $\hat{\theta} \in (\mathbb{R}^+)^3$ we defined $\mu_N^{ss}(\hat{\theta})$ as the sequence of unique invariant measures for the irreducible dynamics defined by (2.10). The hydrostatic principle states that this sequence is associated to the unique stationary solution of the hydrodynamic equation, if existence and uniqueness of such a solution hold. For the proof, we were inspired by [9] and the key argument relies on the convergence of all the trajectories satisfying the hydrodynamic equation to the unique stationary profile of these equations. In [9], the convergence of trajectories is established thanks to a comparison principle. The difficulty here is that we are dealing with a system of coupled equations and we need to define a specific order for which such a comparison principle holds. Now in [17], it has been proved that at the microscopic level, the interacting particle system is attractive only for the following order:

$$2 < 0 < 3 < 1. \quad (4.1)$$

That means that given two configurations $\eta \leq \tilde{\eta}$, it is possible to build a coupling between $(\eta_t)_{t \geq 0}$ and $(\tilde{\eta}_t)_{t \geq 0}$ where both these processes evolve according to the dynamics given by (2.10), such that $\eta_0 \leq \tilde{\eta}_0$ and almost surely, for all $t \geq 0$, $\eta_t \leq \tilde{\eta}_t$ pointwise in the sense of (4.1). Note that using the main result from [4], one can show that the system remains attractive when adding an exchange and reservoir dynamics. It is then natural to think that attractiveness also holds at the macroscopic level through a comparison principle. A comparison principle means that if two profiles are such that at a certain time, one is smaller than the other almost everywhere, then the same is true at any later time. Considering the microscopic order (4.1) it is natural to consider that the largest state at the macroscopic level corresponds to $(\rho_1 = 1, \rho_2 = 0, \rho_3 = 0)$ and the smallest state to $(\rho_2 = 1, \rho_1 = \rho_3 = 0)$. We will work under the following change of coordinates:

$$\begin{cases} \rho_1 \\ T := \rho_1 + \rho_3 \\ R := 1 - (\rho_2 + \rho_3) \end{cases} \quad (4.2)$$

which is consistent with the fact that $(1, 1, 1)$ corresponds to the largest profile $(\rho_1 = 1, \rho_2 = 0, \rho_3 = 0)$ and $(0, 0, 0)$ with the lowest one $(\rho_2 = 1, \rho_1 = \rho_3 = 0)$. In the sequel, we will say that given two profiles $\widehat{\rho}$ and $\widehat{\phi}$, $\widehat{\rho} \leq \widehat{\phi}$ if:

$$\begin{cases} \rho_1 \leq \phi_1 \\ \rho_1 + \rho_3 \leq \phi_1 + \phi_3 \\ 1 - (\rho_2 + \rho_3) \leq 1 - (\phi_2 + \phi_3) \end{cases} \quad (4.3)$$

almost everywhere. We prove a comparison principle under that change of coordinates (see Lemma 3). As previously, since we are working in any dimension $d \geq 1$ with mixed boundary conditions, some care must be taken to deal with the integral terms on Γ when proving the comparison principle. For that, we strongly rely on analytical tools stated in [21].

Under the change of coordinates (4.2), the coupled equations in the bulk become, :

$$\begin{cases} \partial_t \rho_1 = D\Delta \rho_1 + F_1(\rho_1, T, R) \\ \partial_t T = D\Delta T + H(\rho_1, T, R) \\ \partial_t R = D\Delta R + J(R) \end{cases} \quad (4.4)$$

with

$$\begin{cases} F_1(\rho_1, T, R) = 2d[(\lambda_1 - \lambda_2)\rho_1 + \lambda_2 T](R - \rho_1) + T - (r + 2)\rho_1 \\ H(\rho_1, T, R) = 2d[(\lambda_1 - \lambda_2)\rho_1 + \lambda_2 T](1 - T) - T \\ J(R) = -(r + 1)R + 1. \end{cases} \quad (4.5)$$

We will see that the comparison principle stated and proved in Lemma 3 yields the following Theorem which is used to prove Theorem 2.

Theorem 6. *Suppose that conditions (H_1) hold. Then, there exists a unique stationary solution $\bar{\rho}^{D,R}$, resp. $\bar{\rho}^{N,R}$ of (2.20), resp. (2.24). Furthermore, for any solution $\widehat{\rho}^{D,R}$, resp. $\widehat{\rho}^{N,R}$ to the boundary value problem (2.20), resp. (2.24)*

$$\lim_{t \rightarrow \infty} \sum_{i=1}^3 \|\rho_i^{D,R}(t, \cdot) - \bar{\rho}_i^{D,R}(\cdot)\|_1 = 0, \quad (4.6)$$

resp.

$$\lim_{t \rightarrow \infty} \sum_{i=1}^3 \|\rho_i^{N,R}(t, \cdot) - \bar{\rho}_i^{N,R}(\cdot)\|_1 = 0. \quad (4.7)$$

Note that this result can be equivalently formulated in the change of coordinates (4.2) and we will prove it in that setting in the next subsection.

Remark 4. *One could ask if conditions on the parameters are necessary to establish existence and uniqueness of the stationary solution of the hydrodynamic equation. Could we not generalize the result to all parameters? In order to answer that, we simulated the solutions to the equation in the (Neumann ; Neumann) regime for which the constant profile $(\rho_1 = 0, \rho_2 = \frac{r}{r+1}, \rho_3 = 0)$ is stationary. Indeed,*

$$F_1\left(0, \frac{r}{r+1}, 0\right) = F_2\left(0, \frac{r}{r+1}, 0\right) = F_3\left(0, \frac{r}{r+1}, 0\right) = 0$$

and it corresponds to the extinction regime, that is, there are no more wild insects. We observed (see below in the Appendix B) that in dimensions 1, for parameters $\lambda_1 = 1, \lambda_2 = 0.75$ and $D = r = 1$, for which conditions (H_1) are not satisfied, the solution of the hydrodynamic equation starting from $\rho_1 = 1, \rho_2 = \rho_3 = 0$ converges to a constant profile which is not $(0, \frac{r}{r+1}, 0)$ so uniqueness does not hold. Simulations confirm that Theorem 6 does not hold in all generality and that conditions on the parameters are necessary, although conditions (H_1) might not be the optimal ones.

4.1 Proof of the hydrostatic principle

Let us prove Theorem 2. We prove the first point, the second one follows in the same way. Denote $\mathcal{A}_T \subset D([0, T], (\mathcal{M}^+)^3)$ the set of trajectories $\{\widehat{\rho}(t, u)du, 0 \leq t \leq T\}$ whose density $\widehat{\rho} = (\rho_1, \rho_2, \rho_3)$ satisfies conditions (2.21) and (2.23) of the definition of a weak solution of (2.20) for some initial profile $\widehat{\rho}_0$. Consider $Q_{ss}^*(\widehat{\theta})$ a limit point of the sequence $(Q_{\mu_N^{ss}(\widehat{\theta})}^N)_{N \geq 1}$ associated to the invariant measures. By Theorem 1,

$$Q_{ss}^*(\widehat{\theta})(\mathcal{A}_T) = 1. \quad (4.8)$$

Now consider $Q_{ss}^{N_k}(\widehat{\theta})$ a subconverging sequence of $(Q_{\mu_N^{ss}(\widehat{\theta})}^N)_{N \geq 1}$. By stationarity of $\mu_N^{ss}(\widehat{\theta})$

$$\mathbb{E}_{Q_{ss}^{N_k}(\widehat{\theta})} \left(\left| \langle \widehat{\pi}^N, \widehat{G} \rangle - \langle \bar{\rho}, \widehat{G} \rangle \right| \right) = \mathbb{E}_{Q_{ss}^{N_k}(\widehat{\theta})} \left(\left| \langle \widehat{\pi}_T^N, \widehat{G} \rangle - \langle \bar{\rho}, \widehat{G} \rangle \right| \right) \quad (4.9)$$

and

$$\begin{aligned} \lim_{k \rightarrow \infty} \mathbb{E}_{Q_{ss}^{N_k}(\widehat{\theta})} \left(\left| \langle \widehat{\pi}_T^N, \widehat{G} \rangle - \langle \bar{\rho}, \widehat{G} \rangle \right| \right) &= \mathbb{E}_{Q_{ss}^*(\widehat{\theta})} \left(\left| \langle \widehat{\pi}_T, \widehat{G} \rangle - \langle \bar{\rho}, \widehat{G} \rangle \right| \mathbf{1}_{\mathcal{A}_T} \right) \\ &\leq \sum_{i=1}^3 \|G_i\|_{\infty} \sup_{\widehat{\rho}} \left(\sum_{i=1}^3 \|\rho_i(T, \cdot) - \bar{\rho}_i(\cdot)\|_1 \right) \end{aligned} \quad (4.10)$$

where the supremum is taken over trajectories which are solutions to (2.20). Then, one concludes thanks to (4.6) in Theorem 6.

4.2 Proof of Theorem 6

In order to prove Theorem 6 we first establish a comparison principle (Lemma 3). Then, we show that the difference between the largest solution and the smallest solution vanishes (Lemma 4). Using an integration by parts, it is useful to rewrite the weak formulations (2.22) and (2.22), in the following suitable forms: for any $0 \leq \tau \leq t \leq T$, for any $G \in \mathcal{C}^2([0, T] \times B)$,

$$\begin{aligned} \langle \widehat{\rho}_t, \widehat{G}_t \rangle - \langle \widehat{\rho}_\tau, \widehat{G}_\tau \rangle &= \int_{\tau}^t \langle \widehat{\rho}_s, \partial_s \widehat{G}_s \rangle ds - D \int_{\tau}^t \int_B (\nabla \widehat{\rho}_s \cdot \nabla \widehat{G}_s)(r) dr ds \\ &\quad - \int_{\tau}^t \langle \widehat{F}(\widehat{\rho}_s), \widehat{G}_s \rangle ds - D \sum_{i=1}^3 \int_{\tau}^t \int_{\Gamma^-} b_i(r) (\partial_{e_i} G_{i,s})(r) n_1(r) \cdot dS(r) ds \\ &\quad + \sum_{i=1}^3 \int_{\tau}^t \int_{\Gamma^+} G_i(r) (b_i(r) - \rho_i(s, r)) n_1(r) \cdot dS(r) ds = 0, \end{aligned} \quad (4.11)$$

and

$$\begin{aligned} \langle \widehat{\rho}_t, \widehat{G}_t \rangle - \langle \widehat{\rho}_\tau, \widehat{G}_\tau \rangle &= \int_{\tau}^t \langle \widehat{\rho}_s, \partial_s \widehat{G}_s \rangle ds - D \int_{\tau}^t \int_B (\nabla \widehat{\rho}_s \cdot \nabla \widehat{G}_s)(r) dr ds \\ &\quad - \int_{\tau}^t \langle \widehat{F}(\widehat{\rho}_s), \widehat{G}_s \rangle ds - \int_{\Gamma^+} G_i(r) (b_i(r) - \rho_i(s, r)) n_1(r) \cdot dS(r) ds = 0. \end{aligned} \quad (4.12)$$

Lemma 3. Consider $\widehat{\rho}_0^1$ and $\widehat{\rho}_0^2$ two initial profiles.

- Denote $\widehat{\rho}_t^1$ resp. $\widehat{\rho}_t^2$ the solutions to the (Dirichlet ; Robin) boundary problem (2.20) associated to each of those initial profiles. Assume that there is an $s \geq 0$ such that almost surely (in the Lebesgue measure sense), $\rho_1^1(s, u) \leq \rho_1^2(s, u)$, $T^1(s, u) \leq T^2(s, u)$ and $R^1(s, u) \leq R^2(s, u)$. Then, for all $s \geq t$, $\rho_1^1(t, u) \leq \rho_1^2(t, u)$, $T^1(t, u) \leq T^2(t, u)$ and $R^1(t, u) \leq R^2(t, u)$ almost surely.

- The same result holds when $\widehat{\rho}_t^1$ resp. $\widehat{\rho}_t^2$ are two solutions to the (Neumann ; Robin) boundary problem (2.24).

Note that Lemma 3 holds for all parameters $r, \lambda_1, \lambda_2, d$ and D , regardless of conditions (H_1) .

Proof. We prove the first point and the proof of the second one follows in the same way. Introduce

$$\begin{aligned} A(t) &= \int_B (\rho_1^1 - \rho_1^2)_+^2(t, u) du + \int_B (T^1 - T^2)_+^2(t, u) du + \int_B (R^1 - R^2)_+^2(t, u) du \\ &:= A_1(t) + A_2(t) + A_3(t) \end{aligned} \quad (4.13)$$

where x_+ denotes $\max(x, 0)$, the positive part of x . We show that $A(t) = 0$ for all $t \geq s$. Using the weak formulation (2.22) of the solution of the (Dirichlet ; Robin) boundary problem and using Lemma 7.3 and Remark 7.5 in [21], we get:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} A_1(t) &= \frac{1}{2} \frac{d}{dt} \int_B (\rho_1^1 - \rho_1^2)_+^2(t, u) du = -D \int_B \nabla(\rho_1^1 - \rho_1^2) \nabla(\rho_1^1 - \rho_1^2)_+(t, u) du \\ &+ \int_B (F_1(\widehat{\rho}^1) - F_2(\widehat{\rho}^2)) (\rho_1^1 - \rho_1^2)_+(t, u) du - \int_{\Gamma^+} (\rho_1^1 - \rho_1^2)_+^2(t, u) du. \end{aligned} \quad (4.14)$$

Using that $\nabla((\rho_1^1 - \rho_1^2)_+) = \mathbb{1}_{(\rho_1^1 - \rho_1^2) \geq 0} \nabla(\rho_1^1 - \rho_1^2)$ and that $\int_B (\rho_1^1 - \rho_1^2)_+^2(0, u) du = 0$, we are left with:

$$\begin{aligned} \frac{1}{2} \int_B (\rho_1^1 - \rho_1^2)_+^2(t, u) du &\leq - \int_0^t \int_B D \mathbb{1}_{(\rho_1^1 - \rho_1^2) \geq 0} \|\nabla(\rho_1^1 - \rho_1^2)\|_2^2(r, u) dudr \\ &+ \int_0^t \int_B (F_1(\widehat{\rho}^1) - F_2(\widehat{\rho}^2)) ((\rho_1^1 - \rho_1^2)_+)(r, u) dudr. \end{aligned} \quad (4.15)$$

Proceeding in the same way for $\int_B (T^1 - T^2)_+^2(t, u) du$ and $\int_B (R^1 - R^2)_+^2(t, u) du$ we get:

$$\begin{aligned} \frac{1}{2} \int_B (T^1 - T^2)_+^2(t, u) du &\leq - \int_0^t \int_B D \mathbb{1}_{(T^1 - T^2) \geq 0} \|\nabla(T^1 - T^2)\|_2^2(r, u) dudr \\ &+ \int_0^t \int_B (H(\widehat{\rho}^1) - H(\widehat{\rho}^2)) (T^1 - T^2)_+(r, u) dudr \end{aligned} \quad (4.16)$$

and

$$\begin{aligned} \frac{1}{2} \int_B (R^1 - R^2)_+^2(t, u) du &\leq - \int_0^t \int_B D \mathbb{1}_{(R^1 - R^2) \geq 0} \|\nabla(R^1 - R^2)\|_2^2(r, u) dudr \\ &+ \int_0^t \int_B (J(R^1) - J(R^2)) (R^1 - R^2)_+(r, u) dudr. \end{aligned} \quad (4.17)$$

Therefore,

$$\begin{aligned} \frac{1}{2} A(t) &\leq \int_0^t \int_B (F_1(\widehat{\rho}^1) - F_2(\widehat{\rho}^2)) (\rho_1^1 - \rho_1^2)_+(r, u) dudr \\ &+ \int_0^t \int_B (H(\widehat{\rho}^1) - H(\widehat{\rho}^2)) (T^1 - T^2)_+(r, u) dudr + \int_0^t \int_B (J(R^1) - J(R^2)) (R^1 - R^2)_+(r, u) dudr. \end{aligned} \quad (4.18)$$

Now let us use the explicit expressions of F_1 , J and H . We also use the following inequality: for any $C \geq 0$, $x, y \in \mathbb{R}$,

$$Cxy_+ \leq Cx_+y_+. \quad (4.19)$$

In order to avoid confusions, a squared term will always be put between brackets, while, for instance ρ_1^2 refers to the first coordinate of $\widehat{\rho}^2$. We will denote C a constant which depends on $\lambda_1, \lambda_2, r, d$ with values possibly changing from one line to the next.

$$\begin{aligned}
(F_1(\widehat{\rho}^1) - F_2(\widehat{\rho}^2))(\rho_1^1 - \rho_1^2)_+ &= [2d(\lambda_1 - \lambda_2)(R^2 - \rho_1^1 - \rho_1^2) - 2d\lambda_1 T^1 - (r + 2)](\rho_1^1 - \rho_1^2)_+^2 \\
&+ [2d\lambda_1 \rho_1^1 + 2d\lambda_2 \rho_3^1](R^1 - R^2)(\rho_1^1 - \rho_1^2)_+ + [1 + 2d\lambda_2(1 - \rho_2^2 - \rho_3^3 - \rho_1^2)](T^1 - T^2)(\rho_1^1 - \rho_1^2)_+ \\
&\leq C(\rho_1^1 - \rho_1^2)_+^2 + [2d\lambda_1 \rho_1^1 + 2d\lambda_2 \rho_3^1](R^1 - R^2)_+(\rho_1^1 - \rho_1^2)_+ \\
&+ [1 + 2d\lambda_2(1 - \rho_2^2 - \rho_3^3 - \rho_1^2)](T^1 - T^2)_+(\rho_1^1 - \rho_1^2)_+,
\end{aligned} \tag{4.20}$$

where we used (4.19) and the fact that $2d\lambda_1 \rho_1^1 + 2d\lambda_2 \rho_3^1 \geq 0$ and $1 + 2d\lambda_2(1 - \rho_2^2 - \rho_3^3 - \rho_1^2) \geq 0$ in the last line.

$$\begin{aligned}
(H(\widehat{\rho}^1) - H(\widehat{\rho}^2))(T^1 - T^2)_+ &= [2d\lambda_2 - 2d(\lambda_1 - \lambda_2)\rho_1^1 - 2d\lambda_2((T^2)^2 + (T^1)^2) - 1](T^1 - T^2)_+^2 \\
&+ [2d(\lambda_1 - \lambda_2)(1 - \rho_1^2 - \rho_3^3)](\rho_1^1 - \rho_1^2)(T^1 - T^2)_+ \\
&\leq C(T^1 - T^2)_+^2 + [2d(\lambda_1 - \lambda_2)(1 - \rho_1^2 - \rho_3^3)](\rho_1^1 - \rho_1^2)_+(T^1 - T^2)_+
\end{aligned} \tag{4.21}$$

where again, we used (4.19) in the last line, the fact that $\lambda_1 \geq \lambda_2$ and that $(1 - \rho_1^2 - \rho_3^3) \geq 0$. Finally,

$$(J(R^1) - J(R^2))(R^1 - R^2)_+ = -(r + 1)(R^1 - R^2)_+^2. \tag{4.22}$$

Collecting (4.20), (4.21) and (4.22) we are left with

$$\frac{1}{2}A(t) \leq C \int_0^t \int_B \left((\rho_1^1 - \rho_1^2)_+^2(r, u) + (T^1 - T^2)_+^2(r, u) + (R^1 - R^2)_+^2(r, u) \right) dudr = C \int_0^t A(r)dr \tag{4.23}$$

where C is a constant which depends on $\lambda_1, \lambda_2, r, d$ and by Grönwall's lemma, $A(t) = 0$. \square

Corollary 1. Denote $\widehat{\rho}^0 = (\rho_1^0, T^0, R^0)$, resp. $\widehat{\rho}^1 = (\rho_1^1, T^1, R^1)$ the weak solution of (4.4) with (Dirichlet ; Robin) boundary conditions and initial data $\rho_1^0 = T^0 = R^0 = 0$, resp. $\rho_1^1 = T^1 = R^1 = 1$. Then for every $t \geq s$, $\rho_1^0(s, \cdot) \leq \rho_1^0(t, \cdot)$, $T^0(s, \cdot) \leq T^0(t, \cdot)$ and $R^0(s, \cdot) \leq R^0(t, \cdot)$, resp. $\rho_1^1(s, \cdot) \geq \rho_1^1(t, \cdot)$, $T^1(s, \cdot) \geq T^1(t, \cdot)$ and $R^1(s, \cdot) \geq R^1(t, \cdot)$ almost surely. Furthermore, any other solution (ρ_1, T, R) of (4.4) with (Dirichlet ; Robin) boundary conditions satisfies: $\rho_1^0 \leq \rho_1 \leq \rho_1^1$, $T^0 \leq T \leq T^1$ and $R^0 \leq R \leq R^1$ almost surely.

The same result holds for $\widehat{\rho}^0 = (\rho_1^0, T^0, R^0)$ resp. $\widehat{\rho}^1 = (\rho_1^1, T^1, R^1)$ the weak solution of (4.4) with (Neumann ; Robin) boundary conditions and initial data $\rho_1^0 = T^0 = R^0 = 0$, resp. $\rho_1^1 = T^1 = R^1 = 1$.

Proof. We prove the result for the (Dirichlet ; Robin) boundary problem and for $\widehat{\rho}^0$. The proof is the same for $\widehat{\rho}^1$ and for the (Neumann ; Robin) case. Fix $s \geq 0$ and consider $\tau_s \widehat{\rho}^0 : (t, u) \mapsto \widehat{\rho}^0(t + s, u)$. $\tau_s \widehat{\rho}^0$ is the solution of (4.4) with initial condition $u \mapsto \widehat{\rho}^0(s, u)$ and almost surely in B , $\tau_s \widehat{\rho}^0(0, u) \geq (0, 0, 0) = \widehat{\rho}^0(0, u)$. Applying Lemma 3 to $\tau_s \widehat{\rho}^0$ and $\widehat{\rho}^0$ with $s = 0$ and $t = t - s$ yields $\widehat{\rho}^0(t, u) \geq \widehat{\rho}^0(s, u)$ almost surely. \square

Lemma 4. Assume conditions (H_1) are satisfied.

- Denote $\widehat{\rho}^0 = (\rho_1^0, T^0, R^0)$, resp. $\widehat{\rho}^1 = (\rho_1^1, T^1, R^1)$ the weak solution of (4.4) with (Dirichlet ; Robin) boundary conditions and with initial data $(0, 0, 0)$, resp. $(1, 1, 1)$. Then,

$$\lim_{t \rightarrow \infty} \sum_{i=1}^3 \int_B \left(|\rho_1^i(t, u) - \rho_1^0(t, u)| + |T^i(t, u) - T^0(t, u)| + |R^i(t, u) - R^0(t, u)| \right) du = 0. \tag{4.24}$$

- Denote $\widehat{\rho}^0 = (\rho_1^0, T^0, R^0)$, resp. $\widehat{\rho}^1 = (\rho_1^1, T^1, R^1)$ the weak solution of (4.4) with (Neumann ; Robin) boundary conditions and with initial data $(0, 0, 0)$, resp. $(1, 1, 1)$. Then,

$$\lim_{t \rightarrow \infty} \sum_{i=1}^3 \int_B \left(|\rho_1^i(t, u) - \rho_1^0(t, u)| + |T^i(t, u) - T^0(t, u)| + |R^i(t, u) - R^0(t, u)| \right) du = 0. \tag{4.25}$$

Proof. We start with the proof of the (Dirichlet ; Robin) regime. It is enough to show that

$$\lim_{t \rightarrow \infty} \left(\|\rho_1^1(t, \cdot) - \rho_1^0(t, \cdot)\|_2^2 + \|T^1(t, \cdot) - T^0(t, \cdot)\|_2^2 + \|R^1(t, \cdot) - R^0(t, \cdot)\|_2^2 \right) = 0. \quad (4.26)$$

Consider the eigenvalue problem for the Laplacian (3.65) and the countable system $\{W_n, \gamma_n, n \geq 1\}$ of eigensolutions for that problem. For $n \geq 1$ introduce

$$\begin{aligned} K_n(t) &= \sum_{k=1}^n |\langle R^1(t, \cdot) - R^0(t, \cdot), W_k \rangle|^2 + \sum_{k=1}^n |\langle \rho_1^1(t, \cdot) - \rho_1^0(t, \cdot), W_k \rangle|^2 + \sum_{k=1}^n |\langle T^1(t, \cdot) - T^0(t, \cdot), W_k \rangle|^2 \\ &=: A_n(t) + B_n(t) + C_n(t). \end{aligned} \quad (4.27)$$

Recall that by (3.53), one has

$$A_n(t) \xrightarrow{n \rightarrow \infty} \|R^1(t, \cdot) - R^0(t, \cdot)\|_2^2 =: A(t), \quad B_n(t) \xrightarrow{n \rightarrow \infty} \|\rho_1^1(t, \cdot) - \rho_1^0(t, \cdot)\|_2^2 =: B(t),$$

and

$$C_n(t) \xrightarrow{n \rightarrow \infty} \|T^1(t, \cdot) - T^0(t, \cdot)\|_2^2 =: C(t).$$

Let us first prove that $\lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} A_n(t) = 0$. A_n is time differentiable and the weak formulation of a solution of (4.4) with (Dirichlet ; Robin) boundary conditions yields,

$$\begin{aligned} A'_n(t) &= -2 \sum_{k=1}^n (D\gamma_k + r + 1) |\langle R_t^1 - R_t^0, W_k \rangle|^2 \\ &\quad - 2 \sum_{k=1}^n \langle R_t^1 - R_t^0, W_k \rangle \int_{\Gamma_+} (R_t^1 - R_t^0)(r) W_k(r) n_1(r) dS(r). \end{aligned} \quad (4.28)$$

Integrating this between 0 and T and using the Cauchy-Schwarz inequality twice yields

$$\begin{aligned} A_n(0) - A_n(T) &\geq \int_0^T \sum_{k=1}^n 2(D\gamma_k + r + 1) |\langle R_t^1 - R_t^0, W_k \rangle|^2 dt \\ &\quad - 2 \sqrt{\int_0^T \sum_{k=1}^n |\langle R_t^1 - R_t^0, W_k \rangle|^2 dt} \sqrt{\int_0^T \sum_{k=1}^n \left(\int_{\Gamma_+} (R_t^1 - R_t^0)(r) \check{W}_k(r) n_1(r) dS(r) \right)^2 dt}. \end{aligned}$$

Taking $n \rightarrow \infty$ and using (3.54) and (3.55) using the W_k 's and \check{W}_k 's instead of the V_k 's and \check{V}_k 's we get

$$\begin{aligned} A(0) &\geq 2(r+1) \int_0^T A(t) dt + 2D \int_0^T \tilde{A}(t) dt - 2 \sqrt{\int_0^T A(t) dt} \sqrt{\int_0^T \|R_t^1 - R_t^0\|_{L^2(\Gamma)}^2 dt} \\ &\geq 2(r+1) \int_0^T A(t) dt + 2D \int_0^T \tilde{A}(t) dt - 2 \sqrt{\int_0^T A(t) dt} \sqrt{\int_0^T A(t) dt + \int_0^T \tilde{A}(t) dt} \\ &\geq 2(r+1) \int_0^T A(t) dt + 2D \int_0^T \tilde{A}(t) dt - 2 \left(\int_0^T A(t) dt + \int_0^T \tilde{A}(t) dt \right) \\ &\geq 2r \int_0^T A(t) dt + 2(D-1) \int_0^T \tilde{A}(t) dt \end{aligned}$$

where $\tilde{A}(t) = \|\nabla(R_t^1 - R_t^0)\|_{L^2}^2$ and where we used the trace inequality (3.51) in the second inequality. Taking $T \rightarrow \infty$, and using that $D \geq 1$ we get that

$$\int_0^\infty \|R_t^1 - R_t^0\|_2^2 dt < \infty.$$

By Corollary 1, R^1 is almost surely decreasing and R^0 increasing therefore $R_t^1 - R_t^0$ is almost surely decreasing and the above inequality implies

$$\|R_t^1 - R_t^0\|_2^2 \xrightarrow[t \rightarrow \infty]{} 0.$$

We are now left to show that

$$\lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} [B_n(t) + C_n(t)] = 0. \quad (4.29)$$

We proceed following the same steps as for A_n .

$$\begin{aligned} B'_n(t) &= -2D \sum_{k=1}^n \gamma_k |\langle \rho_{1,t}^1 - \rho_{1,t}^0, W_k \rangle|^2 + 2 \sum_{k=1}^n \langle F_1(\hat{\rho}_t^1) - F_1(\hat{\rho}_t^0), W_k \rangle \langle \rho_{1,t}^1 - \rho_{1,t}^0, W_k \rangle \\ &\quad - 2 \sum_{k=1}^n \langle \rho_{1,t}^1 - \rho_{1,t}^0, W_k \rangle \int_{\Gamma_+} (\rho_{1,t}^1 - \rho_{1,t}^0)(r) \check{W}_k(r) n_1(r) dS(r). \end{aligned} \quad (4.30)$$

To lighten notations we will not write the subscript t in the computations. Let us compute the second term.

$$\begin{aligned} &\sum_{k=1}^n \langle F_1(\hat{\rho}^1) - F_1(\hat{\rho}^0), W_k \rangle \langle \rho_1^1 - \rho_1^0, W_k \rangle = 2d(\lambda_1 - \lambda_2) \sum_{k=1}^n \langle \rho_1^1(R^1 - R^0), W_k \rangle \langle \rho_1^1 - \rho_1^0, W_k \rangle \\ &+ 2d(\lambda_1 - \lambda_2) \sum_{k=1}^n \langle R^0(\rho_1^1 - \rho_1^0), W_k \rangle \langle \rho_1^1 - \rho_1^0, W_k \rangle - 2d(\lambda_1 - \lambda_2) \sum_{k=1}^n \langle (\rho_1^1)^2 + (\rho_1^0)^2, W_k \rangle \langle \rho_1^1 - \rho_1^0, W_k \rangle \\ &+ 2d\lambda_2 \sum_{k=1}^n \langle R^1(T^1 - T^0), W_k \rangle \langle \rho_1^1 - \rho_1^0, W_k \rangle + 2d\lambda_2 \sum_{k=1}^n \langle T^0(R^1 - R^0), W_k \rangle \langle \rho_1^1 - \rho_1^0, W_k \rangle \\ &- 2d\lambda_2 \sum_{k=1}^n \langle T^1(\rho_1^1 - \rho_1^0), W_k \rangle \langle \rho_1^1 - \rho_1^0, W_k \rangle - 2d\lambda_2 \sum_{k=1}^n \langle \rho_1^0(T^1 - T^0), W_k \rangle \langle \rho_1^1 - \rho_1^0, W_k \rangle \\ &+ \sum_{k=1}^n \langle T^1 - T^0, W_k \rangle \langle \rho_1^1 - \rho_1^0, W_k \rangle - (r+2) \sum_{k=1}^n |\langle \rho_1^1 - \rho_1^0, W_k \rangle|^2. \end{aligned} \quad (4.31)$$

Using Lemma 3 and the Cauchy-Schwarz inequality, we get:

$$\begin{aligned} -\frac{1}{2}B'_n(t) &\geq \sum_{k=1}^n [D\gamma_k + r + 2 - 2d(\lambda_1 - \lambda_2)] |\langle \rho_1^1 - \rho_1^0, W_k \rangle|^2 \\ &\quad - (1 + 2d\lambda_2) \sum_{k=1}^n \langle T^1 - T^0, W_k \rangle \langle \rho_1^1 - \rho_1^0, W_k \rangle \\ &\quad - 2d\lambda_1 \sqrt{A_n(t)} \sqrt{B_n(t)} - \sqrt{B_n(t)} \sqrt{\sum_{k=1}^n \left(\int_{\Gamma_+} (\rho_1^1 - \rho_1^0)(r) \check{W}_k(r) n_1(r) dS(r) \right)^2}. \end{aligned}$$

Integrating this between 0 and T and using the Cauchy-Schwarz inequality we are left with

$$\begin{aligned} \frac{1}{2}(B_n(0) - B_n(T)) &\geq \int_0^T \sum_{k=1}^n [D\gamma_k + r + 2 - 2d(\lambda_1 - \lambda_2)] |\langle \rho_1^1 - \rho_1^0, W_k \rangle|^2 dt \\ &\quad - (1 + 2d\lambda_2) \int_0^T \sqrt{B_n(t)} \sqrt{C_n(t)} dt - 2d\lambda_1 \int_0^T \sqrt{A_n(t)} \sqrt{B_n(t)} dt \\ &\quad - \sqrt{\int_0^T B_n(t) dt} \sqrt{\int_0^T \sum_{k=1}^n \left(\int_{\Gamma_+} (\rho_1^1 - \rho_1^0)(r) \check{W}_k(r) n_1(r) dS(r) \right)^2 dt}. \end{aligned} \quad (4.32)$$

Now

$$\begin{aligned}
C'_n(t) &= -2D \sum_{k=1}^n \gamma_k |\langle T^1 - T^0, W_k \rangle|^2 + 2 \sum_{k=1}^n \langle H(\hat{\rho}^1) - H(\hat{\rho}^0), W_k \rangle \langle T^1 - T^0, W_k \rangle \\
&\quad - 2 \sum_{k=1}^n \langle T^1 - T^0, W_k \rangle \int_{\Gamma_+} (T^1 - T^0)(r) \check{W}_k(r) n_1(r) dS(r).
\end{aligned} \tag{4.33}$$

Again, we compute the second term using the explicit expression of H :

$$\begin{aligned}
&\sum_{k=1}^n \langle H(\hat{\rho}^1) - H(\hat{\rho}^0), W_k \rangle \langle \rho_1^1 - \rho_1^0, W_k \rangle = 2d(\lambda_1 - \lambda_2) \sum_{k=1}^n \langle \rho_1^1 - \rho_1^0, W_k \rangle \langle T^1 - T^0, W_k \rangle \\
&+ (2d\lambda_2 - 1) \sum_{k=1}^n |\langle T^1 - T^0, W_k \rangle|^2 - 2d(\lambda_1 - \lambda_2) \sum_{k=1}^n \langle \rho_1^1 (T^1 - T^0), W_k \rangle \langle T^1 - T^0, W_k \rangle \\
&- 2d(\lambda_1 - \lambda_2) \sum_{k=1}^n \langle T^0(\rho_1^1 - \rho_1^0), W_k \rangle \langle T^1 - T^0, W_k \rangle - 2d\lambda_2 \sum_{k=1}^n \langle (T^1)^2 - (T^0)^2, W_k \rangle \langle T^1 - T^0, W_k \rangle.
\end{aligned} \tag{4.34}$$

Using Lemma 3 and the Cauchy-Schwarz inequality, we get:

$$\begin{aligned}
-\frac{1}{2}C'_n(t) &\geq \sum_{k=1}^n [D\gamma_k + 1 - 2d\lambda_2] |\langle T^1 - T^0, W_k \rangle|^2 - \sqrt{C_n(t)} \sqrt{\sum_{k=1}^n \left(\int_{\Gamma_+} (T^1 - T^0)(r) \check{W}_k(r) n_1(r) dS(r) \right)^2} \\
&\quad - 2d(\lambda_1 - \lambda_2) \sqrt{C_n(t)} \sqrt{B_n(t)}.
\end{aligned}$$

Integrating this between 0 and T and using the Cauchy-Schwarz inequality we are left with:

$$\begin{aligned}
\frac{1}{2}(C_n(0) - C_n(T)) &\geq \int_0^T \sum_{k=1}^n [D\gamma_k + 1 - 2d\lambda_2] |\langle T^1 - T^0, W_k \rangle|^2 dt - 2d(\lambda_1 - \lambda_2) \int_0^T \sqrt{C_n(t)} \sqrt{B_n(t)} dt \\
&\quad - \sqrt{\int_0^T C_n(t) dt} \sqrt{\int_0^T \sum_{k=1}^n \left(\int_{\Gamma_+} (T^1 - T^0)(r) \check{W}_k(r) n_1(r) dS(r) \right)^2 dt}.
\end{aligned} \tag{4.35}$$

Summing inequalities (4.32) and (4.35), using that B_n is uniformly bounded by a constant C_1 and C_n by a constant C_2 , we obtain

$$\begin{aligned}
\frac{1}{2}(B_n(0) - B_n(T) + C_n(0) - C_n(T)) &\geq \int_0^T \sum_{k=1}^n [D\gamma_k + r + 2 - 2d(\lambda_1 - \lambda_2)] |\langle \rho_1^1 - \rho_1^0, W_k \rangle|^2 dt \\
&\quad - \max \left(\int_0^T B_n(t) dt, \int_0^T \sum_{k=1}^n \left(\int_{\Gamma_+} (\rho_1^1 - \rho_1^0)(r) \check{W}_k(r) n_1(r) dS(r) \right)^2 dt \right) \\
&\quad + \int_0^T \sum_{k=1}^n [D\gamma_k + 1 - 2d\lambda_2] |\langle T^1 - T^0, W_k \rangle|^2 dt \\
&\quad - \max \left(\int_0^T C_n(t) dt, \int_0^T \sum_{k=1}^n \left(\int_{\Gamma_+} (T^1 - T^0)(r) \check{W}_k(r) n_1(r) dS(r) \right)^2 dt \right) \\
&\quad - 2dC_1\lambda_1 \sqrt{\int_0^T A_n(t) dt} - (1 + 2d\lambda_1)C_2 \sqrt{\int_0^T B_n(t) dt}
\end{aligned} \tag{4.36}$$

Taking n to infinity and using the dominated convergence theorem as well as the trace inequality (Theorem 3) we get:

$$\begin{aligned}
& \frac{1}{2}(B(0) - B(T) + C(0) - C(T)) \geq \min \left([D\gamma_1 + r + 1 - 2d(\lambda_1 - \lambda_2)] \int_0^T B(t)dt \right. \\
& - (2d\lambda_2 + 1)C_2 \sqrt{\int_0^T B(t)dt}, \quad (D - 1) \int_0^T \|\nabla(\rho_1^1 - \rho_1^0)\|^2 dt + [r + 1 - 2d(\lambda_1 - \lambda_2)] \int_0^T B(t)dt \\
& - (2d\lambda_2 + 1)C_2 \sqrt{\int_0^T B(t)dt} + \\
& \left. + \min \left([D\gamma_1 + 1 - 2d\lambda_2] \int_0^T C(t)dt, (D - 1) \int_0^T \|\nabla(T^1 - T^0)\|^2 dt + (1 - 2d\lambda_2) \int_0^T C(t)dt \right) \right). \tag{4.37}
\end{aligned}$$

Since conditions (H_1) hold, all the factors between the time integrals $\int_0^T B(t)dt$ and $\int_0^T C(t)dt$ are strictly positive and inequality (4.37) implies that

$$\int_0^\infty B(t)dt < \infty, \quad \text{and} \quad \int_0^\infty C(t)dt < \infty.$$

Again, by Corollary 1, ρ_1^1 and T^1 are almost surely decreasing and ρ_1^0 and T^0 increasing, therefore $\rho_1^1 - \rho_1^0$ and $T^1 - T^0$ are almost surely decreasing and the above inequalities imply

$$\|\rho_{1,t}^1 - \rho_{1,t}^0\|_2^2 \xrightarrow[t \rightarrow \infty]{} 0, \quad \text{and} \quad \|T_t^1 - T_t^0\|_2^2 \xrightarrow[t \rightarrow \infty]{} 0.$$

For the proof in the (Neumann ; Robin) regime, one proceeds in the same way, but decomposing the difference between $\widehat{\rho}^1$ and $\widehat{\rho}^0$ on the basis $(V_k)_{k \geq 1}$. \square

Now, we are able to prove Theorem 6.

Proof. Again, we focus on the (Dirichlet ; Robin) regime and the proof is the same for all the others. As said before, it is enough to prove uniqueness of a solution of

$$\begin{cases} D\Delta\rho_1 + F_1(\rho_1, T, R) = 0, & \rho_1|_{\Gamma^-} = b_1(\cdot), & \partial_{e_1}\rho_1(t, \cdot)|_{\Gamma^+} = \frac{1}{D}(b_1 - \rho_1)|_{\Gamma^+} \\ D\Delta T + H(\rho_1, T, R) = 0, & T|_{\Gamma^-} = b_1(\cdot) + b_3(\cdot), & \partial_{e_1}T(t, \cdot)|_{\Gamma^+} = \frac{1}{D}(b_1 + b_3 - \rho_1 - \rho_3)|_{\Gamma^+} \\ D\Delta R + J(R) = 0, & R|_{\Gamma^-} = 1 - b_2(\cdot) - b_3(\cdot), & \partial_{e_1}R(t, \cdot)|_{\Gamma^+} = \frac{1}{D}(\rho_2 + \rho_3 - b_1 - b_3)|_{\Gamma^+}. \end{cases} \tag{4.38}$$

(i) *Existence:* For $n \in \mathbb{N}$, define

$$U_n^0 = \{u \in B, \rho_1^0(n, \cdot) \leq \rho_1^0(n+1, \cdot), T^0(n, \cdot) \leq T^0(n+1, \cdot), R^0(n, \cdot) \leq R^0(n+1, \cdot)\} \tag{4.39}$$

and

$$U_n^1 = \{u \in B, \rho_1^1(n, \cdot) \leq \rho_1^1(n+1, \cdot), T^1(n, \cdot) \leq T^1(n+1, \cdot), R^1(n, \cdot) \leq R^1(n+1, \cdot)\}. \tag{4.40}$$

By Corollary 1, the above sets are almost sure and so is $U := \bigcap_{n \geq 0} (U_n^0 \cap U_n^1)$. On U , the sequence of profiles $\{\widehat{\rho}^1(n, \cdot), n \geq 1\}$ (resp. $\{\widehat{\rho}^0(n, \cdot), n \geq 1\}$) decreases (resp. increases) to a limit that we denote $\widehat{\rho}^+(\cdot) = (\rho_1^+(\cdot), T^+(\cdot), R^+(\cdot))$ (resp. $\widehat{\rho}^-(\cdot) = (\rho_1^-(\cdot), T^-(\cdot), R^-(\cdot))$). By Lemma 4, $\widehat{\rho}^+ = \widehat{\rho}^-$ everywhere on U so almost surely on B . Denote this profile $\underline{\rho}$ and consider $\underline{\rho}(t, \cdot)$ the solution to (4.38) with initial condition $\underline{\rho}$. Since for all $t \geq 0$, $\widehat{\rho}^0(t, \cdot) \leq \underline{\rho}(\cdot) \leq \widehat{\rho}^1(t, \cdot)$ almost surely, by Lemma 3 we have that for every $s, t \geq 0$, $\widehat{\rho}^0(t+s, \cdot) \leq \underline{\rho}(s, \cdot) \leq \widehat{\rho}^1(t+s, \cdot)$ almost surely and letting $t \rightarrow \infty$ we get that $\underline{\rho}(s, \cdot) = \underline{\rho}(\cdot)$ for all s so $\underline{\rho}$ is a solution of (4.38).

- (ii) *Uniqueness*: Note that by Lemma 3 and Corollary 1, for any profiles $\widehat{\rho}^a = (\rho_1^a, T^a, R^a)$ and $\widehat{\rho}^b = (\rho_1^b, T^b, R^b)$ satisfying (4.4) with any initial condition, for every $t > 0$

$$\begin{aligned} & \int_B \left(|\rho_1^a(t, u) - \rho_1^b(t, u)| + |T^a(t, u) - T^b(t, u)| + |R^a(t, u) - R^b(t, u)| \right) du \\ & \leq \int_B \left(|\rho_1^1(t, u) - \rho_1^0(t, u)| + |T^1(t, u) - T^0(t, u)| + |R^1(t, u) - R^0(t, u)| \right) du. \end{aligned} \quad (4.41)$$

Applying (4.41) to two stationary solutions and using Lemma 4, one gets uniqueness.

As said before, existence and uniqueness of a solution $\underline{\rho}$ of (4.38) yields existence and uniqueness of the stationary solution of (2.20). Similarly, the proof of (4.6) comes from the fact that

$$\begin{aligned} & \int_B \left(|\rho_1(t, u) - \underline{\rho}_1(u)| + |T(t, u) - \underline{T}(u)| + |R(t, u) - \underline{R}(u)| \right) du \\ & \leq \int_B \left(|\rho_1^1(t, u) - \rho_1^0(t, u)| + |T^1(t, u) - T^0(t, u)| + |R^1(t, u) - R^0(t, u)| \right) du \end{aligned} \quad (4.42)$$

where again, we applied (4.41) and the fact that the right hand side term converges to 0. \square

A Change of variable formulas

The following change of variable formulas have been established in [18, Section 5.2]. Recall that for $i, j \in \{0, 1, 2, 3\}$ and $x, y \in B_N$, $v_j(x/N) = \log(\alpha_j(x/N))$, and

$$R_{i,j}^{x,y}(\widehat{\alpha}) = \exp \left((v_j(y/N) - v_j(x/N)) - (v_i(y/N) - v_i(x/N)) \right) - 1. \quad (A.1)$$

Note that $R_{i,j}^{x,y}(\widehat{\alpha}) = O(N^{-1})$. Consider $f : \widehat{\Sigma}_N \rightarrow \mathbb{R}$ and $x, y \in B_N$.

- (i) For $(i, j) \in \{0, 1, 2, 3\}^2$ such that $i \neq j$,

$$\int_{\widehat{\Sigma}_N} \eta_i(x) \eta_j(y) f(\xi^{x,y}, \omega^{x,y}) d\nu_{\widehat{\alpha}}^N(\xi, \omega) = \int_{\widehat{\Sigma}_N} \eta_j(x) \eta_i(y) (R_{i,j}^{x,y}(\widehat{\alpha}) + 1) f(\xi, \omega) d\nu_{\widehat{\alpha}}^N(\xi, \omega). \quad (A.2)$$

- (ii) For $(i, j) \in \{(1, 2), (2, 1), (3, 0), (0, 3)\}$,

$$\int_{\widehat{\Sigma}_N} \eta_i(x) f(\sigma^x \xi, \sigma^x \omega) d\nu_{\widehat{\alpha}}^N(\xi, \omega) = \int_{\widehat{\Sigma}_N} \eta_j(x) \exp(v_i(x/N) - v_j(x/N)) f(\xi, \omega) d\nu_{\widehat{\alpha}}^N(\xi, \omega). \quad (A.3)$$

- (iii) For $(i, j) \in \{(1, 0), (0, 1), (3, 2), (2, 3)\}$,

$$\int \eta_i(x) f(\sigma^x \xi, \omega) d\nu_{\widehat{\alpha}}^N(\xi, \omega) = \int \eta_j(x) \exp(v_i(x/N) - v_j(x/N)) f(\xi, \omega) d\nu_{\widehat{\alpha}}^N(\xi, \omega). \quad (A.4)$$

- (iv) For $(i, j) \in \{(1, 3), (3, 1), (2, 0), (0, 2)\}$,

$$\int \eta_i(x) f(\xi, \sigma^x \omega) d\nu_{\widehat{\alpha}}^N(\xi, \omega) = \int \eta_j(x) \exp(v_i(x/N) - v_j(x/N)) f(\xi, \omega) d\nu_{\widehat{\alpha}}^N(\xi, \omega). \quad (A.5)$$

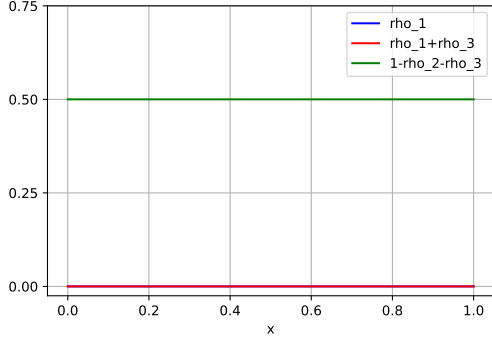


Figure 2: $\lambda_1 = 0.75$, $\lambda_2 = 0.25$, $\widehat{\rho}(0, \cdot) = (1, 0, 0)$.

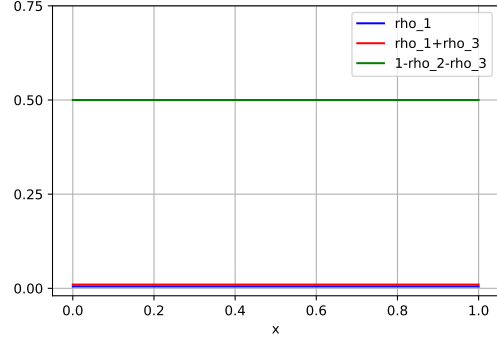


Figure 3: $\lambda_1 = 0.75$, $\lambda_2 = 0.25$, $\widehat{\rho}(0, \cdot) = (0, 1, 0)$.

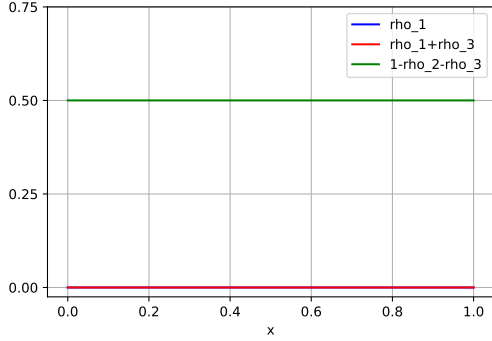


Figure 4: $\lambda_1 = 1$, $\lambda_2 = 0.75$, $\widehat{\rho}(0, \cdot) = (1, 0, 0)$.

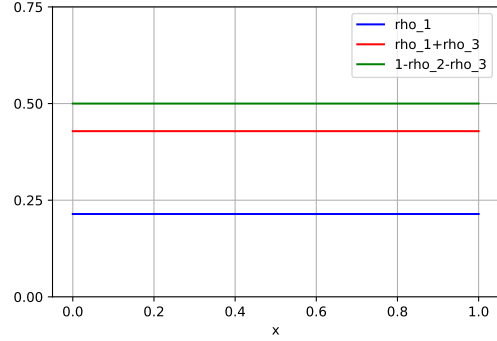


Figure 5: $\lambda_1 = 1$, $\lambda_2 = 0.75$, $\widehat{\rho}(0, \cdot) = (0, 1, 0)$.

B Simulations

The hydrodynamic equations have been simulated with an Euler explicit scheme, in dimension 1 and spatial domain $[0, 1]$, with Neumann boundary conditions. The fixed parameters chosen for the simulations are:

- Time horizon: $T=100$
- Time subdivision: $\delta_T = 5 \cdot 10^5$
- Space subdivision: $\delta_x = 100$
- $r = 1$ and $D = 1$.

In the first simulation we took $\lambda_1 = 0.75$ and $\lambda_2 = 0.25$. The conditions (H_1) are then satisfied and in Figures 2 and 3 we present the limiting profiles starting from the initial condition $(\rho_1 = 1, \rho_2 = 0, \rho_3 = 0)$, resp. $(\rho_1 = 0, \rho_2 = 1, \rho_3 = 0)$ in the system of coordinates $(\rho_1, \rho_1 + \rho_3, 1 - \rho_2 - \rho_3)$ and we observe that both limiting profiles coincide. In the second simulation we took $\lambda_1 = 1$ and $\lambda_2 = 0.75$. The conditions (H_1) are not satisfied and in Figures 4 and 5 we present the limiting profiles starting from the initial condition $(\rho_1 = 1, \rho_2 = 0, \rho_3 = 0)$, resp. $(\rho_1 = 0, \rho_2 = 1, \rho_3 = 0)$ and we observe that both limiting profiles do not coincide.

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