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Processing distortion models: A comparative study

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29 of different models under merging and combination, reminding the basics of each
 30 distortion model in the corresponding section. More precisely:

- 31 • Section 3 deals with the *Pari mutuel model* (PMM), that originates from
 32 betting schemes on horse-racing. It has been studied from the point of view
 33 of imprecise probabilities in [29, 40, 53], and in [30] as a distortion model.
- 34 • Section 4 deals with the *Linear-Vacuous* (LV) *model*, that consists in taking
 35 a mixture between a precise probability measure and the set of all possible
 36 distributions. This model has been used for instance in robust statistics [21].
 37 From the point of view of imprecise probabilities it was studied in [53], and
 38 as a distortion model in [30, Sec. 5]. Moreover, it corresponds to the well-
 39 known and basic discounting approach in evidence theory.
- 40 • Section 5 deals with the *constant-odds ratio* (COR) *model*, that has the
 41 advantage of being stable under Bayesian updating. The constant odds
 42 ratio was given a behavioural interpretation in [53, Sec. 2.9.4]. We refer to
 43 [4, 5, 41, 50] for some applications of this model, and to [30, Sec. 6] for a
 44 detailed study of some of its properties as a distortion model.
- 45 • Section 6 deals with the *total variation* (TV) *model*, which is the distortion
 46 model induced by the *total variation* distance, i.e., the maximum absolute
 47 difference between two probability measures. We refer to [53, Sec. 3.2.4],
 48 [40, Sec. 3.2] and [20], [31, Sec. 2] for some studies of this distortion model.
 49 The total variation distance has also been used in other setting, such as to
 50 find consensus between various probabilities [43] (a task close in spirit to
 51 the one of information merging).
- 52 • Section 7 deals with the *Kolmogorov* (K) *model*, a distortion model induced
 53 by the Kolmogorov distance between cumulative distributions. It is con-
 54 nected to imprecise cumulative distribution functions, also called *p-boxes*.
 55 This distortion model was analysed in detail in [31, Sec. 3].
- 56 • Section 8 deals with the L_1 *model*, a distortion model induced by the L_1
 57 distance. While this distance has been used in robust statistics [42], it was
 58 studied as an imprecise model for the first time in [31, Sec. 4].

59 Finally, in Section 9 we provide our final comments and remarks.

60

2. PRELIMINARY CONCEPTS

61 This section introduces the chosen notations, as well as the necessary general
 62 elements on distortion models and their processing. Readers interested in further
 63 details about those models can refer to [30, 31].

64 **2.1. Notation and basic notions about probability.** We consider in this paper
 65 finite possibility spaces, that will be denoted by \mathcal{X} , \mathcal{Y} or their product space $\mathcal{X} \times \mathcal{Y}$.
 66 We denote by $\mathcal{P}(\mathcal{X})$ the power set of a space \mathcal{X} , by $\mathbb{P}(\mathcal{X})$ the set of probability
 67 measures on \mathcal{X} , and by $\mathbb{P}^*(\mathcal{X})$ the set of probability measures P satisfying $P(A) \in$
 68 $(0, 1)$ for any $A \neq \emptyset, \mathcal{X}$.

69 Whenever $\mathcal{X} = \{x_1, \dots, x_n\}$ is equipped with a total order, we will assume that
 70 $x_1 < \dots < x_n$. In that case, we denote by F_P the cumulative distribution function
 71 (cdf, for short) associated with the probability measure P , given by

$$F_P(x_i) = P(\{x_1, \dots, x_i\}) \text{ for every } i = 1, \dots, n.$$

72 When we deal with two ordered spaces \mathcal{X} and $\mathcal{Y} = \{y_1, \dots, y_m\}$ ($y_1 < \dots < y_m$),
 73 and we consider the product space $\mathcal{X} \times \mathcal{Y}$, every $P \in \mathbb{P}(\mathcal{X} \times \mathcal{Y})$ has an associated

74 bivariate cdf F_P given by

$$F_P(x_i, y_j) = P(\{x_1, \dots, x_i\} \times \{y_1, \dots, y_j\}) \quad \forall i = 1, \dots, n, \quad \forall j = 1, \dots, m.$$

Given a probability measure $P \in \mathbb{P}(\mathcal{X} \times \mathcal{Y})$ or its associated bivariate cdf F_P , we denote by $P^{\mathcal{X}}, F^{\mathcal{X}}$ and $P^{\mathcal{Y}}, F^{\mathcal{Y}}$ its \mathcal{X} and \mathcal{Y} marginals, respectively, given by:

$$P^{\mathcal{X}}(A) = P(A \times \mathcal{Y}) \quad \forall A \subseteq \mathcal{X}, \quad F^{\mathcal{X}}(x_i) = F_P(x_i, y_m) \quad \forall i = 1, \dots, n.$$

$$P^{\mathcal{Y}}(B) = P(\mathcal{X} \times B) \quad \forall B \subseteq \mathcal{Y}, \quad F^{\mathcal{Y}}(y_j) = F_P(x_n, y_j) \quad \forall j = 1, \dots, m.$$

75 Also, every bivariate cdf F_P satisfies the Fréchet-Hoeffding inequalities:

$$\max \{F^{\mathcal{X}}(x_i) + F^{\mathcal{Y}}(y_j) - 1, 0\} \leq F_P(x_i, y_j) \leq \min \{F^{\mathcal{X}}(x_i), F^{\mathcal{Y}}(y_j)\}$$

76 for every $i = 1, \dots, n$ and $j = 1, \dots, m$.

77 **2.2. Imprecise probabilities.** Let us introduce the main notions from the theory
78 of imprecise probabilities that we shall use in this paper. We refer to [2, 47, 53] for
79 a deeper discussion of this theory.

A *lower probability* on a possibility space \mathcal{X} is a function $\underline{P} : \mathcal{P}(\mathcal{X}) \rightarrow [0, 1]$ that is monotone ($A \subseteq B$ implies $\underline{P}(A) \leq \underline{P}(B)$) and normalised ($\underline{P}(\emptyset) = 0, \underline{P}(\mathcal{X}) = 1$). To any lower probability, we can associate a *credal set*, which is a closed and convex set of probability measures defined as:

$$\mathcal{M}(\underline{P}) := \{P \in \mathbb{P}(\mathcal{X}) \mid P(A) \geq \underline{P}(A) \quad \forall A \subseteq \mathcal{X}\}.$$

\underline{P} is called *coherent* if and only if $\mathcal{M}(\underline{P})$ is non-empty and $\underline{P}(A) = \min_{P \in \mathcal{M}(\underline{P})} P(A)$ for every $A \subseteq \mathcal{X}$. We will assume that all the lower probabilities we consider in this paper satisfy this consistency requirement. Since the credal set $\mathcal{M}(\underline{P})$ is closed and convex, it is determined by its *extreme points*, which are those probability measures $P \in \mathbb{P}(\mathcal{X})$ such that

$$(\forall P_1, P_2 \in \mathcal{M}(\underline{P}), \alpha \in (0, 1))(P = \alpha P_1 + (1 - \alpha)P_2 \Rightarrow P_1 = P_2 = P).$$

80 We can associate with a coherent lower probability \underline{P} its conjugate *coherent*
81 *upper probability*, given by $\overline{P}(A) = 1 - \underline{P}(A^c)$ for every $A \subseteq \mathcal{X}$. In fact, every
82 probability measure $P \in \mathcal{M}(\underline{P})$ also satisfies $P(A) \leq \overline{P}(A)$ for every $A \subseteq \mathcal{X}$.

83 A more general notion than lower probability is that of *lower prevision*. A *gamble*
84 on \mathcal{X} is a real-valued function $f : \mathcal{X} \rightarrow \mathbb{R}$, and the set of all the gambles on \mathcal{X} is
85 denoted by $\mathcal{L}(\mathcal{X})$. A *lower prevision* is a map $\underline{P} : \mathcal{L}(\mathcal{X}) \rightarrow \mathbb{R}$. Its associated credal
86 set is

$$\mathcal{M}(\underline{P}) = \{P \in \mathbb{P}(\mathcal{X}) \mid P(f) \geq \underline{P}(f) \quad \forall f \in \mathcal{L}(\mathcal{X})\},$$

87 where, in order to ease the notation, we are using the same symbol to denote a
88 probability measure P and its associated expectation operator. A lower prevision
89 \underline{P} is called *coherent* if and only if $\mathcal{M}(\underline{P})$ is non-empty and $\underline{P}(f) = \min_{P \in \mathcal{M}(\underline{P})} P(f)$
90 for every gamble $f \in \mathcal{L}(\mathcal{X})$. Given a coherent lower prevision \underline{P} , its associated
91 conjugate *coherent upper prevision* is given by $\overline{P}(f) = -\underline{P}(-f)$ for any $f \in \mathcal{L}(\mathcal{X})$.

92 When a coherent lower prevision \underline{P} is restricted to indicators² of events, i.e., we
93 restrict to the set of gambles $\{I_A \mid A \subseteq \mathcal{X}\}$, the coherent lower prevision becomes
94 a coherent lower probability, where we use the notation $\underline{P}(A)$ for $\underline{P}(I_A)$. However,
95 two different coherent lower previsions may have the same restriction to indicators
96 of events, and as a consequence may induce the same coherent lower probability.

²Recall that the indicator I_A of an event A is the gamble that takes the value 1 on the elements of A and 0 elsewhere.

97 A subfamily of particular interest within coherent lower probabilities is given by
 98 those that satisfy 2-monotonicity:

Definition 1. A lower probability $\underline{P} : \mathcal{P}(\mathcal{X}) \rightarrow [0, 1]$ is 2-monotone when it satisfies

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B) \quad \forall A, B \subseteq \mathcal{X}.$$

99 We refer to [14, 51] for a detailed account of these models. In particular, they
 100 include those associated with probability boxes. Assuming that \mathcal{X} is equipped with
 101 a total order, a (univariate) *probability box* [19] is a pair of cdfs $\underline{F}, \overline{F} : \mathcal{X} \rightarrow [0, 1]$,
 102 called lower and upper cdfs, satisfying $\underline{F} \leq \overline{F}$. A p-box $(\underline{F}, \overline{F})$ defines a credal set
 103 $\mathcal{M}(\underline{F}, \overline{F})$ by:

$$\mathcal{M}(\underline{F}, \overline{F}) = \{P \in \mathbb{P}(\mathcal{X}) \mid \underline{F} \leq F_P \leq \overline{F}\}.$$

104 The lower and upper envelopes of $\mathcal{M}(\underline{F}, \overline{F})$, $\underline{P}_{(\underline{F}, \overline{F})}$ and $\overline{P}_{(\underline{F}, \overline{F})}$, are conjugate
 105 coherent lower and upper probabilities satisfying

$$\underline{P}_{(\underline{F}, \overline{F})}(\{x_1, \dots, x_i\}) = \underline{F}(x_i) \text{ and } \overline{P}_{(\underline{F}, \overline{F})}(\{x_1, \dots, x_i\}) = \overline{F}(x_i) \quad \forall i = 1, \dots, n.$$

106 These coherent lower and upper probabilities can be computed following the results
 107 in [48], where it was proven that $\underline{P}_{(\underline{F}, \overline{F})}$ is 2-monotone (in fact, it satisfies the
 108 stronger property of complete monotonicity). We refer to [17, 27, 48, 49] for detailed
 109 studies about (univariate) p-boxes.

110 **2.3. Distortion models.** In this paper, our focus is on a family of imprecise prob-
 111 ability models that are usually referred to as *distortion models* [6, 9, 21]. They can
 112 arise by considering a neighbourhood model around some probability measure using
 113 some distorting function d and some distortion factor $\delta > 0$ (as in [22, 41, 50]),
 114 or making a transformation of a given (lower) probability (as in [7, 10, 44]). We
 115 showed in [30, Prop. 2] that this second approach can be embedded in the first,
 116 and for this reason we consider here distortion models defined in terms of neigh-
 117 bourhoods. Given a distorting function $d : \mathbb{P}(\mathcal{X}) \times \mathbb{P}(\mathcal{X}) \rightarrow [0, \infty)$, a distortion
 118 parameter $\delta > 0$ and a fixed probability measure $P_0 \in \mathbb{P}(\mathcal{X})$, we can define the
 119 following set of probabilities:

$$B_d^\delta(P_0) = \{P \in \mathbb{P}(\mathcal{X}) \mid d(P, P_0) \leq \delta\}.$$

120 Whenever d is convex and continuous, $B_d^\delta(P_0)$ is a convex and closed set of proba-
 121 bilities [30, Prop. 1]. This means that if we consider its lower envelope:

$$\underline{P}_d(f) = \min \{P(f) \mid P \in B_d^\delta(P_0)\} \quad \forall f \in \mathcal{L}(\mathcal{X}),$$

122 the credal sets $\mathcal{M}(\underline{P}_d)$ and $B_d^\delta(P_0)$ coincide, and \underline{P}_d is a coherent lower prevision.

123 For the sake of simplicity, in [30, 31] we assumed that $P_0 \in \mathbb{P}^*(\mathcal{X})$, i.e. P_0 is
 124 strictly positive for every non-empty event, and that also δ is small enough so that
 125 $B_d^\delta(P_0) \subseteq \mathbb{P}^*(\mathcal{X})$. In this paper, we also assume that this simplifying hypothesis
 126 holds.

127 **2.4. Processing imprecise probabilistic models.** One of the criteria that may
 128 be used in order to choose one uncertainty model over another is that it is closed
 129 under a number of operations that we may perform. These operations may arise for
 130 instance from the combination of different sources of information, the extension to
 131 a different domain, or the updating under the presence of new information. Next,
 132 we introduce the procedures that shall be analysed in this paper.

133 2.4.1. *Merging.* The first operation we shall consider in this paper is that of *merg-*
 134 *ing.* By this, we refer to the procedure where we aggregate a number of belief
 135 models, defined on the same domain \mathcal{X} , into a unique one. These models may arise
 136 as the opinion of different experts or several data sources, for instance. The prob-
 137 lem of aggregating imprecise beliefs has been analysed from the axiomatic point of
 138 view by Walley [52]. Other relevant works on this topic are [36, 37].

139 In this paper, we shall focus on the three most fundamental merging procedures:
 140 those of *conjunction*, *disjunction* and *convex mixture*. If we model our beliefs in
 141 terms of two credal sets $\mathcal{M}_1, \mathcal{M}_2$, they will produce the sets $\mathcal{M}_1 \cap \mathcal{M}_2$, $\mathcal{M}_1 \cup \mathcal{M}_2$
 142 and $\epsilon\mathcal{M}_1 + (1 - \epsilon)\mathcal{M}_2 = \{\epsilon P_1 + (1 - \epsilon)P_2 | P_i \in \mathcal{M}_i\}$ with $\epsilon \in [0, 1]$, respectively.

143 In terms of the lower probabilities associated with these sets, it should be noted
 144 that, while $\mathcal{M}_1 \cup \mathcal{M}_2$ is not convex in general, its lower envelope, that coincides with
 145 the lower envelope of its convex hull $ch(\mathcal{M}_1 \cup \mathcal{M}_2)$, is given by $\underline{P} := \min\{\underline{P}_1, \underline{P}_2\}$,
 146 where $\underline{P}_1, \underline{P}_2$ denote the lower envelopes of $\mathcal{M}_1, \mathcal{M}_2$, respectively. Since we fo-
 147 cus in this paper on lower probabilities and previsions, we can restrict ourselves
 148 to³ $ch(\mathcal{M}_1 \cup \mathcal{M}_2)$. While this disjunction will always determine a coherent lower
 149 probability by considering its associated lower envelope of events, the latter may
 150 not belong to the same family as the original models $\underline{P}_1, \underline{P}_2$. In that case, one
 151 possibility would be to consider an outer approximation; in this respect, our earlier
 152 work in [24, 33, 34] shall be useful.

153 In contrast, while $\mathcal{M}_1 \cap \mathcal{M}_2$ is convex, its lower envelope \underline{P} will dominate in
 154 general $\max\{\underline{P}_1, \underline{P}_2\}$. It is not difficult to see that for any linear prevision P it
 155 holds that $P \in \mathcal{M}_1 \cap \mathcal{M}_2$ if and only if $P \geq \max\{\underline{P}_1, \underline{P}_2\}$. This means that \underline{P}
 156 is the *natural extension* (the smallest dominating coherent lower probability) of
 157 $\max\{\underline{P}_1, \underline{P}_2\}$. A sufficient condition for the equality between them is precisely the
 158 convexity of $\mathcal{M}_1 \cup \mathcal{M}_2$, as shown in [54, Thm. 6]. The equality between the lower
 159 envelope of $\mathcal{M}_1 \cap \mathcal{M}_2$ and $\max\{\underline{P}_1, \underline{P}_2\}$ was investigated in the case of possibility
 160 measures in [25], and it will be analysed here for distortion models.

161 Finally, $\epsilon\mathcal{M}_1 + (1 - \epsilon)\mathcal{M}_2$ is always convex, and its lower envelope is such that
 162 $\underline{P} := \epsilon\underline{P}_1 + (1 - \epsilon)\underline{P}_2$.

163 2.4.2. *Marginal and joint models in multivariate settings.* Another relevant scenario
 164 is the restriction of the model to a smaller domain or its extension to a larger one.
 165 In this paper, we shall focus on the case where our possibility space is the product
 166 $\mathcal{X} \times \mathcal{Y}$ of two finite spaces. In that case, we may move from the joint model to the
 167 marginals, or viceversa.

168 Marginalisation

169 In this first case, given a joint model $\underline{P}^{\mathcal{X}, \mathcal{Y}}$ defined on the space $\mathcal{X} \times \mathcal{Y}$, we
 170 can consider the marginal models $\underline{P}^{\mathcal{X}}$ and $\underline{P}^{\mathcal{Y}}$, defined on \mathcal{X} and \mathcal{Y} , respectively.
 171 Their corresponding credal sets $\mathcal{M}(\underline{P}^{\mathcal{X}})$ and $\mathcal{M}(\underline{P}^{\mathcal{Y}})$ are formed by the \mathcal{X} - and
 172 \mathcal{Y} -projections of the probability measures in $\mathcal{M}(\underline{P}^{\mathcal{X}, \mathcal{Y}})$, respectively.

³It should however be noted that convexity is not desirable in every situation: for instance Seidenfeld *et al.* [45] show in a decision-making context that when considering pairs of imprecise probabilities and utilities, one may have to let go of convexity.

173 Independent products

174 Conversely, we may start from two marginal models $\underline{P}^{\mathcal{X}}$ and $\underline{P}^{\mathcal{Y}}$ defined on \mathcal{X}
 175 and \mathcal{Y} , respectively, and build a joint model on $\mathcal{X} \times \mathcal{Y}$ that is compatible with
 176 them. When the sources are assumed to be independent, this leads us to consider
 177 an *independent product*. Out of the several extensions of the notion of independence
 178 to the imprecise case [11], we consider in this paper the *strong product* of $\underline{P}^{\mathcal{X}}$ and
 179 $\underline{P}^{\mathcal{Y}}$, that we shall denote $\underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}$. It is the lower envelope of

$$\mathcal{M}(\underline{P}^{\mathcal{X}}) \boxtimes \mathcal{M}(\underline{P}^{\mathcal{Y}}) = \{P^{\mathcal{X}} \times P^{\mathcal{Y}} \mid P^{\mathcal{X}} \in \mathcal{M}(\underline{P}^{\mathcal{X}}), P^{\mathcal{Y}} \in \mathcal{M}(\underline{P}^{\mathcal{Y}})\}, \quad (1)$$

where $P^{\mathcal{X}} \times P^{\mathcal{Y}}$ denotes the joint probability defined using the marginals $P^{\mathcal{X}}$ and $P^{\mathcal{Y}}$
 through stochastic independence. The strong product $\underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}$ and its conjugate
 $\overline{P}^{\mathcal{X}} \boxtimes \overline{P}^{\mathcal{Y}}$ satisfy the following factorisation properties on events:

$$\begin{aligned} \underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}(A \times B) &= \underline{P}^{\mathcal{X}}(A) \cdot \underline{P}^{\mathcal{Y}}(B) \quad \text{for any } A \subseteq \mathcal{X}, B \subseteq \mathcal{Y}. & (2) \\ \overline{P}^{\mathcal{X}} \boxtimes \overline{P}^{\mathcal{Y}}(A \times B) &= \overline{P}^{\mathcal{X}}(A) \cdot \overline{P}^{\mathcal{Y}}(B) \quad \text{for any } A \subseteq \mathcal{X}, B \subseteq \mathcal{Y}. \end{aligned}$$

180 Natural extension of marginal models

Alternatively, given two marginal models $\underline{P}^{\mathcal{X}}$ and $\underline{P}^{\mathcal{Y}}$ on \mathcal{X} and \mathcal{Y} , respectively,
 we may look for the most conservative joint model in $\mathcal{X} \times \mathcal{Y}$ with these given
 marginals, imposing no dependence assumption whatsoever. Using Walley's termi-
 nology [53], this corresponds to considering the *natural extension* \underline{E} of the coherent
 lower probability \underline{P} that is defined on $\{A \times \mathcal{Y} : A \subseteq \mathcal{X}\} \cup \{\mathcal{X} \times B : B \subseteq \mathcal{Y}\}$ by

$$\underline{P}(A \times \mathcal{Y}) = \underline{P}^{\mathcal{X}}(A) \quad \text{and} \quad \underline{P}(\mathcal{X} \times B) = \underline{P}^{\mathcal{Y}}(B).$$

181 It can be equivalently obtained as the lower envelope of the credal set given by
 182 those probabilities whose marginals are compatible with the information provided
 183 by $\underline{P}^{\mathcal{X}}$ and $\underline{P}^{\mathcal{Y}}$:

$$\mathcal{E}(\underline{P}^{\mathcal{X}}, \underline{P}^{\mathcal{Y}}) = \left\{ P \in \mathbb{P}(\mathcal{X} \times \mathcal{Y}) \mid P^{\mathcal{X}} \in \mathcal{M}(\underline{P}^{\mathcal{X}}), P^{\mathcal{Y}} \in \mathcal{M}(\underline{P}^{\mathcal{Y}}) \right\}. \quad (3)$$

If we consider the upper envelope \overline{E} of this set we obtain the conjugate upper
 prevision of \underline{E} , that corresponds to the (upper) natural extension of the coherent
 upper probability \overline{P} given by

$$\overline{P}(A \times \mathcal{Y}) = \overline{P}^{\mathcal{X}}(A) \quad \text{and} \quad \overline{P}(\mathcal{X} \times B) = \overline{P}^{\mathcal{Y}}(B).$$

184 Equivalently,

$$\underline{E}(C) = \inf_{P \in \mathcal{E}(\underline{P}^{\mathcal{X}}, \underline{P}^{\mathcal{Y}})} P(C), \quad \overline{E}(C) = \sup_{P \in \mathcal{E}(\underline{P}^{\mathcal{X}}, \underline{P}^{\mathcal{Y}})} P(C) \quad \forall C \subseteq \mathcal{X} \times \mathcal{Y}. \quad (4)$$

185 The study of a joint distribution with given marginals has received a long standing
 186 attention in the literature; see for instance [1, 23, 26, 46]. Its existence is trivial in
 187 situations like the one considered in this paper: where the marginals are established
 188 in disjoint sets of variables. In that case, we can use the techniques of natural
 189 extension to determine the lower and upper envelopes of all such joints. Our next
 190 proposition gives the expression of $\underline{E}, \overline{E}$ on Cartesian products of events:

Proposition 1. *Let $\underline{P}^{\mathcal{X}}$ and $\underline{P}^{\mathcal{Y}}$ be two coherent lower probabilities on \mathcal{X} and \mathcal{Y} , respectively, with conjugates $\overline{P}^{\mathcal{X}}$ and $\overline{P}^{\mathcal{Y}}$. Then:*

$$\underline{E}(A \times B) = \max \{ \underline{P}^{\mathcal{X}}(A) + \underline{P}^{\mathcal{Y}}(B) - 1, 0 \}, \quad (5)$$

$$\overline{E}(A \times B) = \min \{ \overline{P}^{\mathcal{X}}(A), \overline{P}^{\mathcal{Y}}(B) \} \quad \forall A \subseteq \mathcal{X}, B \subseteq \mathcal{Y}. \quad (6)$$

191 *Proof.* This is a consequence of the result stated by Walley in [53, Sec. 3.1.1] for
 192 intersections of events. \square

In the particular case where we start with precise marginals $P_0^{\mathcal{X}}$ and $P_0^{\mathcal{Y}}$ on \mathcal{X}, \mathcal{Y} , we obtain

$$\underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}}(A \times B) = \max \{ P_0^{\mathcal{X}}(A) + P_0^{\mathcal{Y}}(B) - 1, 0 \}, \quad (7)$$

$$\overline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}}(A \times B) = \min \{ P_0^{\mathcal{X}}(A), P_0^{\mathcal{Y}}(B) \} \quad \forall A \subseteq \mathcal{X}, B \subseteq \mathcal{Y}. \quad (8)$$

193 From Proposition 1 we obtain a simple procedure for computing the natural extension
 194 in Equation (3) for events of the type $A \times B$, which are simply the Fréchet-
 195 Hoeffding bounds. One may think that the expressions in Equations (5) and (6)
 196 also hold when considering any event $C \subseteq \mathcal{X} \times \mathcal{Y}$, and decomposing it into its \mathcal{X} -
 197 and \mathcal{Y} -projections. However, our next example shows that this is not always the
 198 case.

199 **Example 1.** *Let $\mathcal{X} = \{x_1, x_2, x_3\}$ and $\mathcal{Y} = \{y_1, y_2, y_3\}$ and let the coherent conju-*
 200 *gate lower and upper probabilities be given by:*

A	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
$\underline{P}^{\mathcal{X}}(A)$	0.1	0.2	0.5	0.4	0.7	0.8
$\overline{P}^{\mathcal{X}}(A)$	0.2	0.3	0.6	0.5	0.8	0.9
B	$\{y_1\}$	$\{y_2\}$	$\{y_3\}$	$\{y_1, y_2\}$	$\{y_1, y_3\}$	$\{y_2, y_3\}$
$\underline{P}^{\mathcal{Y}}(B)$	0.1	0.2	0.4	0.4	0.7	0.7
$\overline{P}^{\mathcal{Y}}(B)$	0.3	0.3	0.6	0.6	0.8	0.9

201 *Consider the event $C_1 = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$, whose projections are $C_1^{\mathcal{X}} =$*
 202 *\mathcal{X} , $C_1^{\mathcal{Y}} = \mathcal{Y}$, and the probability mass function P given by*

y_3	0	0	0.4
y_2	0	0.1	0.2
y_1	0.2	0.1	0
$P(\{(x_i, y_j)\})$	x_1	x_2	x_3

203 *Its marginals dominate $\underline{P}^{\mathcal{X}}, \underline{P}^{\mathcal{Y}}$, respectively. As a consequence, $\underline{E}(C_1) \leq P(C_1) =$*
 204 *0.7, while*

$$\max \{ \underline{P}^{\mathcal{X}}(C_1^{\mathcal{X}}) + \underline{P}^{\mathcal{Y}}(C_1^{\mathcal{Y}}) - 1, 0 \} = 1. \blacklozenge$$

205 **2.5. Aim of the paper.** In [30, 31], we carried out a comparative analysis of six
 206 distortion models: the pari mutuel, linear vacuous, constant odds ratio, total vari-
 207 ation, Kolmogorov and L_1 distance models. The aim was to give guidelines about
 208 which model performs better on some respect or which one is more appropriate
 209 in each particular scenario. Specifically, we analysed the following features: (i)
 210 the amount of imprecision present in the model once the probability measure P_0
 211 and the distortion factor $\delta > 0$ are fixed; (ii) the properties of the associated lower

212 probability as a non-additive measure; (iii) the complexity of their associated neigh-
 213 bourhood models, in terms of the number of extreme points; and (iv) the behaviour
 214 of the model under conditioning.

215 Our goal in this paper is to complement the analysis performed in [31, Sec. 5]
 216 by investigating the behaviour of the different families of distortion models under
 217 the procedures described in Section 2.4. Specifically, we shall tackle the following
 218 problems:

219 **Merging:** We first analyse if the distortion models are closed under merging.
 220 For this aim, we consider two distortion models $B_d^{\delta_1}(P_0^1) \subseteq \mathbb{P}^*(\mathcal{X})$ and
 221 $B_d^{\delta_2}(P_0^2) \subseteq \mathbb{P}^*(\mathcal{X})$ in some specific family. Our aim is to know whether
 222 their conjunction $B_d^{\delta_1}(P_0^1) \cap B_d^{\delta_2}(P_0^2)$, their disjunction $B_d^{\delta_1}(P_0^1) \cup B_d^{\delta_2}(P_0^2)$
 223 or their mixture $\epsilon B_d^{\delta_1}(P_0^1) + (1 - \epsilon)B_d^{\delta_2}(P_0^2)$ belong to the same family,
 224 in the sense that it is equal to $B_d^{\delta^*}(P_0^*)$ for some appropriate δ^* and P_0^* .
 225 As we shall see, this is almost always the case for the convex mixture⁴,
 226 sometimes the case for the conjunction, and never for the disjunction. In
 227 this last case, we might then consider the convex hull of the disjunction
 228 $ch(B_d^{\delta_1}(P_0^1) \cup B_d^{\delta_2}(P_0^2))$ and investigate whether it has a unique outer ap-
 229 proximation in the same family [8, 33, 34]. By an *outer approximation* of
 230 a coherent lower probability \underline{P} in some family \mathcal{C} we mean some $\underline{Q} \in \mathcal{C}$ such
 231 that $\underline{Q} \leq \underline{P}$. We will say that the outer approximation is *undominated*
 232 when there is no $\underline{Q}' \in \mathcal{C}$ such that $\underline{Q} \leq \underline{Q}' \leq \underline{P}$.

233 Note that since we are assuming that $B_d^{\delta_1}(P_0^1)$ and $B_d^{\delta_2}(P_0^2)$ are included
 234 in $\mathbb{P}^*(\mathcal{X})$, their convex mixture and their intersection will be also included
 235 in $\mathbb{P}^*(\mathcal{X})$. However, an undominated outer approximation of the disjunction
 236 need not be included in $\mathbb{P}^*(\mathcal{X})$.

237 **Marginalisation:** Given a distortion model $B_d^\delta(P_0^{\mathcal{X},\mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{X} \times \mathcal{Y})$ with associ-
 238 ated lower prevision \underline{P}_d , we want to know whether the marginal models $\underline{P}_d^{\mathcal{X}}$
 239 and $\underline{P}_d^{\mathcal{Y}}$ correspond to distortion models of the same family on $\mathcal{P}(\mathcal{X})$ and
 240 $\mathcal{P}(\mathcal{Y})$, respectively. In other words, we want to know if $\mathcal{M}(\underline{P}_d^{\mathcal{X}}) = B_d^\delta(P_0^{\mathcal{X}})$
 241 and $\mathcal{M}(\underline{P}_d^{\mathcal{Y}}) = B_d^\delta(P_0^{\mathcal{Y}})$.

242 **Independent products:** Consider two distortion models $B_d^\delta(P_0^{\mathcal{X}}) \subseteq \mathbb{P}^*(\mathcal{X})$ and
 243 $B_d^\delta(P_0^{\mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{Y})$ with the same distortion parameter. We want to build
 244 a joint model using independence, and we want to know whether the joint
 245 model belongs to the same family. Two solutions are then possible in the
 246 setting of this study:

- 247 • Combine $P_0^{\mathcal{X}}$ and $P_0^{\mathcal{Y}}$ into a joint probability $P_0^{\mathcal{X},\mathcal{Y}} \in \mathbb{P}^*(\mathcal{X} \times \mathcal{Y})$ and
 248 apply the distortion to it. In this way, we obtain the distortion model
 249 $B_d^\delta(P_0^{\mathcal{X},\mathcal{Y}})$. We shall denote by $\underline{P}^{\mathcal{X} \times \mathcal{Y}}$ and $\overline{P}^{\mathcal{X} \times \mathcal{Y}}$ the resulting lower
 250 and upper probabilities obtained as the lower and upper envelope of
 251 the credal set $B_d^\delta(P_0^{\mathcal{X},\mathcal{Y}})$, respectively.
- 252 • Consider the distortion models $B_d^\delta(P_0^{\mathcal{X}}) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_d^\delta(P_0^{\mathcal{Y}}) \subseteq$
 253 $\mathbb{P}^*(\mathcal{Y})$ and combine them using the strong product. By Equation (1),
 254 this produces the set $\mathcal{M}(\underline{P}^{\mathcal{X}}) \boxtimes \mathcal{M}(\underline{P}^{\mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{X} \times \mathcal{Y})$, whose lower

⁴The constant odds ratio model is an exception.

255 and upper envelopes are the lower and upper probabilities $\underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}$
 256 and $\overline{P}^{\mathcal{X}} \boxtimes \overline{P}^{\mathcal{Y}}$.

257 We wonder if the lower envelopes of $B_d^\delta(P_0^{\mathcal{X}, \mathcal{Y}})$ and $\mathcal{M}(\underline{P}^{\mathcal{X}}) \boxtimes \mathcal{M}(\underline{P}^{\mathcal{Y}})$
 258 coincide, or in case they do not, whether there is a dominance relationship
 259 between them, meaning that one of the procedures is more precise than the
 260 other. In other words, we shall analyse whether there is some dominance
 261 relation between $\underline{P}^{\mathcal{X} \times \mathcal{Y}}$ and $\underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}$.

262 We will also check whether the different models are closed under the
 263 operation of independent product, that is, whether $\underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}$ is also a
 264 distortion model of the same family.

265 **Natural extension:** Consider two marginal distortion models $B_d^{\delta \mathcal{X}}(P_0^{\mathcal{X}}) \subseteq \mathbb{P}^*(\mathcal{X})$
 266 and $B_d^{\delta \mathcal{Y}}(P_0^{\mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{Y})$ with associated lower probabilities $\underline{P}^{\mathcal{X}}$ and $\underline{P}^{\mathcal{Y}}$.
 267 We want to compute their least committal extension to $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$, i.e. the
 268 natural extension given in Equation (3). In this case, the credal set of the
 269 natural extension is

$$\mathcal{E}(\underline{P}^{\mathcal{X}}, \underline{P}^{\mathcal{Y}}) = \{P \in \mathbb{P}(\mathcal{X} \times \mathcal{Y}) \mid P^{\mathcal{X}} \in B_d^{\delta \mathcal{X}}(P_0^{\mathcal{X}}), P^{\mathcal{Y}} \in B_d^{\delta \mathcal{Y}}(P_0^{\mathcal{Y}})\}.$$

270 We already know the expression of \underline{E} and its conjugate \overline{E} on events like
 271 $A \times B$ (see Equations (5) and (6)). We wonder whether we can give a simple
 272 expression of $\mathcal{E}(\underline{P}^{\mathcal{X}}, \underline{P}^{\mathcal{Y}})$, \underline{E} and \overline{E} and also whether $\mathcal{E}(\underline{P}^{\mathcal{X}}, \underline{P}^{\mathcal{Y}})$ is also the
 273 credal set of a distortion model of the same family.

274 Note that, although $B_d^{\delta \mathcal{X}}(P_0^{\mathcal{X}})$ and $B_d^{\delta \mathcal{Y}}(P_0^{\mathcal{Y}})$ are included in $\mathbb{P}^*(\mathcal{X})$ and
 275 $\mathbb{P}^*(\mathcal{Y})$, respectively, we cannot guarantee the resulting model $\mathcal{E}(\underline{P}^{\mathcal{X}}, \underline{P}^{\mathcal{Y}})$
 276 to be included in $\mathbb{P}^*(\mathcal{X} \times \mathcal{Y})$.

277 From now on, we devote one section to each of the six distortion models mentioned
 278 in the introduction and we analyse their behaviour under the previous operations.

279 3. PARI MUTUEL MODEL

280 The first distortion model we analyse in this paper is the pari mutuel model
 281 (PMM, for short):

Definition 2. *Given a probability measure P_0 and a distortion factor $\delta > 0$, the associated pari mutuel model is determined by the following lower and upper probabilities:*

$$\begin{aligned} \underline{P}_{PMM}(A) &= \max\{(1 + \delta)P_0(A) - \delta, 0\}, \\ \overline{P}_{PMM}(A) &= \min\{(1 + \delta)P_0(A), 1\} \quad \forall A \subseteq \mathcal{X}. \end{aligned}$$

282 Since we are assuming that our initial model satisfies $P_0 \in \mathbb{P}^*(\mathcal{X})$ and that its
 283 lower probability \underline{P}_{PMM} takes strictly positive values on non-empty events, the
 284 previous expressions simplify to:

$$\underline{P}_{PMM}(A) = (1 + \delta)P_0(A) - \delta, \quad \overline{P}_{PMM}(A) = (1 + \delta)P_0(A) \quad \forall A \neq \emptyset, \mathcal{X},$$

285 and taking the trivial values 0 and 1 for \emptyset and \mathcal{X} , respectively.

286 It was shown in [30, Thm. 5] that the credal set $\mathcal{M}(\underline{P}_{PMM})$ coincides with
 287 $B_{d_{PMM}}^\delta(P_0)$, where $d_{PMM} : \mathbb{P}^*(\mathcal{X}) \times \mathbb{P}^*(\mathcal{X}) \rightarrow [0, \infty)$ is the distorting function
 288 given by

$$d_{PMM}(P, Q) = \max_{A \subseteq \mathcal{X}} \frac{Q(A) - P(A)}{1 - Q(A)}.$$

289 Next, we complement the work in [30] by investigating the behaviour of the family
 290 of pari mutuel models under a number of operations.

291 **3.1. Merging.** Let us first study the behaviour of the PMM under merging oper-
 292 ators.

293 Conjunction

294 We start by analysing the conjunction of PMMs. Given two neighbourhood
 295 models $B_{d_{PMM}}^{\delta_1}(P_0^1) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{d_{PMM}}^{\delta_2}(P_0^2) \subseteq \mathbb{P}^*(\mathcal{X})$, it was established in [29,
 296 Prop. 12] that their intersection is non-empty iff

$$\sum_{x \in \mathcal{X}} \min \{(1 + \delta_1)P_0^1(\{x\}), (1 + \delta_2)P_0^2(\{x\}), 1\} \geq 1. \quad (9)$$

This intersection is the PMM $B_{d_{PMM}}^{\delta^\cap}(P_0^\cap)$, for

$$\delta^\cap = \left(\sum_{x \in \mathcal{X}} \min \{(1 + \delta_1)P_0^1(\{x\}), (1 + \delta_2)P_0^2(\{x\})\} \right) - 1, \quad \text{and}$$

$$P_0^\cap(\{x\}) = \frac{\min \{(1 + \delta_1)P_0^1(\{x\}), (1 + \delta_2)P_0^2(\{x\})\}}{1 + \delta^\cap} \quad \forall x \in \mathcal{X}.$$

297 Disjunction

298 Regarding the disjunction, the convex hull of $B_{d_{PMM}}^{\delta_1}(P_0^1) \cup B_{d_{PMM}}^{\delta_2}(P_0^2)$ will not
 299 be in general a PMM, as we show in the following example.

300 **Example 2.** Consider $P_0^1 = (0.5, 0.3, 0.2)$, $P_0^2 = (0.3, 0.5, 0.2)$ and $\delta_1 = \delta_2 = 0.1$.
 301 The associated PMMs \underline{P}_{PMM_1} , \underline{P}_{PMM_2} and their disjunction \underline{P}^\cup are given in the
 302 following table:

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
\underline{P}_{PMM_1}	0.45	0.23	0.12	0.78	0.67	0.45
\underline{P}_{PMM_2}	0.23	0.45	0.12	0.78	0.45	0.67
\underline{P}^\cup	0.23	0.23	0.12	0.78	0.45	0.45

303 If it was $\mathcal{M}(\underline{P}^\cup) = B_{d_{PMM}}^\delta(P_0)$ for some P_0, δ , then we would obtain

$$\sum_{x \in \mathcal{X}} \underline{P}^\cup(\{x\}) = 0.58 = 1 - 2\delta \Rightarrow \delta = 0.21;$$

304 on the other hand, the equality

$$0.45 = \underline{P}^\cup(\{x_2, x_3\}) = (1 + \delta)P_0(\{x_2, x_3\}) - \delta = \underline{P}^\cup(\{x_2\}) + \underline{P}^\cup(\{x_3\}) + \delta,$$

305 means that it should be $\delta = 0.1$. Thus, \underline{P}^\cup is not a PMM. \blacklozenge

Interestingly, this disjunction has a unique undominated outer-approximation that is a PMM [33, Prop. 7]. It is given by the model $B_{d_{PMM}}^{\delta^\cup}(P_0^\cup)$ such that:

$$\delta^\cup = \left(\sum_{x \in \mathcal{X}} \max \{ (1 + \delta_1)P_0^1(\{x\}), (1 + \delta_2)P_0^2(\{x\}) \} \right) - 1, \quad \text{and}$$

$$P_0^\cup(\{x\}) = \frac{\max \{ (1 + \delta_1)P_0^1(\{x\}), (1 + \delta_2)P_0^2(\{x\}) \}}{1 + \delta^\cup} \quad \forall x \in \mathcal{X}.$$

306 This is the greatest (in terms of \underline{P}), or more informative, PMM whose associated
307 neighbourhood includes the disjunction $B_{d_{PMM}}^{\delta_1}(P_0^1) \cup B_{d_{PMM}}^{\delta_2}(P_0^2)$.

308 Convex mixture

The mixture operation was studied in [33, Sec. 5.1], where it was shown that the convex mixture of two PMM is again a PMM, given by $B_{d_{PMM}}^{\delta_\epsilon}(P_0^\epsilon)$ where

$$1 + \delta_\epsilon = \epsilon(1 + \delta_1) + (1 - \epsilon)(1 + \delta_2) \quad \text{and}$$

$$P_0^\epsilon(\{x\}) = \frac{\epsilon(1 + \delta_1)P_0^1(\{x\}) + (1 - \epsilon)(1 + \delta_2)P_0^2(\{x\})}{1 + \delta_\epsilon} \quad \forall x \in \mathcal{X}.$$

309 **3.2. Multivariate setting.** Let us now look at the behaviour of the PMM in a
310 multivariate setting.

311 Marginalisation

312 In [29, Sec. 6.2], it was shown that the marginal lower probability $\underline{P}^\mathcal{X}$ obtained
313 from a joint PMM $B_{d_{PMM}}^\delta(P_0^{\mathcal{X}, \mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{X} \times \mathcal{Y})$ is again a PMM $B_{d_{PMM}}^\delta(P_0^\mathcal{X}) \subseteq$
314 $\mathbb{P}^*(\mathcal{X})$ with $P_0^\mathcal{X}$ the marginal probability of $P_0^{\mathcal{X}, \mathcal{Y}}$ on \mathcal{X} and the same distortion
315 factor.

316 Independent products

When going from marginal models $P_0^\mathcal{X}$ and $P_0^\mathcal{Y}$ to a joint one under the assumption of independence, it can be seen that there is no dominance relationship between $\underline{P}_{PMM}^{\mathcal{X} \times \mathcal{Y}}$ (combine through stochastic independence then distort) and $\underline{P}_{PMM}^\mathcal{X} \boxtimes \underline{P}_{PMM}^\mathcal{Y}$ (distort then combine through strong independence). To see this, note that on the one hand for the Cartesian product of events A, B , it holds that:

$$\begin{aligned} \overline{P}_{PMM}^{\mathcal{X} \times \mathcal{Y}}(A \times B) &= (1 + \delta)P_0^\mathcal{X}(A)P_0^\mathcal{Y}(B) \\ &\leq (1 + \delta)P_0^\mathcal{X}(A)(1 + \delta)P_0^\mathcal{Y}(B) = \overline{P}_{PMM}^\mathcal{X} \boxtimes \overline{P}_{PMM}^\mathcal{Y}(A \times B), \end{aligned} \quad (10)$$

317 where the inequality is strict whenever we consider non-trivial events A, B , i.e.
318 $P_0^\mathcal{X}(A), P_0^\mathcal{Y}(B) \in (0, 1)$. On the other hand, for events E that are not products, the
319 relationship between $\overline{P}_{PMM}^{\mathcal{X} \times \mathcal{Y}}(E)$ and $\overline{P}_{PMM}^\mathcal{X} \boxtimes \overline{P}_{PMM}^\mathcal{Y}(E)$ can be the reverse one,
320 as we show in our next example:

321 **Example 3.** Consider the spaces $\mathcal{X} = \{x_1, x_2\}$ and $\mathcal{Y} = \{y_1, y_2\}$ and the probability
322 measures $P_0^\mathcal{X}$ and $P_0^\mathcal{Y}$ given by:

$$P_0^\mathcal{X}(\{x_1\}) = 0.3, \quad P_0^\mathcal{X}(\{x_2\}) = 0.7, \quad P_0^\mathcal{Y}(\{y_1\}) = P_0^\mathcal{Y}(\{y_2\}) = 0.5,$$

and let $\delta = 0.1$. Given the event $E_1 = \{(x_2, y_2)\}^c$, it holds that:

$$\begin{aligned} \bar{P}_{PMM}^{\mathcal{X} \times \mathcal{Y}}(E_1) &= 1 - \underline{P}_{PMM}^{\mathcal{X} \times \mathcal{Y}}(\{(x_2, y_2)\}) = 0.715 \\ &> \bar{P}_{PMM}^{\mathcal{X}} \boxtimes \bar{P}_{PMM}^{\mathcal{Y}}(E_1) = 1 - \underline{P}_{PMM}^{\mathcal{X}} \boxtimes \underline{P}_{PMM}^{\mathcal{Y}}(\{(x_2, y_2)\}) \\ &= 1 - \underline{P}_{PMM}^{\mathcal{X}}(\{x_2\}) \underline{P}_{PMM}^{\mathcal{Y}}(\{y_2\}) = 0.6985. \end{aligned}$$

323 On the other hand, if we consider the events $A = \{x_1\}$ and $B = \{y_1\}$, we obtain
 324 $\bar{P}_{PMM}^{\mathcal{X} \times \mathcal{Y}}(A \times B) = 0.165$ and $\bar{P}_{PMM}^{\mathcal{X}} \boxtimes \bar{P}_{PMM}^{\mathcal{Y}}(A \times B) = 0.1815$, showing that the
 325 inequality in Equation (10) may be strict.

326 Therefore, there is not a dominance relationship between $\underline{P}_{PMM}^{\mathcal{X} \times \mathcal{Y}}$ and $\underline{P}_{PMM}^{\mathcal{X}} \boxtimes$
 327 $\underline{P}_{PMM}^{\mathcal{Y}}$ (or equivalently between $\bar{P}_{PMM}^{\mathcal{X} \times \mathcal{Y}}$ and $\bar{P}_{PMM}^{\mathcal{X}} \boxtimes \bar{P}_{PMM}^{\mathcal{Y}}$). \blacklozenge

328 We may also wonder whether the family of PMM is closed under the strong
 329 product, or in other words if $\underline{P}_{PMM}^{\mathcal{X}} \boxtimes \underline{P}_{PMM}^{\mathcal{Y}}$ is still a PMM. The next example
 330 shows that this is not the case.

Example 4. Consider the setting of Example 3 and the events $\{(x_1, y_1)\}, \{(x_1, y_2)\}$
 and $\{x_1\} \times \mathcal{Y}$. We obtain

$$\bar{P}_{PMM}^{\mathcal{X}} \boxtimes \bar{P}_{PMM}^{\mathcal{Y}}(\{(x_1, y_1)\}) = \bar{P}_{PMM}^{\mathcal{X}} \boxtimes \bar{P}_{PMM}^{\mathcal{Y}}(\{(x_1, y_2)\}) = 0.33 \cdot 0.55 = 0.1815$$

while

$$\bar{P}_{PMM}^{\mathcal{X}} \boxtimes \bar{P}_{PMM}^{\mathcal{Y}}(\{x_1\} \times \mathcal{Y}) = \bar{P}_{PMM}^{\mathcal{X}}(\{x_1\}) = 0.33.$$

331 Since $0.33 \neq 2 \cdot 0.1815$ and any PMM satisfies $\bar{P}(A \cup B) = \bar{P}(A) + \bar{P}(B)$ if $A \cup B \subset \mathcal{X}$
 332 and $A \cap B = \emptyset$, we conclude that $\bar{P}_{PMM}^{\mathcal{X}} \boxtimes \bar{P}_{PMM}^{\mathcal{Y}}$ is not a PMM. \blacklozenge

333 Natural extension of marginal models

Consider the lower and upper probabilities that are the lower and upper envelopes of $B_{d_{PMM}}^{\delta_{\mathcal{X}}}(P_0^{\mathcal{X}}) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{d_{PMM}}^{\delta_{\mathcal{Y}}}(P_0^{\mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{Y})$. Using Equations (5) and (6), we can give the form of \underline{E}_{PMM} and \bar{E}_{PMM} for the events $A \times B \neq \emptyset$, for $A \subset \mathcal{X}, B \subset \mathcal{Y}$:

$$\begin{aligned} \bar{E}_{PMM}(A \times B) &= \min \left\{ \bar{P}_{PMM}^{\mathcal{X}}(A), \bar{P}_{PMM}^{\mathcal{Y}}(B) \right\} \\ &= \min \left\{ (1 + \delta_{\mathcal{X}})P_0^{\mathcal{X}}(A), (1 + \delta_{\mathcal{Y}})P_0^{\mathcal{Y}}(B), 1 \right\}. \\ \underline{E}_{PMM}(A \times B) &= \max \left\{ \underline{P}_{PMM}^{\mathcal{X}}(A) + \underline{P}_{PMM}^{\mathcal{Y}}(B) - 1, 0 \right\} \\ &= \max \left\{ (1 + \delta_{\mathcal{X}})P_0^{\mathcal{X}}(A) - \delta_{\mathcal{X}} + (1 + \delta_{\mathcal{Y}})P_0^{\mathcal{Y}}(B) - \delta_{\mathcal{Y}} - 1, 0 \right\}. \end{aligned}$$

Moreover, when the distortion parameters coincide, $\delta_{\mathcal{X}} = \delta_{\mathcal{Y}} = \delta$, the previous equations become:

$$\begin{aligned} \bar{E}_{PMM}(A \times B) &= \min \left\{ (1 + \delta)P_0^{\mathcal{X}}(A), (1 + \delta)P_0^{\mathcal{Y}}(B), 1 \right\} \\ &= \min \left\{ 1, (1 + \delta) \min \left\{ P_0^{\mathcal{X}}(A), P_0^{\mathcal{Y}}(B) \right\} \right\}. \end{aligned} \quad (11)$$

$$\begin{aligned} \underline{E}_{PMM}(A \times B) &= \max \left\{ (1 + \delta)(P_0^{\mathcal{X}}(A) + P_0^{\mathcal{Y}}(B)) - 2\delta - 1, 0 \right\} \\ &= \max \left\{ (1 + \delta)(P_0^{\mathcal{X}}(A) + P_0^{\mathcal{Y}}(B) - 1) - \delta, 0 \right\} \\ &= (1 + \delta) \max \left\{ P_0^{\mathcal{X}}(A) + P_0^{\mathcal{Y}}(B) - 1, \frac{\delta}{1 + \delta} \right\} - \delta. \end{aligned} \quad (12)$$

334 We note that the above expressions for $\bar{E}_{PMM}(A \times B)$ and $\underline{E}_{PMM}(A \times B)$ recall
 335 those of the upper and lower probabilities of a PMM.

336 Even if the expressions in Equations (11) and (12) are only valid for the events
 337 of the type $A \times B$, one may think that the natural extension is somehow related to
 338 a PMM. Our next result shows that indeed such a connection can be established.

339 **Theorem 2.** *Let $B_{d_{PMM}}^\delta(P_0^\mathcal{X}) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{d_{PMM}}^\delta(P_0^\mathcal{Y}) \subseteq \mathbb{P}^*(\mathcal{Y})$ be two PMMs
 340 with associated lower and upper probabilities $\underline{P}_{PMM}^\mathcal{X}, \bar{P}_{PMM}^\mathcal{X}$ and $\underline{P}_{PMM}^\mathcal{Y}, \bar{P}_{PMM}^\mathcal{Y}$.
 341 Then, the credal set of the natural extension defined in Equation (3) can be expressed
 342 as:*

$$\mathcal{E}(\underline{P}_{PMM}^\mathcal{X}, \underline{P}_{PMM}^\mathcal{Y}) = \left\{ P \in \mathbb{P}(\mathcal{X} \times \mathcal{Y}) \mid P \leq (1 + \delta) \bar{E}_{P_0^\mathcal{X}, P_0^\mathcal{Y}} \right\}; \quad (13)$$

343 equivalently,

$$\bar{E}_{PMM}(C) = \min \{ (1 + \delta) \bar{E}_{P_0^\mathcal{X}, P_0^\mathcal{Y}}(C), 1 \} \quad \forall C \subseteq \mathcal{X} \times \mathcal{Y}, \quad (14)$$

344 where $\bar{E}_{P_0^\mathcal{X}, P_0^\mathcal{Y}}$ corresponds to the upper envelope of the credal set in Equation (3)
 345 applied to the particular case of precise marginals $P_0^\mathcal{X}, P_0^\mathcal{Y}$.

Proof. Consider first of all P satisfying $P \leq (1 + \delta) \bar{E}_{P_0^\mathcal{X}, P_0^\mathcal{Y}}$, and let us prove that
 $P \in \mathcal{E}(\underline{P}_{PMM}^\mathcal{X}, \underline{P}_{PMM}^\mathcal{Y})$. Since

$$\begin{aligned} P(A \times \mathcal{Y}) &\leq (1 + \delta) \bar{E}_{P_0^\mathcal{X}, P_0^\mathcal{Y}}(A \times \mathcal{Y}) \\ &= (1 + \delta) \min \{ P_0^\mathcal{X}(A), P_0^\mathcal{Y}(\mathcal{Y}) \} = (1 + \delta) P_0^\mathcal{X}(A) \quad \forall A \subseteq \mathcal{X}, \\ P(\mathcal{X} \times B) &\leq (1 + \delta) \bar{E}_{P_0^\mathcal{X}, P_0^\mathcal{Y}}(\mathcal{X} \times B) \\ &= (1 + \delta) \min \{ P_0^\mathcal{X}(\mathcal{X}), P_0^\mathcal{Y}(B) \} = (1 + \delta) P_0^\mathcal{Y}(B) \quad \forall B \subseteq \mathcal{Y}, \end{aligned}$$

346 we deduce that the \mathcal{X} and \mathcal{Y} marginals of P are dominated by the upper proba-
 347 bility determined by $\bar{P}_{PMM}^\mathcal{X}$ and $\bar{P}_{PMM}^\mathcal{Y}$, and thus that $P \in \mathcal{E}(\underline{P}_{PMM}^\mathcal{X}, \underline{P}_{PMM}^\mathcal{Y})$.
 348 Therefore,

$$\mathcal{E}(\underline{P}_{PMM}^\mathcal{X}, \underline{P}_{PMM}^\mathcal{Y}) \supseteq \left\{ P \in \mathbb{P}(\mathcal{X} \times \mathcal{Y}) \mid P \leq (1 + \delta) \bar{E}_{P_0^\mathcal{X}, P_0^\mathcal{Y}} \right\},$$

349 or equivalently

$$\bar{E}_{PMM}(C) \geq \min \{ (1 + \delta) \bar{E}_{P_0^\mathcal{X}, P_0^\mathcal{Y}}(C), 1 \} \quad \forall C \subseteq \mathcal{X} \times \mathcal{Y}. \quad (15)$$

350 Fix now $C \subseteq \mathcal{X} \times \mathcal{Y}$, take $P \in \mathcal{E}(\underline{P}_{PMM}^\mathcal{X}, \underline{P}_{PMM}^\mathcal{Y})$ such that $P(C) = \bar{E}_{PMM}(C)$,
 351 and denote by $P^\mathcal{X}$ and $P^\mathcal{Y}$ its marginals.

Let us define the probability measures $Q^\mathcal{X}, Q^\mathcal{Y}$ by means of the equalities

$$Q^\mathcal{X}(\{x\}) = \frac{(1 + \delta) P_0^\mathcal{X}(\{x\}) - P^\mathcal{X}(\{x\})}{\delta}, \quad Q^\mathcal{Y}(\{y\}) = \frac{(1 + \delta) P_0^\mathcal{Y}(\{y\}) - P^\mathcal{Y}(\{y\})}{\delta}.$$

Note that by construction

$$\sum_{x \in \mathcal{X}} Q^\mathcal{X}(\{x\}) = \frac{1 + \delta - 1}{\delta} = 1,$$

and also $Q^\mathcal{X}(\{x\}) \geq 0$ because $P^\mathcal{X} \leq (1 + \delta) P_0^\mathcal{X}$. Therefore, $Q^\mathcal{X}$ is a probability
 measure on \mathcal{X} . Similar considerations imply that $Q^\mathcal{Y}$ is a probability measure on \mathcal{Y} .
 Let $Q := Q^\mathcal{X} \times Q^\mathcal{Y}$ denote their independent product, and consider the probability
 measure P' given by

$$P' = \frac{1}{1 + \delta} P + \frac{\delta}{1 + \delta} Q.$$

P' is a probability measure because it is a convex combination of probability measures. Moreover, for any $x \in \mathcal{X}$:

$$\begin{aligned} \sum_{y \in \mathcal{Y}} P'(\{(x, y)\}) &= \sum_{y \in \mathcal{Y}} \left(\frac{1}{1 + \delta} P(\{(x, y)\}) + \frac{\delta}{1 + \delta} Q(\{(x, y)\}) \right) \\ &= \frac{P^{\mathcal{X}}(\{x\})}{1 + \delta} + \frac{\delta}{1 + \delta} Q^{\mathcal{X}}(\{x\}) \\ &= \frac{P^{\mathcal{X}}(\{x\})}{1 + \delta} + \frac{(1 + \delta)P_0^{\mathcal{X}}(\{x\}) - P^{\mathcal{X}}(\{x\})}{1 + \delta} = P_0^{\mathcal{X}}(\{x\}). \end{aligned}$$

Similarly, the \mathcal{Y} -marginal of P' coincides with $P_0^{\mathcal{Y}}$. As a consequence, $P' \in \mathcal{E}(P_0^{\mathcal{X}}, P_0^{\mathcal{Y}})$, whence $P'(C) \leq \bar{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}}(C)$. Moreover

$$\begin{aligned} P'(C) &= \sum_{(x, y) \in C} P'(\{(x, y)\}) \geq \frac{1}{1 + \delta} \sum_{(x, y) \in C} P(\{(x, y)\}) \\ &= \frac{1}{1 + \delta} P(C) = \frac{1}{1 + \delta} \bar{E}_{PMM}(C), \end{aligned}$$

352 so we deduce that

$$\bar{E}_{PMM}(C) \leq \min \left\{ (1 + \delta) \bar{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}}(C), 1 \right\}. \quad (16)$$

353 By putting together Equations (15) and (16) we conclude that Equations (13)
354 and (14) hold. \square

355 This result shows that the procedures of natural extension and the distortion
356 produced by the PMM commute, in the sense that the natural extension of the
357 coherent upper probability determined by the two marginal PMMs can also be
358 obtained as a PMM starting from the joint probability the two marginals determine
359 on the product events⁵. This is illustrated in Figure 1.

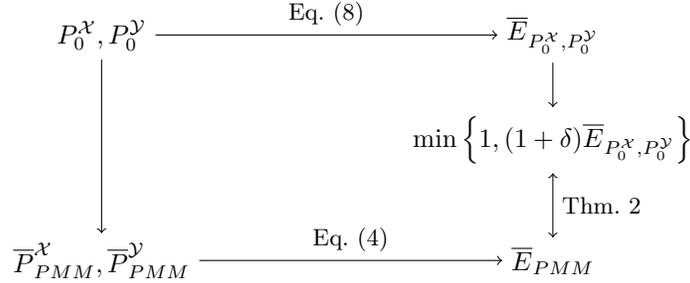


FIGURE 1. Graphical representation of the natural extension of two PMMs.

⁵This second approach is reminiscent of that considered by Moral in [35] for the distortion of credal sets (there with the name *discounting*): he distorts each of the elements in the initial credal sets, and then takes the closure of the union of those credal sets obtained. In particular, he investigated the cases of the total variation distance and the linear vacuous mixtures we shall consider later on in this paper.

360

4. LINEAR VACUOUS MIXTURES

361 Our next model is the so-called ϵ -contamination model, or linear vacuous mixture
362 (LV, for short):

363 **Definition 3.** *Given a probability measure P_0 and a distortion factor $\delta \in (0, 1)$,*
364 *its associated linear vacuous mixture is given by the following conjugate lower and*
365 *upper probabilities:*

$$\underline{P}_{LV}(A) = (1 - \delta)P_0(A), \quad \overline{P}_{LV}(A) = (1 - \delta)P_0(A) + \delta \quad \forall A \neq \emptyset, \mathcal{X},$$

366 $\underline{P}_{LV}(\emptyset) = \overline{P}_{LV}(\emptyset) = 0$ and $\underline{P}_{LV}(\mathcal{X}) = \overline{P}_{LV}(\mathcal{X}) = 1$.

367 The credal set $\mathcal{M}(\underline{P}_{LV})$ coincides with $B_{d_{LV}}^\delta(P_0)$, where $d_{LV} : \mathbb{P}^*(\mathcal{X}) \times \mathbb{P}^*(\mathcal{X}) \rightarrow$
368 $[0, \infty)$ is the distorting function given by [30, Thm. 9]:

$$d_{LV}(P, Q) = \max_{A \neq \emptyset} \frac{Q(A) - P(A)}{Q(A)}.$$

369 Let us analyse the behaviour of the LV model under the different operations intro-
370 duced in Section 2.4.

371 **4.1. Merging.** Let us first look at the behaviour of LV models under merging.

372 **Conjunction**

373 Similarly to the PMM, the intersection of two LV models is again a LV model,
374 when this intersection is non-empty.

375 **Proposition 3.** *Given the distortion models $B_{d_{LV}}^{\delta_1}(P_0^1) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{d_{LV}}^{\delta_2}(P_0^2) \subseteq$
376 $\mathbb{P}^*(\mathcal{X})$, the set $B_{d_{LV}}^{\delta_1}(P_0^1) \cap B_{d_{LV}}^{\delta_2}(P_0^2)$ is non-empty if and only if*

$$\sum_{x \in \mathcal{X}} \max \{ (1 - \delta_1)P_0^1(\{x\}), (1 - \delta_2)P_0^2(\{x\}) \} \leq 1. \quad (17)$$

*In that case, it is induced by the LV model generated by the following probability
measure P_0^\cap and the distortion parameter δ^\cap :*

$$\delta^\cap = 1 - \sum_{x \in \mathcal{X}} \max \{ (1 - \delta_1)P_0^1(\{x\}), (1 - \delta_2)P_0^2(\{x\}) \}, \quad \text{and}$$

$$P_0^\cap(\{x\}) = \frac{\max \{ (1 - \delta_1)P_0^1(\{x\}), (1 - \delta_2)P_0^2(\{x\}) \}}{1 - \delta^\cap} \quad \forall x \in \mathcal{X}.$$

377 *Proof.* It suffices to notice [30, Sec. 5.1] that a linear vacuous model is equivalent to
378 specific probability intervals that are only lower bounded, i.e. given P_0, δ , the credal
379 set $B_{LV}^\delta(P_0)$ is the set of those probability measures P satisfying the constraints

$$(1 - \delta)P_0(\{x\}) \leq P(\{x\}) \quad \forall x \in \mathcal{X}.$$

380 It is then known [13, Sec. 3.2] that the intersection of two such models $B_{d_{LV}}^{\delta_1}(P_0^1) \cap$
381 $B_{d_{LV}}^{\delta_2}(P_0^2)$ corresponds to the probability interval whose lower bounds are the maxi-
382 mum of their respective lower bounds. From this, it follows that this conjunction is
383 a linear vacuous model when this intersection is non-empty. Equation (17) follows
384 then from [13, Eq. (2)]. \square

385 **Disjunction**

386 Regarding the disjunction, the convex hull of $B_{d_{LV}}^{\delta_1}(P_0^1) \cup B_{d_{LV}}^{\delta_2}(P_0^2)$ will in general
387 not be a LV model, not even when $\delta_1 = \delta_2$ as we show in next example.

388 **Example 5.** *As in Example 2, take the probability measures $P_0^1 = (0.5, 0.3, 0.2)$
389 and $P_0^2 = (0.3, 0.5, 0.2)$ and the distortion factor $\delta_1 = \delta_2 = 0.1$. The associated LV
390 models $\underline{P}_{LV_1}, \underline{P}_{LV_2}$ and their disjunction \underline{P}_{LV}^\cup are given in the following table:*

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
\underline{P}_{LV_1}	0.45	0.27	0.18	0.72	0.63	0.45
\underline{P}_{LV_2}	0.27	0.45	0.18	0.72	0.45	0.63
\underline{P}_{LV}^\cup	0.27	0.27	0.18	0.72	0.45	0.45

391 *If there was some probability measure P_0 and $\delta > 0$ such that $\mathcal{M}(\underline{P}_{LV}) = B_{d_{LV}}^\delta(P_0)$,*
392 *then we would obtain*

$$\underline{P}_{LV}^\cup(\{x_1, x_2\}) = (1 - \delta)P_0(\{x_1, x_2\}) = \underline{P}_{LV}^\cup(\{x_1\}) + \underline{P}_{LV}^\cup(\{x_2\}),$$

393 *which does not hold. As a consequence, the disjunction of $B_{d_{LV}}^{\delta_1}(P_0^1) \cup B_{d_{LV}}^{\delta_2}(P_0^2)$
394 does not produce a LV model. \blacklozenge*

This disjunction has a unique undominated outer approximation as a LV model, since the greatest LV outer approximation (in terms of \underline{P}) of any given credal set is unique [33, Prop. 8]. This undominated outer approximation is the model $B_{d_{LV}}^{\delta^\cup}(P_0^\cup)$ such that

$$\delta^\cup = 1 - \left(\sum_{x \in \mathcal{X}} \min \{ (1 - \delta_1)P_0^1(\{x\}), (1 - \delta_2)P_0^2(\{x\}) \} \right), \quad \text{and}$$

$$P_0^\cup(\{x\}) = \frac{\min \{ (1 - \delta_1)P_0^1(\{x\}), (1 - \delta_2)P_0^2(\{x\}) \}}{1 - \delta^\cup} \quad \forall x \in \mathcal{X}.$$

395 **Convex mixture**

396 The mixture of two LV models, that is, the computation of $B_{d_{LV}}^{\delta^\epsilon}(P_0^\epsilon)$ for a given
397 $\epsilon \in (0, 1)$ can be established by a reasoning similar to the one done for the PMM
398 in [33, Sec. 5.1]. In particular, using in a straightforward way results established
399 for probability intervals [36], $B_{d_{LV}}^{\delta^\epsilon}(P_0^\epsilon)$ is given by the probability measures P
400 satisfying

$$\epsilon(1 - \delta_1)P_0^1(\{x\}) + (1 - \epsilon)(1 - \delta_2)P_0^2(\{x\}) \leq P(\{x\}) \quad \forall x \in \mathcal{X}.$$

From this, we deduce that

$$\begin{aligned} 1 - \delta^\epsilon &= \sum_{x \in \mathcal{X}} \left(\epsilon(1 - \delta_1)P_0^1(\{x\}) + (1 - \epsilon)(1 - \delta_2)P_0^2(\{x\}) \right) \\ &= \epsilon(1 - \delta_1) \sum_{x \in \mathcal{X}} P_0^1(\{x\}) + (1 - \epsilon)(1 - \delta_2) \sum_{x \in \mathcal{X}} P_0^2(\{x\}) \\ &= \epsilon(1 - \delta_1) + (1 - \epsilon)(1 - \delta_2), \end{aligned}$$

and

$$P_0^\epsilon(\{x\}) = \frac{\epsilon(1 - \delta_1)P_0^1(\{x\}) + (1 - \epsilon)(1 - \delta_2)P_0^2(\{x\})}{1 - \delta_\epsilon} \quad \forall x \in \mathcal{X}.$$

401 **4.2. Multivariate setting.** Let us now look at the behaviour of the LV in a mul-
402 tivariate setting.

403 Marginalisation

404 We first show that the marginal model of a joint LV is again a LV model, with
405 the same distortion factor δ applied to the marginal probability.

406 **Proposition 4.** *Consider the distortion model $B_{d_{LV}}^\delta(P_0^{\mathcal{X}, \mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{X} \times \mathcal{Y})$ and its
407 induced lower probability $\underline{P}_{LV}^{\mathcal{X}}$. Then, the marginal model $\underline{P}_{LV}^{\mathcal{X}}$ induces the credal
408 set $B_{d_{LV}}^\delta(P_0^{\mathcal{X}})$ with $P_0^{\mathcal{X}}$ the marginal probability of $P_0^{\mathcal{X}, \mathcal{Y}}$ on \mathcal{X} .*

Proof. Using again that, from [30, Sec. 5.1] $B_{d_{LV}}^\delta(P_0^{\mathcal{X}, \mathcal{Y}})$ is defined by lower bounds
($1 - \delta$) $P_0^{\mathcal{X}, \mathcal{Y}}(\{(x, y)\})$ on singletons, it is sufficient to notice that the marginal model
on \mathcal{X} is described by the constraints

$$P(\{x\}) = \sum_{y \in \mathcal{Y}} P(\{(x, y)\}) \geq \sum_{y \in \mathcal{Y}} (1 - \delta)P_0^{\mathcal{X}, \mathcal{Y}}(\{(x, y)\}) = (1 - \delta)P_0^{\mathcal{X}}(\{x\}).$$

409 As a consequence, it is a LV model. \square

410 Independent products

Regarding the problem of going from marginal models $P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}$ to multivariate
ones, we can first notice that on Cartesian products of events, we have

$$\begin{aligned} \underline{P}_{LV}^{\mathcal{X} \times \mathcal{Y}}(A \times B) &= (1 - \delta)P_0^{\mathcal{X}}(A)P_0^{\mathcal{Y}}(B) \\ &\geq (1 - \delta)P_0^{\mathcal{X}}(A)(1 - \delta)P_0^{\mathcal{Y}}(B) = \underline{P}_{LV}^{\mathcal{X}} \boxtimes \underline{P}_{LV}^{\mathcal{Y}}(A \times B) \end{aligned} \quad (18)$$

411 where last equality follows from the factorization property in Equation (2). Note
412 that the inequality is strict for any $\delta > 0$. We may then wonder if $\underline{P}_{LV}^{\mathcal{X} \times \mathcal{Y}} \geq$
413 $\underline{P}_{LV}^{\mathcal{X}} \boxtimes \underline{P}_{LV}^{\mathcal{Y}}$ in general. The next example shows that this is not the case, and
414 hence that we have no dominance relation between the joint models $\underline{P}_{LV}^{\mathcal{X} \times \mathcal{Y}}$ and
415 $\underline{P}_{LV}^{\mathcal{X}} \boxtimes \underline{P}_{LV}^{\mathcal{Y}}$.

Example 6. *Let us continue with Example 3. Given $E_1 = \{(x_2, y_2)\}^c$, we obtain*

$$\begin{aligned} \underline{P}_{LV}^{\mathcal{X} \times \mathcal{Y}}(E_1) &= 1 - \overline{P}_{LV}^{\mathcal{X} \times \mathcal{Y}}(\{(x_2, y_2)\}) = 0.585 \\ &< \underline{P}_{LV}^{\mathcal{X}} \boxtimes \underline{P}_{LV}^{\mathcal{Y}}(E_1) = 1 - \overline{P}_{LV}^{\mathcal{X}} \boxtimes \overline{P}_{LV}^{\mathcal{Y}}(\{(x_2, y_2)\}) \\ &= 1 - \overline{P}_{LV}^{\mathcal{X}}(\{x_2\})\overline{P}_{LV}^{\mathcal{Y}}(\{y_2\}) = 0.5985, \end{aligned}$$

416 and therefore it cannot be $B_{d_{LV}}^\delta(P_0^{\mathcal{X}, \mathcal{Y}}) \subseteq \mathcal{M}(\underline{P}_{LV}^{\mathcal{X}}) \boxtimes \mathcal{M}(\underline{P}_{LV}^{\mathcal{Y}})$.

417 On the other hand, taking the events $A = \{x_1\}$ and $B = \{y_1\}$, we obtain
418 $\underline{P}_{LV}^{\mathcal{X} \times \mathcal{Y}}(A \times B) = 0.135$ and $\underline{P}_{LV}^{\mathcal{X}}(A) \boxtimes \underline{P}_{LV}^{\mathcal{Y}}(B) = 0.1215$, showing that the in-
419 equality in Equation (18) may be strict. We therefore conclude that there is no
420 dominance relationship between $\underline{P}_{LV}^{\mathcal{X} \times \mathcal{Y}}$ and $\underline{P}_{LV}^{\mathcal{X}} \boxtimes \underline{P}_{LV}^{\mathcal{Y}}$. \blacklozenge

421 Our next example shows that the family of LV models is not closed under strong
422 products.

Example 7. Consider the same probability measures and distortion factor as in Example 3, and the events $\{(x_1, y_1)\}$, $\{(x_1, y_2)\}$ and $\{x_1\} \times \mathcal{Y}$. Then

$$\underline{P}_{LV}^{\mathcal{X}} \boxtimes \underline{P}_{LV}^{\mathcal{Y}}(\{(x_1, y_1)\}) = \underline{P}_{LV}^{\mathcal{X}} \boxtimes \underline{P}_{LV}^{\mathcal{Y}}(\{(x_1, y_2)\}) = 0.27 \cdot 0.45 = 0.1215$$

and

$$\underline{P}_{LV}^{\mathcal{X}} \boxtimes \underline{P}_{LV}^{\mathcal{Y}}(\{x_1\} \times \mathcal{Y}) = \underline{P}_{PMM}^{\mathcal{X}}(\{x_1\}) = 0.27.$$

423 The fact that $0.27 \neq 2 \cdot 0.1215$ contradicts the fact that a LV model should satisfy
424 $\underline{P}(A \cup B) = \underline{P}(A) + \underline{P}(B)$ if $A \cap B = \emptyset$ and $\min(\underline{P}(A), \underline{P}(B)) > 0$; thus, $\underline{P}_{LV}^{\mathcal{X}} \boxtimes \underline{P}_{LV}^{\mathcal{Y}}$
425 is not a LV model. \blacklozenge

426 Natural extension of marginal models

We now consider the lower and upper probabilities that are the lower and upper envelopes of $B_{dLV}^{\delta_{\mathcal{X}}}(P_0^{\mathcal{X}}) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{dLV}^{\delta_{\mathcal{Y}}}(P_0^{\mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{Y})$. Using Equations (5) and (6), we can give the form of \underline{E}_{LV} and \overline{E}_{LV} for the events $A \times B$, for $A \subset \mathcal{X}$, $B \subset \mathcal{Y}$:

$$\begin{aligned} \underline{E}_{LV}(A \times B) &= \max \{ \underline{P}_{LV}^{\mathcal{X}}(A) + \underline{P}_{LV}^{\mathcal{Y}}(B) - 1, 0 \} \\ &= \max \{ (1 - \delta_{\mathcal{X}})P_0^{\mathcal{X}}(A) + (1 - \delta_{\mathcal{Y}})P_0^{\mathcal{Y}}(B) - 1, 0 \}, \\ \overline{E}_{LV}(A \times B) &= \min \{ \overline{P}_{LV}^{\mathcal{X}}(A), \overline{P}_{LV}^{\mathcal{Y}}(B) \} \\ &= \min \{ (1 - \delta_{\mathcal{X}})P_0^{\mathcal{X}}(A) + \delta_{\mathcal{X}}, (1 - \delta_{\mathcal{Y}})P_0^{\mathcal{Y}}(B) + \delta_{\mathcal{Y}} \}. \end{aligned}$$

When the distortion parameters coincide, $\delta_{\mathcal{X}} = \delta_{\mathcal{Y}} = \delta$, the previous expressions simplify to:

$$\begin{aligned} \underline{E}_{LV}(A \times B) &= \max \{ (1 - \delta)P_0^{\mathcal{X}}(A) + (1 - \delta)P_0^{\mathcal{Y}}(B) - 1, 0 \} \\ &= \max \{ (1 - \delta)(P_0^{\mathcal{X}}(A) + P_0^{\mathcal{Y}}(B)) - 1, 0 \} \\ &= \max \left\{ (1 - \delta) \left(P_0^{\mathcal{X}}(A) + P_0^{\mathcal{Y}}(B) - \frac{1}{1 - \delta} \right), 0 \right\} \\ &= (1 - \delta) \max \left\{ P_0^{\mathcal{X}}(A) + P_0^{\mathcal{Y}}(B) - \frac{1}{1 - \delta}, 0 \right\}. \end{aligned} \quad (19)$$

$$\begin{aligned} \overline{E}_{LV}(A \times B) &= \min \{ (1 - \delta)P_0^{\mathcal{X}}(A) + \delta, (1 - \delta)P_0^{\mathcal{Y}}(B) + \delta \} \\ &= (1 - \delta) \min \{ P_0^{\mathcal{X}}(A), P_0^{\mathcal{Y}}(B) \} + \delta. \end{aligned} \quad (20)$$

The expressions in Equations (19) and (20) are somewhat similar to the lower and upper probabilities of a LV model. However, unlike what happened in the case of the PMM, the equality

$$\underline{E}_{LV} = (1 - \delta)\underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}}$$

427 does not hold:

Example 8. Consider the probability measures from Example 3, and let $\delta = 0.2$. Then Equation (19) gives

$$\underline{E}_{LV}(\{(x_2, y_2)\}) = \max\{0.8 \cdot 0.7 + 0.8 \cdot 0.5 - 1, 0\} = 0,$$

while from Equation (7) we obtain

$$\underline{E}_{P_0^x, P_0^y}(\{(x_2, y_2)\}) = \max\{0.7 + 0.5 - 1, 0\} = 0.2,$$

428 meaning that $(1 - \delta)\underline{E}_{P_0^x, P_0^y}(\{(x_2, y_2)\}) = 0.16$. \blacklozenge

429

5. CONSTANT ODDS RATIO

430

Our next distortion model is the constant odds ratio model (COR, for short):

431

Definition 4. Given a probability measure P_0 and a distortion factor $\delta \in (0, 1)$, the associated constant odds ratio model is the coherent lower prevision \underline{P}_{COR} that, on any gamble f , is defined as the unique solution to the implicit equation:

432

$$(1 - \delta)P_0((f - \underline{P}_{COR}(f))^+) = P_0((f - \underline{P}_{COR}(f))^-), \quad (21)$$

434

where $g^+ = \max\{g, 0\}$ and $g^- = \max\{-g, 0\}$.

435

While Equation (21) does not have an explicit expression, the restriction to (indicators of) events of the constant odds ratio can be more easily computed as:

436

$$\underline{P}_{COR}(A) = \frac{(1 - \delta)P_0(A)}{1 - \delta P_0(A)} \quad \forall A \subseteq \mathcal{X}. \quad (22)$$

437

When $P_0 \in \mathbb{P}^*(\mathcal{X})$ and δ is small enough, the credal set $\mathcal{M}(\underline{P}_{COR})$ coincides with $B_{d_{COR}}^\delta(P_0)$, where $d_{COR} : \mathbb{P}^*(\mathcal{X}) \times \mathbb{P}^*(\mathcal{X}) \rightarrow [0, \infty)$ is the distorting function given by [30, Thm. 14]:

438

$$d_{COR}(P, Q) = \max_{A, B \neq \emptyset} \left\{ 1 - \frac{P(A) \cdot Q(B)}{P(B) \cdot Q(A)} \right\}.$$

440

Also, the credal set $\mathcal{M}(\underline{P}_{COR})$ can be expressed as [53, Sec. 3.3.5]:

$$\mathcal{M}(\underline{P}_{COR}) = \{P \in \mathbb{P}(\mathcal{X}) \mid P(A)P_0(B) \geq (1 - \delta)P_0(A)P(B) \quad \forall A, B \subseteq \mathcal{X}\}. \quad (23)$$

441

Also, the COR model is more informative than the PMM and the LV models, in the sense that once we fix P_0 and δ , it holds that $B_{d_{COR}}^\delta(P_0) \subseteq B_{d_{PMM}}^\delta(P_0) \cap B_{d_{LV}}^\delta(P_0)$ (see [53, Sec. 4.6.5] for more comments in this direction).

442

443

5.1. Merging. Let us first look at the behaviour of the family of COR models under merging.

444

445

Conjunction

446

Unlike the PMM and LV models, the intersection of two constant odds ratio models is not a COR model in general, as next example shows.

447

448

Example 9. Consider $\mathcal{M}_1 = B_{d_{COR}}^{\delta_1}(P_0^1) \subseteq \mathbb{P}^*(\mathcal{X})$ with $P_0^1 = (0.5, 0.3, 0.2)$ and $\delta_1 = 0.2$, and $\mathcal{M}_2 = B_{d_{COR}}^{\delta_2}(P_0^2) \subseteq \mathbb{P}^*(\mathcal{X})$ such that $P_0^2 = (0.35, 0.3, 0.35)$ with $\delta_2 = 0.5$. From Equation (23), the ratio $P(\{x_1\})/P(\{x_3\})$ is constrained by the inequalities

449

450

451

$$3.125 \geq \frac{P(\{x_1\})}{P(\{x_3\})} \geq 2, \quad 2 \geq \frac{P(\{x_1\})}{P(\{x_3\})} \geq 0.5,$$

respectively for \mathcal{M}_1 and \mathcal{M}_2 . From this, we can deduce that any $P \in \mathcal{M}_1 \cap \mathcal{M}_2$ must satisfy the constraint $\frac{P(\{x_1\})}{P(\{x_3\})} = 2$. As a consequence, the credal set $\mathcal{M}_1 \cap \mathcal{M}_2$ has at most two extreme points:

$$\left(\frac{2(1 - \underline{P}(\{x_2\}))}{3}, \underline{P}(\{x_2\}), \frac{1 - \underline{P}(\{x_2\})}{3} \right)$$

and

$$\left(\frac{2(1 - \bar{P}(\{x_2\}))}{3}, \bar{P}(\{x_2\}), \frac{1 - \bar{P}(\{x_2\})}{3} \right),$$

452 where \underline{P}, \bar{P} denote the lower and upper probabilities associated with $\mathcal{M}_1 \cap \mathcal{M}_2$. Since
 453 it was proven in [30, Prop. 13] that the number of extreme points of the credal set
 454 of a COR model is equal to $2^n - 2$, where n is the cardinality of \mathcal{X} , this implies
 455 that the conjunction $\mathcal{M}_1 \cap \mathcal{M}_2$ does not determine a constant odds-ratio model, as
 456 it contains less than $2^n - 2 = 6$ extreme points. \blacklozenge

457 For the PMM and LV models, we had easy ways to check whether a conjunction
 458 was empty (Equations (9) and (17), respectively). This resulted from the fact that
 459 their conjunctions are specific probability intervals, which are models for which
 460 checking non-emptiness is easy. This is not the case for the COR model, that is
 461 not closed under conjunction. A possibility is to use the constraints induced by the
 462 models $\mathcal{M}_1 = B_{dCOR}^{\delta_1}(P_0^1)$ and $\mathcal{M}_2 = B_{dCOR}^{\delta_2}(P_0^2)$, to check that taken together they
 463 still have a solution (i.e., that there is at least a probability P within $\mathcal{M}_1 \cap \mathcal{M}_2$).
 464 However, as those are defined implicitly by Equation (21), they would have to be
 465 made explicit, for instance by enumerating the extreme points induced by $\mathcal{M}_1, \mathcal{M}_2$
 466 (see for example [30, Prop. 12]) and extracting the corresponding constraints.

467 Disjunction

468 Similarly, the disjunction of two COR models will not produce a COR model in
 469 general, not even when $\delta_1 = \delta_2$:

470 **Example 10.** Consider $P_0^1 = (0.4, 0.3, 0.3), P_0^2 = (0.3, 0.4, 0.3)$ and $\delta_1 = \delta_2 =$
 471 0.1 . Using Equation (22), the associated COR models $\underline{P}_{COR_1}, \underline{P}_{COR_2}$ and their
 472 disjunction $\underline{P}^\cup = \min\{\underline{P}_{COR_1}, \underline{P}_{COR_2}\}$ are given in the following table:

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
\underline{P}_{COR_1}	3/8	27/97	27/97	21/31	21/31	27/47
\underline{P}_{COR_2}	27/97	3/8	27/97	21/31	27/47	21/31
\underline{P}^\cup	27/97	27/97	27/97	21/31	27/47	27/47

473 If \underline{P}^\cup was a COR model, i.e. if $\mathcal{M}(\underline{P}^\cup) = B_{dCOR}^\delta(P_0)$ for some P_0 and δ , since
 474 $\underline{P}^\cup(\{x_1\}) = \underline{P}^\cup(\{x_2\}) = \underline{P}^\cup(\{x_3\})$, it should be

$$P_0(\{x_1\}) = P_0(\{x_2\}) = P_0(\{x_3\}) = \frac{1}{3}.$$

475 But in that case, regardless of the value of δ , \underline{P}^\cup must take the same value for all
 476 the events of cardinality two, a contradiction.

477 This example also allows us to show that \underline{P}^\cup does not have a unique undominated
 478 outer approximation in terms of COR models. Consider $P_A = (31/80, 31/80, 18/80)$,
 479 $P_B = (35/124, 35/124, 27/62)$, $\delta_A = 121/310$, $\delta_B = \frac{1}{2}$ and the COR models $B_{dCOR}^{\delta_A}(P_A)$
 480 and $B_{dCOR}^{\delta_B}(P_B)$ they induce. These produce the following coherent lower probabili-
 481 ties:

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
\underline{P}_{COR_A}	27/97	27/97	1701/11311	21/31	9261/18871	9261/18871
\underline{P}_{COR_B}	35/213	35/213	27/97	35/89	89/159	89/159

482 Both \underline{P}_{COR_A} and \underline{P}_{COR_B} are outer approximations of \underline{P}^\cup . Moreover, if there was
 483 a unique undominated outer approximation \underline{Q}_{COR} of \underline{P}^\cup in terms of COR models,
 484 then it should be $\underline{P}_{COR_A}, \underline{P}_{COR_B} \leq \underline{Q}_{COR}$, that implies

$$\underline{Q}_{COR}(\{x_i\}) = \underline{P}^\cup(\{x_i\}) = \frac{27}{97} \quad \text{for } i = 1, 2, 3,$$

485 meaning that \underline{Q}_{COR} is defined through $P_0 = (1/3, 1/3, 1/3)$ and $\delta = 8/35$. However,
 486 on the event $\{x_1, x_3\}$ this distortion model satisfies

$$\underline{Q}_{COR}(\{x_1, x_3\}) = \frac{(1 - \delta)P_0(\{x_1, x_3\})}{1 - \delta P_0(\{x_1, x_3\})} = \frac{54}{89} > \frac{27}{47} = \underline{P}^\cup(\{x_1, x_3\}),$$

487 so \underline{Q}_{COR} is not an outer approximation of \underline{P}^\cup . \blacklozenge

488 We therefore conclude that the COR model is neither preserved by conjunction
 489 nor by disjunction, and also that its disjunction has not a unique undominated
 490 outer approximation.

491 Convex mixture

492 As for the previous models, given the fact that two COR models $B_{d_{COR}}^{\delta_1}(P_0^1) \subseteq$
 493 $\mathbb{P}^*(\mathcal{X})$ and $B_{d_{COR}}^{\delta_2}(P_0^2) \subseteq \mathbb{P}^*(\mathcal{X})$ are described by the same set of constraints over
 494 $P^{(A)/P^{(B)}}$, their convex mixture is a credal set described by the constraints

$$\frac{P(A)}{P(B)} \geq \epsilon(1 - \delta_1) \frac{P_1(A)}{P_1(B)} + (1 - \epsilon)(1 - \delta_2) \frac{P_2(A)}{P_2(B)}.$$

495 However, next example shows that such constraints will not lead, in general, to a
 496 COR model.

497 **Example 11.** Consider $P_0^1 = (1/4, 1/4, 1/2)$, $P_0^2 = (1/2, 1/4, 1/4)$ and $\delta_1 = \delta_2 = 0.5$.
 498 Using Equation (22), the associated COR models $\underline{P}_{COR_1}, \underline{P}_{COR_2}$ and their average
 499 $\underline{P}^{0.5}$ obtained for $\epsilon = 0.5$ are given in the following table:

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
\underline{P}_{COR_1}	1/7	1/7	1/3	1/3	3/5	3/5
\underline{P}_{COR_2}	1/3	1/7	1/7	3/5	3/5	1/3
$\underline{P}^{0.5}$	5/21	1/7	5/21	7/15	3/5	7/15

500 Should \underline{P}^ϵ be the lower probability of a COR model $B_{d_{COR}}^{\delta_\epsilon}(P_0^\epsilon)$, we should have
 501 $P_0^\epsilon(\{x_1\}) = P_0^\epsilon(\{x_3\}) = p$, hence $P_0^\epsilon(\{x_2\}) = 1 - 2p$. Using this observation
 502 and Equation (22) on events $\{x_1\}$ and $\{x_1, x_3\}$, we should have $\delta_\epsilon = 13/28$ and
 503 $p = 7/19$, and applying again Equation (22) with these values on $\{x_2\}$ would give
 504 $\underline{P}(\{x_2\}) = 75/467$ for COR model, which is close but not equal to the $1/7$ reported
 505 in the table. \blacklozenge

506 The COR model is therefore also not preserved under convex mixture, making
 507 it a not very convenient model when having to merge distortion models.

508 5.2. Multivariate setting.

509 Marginalisation

510 As for the PMM and LV models, we can show that the marginal distribution of
511 a joint constant odds ratio model is also a constant odds ratio model.

512 **Proposition 5.** *Consider the distortion model $B_{d_{COR}}^\delta(P_0^{\mathcal{X},\mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{X} \times \mathcal{Y})$ and its
513 induced lower prevision \underline{P}_{COR} . Then, the marginal model $\underline{P}_{COR}^\mathcal{X}$ induces the credal
514 set $B_{d_{COR}}^\delta(P_0^\mathcal{X}) \subseteq \mathbb{P}^*(\mathcal{X})$ with $P_0^\mathcal{X}$ the marginal probability of $P_0^{\mathcal{X},\mathcal{Y}}$ on \mathcal{X} .*

Proof. By definition, $\underline{P}_{COR}(f)$ is determined as the unique solution of the equation:

$$(1 - \delta)P_0\left((f - \underline{P}_{COR}(f))^+\right) = P_0\left((f - \underline{P}_{COR}(f))^-\right).$$

515 Consider a gamble f that only depends on the values in \mathcal{X} , in the sense that
516 $f(x, y) = f(x, y')$ for every $y \neq y' \in \mathcal{Y}$ and every $x \in \mathcal{X}$, and let us define the
517 gamble $f_\mathcal{X} : \mathcal{X} \rightarrow \mathbb{R}$ by $f_\mathcal{X}(x) = f(x, y)$. Then $\underline{P}_{COR}^\mathcal{X}(f_\mathcal{X}) = \underline{P}_{COR}(f)$. Moreover,

$$(1 - \delta)P_0\left((f_\mathcal{X} - \underline{P}_{COR}^\mathcal{X}(f_\mathcal{X}))^+\right) = (1 - \delta)P_0^\mathcal{X}\left((f_\mathcal{X} - \underline{P}_{COR}^\mathcal{X}(f_\mathcal{X}))^+\right),$$

518 and also:

$$P_0\left((f_\mathcal{X} - \underline{P}_{COR}^\mathcal{X}(f_\mathcal{X}))^-\right) = P_0^\mathcal{X}\left((f_\mathcal{X} - \underline{P}_{COR}^\mathcal{X}(f_\mathcal{X}))^-\right),$$

because $(f_\mathcal{X} - \underline{P}_{COR}^\mathcal{X}(f_\mathcal{X}))^+$ only depends on the values of $x \in \mathcal{X}$. This means
that $\underline{P}_{COR}^\mathcal{X}(f_\mathcal{X})$ is the unique solution of the equation:

$$(1 - \delta)P_0^\mathcal{X}\left((f_\mathcal{X} - \underline{P}_{COR}^\mathcal{X}(f_\mathcal{X}))^+\right) = P_0^\mathcal{X}\left((f_\mathcal{X} - \underline{P}_{COR}^\mathcal{X}(f_\mathcal{X}))^-\right),$$

519 meaning that $\underline{P}_{COR}^\mathcal{X}$ is a constant odds ratio with respect to the parameter δ and
520 probability $P_0^\mathcal{X}$ (the restriction of P_0 to the first component). \square

521 Independent products

Consider now the marginal models $B_{d_{COR}}^\delta(P_0^\mathcal{X}) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{d_{COR}}^\delta(P_0^\mathcal{Y}) \subseteq \mathbb{P}^*(\mathcal{Y})$. Regarding the problem of going from marginal models $P_0^\mathcal{X}, P_0^\mathcal{Y}$ to joint ones, we can first notice that on Cartesian products of events, we have

$$\begin{aligned} \underline{P}_{COR}^{\mathcal{X} \times \mathcal{Y}}(A \times B) &= \frac{(1 - \delta)P_0^\mathcal{X}(A)P_0^\mathcal{Y}(B)}{1 - \delta P_0^\mathcal{X}(A)P_0^\mathcal{Y}(B)} \\ &\geq \frac{(1 - \delta)P_0^\mathcal{X}(A)(1 - \delta)P_0^\mathcal{Y}(B)}{(1 - \delta P_0^\mathcal{X}(A))(1 - \delta P_0^\mathcal{Y}(B))} = \underline{P}_{COR}^\mathcal{X} \boxtimes \underline{P}_{COR}^\mathcal{Y}(A \times B) \end{aligned} \quad (24)$$

522 where the last equality follows from the factorisation property in Equation (2).
523 We can then wonder if $\underline{P}_{COR}^{\mathcal{X} \times \mathcal{Y}}(C) \geq \underline{P}_{COR}^\mathcal{X} \boxtimes \underline{P}_{COR}^\mathcal{Y}(C)$ for any event $C \subseteq \mathcal{X} \times \mathcal{Y}$.
524 The next example shows that this is not the case, and that the inequality in
525 Equation (24) may be strict; therefore it shows that there is no dominance relation
526 between $\underline{P}_{COR}^{\mathcal{X} \times \mathcal{Y}}$ and $\underline{P}_{COR}^\mathcal{X} \boxtimes \underline{P}_{COR}^\mathcal{Y}$.

Example 12. *Consider our running Example 3. Given $E_2 = \{(x_1, y_1), (x_2, y_2)\}$,
we obtain*

$$\underline{P}_{COR}^{\mathcal{X} \times \mathcal{Y}}(E_2) = 0.4737 < \underline{P}_{COR}^\mathcal{X} \boxtimes \underline{P}_{COR}^\mathcal{Y}(E_2) = 0.4883;$$

to see the last equality, note that

$$\begin{aligned} & \underline{P}_{COR}^{\mathcal{X}} \boxtimes \underline{P}_{COR}^{\mathcal{Y}}(E_2) \\ &= \min_{P_{\mathcal{X}} \in \mathcal{M}(\underline{P}_{COR}^{\mathcal{X}}), P_{\mathcal{Y}} \in \mathcal{M}(\underline{P}_{COR}^{\mathcal{Y}})} (P_{\mathcal{X}}(\{x_1\})P_{\mathcal{Y}}(\{y_1\}) + P_{\mathcal{X}}(\{x_2\})P_{\mathcal{Y}}(\{y_2\})), \end{aligned}$$

527 that $\mathcal{M}(\underline{P}_{COR}^{\mathcal{X}})$ consists of the probabilities $P_{\mathcal{X}}$ for which $P_{\mathcal{X}}(\{x_1\}) \in [\frac{27}{97}, \frac{10}{31}]$ and,
528 similarly, that $\mathcal{M}(\underline{P}_{COR}^{\mathcal{Y}})$ consists of the probabilities $P_{\mathcal{Y}}$ for which $P_{\mathcal{Y}}(\{y_1\}) \in$
529 $[\frac{9}{19}, \frac{10}{19}]$; the minimum in the equation above is obtained for $P_{\mathcal{X}}(\{x_1\}) = \frac{27}{97}$ and
530 $P_{\mathcal{Y}}(\{y_1\}) = \frac{10}{19}$, yielding

$$\frac{27}{97} \cdot \frac{10}{19} + \frac{70}{97} \cdot \frac{9}{19} = \frac{900}{1843} = 0.4883.$$

531 Therefore, $\underline{P}_{COR}^{\mathcal{X} \times \mathcal{Y}}$ does not dominate $\underline{P}_{COR}^{\mathcal{X}} \boxtimes \underline{P}_{COR}^{\mathcal{Y}}$ on all events.

532 Finally, taking the events $A = \{x_1\}$ and $B = \{y_1\}$, it holds that $\underline{P}_{COR}^{\mathcal{X} \times \mathcal{Y}}(A \times$
533 $B) \approx 0.137$, while $\underline{P}_{COR}^{\mathcal{X}} \boxtimes \underline{P}_{COR}^{\mathcal{Y}}(A \times B) \approx 0.132$, showing that the inequality in
534 Equation (24) may be strict. \blacklozenge

535 On the other hand, the family of COR models is not closed under strong prod-
536 ucts.

537 **Example 13.** Consider again the running Example 3. Let $E_2 = \{(x_1, y_1), (x_2, y_2)\}$
538 and $E_3 = \{x_1\} \times \mathcal{Y}$. Then

$$\underline{P}_{COR}^{\mathcal{X}} \boxtimes \underline{P}_{COR}^{\mathcal{Y}}(E_2) = 0.4883 \text{ and } \underline{P}_{COR}^{\mathcal{X}} \boxtimes \underline{P}_{COR}^{\mathcal{Y}}(E_3) = \underline{P}_{COR}^{\mathcal{X}}(\{x_1\}) = \frac{27}{97},$$

where the first equality follows from Example 12. On the other hand,

$$\underline{P}_{COR}^{\mathcal{X}} \boxtimes \underline{P}_{COR}^{\mathcal{Y}}(E_2 \cup E_3) = 1 - \overline{P}_{COR}^{\mathcal{X}} \boxtimes \overline{P}_{COR}^{\mathcal{Y}}(\{(x_2, y_1)\}) = 1 - \frac{70}{97} \cdot \frac{10}{19} = 0.6202$$

$$\underline{P}_{COR}^{\mathcal{X}} \boxtimes \underline{P}_{COR}^{\mathcal{Y}}(E_2 \cap E_3) = \underline{P}_{COR}^{\mathcal{X}} \boxtimes \underline{P}_{COR}^{\mathcal{Y}}(\{(x_1, y_1)\}) = \frac{27}{97} \cdot \frac{9}{19} = 0.1319,$$

whence

$$\begin{aligned} & \underline{P}_{COR}^{\mathcal{X}} \boxtimes \underline{P}_{COR}^{\mathcal{Y}}(E_2 \cup E_3) + \underline{P}_{COR}^{\mathcal{X}} \boxtimes \underline{P}_{COR}^{\mathcal{Y}}(E_2 \cap E_3) = 0.7520 \\ & < \underline{P}_{COR}^{\mathcal{X}} \boxtimes \underline{P}_{COR}^{\mathcal{Y}}(E_2) + \underline{P}_{COR}^{\mathcal{X}} \boxtimes \underline{P}_{COR}^{\mathcal{Y}}(E_3) = 0.7667 \end{aligned}$$

539 contradicting the fact that $\underline{P}_{COR}^{\mathcal{X}} \boxtimes \underline{P}_{COR}^{\mathcal{Y}}$ should be 2-monotone on events if it was
540 a COR model. \blacklozenge

541 Natural extension of marginal models

542 We have already mentioned that there is not an explicit equation for the lower
543 prevision of the COR model in gambles (see Equation (21)), and it can only be
544 given for events (see Equation (22)). This hinders a bit the computation of the
545 natural extension of this model. In addition, even if we consider only the values in
546 events, this model is more difficult to handle than the PMM or the LV.

First of all, using Equations (5) and (6), we give the explicit form of the lower and upper natural extension on the Cartesian products $A \times B$, for $A \subseteq \mathcal{X}$, $B \subseteq \mathcal{Y}$:

$$\begin{aligned}\overline{E}_{COR}(A \times B) &= \min \left\{ \frac{(1 - \delta_{\mathcal{X}})P_0^{\mathcal{X}}(A)}{1 - \delta_{\mathcal{X}}P_0^{\mathcal{X}}(A)}, \frac{(1 - \delta_{\mathcal{Y}})P_0^{\mathcal{Y}}(B)}{1 - \delta_{\mathcal{Y}}P_0^{\mathcal{Y}}(B)} \right\}. \\ \underline{E}_{COR}(A \times B) &= \max \left\{ \frac{(1 - \delta_{\mathcal{X}})P_0^{\mathcal{X}}(A)}{1 - \delta_{\mathcal{X}}P_0^{\mathcal{X}}(A)} + \frac{(1 - \delta_{\mathcal{Y}})P_0^{\mathcal{Y}}(B)}{1 - \delta_{\mathcal{Y}}P_0^{\mathcal{Y}}(B)} - 1, 0 \right\}.\end{aligned}$$

When the distortion parameters coincide, $\delta_{\mathcal{X}} = \delta_{\mathcal{Y}} = \delta$, these two equations become:

$$\begin{aligned}\overline{E}_{COR}(A \times B) &= \min \left\{ \frac{(1 - \delta)P_0^{\mathcal{X}}(A)}{1 - \delta P_0^{\mathcal{X}}(A)}, \frac{(1 - \delta)P_0^{\mathcal{Y}}(B)}{1 - \delta P_0^{\mathcal{Y}}(B)} \right\} \\ &= (1 - \delta) \min \left\{ \frac{P_0^{\mathcal{X}}(A)}{1 - \delta P_0^{\mathcal{X}}(A)}, \frac{P_0^{\mathcal{Y}}(B)}{1 - \delta P_0^{\mathcal{Y}}(B)} \right\}. \\ \underline{E}_{COR}(A \times B) &= \max \left\{ \frac{(1 - \delta)P_0^{\mathcal{X}}(A)}{1 - \delta P_0^{\mathcal{X}}(A)} + \frac{(1 - \delta)P_0^{\mathcal{Y}}(B)}{1 - \delta P_0^{\mathcal{Y}}(B)} - 1, 0 \right\}\end{aligned}$$

547 Although these expressions do not seem to resemble a COR model, we may wonder
548 if, similarly to what happened with the PMM (see Theorem 2), the equality
549 $\mathcal{E}(\underline{P}_{COR}^{\mathcal{X}}, \underline{P}_{COR}^{\mathcal{Y}}) = B_{d_{COR}}(\underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}})$ holds. As we see next, this is not the case.

Example 14. Consider our running Example 3. Given $E_4 = \{(x_2, y_2)\}$, we obtain

$$\underline{E}_{COR}(E_4) = \max\{0.6774 + 0.4737 - 1, 0\} = 0.1511.$$

550 On the other hand, from Equation (7) we have that $\underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}}(E_4) = 0.2$, whence

$$\frac{(1 - \delta)\underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}}(E_4)}{1 - \delta\underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}}(E_4)} = 0.1837.$$

551 Thus, the two values do not coincide. \blacklozenge

552 6. TOTAL VARIATION MODEL

The last three distortion models we shall analyse in this paper are defined directly from some distance between probability measures. The first of them is the total variation model (TV, for short): given two probability measures P, Q , their total variation distance is

$$d_{TV}(P, Q) = \max_{A \subseteq \mathcal{X}} |P(A) - Q(A)|.$$

553 By taking the lower and upper envelopes of the neighbourhood model it produces,
554 we obtain the following:

555 **Definition 5.** Given a probability measure P_0 and a distortion factor $\delta > 0$, the
556 total variation model is given by the following lower and upper probabilities:

$$\underline{P}_{TV}(A) = \max\{P_0(A) - \delta, 0\}, \quad \overline{P}_{TV}(A) = \min\{P_0(A) + \delta, 1\} \quad \forall A \subseteq \mathcal{X}. \quad (25)$$

557 Since we are assuming that $P_0 \in \mathbb{P}^*(\mathcal{X})$ and that δ is small enough so that
558 $B_{d_{TV}}^{\delta}(P_0) \subseteq \mathbb{P}^*(\mathcal{X})$, Equation (25) simplifies to:

$$\underline{P}_{TV}(A) = P_0(A) - \delta, \quad \overline{P}_{TV}(A) = P_0(A) + \delta \quad \forall A \neq \emptyset, \mathcal{X}. \quad (26)$$

559 **6.1. Merging.** Let us now consider the problem of merging two TV models.

560 **Conjunction and disjunction**

561 Our next example shows that neither the conjunction nor the disjunction of two
 562 such models will produce in general a total variation model.

563 **Example 15.** From Equation (26), we know that a TV model satisfies, for any
 564 event A such that $\bar{P}_{TV}(A), \underline{P}_{TV}(A) \in (0, 1)$, the following equality:

$$\bar{P}_{TV}(A) - \underline{P}_{TV}(A) = (P_0(A) + \delta) - (P_0(A) - \delta) = 2\delta.$$

565 In particular, since we are assuming that $B_{d_{TV}}^\delta(P_0) \subseteq \mathbb{P}^*(P_0)$, this equality holds
 566 for any $A \neq \emptyset, \mathcal{X}$. Let us use this to derive that the family of TV models is not
 567 closed under conjunction or disjunction.

568 Let $B_{d_{TV}}^{\delta_1}(P_0^1)$ be induced by $P_0^1 = (0.41, 0.37, 0.22)$ and $\delta_1 = 0.12$, and $B_{d_{TV}}^{\delta_2}(P_0^2)$
 569 be determined by $P_0^2 = (0.37, 0.41, 0.22)$ and $\delta_2 = 0.12$. The lower probabilities \underline{P}_{TV_1}
 570 and \underline{P}_{TV_2} , their conjunction \underline{P}^\cap and disjunction \underline{P}^\cup , are given by:

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
\underline{P}_{TV_1}	0.29	0.25	0.1	0.66	0.51	0.47
\underline{P}_{TV_2}	0.25	0.29	0.1	0.66	0.47	0.51
\underline{P}^\cap	0.29	0.29	0.1	0.66	0.51	0.51
\underline{P}^\cup	0.25	0.25	0.1	0.66	0.47	0.47

571 For the third line, use that \underline{P}^\cap is the natural extension of $\max\{\underline{P}_{TV_1}, \underline{P}_{TV_2}\}$, i.e.,
 572 the smallest coherent lower probability that dominates $\max\{\underline{P}_{TV_1}, \underline{P}_{TV_2}\}$; but the
 573 latter is coherent since it is the lower envelope of

$$(0.29, 0.37, 0.34), (0.37, 0.29, 0.34), (0.41, 0.49, 0.1), (0.49, 0.41, 0.1),$$

and therefore it coincides with \underline{P}^\cap . We observe that

$$\bar{P}^\cap(\{x_1\}) - \underline{P}^\cap(\{x_1\}) = 0.2 \neq 0.24 = \bar{P}^\cap(\{x_3\}) - \underline{P}^\cap(\{x_3\}),$$

$$\bar{P}^\cup(\{x_1\}) - \underline{P}^\cup(\{x_1\}) = 0.28 \neq 0.24 = \bar{P}^\cup(\{x_3\}) - \underline{P}^\cup(\{x_3\}),$$

574 concluding that neither \underline{P}^\cap nor \underline{P}^\cup are TV models. \blacklozenge

575 As we know that a TV model is described by a 2-monotone lower probability [31,
 576 Prop. 4], a simple way to check whether the intersection of two TV models is non-
 577 empty is simply to take the constraints (26) induced by both models $B_{d_{TV}}^{\delta_1}(P_0^1)$ and
 578 $B_{d_{TV}}^{\delta_2}(P_0^2)$, and to check whether they have a solution. This can be achieved through
 579 standard linear programming, with the caveat that the number of constraints will
 580 increase exponentially with n .

581 The same example allows us to show that the disjunction does not have a unique
 582 undominated outer approximation:

Example 16. Consider the model \underline{P}^\cup from the previous example, and let us con-
 sider the TV models $B_{d_{TV}}^{\delta_A}(P_0^A)$ and $B_{d_{TV}}^{\delta_B}(P_0^B)$, where $P_A = (0.31, 0.31, 0.38)$, $P_B =$
 $(0.41, 0.41, 0.18)$, $\delta_A = 0.28$ and $\delta_B = 0.16$. The lower probabilities $\underline{P}_{TV_A}, \underline{P}_{TV_B}$
 are given by:

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
\underline{P}_{TV_A}	0.03	0.03	0.1	0.34	0.41	0.41
\underline{P}_{TV_B}	0.25	0.25	0.02	0.66	0.43	0.43

583 Both $\underline{P}_{TV_A}, \underline{P}_{TV_B}$ are outer approximations of \underline{P}_{TV}^\cup in the TV family. If there
 584 was a unique undominated outer approximation of \underline{P}^\cup , denoted by $B_{d_{TV}}^\delta(P_0)$ and
 585 with associated lower probability \underline{Q}_{TV} , then $\underline{P}_{TV_A}, \underline{P}_{TV_B} \leq \underline{Q}_{TV} \leq \underline{P}^\cup$, whence
 586 $\underline{Q}_{TV}(\{x_1\}) = \underline{P}^\cup(\{x_1\}) = 0.25$, $\underline{Q}_{TV}(\{x_2\}) = \underline{P}^\cup(\{x_2\}) = 0.25$, $\underline{Q}_{TV}(\{x_3\}) =$
 587 $\underline{P}^\cup(\{x_3\}) = 0.1$. Therefore,

$$\underline{Q}_{TV}(\{x_1\}) + \underline{Q}_{TV}(\{x_2\}) + \underline{Q}_{TV}(\{x_3\}) = 1 - 3\delta = 0.6,$$

588 whence $\delta = 0.4/3$ and $P_0 = (0.25 + \delta, 0.25 + \delta, 0.1 + \delta)$. However, this means that:

$$\underline{Q}_{TV}(\{x_1, x_3\}) = P_0(\{x_1, x_3\}) - \delta = 0.35 + \delta > 0.47 = \underline{P}^\cup(\{x_1, x_3\}),$$

589 a contradiction. \blacklozenge

590 Convex mixture

It is rather direct to check that the convex mixture of two TV models $B_{d_{TV}}^{\delta_1}(P_0^1)$
 and $B_{d_{TV}}^{\delta_2}(P_0^2)$ is again a TV model as we have

$$\epsilon \underline{P}_{TV}^1(A) + (1 - \epsilon) \underline{P}_{TV}^2(A) = \epsilon P_0^1(A) + (1 - \epsilon) P_0^2(A) - \epsilon \delta_1 - (1 - \epsilon) \delta_2$$

which are lower probabilities induced by the TV model $B_{d_{TV}}^{\delta_\epsilon}(P_0^\epsilon)$ with

$$\delta_\epsilon = \epsilon \delta_1 + (1 - \epsilon) \delta_2 \quad \text{and} \quad P_0^\epsilon(\{x\}) = \epsilon P_0^1(\{x\}) + (1 - \epsilon) P_0^2(\{x\}) \quad \forall x \in \mathcal{X}.$$

591 6.2. Multivariate setting.

592 Marginalisation

593 Let us now look at the behaviour of the TV model in a multivariate setting. We
 594 can first show that the marginal model of a joint TV model is again a TV model,
 595 with the same distortion factor δ applied to the marginal probability.

596 **Proposition 6.** Consider the distortion model $B_{d_{TV}}^\delta(P_0^{\mathcal{X}, \mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{X} \times \mathcal{Y})$ and its
 597 associated lower probability \underline{P}_{TV} . Then, the marginal model $\underline{P}_{TV}^{\mathcal{X}}$ induces the credal
 598 set $B_{d_{TV}}^\delta(P_0^{\mathcal{X}}) \subseteq \mathbb{P}^*(\mathcal{X})$ with $P_0^{\mathcal{X}}$ the marginal probability of $P_0^{\mathcal{X}, \mathcal{Y}}$ on \mathcal{X} .

599 *Proof.* From Equation (25), we obtain that for every non-empty $B \subset \mathcal{X}$

$$\underline{P}^{\mathcal{X}, \mathcal{Y}}(B \times \mathcal{Y}) = P_0^{\mathcal{X}, \mathcal{Y}}(B \times \mathcal{Y}) - \delta = P_0^{\mathcal{X}}(B) - \delta,$$

600 the last term being the lower probability induced by $B_{d_{TV}}^\delta(P_0^{\mathcal{X}})$. \square

601 Independent products

Consider now two marginal models $B_{d_{TV}}^\delta(P_0^{\mathcal{X}}) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{d_{TV}}^\delta(P_0^{\mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{Y})$.
 Regarding the problem of going from marginal models $P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}$ to joint ones, we
 can first notice that on Cartesian products of events, we have

$$\begin{aligned} \underline{P}_{TV}^{\mathcal{X} \times \mathcal{Y}}(A \times B) &= P_0^{\mathcal{X}}(A) P_0^{\mathcal{Y}}(B) - \delta, \text{ while} \\ \underline{P}_{TV}^{\mathcal{X}} \boxtimes \underline{P}_{TV}^{\mathcal{Y}}(A \times B) &= (P_0^{\mathcal{X}}(A) - \delta)(P_0^{\mathcal{Y}}(B) - \delta) \end{aligned}$$

602 where the last equality follows from the factorization property in Equation (2).

603 Clearly, the equality $\underline{P}_{TV}^{\mathcal{X} \times \mathcal{Y}}(A \times B) = \underline{P}_{TV}^{\mathcal{X}} \boxtimes \underline{P}_{TV}^{\mathcal{Y}}(A \times B)$ does not generally

604 hold. We show in our next example that the inequality can go both ways (either
605 $\underline{P}_{TV}^{\mathcal{X} \times \mathcal{Y}}(A \times B) < \underline{P}_{TV}^{\mathcal{X}} \boxtimes \underline{P}_{TV}^{\mathcal{Y}}(A \times B)$ or the reverse).

606 **Example 17.** Take the distortion models $B_{d_{TV}}^{\delta}(P_0^{\mathcal{X}}) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{d_{TV}}^{\delta}(P_0^{\mathcal{Y}}) \subseteq$
607 $\mathbb{P}^*(\mathcal{Y})$ induced by the probability measures $P_0^{\mathcal{X}}$ and $P_0^{\mathcal{Y}}$ and the distortion factor
608 $\delta = 0.1$. On the one hand, assume that there are some events $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$
609 such that $P_0^{\mathcal{X}}(A) = P_0^{\mathcal{Y}}(B) = 0.5$. We obtain that:

$$\underline{P}_{TV}^{\mathcal{X} \times \mathcal{Y}}(A \times B) = 0.15 < 0.16 = \underline{P}_{TV}^{\mathcal{X}} \boxtimes \underline{P}_{TV}^{\mathcal{Y}}(A \times B).$$

610 On the other hand, assume that there are $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$ satisfying $P_0^{\mathcal{X}}(A) =$
611 $P_0^{\mathcal{Y}}(B) = 0.6$. We obtain that:

$$\underline{P}_{TV}^{\mathcal{X} \times \mathcal{Y}}(A \times B) = 0.26 > 0.25 = \underline{P}_{TV}^{\mathcal{X}} \boxtimes \underline{P}_{TV}^{\mathcal{Y}}(A \times B).$$

612 Therefore, there is no dominance relationship between the lower probabilities ob-
613 tained with the two approaches. \blacklozenge

614 Let us now show through an example that the TV model is not closed under
615 strong products.

Example 18. Consider again the setting of our running Example 3. Given the
events $\{(x_1, y_1)\}$ and $\{x_1\} \times \mathcal{Y}$,

$$\underline{P}_{TV}^{\mathcal{X}} \boxtimes \underline{P}_{TV}^{\mathcal{Y}}(\{(x_1, y_1)\}) = 0.2 \cdot 0.4 = 0.08,$$

$$\overline{P}_{TV}^{\mathcal{X}} \boxtimes \overline{P}_{TV}^{\mathcal{Y}}(\{(x_1, y_1)\}) = 0.4 \cdot 0.6 = 0.24,$$

$$\underline{P}_{TV}^{\mathcal{X}} \boxtimes \underline{P}_{TV}^{\mathcal{Y}}(\{x_1\} \times \mathcal{Y}) = \underline{P}_{TV}^{\mathcal{X}}(\{x_1\}) = 0.2,$$

$$\overline{P}_{TV}^{\mathcal{X}} \boxtimes \overline{P}_{TV}^{\mathcal{Y}}(\{x_1\} \times \mathcal{Y}) = \overline{P}_{TV}^{\mathcal{X}}(\{x_1\}) = 0.4.$$

616 Since the differences between the upper and lower probabilities of the two events do
617 not coincide (0.16 and 0.2, respectively) and they are all strictly positive, we deduce
618 that $\underline{P}_{TV}^{\mathcal{X}} \boxtimes \underline{P}_{TV}^{\mathcal{Y}}$ is not a TV model. \blacklozenge

619 Natural extension of marginal models

Consider now two TV models $B_{d_{TV}}^{\delta_{\mathcal{X}}}(P_0^{\mathcal{X}}) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{d_{TV}}^{\delta_{\mathcal{Y}}}(P_0^{\mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{Y})$.
Using Equations (5) and (6), we can give the form of the natural extension in the
events $A \times B$ for $A \subset \mathcal{X}$ and $B \subset \mathcal{Y}$:

$$\begin{aligned} \underline{E}_{TV}(A \times B) &= \max \{ \underline{P}_{TV}^{\mathcal{X}}(A) + \underline{P}_{TV}^{\mathcal{Y}}(B) - 1, 0 \} \\ &= \max \{ P_0^{\mathcal{X}}(A) + P_0^{\mathcal{Y}}(B) - 1 - (\delta_{\mathcal{X}} + \delta_{\mathcal{Y}}), 0 \}. \end{aligned}$$

$$\overline{E}_{TV}(A \times B) = \min \{ \overline{P}_{TV}^{\mathcal{X}}(A), \overline{P}_{TV}^{\mathcal{Y}}(B) \} = \min \{ P_0^{\mathcal{X}}(A) + \delta_{\mathcal{X}}, P_0^{\mathcal{Y}}(B) + \delta_{\mathcal{Y}} \}.$$

When the distortion parameters coincide, $\delta_{\mathcal{X}} = \delta_{\mathcal{Y}} = \delta$, these expressions simplify
to:

$$\begin{aligned} \underline{E}_{TV}(A \times B) &= \max \{ P_0^{\mathcal{X}}(A) + P_0^{\mathcal{Y}}(B) - 1 - 2\delta, 0 \} \\ &= \max \{ P_0^{\mathcal{X}}(A) + P_0^{\mathcal{Y}}(B) - 1, 2\delta \} - 2\delta, \end{aligned} \quad (27)$$

$$\overline{E}_{TV}(A \times B) = \min \{ P_0^{\mathcal{X}}(A) + \delta, P_0^{\mathcal{Y}}(B) + \delta \} = \min \{ P_0^{\mathcal{X}}(A), P_0^{\mathcal{Y}}(B) \} + \delta. \quad (28)$$

620 The lower and upper natural extension have a similar form as a TV model. How-
621 ever, the lower bound of the natural extension in Equation (27) has a distortion

622 parameter of 2δ , while the upper bound of the natural extension in Equation (28)
 623 has a distortion parameter of δ . This fact suggests that, as happened with the
 624 COR model, the natural extension of the TV model can neither be expressed as
 625 $\mathcal{E}(\underline{P}_{TV}^{\mathcal{X}}, \underline{P}_{TV}^{\mathcal{Y}}) = B_{d_{TV}}^{\delta}(\underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}})$ nor as $\mathcal{E}(\underline{P}_{TV}^{\mathcal{X}}, \underline{P}_{TV}^{\mathcal{Y}}) = B_{d_{TV}}^{2\delta}(\underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}})$. This is
 626 illustrated in our next example.

Example 19. Consider the spaces \mathcal{X} and \mathcal{Y} and the probabilities $P_0^{\mathcal{X}}$ and $P_0^{\mathcal{Y}}$
 from Example 3. Given the event $E_4 = \{(x_2, y_2)\}$, we deduce from Equations (27)
 and (7) that

$$\begin{aligned} \underline{E}_{TV}(\{(x_2, y_2)\}) &= \max\{P_0^{\mathcal{X}}(\{x_2\}) + P_0^{\mathcal{Y}}(\{y_2\}) - 1 - 2\delta, 0\} \\ &= 0.2 - 2\delta = \underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}}(\{(x_2, y_2)\}) - 2\delta \end{aligned}$$

for every $\delta \in (0, 0.1)$. On the other hand, if we consider the event $E_1 = \{(x_2, y_2)\}^c$,
 we deduce from Equations (28) and (8) that

$$\begin{aligned} \underline{E}_{TV}(\{(x_2, y_2)\}^c) &= 1 - \overline{E}_{TV}(\{(x_2, y_2)\}) = 0.5 - \delta \quad \text{while} \\ \underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}}(\{(x_2, y_2)\}^c) &= 1 - \overline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}}(\{(x_2, y_2)\}) = 0.5 \end{aligned}$$

627 meaning that we should distort $\underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}}$ by δ . We conclude from this that \underline{E}_{TV} is
 628 not a TV model starting from $\underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}}$.

To see that it is not a TV model starting from any probability measure on $\mathcal{X} \times \mathcal{Y}$,
 recall that, if that was the case, it should be $\overline{E}_{TV}(C) - \underline{E}_{TV}(C) = 2\delta$ whenever
 $0 < \underline{E}_{TV}(C) < \overline{E}_{TV}(C) < 1$. But in this case we have

$$\begin{aligned} \overline{E}_{TV}(\{(x_2, y_2)\}) - \underline{E}_{TV}(\{(x_2, y_2)\}) &= 0.5 + \delta - (0.2 - 2\delta) = 0.3 + 3\delta \\ \overline{E}_{TV}(\mathcal{X} \times \{y_2\}) - \underline{E}_{TV}(\mathcal{X} \times \{y_2\}) &= 0.5 + \delta - (0.5 - 2\delta) = 3\delta, \end{aligned}$$

629 a contradiction. \blacklozenge

7. KOLMOGOROV MODEL

630 Our next distortion model is very much related to the total variation, because it
 631 can be regarded as the case when instead of comparing the probability measures we
 632 compare their associated distribution functions. It is referred to as Kolmogorov's
 633 distortion model (K model, for short).
 634

635 Assuming that \mathcal{X} is an ordered space, the Kolmogorov distance between two
 636 probability measures is defined by

$$d_K(P, Q) = \max_{x \in \mathcal{X}} |F_P(x) - F_Q(x)|.$$

637 Following our initial assumption that $P_0 \in \mathbb{P}^*(\mathcal{X})$ and that δ is small enough so
 638 that $B_{d_K}^{\delta}(P_0) \subseteq \mathbb{P}^*(\mathcal{X})$, the credal set $B_{d_K}^{\delta}(P_0)$ can be expressed as:

$$B_{d_K}^{\delta}(P_0) = \{P \in \mathbb{P}(\mathcal{X}) \mid \underline{F}_K(x) \leq F_P(x) \leq \overline{F}_K(x) \forall x \in \mathcal{X}\},$$

where \underline{F}_K and \overline{F}_K are given by:

$$\begin{aligned} \underline{F}_K(x_i) &= \max\{0, F_{P_0}(x_i) - \delta\} = F_{P_0}(x_i) - \delta \\ \overline{F}_K(x_i) &= \min\{1, F_{P_0}(x_i) + \delta\} = F_{P_0}(x_i) + \delta \quad \forall i = 1, \dots, n-1 \end{aligned}$$

639 since $B_{d_K}^{\delta}(P_0) \subseteq \mathbb{P}^*(P_0)$, and $\underline{F}_K(x_n) = \overline{F}_K(x_n) = 1$. This means that $B_{d_K}^{\delta}(P_0) =$
 640 $\mathcal{M}(\underline{F}_K, \overline{F}_K)$, so it coincides with the credal set of a p-box. We will denote as \underline{P}_K

641 and \bar{P}_K the lower and upper probabilities associated with $B_{d_K}^\delta(P_0)$. We refer to
 642 [31, Sec. 3] for a study of the Kolmogorov model.

643 **7.1. Merging.**

644 **Conjunction**

645 We start studying the behaviour of the Kolmogorov model under conjunction.
 646 Next example demonstrates that the conjunction of two K models, while being a
 647 p-box, does not necessarily correspond to a p-box induced by a K model.

648 **Example 20.** For the model induced by the Kolmogorov distance and for those
 649 events A of the type $\{x_1, x_2, \dots, x_k\}$ such that $0 < \underline{P}_K(A) \leq \bar{P}_K(A) < 1$, we have
 650 that $\bar{P}_K(A) - \underline{P}_K(A) = 2\delta$. Let us now show that this is not necessarily the case
 651 for their conjunction.

Consider a three-element space $\mathcal{X} = \{x_1, x_2, x_3\}$, $P_0^1 = (0.25, 0.25, 0.5)$, $\delta_1 =$
 0.05 and consider the associated K model $B_{d_K}^{\delta_1}(P_0^1) \subseteq \mathbb{P}^*(\mathcal{X})$. On the other hand,
 let $B_{d_K}^{\delta_2}(P_0^2) \subseteq \mathbb{P}^*(\mathcal{X})$ be the K model induced by the probability measure $P_0^2 =$
 $(0.15, 0.35, 0.5)$ and $\delta_2 = 0.05$. Since the conjunction of two p-boxes $(\underline{F}_1, \bar{F}_1)$ and
 $(\underline{F}_2, \bar{F}_2)$ is the p-box $(\max\{\underline{F}_1, \underline{F}_2\}, \min\{\bar{F}_1, \bar{F}_2\})$, we have that

$$\begin{aligned} \min\{\bar{F}_{K_1}(x_1), \bar{F}_{K_2}(x_1)\} - \max\{\underline{F}_{K_1}(x_1), \underline{F}_{K_2}(x_1)\} &= 0.2 - 0.2 = 0, \text{ and} \\ \min\{\bar{F}_{K_1}(x_2), \bar{F}_{K_2}(x_2)\} - \max\{\underline{F}_{K_1}(x_2), \underline{F}_{K_2}(x_2)\} &= 0.55 - 0.45 = 0.1, \end{aligned}$$

652 meaning that this conjunction is not a distortion model induced by the Kolmogorov
 653 distance. \blacklozenge

However, despite the fact that the K model is not closed under conjunction,
 this conjunction still remains a p-box [16]. Among other things, this means that
 we have a straightforward way to check whether the conjunction of two models
 $\mathcal{M}_1 = B_{d_K}^{\delta_1}(P_0^1)$ and $\mathcal{M}_2 = B_{d_K}^{\delta_2}(P_0^2)$ is non-empty. We have that $\mathcal{M}_1 \cap \mathcal{M}_2 \neq \emptyset$ if
 and only if

$$\min\{\bar{F}_{K_1}(x_i), \bar{F}_{K_2}(x_i)\} - \max\{\underline{F}_{K_1}(x_i), \underline{F}_{K_2}(x_i)\} \geq 0 \quad \forall x_i \in \mathcal{X},$$

654 which is very easy to check.

655 **Disjunction**

656 The family of Kolmogorov models is not closed under disjunction:

657 **Example 21.** Consider $\mathcal{X} = \{x_1, x_2, x_3\}$ and the probability measures $P_0^1 =$
 658 $(0.5, 0.3, 0.2)$, $P_0^2 = (0.3, 0.5, 0.2)$ and $\delta_1 = \delta_2 = 0.05$. The lower probabilities
 659 $\underline{P}_{K_1}, \underline{P}_{K_2}$ associated with the K models $B_{d_K}^{\delta_1}(P_0^1) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{d_K}^{\delta_2}(P_0^2) \subseteq \mathbb{P}^*(\mathcal{X})$
 660 and their disjunction \underline{P}^\cup , are given in the following table:

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
\underline{P}_{K_1}	0.45	0.2	0.15	0.75	0.6	0.45
\underline{P}_{K_2}	0.25	0.4	0.15	0.75	0.4	0.65
\underline{P}^\cup	0.25	0.2	0.15	0.75	0.4	0.45

If \underline{P}^\cup was the K model associated with a distribution function F_0 and a distortion factor δ , its associated p -box $(\underline{F}, \overline{F})$ would be determined by the constraints

$$\begin{aligned} [\underline{F}(x_1), \overline{F}(x_1)] &= [F_0(x_1) - \delta, F_0(x_1) + \delta] = [0.25, 0.55] \Rightarrow F_0(x_1) = 0.4, \delta = 0.15, \\ [\underline{F}(x_2), \overline{F}(x_2)] &= [F_0(x_2) - \delta, F_0(x_2) + \delta] = [0.75, 0.85] \Rightarrow F_0(x_2) = 0.8, \delta = 0.05. \end{aligned}$$

661 Thus, \underline{P}_K^\cup is not a K model. To see that moreover it does not have a unique
 662 undominated outer approximation in the family of K models, consider the K models
 663 $B_{d_K}^{\delta_A}(P_A)$ and $B_{d_K}^{\delta_B}(P_B)$ induced by $P_A = (0.4, 0.33, 0.27)$, $P_B = (0.4, 0.36, 0.24)$ and
 664 $\delta_A = \delta_B = 0.16$. These produce the following lower probabilities:

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
\underline{P}_{K_A}	0.24	0.01	0.11	0.57	0.35	0.44
\underline{P}_{K_B}	0.24	0.04	0.08	0.6	0.32	0.44

665 Thus, $\underline{P}_{K_A}, \underline{P}_{K_B}$ are outer approximations of \underline{P}_K^\cup . If there was a unique undomi-
 666 nated outer approximation, then there should be another probability measure P' and
 667 $\delta' > 0$ such that either $\underline{P}_{K_A} \preceq P' \leq \underline{P}_K^\cup$ and $\underline{P}_{K_B} \preceq P' \leq \underline{P}_K^\cup$, where \underline{P}' denotes
 668 the lower envelope of $B_{d_K}^{\delta'}(P')$. However, this means that:

- 669 • $\overline{P}'(\{x_1\}) - \underline{P}'(\{x_1\}) \geq \overline{P}^\cup(\{x_1\}) - \underline{P}^\cup(\{x_1\}) = 0.55 - 0.25 = 0.3$, which
 670 implies that $\delta' = \frac{1}{2} (\overline{P}'(\{x_1\}) - \underline{P}'(\{x_1\})) \geq 0.15$.
- The same reasoning with the event $\{x_1, x_2\}$ leads to:

$$\begin{aligned} &\overline{P}'(\{x_1, x_2\}) - \underline{P}'(\{x_1, x_2\}) \\ &\leq \min\{\overline{P}_{K_A}(\{x_1, x_2\}), \overline{P}_{K_B}(\{x_1, x_2\})\} - \max\{\underline{P}_{K_A}(\{x_1, x_2\}), \underline{P}_{K_B}(\{x_1, x_2\})\} \\ &\leq 0.89 - 0.6 = 0.29, \end{aligned}$$

671 which implies that $\delta' = \frac{1}{2} (\overline{P}'(\{x_1, x_2\}) - \underline{P}'(\{x_1, x_2\})) \leq 0.145$.

672 This is a contradiction, from which we deduce that there is not a unique undomi-
 673 nated outer approximation. \blacklozenge

674 In fact, it is worth noting that the disjunction of two Kolmogorov models is not
 675 always a p -box, as we show in this example:

676 **Example 22.** Let $\mathcal{X} = \{x_1, x_2, x_3\}$, and consider the distortion models $B_{d_K}^{\delta_1}(P_0^1) \subseteq$
 677 $\mathbb{P}^*(\mathcal{X})$ and $B_{d_K}^{\delta_2}(P_0^2) \subseteq \mathbb{P}^*(\mathcal{X})$ given by the probability measures $P_0^1 = (0.3, 0.3, 0.4)$,
 678 $P_0^2 = (0.6, 0.3, 0.1)$ and the distortion factors $\delta_1 = \delta_2 = 0.05$. They induce the K
 679 models whose associated p -boxes $(\underline{F}_1, \overline{F}_1)$ and $(\underline{F}_2, \overline{F}_2)$ are:

	x_1	x_2	x_3		x_1	x_2	x_3
\underline{F}_1	0.25	0.55	1	\underline{F}_2	0.55	0.85	1
\overline{F}_1	0.35	0.65	1	\overline{F}_2	0.65	0.95	1

680 Then the disjunction $B_{d_K}^{\delta_1}(P_0^1) \cup B_{d_K}^{\delta_2}(P_0^2)$ determines the lower and upper cdfs:

	x_1	x_2	x_3
\underline{F}^\cup	0.25	0.55	1
\overline{F}^\cup	0.65	0.95	1

681 Thus, the cdf associated with $P = (0.25, 0.7, 0.05)$ would belong to $\mathcal{M}(\underline{F}^\cup, \overline{F}^\cup)$ but
 682 not to $B_{d_K}^{\delta_1}(P_0^1) \cup B_{d_K}^{\delta_2}(P_0^2)$. \blacklozenge

683 We conclude that the behaviour of the Kolmogorov model is quite inadequate
 684 when taking its intersection or union, because it is not preserved by conjunction or
 685 disjunction, but also the disjunction is not even a p-box.

686 Convex mixture

As for the total variation model, it is rather direct to check that the convex mixture of two K models $B_{d_K}^{\delta_1}(P_0^1)$ and $B_{d_K}^{\delta_2}(P_0^2)$ is again a K model. Indeed, since the convex combination of two p-boxes is simply the convex combination of the lower and upper cdfs, we have

$$\begin{aligned} \epsilon \underline{F}_{K_1}(x) + (1 - \epsilon) \underline{F}_{K_2}(x) &= \epsilon F_{P_0^1}(x) + (1 - \epsilon) F_{P_0^2}(x) - \epsilon \delta_1 - (1 - \epsilon) \delta_2 \\ \epsilon \overline{F}_{K_1}(x) + (1 - \epsilon) \overline{F}_{K_2}(x) &= \epsilon F_{P_0^1}(x) + (1 - \epsilon) F_{P_0^2}(x) + \epsilon \delta_1 + (1 - \epsilon) \delta_2 \end{aligned}$$

687 which are the cdfs induced by the K model $B_{d_K}^{\delta_\epsilon}(P_0^\epsilon)$ with $\delta_\epsilon = \epsilon \delta_1 + (1 - \epsilon) \delta_2$ and

$$F_{P_0^\epsilon}(x) = \epsilon F_{P_0^1}(x) + (1 - \epsilon) F_{P_0^2}(x) \quad \forall x \in \mathcal{X}.$$

688 7.2. Multivariate setting.

689 Marginalisation

690 In order to study the Kolmogorov model in a multivariate setting, we first need
 691 to provide its definition in this context. For this aim, we are now dealing with two
 692 ordered spaces $\mathcal{X} = \{x_1, \dots, x_n\}$ and $\mathcal{Y} = \{y_1, \dots, y_m\}$. A bivariate p-box [39] is a
 693 pair of component-wise increasing functions $\underline{F}_{\mathcal{X}, \mathcal{Y}}, \overline{F}_{\mathcal{X}, \mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \rightarrow [0, 1]$ satisfying
 694 $\underline{F}_{\mathcal{X}, \mathcal{Y}} \leq \overline{F}_{\mathcal{X}, \mathcal{Y}}$ and $\underline{F}_{\mathcal{X}, \mathcal{Y}}(x_n, y_m) = \overline{F}_{\mathcal{X}, \mathcal{Y}}(x_n, y_m) = 1$. A bivariate p-box defines a
 695 credal set by:

$$\mathcal{M}(\underline{F}_{\mathcal{X}, \mathcal{Y}}, \overline{F}_{\mathcal{X}, \mathcal{Y}}) = \{P \in \mathbb{P}(\mathcal{X} \times \mathcal{Y}) \mid \underline{F} \leq F_P \leq \overline{F}\}.$$

696 In contrast with the univariate case, the credal set $\mathcal{M}(\underline{F}_{\mathcal{X}, \mathcal{Y}}, \overline{F}_{\mathcal{X}, \mathcal{Y}})$ may be empty.
 697 When it is non-empty, we can define a coherent lower and upper probability by
 698 taking lower and upper envelopes.

A bivariate p-box $(\underline{F}_{\mathcal{X}, \mathcal{Y}}, \overline{F}_{\mathcal{X}, \mathcal{Y}})$ defines marginal (univariate) p-boxes $(\underline{F}_{\mathcal{X}}, \overline{F}_{\mathcal{X}})$ and $(\underline{F}_{\mathcal{Y}}, \overline{F}_{\mathcal{Y}})$, respectively in \mathcal{X} and \mathcal{Y} , by:

$$\begin{aligned} \underline{F}_{\mathcal{X}}(x_i) &= \underline{F}_{\mathcal{X}, \mathcal{Y}}(x_i, y_m), & \overline{F}_{\mathcal{X}}(x_i) &= \overline{F}_{\mathcal{X}, \mathcal{Y}}(x_i, y_m), & \forall i = 1, \dots, n. \\ \underline{F}_{\mathcal{Y}}(y_j) &= \underline{F}_{\mathcal{X}, \mathcal{Y}}(x_n, y_j), & \overline{F}_{\mathcal{Y}}(y_j) &= \overline{F}_{\mathcal{X}, \mathcal{Y}}(x_n, y_j), & \forall j = 1, \dots, m. \end{aligned}$$

699 In that case, the credal sets $\mathcal{M}(\underline{F}_{\mathcal{X}}, \overline{F}_{\mathcal{X}})$ and $\mathcal{M}(\underline{F}_{\mathcal{Y}}, \overline{F}_{\mathcal{Y}})$ coincide with the \mathcal{X}
 700 and \mathcal{Y} projections, respectively, of the probability measures in $\mathcal{M}(\underline{F}_{\mathcal{X}, \mathcal{Y}}, \overline{F}_{\mathcal{X}, \mathcal{Y}})$.
 701 We refer to [28, 39] for some studies about bivariate p-boxes, and to [32] for some
 702 comments on the connection between uni- and bivariate p-boxes.

703 Given the ordered spaces \mathcal{X} and \mathcal{Y} , we define the K model for a bivariate cdf
 704 $F_{P_0}(x, y)$ as the credal set induced by the bivariate p-box [39]

$$\underline{F}(x, y) = \max\{F_{P_0}(x, y) - \delta, 0\}, \quad \overline{F}(x, y) = \min\{F_{P_0}(x, y) + \delta, 1\} \quad (29)$$

705 for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$. We then get the following result regarding the marginals
706 of a bivariate K model:

707 **Proposition 7.** *Consider the model $B_{d_K}^\delta(P_0^{\mathcal{X}, \mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{X} \times \mathcal{Y})$ and its associated
708 lower prevision \underline{P}_K . Then, the marginal model $\underline{P}_K^{\mathcal{X}}$ induces the credal set $B_{d_K}^\delta(P_0^{\mathcal{X}})$
709 with $P_0^{\mathcal{X}}$ the marginal probability of $P_0^{\mathcal{X}, \mathcal{Y}}$ on \mathcal{X} .*

Proof. Let us first notice that the bivariate p-box defined by Equation (29) induces a coherent lower probability, as it corresponds to the projection over events of the kind $\{x_1, \dots, x_i\} \times \{y_1, \dots, y_j\}$ of the coherent lower probability induced by $B_{d_{TV}}^\delta(P_0^{\mathcal{X}, \mathcal{Y}})$. We can then easily check that the marginal p-box $(\underline{F}_{\mathcal{X}}, \overline{F}_{\mathcal{X}})$ satisfies:

$$\underline{F}_{\mathcal{X}}(x_i) = \underline{F}(x_i, y_m) = F_{P_0^{\mathcal{X}, \mathcal{Y}}}(x_i, y_m) - \delta = F_{P_0^{\mathcal{X}}}(x_i) - \delta \quad \forall i = 1, \dots, n-1$$

that is the lower cdf induced by $B_{d_K}^\delta(P_0^{\mathcal{X}})$. Similarly,

$$\overline{F}_{\mathcal{X}}(x_i) = \overline{F}(x_i, y_m) = F_{P_0^{\mathcal{X}, \mathcal{Y}}}(x_i, y_m) + \delta = F_{P_0^{\mathcal{X}}}(x_i) + \delta \quad \forall i = 1, \dots, n-1.$$

710 Therefore the marginal model also belongs to the Kolmogorov family. \square

711 Independent products

Consider now two marginal K models $B_{d_K}^\delta(P_0^{\mathcal{X}}) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{d_K}^\delta(P_0^{\mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{Y})$. Regarding the problem of going from marginal models $P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}$ to joint ones, we can first notice that on the Cartesian product of the events $A_i = \{x_1, \dots, x_i\}$ and $B_j = \{y_1, \dots, y_j\}$ (with $i < n$ and $j < m$), we have

$$\begin{aligned} \underline{P}_K^{\mathcal{X} \times \mathcal{Y}}(A_i \times B_j) &= \underline{F}^{\mathcal{X} \times \mathcal{Y}}(x_i, y_j) = F_{P_0}(x_i, y_j) - \delta, \text{ while} \\ \underline{P}_K^{\mathcal{X}} \boxtimes \underline{P}_K^{\mathcal{Y}}(A_i \times B_j) &= (F_{P_0^{\mathcal{X}}}(x_i) - \delta)(F_{P_0^{\mathcal{Y}}}(y_j) - \delta) \\ &= F_{P_0}(x_i, y_j) - \delta(F_{P_0^{\mathcal{X}}}(x_i) + F_{P_0^{\mathcal{Y}}}(y_j) - \delta) \end{aligned}$$

712 where last equation follows from the factorization property in Equation (2). Hence,
713 the equality $\underline{P}_K^{\mathcal{X} \times \mathcal{Y}}(A_i \times B_j) = \underline{P}_K^{\mathcal{X}} \boxtimes \underline{P}_K^{\mathcal{Y}}(A_i \times B_j)$ may not hold. The inequality can
714 go both ways (either $\underline{P}_K^{\mathcal{X} \times \mathcal{Y}}(A \times B) < \underline{P}_K^{\mathcal{X}} \boxtimes \underline{P}_K^{\mathcal{Y}}(A \times B)$ or the reverse) depending
715 on the value of δ , as we show next.

716 **Example 23.** *Consider the ordered spaces $\mathcal{X} = \{x_1, x_2, x_3\}$ and $\mathcal{Y} = \{y_1, y_2\}$, and
717 the probability measures $P_0^{\mathcal{X}}$ and $P_0^{\mathcal{Y}}$ given by:*

$$P_0^{\mathcal{X}}(\{x_1\}) = 0.3, P_0^{\mathcal{X}}(\{x_2\}) = 0.5, P_0^{\mathcal{X}}(\{x_3\}) = 0.2, P_0^{\mathcal{Y}}(\{y_1\}) = P_0^{\mathcal{Y}}(\{y_2\}) = 0.5.$$

Consider $\delta = 0.1$, the K models $B_{d_K}^\delta(P_0^{\mathcal{X}}) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{d_K}^\delta(P_0^{\mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{Y})$ and their associated lower probabilities $\underline{P}_K^{\mathcal{X}}$ and $\underline{P}_K^{\mathcal{Y}}$. Then we obtain on the one hand

$$\begin{aligned} \underline{P}_K^{\mathcal{X} \times \mathcal{Y}}(\{x_1\} \times \{y_1\}) &= 0.15 - \delta = 0.05 < 0.08 = (0.3 - \delta)(0.5 - \delta) \\ &= \underline{P}_K^{\mathcal{X}}(\{x_1\})\underline{P}_K^{\mathcal{Y}}(\{y_1\}) = \underline{P}_K^{\mathcal{X}} \boxtimes \underline{P}_K^{\mathcal{Y}}(\{x_1\} \times \{y_1\}), \end{aligned}$$

while on the other hand:

$$\begin{aligned} \underline{P}_K^{\mathcal{X} \times \mathcal{Y}}(\{x_1, x_2\} \times \{y_1\}) &= 0.4 - \delta = 0.3 > 0.28 = (0.8 - \delta)(0.5 - \delta) \\ &= \underline{P}_K^{\mathcal{X}}(\{x_1, x_2\})\underline{P}_K^{\mathcal{Y}}(\{y_1\}) = \underline{P}_K^{\mathcal{X}} \boxtimes \underline{P}_K^{\mathcal{Y}}(\{x_1, x_2\} \times \{y_1\}). \end{aligned}$$

718 We conclude that there is no dominance relationship between the coherent lower
719 probabilities obtained by the two approaches. \blacklozenge

720 Let us now show that, as with the other models, K models are not closed under
721 strong products.

722 **Example 24.** Consider the same setting of Example 23. The K models are the
723 p-boxes $(\underline{F}_{\mathcal{X}}, \overline{F}_{\mathcal{X}})$, $(\underline{F}_{\mathcal{Y}}, \overline{F}_{\mathcal{Y}})$:

\mathcal{X}	x_1	x_2	x_3	\mathcal{Y}	y_1	y_2
$\underline{F}_{\mathcal{X}}$	0.2	0.7	1	$\underline{F}_{\mathcal{Y}}$	0.4	1
$\overline{F}_{\mathcal{X}}$	0.4	0.9	1	$\overline{F}_{\mathcal{Y}}$	0.6	1

724 Using the factorisation property of the strong product, these generate the following
725 joint bounds for the events $\{x_1, \dots, x_i\} \times \{y_1, \dots, y_j\}$:

	$\{x_1\} \times \{y_1\}$	$\{x_1\} \times \mathcal{Y}$	$\{x_1, x_2\} \times \{y_1\}$	$\{x_1, x_2\} \times \mathcal{Y}$	$\mathcal{X} \times \{y_1\}$
$P_K^{\mathcal{X}} \boxtimes P_K^{\mathcal{Y}}$	0.08	0.2	0.28	0.7	0.4
$\overline{P}_K^{\mathcal{X}} \boxtimes \overline{P}_K^{\mathcal{Y}}$	0.24	0.4	0.54	0.9	0.6

726 If this was a K model, than the differences between the upper and lower probabilities
727 for these events should be constant, which is not the case here. \blacklozenge

728 **Natural extension of marginal models**

729 Consider now two K models $B_{d_K}^{\delta_{\mathcal{X}}}(P_0^{\mathcal{X}})$ and $B_{d_K}^{\delta_{\mathcal{Y}}}(P_0^{\mathcal{Y}})$, and let us focus on the
730 problem of applying to them the natural extension. We already know that these K
731 models are equivalent to the univariate p-boxes $(\underline{F}_{\mathcal{X}}, \overline{F}_{\mathcal{X}})$ and $(\underline{F}_{\mathcal{Y}}, \overline{F}_{\mathcal{Y}})$. Thus, we
732 can use the results from [32, Sec. 3.2], where we studied the natural extension of two
733 p-boxes. In particular, in [32, Prop. 5] we proved that $\mathcal{E}((\underline{F}_{\mathcal{X}}, \overline{F}_{\mathcal{X}}), (\underline{F}_{\mathcal{Y}}, \overline{F}_{\mathcal{Y}})) =$
734 $\mathcal{M}(\underline{F}, \overline{F})$, where $(\underline{F}, \overline{F})$ is the bivariate p-box given, for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$, by:

$$\underline{F}(x, y) = \max \{ \underline{F}_{\mathcal{X}}(x) + \underline{F}_{\mathcal{Y}}(y) - 1, 0 \}, \quad \overline{F}(x, y) = \min \{ \overline{F}_{\mathcal{X}}(x), \overline{F}_{\mathcal{Y}}(y) \}. \quad (30)$$

735 Then, the natural extension of two K models is again a p-box. However, our next
736 example shows that this natural extension is not a K model.

737 **Example 25.** Consider the spaces \mathcal{X} and \mathcal{Y} and the probabilities $P_0^{\mathcal{X}}$ and $P_0^{\mathcal{Y}}$
738 from Example 23 and the associated p-boxes detailed in Example 24. Applying
739 Equation (30), their natural extension is the p-box $(\underline{F}, \overline{F})$:

y_2	[0.2, 0.4]	[0.7, 0.9]	[1, 1]
y_1	[0, 0.4]	[0.1, 0.6]	[0.4, 0.6]
$[\underline{F}(x_i, y_j), \overline{F}(x_i, y_j)]$	x_1	x_2	x_3

740 We can see that:

$$\overline{F}(x_1, y_2) - \underline{F}(x_1, y_2) = 0.2, \quad \overline{F}(x_2, y_1) - \underline{F}(x_2, y_1) = 0.5.$$

741 Since the differences between \overline{F} and \underline{F} are not constant, we conclude that $(\underline{F}, \overline{F})$
742 does not correspond to a K model. \blacklozenge

743 We conclude that, even if there is a simple formula for computing the natural
744 extension (Equation (30)), the latter is not a Kolmogorov model.

745

8. L_1 DISTORTION MODEL

We conclude our investigation by considering the distortion model associated with the L_1 distance. Given two probability measures P, Q , their L_1 distance is

$$d_{L_1}(P, Q) = \sum_{A \subseteq \mathcal{X}} |P(A) - Q(A)|.$$

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In order to alleviate the notation, we shall also denote it by d_1 . This distance has been used in robust statistics in [42]. When $P_0 \in \mathbb{P}^*(\mathcal{X})$ and δ is small enough, it induces the credal set $B_{d_1}^\delta(P_0) \subseteq \mathbb{P}^*(\mathcal{X})$ whose associated lower and upper probabilities are [31, Thm. 11]:

$$\underline{P}_{L_1}(A) = P_0(A) - \frac{\delta}{\varphi(|\mathcal{X}|, |A|)} \quad \forall A \neq \emptyset, \mathcal{X}, \quad (31)$$

750 where

$$\varphi(n, k) = \sum_{l=0}^k \binom{k}{l} \sum_{j=0}^{n-k} \binom{n-k}{j} \left| \frac{l}{k} - \frac{j}{n-k} \right| \quad \forall k = 1, \dots, n.$$

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8.1. Merging. Let us analyse the behaviour of the L_1 model under conjunction, disjunction and convex mixture.

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Conjunction and disjunction

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In general, the conjunction of two L_1 models will not lead to a new L_1 model. In fact, observe that in the case of ternary spaces, we can establish a relationship between the total variation and L_1 models. In that case, $\varphi(3, 1) = \varphi(3, 2) = 4$, whence from Equation (31) \underline{P}_{L_1} is given by:

$$\underline{P}_{L_1}(A) = P_0(A) - \frac{\delta}{4} \quad \forall A \neq \emptyset, \mathcal{X}.$$

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Also, from Equation (25), the total variation model generated by the probability measure P_0 and the distortion parameter $\frac{\delta}{4}$ is given by:

$$\underline{P}_{TV}(A) = P_0(A) - \frac{\delta}{4} \quad \forall A \neq \emptyset, \mathcal{X}.$$

Since we are assuming that δ is small enough such that $\underline{P}_{d_1}(A) > 0$ for every $A \neq \emptyset$, we conclude that in cardinality three $B_{d_1}^\delta(P_0) = B_{d_{TV}}^{\delta/4}(P_0)$. Thus, if we consider $\mathcal{X} = \{x_1, x_2, x_3\}$, $P_0^1 = (0.41, 0.37, 0.22)$, $P_0^2 = (0.37, 0.41, 0.22)$ and $\delta_1 = \delta_2 = 0.48$, we obtain that

$$B_{d_1}^{0.48}(P_0^1) = B_{d_{TV}}^{0.12}(P_0^1) \text{ and } B_{d_1}^{0.48}(P_0^2) = B_{d_{TV}}^{0.12}(P_0^2).$$

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Using Example 15, we conclude that the family of L_1 models is not closed under conjunction nor disjunction.

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Moreover, taking into account this same connection between the TV and the L_1 models on ternary spaces as well as Example 16, we conclude that the disjunction of two L_1 models does not possess a unique undominated outer approximation in the L_1 family.

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Several facts indicate that checking in which cases the conjunction of two L_1 models is a L_1 model is a difficult task, for a number of reasons: (i) a L_1 model is not determined in general by the lower probability it defines on events, in the sense that a probability measure P may dominate the lower envelope of $B_{d_1}^\delta(P_0)$ on any

770 event but still do not belong to $B_{d_1}^\delta(P_0)$; (ii) it is not described by an explicit lower
 771 prevision; and (iii) even enumerating the extreme points of $B_{d_1}(P_0)$ is to this point
 772 an open issue [31].

773 **Convex mixture**

Again, it is rather direct to check that, if we consider their restriction on events,
 the convex mixture of two lower probabilities associated to L_1 models $B_{d_1}^{\delta_1}(P_0^1)$ and
 $B_{d_1}^{\delta_2}(P_0^2)$ is again a lower probability associated to an L_1 model, as we have

$$\epsilon \underline{P}_{L_1}^{\delta_1}(A) + (1 - \epsilon) \underline{P}_{L_1}^{\delta_2}(A) = \epsilon P_0^1(A) + (1 - \epsilon) P_0^2(A) - \epsilon \frac{\delta_1}{\varphi(|\mathcal{X}|, |A|)} - (1 - \epsilon) \frac{\delta_2}{\varphi(|\mathcal{X}|, |A|)}$$

which are lower probabilities induced by the L_1 model $B_{d_1}^{\delta_\epsilon}(P_0^\epsilon)$ with $\delta_\epsilon = \epsilon\delta_1 + (1 - \epsilon)\delta_2$ and

$$P_0^\epsilon(\{x\}) = \epsilon P_0^1(\{x\}) + (1 - \epsilon) P_0^2(\{x\}) \quad \forall x \in \mathcal{X}.$$

774 Since L_1 models can be described by their lower probabilities whenever $n \leq 11$, this
 775 is sufficient to show that in those cases the model is closed under convex mixture.
 776 The case $n > 11$, for which we have no simple and explicit description of the lower
 777 envelope of the L_1 model, remains an open problem.

778 **8.2. Multivariate setting.**

779 **Marginalisation**

780 For the multivariate case, let us first look at the marginals of a joint L_1 model
 781 $B_{d_1}^\delta(P_0^{\mathcal{X}, \mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{X} \times \mathcal{Y})$. Somewhat surprisingly, the marginal model of a L_1 model
 782 is not a L_1 model, as our next example shows.

Example 26. Let $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$, $\mathcal{Y} = \{y_1, y_2\}$, $P_0^{\mathcal{X}, \mathcal{Y}}$ the uniform distribu-
 tion on $\mathcal{X} \times \mathcal{Y}$ and consider the distortion parameter $\delta = 0.1$. Then:

$$\begin{aligned} \underline{P}_{L_1}^{\mathcal{X}}(\{x_i\}) &= \underline{P}_{L_1}(\{x_i\} \times \mathcal{Y}) = P_0(\{x_i\} \times \mathcal{Y}) - \frac{0.1}{\varphi(8, 2)} = \frac{1}{4} - \frac{0.1}{84} \\ &= P_0^{\mathcal{X}}(\{x_i\}) - \frac{\delta_{\mathcal{X}}}{\varphi(4, 1)} = \frac{1}{4} - \frac{\delta_{\mathcal{X}}}{8} \Rightarrow \delta_{\mathcal{X}} = \frac{0.1}{10.5}. \\ \underline{P}_{L_1}^{\mathcal{X}}(\{x_i, x_j\}) &= \underline{P}_{L_1}(\{x_i, x_j\} \times \mathcal{Y}) = P_0(\{x_i, x_j\} \times \mathcal{Y}) - \frac{0.1}{\varphi(8, 4)} = \frac{1}{2} - \frac{0.1}{70} \\ &= P_0^{\mathcal{X}}(\{x_i, x_j\}) - \frac{\delta_{\mathcal{X}}}{\varphi(4, 2)} = \frac{1}{2} - \frac{\delta_{\mathcal{X}}}{6} \Rightarrow \delta_{\mathcal{X}} = \frac{0.3}{35}. \end{aligned}$$

783 Since $\frac{0.1}{10.5} \neq \frac{0.3}{35}$, we conclude that the marginal model does not belong to the L_1
 784 family. \blacklozenge

785 **Independent products**

786 We now investigate what happens when building a joint from marginal L_1 models
 787 $B_{d_1}^\delta(P_0^{\mathcal{X}}) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{d_1}^\delta(P_0^{\mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{Y})$. Again, on Cartesian products of events,
 788 we have

$$\underline{P}_{L_1}^{\mathcal{X} \times \mathcal{Y}}(A \times B) = P_0^{\mathcal{X} \times \mathcal{Y}}(A \times B) - \frac{\delta}{\varphi(|\mathcal{X} \times \mathcal{Y}|, |A \times B|)},$$

789 while

$$\underline{P}_{L_1}^{\mathcal{X}} \boxtimes \underline{P}_{L_1}^{\mathcal{Y}}(A \times B) = \left(P_0^{\mathcal{X}}(A) - \frac{\delta}{\varphi(|\mathcal{X}|, |A|)} \right) \left(P_0^{\mathcal{Y}}(B) - \frac{\delta}{\varphi(|\mathcal{Y}|, |B|)} \right).$$

790 In that case, it may happen that $\underline{P}_{L_1}^{\mathcal{X} \times \mathcal{Y}}(A \times B) \geq \underline{P}_{L_1}^{\mathcal{X}} \boxtimes \underline{P}_{L_1}^{\mathcal{Y}}(A \times B)$ or $\underline{P}_{L_1}^{\mathcal{X} \times \mathcal{Y}}(A \times$
791 $B) \leq \underline{P}_{L_1}^{\mathcal{X}} \boxtimes \underline{P}_{L_1}^{\mathcal{Y}}(A \times B)$, even with strict inequality, as we show in next example.

792 **Example 27.** Consider \mathcal{X} and \mathcal{Y} such that $|\mathcal{X}| = |\mathcal{Y}| = 2$, P_0 the uniform distribu-
793 tion on $\mathcal{X} \times \mathcal{Y}$ and A, B two singletons. The connection between $\underline{P}_{L_1}^{\mathcal{X} \times \mathcal{Y}}(A \times B)$ and
794 $\underline{P}_{L_1}^{\mathcal{X}} \boxtimes \underline{P}_{L_1}^{\mathcal{Y}}(A \times B)$ depends on whether $\delta > \frac{3}{2}$ or $\delta < \frac{3}{2}$. The reason is that if $A \subseteq \mathcal{X}$
795 and $B \subseteq \mathcal{Y}$ are singletons, $\varphi(|\mathcal{X}|, |A|) = \varphi(|\mathcal{Y}|, |B|) = 2$, $\varphi(|\mathcal{X} \times \mathcal{Y}|, |A \times B|) =$
796 $\varphi(4, 1) = 8$, $P_0(A \times B) = \frac{1}{4}$ and $P_0^{\mathcal{X}}(A) = P_0^{\mathcal{Y}}(B) = \frac{1}{2}$. Hence:

$$\underline{P}_{L_1}^{\mathcal{X} \times \mathcal{Y}}(A \times B) = \frac{1}{4} - \frac{\delta}{8}, \quad \underline{P}_{L_1}^{\mathcal{X}} \boxtimes \underline{P}_{L_1}^{\mathcal{Y}}(A \times B) = \left(\frac{1}{2} - \frac{\delta}{2} \right) \cdot \left(\frac{1}{2} - \frac{\delta}{2} \right).$$

797 Operating, we obtain that $\underline{P}_{L_1}^{\mathcal{X} \times \mathcal{Y}}(A \times B) > \underline{P}_{L_1}^{\mathcal{X}} \boxtimes \underline{P}_{L_1}^{\mathcal{Y}}(A \times B)$ if and only if $\delta < \frac{3}{2}$
798 and $\underline{P}_{L_1}^{\mathcal{X} \times \mathcal{Y}}(A \times B) < \underline{P}_{L_1}^{\mathcal{X}} \boxtimes \underline{P}_{L_1}^{\mathcal{Y}}(A \times B)$ if and only if $\delta > \frac{3}{2}$. Therefore, in general
799 there is no dominance relation between the two approaches. \blacklozenge

800 Let us now show through an example that the L_1 model is also not closed under
801 strong products.

Example 28. Let us consider the case where $\mathcal{X} = \{x_1, x_2\}$ and $\mathcal{Y} = \{y_1, y_2\}$ with
 $P_0^{\mathcal{X}}$ and $P_0^{\mathcal{Y}}$ uniform with some δ . Let us now assume that $\underline{P}_{L_1}^{\mathcal{X}} \boxtimes \underline{P}_{L_1}^{\mathcal{Y}}$ is a L_1 model
with some δ^* . Due to the factorization property, we have

$$\underline{P}_{L_1}^{\mathcal{X}} \boxtimes \underline{P}_{L_1}^{\mathcal{Y}}(\{(x_i, y_j)\}) = \left(0.5 - \frac{\delta}{2} \right) \left(0.5 - \frac{\delta}{2} \right) \text{ for } i, j \in \{1, 2\}$$

802 and since by assumption this should also be a L_1 model with a uniform distribution,
803 we should also have

$$\underline{P}_{L_1}^{\mathcal{X}} \boxtimes \underline{P}_{L_1}^{\mathcal{Y}}(\{(x_i, y_j)\}) = \frac{1}{4} - \frac{\delta^*}{\varphi(4, 1)} = \frac{1}{4} - \frac{\delta^*}{8}.$$

804 Fixing $\delta = 0.1$, the two equalities lead to

$$\underline{P}_{L_1}^{\mathcal{X}} \boxtimes \underline{P}_{L_1}^{\mathcal{Y}}(\{(x_i, y_j)\}) = 0.45^2 = \frac{1}{4} - \frac{\delta^*}{8},$$

805 which gives us $\delta^* = 0.38$ for the joint model $\underline{P}_{L_1}^{\mathcal{X}} \boxtimes \underline{P}_{L_1}^{\mathcal{Y}}$. However, on the event
806 $\{x_1\} \times \mathcal{Y}$ this gives $\underline{P}_{L_1}^{\mathcal{X}} \boxtimes \underline{P}_{L_1}^{\mathcal{Y}}(\{x_1\} \times \mathcal{Y}) = 0.45$, which is different from $1/2 -$
807 $\delta^*/\varphi(4, 2) = 0.437$. \blacklozenge

808 Natural extension of marginal models

Consider now two L_1 models $B_{d_1}^{\delta_{\mathcal{X}}}(P_0^{\mathcal{X}}) \subseteq \mathbb{P}^*(\mathcal{X})$ and $B_{d_1}^{\delta_{\mathcal{Y}}}(P_0^{\mathcal{Y}}) \subseteq \mathbb{P}^*(\mathcal{Y})$ in \mathcal{X}
and \mathcal{Y} , respectively. Using Equations (5) and (6), we can give the form of the lower
and upper bounds of their natural extension in the events $A \times B$:

$$\underline{E}_{L_1}(A \times B) = \max \left\{ P_0^{\mathcal{X}}(A) - \frac{\delta_{\mathcal{X}}}{\varphi(|\mathcal{X}|, |A|)} + P_0^{\mathcal{Y}}(B) - \frac{\delta_{\mathcal{Y}}}{\varphi(|\mathcal{Y}|, |B|)} - 1, 0 \right\}.$$

$$\overline{E}_{L_1}(A \times B) = \min \left\{ P_0^{\mathcal{X}}(A) + \frac{\delta_{\mathcal{X}}}{\varphi(|\mathcal{X}|, |A|)}, P_0^{\mathcal{Y}}(B) + \frac{\delta_{\mathcal{Y}}}{\varphi(|\mathcal{Y}|, |B|)} \right\}.$$

When the distortion parameters coincide, these expressions simplify to:

$$\underline{E}_{L_1}(A \times B) = \max \left\{ P_0^{\mathcal{X}}(A) + P_0^{\mathcal{Y}}(B) - 1 - \delta \left(\frac{1}{\varphi(|\mathcal{X}|, |A|)} + \frac{1}{\varphi(|\mathcal{Y}|, |B|)} \right), 0 \right\}.$$

$$\overline{E}_{L_1}(A \times B) = \min \left\{ P_0^{\mathcal{X}}(A) + \frac{\delta}{\varphi(|\mathcal{X}|, |A|)}, P_0^{\mathcal{Y}}(B) + \frac{\delta}{\varphi(|\mathcal{Y}|, |B|)} \right\}.$$

809 Note that none of the equations can be simplified, because even if the distortion pa-
 810 rameters coincide, the expressions depend on the values $\varphi(|\mathcal{X}|, |A|)$ and $\varphi(|\mathcal{Y}|, |B|)$.

811 Taking this into account, we deduce that in general $\mathcal{E}(\underline{P}_{L_1}^{\mathcal{X}}, \underline{P}_{L_1}^{\mathcal{Y}})$ cannot be ex-
 812 pressed as $B_{d_1}^{\delta}(\underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}})$: it suffices to note that, since \mathcal{X}, \mathcal{Y} are binary spaces,
 813 $B_{d_1}^{\delta}(P_0) = B_{d_{TV}}^{\delta/2}(P_0)$ (see for instance [31, p.646]), so we can use the same Ex-
 814 ample 19 for the TV model to deduce that we do not have the equality between
 815 $\mathcal{E}(\underline{P}_{L_1}^{\mathcal{X}}, \underline{P}_{L_1}^{\mathcal{Y}})$ and $B_{d_1}^{\delta}(\underline{E}_{P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}})$.

816 **9. CONCLUSIONS**

817 The variety of distortion models present in the literature makes it interesting
 818 to develop tools to compare their behaviour in a number of situations, so as to
 819 be able to choose the most appropriate model in each scenario. In this paper, we
 820 have complemented our earlier work in [30, 31] and compared six different distort-
 821 tion models by determining (i) if they are closed under conjunction, disjunction or
 822 convex mixture; (ii) whether there is a unique procedure to build an independent
 823 product; (iii) if they are closed under marginalisation; and (iv) whether the proce-
 824 dures of distortion and natural extension commute. Tables 1 and 2 summarise our
 825 results. From these tables, it is clear that the PMM and LV models are the most
 826 stable, and that the least stables are the COR and L_1 models, respectively from
 827 the merging and multi-variate point of view.

	Conjunction	Disjunction	Mixture	Unique OA?
PMM	YES [29, Prop.12]	NO (Ex.2)	YES	YES [29, Prop.12]
LV	YES (Prop.3)	NO (Ex.5)	YES	YES [33, Prop.8]
COR	NO (Ex.9)	NO (Ex.10)	NO (Ex.11)	NO (Ex.10)
TV	NO (Ex.15)	NO (Ex.15)	YES	NO (Ex.16)
K	NO (Ex.20)	NO (Ex.21)	YES	NO (Ex.21)
L_1	NO (Ex.15)	NO (Ex.15)	YES	NO (Ex.16)

TABLE 1. Behaviour of the neighbourhood models under conjunc-
 tion, disjunction and convex mixture.

828 In the case of the natural extension, we should remark that, strictly speaking,
 829 the natural extension of two marginal pari mutuel models is only a PMM if we
 830 regard it as a PMM-distortion like of a lower probability, but not in the sense of
 831 Definition 2; see Theorem 2 for more details.

832 Beyond these comparisons, there are a few global remarks that we find interest-
 833 ing:

- 834 • Regarding merging, the union of two convex sets \mathcal{M}_1 and \mathcal{M}_2 will not be
 835 convex in general [54, Thm. 6], and it is common to consider the convex
 836 hull $ch(\mathcal{M}_1 \cup \mathcal{M}_2)$ of the union. None of the distortion models considered

	Marginalising	Strong Product	Natural Extension
PMM	YES [29, Sec.6.2]	NO (Ex.4)	YES (Thm.2)
LV	YES (Prop.4)	NO (Ex.7)	NO (Ex.8)
COR	YES (Prop.5)	NO (Ex.13)	NO (Ex.14)
TV	YES (Prop.6)	NO (Ex.18)	NO (Ex.19)
K	YES (Prop.7)	NO (Ex.24)	NO (Ex.25)
L_1	NO (Ex.26)	NO (Ex.28)	NO (Ex.19)

TABLE 2. Behaviour of the neighbourhood models under different operations.

837 in this study is closed under disjunction; we may then consider the problem
838 of outer approximating this disjunction within that family. However, this
839 outer approximation is not unique [33]. The PMM and LV models, being
840 special instances of probability intervals, are remarkable exceptions, as for
841 any set \mathcal{M} , its greatest outer-approximation in terms of the pari mutuel or
842 the linear vacuous is unique [33].

- 843 • The results regarding the problem of constructing a joint model, and the
844 relation between the two approaches considered here (combine then distort
845 vs. distort then combine), are valid for other independence notions
846 from imprecise probability [11]. Indeed, most of them (including epistemic
847 independence and random set independence, for instance) also satisfy Equation
848 (2), hence the inequality concerning events of the kind $A \times B$ remains
849 true for them. As this factorisation property is also true for lower proba-
850 bilities, the various examples and discussion given for the different models
851 also apply to them. This may be an important issue when having to choose
852 whether one should first combine then discount, or discount then combine.
853 We can nevertheless observe that there is essentially one way to apply the
854 first option (using stochastic independence), and many to apply the second
855 (as one has to choose an adequate notion of independence).

856 Taking these results into account, it may be interesting to look at the problem
857 from a different angle: to characterise those distortion models that are closed under
858 merging operations in terms of the properties of the distorting function d . While
859 this is left as future research, we can give a number of preliminary comments. On
860 the one hand, it may be useful to consider some of the properties from [30, 31], where
861 we characterised those distorting functions d determining probability intervals, for
862 which it may be easier to analyse their conjunction and disjunction, using results
863 from de Campos et al. [13]. Note nevertheless that whether those probability
864 intervals will remain distortion models is not guaranteed in general. With respect
865 to the natural extension, we think that Proposition 1 should be useful in this regard;
866 and concerning independent products, we conjecture that the equality between the
867 two approaches considered in the paper will only hold in very particular cases.

868 Our work in this paper may be extended in a number of ways: on the one hand,
869 we may consider other distortion models, such as those based on divergences such as
870 Kullback-Leibler [12, 35] or the recently introduced nearly-linear models [10]; on the
871 other hand, we may consider other models of merging [52] or of independence [11];
872 and we may take the approach one step further and consider distorted credal sets,
873 considering the ideas put forward by Moral in [35].

874

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883

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