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Abstract

"Do Carroll particles move?" The answer depends on the characteristics of the particle such as its mass, spin, electric charge, and magnetic moment. A massive Carroll particle (closely related to fractons) does not move; its immobility follows from the Carroll boost symmetry. A strictly massless Carroll particle may propagate by following the Hall law consistently with partial breaking of their Carroll boost symmetry. In $d = 2$ space dimensions, the Carroll group has a two-fold central extension which allows us to generalize the dynamics to massive and massless particles, including anyons. The anyonic spin and magnetic moment combine with the doubly-extended structure parameterized by two Casimir invariants interpreted as intrinsic magnetization and non-commutativity parameter. The extended Carroll particle subjected to an electromagnetic background field moves following a generalized (anomalous) Hall law which includes a Zeeman force. Our theory is illustrated by massless, uncharged anyons with doubly-centrally extended structure (we call "exotic photons") which move on the horizon of a Kerr-Newman Black Hole, giving rise to an anyonic spin-Hall Effect.

Key words: Carroll symmetry, Hall motions, fractons, anyonic spin-Hall effects, motion on Black Hole horizon.

PACS numbers: 04.20.-q Classical general relativity;

Contents

I. INTRODUCTION

The Carroll algebra is a contraction of the Poincaré algebra in which the speed of light is sent to zero [2–4]. In this limit *particles do not move*, because they would otherwise violate the causality constraint imposed by special relativity [2–7].

Interest in Carrollian dynamics has long been delayed, precisely, due to this "no-go" statement. The situation started to change after physical applications were found [8–10]. The celebrated Bondi-Metzner-Sachs (BMS) group of General Relativity [11] is, for example, a conformal extension of the Carroll group [12–16]. Carrollian hydrodynamics has attracted considerable recent attention [17–19]. Further applications include plane gravitational waves and the "memory effect" [20–23, 25]. The geometric approach is nicely complemented from the Lie algebra point of view [26].

A recent development which overturns the negative assessment mentioned above comes from *condensed matter physics*. As it will be explained in sec.III B, the *limited mobility* of quasiparticles called *fractons* [24, 27] follows from that they are indeed Carroll particles [30]. Incredibly, the "original sin" of Carrollian physics has turned into a *virtue*!

A sprain to the "no-motion-for-Carroll" tenet comes from recent study of an "exotic photon" i.e., a massless Carroll particle with spin, magnetic moment but with no electric charge on the *horizon of a Kerr-Newman black hole* (see sec. IV F). In [31, 32] we found that an "exotic photon" *can move*, namely by following a *Hall-type law*.

The particular interest of our "exotic photon" is underlined by that they can in principle be imitated using appropriate meta-materials and thus studied in laboratories [25, 33].

From the theoretical point of view, the celebrated *move-or-not-to-move-dilemma* is remarkably related to the behavior under Carroll boosts,

$$\boldsymbol{x} \to \boldsymbol{x}, \qquad s \to s - \mathbf{b} \cdot \boldsymbol{x}\,, \tag{I.1}$$

where $\boldsymbol{x}$ is the position and s is Carrollian time [5, 30, 32].

This paper is devoted to a comprehensive study of Carroll dynamics and its applications to general relativity and condensed matter physics.

Historically, the archetype of Hall-type motions arises when a charged particle is put into combined perpendicular electric field and magnetic fields in such a way that the electric and the Lorentz forces compensate each other. This requires the particle to move forcelessly with the Hall velocity [34–37],

$$\left(\frac{dx^i}{dt}\right)^{\text{Hall}} = \epsilon_{ij}\frac{E_j}{B}\,, \qquad i,j=1,2, \qquad B\equiv B_3\,, \tag{I.2}$$

where the motion is considered in a $d=2$ plane normal to the direction of magnetic field $\boldsymbol{B}=(0,0,B_3)$. To fix our terminology, we shall agree that a Hall motion is one which satisfies relation (I.2).

Many years after Hall's original discovery, an anomalous Hall Effect (i.e. one with no magnetic field) was observed in certain ferromagnetic crystals and explained by an anomalous current [38]. A similar effect was proposed, even later, for massive particles with spin [39] and then further extended to light [40–42] and to chiral fermions [43–45]. Spin-Hall effects in curved space were considered in [46–49].

Reviewing 150 years of research of Hall-type Effects goes beyond our scope here. The interested reader is advised the literature, see, e.g. [35–37]. Here we just mention that a semi-classical explanation based on the two-parameter "exotic" *central extension* of the planar Galilei group [50–54] was proposed using a Berry phase–extended framework [55–57].

The rôle of *central extensions* for physics has been first recognized by Bargmann in his seminal paper [58]: for a massive non-relativistic system it is not the Galilei group itself but its 1-*parameter central extension* by the mass (called the Bargmann group [59]) which is physically relevant.

Unexpectedly, the dynamics is indeed even richer in the plane: Galilean systems admit a second ("exotic") central extension [1, 50, 60, 61] which yields *non-commutating position coordinates* [1, 51, 52, 60–62],

$$\{x,y\} = \frac{\kappa}{m^2} = \theta\,, \tag{I.3}$$

where κ is the extension parameter.

Returning to Carroll, a curious fact is that while in $d\geq 3$ space dimensions the group has no nontrivial central extension, in $d=2$ it does admit one with *two central parameters*, κ_{exo} and κ_{mag} [6, 31, 63] [1]. The doubly extended Carroll group will be denoted by $\widetilde{C}$. The

[1] We changed our notation with respect to Ref. [31]: $q_1 \to -\frac{1}{2}\kappa_{\text{mag}}$ and $q_2 \to +\frac{1}{2}\kappa_{\text{exo}}$. In this paper we

associated dynamics has been worked out in [31].

In our investigations we are led by the remarkable analogy (called in [5] a "duality") between extended planar Carrollian dynamics outlined in Sec. IV below with its "exotic" Galilean counterpart [50–53], highlighted by that the central charges combine, for both cases, with the external magnetic field, B, into an effective mass:

$$m^* = \begin{cases} m\,(1-\theta\,eB) & \text{Galilei} \\ m\left(1-\dfrac{\kappa_{\rm exo}}{m^2}\,B^*\right) & \text{Carroll} \end{cases} \tag{I.4}$$

where

$$B^* = eB + \kappa_{\rm mag}\,. \tag{I.5}$$

The comparison of these equations allow us to clarify the physical sense of the double-fold central extension: the parameter $\kappa_{\rm exo}$ in the Carrollian dynamics corresponds to the noncommutativity parameter θ of its Galilean counterpart (with the identification $\theta \leftrightarrow \kappa_{\rm exo}/m^2$) while the parameter $\kappa_{\rm mag}$ generates a magnetic-field background by shifting the physical background field[2] $eB \to B^* = eB + \kappa_{\rm mag}$.

The relationship between B^* and B in Eq. (I.5) is reminiscent the relationship between the magnetic fields $\boldsymbol{B}$ and $\boldsymbol{H}$, and the magnetization $\boldsymbol{M}$ inside a media: $\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{M}$, where we used Gaussian units. Within this interpretation, the parameter $\kappa_{\rm mag}$ corresponds to the z-component of the intrinsic magnetization of the material, $\kappa_{\rm mag} = 4\pi M_z$.

The double-extension of the Carroll system has a peculiar relation with the anomalous Hall Effect (AHE) which emerges in the absence of the background magnetic field B [38]. One could naively think that AHE is caused by the intrinsic magnetization $\kappa_{\rm mag}$ which could play the role of a background magnetic B inside the system even when the external magnetic field vanishes, $B = 0$.

However, the Casimir invariant $\kappa_{\rm mag}$ is *not* the origin of the anomalous Hall effect. It is the other invariant, $\kappa_{\rm exo}$, which is responsible for the AHE: it generates the non-commutativity in the phase space, (I.3) with $\theta \neq 0$, which induces an anomalous velocity/momentum relation (II.17a) and leads, in turn to, Hall-like motion even in the absence of the background magnetic field [see [43, 44, 51–53, 56] and also Eqs. (IV.28) below].

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provide also a "dictionary" from Souriau's notations to a more commun language [64].

[2] Notice that the field B^* in Eq. (I.4) is defined without the standard electric charge factor e. This intentional omission keeps the field B^* finite in the neutral-charge limit $e \to 0$.

In the solid-state language, the AHE is produced by the Berry curvature of occupied electronic bands which appears due to the topology of Fermi surfaces in particular ferromagnets when spin-orbit coupling is included [65].

When the effective mass vanishes, $m^* = 0$, the system becomes singular and requires Hamiltonian (alias "Faddeev-Jackiw") reduction [51–53, 66]. In the Galilean case the only allowed motions then obey the Hall law (I.2). The Carroll case is more subtle, as it is explained in sections IV and VI.

Our paper is organized as follows. After a short reminder of the Souriau framework, we recall how "exotic" (i.e., doubly extended) Galilean particles [50–52, 60, 61] are constructed by the Kirillov-Kostant-Souriau (KKS) orbit method [59, 67, 68] applied to the Bargmann group and conveniently studied in the "Eisenhart-Duval" framework [69–74].

Sec.III is devoted to non-centrally-extended Carroll particles [2–9]. The "no-motion-for-fractons" statement [24, 30] follows at once from their relation to our massive Carrollian model III B. The wave equation of free fractons, Eqn. (III.26) below, is indeed identical to the Carrollian one studied in [31].

Doubly-extended Carroll particles [6, 31, 32, 63] are introduced in sec IV and then extended to massless particles. Theorems III.1-IV.7 address the recurrent "moving or not" question with answers listed in Tables I and II, and illustrated in sec.IV F. The Carrollian model is related to relativistic anyons in sec.V. Sect.VI illustrates the general results by a comprehensive study of motion on the *horizon of a black hole* [16, 31, 32].

II. GALILEAN DYNAMICS

We start with a brief outline of the Souriau approach [59], spelled out in pedestrian terms. The fundamental object is a "presymplectic" form i.e., a closed 2-form of constant rank σ we call the *Souriau form*[3] , which is defined on a manifold $\mathcal{E}$ called the "evolution space". The classical motions are determined by the kernel of σ. The closedness property $d\sigma = 0$ implies that $\sigma = d\varpi$ locally, and the "Cartan 1-form" ϖ may be used for a variational interpretation [59, 76].

[3] Souriau calls his 2-form σ the "Lagrange form". He claims indeed to have found his framework hidden in sect. V. of the second (1811) edition of "Mécanique Analytique" of Lagrange [75]. A physically relevant example with a 3-dimensional kernel is given by *massless relativistic particles with spin* [43, 46, 59].

For a non-relativistic system, for example, $\ker\sigma$ can be 1-dimensional, parametrized by non-relativistic time, t. The restriction to $t = t_0 = \text{const.}$ of the evolution space, $\mathcal{M}_0 \equiv \mathcal{M}_{t_0}$, is the "phase space at t_0" and the restriction of σ to $\mathcal{M}_0$ is then a 2-form $\Omega_0 = \sigma_{|\mathcal{M}_0}$ which we call the "symplectic form at t_0". In a somewhat sloppy way, the choice of t_0 is neglected and one speaks of "the" phase space $\mathcal{M}$ endowed "the" symplectic form Ω.[4]

After the tacit choice of t_0 (say $t_0 = 0$), the Souriau 2-form can be split as

$$\sigma = \Omega - d\mathscr{H} \wedge dt\,, \tag{II.1}$$

where $\mathscr{H}$ is the Hamiltonian. Note that while this decomposition does depend on the choice of t_0, different choices generate equivalent Hamiltonian structures.

The next step is to introduce minimal coupling to an electromagnetic field represented by a closed 2-form on space-time, $\mathbb{F} = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = E_i dx^i \wedge dt + \frac{1}{2}F_{ij}dx^i \wedge dx^j$, where the spatial components of the field strength tensor encode the magnetic field, $F_{ij} = \epsilon_{ijk}B_k$. Then Souriau's rule is to add $\mathbb{F}$ to the free form $\sigma_0 = d\varpi_0$,

$$\sigma_0 \to \sigma = \sigma_0 + e\mathbb{F}\,, \tag{II.2}$$

where e is the electric charge. For a free non-relativistic particle with mass m, for example, $\sigma_0 = \Omega_0 - d\mathscr{H}_0 \wedge dt$ and $\mathscr{H}_0 = \boldsymbol{p}^2/2m$. In the static case, adding $e\mathbb{F}$ to σ_0 as in (II.2) modifies both the symplectic form and the Hamiltonian, respectively,

$$\Omega_0 \to \Omega = \Omega_0 + \tfrac{e}{2}F_{ij}dx^i \wedge dx^j \quad \text{and} \quad \mathscr{H}_0 \to \mathscr{H}_{em} = \frac{\boldsymbol{p}^2}{2m} + eV\,. \tag{II.3}$$

The vector gauge field $\boldsymbol{A}$ and the electrostatic potential V determine the magnetic and electric fields,[5]

$$\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}\,, \qquad \boldsymbol{\nabla}V = -\mathbf{E}\,, \tag{II.4}$$

respectively.

Poisson brackets can be introduced when $\Omega = \frac{1}{2}\Omega_{\alpha\beta}dz^\alpha \wedge dz^\beta$ is regular (where the z_α are coordinates on "the" phase space, $\mathcal{M}$). They involve the inverse matrix $\Omega^{\alpha\beta}$, $\Omega_{\alpha\beta}\Omega^{\beta\gamma} = \delta^\gamma_\alpha$,

$$\{f, g\} = \Omega^{\alpha\beta}\frac{\partial f}{\partial z^\alpha}\frac{\partial g}{\partial z^\beta}\,. \tag{II.5}$$

[4] Notice that $\sigma = \pi^*\Omega$, where π is the projection from the evolution space $\mathcal{E}$ onto $\mathcal{E}/\ker\sigma$ called the "space of motions" [59].

[5] Space and time are treated symmetrically in (II.2) but not in (II.3), which is in general *different* from the naive "minimal coupling" rule $p_\mu \to p_\mu - eA_\mu$ with $A^\mu = (A^0 \equiv V, \boldsymbol{A})$. A counter-example is provided, e.g., by an "exotic" Galilean particle [51, 52] see Sec. II B.

A. The massive Galilean model

For the sake of comparison and also for later use, we spell out how the massive Galilean model is obtained. Instead of Souriau's original approach [59], we start with the Bargmann group, i.e. the 1-parameter (non-trivial) central extension of the Galilei group by the mass [58] denoted by $\widetilde{G}$, represented by the matrices,

$$\begin{pmatrix} A_{ij} & \mathbf{b} & 0 & \mathbf{c} \\ 0 & 1 & 0 & h \\ -b_i A_{ji} & -\mathbf{b}^2/2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{II.6}$$

with $(A_{ij}) \in \mathrm{SO}(d)$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d, h, f \in \mathbb{R}$ representing rotations, boosts, space translations, time translations, and "vertical" translations along the central extension. Then the *coadjoint orbit method* of Kirillov, Kostant, and of Souriau (KKS) can be used to build the evolution space $\mathcal{E}$ [59, 67, 68].

In the Galilean case a non-relativistic spinless massive particle with internal energy E_0 and mass m is obtained by contracting the Maurer-Cartan form $g^{-1}dg$ with the basepoint $\mu_0 = (0, 0, 0, E_0, m)$ chosen in the dual Lie algebra. The KKS method then yields the Cartan resp. Souriau forms,

$$\varpi_0^{\mathrm{Gal}} = \mu_0 \cdot \Theta = -E_0 dt - m ds + \boldsymbol{p} \cdot d\boldsymbol{x} - \frac{\boldsymbol{p}^2}{2m}\, dt\,, \tag{II.7a}$$

$$\sigma_0^{\mathrm{Gal}} = d\boldsymbol{p} \wedge \Big(d\boldsymbol{x} - \frac{\boldsymbol{p}}{m} dt \Big)\,. \tag{II.7b}$$

These forms are defined on the extended evolution space with coordinates $\big(\boldsymbol{x}, \boldsymbol{p}, t, s\big)$, where t is non-relativistic time and s is the "vertical coordinate" along the central extension.

The equations of motion are given by the kernel of the Souriau form [59]; for (II.7b) one of these equations requires, in particular, the *velocity relation*

$$\frac{d\boldsymbol{x}}{dt} = \frac{\boldsymbol{p}}{m} \tag{II.8}$$

to be satisfied. The construction is completed by applying the coupling rule (II.2). Massive nonrelativistic systems are conveniently studied in the "Bargmann" framework [69–71, 73, 74].

The well-known conserved quantities associated with Galilean symmetry will not be reproduced here with the exception of the boost momentum,

$$\mathbf{g}^{\mathrm{Gal}} = m\Big(\boldsymbol{x} - \frac{\boldsymbol{p}}{m}t\Big)\,, \tag{II.9}$$

we record for later comparison with (III.10b), (III.46) and (IV.3b).

We mention for completeness that by changing the basepoint μ_0 the Souriau model can be extended to: (i) spinning particles [40, 59, 76] and (ii) to *massless particles*, in both the Relativistic and Galilean frameworks [59] including curved space. Light which propagates in an optical medium can, in particular, be viewed as a massless particle which moves in a curved manifold [41, 42, 47, 77]. The optical index of the medium, $n(\boldsymbol{x})$, allows us to define a metric

$$g_{ij} = n^2(\boldsymbol{x})\,\delta_{ij} \tag{II.10}$$

on space such that the gradient, $\boldsymbol{\nabla}(1/n)$, plays the rôle of an effective "electric" force. When $n = \text{const.}$ the motion is free and instantaneous (as seen above). For light with spin s, the non-uniformoptical index $n = n(\boldsymbol{x})$ yields an anomalous velocity term with approximate equations of motion Eqn. # (14) of [41],

$$\dot{\boldsymbol{r}} \approx \boldsymbol{p} - \frac{s}{k}\,\boldsymbol{\nabla}\left(\frac{1}{n}\right) \times \boldsymbol{p}, \qquad \dot{\boldsymbol{p}} \approx -n^3k^2\boldsymbol{\nabla}\left(\frac{1}{n}\right)\,, \tag{II.11}$$

consistently with [40].

B. Exotic Galilean dynamics in the plane

In the plane the Galilei group has been known to admit a second, central extension [50] which allows for an extended "exotic" dynamics [51–54, 60, 61]. Now we spell out the Souriau framework for the "exotic" (i.e. doubly-centrally extended) Galilean particle with mass m subjected to the background electromagnetic field. Setting $\theta = \kappa/m^2$ as in (I.3), the "exotic" Souriau 2-form

$$\sigma^{\mathrm{Gal}}_{\mathrm{exo}} = \underbrace{(dp_i - eE_i dt) \wedge (dx^i - \frac{p_i}{m}dt) \;+\; \tfrac{1}{2}eB\,\epsilon_{ij}\,dx^i \wedge dx^j}_{\sigma^{\mathrm{Gal}}_{em}} + \underbrace{\theta\,\epsilon_{ij}\,dp^i \wedge dp^j}_{exotic} \tag{II.12}$$

is defined on the evolution space $\mathcal{E} = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} = \Big\{\boldsymbol{x}, \boldsymbol{p}, t\Big\}$. It can be split as $\sigma^{\rm Gal}_{\rm exo} = \Omega^{\rm Gal}_{\rm exo} - d\mathcal{H} \wedge dt$, where

$$\Omega^{\rm Gal}_{\rm exo} = \underbrace{d\boldsymbol{p} \wedge d\boldsymbol{x} + \tfrac{e}{2} F_{ij} dx^i \wedge dx^j}_{\Omega_{em}} + \underbrace{\frac{\kappa}{2m^2}\, \epsilon_{ij}\, dp^i \wedge dp^j}_{exotic} \,, \qquad \mathscr{H}_{em} = \frac{\boldsymbol{p}^2}{2m} + eV \,. \tag{II.13}$$

The system is regular when the determinant of the symplectic form does not vanish,

$$\det(\Omega^{\rm Gal}_{ij}) = \left(\frac{m^*}{m}\right)^2 \neq 0 \,, \quad \text{where} \quad m^* \equiv m^*_G = m\big(1 - \theta\, eB\big) \,. \tag{II.14}$$

Here m^* is an *effective mass* [51, 52]. Dropping the suffix G, the Poisson brackets associated with (II.12) are, for $m^* \neq 0$,

$$\{x_i, x_j\} = \frac{m}{m^*}\, \theta \epsilon_{ij} \qquad \{x_i, p_j\} = \frac{m}{m^*}\, \delta_{ij}, \qquad \{p_i, p_j\} = \frac{m}{m^*}\, eB\, \epsilon_{ij} \,, \tag{II.15}$$

or equivalently,

$$\{x_i, x_j\} = \left(\frac{1}{1 - \theta eB}\right) \theta \epsilon_{ij}, \quad \{x_i, p_j\} = \left(\frac{1}{1 - \theta eB}\right) \delta_{ij}, \quad \{p_i, p_j\} = \left(\frac{1}{1 - \theta eB}\right) eB \epsilon_{ij} \,. \tag{II.16}$$

In particular, the coordinates do not commute when $\theta \neq 0$. The associated Hamilton equations are

$$\frac{m^*}{m} \dot{x}_i = \frac{p_i}{m} - \theta \epsilon_{ij} E_j \,, \tag{II.17a}$$

$$\dot{p}_i = eE_i + eB\epsilon_{ij} \dot{x}^j \,. \tag{II.17b}$$

Note the velocity and the momentum cease to be parallel when $\theta\, \mathbf{E} \neq 0$. For $\theta = 0$ we recover the usual expressions for a charged particle in a constant magnetic field and for $B = 0$ we get the exotic expression #(2.3) in [52]. The system becomes singular when the effective mass vanishes, $m^* = 0$, and Hamiltonian (alias "Faddeev-Jackiw") reduction [52, 66] yields [51–53]. The behavior when the effective mass changes sign is studied conveniently using chiral decomposition [78, 79]. We have:

Theorem II.1. *Eqn.* (II.17) *implies that when its effective mass* (II.14) *vanishes,*

$$m^* = 0 \,, \tag{II.18}$$

then an "exotic" Galilean particle follows the Hall law (I.2).

The Souriau form (II.12) is locally $\sigma^{\rm Gal}_{\rm exo} = d\varpi^{\rm Gal}_{\rm exo}$, where the Cartan 1-form $\varpi^{\rm Gal}_{\rm exo}$ corresponds to the "exotic" phase space Lagrangian [52] [6],

$$\mathscr{L}^{\rm Gal}_{\rm exo} = \underbrace{(\boldsymbol{p} - e\mathbf{A}\,)\cdot\frac{d\boldsymbol{x}}{dt} - \mathscr{H}_{em}\,dt}_{\mathscr{L}_{em}} + \underbrace{\frac{\theta}{2}\,\boldsymbol{p}\times\frac{d\boldsymbol{p}}{dt}}_{exotic}\,. \tag{II.19}$$

We record for further reference that in a constant B-field the *conserved angular momentum* is [52]

$$\ell = \boldsymbol{x}\times\boldsymbol{p} + \tfrac{1}{2}eB\boldsymbol{x}^2 + \tfrac{1}{2}\theta\boldsymbol{p}^2 + \chi\,, \tag{II.20}$$

where the real constant χ is identified as the *anyonic spin* [80–82]. The relation to planar vortex dynamics [83] will be studied elsewhere.

C. Galilean anyons with spin-field coupling

The planar model can be further extended. In $d = 2$ spatial dimensions, the anyonic spin is just a constant χ, and a term involving the spin-field Hamiltonian[7],

$$\mathscr{H}_{any} = -\mu\chi B\,, \tag{II.21}$$

where μ is a magnetic momentum can be added to the spinless expression (II.12). Then we end up with the Souriau form of a massive, exotic anyon with spin in an em field,

$$\sigma^{\rm Gal} = \underbrace{(dp_i - eE_i dt)\wedge(dx^i - \frac{p^i}{m}dt)}_{electric} + \underbrace{\tfrac{1}{2}eB\,\epsilon_{ij}\,dx^i\wedge dx^j}_{magnetic} + \underbrace{\frac{\theta}{2}\,\epsilon_{ij}\,dp^i\wedge dp^j}_{exotic} + \underbrace{\mu\chi dB\wedge dt}_{spin-field}\,. \tag{II.22}$$

The new term merely shifts the force,

$$eE_i \to E^*_i = eE_i + \mu\chi\partial_i B, \tag{II.23}$$

and the equations of motion become again (II.17) up to this shift.

Note the important role which is played by the magnetic field B in Eq. (II.23). The gradient of B behaves as an effective electric field which adds up to the background electric

[6] In $d = 2$ spatial dimensions, the vector product is a pseudoscalar, $\mathbf{a}\times\mathbf{b} = \epsilon_{ij}a^i b^j$.

[7] For an "exotic photon", the spin-field term (II.21) plays a particular role on the horizon, see Sec. VI. Another example is studied in [84].

field $\boldsymbol{E}$ in Eq. (II.23). The effective electric field appears from the interaction of the anyonic spin with the background magnetic field described by the Hamiltonian (II.21). The spin-field interaction plays the same role as the electrostatic potential eV in the Hamiltonian (II.3) for a usual electrically charged particle. This analogy identifies the effective electrostatic potential for the anyons, $eV_{eff} = -\mu\chi B$. The inhomogeneities in the latter produce the force on the anyon which can be described, according to Eq. (II.4), via an effective electric field $\boldsymbol{E}_{\text{eff}} = (\mu\chi/e)\boldsymbol{\nabla}B$.

When the effective mass vanishes, $m^* = 0$, the system becomes once again singular. However then the magnetic field, $B = B_{crit} = m^2/e\kappa$, is necessarily *constant* and therefore the new, μ-term drops out: the anyonic spin has *no impact* on the (Hall) motions.

Massless Galilean particle models can also be constructed by the KKS method by choosing appropriately the base point μ_0 in the dual Lie algebra. They can be spinless ("Fermat particle"– see sec.III C), or can also carry spin [41, 42, 59]. The same method works also in the relativistic case. Such a particle can be coupled to a background metric [46, 47, 49].

The "exotic" model introduced in [51, 52] has anomalous gyromagnetic factor $g = 0$. More general cases are studied in [85]. An optical medium can be viewed also as a special metric [42]; then for massless particles we get a *Spin-Hall effect for light* [40–43].

III. CARROLL DYNAMICS

After our brief review of the Galilean case, henceforth we focus our attention at Carroll particles which are indeed our main objects of our interest. The label $\{\cdot\}^{\text{Carr}}$ analogous to $\{\cdot\}^{\text{Gal}}$ will be omitted; all our formulae refer to the Carroll case in what follows.

Let us first recall that a *Carroll structure* [5, 8, 9, 13, 14, 71] is a triple $(\mathcal{M}, g, \xi)$ composed of:

1. a manifold $\mathcal{M}$ of dimension $d+1$;
2. a degenerate twice symmetric covariant tensor $(g_{\mu\nu})$ such that $\dim\ker(g_{\mu\nu}) = 1$;
3. a nowhere vanishing vector field (ξ^μ) in the kernel, $g_{\mu\nu}\xi^\nu = 0$.

Its "isometries" – i.e., transformations which preserve both $(g_{\mu\nu})$ and ξ – span an infinite-dimensional group. A finite dimensional group is obtained if, mimicking the Galilean case

[86], a connection is chosen. A *strong Carroll structure* is then obtained by completing the above definition to a quadruple $(\mathcal{M}, g_{\mu\nu}, \xi^\mu, \nabla)$ with:

4. a (non unique) connection ∇ compatible with both the "metric" $(g_{\mu\nu})$ and the vector field ξ^μ, i.e. $\nabla(g_{\mu\nu}) = 0$ and $\nabla_\mu \xi^\mu = 0$.

A Carroll manifold is obtained, for example, by constraining a Bargmann space [69–72] to a $t = t_0 = \text{const.}$ submanifold [5]. More generally, any null-submanifold is a Carroll manifold [73, 74]. The isometries of the flat strong Carroll structure are conveniently represented by $(d+2) \times (d+2)$ matrices,

$$\begin{pmatrix} A^i{}_j & 0 & c^i \\ -b_k A^k{}_j & 1 & f \\ 0 & 0 & 1 \end{pmatrix}, \tag{III.1}$$

where $(A_{ij}) \in \mathrm{SO}(d)$ is a rotation, $\mathbf{c}$ is a space translation, $\mathbf{b}$ is a boost, and f is a "Carrollian time translation". The matrix acts on the Carrollian coordinates by matrix multiplication,

$$\begin{pmatrix} \boldsymbol{x} \\ s \\ 1 \end{pmatrix} \to \begin{pmatrix} A\boldsymbol{x} + \mathbf{c} \\ s - \mathbf{b} \cdot A\boldsymbol{x} + f \\ 1 \end{pmatrix}, \tag{III.2}$$

where we denoted Carrollian time by s to distinguish it from Galilean time, t.[8] The implementation (III.2) highlights the characteristic feature of Carroll structures: unlike in the Galilean case, boosts leave the position invariant and shift instead Carrollian time, s, cf. (I.1).

The dynamics of a Carroll particle in an electromagnetic field can be derived in two steps as outlined in section II: first a free model is constructed by the KKS method II A applied to the Carroll group [5, 6, 14, 31, 59], which is then coupled to the em field by the rule (II.2).

An element $g = g(A, \mathbf{b}, \mathbf{c}, f)$ of the Carroll group is parametrized by $\frac{d(d+3)}{2} + 1$ numbers. Dual to them are the "moments" [59], which are the physical parameters that describe the state of a particle at a point. A moment is $\mu = \mu(\ell, \mathbf{g}, \boldsymbol{\pi}, m)$ where $\ell \in \mathfrak{so}(d)^*$ is the angular momentum, $\mathbf{g}$ represents the boost momentum [center of mass], $\boldsymbol{\pi}$ is the conserved momentum, and m is the mass of the particle. Notice that the mass is not an externally given quantity but a *moment* and a *Casimir invariant* associated now *not* with a central

[8] Our notation is consistent with the Bargmann framework [69–72].

extension as for Galilei/Bargmann theory, but with (Carrollian) *"time" translations* along Carrollian time s. It is a sort of "Carrollian energy" cf. (III.3).

The word "momentum" has indeed three related but somewhat different meanings here. Firstly, one has $\boldsymbol{p}$, which we define to be always $m\boldsymbol{v}$. Secondly there is the *canonical* or *generalized* momentum, defined as $\mathbf{P} = \partial\mathscr{L}/\partial\boldsymbol{x}'$. For a non relativistic particle in a magnetic field, for instance, $\mathbf{P} = m\boldsymbol{v} - e\mathbf{A}$. At last, "momentum" is also the *conserved quantity* dual to space translation symmetry (a "moment" in Souriau's terminology [59]), which we shall call here *"impulsion"* and denote by $\boldsymbol{\pi}$. The three definitions coincide in some simple cases but not always, see, e.g., (IV.3).

The dynamical description for Carroll particles is readily found by embedding the Carroll group C (III.1) as a subgroup of the Bargmann group $\widetilde{G}$ (II.6) by *"freezing out"* (Galilean) *time translations* [5, 22],

$$h = 0\,. \tag{III.3}$$

The moments allow us, e.g., to define Casimir invariants. As said above, the first one is m, the *mass* of the particle.

Carroll particles can also carry spin, defined as an additional Casimir invariant. However the planar case discussed in Sec. IV will be sufficient for our purposes here.

A. Massive Carroll particles

We consider first the free massive case $m \neq 0$ [with no spin]. The dt -terms in (II.7) are switched off and we end up (consistently with # (A.9) and (A.10) in [5]), with,

$$\varpi_0 = \boldsymbol{p}\cdot d\boldsymbol{x} - m ds\,, \tag{III.4a}$$

$$\sigma_0 = d\varpi_0 = d\boldsymbol{p}\wedge d\boldsymbol{x}\,. \tag{III.4b}$$

Note that Carrollian time, s, does not appear in the Souriau form. These forms are defined on "Carroll evolution space"

$$\mathcal{E} = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} = \Big\{\boldsymbol{x}, \boldsymbol{p}, s\Big\}\,. \tag{III.5}$$

Compared to the Galilean theory of sec.II A, not only E_0, the internal energy is lost but, even more importantly, the usual *kinetic term* $(\boldsymbol{p}^2/2m)\,dt$ in (II.3) of the Galilean theory

is *missing* from the Cartan form (III.4a) (or equivalently, from the Hamiltonian $\mathscr{H}$ (III.6) and/or from the $\mathscr{L}$ Lagrangian (III.11)).

Note that the usual $-(\boldsymbol{p}/m)ds$ is "missing" from behind the $d\boldsymbol{x}$ in σ_0, *leaving* $\boldsymbol{p}$ *and* $d\boldsymbol{x}/ds$ *unrelated*: the Galilean velocity relation (II.8) is lost. This difference between Galilei/Bargmann and Carroll systems would not be altered by adding further (electromagnetic or central extension) terms.

Splitting the Souriau form into symplectic form and Hamiltonian as in (II.1),

$$\boxed{\sigma_0 = \Omega_0 - d\mathscr{H}_0 \wedge ds \quad \text{with} \quad \Omega_0 = d\boldsymbol{p} \wedge d\boldsymbol{x} \quad \text{and} \quad \mathscr{H}_0 \equiv 0\,,} \tag{III.6}$$

shows that a free massive Carroll particle has an *identically zero Hamiltonian*. The corresponding Hamilton equations read as follows:

$$(x^i)' = \{x^i, \mathscr{H}_0\} = 0, \qquad p_i' = \{p_i, \mathscr{H}_0\} = 0\,, \tag{III.7}$$

where the prime,

$$\{\,\cdot\,\}' = d/ds\,, \tag{III.8}$$

denotes a derivative with respect to Carrollian time. Equations (III.7) imply therefore:

Theorem III.1. *A massive free Carroll particle does not move,*

$$\boldsymbol{x}(s) = \boldsymbol{x}_0, \qquad \boldsymbol{p}(s) = \boldsymbol{p}(0)\,, \tag{III.9}$$

cf. (A.11) in [5].

A free massive particle is a Carroll-symmetric by construction. The associated conserved quantities (Souriau's "moment" [59]) in $d = 2$ spatial dimensions are as follows:

$$\ell = \boldsymbol{x} \times \boldsymbol{p} \qquad \text{angular momentum,} \tag{III.10a}$$

$$\mathbf{g} = m\boldsymbol{x} \qquad \text{boost momentum,} \tag{III.10b}$$

$$\boldsymbol{\pi} = \boldsymbol{p} \qquad \text{linear momentum.} \tag{III.10c}$$

Comparison with the Galilean boosts in (II.9) shows that the difference comes, once again, from the decoupling of position and momentum. These expressions will later be compared with (IV.3). See also [30].

Corollary 1. *The conservation of the boost momentum* $\mathbf{g}$ *in* (III.10b) *implies, consistently with Theorem III.1, that a massive particle with Carroll boost symmetry can not move.*

Lagrangian formulation

The Carrollian dynamics can also be described by a Lagrangian. In the free massive case, viewing $\boldsymbol{p}$ and $\boldsymbol{x}'$ as independent variables on the evolution space $\mathcal{E}$, the Euler-Lagrange equation of the system

$$\mathscr{L}_0 = \boldsymbol{p} \cdot \boldsymbol{x}' \,, \tag{III.11}$$

confirm the "no motion" conclusion (III.9).

Coupling our Carroll particle to an electromagnetic (EM) field by the rule (II.2) amounts to

$$\boldsymbol{p} \to \boldsymbol{p} - e\mathbf{A} \quad \text{and} \quad 0 \equiv \mathscr{H}_0 \to \mathscr{H}_{em} = eV \tag{III.12}$$

which yields, in Lagrangian form,

$$\mathscr{L}_{em} = (\boldsymbol{p} - e\mathbf{A}) \cdot \boldsymbol{x}' - eV \,. \tag{III.13}$$

Variation w.r.t. $\boldsymbol{p}$ and $\boldsymbol{x}$ viewed as independent variables implies the equations of motion # (3.21a-b) of Ref. [31],

$$\boldsymbol{x}' = 0 \,, \qquad \boldsymbol{p}' = e\mathbf{E} \,. \tag{III.14}$$

Note that the magnetic field does not play any role in the derivation. To understand how this comes about, note that the variational equations are,

$$(x^i)' = 0 \quad \text{and} \quad p_i' = eE_i + eB\epsilon_{ij}(x^j)' \,. \tag{III.15}$$

The first equation here switches off the Lorentz-force in the second one. In words: "no motion" implies "no Lorentz force", allowing us to state the *"no-motion theorem"* which generalises Theorem III.1:

Theorem III.2. *A* massive *charged Carroll particle* does not move *in the electromagnetic background field :*

$$\boldsymbol{x} = \boldsymbol{x}_0 = \text{const.} \tag{III.16}$$

We notice that while the presence of the electric field $\mathbf{E} \neq 0$ produces the change in the momentum $\boldsymbol{p}$ of the particle, it does not affect its position, $\boldsymbol{x}$, in the course of the evolution of the system with Carrollean time s.

For completeness we record also the Souriau form obtained by the rule (II.2),

$$\sigma_{em} = \sigma_0 + e\mathbb{F} = d\boldsymbol{p} \wedge d\boldsymbol{x} + \tfrac{e}{2} F_{ij} dx^i \wedge dx^j + eE_i dx^i \wedge ds\,, \tag{III.17}$$

whose kernel yields again (III.14)[9].

B. Fractons

The Carrollian particles are closely related to *fractons* which, associated with immobile (or partially mobile) quasiparticles in condensed matter systems, attract much current attention [24, 27, 30]. The fracton phase of matter can be described by complex scalar field theories endowed with dipole symmetry [24].[10] The simplest Lagrangian which describes the interacting fractons has the following form:

$$\mathscr{L} = \underbrace{|\partial_s \Phi|^2 - m^2 |\Phi|^2}_{\mathscr{L}_0} - \lambda \underbrace{(\partial_i \Phi \partial_j \Phi - \Phi \partial_i \partial_j \Phi)(\partial_i \Phi^* \partial_j \Phi^* - \Phi^* \partial_i \partial_j \Phi^*)}_{self-interaction\ \mathscr{L}_{int}}\,, \tag{III.18}$$

where $\Phi(\boldsymbol{x}, s)$ is a complex scalar field [30]. We denote, for reasons which will be clear later, the time variable by s. The model (III.18) possess the Aristotelian spacetime symmetries since it is invariant under spacetime translations and spatial rotations but not under the boosts.

In the absence of interactions, $\lambda = 0$, the model (III.18) possess the trivial energy spectrum:

$$\varepsilon_{\boldsymbol{k}} = m\,, \tag{III.19}$$

implying that the group velocity of the excitations is zero:

$$\boldsymbol{v}_{\boldsymbol{k}} = \frac{\partial \varepsilon_{\boldsymbol{k}}}{\partial \boldsymbol{k}} = 0. \tag{III.20}$$

Similarly to the massive Carrollian particles, the quasiparticles described by the free Lagrangian $\mathscr{L}_0$ in Eq. (III.18) do not move [24, 27, 30].

---

[9] We also get $\mathbf{E} \cdot \boldsymbol{x}' = 0$ i.e., that the velocity is perpendicular to the electric field. However this is reduced to an identity because of the first relation in (III.14). The spinning Carroll equations in $d = 3$ spatial dimensions were first presented in [31].

[10] The term "fractons", which is now widely accepted to be associated with immobile excitations in condensed matter systems, has been earlier used to denote particles that carry fractional charges [28] and collective quantized phonon-like excitations on a substrate with a fractal structure [29]. These two objects should not be confused with the immobile excitations which are also called "fractons" [24, 27, 30].

The independence of the particle energy level on its momentum is the characteristic of the so-called flat bands, the subject which is under staring interest in condensed matter physics. The flat bands appear in the magic-angle bilayer graphene [104], which hosts unconventional superconductivity at low temperatures [105]. The fermionic models and the relation to Carroll physics of flat bands has been very recently discussed Ref. [103].

The full Lagrangian (III.18) is invariant under rigid and linear-in-the-position rotations of the phase of Φ,

$$\Phi\left(\boldsymbol{x}, s\right) \rightarrow \widehat{\Phi}\left(\boldsymbol{x}, s\right)=e^{i\left(\beta_0-\vec{\boldsymbol{\beta}}_1 \cdot \vec{\boldsymbol{x}}\right)} \Phi\left(\boldsymbol{x}, s\right) \,, \tag{III.21}$$

or infinitesimally,

$$\delta \Phi(\boldsymbol{x}, s)= \quad i \beta_0 \, \Phi(\boldsymbol{x}, s), \tag{III.22a}$$

$$\delta \Phi(\boldsymbol{x}, s)=-i\left(\boldsymbol{\beta}_1 \cdot \boldsymbol{x}\right) \Phi(\boldsymbol{x}, s) \tag{III.22b}$$

where we separated the transformations parameterized by the coordinate-independent parameters β_0 and $\boldsymbol{\beta}_1$. We have $|\widehat{\Phi}|^2=|\Phi|^2$ and $|\partial_s \widehat{\Phi}|^2=|\partial_s \Phi|^2$ from which we infer that the free Lagrangian $\mathscr{L}_0$ is left invariant.

Let us now recall the Noether theorem for a general scalar field theory: if the action $\mathscr{S}=\int \mathscr{L}\left(\phi\right) \sqrt{g} d^{D+1} \boldsymbol{x}$ (where g is the determinant of the metric on the manifold upon which ϕ is defined) changes under a transformation $\phi \rightarrow \phi+\delta \phi$ by a surface term, $\delta \mathscr{S}=\int \partial_\alpha K^\alpha d^{d+1} \boldsymbol{x}$, then we have the conserved current

$$J^\alpha=-\frac{\delta\left(\sqrt{g} \mathscr{L}\right)}{\delta\left(\partial_\alpha \phi\right)} \delta \phi+K^\alpha, \qquad \partial_\alpha J^\alpha=0 \,, \tag{III.23}$$

and thus a conserved charge,

$$Q=\int\left(-\frac{\delta\left(\sqrt{g} \mathscr{L}\right)}{\delta\left(\partial_0 \phi\right)} \delta \phi+K^0\right) d^3 \vec{x} \,. \tag{III.24}$$

Applied to the free Lagrangian $\mathscr{L}_0$ (III.18) yields the conserved quantities which correspond to the symmetry transformations (III.22a) and (III.22b), respectively,

$$q= \quad \frac{i}{2} \int\left(\Phi^* \, \partial_s \Phi-\Phi \, \partial_s \Phi^*\right) d^3 \boldsymbol{x} \,, \tag{III.25a}$$

$$\mathbf{g}=-\frac{i}{2} \int\left(\partial_s \Phi^* \boldsymbol{x} \Phi-\partial_s \Phi \, \boldsymbol{x} \Phi^*\right) d^3 \boldsymbol{x} \,, \tag{III.25b}$$

which are interpreted as the *electric charge* and the *dipole moment* carried by the field Φ.

To get more insight, we note that variation of $\mathscr{L}_0$ yields the truncated Klein-Gordon equation

$$\partial_s^2\Phi = -m^2\Phi \tag{III.26}$$

from which the kinetic term $|\boldsymbol{\nabla}\Phi|^2$ is missing[11]. Here we recognize the free quantum equation of a Carroll particle [31] mentioned in the Introduction.

The surprising forms (III.18) and (III.26) can actually be derived from the relativistic theory by a contraction familiar from the Carroll theory [5, 31]. Let us indeed start with the Klein-Gordon equation for a free relativistic particle of mass m in $1+d$ dimensions with coordinates $x^0, x^1 \dots x^d$ and metric $(-1, \underbrace{1, \dots, 1}_{d})$,

$$\partial_\mu\partial^\mu\Phi - m^2C^2\Phi = 0\,, \tag{III.27}$$

where the parameter C has a dimension of velocity and is, in general, different from the physical speed of light c. Introducing Carroll time, s, by putting $x^0 = s/C$ [5, 31] (III.27) becomes,

$$\Big[\frac{1}{C^2}\triangle - \partial_s^2 - m^2\Big]\Phi = 0\,. \tag{III.28}$$

Then letting $C \to \infty$ we end up with (III.26). Alternatively, we rewrite the Klein-Gordon Lagrangian in terms of Carroll time s,

$$\mathscr{L}_{KG} = C^2\Big(\partial_s\Phi\partial_s\Phi^* + \frac{1}{C^2}|\boldsymbol{\nabla}\Phi|^2 - m^2\Phi\Phi^*\Big)\,. \tag{III.29}$$

The overall factor C^2 can be dropped and the $C \to \infty$ limit of the bracketed quantity is $\mathscr{L}_0$ in (III.18).

We now relate fractons to Carroll particles through the relativistic Klein-Gordon theory by passing through the Bargmann space. We start with the free case. Eqn. (III.26) can indeed be solved as[12],

$$\Phi(\boldsymbol{x}, s) = e^{ims}\Psi(\boldsymbol{x})\,, \tag{III.30}$$

where $\Psi(\boldsymbol{x})$ is an arbitrary function of $\boldsymbol{x}$ only. We then note that (III.30) is defined on the Carroll slice $\mathcal{C} = \Big\{(\boldsymbol{x}, t=0, s)\Big\}$ (naturally extended to Bargmann space) as an equivariant function w.r.t. Carroll time, $\partial_s\Phi = im\Phi$.

[11] The absence of $|\boldsymbol{\nabla}\Phi|^2$ from the Lagrangian (III.18) and/or space derivatives in (III.26) is reminiscent of the absence of the kinetic term $\boldsymbol{p}^2/2m$ in the classical Carroll dynamics and anticipates the analogy to be discussed below.

[12] We choose positive mass solutions, $m > 0$.

The subgroup of the Bargmann group which leaves the slice $\mathcal{C}$ invariant is the Carroll group, (III.2). Viewing (III.30) as defined on Bargmann space, the action of the Carroll group on $\mathcal{C}$ induces,

$$\Phi(\boldsymbol{x}, s) \to e^{-im(\mathbf{b}\cdot(\boldsymbol{x}-\mathbf{c})+f)}\Phi\big(A^{-1}(\boldsymbol{x}-\mathbf{c}), s\big)\,. \tag{III.31}$$

For $A = \mathrm{Id}$, $\mathbf{c} = 0$, $f = 0$, the infinitesimal version of (III.31) is the Carroll boosts (III.22b). Dipole moment conservation thus follows from Carroll boost invariance of the massive Carroll model [2, 5, 30]. Therefore the "*fractons do not move*" statement is a rephrasing of our Theorem III.1, itself consistent with the conservation of $\mathbf{g}$ in (III.10b).

The free Lagrangian (III.18) is indeed invariant under BMS-like symmetries [30]. The transformation

$$\Phi\left(\boldsymbol{x}, s\right) \to \widehat{\Phi}\left(\boldsymbol{x}, s\right) = e^{iT(\boldsymbol{x})}\Phi\left(\boldsymbol{x}, s\right) \tag{III.32}$$

leaves indeed $\mathscr{L}_0$ manifestly invariant for any supertranslation $T(\boldsymbol{x})$.

We proceed by adding yet another remark about the celebrated "no-motion" property. Let us observe in fact that for any solution (III.30) of the wave equation (III.26) the particle density is independent of Carrollian time,

$$|\Phi(\boldsymbol{x}, s)|^2 = |\Psi(\boldsymbol{x})|^2\,, \tag{III.33}$$

which is indeed the field-theoretical analog of immobility.

This same conclusion can also be obtained by looking at the conserved dipole moment alias Carroll boost moment $\mathbf{g}$ in (III.25b). Evaluated on a solution (III.30), we get the field-theoretical analog of (III.10b),

$$\mathbf{g} = m\int |\Psi(\boldsymbol{x})|^2 \boldsymbol{x}\, d^2\boldsymbol{x}\,, \tag{III.34}$$

which does not depend on s. For completeness, we record also the conserved charge (III.25a),

$$q = -m\int |\Phi|^2 d\boldsymbol{x} = -m\int |\Psi(\boldsymbol{x})|^2 d\boldsymbol{x}\,. \tag{III.35}$$

Turning to the full Lagrangian (III.18), we note that the self-interaction term $\mathscr{L}_{int}$ is invariant under both transformations in (III.21),

$$\mathscr{L}_{int}(\Phi) = \mathscr{L}_{int}(\widehat{\Phi})\,. \tag{III.36}$$

Thus (III.22) are symmetries for (III.18), as said in Ref. [30]. Then the Noether theorem shows that the free expressions (III.25) are indeed correct for the full dynamics.

Let us now consider how the free fractons described by the Lagrangian $\mathscr{L}_0$, Eq. (III.18) with $\lambda = 0$, move when coupled to the background electromagnetic field. One immediately recognizes that due to the absence of the spatial derivative $\boldsymbol{\nabla}$ in the free Lagrangian $\mathscr{L}_0$, the vector part of the gauge potential, $\boldsymbol{A}$, does not enter the Lagrangian picture. The electrostatic potential $V = V(\boldsymbol{x})$ does appear via the formal extension of the derivative, $\partial_s \to \partial_s - ieV(\boldsymbol{x})$, but it only modifies the phase of the solution (III.30):

$$\Phi(\boldsymbol{x}, s) = e^{i(m+eV(\boldsymbol{x}))s}\Psi(\boldsymbol{x})\,, \tag{III.37}$$

without affecting the particle density (III.33). Therefore, in the simplest Lagrangian picture (III.18), free fractons do not perform a Hall-like motion neither in the massless nor in the massive case. This is the field-theoretical version of the classical "no-motion".

Then one can wonder if the BMS-type higher symmetry survives the addition of the $\lambda \neq 0$-term in (III.18). Plugging the Ansatz (III.32) into (III.18), we readily see that

$$\mathscr{L} \to \widehat{\mathscr{L}} = \mathscr{L} + \lambda\Phi^4(\partial_i\partial_j T)(\partial_i\partial_j T)\,. \tag{III.38}$$

Not being a total derivative, the shift breaks the symmetry, unless

$$\partial_i\partial_j T = 0 \tag{III.39}$$

for all i, j. Thus only terms up to linear order in $\boldsymbol{x}$, $T(\boldsymbol{x}) = \beta_0 - \vec{\boldsymbol{\beta}}_1 \cdot \vec{\boldsymbol{x}}$ survive in general, as in (III.21). Similarly, a dilation, $\Phi \to \widehat{\Phi} = \Lambda\Phi$, $\Lambda = \text{const.}$ breaks the scale invariance of the free Lagrangian,

$$\mathscr{L}_0 \to \widehat{\mathscr{L}_0} = \Lambda^2\mathscr{L}_0 \quad \text{but} \quad \mathscr{L}_{int} \quad \Rightarrow \quad \Lambda^4\mathscr{L}_{int}\,. \tag{III.40}$$

C. Massless Carroll particles

Massless Carroll particles can also be constructed along the KKS lines [14]. There are in fact, two classes. The conserved norm of their momentum $|\boldsymbol{\pi}|$ is promoted to a Casimir invariant. However massless particles with zero or with non-zero momentum behave differently, providing us with two different classes of massless Carroll particles as will be seen below.

Consider first *massless particles with non-zero conserved momentum*,

$$\boldsymbol{\pi} \neq 0\,. \tag{III.41}$$

Then the KKS algorithm yields the spinless "*Fermat particles*" of geometrical optics — which are indeed both Galilean *and* relativistic [14, 41, 42, 59]. This is because the coadjoint orbits of interest for us of the Galilei, Poincaré and Carroll groups are in fact those of their common Euclidean subgroup, obtained when boosts and time translations are dropped [42]. *Geometric Optics is Euclidean* — both in direct and historical sense [87].

"Spinless light" [alias Fermat particle] *does move*: it propagates instantaneously along the light rays of geometrical optics. In detail, the geometrical model of Ref. [59] describes a light ray by a pair $(\boldsymbol{x}, \boldsymbol{u})$, where $\boldsymbol{x}$ is an arbitrary point on the ray and $\boldsymbol{u}$ is a unit vector such that $\boldsymbol{\pi} = k\boldsymbol{u}$ is oriented along the ray. The "evolution space" $\mathcal{E} = \big\{(\boldsymbol{x}, \boldsymbol{u})\big\}$, is thus 5-dimensional.

Coadjoint action of the euclidean group.

An element $(A, \mathbf{c})$ of the euclidean group acts on $\mu_0 = (\boldsymbol{\ell}, \boldsymbol{\pi})$ in the dual euclidean algebra $\mathfrak{e}^*$ labeled as,

$$\boldsymbol{\ell} \to A\boldsymbol{\ell} - \mathbf{c} \times A\boldsymbol{\pi}\,, \qquad \boldsymbol{\pi} \to A\boldsymbol{\pi}\,. \tag{III.42}$$

The Casimir invariants are,

$$|\boldsymbol{\pi}| = k, \qquad \boldsymbol{\ell} \cdot \frac{\boldsymbol{\pi}}{k} = j\,, \tag{III.43}$$

where the non-zero constant k is Souriau's "color" [59], which corresponds to the absolute value of the conserved momentum (our "impulsion"). j is the spin of the particle. Choosing the basepoint $\mu_0 = (0, \boldsymbol{\pi}) \in \mathfrak{e}^*$, the KKS algorithm [59, 67, 68] endows $\mathcal{E}$ with the Cartan resp. Souriau forms[13] [42, 59],

$$\varpi_0 = k\,\boldsymbol{u} \cdot d\boldsymbol{x}\,, \qquad \sigma_0 = k\,d\boldsymbol{u} \wedge d\boldsymbol{x}\,. \tag{III.44}$$

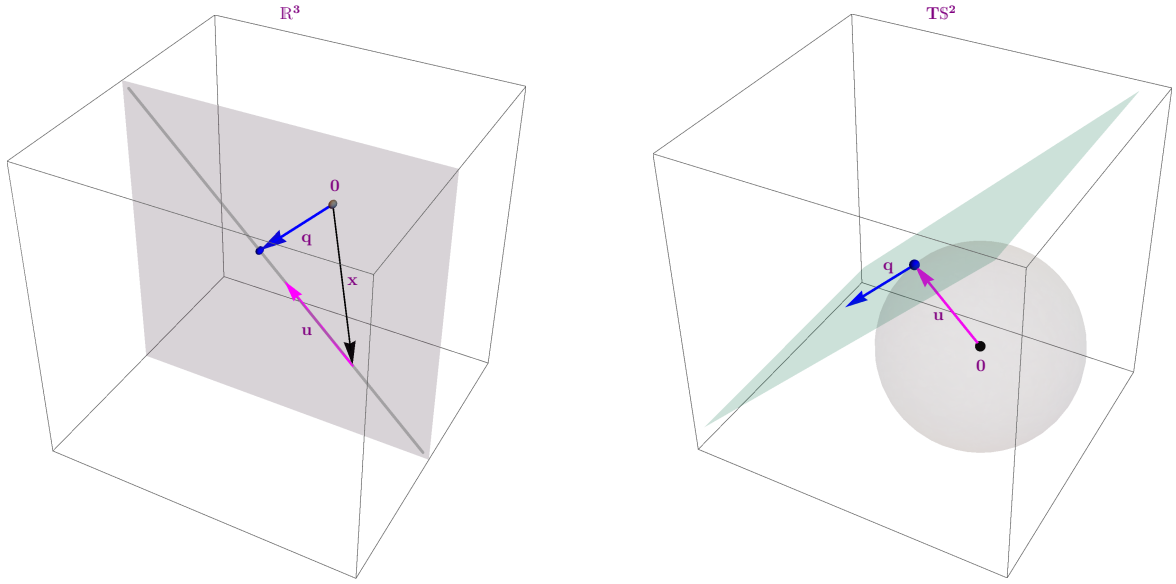
A light ray is determined by the characteristic foliation of σ_0 in (III.44), $\delta\boldsymbol{u} = 0$ and $\delta\boldsymbol{x} \propto \boldsymbol{u}$, which is indeed a straight line oriented along $\boldsymbol{u}$. The description by $(\boldsymbol{u}, \boldsymbol{x})$ is however redundant: $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ lie on same ray if they differ by a multiple of $\boldsymbol{u}$. A light ray

[13] Equivalently, with the Lagrangian $\mathscr{L}_0 = \boldsymbol{\pi} \cdot \boldsymbol{x}' = k\boldsymbol{u} \cdot \boldsymbol{x}'$, cf. (III.11).

is thus labeled ultimately by the two orthogonal vectors,

$$\boldsymbol{u} \quad \text{and} \quad \boldsymbol{q} = \boldsymbol{x} - (\boldsymbol{u}\cdot\boldsymbol{x})\boldsymbol{u}\,, \qquad \boldsymbol{u}\cdot\boldsymbol{q} = 0. \tag{III.45}$$

The "space of motions" of a Fermat particle is thus $\mathcal{M} = T\mathbb{S}^2$, the tangent bundle of the 2-sphere, depicted in Fig. 1. In conclusion:



(a) (b)

FIG. 1: *(a) A Fermat particle in flat space moves along oriented straight lines. The unit vector* $\boldsymbol{u}$ *gives its direction and the vector* $\boldsymbol{q}$ *points to its closest point to a (chosen) origin.* $\boldsymbol{x}$ *is any point on the ray. (b) The space of motions is thus the tangent bundle of the unit 2-sphere, endowed with k-times its canonical symplectic form.*

Theorem III.3. *A* free, mass, spin and charge-less *Carroll particle is equivalent*[14] *to the previously considered euclidean "Fermat particle" and* moves, *instantaneously, along the oriented straight lines of geometrical optics.*

An intuitive explanation is that there is no canonical quantity with the dimension of velocity in that theory, so either they do not move at all, or they move instantaneously. Since we know that they do move (the trajectory is a ray), it has to be instantaneous.

Another way to see how such "spinless light" may escape the "no-motion-for-Carroll" tenet is to recall that Carroll boosts, (I.1), leave the free Souriau-form σ_0 in (III.44) obviously

[14] Up to interchanging Galilean and Carrollian times, $t \leftrightarrow s$.

invariant; the associated conserved quantity is [14],

$$\mathbf{g} \equiv 0 \tag{III.46}$$

cf. eqn. # (VI.17) of [14], consistently with putting $m = 0$ into the conserved quantities (III.10). Thus, unlike in the massive case, we can *not* now infer from the conservation of the boost momentum that $\boldsymbol{x}$ is fixed [30] — consistently with what we had found above.

Extension to spin in $d \geq 3$ space dimensions (which allows us to study the "spin-Hall effect for light", see [40–42]) is not considered in this paper.

Anyonic spin in $d = 2$ will be studied in sec. IV. The description of Carroll particles can be extended to curved space [7].

Coupling to a background electromagnetic field.

A free particle constructed by the KKS method can be coupled to an electromagnetic field by the rule (II.2). The Carrollian version can be obtained by taking the Carrollian limit [88]. One starts with the relativistic electromagnetic tensor

$$\mathbb{F} = F_{ij} dx^i \wedge dx^j + \frac{E_i}{c} dx^i \wedge dx^0 \,. \tag{III.47}$$

However, rescaling the electric field as $E_i \mapsto \widetilde{E}_i = E_i/(cC)$ one gets, after introducing Carrollian time by $x^0 = s/C$ (so that $C \to \infty$ is the Carrollian limit),

$$\mathbb{F} = F_{ij} dx^i \wedge dx^j + \widetilde{E_i} dx^i \wedge ds \tag{III.48}$$

where Carroll time, s, was replaced the usual time coordinate[15].

The electric and magnetic fields defined above satisfy the Carroll-Maxwell laws [5], either in their electric version,

$$\begin{cases} \epsilon_{ijk} \partial_j E^k + \partial_s B^i = 0 \,, & \partial_i B^i = 0 \\ \partial_s E^i = 0 \,, & \partial_i E^i = 0 \end{cases} \qquad \text{electric} \tag{III.49}$$

or in their magnetic version,

$$\begin{cases} \partial_s B^i = 0 \,, & \partial_i B^i = 0 \\ \epsilon_{ijk} \partial_j B^k - \partial_s E^i = 0 \,, & \partial_i E^i = 0 \end{cases} \qquad \text{magnetic} \tag{III.50}$$

respectively [16].

[15] The same result is obtained [5, 73] by switching to light-cone coordinates u and v. Then the restriction to the $u =$ const. null hypersurface is again (III.48) with $\widetilde{E}_i = E_i - F_{iz}$.

[16] When compared to Galilean electromagnetism [5, 88] the words "electric" and "magnetic" are inter-

On may wonder then what would happen when we couple (hypothetical) massless Carroll particle with non-zero "color" k in (III.43), and endowed also with an electric charge to electromagnetism . Such particles would be described by the Souriau 2-form

$$\sigma = k\, du_i \wedge dx^i + \frac{e}{2} F_{ij} dx^i \wedge dx^j + eE_i dx^i \wedge ds\,, \tag{III.51}$$

whose integral leaves $(\delta \boldsymbol{x}, \delta \boldsymbol{u}, \delta s) \in \ker \sigma$ are given by,

$$\begin{cases} eE_i \delta x^i = 0\,, \\ \delta \boldsymbol{x} \propto \boldsymbol{u}\,, \\ k\delta \boldsymbol{u} - e\mathbf{E}\delta s + e\mathbf{B} \times \delta \boldsymbol{x} = 0\,, \end{cases} \tag{III.52}$$

augmented with the constraint $|\boldsymbol{u}| = 1$.

Such a model looks similar to a massive, non relativistic charged particle in a background electromagnetic field, — but with the color k playing the role of a mass, and $\boldsymbol{u}$ playing that of its momentum. The only formal difference is that here $\boldsymbol{u}$ is a unit vector.

The upper two relations in (III.52) imply that the propagation is orthogonal to the electric field,

$$E_i u^i = 0\,. \tag{III.53}$$

The system (III.52) has several kinds of motions. First, one should note that such particles cannot exist in a pure electric field, as the equations of motion become inconsistent for $k = \text{const.}$. Then, as for standard Newtonian mechanics of charged particles subject to the Lorentz force in a pure magnetic field, we get helical motion. The motion becomes straight if the initial velocity is parallel to the magnetic field while the helical motion degenerates into a circle if the initial velocity is orthogonal to it.

While pure electric fields are forbidden, the system (III.52) admits Hall-type solutions. To see this let us assume, for example, that the fields $\mathbf{E} = \text{const.}$ and $B = \text{const.}$ are as in the usual Hall scenario where $\mathbf{E}$ is in the plane and B is the third component of the magentic field perpendicular to the plane. Let us assume, moreover, that the ray's direction is fixed,

$$\boldsymbol{u} = \text{const.} \neq 0\,. \tag{III.54}$$

changed, underlining the dual nature of Carrollian vs Galilean physics. These equations are written in $d = 3$ spatial dimensions.

Then from the third equation in (III.52) we deduce that

$$\frac{\delta x^i}{\delta s} = \epsilon^{ij}\frac{E_j}{B}, \tag{III.55}$$

which means Hall motion. We notice also that for vanishing electric field, $\mathbf{E} = 0$ but $B \neq 0$, we get the circular motion.

D. Immobility and Carroll boost symmetry

(Im)mobility and Carroll symmetry are closely related [2, 5, 30], as said before. We first recall the description by a $2d+1$ dimensional evolution space $\mathcal{E} = \big\{(\boldsymbol{x}, \boldsymbol{v}, s)\big\}$, which is invariant under Carroll boosts. The Cartan 1-form is [17],

$$\varpi = A_i(\boldsymbol{x}, \boldsymbol{v})dx^i + B_i(\boldsymbol{x}, \boldsymbol{v})dv^i - \mathscr{H}(\boldsymbol{x}, \boldsymbol{v}, s)ds\,, \tag{III.56}$$

which is the most general 1-form on $\mathcal{E}$ with A and B do not depend explicitly on time. Then Carroll boost-invariance requires the Cartan 1-form to obey $L_X\varpi = df$ for some f and $X = \beta_i x^i \partial_s$, implying $dL_X\varpi = 0$. Then

$$dL_X\varpi = \beta_j\left(\partial_{x^i}\mathscr{H}dx^j \wedge dx^i - \partial_{v^i}\mathscr{H}dx^j \wedge dv^i\right) + \boldsymbol{\beta}\cdot\boldsymbol{x}\big(\partial_s\partial_{x^i}\mathscr{H}ds \wedge dx^i + \partial_s\partial_{v^i}\mathscr{H}ds \wedge dv^i\big) \tag{III.57}$$

which vanishes if [18]

$$\partial_{x^i}\mathscr{H} = 0 \quad \text{and} \quad \partial_{v^i}\mathscr{H} = 0\,. \tag{III.58}$$

Thus $\mathscr{H}(s)ds$ is exact and can be dropped (i.e., $\mathscr{H}$ is gauge-equivalent to $\mathscr{H}_0 = 0$). The system can thus be described by a Carroll-boosts-invariant Cartan 1-form. Thus assuming that the system is regular, the equations of motion are,

$$\frac{d\boldsymbol{x}}{ds} = 0 \quad \text{and} \quad \frac{d\boldsymbol{v}}{ds} = 0\,.$$

In conclusion, the invariance under Carroll boosts prevents motion [2, 5, 30].

Non-trivial motion requires to introduce an external electromagnetic field, which breaks the Carroll boost invariance.

[17] In a more standard language, $\mathscr{H}$ here is in fact the Hamiltonian.

[18] Note that the second condition is the one which forbids the kinetic $m\boldsymbol{v}^2/2$ term in the Lagrangian of Carroll systems.

Remarkably, the *limited mobility of Carroll particles is related to partial breaking of the Carroll symmetry.* Indeed, if one considers a planar system with a broken Carroll boost symmetry in (say the x) direction, then we see from (III.57) that $\mathscr{H}$ must not depend on the orthogonal (i.e. y) variable (as well as satisfy some restrictions on the time dependence of $\mathscr{H}$). There is no restriction on $\mathscr{H}$ in the remaining (i.e. x) direction, leaving the possibility for motion in that direction.

The allowed resp. forbidden directions can also be identified using the conserved quantities. For the example considered above, the conserved quantity associated to the remaining boost symmetry is $i_X\varpi = \beta_x x\mathscr{H}(x)$ which forbids motion in the unbroken x-boost direction. Motion is instead allowed in turn along the broken Carroll boost in the y-direction. In conclusion,

Theorem III.4. *A regular dynamical system which is invariant under Carroll boosts in at least 2 directions cannot move. If all but one Carroll boosts are broken, motion is forbidden in the unbroken, but is possible in the broken boosts' directions.*

	free Carroll no charge $e = 0$	charged $e \neq 0$ in EM field	in curved metric
$m \neq 0$	no motion	no motion	no motion
$m = 0, k = 0$	no motion	Hall motions	no motion
$m = 0, k \neq 0$ Fermat particle	instantaneous motion straight lines	helical/Hall motions	geometric optics

TABLE I: *Motion of an unextended Carroll particle with vanishing spin, $s = 0$.*

E. "No-particle" motion

So far we assumed that the momentum $\boldsymbol{\pi}$ does not vanish. However a curious degenerate case arises for

$$\boldsymbol{\pi} = 0, \tag{III.59}$$

for which the KKS algorithm would yield *identically zero* Cartan (and thus Souriau) forms,

$$\varpi_{m=0} \equiv 0 \quad \text{and} \quad \sigma_{m=0} \equiv 0\,, \tag{III.60}$$

respectively, — and thus *no dynamics* at all. However coupling "it" naively to electromagnetism by the rule (II.2) yields ... a Carroll dynamics with a "purely electromagnetic Souriau form ,

$$\sigma^{(0)} = \frac{e}{2} F_{ij} dx^i \wedge dx^j + eE_i dx^i \wedge ds, \tag{III.61}$$

whose characteristic foliation, in $2+1$ dimensions, yields neverthess non-trivial equations of motion. Remarkably, the electric charge drops out (as long as it does not vanish), leaving us with

Corollary 2. *Coupling minimally a charged degenerate "no-particle" with vanishing conserved momentum, $\boldsymbol{\pi} = 0$ (III.59) and therefore no free dynamics, (III.60), the system will move by following the Hall law (I.2):*

$$(x^i)' = \epsilon_{ij} \frac{E^j}{B} \,. \tag{III.62}$$

How can a "no-particle" have non-trivial motion ? The mystery is clarified by observing that belonging to the kernel of $\sigma^{(0)}$ means simply that the *combined forces vanish along the trajectories* and perpendicularly to the "electric" direction the motion is thus "free".

Alternatively, from the symmetry point of view, a Carroll boost $X = \boldsymbol{\beta} \cdot \boldsymbol{x}\, \partial_s$ with $\boldsymbol{\beta} = (\beta_x, \beta_y)$ is implemented on the Souriau 2-form (III.61) as,

$$L_X \sigma^{(0)} = (\mathbf{E} \times \boldsymbol{\beta}) dx \wedge dy \,, \tag{III.63}$$

whose vanishing is consistent with Thm III.4 above.

The weird-looking "naked" electromagnetic Souriau form (III.61) can actually be derived from the massive 2-form (III.17). Replacing $\boldsymbol{p}$ by $m\boldsymbol{v}$ we get,

$$\sigma^{(m)} = m d\boldsymbol{v} \wedge d\boldsymbol{x} + \tfrac{e}{2} F_{ij} dx^i \wedge dx^j + eE_i dx^i \wedge ds \,. \tag{III.64}$$

Then letting here $m \to 0$ switches off the first term, leaving us just with (III.61). Using a more conventional language, putting $\boldsymbol{p} = m\boldsymbol{v}$ into the massive Carroll Lagrangian with EM coupling (III.13) to get

$$\mathscr{L}_{em} = (m\boldsymbol{v} - e\mathbf{A}) \cdot \boldsymbol{x}' - eV \,, \tag{III.65}$$

whose variational equations are, instead of (III.15)

$$m(x^i)' = 0 \quad \text{and} \quad mv_i' = eE_i + eB\epsilon_{ij}(x^j)' \,. \tag{III.66}$$

Then letting here $m \to 0$ switches off the first equation *leaving* $(x^i)'$ *undetermined*, while the second one becomes the Hall law, (III.62).

F. Carroll "no motion": interpretation through General Relativity

Carroll structures implying Carroll dynamics are plentiful in General Relativity: in fact, any null hypersurface of a Lorentzian spacetime carries a Carroll structure [5, 73, 74].

Finding null hypersurfaces amounts to finding null geodesics; Carroll structures are thus characterized by the motions of massless particles. Let us look at such particles in flat spacetime. Their geodesic motion is given by the set of first order equations,

$$\dot{X}^\mu = P^\mu \,, \tag{III.67}$$

$$\dot{P}^\mu = 0 \,, \tag{III.68}$$

where the dot denotes the derivative with respect to some affine parameter. Pick coordinates (x, y, z, t) such that the momentum for outgoing massless particles be $P = (0, 0, 1, 1)$, yielding the spacetime diagram in figure 2.

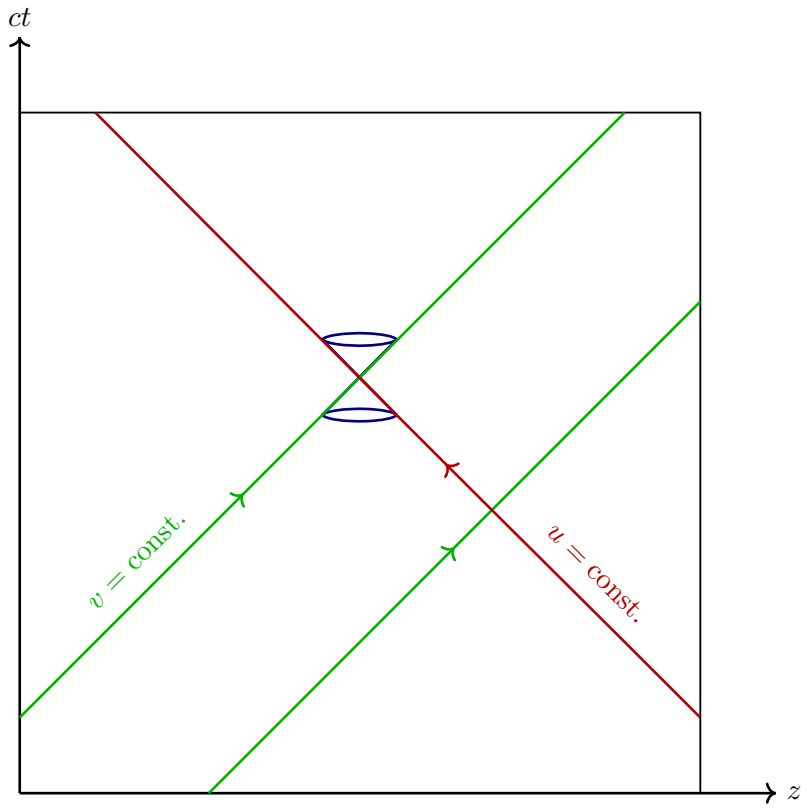


FIG. 2: Spacetime diagram. The light-cone is in **blue**. Photon trajectories are either ***outgoing*** ($\dot{z} > 0$) depicted in **green** or ***ingoing*** depicted in **red**.

In- and outgoing null geodesics define new coordinates that are well suited for the study

of null trajectories. Let us then introduce null coordinates u and v such that

$$u = \frac{ct + z}{2}\,, \qquad v = \frac{ct - z}{2}\,. \tag{III.69}$$

The next step is to compute the geodesic equation of motion (III.67) for a massless particle, for an "outgoing" particle with momentum P chosen as above, for instance. We readily see from (III.69) that $\dot{u} = 1$ and $\dot{v} = 0$, meaning that outgoing massless particles have their trajectories along a $v =$ const. hypersurface, while u is a natural evolution parameter. Conversely, an ingoing massless particle would follow $u =$ const. hypersurfaces, with v the evolution parameter. The situation is depicted in figure 2. The other equations of an outgoing particle are, from (III.67), given by,

$$\frac{dx}{du} = 0\,, \qquad \frac{dy}{du} = 0\,, \tag{III.70}$$

where we used u as the evolution parameter.

What is said above allows us to understand why the "no-motion" statement for Carroll particles is so natural: the $v =$ const. hypersurfaces define the Carroll structure. Such a structure would be described by 2 spatial coordinates x and y and by another one "along" the null direction, u. Geometrically, u would be what we called the "Carroll time". The outgoing massless particles stay on these Carroll structures, and their equations of motion are thus induced from the ambient spacetime (III.70). Thus there is "no motion" in the (x, y) plane, as the induced momentum vanishes in the spatial directions, as expected for Carroll dynamics.

For the Carroll structures that are found as null hypersurfaces in lorentzian spacetimes, the intrinsic statement that Carroll particles do not move mean simply that particles move in the null direction that defines the hypersurface, i.e. , that it follows a null geodesic in the ambient spacetime. The massless particle has all its momentum directed toward the null direction, and has no spare momentum left to deviate in the other directions. In that sense, the statement of the Red Queen makes perfect sense for Carrollian physics.

So in a sense, we can think of Carroll structures/dynamics as an effective description of phenomena which follow null geodesics. This also implies that finding Carrollian motion means finding potential deviations from null geodesics.

G. On the role of the mass

Comparison of the behaviors in the massive/massless cases yields further insight: Eqn. (III.14) implies that the mass $m \neq 0$ "glues" our particle to a fixed position. However in the massless limit the particle is "liberated" and moves as in (III.62). The system has obviously the half-broken Carrollian boost symmetry found earlier in Theorem III.4. This non-analytical behavior of the mass-induced spatial "confinement" of the Carroll particle has similar features with the conventional Hall effect (I.2) in two spatial dimensions.

The motion of a particle with a nonvanishing (even infinitesimally small, but nonzero) electric charge $e \neq 0$ is radically different from that of a particle with an exactly zero charge $e = 0$. In the first case, the electromagnetic field generates the Hall motion of the particle (I.2), while in the second case, the background field does not affect the motion at all. Moreover, despite the paramount requirement of the nonvanishing electric charge $e \neq 0$, the Hall motion does not depend explicitly on the actual value of the electric charge e (I.2). In the Hall conditions (with perpendicular electric and magnetic fields), the motion is non-analytic in e as $e = 0$ and $e \neq 0$ cases are qualitatively different.

The "no-particle" motion of the Carroll particle is a non-analytical phenomenon but now in the particle mass rather than in the electric charge: the motion is radically different for massive ($m \neq 0$) and massless ($m = 0$) cases. The second similarity appears because the actual value of the mass does not enter the solution of equations of motion for the Carroll particle, similarly to the electric charge in the Hall effect.

IV. DOUBLY-EXTENDED CARROLL DYNAMICS

A. Free massive doubly-extended particle

A fact which escaped attention until recently is that that, in $d = 2$ space dimensions, the Carroll group admits a *two-parameter central extension* [6, 31, 63]. Its central charges shall be denoted by κ_{exo} and κ_{mag}, respectively whose non-vanishing will henceforth be assumed. Such additional charges are allowed by the theory and they may play an important role for planar systems with Carroll symmetry.

The doubly-extended group, $\widetilde{\mathcal{C}}$, can be represented by 6×6 matrices (Eqn # (2.9) in [31])

to be compared with (III.1),

$$\begin{pmatrix} A^i{}_j & 0 & c^i & 0 & \epsilon^i{}_k b^k \\ -b_k A^k{}_j & 1 & f & 0 & A_{\rm exo} \\ 0 & 0 & 1 & 0 & 0 \\ c_k \epsilon^k{}_l A^l{}_j & 0 & A_{\rm mag} & 1 & -(f + \mathbf{b}\cdot\mathbf{c}) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} , \tag{IV.1}$$

where $A_{\rm mag}$ and $A_{\rm exo}$ are new parameters associated with the double extension. The extended algebra has a 8-parameter moment map, $\mu(\ell, \mathbf{g}, \boldsymbol{\pi}, m, \kappa_{\rm mag}, \kappa_{\rm exo})$ with commutators,

$$\begin{aligned} &[J_3, K_i] = \epsilon_{ij} K_j \,, \ [K_i, K_j] = \epsilon_{ij} A_{\rm exo} \,, \ [J_3, P_i] = \epsilon_{ij} P_j \,, \qquad [K_i, P_j] = \delta_{ij} P_0 \,, \\ &[J_3, P_0] = 0 \,, \qquad [K_i, P_0] = 0 \,, \qquad\quad [P_i, P_j] = \epsilon_{ij} A_{\rm mag} \,, \ [P_i, P_0] = 0 \,, \end{aligned} \tag{IV.2}$$

where J_3 is the rotation, the (K_i) are boosts, the (P_i) spatial translations, (P_0) is time translation, and $A_{\rm exo}$ and $A_{\rm mag}$ are the exotic and magnetic extensions, respectively.

More insight into the extended algebra structure is obtained by comparison with previously considered extensions. The one with A_{exo} is plainly the same as the "exotic" extension considered for the planar Galilean algebra [50–52].

On the other hand, the "magnetic" extension which implies the non-commutativity of spatial translations, $[P_i, P_j] = \epsilon_{ij} A_{mag}$, does not appear in the Galilei algebra — while it can appear instead in the extension of the *planar Euclidean algebra*, which is a subalgebra of both of the Carroll and the Galilei algebras. Then one may wonder why does the magnetic extension appear for Carroll, but not for Galilei.

Let us recall that in order to be a Lie algebra, the bracket operation should respect the Jacobi identity, which imposes strong constraints. For the Galilei algebra, the Jacobi identity for a time translation, a boost in one and a spatial translation in the other direction,

$$0 = [P_0, [K_1, P_2]] + [K_1, [P_2, P_0]] + [P_2, [P_0, K_1]] = [P_2, [P_0, K_1]] = [P_2, P_1]$$

requires the magnetic extension to vanishes. For the Carroll algebra instead, this identity is trivially realized because Carroll time translation commutes with boosts.

The free doubly-extended model obtained by the KKS algorithm is, by construction, symmetric w.r.t. the doubly-extended Carroll group $\widetilde{C}$, with associated conserved quantities,

$$\ell = \boldsymbol{x} \times \boldsymbol{p} + \tfrac{1}{2}\theta \boldsymbol{p}^2 + \tfrac{1}{2}\kappa_{\rm mag}\boldsymbol{x}^2 + \chi\,, \qquad \text{angular momentum} \tag{IV.3a}$$

$$g^i = m\Big(x^i + \theta\,\epsilon^{ij}p_j\Big)\,, \qquad \text{boost momentum} \tag{IV.3b}$$

$$\pi_i = p_i - \kappa_{\rm mag}\,\epsilon_{ij}x^j\,, \qquad \text{“impulsion”} \tag{IV.3c}$$

completed with m , $\kappa_{\rm exo}$, $\kappa_{\rm mag}$, cf. # (3.18) in [31] and sec.III for the terminology.

These quantities are analogous to the “exotic” Galilean expressions in a constant magnetic field [51, 52]. The real constant χ in (IV.3a) is Carrollian/anyonic spin, defined in (IV.4). The boost momentum $\mathbf{g}$ (IV.3b) plays a particularly important role, as it will be explained below.

The unusual form of the “impulsion” $\boldsymbol{\pi}$ in (IV.3c) deserves a comment. First we note that $A_i = -\frac{1}{2}(\kappa_{\rm mag}/e)\epsilon_{ij}x^j$ is a vector potential for $\kappa_{\rm mag}$ viewed as a (constant) “internal magnetic field”. Then $\boldsymbol{\pi}$ has the unexpected form $\pi_i = p_i + 2\,e\,A_i$, where $\boldsymbol{p}$ is just a coordinate. However in terms of the *canonical momentum* $P_i = p_i - \frac{1}{2}\kappa_{\rm mag}\epsilon_{ij}x^j$ obtained from the Lagrangian (IV.12), $\pi_i = P_i - \frac{1}{2}\kappa_{\rm mag}\epsilon_{ij}x^j = P_i + e\,A_i$.

Casimirs. When the mass does not vanish, $m \neq 0$, we have four Casimirs [31]: $\kappa_{\rm mag}$, $\kappa_{\rm exo}$, the mass, m, and the *anyonic spin*, χ, defined by,

$$m^*\chi = m^*\ell - \mathbf{g}\times\boldsymbol{\pi} + \frac{\kappa_{\rm exo}}{2m}\boldsymbol{\pi}^2 + \frac{\kappa_{\rm mag}}{2m}\mathbf{g}^2\,, \tag{IV.4}$$

where

$$m^* = m\left(1 - \frac{\kappa_{\rm exo}}{m^2}\kappa_{\rm mag}\right) \tag{IV.5}$$

is an effective mass, analogous to m^*_G in (II.14) in the Galilean case.

Eqn. (IV.4) is valid also for $m^* = 0$ when both the spin χ and the total angular momentum ℓ drop out. In this case, realized for a special relation between the Casimir invariants $\kappa_{\rm mag}\cdot\kappa_{\rm exo} = m^2$, we get:

$$\frac{1}{m}\mathbf{g}\times\boldsymbol{\pi} - \frac{1}{2\kappa_{\rm mag}}\boldsymbol{\pi}^2 - \frac{1}{2\kappa_{\rm exo}}\mathbf{g}^2 = 0\,. \tag{IV.6}$$

Multiplying (IV.4) by $m^2 \neq 0$ and then letting $m \to 0$ yields in turn the expression valid in the massless case

$$\chi = \ell - \frac{\boldsymbol{\pi}^2}{2\kappa_{\rm mag}} - \frac{\mathbf{g}^2}{2\kappa_{\rm exo}}\,, \tag{IV.7}$$

where the Casimir invariants $\kappa_{\rm mag}$ and $\kappa_{\rm exo}$ are assumed to be nonvanishing. Equation (IV.7) implies that the Carrollian anyonic spin χ is given by the angular momentum ℓ with contributions coming from the impulsion $\boldsymbol{\pi}$ and the boost momentum $\mathbf{g}$.

The KKS construction applied to $\widetilde{\mathcal{C}}$ yields, for a particle with mass $m \neq 0$, the Cartan resp. Souriau-forms with two additional terms [31],

$$\varpi_{\rm exo} = \boldsymbol{p}\cdot d\boldsymbol{x} \ + \frac{\theta}{2}\boldsymbol{p}\times \mathrm{d}\boldsymbol{p} \qquad + \ \frac{\kappa_{\rm mag}}{2}\boldsymbol{x}\times \mathrm{d}\boldsymbol{x}\,, \tag{IV.8a}$$

$$\sigma_{\rm exo} = d\boldsymbol{p}\wedge d\boldsymbol{x} + \underbrace{\tfrac{1}{2}\theta\,\epsilon_{ij}dp^i\wedge dp^j}_{exotic} \ + \ \underbrace{\frac{\kappa_{\rm mag}}{2}\,\epsilon_{ij}dx^i\wedge dx^j}_{\text{“}internal\ \ magnetic\text{”}}\,, \tag{IV.8b}$$

defined on Carroll evolution space $\mathcal{E}$ (III.5), cf. # (3.10) in [31]. The constant

$$\theta = \frac{\kappa_{\rm exo}}{m^2} \tag{IV.9}$$

here is interpreted as the non-commutativity parameter, see (IV.11) below.

The terms with $\theta \sim \kappa_{\rm exo}$ in (IV.8) are those of exotic Galilean dynamics, (I.3), and are in fact present also for *relativistic anyons* from which the Galilean model can be deduced as it will be shown in sec. V. The terms with $\kappa_{\rm mag}$ are reminiscent of a (constant) “internal magnetic” field, as mentioned above. Cf. also (II.12).

Eqn. (IV.8) is formally $\Omega_{\rm exo} - d\mathscr{H}_0\wedge ds$ with *zero* Hamiltonian,

$$\boxed{\Omega_{\rm exo} = d\boldsymbol{p}\wedge d\boldsymbol{x} + \frac{\theta}{2}\,\epsilon_{ij}dp^i\wedge dp^j + \frac{\kappa_{\rm mag}}{2}\,\epsilon_{ij}dx^i\wedge dx^j \quad \text{and} \quad \mathscr{H}_0 \equiv 0\,,} \tag{IV.10}$$

cf. (III.6). $\Omega_{\rm exo}$ is a closed and regular (i.e. symplectic) 2-form on the $s = s_0 = $ const. submanifold $\mathbb{R}^2\times\mathbb{R}^2 = \left\{\boldsymbol{x},\boldsymbol{p},s_0\right\}$ of the Carroll evolution space $\mathcal{E}$.

When the *effective mass* m^* is non-zero, $m^* \neq 0$, i.e. $\kappa_{\rm exo}\cdot\kappa_{\rm mag} \neq m^2$, the associated Poisson brackets [6]

$$\begin{aligned}
\left\{x^i, x^j\right\} &= \left(\frac{1}{1-\theta\kappa_{\rm mag}}\right)\theta\epsilon^{ij}\,,\\
\left\{x^j, p_i\right\} &= \left(\frac{1}{1-\theta\kappa_{\rm mag}}\right)\delta_i^j\,,\\
\left\{p_i, p_j\right\} &= \left(\frac{1}{1-\theta\kappa_{\rm mag}}\right)\kappa_{\rm mag}\,\epsilon_{ij}
\end{aligned} \tag{IV.11}$$

are *identical* with Eqn. (II.16) of the "exotic" Galilei case under the replacement $\kappa_{\rm mag} \sim eB$. The first relation implies that the coordinates do not commute when $\kappa_{\rm exo} \neq 0$ and the last one implies noncommuting momenta for $\kappa_{\rm mag} \neq 0$.

Alternatively (and equivalently when $m^* \neq 0$) we can consider the doubly-extended Carroll Lagrangian defined on the evolution space $\mathcal{E}$ in (III.5) which has, two more terms added to the free unextended Carroll Lagrangian $\mathscr{L}_0$ (III.11),

$$\mathscr{L}_{\rm exo} = \underbrace{\boldsymbol{p}\cdot\boldsymbol{x}'}_{\mathscr{L}_0} + \underbrace{\tfrac{1}{2}\theta\,\epsilon_{ij}p^i(p^j)'}_{exotic} + \underbrace{\tfrac{1}{2}\kappa_{\rm mag}\,\epsilon_{ij}x^i(x^j)'}_{magnetic} \quad . \tag{IV.12}$$

The associated variational equations (which are valid without the assumption $m^* \neq 0$)

$$(x^k)' + \theta\,\epsilon^{kl}(p_l)' = 0\,, \qquad (p_i)' - \kappa_{\rm mag}\,\epsilon_{ij}(x^j)' = 0\,, \tag{IV.13}$$

are integrated at once to yield the conserved vectors

$$x^k + \theta\epsilon^{kl}p_l = \text{const.} = Q^k\,, \tag{IV.14a}$$

$$p_i - \kappa_{\rm mag}\epsilon_{ij}x^j = \text{const.} = \pi_i\,. \tag{IV.14b}$$

In particular, the 2-vector $\mathbf{Q} = (Q^k)$ in (IV.14a) is identified as the *guiding center* [37, 79].

The conservation of the boost momentum $\mathbf{g}$ in (IV.3) alone does *not* now imply immobility for $\boldsymbol{x}$ – however it *does imply* that of $\mathbf{Q}$ in (IV.14a). Combining with the conservation of $\boldsymbol{\pi}$ (IV.14b) then yields the pair of decoupled equations which involve the effective mass (IV.5),

$$m^*\,(x_i)' = 0\,, \qquad m^*\,(p_i)' = 0 \tag{IV.15}$$

which allow us to infer:

Theorem IV.1. *A free doubly-extended Carroll particle with non-vanishing effective mass, $m^* \neq 0$ in (IV.5),* does not move.

However when the *effective mass vanishes*,

$$m^* = 0 \quad \text{i.e.} \quad \theta\,\kappa_{\rm mag} = 1 \quad \text{i.e.} \quad \kappa_{\rm exo}\kappa_{\rm mag} = m^2\,, \tag{IV.16}$$

then *no conclusion* can be drawn from (IV.15). For $m^* = 0$ the determinant of the symplectic form[19],

$$\det\left(\Omega_{ij}\right) = \left(\frac{m^*}{m}\right)^2 = (1 - \theta\kappa_{\rm mag})^2 \tag{IV.17}$$

[19] Compare with (II.14).

becomes indeed zero, highlighting the fact that the variational system is singular, requiring "Faddeev-Jackiw" reduction [51, 52, 66]. The dimension of the phase space drops from 4 to 2. The variables $\boldsymbol{x}$ and $\boldsymbol{p}$ become redundant however their (fixed) combination $\mathbf{Q} = (Q^k)$ in (IV.14a) is still a physical quantity. In its terms the reduced free symplectic form is simply

$$\Omega_{red} = \kappa_{\rm mag}\, dQ^1 \wedge dQ^2 = \frac{1}{\theta}\, dQ^1 \wedge dQ^2\,. \tag{IV.18}$$

The reduced Poisson brackets and variational forms are thus

$$\big\{Q_1, Q_2\big\}_{red} = -\theta\,, \qquad \varpi_{red} = \frac{1}{2\theta}\,\epsilon_{ij}\, Q^i dQ^j\,, \qquad \mathscr{L}_{red} = \frac{1}{2\theta}\,\mathbf{Q}\times\mathbf{Q}'\,. \tag{IV.19}$$

The Hamiltonian $\mathscr{H}$ is identically zero and the equations of motion, $Q_i' = \{Q^i, \mathscr{H}\}$, are therefore,

$$Q_i' = 0 \quad \Rightarrow \quad \mathbf{Q} = \mathbf{Q}_0 = \text{const.} \tag{IV.20}$$

consistently with (IV.14a). In conclusion,

Theorem IV.2. *When the effective mass of a massive system vanishes, $m^* = 0$, our free massive doubly-extended particle model becomes singular. The dynamics of $\boldsymbol{x}$ and of $\boldsymbol{p}$* can not be separately determined*, however the guiding center $\mathbf{Q}$ in* (IV.14a)*,* remains fixed. *The conserved quantities in* (IV.3) *are expressed in terms of the guiding center alone,*

$$\ell = \frac{1}{2\theta}\mathbf{Q}^2 \qquad \text{angular momentum} \tag{IV.21a}$$

$$\mathbf{g} = m\mathbf{Q} \qquad \text{boost momentum (center of mass)} \tag{IV.21b}$$

$$\pi_i = -\frac{1}{\theta}\epsilon_{ik}Q^k \qquad \text{"impulsion"} \tag{IV.21c}$$

consistently with the loss of two physical degrees of freedom.

For $m^* = 0$ the (no-)motion of the guiding center, (IV.20), follows directly from the conservation of $\mathbf{g}$ or of $\boldsymbol{\pi}$, themselves proportional to Q^k.

We conclude that extended Carroll symmetry plays for the extended model a role analogous to that of Carroll in the unextended case, with the effective mass m^* (IV.5) replacing the "naked" mass, m.

B. Massless doubly-extended particles

Let us now study the double extension of the massless Carroll particles discussed in section III C. The coadjoint action of the Euclidean group (III.42) extends to one of the doubly

extended Carroll group. The group element $\big(A, \mathbf{b}, \mathbf{c}, f, A_{\rm exo}, A_{\rm mag}\big) \in \widetilde{C}$ is implemented on $\tilde{\mu}_0 = (\ell, \mathbf{g}, \boldsymbol{\pi}, m=0, \kappa_{\rm exo}, \kappa_{\rm mag}) \in \widetilde{\mathfrak{c}}^*$ as,

$$\ell \to \ell - \mathbf{c} \times A\boldsymbol{\pi} + \mathbf{b} \times A\mathbf{g} - \tfrac{1}{2}\kappa_{\rm exo}\,\mathbf{b}^2 - \tfrac{1}{2}\kappa_{\rm mag}\,\mathbf{c}^2\,, \tag{IV.22a}$$

$$g^i \to (A\mathbf{g})^i + \kappa_{\rm exo}\,\epsilon_{ij}b^j \tag{IV.22b}$$

$$\pi_i \to (A\boldsymbol{\pi})_i - \kappa_{\rm mag}\,\epsilon_{ij}c^j\,, \tag{IV.22c}$$

while $m=0, \kappa_{\rm exo}, \kappa_{\rm mag}$ are left invariant, [31]. Having $\kappa_{\rm mag} \neq 0$ in (IV.22b) then implies that the norm of the momentum is *not* a Casimir invariant anymore [31]. We thus have only one kind of massless doubly-extended particle — namely the trivial one associated with $\boldsymbol{\pi} = 0$. Put in another way, the "color" k in a chosen basepoint can be eliminated by a translation which carries $\tilde{\mu}_0$ to the origin of the dual algebra, whose contribution to the KKS symplectic form is therefore obviously zero. Thus the first term in (IV.8a) disappears, leaving us with the "truncated" or "purely exotic" Cartan and Souriau forms,

$$\varpi_{\rm exo} = \frac{\kappa_{\rm exo}}{2}\boldsymbol{v} \times \mathrm{d}\boldsymbol{v} \quad + \quad \tfrac{1}{2}\kappa_{\rm mag}\boldsymbol{x} \times \mathrm{d}\boldsymbol{x}\,, \tag{IV.23a}$$

$$\sigma_{\rm exo} = \underbrace{\tfrac{1}{2}\kappa_{\rm exo}\,\epsilon_{ij}dv^i \wedge dv^j}_{exotic} \quad + \quad \underbrace{\tfrac{1}{2}\kappa_{\rm mag}\,\epsilon_{ij}dx^i \wedge dx^j}_{magnetic}\,, \tag{IV.23b}$$

where we reverted to the $\boldsymbol{v}$-notation since $\boldsymbol{p} = m\boldsymbol{v}$ is not well-defined anymore[20] which further extends the "no-particle" model discussed in subsec. III E. Had we set the extension parameters to zero we would find vanishing forms and *thus no particle trajectories*, as we had observed before in section III C. However keeping the terms induced by the central extension, we *do find "trajectories"*: the equations of motion become trivial and yield *fixed positions*[21],

$$(x^i)' = 0\,, \quad (p_i)' = 0 \quad \Rightarrow \quad \boldsymbol{x}(s) = \boldsymbol{x}_0 = \text{const.}, \quad \boldsymbol{p}(s) = \boldsymbol{p}_0 = \text{const.} \tag{IV.24}$$

Theorem IV.3. *A "purely exotic" free massless doubly-extended Carroll particle* (IV.23) *which has* $\boldsymbol{\pi} = 0$ does not move*: the double extension eliminates the free motions we studied in sec.III C.*

This no-motion conclusion will be modified when the particle is coupled to external fields, see the section below.

[20] The same result is obtained alternatively by putting $\boldsymbol{p} = m\boldsymbol{v}$ in (IV.8a) and then letting $m \to 0$.
[21] The vector $\boldsymbol{u}$ in the Fermat model of sec. III C is now ill-defined: null vectors have no direction.

C. Coupling to a gauge field: eppur si muove

The free model can be coupled to an em field by the rule (II.2). The external and "internal" magnetic fields combine into the generalized magnetic field:

$$B^* = eB + \kappa_{\rm mag}\,, \tag{IV.25}$$

and yield non-commuting coordinates and momenta [31],

$$\sigma_{\rm exo} = dp_i \wedge dx_i + \frac{\theta}{2}\,\epsilon_{ij} dp^i \wedge dp^j + \tfrac{1}{2} B^* \epsilon_{ij} dx^i \wedge dx^j + eE_i dx_i \wedge ds\,, \tag{IV.26a}$$

$$\{x_i, x_j\} = \frac{m}{m^*}\theta\,\epsilon_{ij}\,, \qquad \{x_i, p_j\} = \frac{m}{m^*}\,\delta_{ij}\,, \qquad \{p_i, p_j\} = \frac{m}{m^*}\,B^*\,\epsilon_{ij}\,, \tag{IV.26b}$$

where the Carrollian effective mass which generalizes (IV.5) now involves the combined field (IV.25),

$$m^* = m\,(1 - \theta B^*)\,, \tag{IV.27}$$

assumed not to vanish[22]. For a static scalar potential $V = V(\boldsymbol{x})$, $\mathbf{E} = -\boldsymbol{\nabla} V$, for example, the Hamilton equations are [31]),

$$(x^i)' = -\,\frac{\theta}{1-\theta B^*}\epsilon^{ij} eE_j\,, \tag{IV.28a}$$

$$(p_i)' = \quad \frac{1}{1-\theta B^*}\,eE_i\,. \tag{IV.28b}$$

The position $\boldsymbol{x}$ splits off from $\boldsymbol{p}$ which is coupled to $\boldsymbol{x}$ through the electric field, — but its dynamics does not effect the motion of $\boldsymbol{x}$.

These equations are reminiscent of but different from their Galilean counterparts in (II.17): p_i is missing from the "velocity" relation (IV.28a) which is purely anomalous, whereas the Lorentz force is missing from the purely electric $\boldsymbol{p}$-equation (IV.28b). The external magnetic field is hidden in m^* through B^*.

Theorem IV.4. *The position coordinate $\boldsymbol{x}$ of a massive doubly-extended Carroll particle in a static EM field with non-vanishing effective mass, $m^* \neq 0$,* moves, *namely by following the (anomalous) Hall law,* (IV.28a).

However combining the equations (IV.28) implies,

[22] Compare with (II.12), (II.14) and (II.16) in the Galilean case.

Theorem IV.5. *The guiding center* $Q^i = x^i + \theta\,\epsilon^{ij}p_j$ *in* (IV.14a) *of a doubly-extended particle with* $m^* \neq 0$*, coupled to an electromagnetic field does not move,*

$$Q_i' = 0 \quad \Rightarrow \quad \mathbf{Q} = \mathbf{Q}_0 = \text{const.} \tag{IV.29}$$

When $m^* \neq 0$ the real-space motion is recovered by solving the $\boldsymbol{p}$ equation and using (IV.14a),

$$x^i(s) = Q_0^i - \theta\epsilon^{ij}p_j(s)\,. \tag{IV.30}$$

For constant EM fields s.t. $m^* \neq 0$, for example, the motion is perpendicular to the electric field,

$$x^i(s) = Q_0^i - \epsilon^{ij}\Big(\frac{e\theta}{1-\theta B^*}\Big)E_j s\,. \tag{IV.31}$$

For vanishing effective mass, $m^* = 0$ the $\boldsymbol{x}$-motion can not be recovered as said before (and consistently with (IV.28) and (IV.31)).

Another example is obtained by a radial electric field $\mathbf{E} = f(|\boldsymbol{x}|)\,\hat{\boldsymbol{x}}$, where $f(|\boldsymbol{x}|)$ is a function of the radius in the $d = 2$ plane. Multiplying (IV.28a) by $\hat{\boldsymbol{x}}$ shows that the motion which is circular. In terms of the complex variables $\zeta = x^1 + ix^2$ and $\Pi = p_1 + ip_2$,

$$\zeta(s) = |\boldsymbol{x}_0|\exp[-i\Omega s] \quad \text{where} \quad \Omega = \frac{e\theta}{1-\theta B^*(|\boldsymbol{x}_0|)}\frac{f(|\boldsymbol{x}_0|)}{|\boldsymbol{x}_0|}\,, \tag{IV.32a}$$

$$\Pi(s) = i\frac{|\boldsymbol{x}_0|}{\theta}\exp[-i\Omega s] + \Pi_0 = \frac{i}{\theta}\,\zeta(s) + \Pi_0\,. \tag{IV.32b}$$

Thus the momentum rotates around $\Pi_0 = \text{const.}$ (which can be absorbed into the guiding center) with the same angular velocity Ω as ζ, but with a phase advance by $\pi/2$.

For $m^* = 0$ but $\mathbf{E} \neq 0$ the eqns (IV.28) are ill-defined, however their guiding center, i.e. their combination $\mathbf{Q}$ in (IV.14a) remains fixed when m^* changes sign by sweeping through zero,

$$\frac{m^*}{m}(Q^i)' = \frac{m^*}{m}\left((x^i)' + \theta\,\epsilon^{ij}p_j'\right) = 0 \tag{IV.33}$$

by (IV.29), including at $m^* = 0$. See Fig. 3 for an insight into the "phase transition". The external forces vary the frequency $\Omega = \Omega(|\boldsymbol{x}|)$ but the trajectories remain circular with fixed radius $|\boldsymbol{x}_0|/\theta$ for all values of m^*, cf. [79].

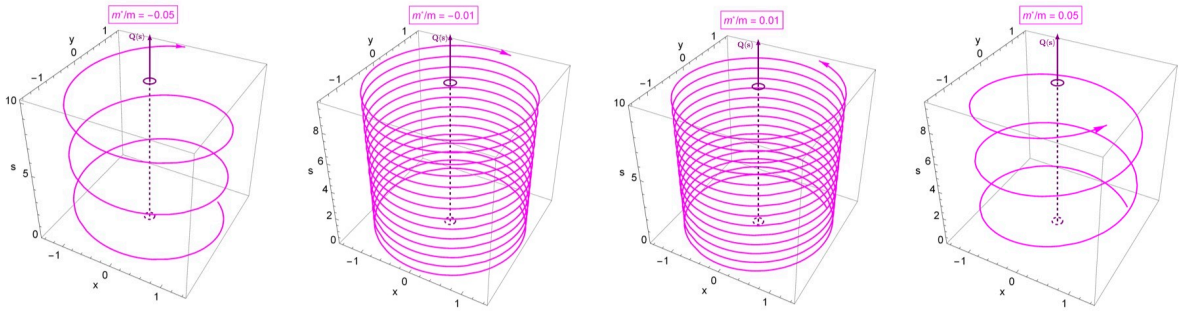


FIG. 3: *Trajectories for unit charge and exotic parameter in a radial electric field* $\mathbf{E} = \boldsymbol{x}/|\boldsymbol{x}|^3$ *in the plane for effective masses passing from negative to positive, unfolded in Carroll time* s*. The rotation around the fixed guiding center* $\mathbf{Q}(s) = \mathbf{Q}_0$ *speeds up when the effective mass increases from* $m^* < 0$*. The motion is instantaneous for* $m^* = 0$ *and changes orientation for* $m^* > 0$*.*

D. Extension to vanishing Carroll mass, $m = 0$

The above results can be extended from massive to *massless* particles, i.e., such that their Carroll mass vanishes, $m = 0$. The clue is to rewrite (IV.28) for $m \neq 0$ as,

$$(x^i)' = \frac{1}{B^* - m^2/\kappa_{\text{exo}}} \epsilon^{ij} e E_j \,, \qquad p_i' = m^2 \frac{eE_i}{m^2 - \kappa_{\text{exo}} B^*} \,. \tag{IV.34}$$

Then letting $m \to 0$ we get:

Theorem IV.6. *A massless, charged doubly-extended Carroll particle in a combined electric and effective magnetic field* $B^* = eB + \kappa_{\text{mag}}$ *in* (IV.25) moves *according to the generalized* Hall law*,*

$$(x^i)' = \epsilon^{ij} \frac{eE_j}{B^*} \quad \textit{and} \quad p_i' = 0 \,. \tag{IV.35}$$

We record for later use that the Hall law above can be derived form the first-order Lagrangian resp. of Souriau form,

$$\mathscr{L}_{2+1} = \tfrac{1}{2} B^* \epsilon_{ij} x^i (x^j)' - eV(\boldsymbol{x}) \,, \tag{IV.36a}$$

$$\sigma_{2+1} = \tfrac{1}{2} B^* \epsilon_{ij} dx^i \wedge dx^j + eE_i dx^i \wedge ds \,. \tag{IV.36b}$$

Returning to the Souriau framework, coupling the free massless particle with "purely exotic dynamics", (IV.23), to an external EM field by the rule (II.2) yields,

$$\sigma_{m=0} = \underbrace{\tfrac{1}{2} \kappa_{\text{exo}}\, \epsilon_{ij} dv^i \wedge dv^j + \tfrac{1}{2} B^* \epsilon_{ij} dx^i \wedge dx^j}_{\Omega_{m=0}} + eE_i dx^i \wedge ds \,. \tag{IV.37}$$

Viewed in another way, rewriting the Souriau form σ_{exo} (IV.26a) in terms of $\boldsymbol{v} = \boldsymbol{p}/m$ and then letting $m \to 0$ suppresses the first term and yields once again (IV.37). The determinant of the symplectic form $\Omega_{m=0}$ in (IV.37) is,

$$\det\left(\Omega_{m=0}\right) = (\kappa_{\text{exo}})^2 (B^*)^2 \,. \tag{IV.38}$$

Thus our previous conditions $\kappa_{\text{exo}} \neq 0$ and $B^* \neq 0$ guarantee that $\Omega_{m=0}$ is regular. However the dynamics trivially *projects* to Carroll spacetime $\{(\boldsymbol{x}, s)\}$: the first term in (IV.37) is switched off, leaving us with (IV.36) and "forgetting" about the "irrelevant" dynamics of $\boldsymbol{p}$.

What can be the physical realization of these particles? Rigorously speaking, there are no strictly massless charged fields in the Standard Model of fundamental interactions in its low-temperature phase, which is relevant to present-day phenomenology [97]. While genuinely massive constituents of the model can be treated as effectively massless particles at very high energies (such as leptons and quarks in heavy ion collisions [98]), they propagate in three spatial dimensions and, therefore, are not relevant for the peculiar features of two-dimensional physics considered in this section.

However, suitable examples do exist in the condensed matter context: massless charged particles in two spatial dimensions were realized in graphene, a two-dimensional crystal made of carbon atoms, discovered experimentally in 2004 [99]. In this material, the dynamics of the low-energy electronic excitations is described by the $2+1$ massless Dirac Hamiltonian,

$$\mathscr{H} = \bar{\psi} i \boldsymbol{\alpha} (\boldsymbol{\nabla} - ie\boldsymbol{A}) \psi + eV \,, \tag{IV.39}$$

where $\boldsymbol{\alpha} = (\gamma^0\gamma^1, \gamma^0\gamma^2)$ and γ^μ are spatial Dirac matrices in $2+1$ dimensions. Being charged, the massless electronic quasiparticles interact with the background electromagnetic field $A^\mu = (V, \boldsymbol{A})$, thus providing us with a system that exhibits equivalences with quantum electrodynamics [100].

In the absence of background electromagnetic fields, the Dirac particle described by the Hamiltonian (IV.39) possesses the 'relativistic' Dirac particle-like dispersion, $\varepsilon_{\boldsymbol{p}} = v_F |\boldsymbol{p}|$, where $\boldsymbol{p} = (p_1, p_2)$ corresponds to a quasimomentum of the particle with the origin at the tip of a Dirac cone. The particles propagate relativistically with the Fermi velocity, v_F, which is typically hundreds of times smaller than the speed of light.

The massless nature of the excitations free-standing graphene is protected by the symmetries of the underlying carbon crystal lattice, thus making the massless nature ($m = 0$)

of these charged excitations ($e \neq 0$) a robust feature of the material [101]. Therefore, the quasiparticles in graphene can be viewed as a charged, doubly-extended Carroll particle in the massless limit described in this section.

E. Carrollian anyons

Carroll particles with anyonic spin, $\chi \neq 0$ in (IV.3), can also be studied along the same lines. The Souriau 2-form $\sigma_{\rm exo}$ (IV.26a) is generalized by following the rule (II.2),

$$\sigma_{any} = dp_i \wedge dx_i + \frac{\theta}{2}\,\epsilon_{ij} dp^i \wedge dp^j + \tfrac{1}{2} eB^* \epsilon_{ij} dx^i \wedge dx^j + E_i^* dx^i \wedge ds\,, \tag{IV.40}$$

where the electric field and the gradient of the magnetic field are again combined (as in (II.23) in the Galilean theory), into an effective electric term [23],

$$E_i^* = eE_i + \mu\chi\partial_i B\,. \tag{IV.41}$$

The equations of motion take the form (IV.28) with $e\mathbf{E}$ replaced by $\mathbf{E}^*$.

The extension to zero mass is obtained along the same lines as above:

Theorem IV.7. *Both massive and massless Carrollian anyons move by following the Hall law* (I.2) *but with the electric and magnetic fields replaced by the effective values* $\mathbf{E}^*$ *and* B^*, (IV.41) *and* (IV.25), *respectively,*

$$\boxed{(x^i)' = \epsilon^{ij}\left(\frac{eE_j + \mu\chi\partial_j B}{eB + \kappa_{\rm mag}}\right) = \epsilon^{ij}\left(\frac{E_j^*}{B^*}\right) \quad \textit{and} \quad p_i' = 0\,.} \tag{IV.42}$$

Note here the double role played by the magnetic field B.

The massless limit can also be discussed in the Souriau framework. We start with (IV.40) with $m \neq 0$ and eliminate $\boldsymbol{p}$ in favor of $\boldsymbol{v} = \boldsymbol{p}/m$. Then letting $m \to 0$ we get a regular Souriau form which, compared to (IV.23b), has an additional "effective electric" term,

$$\sigma_{any} = \tfrac{1}{2}\kappa_{\rm exo}\,\epsilon_{ij} dv^i \wedge dv^j + \tfrac{1}{2} B^* \epsilon_{ij} dx^i \wedge dx^j + E_i^* dx^i \wedge ds\,. \tag{IV.43}$$

The kernel of σ_{any} yields the equation of motion (IV.35) [with p_i replaced by v_i].

[23] A constant shift of the magnetic field, $B \to B + \text{const.}$, produces no effect since the spin coupling involves only the gradient of B.

	free extended Carroll	in EM field	in curved metric
$m \neq 0$	no motion	generalized Hall	no motion
$m = 0$, $e \neq 0$	no motion	generalized Hall	no motion
$m = 0$, $e = 0$, $\mu\chi \neq 0$	no motion	generalized Hall	no motion
$m^* = 0$	$\mathbf{Q}$ fixed, $\boldsymbol{x}$ motion undetermined		

TABLE II: *Motion of doubly-extended Carroll particles in $d = 2$ spatial dimensions.*

The generalized Hall law (IV.42) contains an "anomalous" anyonic contribution activated by the inhomogeneities of the magnetic field (IV.41). For massive particles, the anyonic term can be interpreted as a conventional Zeeman force

$$\boldsymbol{F} = g\mu_B s \boldsymbol{\nabla} B \,, \tag{IV.44}$$

exerted by the spatial gradient of the transverse magnetic field $B \equiv B_z$ on the particle propagating in the plane. Here g is the Landé factor, and μ_B is the Bohr magneton. A comparison of Eqs. (IV.41) and (IV.44) provide us with the identification $e\mu\chi \equiv g\mu_B s$.

In spintronic devices, the electrons with different orientations of their spins $s \equiv s_z$ are split by the Zeeman force (IV.44). This phenomenon lies in the heart of a mesoscopic Stern-Gerlach effect which provides an efficient spin filter for a current traversing a specially fabricated device that incorporates an inhomogeneous magnetic field $B = B_z$ [102].

Thus, two-dimensional Carrollian anyons can be interpreted as spin-polarized charged particles with a nonzero Landé factor.

F. A toy model

Let us recall that an "exotic photon" studied in Ref. [32] is defined by being uncharged, carry magnetic moment, anyonic spin, and have non-vanishing "exotic charges",

$$e = 0, \quad \mu \neq 0, \quad \chi \neq 0 \quad \kappa_{\text{exo}} \neq 0, \quad \kappa_{\text{mag}} \neq 0 \,, \tag{IV.45}$$

which corresponds to the Carroll Hamiltonian of a spinning anyon,

$$\mathscr{H} = -\mu\chi B \,. \tag{IV.46}$$

cf. (II.21). Its motion on the horizon of a black hole will be revisited in Sec. VI.

The exotic photon is a massless, electrically-neutral particle which has spin and possesses a nonvanishing magnetic moment. Although this particle carries a "bosonic" name, it can be of a fermionic or even anyonic nature as well.

A distant physical analogue of the exotic photon in relativistic fundamental field theory can be played by Dirac or Majorana neutrinos which, being almost massless and neutral, could possess a sizable magnetic moment in certain exotic scenarios [89, 90]. A non-vanishing neutrino magnetic moment would have important astrophysical and cosmological consequences [91].

In the condensed matter setup, neutral fermionic excitations appear in strongly correlated electronic systems in Kondo lattice materials such as, for example, the newly found compound $YbIr_3Si_7$. Neutral fermions cannot, evidently, carry electric charge while they can support thermal energy transfer endowing the charge insulator with metallic thermal conducting properties. This type of exotic materials is sensitive to the background magnetic field implying that neutral fermions couple to the magnetic degrees of freedom [92]. The corresponding electrically-neutral massless excitations are remote condensed-matter analogues of the exotic photons proposed in Ref. [32].

Here we illustrate our theory by a massless particle[24], $m = 0$, $e = 0\,, \mu\chi = \kappa_{\rm mag} = 1\,$, put into the electromagnetic field

$$\mathbf{E}(\boldsymbol{x}) = f(|\boldsymbol{x}|)\,\hat{\boldsymbol{x}} \quad \text{and} \quad B(\boldsymbol{x}) = \frac{|\boldsymbol{x}|}{(1+|\boldsymbol{x}|^2)^3}\,. \tag{IV.47}$$

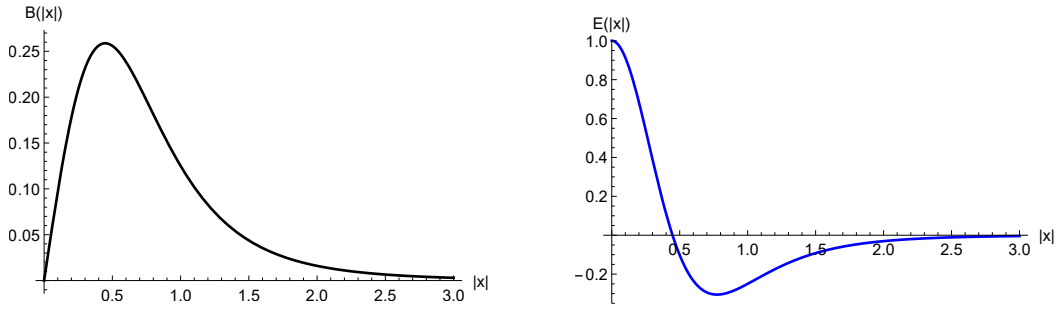
For (IV.45) $\kappa_{\rm exo}$ drops out and (IV.42) reduces to,

$$\boxed{(x^i)' = (\mu\chi)\,\epsilon^{ij}\frac{\partial_j B}{\kappa_{\rm mag}} \quad \text{and} \quad p_i' = 0\,,} \tag{IV.48}$$

which is indeed the key formula we used on the horizon used of a black hole [32]. The external electromagnetic fields are switched off by $e = 0$ but there remains an effective electric field, $\mathbf{E}^* = \mu\chi\boldsymbol{\nabla}B$, coming from the spin-field coupling, (II.21), with $\mu\chi$ behaving as an effective charge and B as an effective potential. The various fields and the Hall motion (IV.48) are shown in 4a-b. Fig. 5 prefigures Fig. 10.

An intuitive picture can be obtained by observing that the motion of our Carroll anyon in

[24] In addition to photons and neutrinos in fundamental particle physics, uncharged and massless quasiparticles also appear in the solid state context as phonon excitations in crystals and fluids.



(a) (b)

FIG. 4: (a) *The planar magnetic field* $B(|\boldsymbol{x}|)$ *in* (IV.47) (b) *induces a radial effective electric field* $\mathbf{E}^* = \boldsymbol{\nabla} B$.

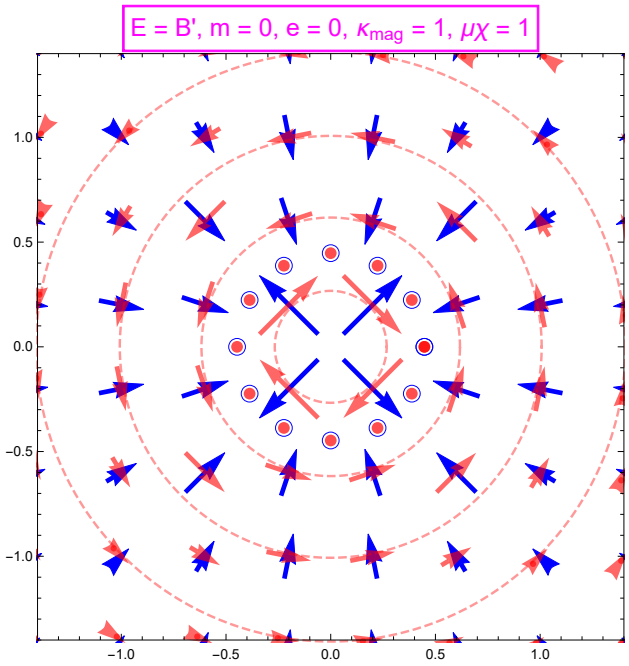


FIG. 5: *For uncharged particles the radial planar magnetic field* B *in* (IV.47) *induces a radial* **effective electric field** $\mathbf{E}^*$. *An "exotic photon" i.e. a mass and chargeless planar Carroll anyon with nonzero spin* χ, *magnetic moment* μ *and magnetic charge* $\kappa_{\rm mag}$, (IV.45), *rotates around the origin along concentric circles, consistently with the* generalized Hall Law, (IV.42). *The rotation stops on a ring of a fixed radius (marked by full* **dots** *in red) where the effective electric field vanishes,* $\mathbf{E}^* = 0$, *and its direction is reversed when* $\mathbf{E}^*$ *changes sign.*

the effective electric field $\mathbf{E}^*$ can can be pictured also as motion o,scrobed to the "Mexican hat" potential $-B(|\boldsymbol{x}|)$ unfolded to three dimensions above the plane as shown in Fig. 6.

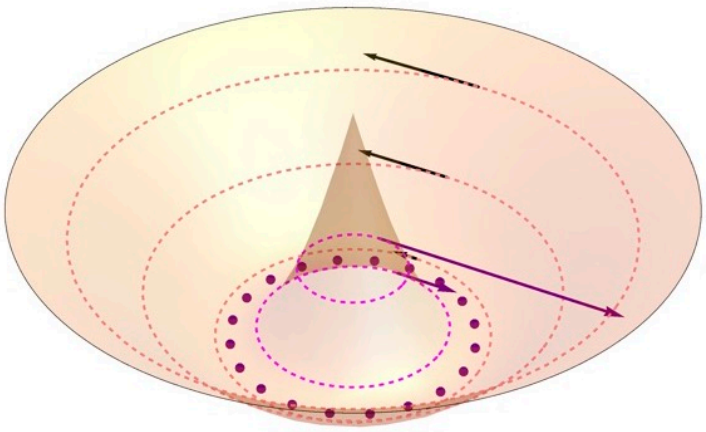


FIG. 6: *The motion of an anyon in the effective electric field* $\mathbf{E}^*$ *can be lifted to "Mexican Hat" potential surface* $-B(|\boldsymbol{x}|)$ *above the plane. The trajectories are horizontal "cords" (circles) starting near the "top" with high velocities and, then decreasing progressively until we arrive at* $|\boldsymbol{x}| = |\boldsymbol{x}_0|$ *where the force vanishes,* $\mathbf{E}^*(|\boldsymbol{x}_0|) = 0$*, and the rotation stops. The "cord" degenerates to fixed points around the "bottom rim". Then, as we climb up the outer "flange" by letting* $|\boldsymbol{x}|$ *increase, the anyon restarts to move with ever increasing velocity — however in the* opposite *direction. The 3D picture projects to Fig. 5.*

For comparison, we consider also a *charged* particle,

$$e = 1 \qquad \kappa_{\text{mag}} = \mu\chi = 1, \qquad \kappa_{\text{exo}} \neq 0\,. \tag{IV.49}$$

As discussed above, such quasi-particles with might be relevant in Condensed Matter physics (see also [27]).

Then we put both particles into the electric same electric field equal to the effective field in the uncharged case,

$$\mathbf{E}(|\boldsymbol{x}|) = \boldsymbol{\nabla} B = \frac{1 - 5|\boldsymbol{x}|^2}{(1 + |\boldsymbol{x}|^2)^4}\,\hat{\boldsymbol{x}}\,. \tag{IV.50}$$

Then (IV.42) reduces to the usual Hall motion (I.2) with position-dependent frequency.

The magnetic and electric fields could be, in fact, chosen also independently, see (IV.25). Pairing, for example, a constant electric field $\mathbf{E} = \mathbf{E}_0 = \text{const.}$, e.g., $e\mathbf{E}_0 = eE_0(1, 0)$ with $B = B(|\boldsymbol{x}|)$ written in (IV.47),

$$\mathbf{E}^* = \mathbf{E}_0 + \mu\chi \frac{1 - 5|\boldsymbol{x}|^2}{(1 + |\boldsymbol{x}|^2)^4}\,\hat{\boldsymbol{x}} \tag{IV.51}$$

yields Fig. 7.

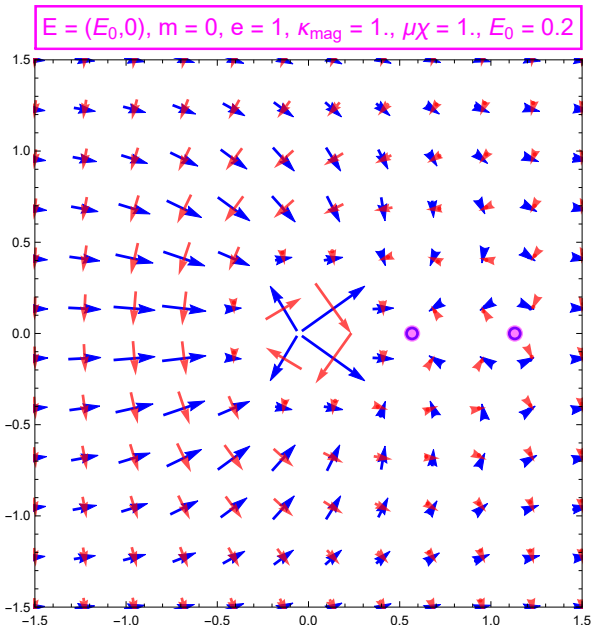


FIG. 7: *Combining a constant external electric field* $\mathbf{E} = \mathbf{E}_0$ *with* $\boldsymbol{\nabla} B$ *induced by the spin-field term* (II.21) *yields* **effective electric field** $\mathbf{E}^*$ (IV.41). **Velocity** *and* **force** *are perpendicular, as required by the Hall law. The number of equilibrium points has been considerably reduced.*

V. CONTRACTION FROM RELATIVISTIC ANYONS

An insight into the physical origin of exotic Galilean symmetry has been provided by its derivation from relativistic anyons [80] by a tricky contraction [81, 82]. Now we study the Carrollian counterpart [2]: one defines the timelike coordinate $x^0 = s/c$ and then sends $c \to \infty$. We start with a massive and spinless particle, described by the Cartan 1-form [59, 82],

$$\varpi = p_\mu dx^\mu = p_i dx^i - m\sqrt{1 + \frac{p^2}{m^2c^2}}\, ds\,. \tag{V.1}$$

Taking the limit $c \to \infty$, we readily recover the Cartan 1-form (III.4) for Carroll,

$$\varpi_0 = p_i dx^i - m ds\,. \tag{V.2}$$

Notice that the Carroll limit is in fact more straightforward than the Galilean one, for which the limit $c \to \infty$ of $mc^2\sqrt{1 + \frac{p^2}{m^2c^2}}\, dt$ requires regularization, achieved in [82] by adding a central extension i.e., working with the Bargmann model.

Now consider a massive relativistic model with spin. Generalizing the Jackiw-Nair ansatz $s = \kappa c^2$ [81], we set $s = \chi + \kappa c^2$, whose Cartan form is (V.2) plus a term describing the spin

state of the particle, $\varpi = \varpi_0 + \varpi_s$. The rather complicated term ϖ_s can be read off from # (4.1) in [82]. The result is seen not depend on the time coordinate. *The Carrollian limit of ϖ_s is therefore the same as in the Galilean case*, and we end up with the Cartan form,

$$\varpi_{any} \equiv \varpi = p_i dx^i - mds + \kappa c^2 d\phi + \chi d\phi + \frac{\theta}{2}\epsilon^{ij} p_i dp_j \,. \tag{V.3}$$

The term $\kappa c^2 d\phi$ here diverges when $c \to \infty$, however its exterior derivative behaves regularly, yielding the exotic part of the symplectic form (IV.10) with $\kappa_{\rm exo} = \kappa$,

$$\Omega_{\rm exo} = dp_i \wedge dx^i + \frac{\kappa}{2m^2}\epsilon^{ij} dp_i \wedge dp_j \,. \tag{V.4}$$

Thus the exotic Carroll charge $\kappa_{\rm exo}$ is deduced from the relativistic anyon model, just as in the Galilean case. The remarkably difference is however that we *do not recover the second charge*, $\kappa_{\rm mag}$ by our contraction.

Some more insight can be gained by regularizing ϖ. We follow [82]. Consider a *trivial* $\mathbb{R}$-central extension to the Poincaré group. Its Cartan 1-form will be $\widetilde{\varpi} = \varpi + \alpha d\theta$. Now, we choose $\alpha = -\kappa c^2$[25] and $\theta = \phi - w/c^2$ for some $w \in \mathbb{R}$. This implies that $\alpha d\theta = -\kappa c^2 d\phi + \kappa dw$, and thus,

$$\widetilde{\varpi} = p_i dx^i - mds + \chi d\phi + \frac{\kappa}{2m^2}\epsilon^{ij} p_i dp_j + \kappa dw \,. \tag{V.5}$$

This expression is to be compared to $\varpi_{\rm exo}$ in (IV.8). The presymplectic structure has gained one dimension, spanned by the parameter w. Such a situation arises in the "non-exotic" Galilean case, where the infinite energy is regularized similarly [60, 82].

In conclusion, the exotic term is recovered (up to a total derivative), but the magnetic term is missing, as it does also from the presymplectic 2-form $d\varpi$, cf. (IV.8).

VI. HALL ON THE HOLE (MOTION ON THE HORIZON)

A genuine physical application of (possibly doubly-extended) Carroll dynamics is provided by motion on the *horizon of a black hole* [15–19]. We first recall the following general properties:

- A black hole is characterized by its mass M, angular momentum J and charge Q:

[25] The quantity α has to be a constant (or depend only on θ) for $\alpha d\theta$ to be exact.

- The horizon $\mathcal{H}$ is a Carroll manifold [15–17];
- The horizon is a null hypersurface, therefore the geodesics constrained to it are necessarily null [94],

$$\epsilon = \tfrac{1}{2}\mathrm{g}_{\mu\nu}\,\dot{x}^{\mu}\dot{x}^{\nu} = 0\,, \tag{VI.1}$$

  see chapters 34.3 and 34.4 in [93].
- The geodesic dynamics can be extended by adding both exotic & spin-field (commonly called spin-orbit) terms discussed in sec.IV.

This section is devoted to the study of an exotic photon defined in (IV.45) whose key dynamical ingredients are, by (IV.43),

$$\varpi_{spec} = \tfrac{1}{2}\kappa_{\rm exo}\,\epsilon_{ij}v^i dv^j + \tfrac{1}{2}\kappa_{\rm mag}\,\epsilon_{ij}x^i dx^j + \mu\chi\, B dx^i ds\,, \tag{VI.2a}$$

$$\sigma_{spec} = \tfrac{1}{2}\kappa_{\rm exo}\,\epsilon_{ij}dv^i \wedge dv^j + \tfrac{1}{2}\kappa_{\rm mag}\,\epsilon_{ij}dx^i \wedge dx^j + \mu\chi\partial_i B dx^i \wedge ds\,, \tag{VI.2b}$$

$$\mathscr{L}_{spec} = \tfrac{1}{2}\kappa_{\rm exo}\,\epsilon_{ij}v^i\frac{dv^j}{ds} + \tfrac{1}{2}\kappa_{\rm mag}\,\epsilon_{ij}x^i\frac{dx^j}{ds} + \mu\chi\, B\,. \tag{VI.2c}$$

A. Schwarzschild Horizons

We first study the simplest black hole, which is the static and spherically symmetric Schwarzschild metric, described by using (ingoing) Eddington-Finkelstein coordinates $\{v, r, \vartheta, \varphi\}$ and natural units with the line element

$$g \equiv g_{\mu\nu}dx^{\mu}dx^{\nu} = -\left(1 - \frac{2M}{r}\right)dv^2 - 2dvdr + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\,, \tag{VI.3}$$

where M is the mass of the Schwarzschild black hole. The Schwarzschild horizon, $\mathcal{H}$, is the hypersurface $r = 2M$, i.e. ,

$$\mathcal{H} = \Big\{r = 2M,\ \vartheta, \varphi, v\Big\} \cong \mathbb{S}^2 \times \mathbb{R}\,. \tag{VI.4}$$

The dv^2 component of the metric (VI.3) induced on $\mathcal{H}$ vanishes, whereas the radial component disappears on the fixed-r hypersurface $\mathcal{H}$, leaving us with the clearly degenerate "metric",

$$\widetilde{g} = g|_{r=2M} = 0 \cdot dv^2 + 4M^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\,. \tag{VI.5}$$

On the other hand,

$$\widetilde{\xi} = \partial_v \equiv \partial_s \tag{VI.6}$$

is a nowhere vanishing vector which belongs to the kernel of $\widetilde{g}$ where, to anticipate the role of Carrollian time that it will play, we renamed v to s.

We conclude that the Schwarzschild horizon $(\mathbb{S}^2 \times \mathbb{R}, \widetilde{g}, \widetilde{\xi})$ has a Carroll structure [16][26], as defined in sec. III.

The vector $\widetilde{\xi}$ (called the generator of the horizon) is also a Killing vector field for the Carroll metric in that $L_{\widetilde{\xi}}\widetilde{g} = 0$. It is in fact a Killing vector field of the original Schwarzschild spacetime, $L_{\widetilde{\xi}}g = 0$, which becomes a null vector on the horizon,

$$g(\widetilde{\xi}, \widetilde{\xi})|_{r=2M} = 0\,. \tag{VI.7}$$

The Schwarzschild horizon provides us with an insight into the peculiarities of Carrollian dynamics. One of them is that the horizon of (stationary) black holes can "trap" photons i.e. , a photon may remain on the horizon forever. As a stationary horizon is always a null hypersurface, and since photons travel in null directions, a photon could move along the null direction defined by the ($\mathbb{R}$ component of the) horizon, thus staying on it forever. In the Schwarzschild case for instance, this amounts to emit a photon right on the horizon and radially outward, see Fig. 8. For details, see *e.g.* [93, §33.6].

"Photon trapping" is a nice thought experiment to understand Carroll geometries. Consider a photon on a Schwarzschild horizon. We can look at its motion (or lack of motion) intrinsically, on the horizon itself. On this surface, the photon is created at some point $(\boldsymbol{x}_0, s_0)$ in (Carroll) time (where "Carroll time" is represented by the $\mathbb{R}$-axis generated by the null vector $\xi = \partial_s$, and $\boldsymbol{x}_0 \in \mathbb{S}^2$). In order to stay on the horizon, the photon must have its momentum directed along the generator ξ of the horizon. Its momentum is then zero in the angular directions, and therefore the photon "stays in place" on $\mathbb{S}^2$. Its "trajectory" is precisely the Carrollian "no-motion", $\boldsymbol{x}(s) = \boldsymbol{x}_0$.

The situation is illustrated by the Eddington-Finkelstein diagram on Fig. 8, cf. [93].

While this intuitive picture on the Schwarzschild horizon seems to imply no motion on this instance of a Carroll structure, the horizon is $2+1$ dimensional and thus *might* accomodate doubly-extended (and potentially non-trivial) dynamics associated with the double central

[26] Any null hypersurface in a Lorentzian spacetime is Carroll [13, 14, 73, 74].

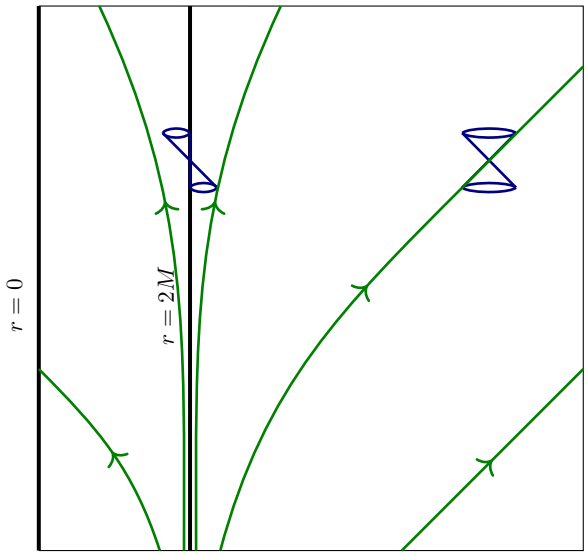


FIG. 8: *Eddington-Finkelstein diagram of a Schwarzschild spacetime, with outgoing light rays* $u = \text{const.}$ *in* **green**, *highlighting key features such as the singularity, the horizon, and different trajectories of outgoing photons. Infinitely far away from the black hole, space is Minkowskian. However, the closer we get, the more the gravitational field bends the light cones inward, resulting in outgoing photons taking an ever longer time to escape from the black hole region. Past the horizon, the light cone is completely bent inward such that the only future-pointing directions are toward the singularity. An outgoing photon emitted right on the horizon would have a vertical trajectory, thus staying on the horizon.*

extension discussed in sec.IV. However, it has been shown in [31] and recalled in section IV C that the two charges κ_{exo} and κ_{mag} only couple to the electromagnetic field but *not* the gravitational field. But the Schwarzschild spacetime carries *no* electromagnetic field. Hence,

Theorem VI.1. *An uncharged massless (exotic or not) particle with purely outgoing momentum on the horizon of a Schwarzschild black hole stays fixed, but does not move.*

This conclusion is valid also for spin-field interaction (IV.35) since $B = 0$

B. Motion on the Kerr-Newman horizon

Eqn. (IV.42) suggests that non-trivial motion may be possible on the horizon of a black hole for a massless particle with magnetic moment $\mu \neq 0$, non-zero anyonic spin $\chi \neq 0$ and

non-zero magnetic charge $\kappa_{\rm mag} \neq 0$, when the black hole has a non-uniform magnetic field $B \neq 0$ on the horizon.

Below we show that these conditions are met for a *Kerr-Newman black hole*, described by its mass M, its angular momentum J, and its charge Q. In terms of Eddington-like coordinates $(v, r, \vartheta, \varphi)$ [95], a Kerr-Newman metric can be written in the form [27],

$$g = -\frac{\Delta}{\Sigma}\left(-dv + \frac{\Sigma}{\Delta}dr + a\sin^2\vartheta d\varphi\right)^2 + \frac{\sin^2\vartheta}{\Sigma}\left(adv - (r^2+a^2)d\varphi\right)^2 + \Sigma d\vartheta^2 + \frac{\Sigma}{\Delta}dr^2 \tag{VI.8a}$$

$$\Sigma = r^2 + a^2\cos^2\vartheta, \quad \Delta = r^2 + a^2 + Q^2 - 2Mr\,, \tag{VI.8b}$$

where $a = J/M$. The (outer) horizon of a Kerr-Newman black hole is the a $r = r_+ = \text{const.}$ hypersurface $\mathcal{H}$ defined by requiring[28]

$$\Delta = 0 \quad \text{i.e.} \quad r = M + \sqrt{M^2 - (a^2+Q^2)} = r_+\,. \tag{VI.9}$$

Then we consider again the $2+1$ dimensional structure [16, 17] whose key ingredients are the induced metric and a vector,

$$\widetilde{g} = g|_{\Delta=0} = \frac{\sin^2\vartheta}{\Sigma}\left(a\,dv - (r_+^2+a^2)d\varphi\right)^2 + \Sigma d\vartheta^2\,, \tag{VI.10a}$$

$$\xi = \partial_v + \Omega_H\partial_\varphi \quad \text{where} \quad \Omega_H = \frac{a}{r_+^2+a^2}\,. \tag{VI.10b}$$

Here Ω_H is the angular velocity of the horizon.

The metric (VI.10a) is again singular, as seen from shifting the coordinates as $(\vartheta, \varphi, v) \mapsto (\vartheta, \widetilde{\varphi} = \varphi - \Omega_H v, v)$,

$$\widetilde{g} = \frac{(r_+^2+a^2)^2\sin^2\vartheta}{\Sigma}d\widetilde{\varphi}^2 + \Sigma d\vartheta^2 \qquad \& \qquad \xi = \partial_v\,. \tag{VI.11}$$

The kernel is generated by the vector ξ which takes now the familiar form, $\widetilde{g}(\xi) = 0$.

Thus we have a degenerate metric and a vector field in its kernel allowing us to conclude that the horizon of a Kerr-Newman black hole carries indeed a Carroll structure $(\mathbb{S}^2 \times \mathbb{R}, \widetilde{g}, \widetilde{\xi})$. Viewed in the original spacetime, ξ is again null on the horizon and generates the ($\mathbb{R}$ component of the) horizon.

[27] The metric and its inverse are regular at $\Delta = 0$, as the dr^2 terms cancel each other out.

[28] In fact, there are two hypersurfaces which satisfying $\Delta = 0$. We are only interested in the behavior at the outer radius r_+ .

Now if we were to study the naïve photon trapping on the horizon of a KN black hole as we did for Schwarzschild, we would arrive at the same negative conclusion: we would find no motion. However the Carroll manifold $\mathcal{H}$ has $2+1$ dimensions and therefore can accommodate the 2-parameter extension yielding an "exotic photon" (IV.45), which may again be coupled to the electromagnetic field and which *can* move, as we will see it below.

In Eddington-like coordinates $(v, r, \vartheta, \varphi)$ [95][29] the electromagnetic tensor in the Kerr-Newman spacetime is given by,

$$F = \frac{2aQr\sin\vartheta\cos\vartheta}{(r^2+a^2\cos^2\vartheta)^2}\Big(adv-(r^2+a^2)d\varphi\Big)\wedge d\vartheta + \frac{Q(r^2-a^2\cos^2\vartheta)}{(r_+^2+a^2\cos^2\vartheta)^2}\Big(a\sin^2\vartheta d\varphi - dv\Big)\wedge dr\,. \tag{VI.12}$$

The magnetic field is conveniently derived by using the Hodge operator which interchanges the electric and magnetic fields, $\star F = B_i dx^i \wedge dv + \ldots$. We then identify the magnetic field induced on the horizon ($r = r_+$) as $B = B^r|_{\Delta=0} = (\star F)^{rv}$, which yields, in our case,

$$B = \left.\frac{F_{\theta\varphi}}{\sqrt{|\det g|}}\right|_{\Delta=0} = \big(2aQr_+(r_+^2+a^2)\big)\,\frac{\cos\vartheta}{(r_+^2+a^2\cos^2\vartheta)^3}\,. \tag{VI.13}$$

We note for later reference that the magnetic field is a function of ϑ only, $B = B(\vartheta)$, as it follows from the axial symmetry of the Kerr-Newman spacetime. Non-vanishing B is obtained for non-zero angular momentum $J = aM$ and charge Q.

The electromagnetic field induced on the horizon,

$$\widetilde{F} = F|_{\Delta=0} = \frac{aQr_+\sin 2\vartheta}{(r_+^2+a^2\cos^2\vartheta)^2}\Big(adv\wedge d\vartheta + \big(r_+^2+a^2\big)\,d\vartheta\wedge d\varphi\Big)\,, \tag{VI.14}$$

seems to have a non-vanishing electric field component,

$$E_\vartheta = \;F_{u\vartheta} = (a^2Qr_+)\,\frac{\sin 2\vartheta}{(r_+^2+a^2\cos^2\vartheta)^2}\,. \tag{VI.15}$$

However switching to co-moving coordinates, (VI.11), the coordinate transformation $\varphi \to \tilde\varphi = \varphi - \Omega_H v$ with Ω_H given in (VI.10b),

$$\widetilde{F} \to \hat{F} = \big(aQr_+(r_+^2+a^2)\big)\,\frac{\sin 2\vartheta}{(r_+^2+a^2\cos^2\vartheta)^2}\,d\vartheta\wedge d\tilde\varphi\,, \tag{VI.16}$$

eliminates the electric field (VI.15): $\mathbf{E} = 0$ on the horizon. Thus we are left with the pure magnetic field (VI.13), which is indeed left invariant under this coordinate transformation.

[29] $0 \le \vartheta \le \pi$ is oriented with $\vartheta = 0$ at the north pole.

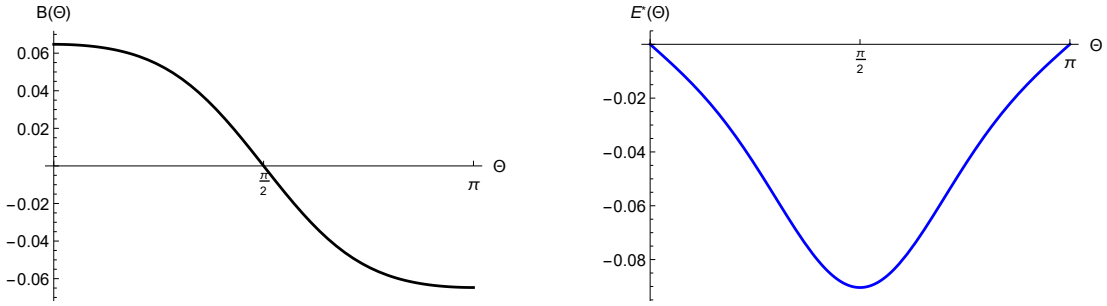
C. Dynamics on the Kerr-Newman black hole horizon

Turning to the dynamics of an "exotic photon" (IV.45) we recall that it has no electric charge, and therefore the terms $e\mathbf{E}$ and eB are switched off, leaving us with the effective fields,

$$B^* = \kappa_{\rm mag}\,, \tag{VI.17a}$$

$$E^*_\vartheta = \mu\chi\,\partial_\vartheta B = -\mu\chi\,\big(2aQr_+(r_+^2+a^2)\big)\,\frac{(r_+^2-5a^2\cos^2\vartheta)}{(r_+^2+a^2\cos^2\vartheta)^4}\sin\vartheta\,, \tag{VI.17b}$$

respectively. The fields B and $\mathbf{E}^* = \boldsymbol{\nabla} B$ are shown on Fig. 9.



(a) (b)

FIG. 9: *(a) The magnetic and (b) the effective electric field, $B = B(\vartheta)$ and $\mathbf{E}^* = \boldsymbol{\nabla} B$, respectively, on the horizon of a Kerr-Newman black hole.*

Thus the general (Hall) equations of motion (IV.42) reduce to[30]

$$\boxed{(x^\vartheta)' = 0, \quad (x^\varphi)' = \underbrace{\big(\mu\chi(2aQr_+)(r_+^2+a^2)\big)\frac{(r_+^2-5a^2\cos^2\vartheta)}{(r_+^2+a^2\cos^2\vartheta)^4}\sin\vartheta}_{E^*_\vartheta} \cdot \underbrace{\big(\frac{1}{\kappa_{\rm mag}}\big)}_{(B^*)^{-1}}} \tag{VI.18}$$

which is the general Hall law with the cast (VI.17). The forces and velocities are depicted in Fig. 10. The trajectories are horizontal ($\vartheta = $ const.) circles; the velocity is determined by E^*_ϑ, shown in Fig. 9b. The angular velocity is maximal on the equator and goes smoothly to zero as we approach the poles. Curiously, the angular velocity depends on the radius of the horizon roughly as r_+^{-3}, which implies higher speed for a smaller black hole.

In conclusion, we get:

[30] Here the coefficient $\mu\chi$ is included into $\mathbf{E}^*$ for convenience, cf. (IV.41).

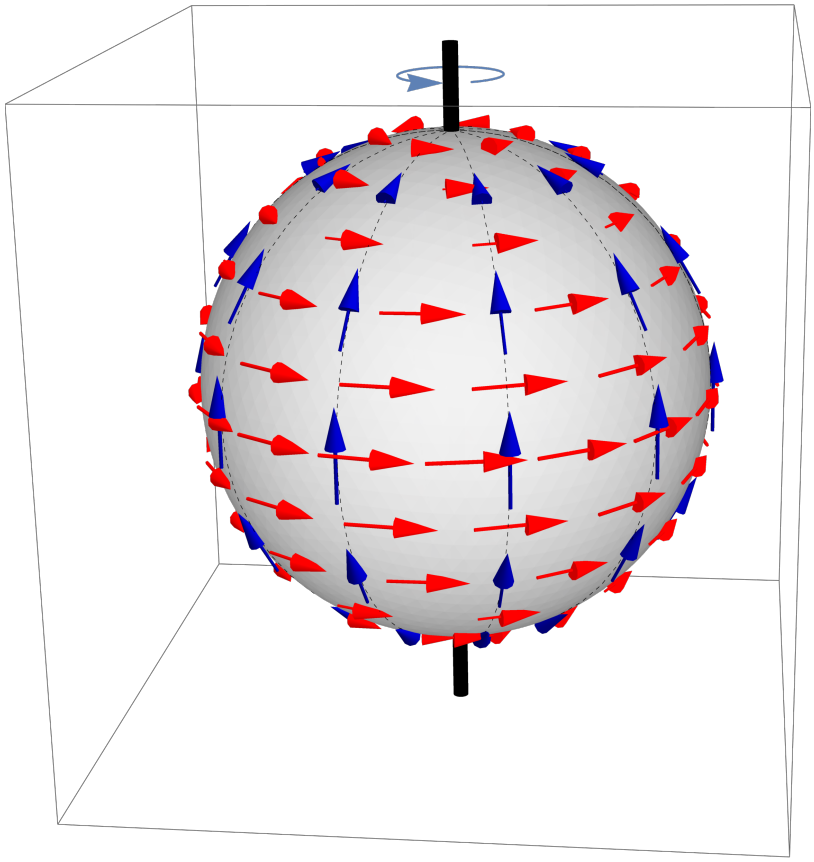


FIG. 10: *The horizon of a Kerr-Newman black hole (topologically a two-sphere) rotates around its vertical axis. For an uncharged particle the usual EM fields are switched off but the spin-field term induces an* **effective electric field** $\mathbf{E}^*$ $\propto \boldsymbol{\nabla} B$ *indicated by vertical* **blue arrows**. *The horizontal* **red arrows** *depict the* **velocity**, (VI.18), *of a massless, uncharged doubly-extended anyon on the horizon. The* **force** *and the* **velocity** *follow the Hall law* (I.2) *which, in the case of the Kerr-Newman black hole, has the form of* (VI.18).

Theorem VI.2. *A massless uncharged anyon with non-vanishing exotic charges* κ_{exo} *and* κ_{mag} *moves on the horizon of a Kerr-Newman black hole by following the Hall law* (VI.18).

The rotation we found is reminiscent of the known frame-dragging by a rotating black hole. It is however *unrelated* to it: frame-dragging is hidden in the coordinates we used in (VI.18). Our "Carroll time" coordinate s is defined indeed through $\partial_s = \partial_v + \Omega_H \partial_\varphi$, see (VI.10b), with Ω_H the angular velocity of the horizon: our coordinates are comoving with the horizon. Equations (VI.13)-(VI.18) and Figs. 9-10 should be compared with those in

flat space, discussed in sec.IV.

One could, theoretically, consider massless particles with electric charge, extending our flat-case study in sect.IV to the curved spaces. Combining the effective fields would yield a distorted version. Massless charged (quasi)particles are considered in condensed matter physics [27].

The general Carrollian equations of motion used to derive (VI.18) depend on the spin-orbit Hamiltonian (II.21), inspired from its flat spacetime form. However we cannot be sure of its the precise expression in curved spacetime until 2+1 dimensional dynamics are derived from an established theory in the ambient spacetime — which does not exists yet, to our knowledge.

One could consider a general Hamiltonian of the form $\mathcal{H} = -\mu\chi\mathscr{B}(\vartheta)$, for some yet unknown $\mathscr{B}(\vartheta)$. It is straightforward to see that the conclusion of this section would not change: there would still be motion on the horizon of the black hole, of the same qualitative form: particles would still display an azimuthal motion.

D. Partially broken Carroll and BMS on the Kerr-Newman horizon

Carroll manifolds have been discussed in connection with BMS symmetry [11] – for the good reason that the BMS group is the conformal extension of the Carroll group [12–14]. Let us briefly summarize how the relation goes. For more details, the reader is advised to consult [13, 14]. The conformal symmetries of a (general) Carroll structure $(\mathcal{C} = \Sigma \times \mathbb{R}, g, \xi)$ are vectorfields on $\mathcal{C}$ s.t.,

$$L_X g = \lambda g, \qquad L_X \xi = \mu\xi \quad \text{with} \quad \lambda + 2\mu = 0\,. \tag{VI.19}$$

Such an X has the form,

$$X = Y + \left(\frac{\lambda}{2}s + \mathscr{T}(x)\right)\partial_s \tag{VI.20}$$

where (x, s) are coordinates on $\mathcal{C}$ and Y is a conformal vector field of (Σ, g_Σ). The $\mathscr{T}(\boldsymbol{x})$ (called "the super translations") are arbitrary functions of the coordinate $\boldsymbol{x}$. So the symmetry group of $(\mathcal{C} = \Sigma \times \mathbb{R}, g, \xi)$ is the semi-direct product $Conf(\Sigma, g_\Sigma) \ltimes \mathcal{T}$ with $\mathcal{T}$ the group of supertranslations $\mathscr{T}$. When λ and thus also μ vanish, such a transformation is a Carrollian isometry, whose form is

$$X = Y + \mathscr{T}(x)\partial_s \tag{VI.21}$$

where Y is a Killing vector of the singular Carroll "metric" g on Σ.

For the round sphere $\Sigma = \mathbb{S}^2$ with its usual metric, for example, one has that $Conf(\mathbb{S}^2, d\Omega^2) = \mathrm{SL}(2,\mathbb{C})$, and thus we get the BMS group: $\mathrm{SL}(2,\mathbb{C}) \ltimes \mathcal{T}$.

For the Kerr-Newman horizon $(\mathbb{S}^2, \widetilde{g})$ with $\widetilde{g}$ as in (VI.11), the conformal isometries are found by solving the system of PDE,

$$2a^2 Y^\vartheta \sin\vartheta\cos\vartheta + (r^2 + a^2\cos^2\vartheta)(\lambda - 2\partial_\vartheta Y^\vartheta) = 0\,, \tag{VI.22}$$

$$2(r^2+a^2)Y^\vartheta\cos\vartheta + (r^2+a^2\cos^2\vartheta)\sin\vartheta(2\partial_{\widetilde{\varphi}}Y^{\widetilde{\varphi}} - \lambda) = 0\,, \tag{VI.23}$$

$$(r^2+a^2)^2\sin^2\vartheta\partial_\vartheta Y^{\widetilde{\varphi}} + (r^2+a^2\cos^2\vartheta)^2\partial_{\widetilde{\varphi}}Y^\vartheta = 0 \tag{VI.24}$$

with $Y^\vartheta, Y^{\widetilde{\varphi}}$ and λ functions of $(\vartheta, \widetilde{\varphi})$. For isometries λ (and thus μ) vanish. The Killing vector of the (spatial part of the) horizon is a rotation around the axis of the black hole [93],

$$Y = \partial_{\widetilde{\varphi}}\,, \tag{VI.25}$$

The spatial isometries of the Kerr-Newman horizon $\mathcal{H}$ are thus the same as that of the full Kerr-Newmann spacetime, $\mathrm{SO}(2) \ltimes \mathcal{T}$, generated by the

$$X = \partial_{\widetilde{\varphi}} + \mathscr{T}(\vartheta, \widetilde{\varphi})\partial_s\,. \tag{VI.26}$$

The symmetries of the equations of motion (IV.42), $(x^i)' = \epsilon^{ij}\left(\frac{E^*_j}{B^*}\right)$, and $p_i' = 0$, spelled out on the horizon are conveniently encoded in the Lagrangian (VI.2c) [or equivalently, in the Cartan form (VI.2a)]:

$$\mathscr{L}_{spec} = \tfrac{1}{2}\kappa_{\mathrm{exo}}\,\epsilon_{ij}v^i\frac{dv^j}{ds} + \tfrac{1}{2}\kappa_{\mathrm{mag}}\,\epsilon_{ij}x^i\frac{dx^j}{ds} + \mu\chi B\,dx^i\,. \tag{VI.27}$$

where $\boldsymbol{x} = (\vartheta, \varphi)$. The magnetic field is a function of ϑ only, $B = B(\vartheta)$ as seen from (VI.13).

Recall now the mechanical version of the Noether theorem : a vector field X is a symmetry if it changes the Cartan form / Lagrangian by a surface term, $\delta\mathscr{L} = df_X$. Then

$$Q_X = \frac{\partial\mathscr{L}}{\partial(x^i)'}X^i - f_X \tag{VI.28}$$

is a conserved quantity[31].

[31] For aficionados of the Souriau approach [59], $L_X\varpi = df_X$ implies that $Q_X = i_X\varpi - f_X$ is conserved.

- The only isometries of the black hole spacetime are "horizontal" rotations generated by $X = \partial_\varphi$ for which we find $\delta\mathscr{L} = -d\big(\kappa_{\rm mag}\vartheta/2\big)$, which combines to yield

$$\pi_\varphi = \kappa_{\rm mag}\vartheta\,. \tag{VI.29}$$

  Its conservation is manifest: trajectories of (VI.18) are "horizontal" ($\vartheta = $ const.). This unusual form agrees with the (3.18c) in [31].

- The zeroth order term of super translations, i.e. a (Carrollian) time-translation $X = \partial_s$ leaves the Lagrangian / Cartan form invariant, providing us with the conserved quantity

$$Q_0 = \mu\chi B = -\mathscr{H}\,, \tag{VI.30}$$

  which is indeed the Hamiltonian (IV.46) of a Carroll anyon.

- For an general super-translation $X = \mathscr{T}(\vartheta,\varphi)\partial_s$ we have, instead,

$$L_X\varpi = \mu\chi B\,\partial_i\mathscr{T}\,dx^i, \tag{VI.31}$$

  which is *not* closed in general,

$$d\big(B\,d\mathscr{T}\big) = dB\wedge d\mathscr{T} \neq 0\,.$$

  Thus the magnetic field breaks the supertranslation symmetry in general. However remembering that $B = B(\vartheta)$ on the horizon, cf. (VI.13), the obstruction vanishes when the supertranslation is "vertical",

$$\mathscr{T} = \mathscr{T}(\vartheta)\,. \tag{VI.32}$$

  For a C-boost for example, the φ-component must vanish, $\mathscr{T}(\vartheta) = -\beta_1^\vartheta\vartheta$. The associated Noether quantity

$$Q_\vartheta = \mu\chi\left(-B\vartheta + \int B d\vartheta\right) \tag{VI.33}$$

  # (V.8) in [32], is a combination of the Carroll Hamiltonian $\mathcal{H}$ (VI.30) and the integral of B. Complicated as it is, Q_ϑ depends on ϑ only, therefore its conservation implies, once again, that the trajectories *must have* $\vartheta = $ const. (as we had found it). On the contrary, φ-boost are broken and thus do not obstruct motion in the φ direction.

- If the supertranslation happens to be induced by the magnetic field itself, e.g.

$$\mathscr{T} \equiv \mathscr{T}_n \propto B^n \tag{VI.34}$$

for some $n = 0, \ldots$, then, we get a conserved quantity for each n,

$$Q_n \propto \big(\mu\chi B\big)^{n+1} = \mathscr{H}^{n+1}. \tag{VI.35}$$

VII. CONCLUSION

Our investigations are devoted to the study of particles with Carroll symmetry, and to their rôle in Hall-type effects.

From the very beginning, a fundamental question has concerned the *mobility* of such a particle [5, 30, 32]. In the massive case, the "no-motion" statement follows either directly from the eqns of motion, (III.7) or alternatively, from the conservation of the quantity (III.10b), associated with Carroll boosts.

After having long limited interest in the subject, the "no-motion property" of Carroll particles [2–4] is now, conversely, *attracting new attention*, namely for quasiparticles called *fractons* (sect. III B), studied in condensed matter physics.

Putting a Carroll particle into an external EM field, the C-boost symmetry (I.1) may be broken in one, but remain unbroken in the perpendicular direction and implying "no-motion" in one, but allowing motion in the perpendicular direction — and realizing the Hall scenario in eqn. (I.2).

The "no-motion" property changes radically for *massless particles* (sec.III C). The conservation of the boost momentum in the free case, (III.46), does not prevent motion any more.

A remarkably fact is that the Carroll group has, in $d = 2$ dimensions, a double central extension [6, 31, 63] with additional parameters we denote by κ_{exo} and κ_{mag}, respectively. Taking them into account, and also that in $d = 2$ we can have anyons with spin χ and magnetic moment μ, provides us with additional terms in their dynamics which may exhibit, under certain circumstances, quite surprising behavior, which is however consistent with the general formula (IV.42) in Theorem IV.7. The motion of extended Carroll particle is subjected to a generalized (anomalous) Hall law which includes a Zeeman force acting on the particle in inhomogeneous fields.

After presenting intuitively appealing toy models in sec.IV F in sec. VI we illustrated our theory in the gravitational context.

Note added. During the final stage to complete this study we were informed about related investigations on magnetized black hole horizons [96].

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