

BOUNDEDNESS OF SCHRÖDINGER OPERATOR IN ENERGY SPACE.

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ABSTRACT. On a complete weighted Riemannian manifold (M^n, g, μ) satisfying the doubling condition and the Poincaré inequalities, we characterize the class of function V such that the Schrödinger operator $\Delta - V$ maps the homogeneous Sobolev space $\overset{\circ}{W}{}^{1,2}(M)$ to its dual space. On Euclidean space, this result is due to Maz'ya and Verbitsky. In the proof of our result, we investigate the weighted L^2 -boundedness of the Hodge projector.

1. INTRODUCTION

Our main result is a generalization of this result of Maz'ya and Verbitsky [MV02]:

Theorem 1.1. *Let V be a distribution on \mathbb{R}^n , $n \geq 3$, then the following properties are equivalent:*

(1) *there is a positive constant \mathbf{A} such that*

$$\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n): |\langle V, \varphi^2 \rangle| \leq \mathbf{A} \int_{\mathbb{R}^n} |d\varphi|^2 dx, \quad (1)$$

(2) *there exists a 1-form $\theta = \sum_j \theta_j dx_j \in L_{loc}^2$ solving $d^*\theta = V$ and a positive constant \mathbf{B} :*

$$\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n): \int_{\mathbb{R}^n} |\theta|^2 \varphi^2 dx \leq \mathbf{B} \int_{\mathbb{R}^n} |d\varphi|^2 dx.$$

Moreover one can chose $\theta = d\Delta^{-1}V$ and the constants \mathbf{A} and $\sqrt{\mathbf{B}}$ are mutually controlled in the sense where if 2) holds with constant \mathbf{B} then 1) holds with $\mathbf{A} = 2\sqrt{\mathbf{B}}$ and there is a positive constant c_n depending only on n such that if 1) holds with constant \mathbf{A} then 2) holds with $\mathbf{B} = c_n \mathbf{A}^2$.

An equivalent formulation of the condition (1) is that

$$\forall \varphi, \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n): |\langle V, \varphi\phi \rangle| \leq \mathbf{A} \|d\varphi\|_{L^2} \|d\phi\|_{L^2}.$$

When (M^n, g, μ) is a complete weighted Riemannian manifold where $d\mu = e^f d\text{vol}_g$ is a smooth measure, we would like to characterize the distributions V on M for which there exists a positive constant \mathbf{A} such that:

$$\forall \varphi, \phi \in \mathcal{C}_c^\infty(M): |\langle V, \varphi\phi \rangle| \leq \mathbf{A} \|d\varphi\|_{L_\mu^2} \|d\phi\|_{L_\mu^2}; \quad (2)$$

where for $\varphi \in \mathcal{C}_c^\infty(M)$, we define

$$\|d\varphi\|_{L_\mu^2}^2 = \int_M |d\varphi|_g^2 d\mu.$$

A first limitation is that (M^n, g, μ) must be non-parabolic:

Definition 1.2. *A complete weighted Riemannian manifold (M^n, g, μ) is said to be parabolic if there is a sequence $\chi_\ell \in \mathcal{C}_c^\infty(M)$ such that*

$$\begin{cases} 0 \leq \chi_\ell \leq 1 & \text{every where on } M \\ \lim_{\ell \rightarrow +\infty} \chi_\ell = 1 & \text{uniformly on compact set} \\ \lim_{\ell \rightarrow +\infty} \int_M |d\chi_\ell|_g^2 d\mu = 0. \end{cases}$$

A complete weighted Riemannian manifold that is not parabolic is said to be non-parabolic.

It is well-known that the Euclidean space \mathbb{R}^n is non-parabolic if and only if $n \geq 3$. If V is a distribution satisfying (2) on a complete parabolic weighted Riemannian manifold, then testing (2), with φ and $\phi = \chi_\ell$ (given by Definition 1.2), we get that $\langle V, \varphi \rangle = 0$; hence the zero distribution is the only one verifying (2).

If (M^n, g, μ) is complete weighted Riemannian manifold, we define its Laplacian Δ through the Green formula:

$$\forall \varphi, \phi \in \mathcal{C}_c^\infty(M): \int_M \langle d\varphi, d\phi \rangle_g d\mu = \int_M \varphi \Delta \phi d\mu,$$

so that our convention is that on the Euclidean space $\Delta = -\sum_j \partial^2/\partial x_j^2$. We will also note Δ to be the unique self-adjoint extension associated to the symmetric operator

$$\Delta: \mathcal{C}_c^\infty(M) \rightarrow L_\mu^2(M).$$

Moreover (M^n, g, μ) is non-parabolic then one can define the homogeneous Sobolev space $\mathring{W}^{1,2}(M)$ to be the completion of $\mathcal{C}_c^\infty(M)$ for the norm $\varphi \mapsto \|d\varphi\|_{L_\mu^2}$ and there is a natural injection $\mathring{W}^{1,2}(M) \hookrightarrow W_{\text{loc}}^{1,2}(M)$. The Green formula implies that Δ extends as a bounded operator between $\mathring{W}^{1,2}(M)$ and its dual $\left(\mathring{W}^{1,2}(M)\right)'$, and a distribution V satisfies (2) if and only if the Schrödinger operator $\Delta + V$ extends to a bounded operator:

$$\Delta + V: \mathring{W}^{1,2}(M) \rightarrow \left(\mathring{W}^{1,2}(M)\right)'.$$

In order to be able to show a generalization of Theorem 1.1, we will assume that (M^n, g, μ) satisfies the doubling condition and the Poincaré inequalities.

Definition 1.3. A complete weighted Riemannian manifold (M^n, g, μ) is said to satisfy the doubling condition if there are positive constants κ, ν such that for any $x \in M$ and any $0 < r \leq R$:

$$\mu(B(x, R)) \leq \kappa \left(\frac{R}{r}\right)^\nu \mu(B(x, r)). \quad (\mathbf{D}_{\kappa, \nu})$$

Remark 1.4. It is well known that if (M^n, g, μ) is non-parabolic and satisfy the doubling condition $(\mathbf{D}_{\kappa, \nu})$ then $\nu > 2$.

Definition 1.5. A complete weighted Riemannian manifold (M^n, g, μ) is said to satisfy the Poincaré inequalities if there is a positive constant λ such that for any geodesic ball B of radius r :

$$\forall \varphi \in \mathcal{C}^1(B): \int_B |\varphi - \varphi_B|^2 d\mu \leq \lambda r^2 \int_B |d\varphi|^2 d\mu, \quad (\mathbf{P}_\lambda)$$

where $\varphi_B = \int_B \varphi d\mu$.

If the Ricci curvature of (M^n, g) is non negative, then (M^n, g, vol_g) satisfies these two conditions. We know that there is a smooth positive function (the heat kernel) $P: (0, +\infty) \times M \times M \rightarrow \mathbb{R}$ such that for any $f \in L_c^2(M)$:

$$(e^{-t\Delta} f)(x) = \int_M P(t, x, y) f(y) d\mu(y).$$

According to [Gri91, SC92], the conjunction of doubling condition $(\mathbf{D}_{\kappa, \nu})$ and of the Poincaré inequalities (\mathbf{P}_λ) is equivalent to uniform upper and lower Gaussian estimate of the heat kernel:

$$\forall x, y \in M, \forall t > 0: \frac{c}{\mu(B(x, \sqrt{t}))} e^{-\frac{d^2(x, y)}{ct}} \leq P(t, x, y) \leq \frac{C}{\mu(B(x, \sqrt{t}))} e^{-\frac{d^2(x, y)}{5t}}.$$

Our main result is then the following:

Theorem A. *Let (M, g, μ) be a complete non-parabolic weighted Riemannian manifold satisfying the doubling condition $(D_{\kappa, \nu})$ and the Poincaré inequalities (P_λ) . A distribution V satisfies the inequality (2) if and only if there is a 1-form $\theta \in L^2_{loc}(T^*M)$ solving $d_\mu^* \theta = V$ and such that*

$$\forall \varphi \in \mathcal{C}_c^\infty(M): \int_M |\theta|^2 \varphi^2 d\mu \leq \mathbf{B} \int_M |d\varphi|_g^2 dx. \quad (3)$$

Moreover the constants \mathbf{A} and $\sqrt{\mathbf{B}}$ are mutually controlled: if (3) holds with constant \mathbf{B} then (2) holds with $\mathbf{A} = 2\sqrt{\mathbf{B}}$ and there is a positive constant c depending only of κ, ν, λ such that if (2) holds with constant \mathbf{A} then (3) holds with $\theta = d\Delta^{-1}V$ and $\mathbf{B} = c\mathbf{A}^2$.

We recall that the equation $d_\mu^* \theta = V$ is equivalent to the fact that

$$\forall \varphi \in \mathcal{C}_c^\infty(M): \int_M \langle \theta, d\varphi \rangle_g d\mu = \langle V, \varphi \rangle.$$

Our proof follows the original arguments of Maz'ya and Verbitsky but for this purpose, we prove some new results about the Hodge projector. To describe this operator let (M, g, μ) be a complete non-parabolic weighted Riemannian manifold, it has a positive Green kernel defined for $x \neq y$ by:

$$G(x, y) = \int_0^{+\infty} P(t, x, y) dt.$$

When $\beta \in \mathcal{C}_c^\infty(T^*M)$ is a smooth compactly supported 1-form, one defines $\Pi\beta$ by

$$\Pi\beta = d\varphi \text{ where } \varphi(x) = \int_M G(x, y) d_\mu^* \beta(y) d\mu(y).$$

The operator Π extends to a bounded operator on $L^2_\mu(T^*M)$ and this extension, which is also noted Π , is the orthogonal projection on the closure of $d\mathcal{C}_c^\infty(M) \subset L^2_\mu(T^*M)$. The operator $\Pi = d\Delta^{-1}d_\mu^*$ is called the Hodge projector.

When ρ is a non negative and non trivial Radon measure with compact supported then the function h_ρ defined by

$$h_\rho(x) = \int_M G(x, y) d\rho(y) \quad (4)$$

is a positive superharmonic function. A crucial result for proving Theorem A is the following

Theorem B. *Let (M, g, μ) be a complete non-parabolic weighted Riemannian manifold, then for any $\delta \in (-1, 1)$ and any non negative Radon measure with compact supported ρ and associated superharmonic function h_ρ defined by (4):*

$$\forall \beta \in \mathcal{C}_c^\infty(T^*M): \int_M |\Pi(\beta)|^2 h_\rho^\delta d\mu \leq \left(\frac{1+|\delta|}{1-|\delta|} \right)^2 \int_M |\beta|^2 h_\rho^\delta d\mu.$$

Moreover if (M, g, μ) satisfies the doubling condition $(D_{\kappa, \nu})$ and the Poincaré inequalities (P_λ) then there are some $\delta_+ > 1$ and $C > 0$ depending only on κ, ν, λ such that for any non negative Radon measure with compact supported ρ and associated superharmonic function h_ρ defined by (4):

$$\forall \beta \in \mathcal{C}_c^\infty(T^*M): \int_M |\Pi(\beta)|^2 h_\rho^{\delta_+} d\mu \leq C \int_M |\beta|^2 h_\rho^{\delta_+} d\mu.$$

Classical result on singular integral implies that on \mathbb{R}^n , the second conclusion of Theorem B holds with δ_+ being any real number strictly less than $n/(n-2)$; this was an important argument in the proof of Maz'ya-Verbitsky (see [MV02, page 284]). The study of the boundedness of the Hodge operator on L^p spaces in relationship with the boundedness of the Riesz transform on L^p spaces is a well developed subject (see for instance [AC05, subsection 2.3]); the question of the boundedness of the Hodge operator on weighted L^p

spaces has been extensively studied by Auscher and Martell [AM07, AM08]. These results do not imply directly the second assertion of B. The starting point is the universal boundedness of the Hodge projector on $L^2_{h^\delta \mu}(T^*M)$ for $\delta \in (-1, 1)$ and we apply a result of Auscher-Martell [AM07] on the weighted Riemannian manifold $(M, g, h^\delta \mu)$; in order to be in position of using this result, we have to prove that those functions h^δ_ρ are A_1 -weights.

Our universal boundedness of the Hodge projector on some weighted L^2 space could be a first step toward a more general investigation of the L^p -weighted boundedness of the Hodge operator.

Another important result for proving Theorem A is a decay estimate on $\Pi(\beta)$ when $\beta \in C_c^\infty(T^*M)$, more precisely, we show that there is some $p > 1$ such that

$$\int_{M \setminus B(o,1)} G^{-p}(o, x) |\Pi(\beta)|^2(x) d\mu(x) < \infty.$$

The next section is devoted to the proof of the first conclusion of Theorem B and we will recall and prove some useful results about Green kernel and superharmonic functions; in section 3, we present various equivalent conditions for an estimate of the type

$$\forall \varphi \in C_c^\infty(M): \int_M q \varphi^2 d\mu \leq C \int_M |d\varphi|_g^2 dx,$$

when q is a non-negative locally integrable function. One of the first results in the Euclidean case is due to Fefferman-Phong [FP82] and there were then many works dealing with this questions in Euclidean spaces or on homogeneous space (see for instance [Fef83, KS86, Maz64, MV95, PW03, SW92, SWZ96, Sch88]; the Riemannian setting was adressed recently in [Lan20]). The second conclusion of Theorem B is proven in section 4 and Theorem A is proven in section 5. In an appendix, we explain how an elegant Euclidean argument of Verbitksy [Ver08] gives trace inequalities on doubling metric measure spaces.

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SOME NOTATIONS

When (M, g, μ) is a weighted complete Riemannian manifold then for any Borel set $\mathcal{B} \subset M$, we will note $\mathbf{1}_{\mathcal{B}}$ the characteristic function of \mathcal{B} ; if moreover $\mu(\mathcal{B}) \neq 0$ and if φ is an integrable function on \mathcal{B} , the mean of φ on \mathcal{B} will be noted by

$$\int_{\mathcal{B}} \varphi = \int_{\mathcal{B}} \varphi d\mu = \frac{1}{\mu(\mathcal{B})} \int_{\mathcal{B}} \varphi d\mu;$$

the subscript c will indicate a space made of compactly support functions or forms, for instance $L^2_c(M), C_c^\infty(M), C_c^\infty(T^*M)$; and the space of locally L^2 fonction (resp. locally $W^{1,2}$ functions) is noted $L^2_{loc}(M)$ (resp. $W^{1,2}_{loc}$).

When $B = B(x, r)$ is a geodesic ball and $\theta > 0$ we will define $\theta B := B(x, \theta r)$. For each $x \in M$ and $\beta \in T^*_x M$ the norm of β is

$$|\beta| = |\beta|_g = \sup_{\xi \in T_x M, g_x(\xi, \xi) = 1} \beta(\xi)$$

and it induces on $T^*_x M$ a scalar product that will be noted $\langle \cdot, \cdot \rangle_g$ or $\langle \cdot, \cdot \rangle$, from now we will omit the g subscript, this induces a scalar product on 1-forms

$$\beta_1, \beta_2 \in C_c^\infty(T^*M): \langle \beta_1, \beta_2 \rangle = \int_M \langle \beta_1(x), \beta_2(x) \rangle_g d\mu(x).$$

The associate Hilbert space collecting all square integrable measurable section of T^*M will be noted $L^2_\mu(T^*M)$. If $\mathcal{U} \subset M$ is a bounded open subset $W^{1,2}_o(\mathcal{U})$ is the closure of

$\mathcal{C}_c^\infty(\mathcal{U})$ in $W_{\text{loc}}^{1,2}(M)$, it is also the completion of $\mathcal{C}_c^\infty(\mathcal{U})$ for the topology induced by the norm $\varphi \mapsto \sqrt{\|\varphi\|_{L_\mu^2}^2 + \|d\varphi\|_{L_\mu^2}^2}$.

2. WEIGHTED INEQUALITIES GIVEN BY EQUILIBRIUM POTENTIALS

In this section, we collect some well-known facts and others useful technical results that leads to a very general weighted boundedness result for the Hodge projector. Thorough all this section, (M, g, μ) is a weighted complete Riemannian manifold that is assumed to be non parabolic.

2.1. Capacity and equilibrium potential. The non parabolicity implies the following properties [Anc90, Gri99]:

Proposition 2.1. *a) For each bounded open set $\mathcal{U} \subset M$, there is a positive constant $C_{\mathcal{U}}$ such that:*

$$\forall \varphi \in \mathcal{C}_c^\infty(M): \int_{\mathcal{U}} \varphi^2 d\mu \leq C_{\mathcal{U}} \int_M |d\varphi|^2 d\mu.$$

b) (M, g, μ) admits a positive minimal Green kernel $G: M \times M \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ such that for any $f \in \mathcal{C}_c^\infty(M)$, a solution of the equation $\Delta u = f$ is given by

$$u(x) = \int_M G(x, y) f(y) d\mu(y). \quad (5)$$

c) For any bounded open subset $\mathcal{V} \subset M$, the Green kernel of the Dirichlet-Laplace operator on \mathcal{V} is noted $G^\mathcal{V}: \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$. Then we have that

$$G(x, y) = \lim_{\mathcal{V} \rightarrow M} G^\mathcal{V}(x, y) = \sup_{\mathcal{V}} G^\mathcal{V}(x, y).$$

The function $G_o(y) = G(o, y)$ is called the Green function with pole at $o \in M$; it is a smooth positive harmonic function on $M \setminus \{o\}$. Proposition 2.1-a) implies that if $\mathring{W}^{1,2}(M)$ is the completion of $\mathcal{C}_c^\infty(M)$ for the norm $\varphi \mapsto \|d\varphi\|_{L_\mu^2}$ then the injection $\mathcal{C}_c^\infty(M) \hookrightarrow \mathring{W}_{\text{loc}}^{1,2}(M)$ extends continuous and provides a natural injection $\mathring{W}^{1,2}(M) \hookrightarrow W_{\text{loc}}^{1,2}(M)$. Moreover for any $f \in L_\mu^2$ with compact support there is a unique $u \in \mathring{W}^{1,2}(M)$ solving the equation $\Delta u = f$ and u is given by the formula (5).

It also implies that for any non empty bounded open subset $\mathcal{U} \subset M$, its *capacity*

$$\text{cap}(\mathcal{U}) = \inf \left\{ \int_M |d\varphi|^2 d\mu, \text{ such that } \varphi \in \mathcal{C}_c^\infty(M) \text{ and } \varphi = 1 \text{ on } \mathcal{U} \right\} \quad (6)$$

is strictly positive and there is a unique $h \in \mathring{W}^{1,2}(M)$ such that

$$\text{cap}(\mathcal{U}) = \int_M |dh|^2 d\mu \text{ and } h = 1 \text{ on } \mathcal{U};$$

moreover $0 < h \leq 1$. This function h is called the *equilibrium potential* of \mathcal{U} , it is a superharmonic function and there is a positive Radon measure (called the *equilibrium measure*) $\nu_{\mathcal{U}}$ supported in \mathcal{U} such that

$$\forall x \in M: h(x) = \int_M G(x, y) d\nu_{\mathcal{U}}(y);$$

the equilibrium measure of \mathcal{U} is also given by the equality (in the distributional sense)

$$d\nu_{\mathcal{U}} = \Delta h.$$

For instance if \mathcal{U} has smooth boundary and if $\vec{n}: \partial\mathcal{U} \rightarrow TM$ is the inward unit normal vector fields and σ the induced measure on $\partial\mathcal{U}$ then the equilibrium measure of \mathcal{U} is given by

$$\nu_{\mathcal{U}}(K) = \int_{\partial\mathcal{U} \cap K} \frac{\partial h}{\partial \vec{n}} d\sigma.$$

We recall:

Definition 2.2. A function $f: M \rightarrow \mathbb{R}$ is said to be superharmonic if for any non negative $\varphi \in C_c^\infty(M)$:

$$\int_M f \Delta \varphi d\mu \geq 0.$$

Notice that for any bounded $f: M \rightarrow \mathbb{R}$ and $\varphi \in C_c^\infty(M)$ we have that

$$\int_M f \Delta \varphi d\mu = \lim_{t \rightarrow 0} \int_M f \frac{\varphi - e^{-t\Delta} \varphi}{t} d\mu = \lim_{t \rightarrow 0} \int_M \frac{f - e^{-t\Delta} f}{t} \varphi d\mu.$$

Hence

Lemma 2.3. A positive bounded function $f: M \rightarrow \mathbb{R}$ is superharmonic if and only if for any $t > 0$:

$$e^{-t\Delta} f \leq f.$$

Notice that the result still holds if we assume that

$$\forall t > 0, \forall x \in M: \int_M P(t, x, y) f(y) d\mu(y) < +\infty.$$

2.2. Some properties of equilibrium potentials.

Proposition 2.4. a) For any $o \in M$ and any $\tau > 1/2$, the Green function with pole at o satisfies $\min(G_o^\tau, 1) \in \mathring{W}^{1,2}(M)$.

b) For any $\tau > 1/2$ and any bounded open set $\mathcal{U} \subset M$ with equilibrium potential h , we have $h^\tau \in \mathring{W}^{1,2}(M)$ moreover

$$\int_M |dh^\tau|^2 d\mu \leq \frac{\tau^2}{2\tau - 1} \text{cap}(\mathcal{U}).$$

Proof. Let Ω_ℓ an increasing sequence of bounded open subset with smooth boundary exhausting M . For large enough ℓ , we have that $o \in \Omega_\ell$ and let $g_\ell = G^{\Omega_\ell}(o, \cdot)$. It is enough to show that there is a constant C , independant of ℓ , such that

$$\int_{\{g_\ell < 1\}} |dg_\ell^\tau|^2 d\mu \leq C.$$

Using the coarea formula, we get that

$$\begin{aligned} \int_{\{g_\ell < 1\}} |dg_\ell^\tau|^2 d\mu &= \tau^2 \int_{\{g_\ell < 1\}} g_\ell^{2\tau-2} |dg_\ell|^2 d\mu \\ &= \tau^2 \int_0^1 x^{2\tau-2} \left(\int_{g_\ell=x} |dg_\ell| d\sigma_x \right) dx \end{aligned}$$

But for any regular value x of g_ℓ , the Green formula indicates that the integral $\int_{g_\ell=x} |dg_\ell| d\sigma_x$ does not depends on x :

$$0 = \int_{y < g_\ell < x} \Delta g_\ell d\mu = \int_{g_\ell=y} |dg_\ell| d\sigma_y - \int_{g_\ell=x} |dg_\ell| d\sigma_x.$$

Letting $x \rightarrow +\infty$ and using the asymptotics of the Green kernel around o , we find that

$$\int_{g_\ell=y} |dg_\ell| d\sigma_y = 1.$$

Hence for any $\tau > 1/2$:

$$\int_{\{g_\ell < 1\}} |dg_\ell^\tau|^2 d\mu = \tau^2 \int_0^1 x^{2\tau-2} dx = \frac{\tau^2}{2\tau - 1}.$$

The proof concerning an equilibrium potential is exactly the same. For ℓ large enough, $\bar{\mathcal{U}} \subset \Omega_\ell$ and we introduce $h_\ell \in W_o^{1,2}(\Omega_\ell)$, the equilibrium potential relative to Ω_ℓ , it is characterized by

$$\int_M |dh_\ell|^2 d\mu = \inf \left\{ \int_M |d\varphi|^2 d\mu, \text{ such that } \varphi \in C_c^\infty(\Omega_\ell) \text{ and } \varphi = 1 \text{ on } \mathcal{U} \right\}.$$

Then h_ℓ is harmonic on $\Omega_\ell \setminus \bar{\mathcal{U}}$ and $\Omega_\ell \setminus \bar{\mathcal{U}} = h_\ell^{-1}((0, 1))$. Similarly for every regular value $x \in (0, 1)$ of h_ℓ , the integral $\int_{h_\ell=x} |dh_\ell| d\sigma_x$ does not depend on x and

$$\int_M |dh_\ell|^2 d\mu = \int_0^1 \left(\int_{h_\ell=x} |dh_\ell| d\sigma_x \right) dx.$$

So that for a.e. $x \in (0, 1)$:

$$\int_{h_\ell=x} |dh_\ell| d\sigma_x = \int_M |dh_\ell|^2 d\mu.$$

and if $\tau > 1/2$ then

$$\int_M |dh_\ell^\tau|^2 d\mu = \tau^2 \int_0^1 x^{2\tau-2} \left(\int_{h_\ell=x} |dh_\ell| d\sigma_x \right) dx = \frac{\tau^2}{2\tau-1} \int_M |dh_\ell|^2 d\mu.$$

It is easy to show that the sequence $(h_\ell)_\ell$ converges to h in $\overset{\circ}{W}^{1,2}(M)$ and this estimate implies that for $\tau > 1/2$, h_ℓ^τ converges weakly to h^τ in $\overset{\circ}{W}^{1,2}(M)$; hence the result. \square

Proposition 2.5. *For any $\tau \in [0, 1]$ and any $o \in M$; G_o^τ is superharmonic. Similarly for any $\tau \in [0, 1]$ and any bounded open subset \mathcal{U} with equilibrium potential h , h^τ is superharmonic.*

Proof. We know that $G_o(x) = \int_0^\infty P(\tau, o, x) d\tau$ hence for any $t > 0$:

$$(e^{-t\Delta} G_o)(x) = \int_t^\infty P(\tau, o, x) d\tau \leq G_o(x).$$

Hence G_o is superharmonic then using Hölder inequality and that $e^{-t\Delta} \mathbf{1}_M \leq \mathbf{1}_M$, we have for any $\tau \in [0, 1]$:

$$e^{-t\Delta} G_o^\tau \leq (e^{-t\Delta} G_o)^\tau \leq G_o^\tau.$$

Hence G_o^τ is also superharmonic. The proof of the second assertion is identical. \square

Remark 2.6. *Using Jensen inequality, we can show that for any superharmonic function h and any concave function Φ , then $\Phi(h)$ is superharmonic.*

2.3. Weighted Hardy type inequalities. The next result is a consequence of the argumentation presented in [Car97].

Proposition 2.7. *If $h: M \rightarrow \mathbb{R}_+$ is the equilibrium potential of some bounded open set, then for any $\delta < 1$ then the following general Hardy type inequality holds:*

$$\forall \varphi \in C_c^\infty(M): \left(\frac{\delta-1}{2} \right)^2 \int_M \frac{|dh|^2}{h^2} \varphi^2 h^\delta d\mu \leq \int_M |d\varphi|^2 h^\delta d\mu. \quad (7)$$

Proof. Let $\phi \in C_c^\infty(M)$ and define $\varphi = h^{\frac{1-\delta}{2}} \phi$, we compute:

$$|d\varphi|^2 = h^{1-\delta} |d\phi|^2 + \left(\frac{\delta-1}{2} \right)^2 \frac{|dh|^2}{h^2} \varphi^2 + (1-\delta) \langle dh, d\phi \rangle h^{-\delta} \phi.$$

Then we get

$$\int_M |d\varphi|^2 h^\delta d\mu \geq \left(\frac{\delta-1}{2} \right)^2 \int_M \frac{|dh|^2}{h^2} \varphi^2 h^\delta d\mu + \frac{1-\delta}{2} \int_M \langle dh, d\phi^2 \rangle d\mu.$$

But h is superharmonic hence if $\delta < 1$ we get that

$$\int_M |d\varphi|^2 h^\delta d\mu \geq \left(\frac{\delta-1}{2}\right)^2 \int_M \frac{|dh|^2}{h^2} \varphi^2 h^\delta d\mu.$$

Then by approximation, we get that this weighted Hardy inequality is valid for any $\varphi \in \mathring{W}_{\text{loc}}^{1,2}(M)$ with compact support. \square

2.4. Weighted L^2 -boundedness of the Hodge projector. Recall that the space of L^2 1-forms has the following *Hodge* orthogonal decomposition:

$$L_\mu^2(T^*M) = \mathcal{H}^1(M, \mu) \oplus \overline{d\mathcal{C}_c^\infty(M)} \oplus \overline{d_\mu^* \mathcal{C}_c^\infty(\Lambda^2 T^*M)},$$

where the closure are taken with respect to the L_μ^2 topology; the operator

$$d_\mu^* : \mathcal{C}_c^\infty(\Lambda^2 T^*M) \rightarrow \mathcal{C}_c^\infty(T^*M)$$

is the formal adjoint of exterior differential operator $d : \mathcal{C}_c^\infty(T^*M) \rightarrow \mathcal{C}_c^\infty(\Lambda^2 T^*M)$:

$$\forall \alpha \in \mathcal{C}_c^\infty(T^*M), \beta \in \mathcal{C}_c^\infty(\Lambda^2 T^*M) : \int_M \langle d\alpha, \beta \rangle d\mu = \int_M \langle \alpha, d_\mu^* \beta \rangle d\mu;$$

and

$$\mathcal{H}^1(M, \mu) = \{ \alpha \in L_\mu^2(T^*M), d\alpha = 0 \text{ and } d_\mu^* \alpha = 0 \}.$$

The Hodge projector $\Pi : L_\mu^2(T^*M) \rightarrow L_\mu^2(T^*M)$ is the L_μ^2 -projector on $\overline{d\mathcal{C}_c^\infty(M)}$. As we assumed that (M, g, μ) is non parabolic for any $\alpha \in \mathcal{C}_c^\infty(T^*M)$, we have

$$\Pi\alpha = d\varphi$$

where φ solves the equation $\Delta\varphi = d_\mu^* \alpha$ that is to say:

$$\varphi(x) = \int_M G(x, y) d_\mu^* \alpha(y) d\mu(y) = \int_M \langle d_y G(x, y), \alpha(y) \rangle d\mu(y).$$

Theorem 2.8. *If $h : M \rightarrow \mathbb{R}_+$ is an equilibrium potential and $\delta \in (-1, 1)$ then for any $\beta \in \mathcal{C}_c^\infty(T^*M)$:*

$$\int_M |\Pi(\beta)|^2 h^\delta d\mu \leq \left(\frac{1+|\delta|}{1-|\delta|}\right)^2 \int_M |\beta|^2 h^\delta d\mu.$$

*In particular $\Pi : \mathcal{C}_c^\infty(T^*M) \rightarrow L_{h^\delta \mu}^2(T^*M)$ has a bounded extension $\Pi : L_{h^\delta \mu}^2(T^*M) \rightarrow L_{h^\delta \mu}^2(T^*M)$.*

Proof. As Π is bounded and selfadjoint on $L_\mu^2(T^*M)$, by duality it is enough to show the result for $\delta \in (0, 1)$. So let $\delta \in (0, 1)$ and $\beta \in \mathcal{C}_c^\infty(T^*M)$ and let $\varphi \in \mathring{W}^{1,2}(M)$ be the solution of the equation $\Delta\varphi = d_\mu^* \beta$. We first show that

$$\int_M |d\varphi|^2 h^\delta d\mu \leq \int_M \varphi \Delta\varphi h^\delta d\mu. \quad (8)$$

Consider a sequence (φ_ℓ) made of smooth compactly supported functions that converges toward φ in $\mathring{W}^{1,2}(M)$. Because h^δ is superharmonic, we get that for all ℓ

$$\int_M h^\delta \Delta\varphi_\ell^2 d\mu \geq 0.$$

But $\frac{1}{2}\Delta\varphi_\ell^2 = \varphi_\ell \Delta\varphi_\ell - |d\varphi_\ell|^2$ hence

$$\int_M |d\varphi_\ell|^2 h^\delta d\mu \leq \int_M \varphi_\ell \Delta\varphi_\ell h^\delta d\mu. \quad (9)$$

As h^δ is a bounded positive function, we get that

$$\int_M |d\varphi|^2 h^\delta d\mu = \lim_\ell \int_M |d\varphi_\ell|^2 h^\delta d\mu. \quad (10)$$

Using

- i) $d(\varphi_\ell h^\delta) = h^\delta d\varphi_\ell + \delta \frac{dh}{h} \varphi_\ell h^\delta$;
- ii) the Hardy inequality (7):

$$\forall \phi \in \mathcal{C}_c^\infty(M): \int_M \frac{|dh|^2}{h^2} \phi^2 d\mu \leq 4 \int_M |d\phi|^2 d\mu;$$

- iii) h is a positive function and takes value in $(0, 1]$,
- we deduce that for any ℓ, k :

$$\|d(\varphi_\ell h^\delta - \varphi_k h^\delta)\|_{L_\mu^2}^2 \leq (8\delta + 2) \|d(\varphi_\ell - \varphi_k)\|_{L_\mu^2}^2.$$

hence $h^\delta \varphi \in \mathring{W}^{1,2}(M)$ and

$$\lim_{\ell \rightarrow +\infty} \|d(\varphi_\ell h^\delta - \varphi h^\delta)\|_{L_\mu^2} = 0.$$

So that we get:

$$\lim_{\ell} \int_M \varphi_\ell \Delta \varphi_\ell h^\delta d\mu = \lim_{\ell} \int_M \langle d\varphi_\ell, d(\varphi_\ell h^\delta) \rangle d\mu = \int_M \langle d\varphi, d(\varphi h^\delta) \rangle d\mu.$$

But by definition we know that

$$\forall \phi \in \mathring{W}^{1,2}(M): \int_M \langle d\varphi, d\phi \rangle d\mu = \int_M d_\mu^* \beta \phi d\mu.$$

Hence

$$\lim_{\ell} \int_M \varphi_\ell \Delta \varphi_\ell h^\delta d\mu = \int_M d_\mu^* \beta (\varphi h^\delta) d\mu = \int_M \Delta \varphi (\varphi h^\delta) d\mu. \quad (11)$$

The inequalities (9), (10) (11) implies the inequalities (8). And using $\Delta \varphi = d_\mu^* \beta$, we deduce that

$$\int_M |d\varphi|^2 h^\delta d\mu \leq \int_M d_\mu^* \beta \varphi h^\delta d\mu.$$

Integrating by parts, we get that

$$\int_M d_\mu^* \beta \varphi h^\delta d\mu = \int_M \langle \beta, d\varphi \rangle h^\delta d\mu + \delta \int_M \langle \beta, dh \rangle \varphi h^{\delta-1} d\mu.$$

Using the Cauchy-Schwartz inequality, we obtain

$$\left| \int_M \langle \beta, d\varphi \rangle h^\delta d\mu \right| \leq \left(\int_M |\beta|^2 h^\delta d\mu \right)^{\frac{1}{2}} \left(\int_M |d\varphi|^2 h^\delta d\mu \right)^{\frac{1}{2}}.$$

And the weighted Hardy inequality (7) implies that

$$\begin{aligned} \left| \int_M \langle \beta, dh \rangle \varphi h^{\delta-1} d\mu \right| &\leq \left(\int_M |\beta|^2 h^\delta d\mu \right)^{\frac{1}{2}} \left(\int_M \frac{|dh|^2}{h^2} |\varphi|^2 h^\delta d\mu \right)^{\frac{1}{2}} \\ &\leq \frac{2}{1-\delta} \left(\int_M |\beta|^2 h^\delta d\mu \right)^{\frac{1}{2}} \left(\int_M |d\varphi|^2 h^\delta d\mu \right)^{\frac{1}{2}}. \end{aligned}$$

So that we eventually obtain

$$\int_M |d\varphi|^2 h^\delta d\mu \leq \left(1 + \frac{2\delta}{1-\delta} \right) \left(\int_M |\beta|^2 h^\delta d\mu \right)^{\frac{1}{2}} \left(\int_M |d\varphi|^2 h^\delta d\mu \right)^{\frac{1}{2}}.$$

□

3. TRACE TYPE INEQUALITIES

3.1. In this section, we collect some general facts about the validity of trace inequality of the type

$$\forall \varphi \in \mathcal{C}_c^\infty(M): \int_M q\varphi^2 d\mu \leq C_1 \int_M |d\varphi|^2 d\mu, \quad (12)$$

where q is a non negative locally integrable function. In order to have the existence of at least one non trivial potential q such that (12) holds, we must suppose that (M, g, μ) is non parabolic and in this case using equality :

$$\forall \varphi \in \mathcal{C}_c^\infty(M): \int_M |d\varphi|^2 d\mu = \langle \varphi, \Delta\varphi \rangle_{L_\mu^2} = \int_M |\Delta^{\frac{1}{2}}\varphi|^2 d\mu$$

we know that the operator $\Delta^{\frac{1}{2}}$ defined with the spectral theorem extends to an isometry

$$\Delta^{\frac{1}{2}}: \overset{\circ}{W}^{1,2}(M) \rightarrow L_\mu^2(M),$$

with inverse $\Delta^{-\frac{1}{2}}: L_\mu^2(M) \rightarrow \overset{\circ}{W}^{1,2}(M)$. Hence the above condition (12) is equivalent to the fact that $\sqrt{q}\Delta^{-\frac{1}{2}}$ (or its adjoint $\Delta^{-\frac{1}{2}}\sqrt{q}$) has a bounded extension on $L_\mu^2(M)$ with operator norm satisfying

$$\left\| \sqrt{q}\Delta^{-\frac{1}{2}} \right\|^2 \leq C_1.$$

3.2. **A general result.** The following result is well known in the Euclidean setting and is mostly due to Maz'ya and Vertbisky [Maz64, MV95], however it is folklore that the proofs can be easily adapted in a much more general setting.

Theorem 3.1. *Let (M, g, μ) be a complete non-parabolic weighted Riemannian manifold and q be a non negative, locally integrable function then the following properties are equivalent*

- i) there is a constant C_1 such that (12) holds;
- ii) the operator $\sqrt{q}\Delta^{-\frac{1}{2}}$ has a bounded extension $\sqrt{q}\Delta^{-\frac{1}{2}}: L_\mu^2(M) \rightarrow L_\mu^2(M)$;
- iii) there is a constant C_3 such that for any bounded open set $\mathcal{U} \subset M$:

$$\int_{\mathcal{U}} q d\mu \leq C_3 \text{cap}(\mathcal{U}); \quad (13)$$

- iv) there is a constant C_4 such that for any bounded open set $\mathcal{U} \subset M$:

$$\int_M \left| \Delta^{-\frac{1}{2}}(q \mathbf{1}_{\mathcal{U}}) \right|^2 d\mu \leq C_4 \int_{\mathcal{U}} q d\mu; \quad (14)$$

- v) there is a constant C_5 such that for any bounded open set $\mathcal{U} \subset M$:

$$\int_M \left| \Delta^{-\frac{1}{2}}(q \mathbf{1}_{\mathcal{U}}) \right|^2 d\mu \leq C_5^2 \text{cap}(\mathcal{U}). \quad (15)$$

Moreover $\left\| \sqrt{q}\Delta^{-\frac{1}{2}} \right\|^2$ and C_1, C_3, C_4, C_5 are mutually controlled.

Proof. We have already explained that **i**) \Leftrightarrow **ii**).

Proof of ii) \Rightarrow iv). Testing the L_μ^2 -boundedness of $\Delta^{-\frac{1}{2}}\sqrt{q} = \left(\sqrt{q}\Delta^{-\frac{1}{2}}\right)^* = T^*$ with $f = \sqrt{q} \mathbf{1}_{\mathcal{U}}$, we get

$$\int_M \left| \Delta^{-\frac{1}{2}}(q \mathbf{1}_{\mathcal{U}}) \right|^2 d\mu \leq \|T\|^2 \int_M (\sqrt{q} \mathbf{1}_{\mathcal{U}})^2 d\mu,$$

Hence we get that that **iv**) holds with $C_4 = \left\| \sqrt{q}\Delta^{-\frac{1}{2}} \right\|^2$.

Proof of iv) \Rightarrow v). We notice that (14) is equivalent to

$$\forall \mathcal{U}, \forall f \in L^2_\mu: \langle (q \mathbf{1}_\mathcal{U}), \Delta^{-\frac{1}{2}} f \rangle_{L^2_\mu} = \langle \Delta^{-\frac{1}{2}} (q \mathbf{1}_\mathcal{U}), f \rangle_{L^2_\mu} \leq \sqrt{C_4} \sqrt{\int_{\mathcal{U}} q d\mu} \|f\|_{L^2_\mu}.$$

That is to say that

$$\forall \mathcal{U}, \forall \varphi \in \mathring{W}^{1,2}(M): \langle q \mathbf{1}_\mathcal{U}, \varphi \rangle_{L^2_\mu} \leq \sqrt{C_4} \sqrt{\int_{\mathcal{U}} q d\mu} \|d\varphi\|_{L^2_\mu}.$$

Then using the definition of the capacity we get that

$$\sqrt{\int_{\mathcal{U}} q d\mu} \leq \sqrt{C_4 \text{cap}(\mathcal{U})}$$

and reporting this estimate in (14) we obtain the inequality (15) with $C_5 = C_4$.

v) \Rightarrow iii) The same argumentation yields that inequality (15) is equivalent to

$$\forall \mathcal{U}, \forall \varphi \in \mathring{W}^{1,2}(M): \langle q \mathbf{1}_\mathcal{U}, \varphi \rangle_{L^2_\mu} \leq C_5 \sqrt{\text{cap}(\mathcal{U})} \|d\varphi\|_{L^2_\mu}.$$

Hence we obtain the inequality (13) with $C_3 = C_5$.

Proof of iii) \Rightarrow i). The argument is the one originally given by Maz'ya. For any $\varphi \in C_c^\infty(M)$ and $t > 0$ we test (13) on $\mathcal{U}_t = \{\varphi^2 > t\}$ and get that

$$\int_{\mathcal{U}_t} q d\mu \leq C_3 \int_{\{\varphi^2 < t^2\}} \frac{|d\varphi|^2}{t^2} d\mu = 4C_3 \int_{\{\varphi^2 < t^2\}} \frac{\varphi^2 |d\varphi|^2}{t^2} d\mu.$$

Then integrating with respect to $t \in (0, +\infty)$ and using the Cavalieri's formula and the Fubini theorem, we get that

$$\begin{aligned} \int_M q \varphi^2 d\mu &= \int_0^\infty \left(\int_{\mathcal{U}_t} q d\mu \right) dt \\ &\leq 4C_3 \int_M \varphi^2 |d\varphi|^2 \left(\int_{\varphi^2}^{+\infty} \frac{dt}{t^2} \right) d\mu \\ &= 4C_3 \int_M |d\varphi|^2 d\mu. \end{aligned}$$

Hence we obtain the inequality (12) with $C_1 = 4C_3$. \square

Remark 3.2. The proof show that the constants $\left\| \sqrt{q} \Delta^{-\frac{1}{2}} \right\|^2$ and C_1, C_3, C_4, C_5 are mutually controlled in the following way:

- If **i)** holds with constant C_1 then **ii)** holds with $\left\| \sqrt{q} \Delta^{-\frac{1}{2}} \right\|^2 \leq C_1$;
- If **ii)** then **iv)** with constant $C_4 = \left\| \sqrt{q} \Delta^{-\frac{1}{2}} \right\|^2$;
- If **iv)** holds with constant C_4 then **v)** holds with constant $C_5 = C_4$;
- If **v)** holds with constant C_5 then **iii)** holds with constant $C_3 = C_5$;
- If **iii)** holds with constant C_3 then **i)** holds with constant $C_1 = 4C_3$.

3.3. With the relative Faber-Krahn inequality. If one assumes moreover some a priori geometric estimates, we can get other sufficient or necessary conditions for the properties given in Theorem 3.1. The first set of conditions are the so called relative Faber-Krahn inequality that has been introduced by Grigor'yan in [Gri94].

Definition 3.3. We say that a complete weighted Riemannian manifold (M, g, μ) satisfies the relative Faber-Krahn inequality if there are positive constants b, ν such that for any

geodesic ball B of radius $r(B)$ and any open subset $\mathcal{U} \subset B$:

$$\frac{b}{r^2(B)} \left(\frac{\mu(\mathcal{U})}{\mu(B)} \right)^{-\frac{2}{\nu}} \leq \lambda_1(\mathcal{U}), \quad (\text{FK}_{b,\nu})$$

where $\lambda_1(\mathcal{U})$ is the lowest eigenvalue of the Dirichlet Laplacian on \mathcal{U} .

According to [Gri94], the relative Faber-Krahn inequality is equivalent to the conjunction of a doubling property $(D_{\kappa,\nu})$ and of the upper Gaussian estimate for the heat kernel

$$\forall x, y \in M, \forall t > 0: P(t, x, y) \leq \frac{C}{\mu(B(x, \sqrt{t}))} e^{-\frac{d^2(x,y)}{5t}}. \quad (\text{GUE})$$

Then using the appendix, we can deduce the following:

Theorem 3.4. *Let (M, g, μ) be a complete non-parabolic weighted Riemannian manifold satisfying the relative Faber-Krahn inequality $(\text{FK}_{b,\nu})$. There is a constant $C(b, \nu)$ depending only on b, ν with the property that if q is non negative locally integrable function such that for some positive A and for any geodesic ball $B \subset M$:*

$$\int_B \left[\int_0^{+\infty} \left(\int_{B(x,r)} q \mathbf{1}_B d\mu \right) dr \right]^2 d\mu(x) \leq A \int_B q d\mu$$

then

$$\forall \varphi \in C_c^\infty(M): \int_M q \varphi^2 d\mu \leq C(b, \nu) A \int_M |d\varphi|^2 d\mu$$

Hint on the proof of Theorem 3.4: According to Theorem A.3 **iv**) \Rightarrow **i**), we get that if \mathcal{K} is the operator whose Schwartz kernel is given by

$$K(x, y) = \int_{d(x,y)}^{+\infty} \frac{dr}{\mu(B(x, r))},$$

that is to say

$$(\mathcal{K}f)(x) = \int_0^{+\infty} \left(\int_{B(x,r)} f d\mu \right) dr$$

then the operator $\mathcal{K}\sqrt{q}$ has a bounded extension to L_μ^2 with

$$\|\mathcal{K}\sqrt{q}\|_{L_\mu^2 \rightarrow L_\mu^2}^2 \leq CA.$$

Proposition A.4 implies that the Schwartz kernel of $\Delta^{-\frac{1}{2}}$ is dominated by the Schwartz kernel of \mathcal{K} :

$$\forall x \neq y \in M: \Delta^{-\frac{1}{2}}(x, y) = \int_0^{+\infty} P(t, x, y) \frac{dt}{\sqrt{\pi t}} \leq C \int_{d(x,y)}^{+\infty} \frac{dr}{\mu(B(x, r))} = K(x, y).$$

Hence the operator $\Delta^{-1}\sqrt{q}$ has also a bounded extension to L_μ^2 with

$$\|\Delta^{-1}\sqrt{q}\|_{L_\mu^2 \rightarrow L_\mu^2}^2 \leq CA.$$

and we conclude with the fact that

$$\|\Delta^{-1}\sqrt{q}\|_{L_\mu^2 \rightarrow L_\mu^2} = \|\sqrt{q}\Delta^{-1}\|_{L_\mu^2 \rightarrow L_\mu^2}$$

and with Theorem 3. **lii**) \Rightarrow **i**). □

3.4. Poincaré inequalities.

3.4.1. The second set of geometric conditions are the so-called doubling condition and the Poincaré inequalities (P_λ) .

Recall that according to [Gri91, SC92, SC02], the conjunction of doubling condition $(D_{\kappa, \nu})$ and of the Poincaré inequalities (P_λ) is equivalent to the upper and lower Gaussian estimate of the heat kernel:

$$\forall x, y \in M, \forall t > 0: \frac{c}{\mu(B(x, \sqrt{t}))} e^{-\frac{d^2(x, y)}{ct}} \leq P(t, x, y) \leq \frac{C}{\mu(B(x, \sqrt{t}))} e^{-\frac{d^2(x, y)}{5t}}. \quad (\text{DUE})$$

Note that the lower bound on the heat kernel and the doubling condition $(D_{\kappa, \nu})$ easily imply

$$\forall o \in M, \forall t > 0 \forall x, y \in B(o, \sqrt{t}): \frac{c'}{\mu(B(o, \sqrt{t}))} \leq P(t, x, y). \quad (16)$$

Hence the doubling condition together with the Poincaré inequalities imply the relative Faber-Krahn inequality but the reciprocal is not true for instance for $n \geq 2$, the connected sum $\mathbb{R}^n \# \mathbb{R}^n$ satisfies the relative Faber-Krahn inequality but not the Poincaré inequalities (P_λ) .

3.4.2. *Harnack inequalities for harmonic functions.* We also know that the conditions $(D_{\kappa, \nu}) + (P_\lambda)$ are also equivalent to Parabolic Harnack inequalities for positive solution of the heat equation [Gri91, SC92, SC02], hence they imply the elliptic Harnack inequalities

Proposition 3.5. *If (M, g, μ) is a complete weighted Riemannian manifold satisfying the doubling condition $(D_{\kappa, \nu})$ and the Poincaré inequalities (P_λ) then there is a constant C depending only on κ, ν, λ such that for any geodesic ball B and any positive harmonic function $\psi: 2B \rightarrow \mathbb{R}_+^*$:*

$$\sup_B \psi \leq C \inf_B \psi.$$

These Harnack inequalities imply more properties for harmonic functions:

Proposition 3.6. *If (M, g, μ) is a complete weighted Riemannian manifold satisfying the doubling condition $(D_{\kappa, \nu})$ and of the Poincaré inequalities (P_λ) then there are constants $C > 0, p_+ > 1, \alpha \in (0, 1]$ depending only on κ, ν, λ such that for any geodesic ball B of radius R and any harmonic function $\psi: 3B \rightarrow \mathbb{R}_+^*$:*

i) For any $y, z \in B$:

$$|\psi(y) - \psi(z)| \leq C \left(\frac{d(y, z)}{R} \right)^\alpha \sup_{x \in 2B} |\psi(x)|,$$

ii) for any $0 < \theta_1 \leq \theta_2 \leq 2$:

$$\theta_1^{2-2\alpha} \int_{\theta_1 B} |d\psi|^2 d\mu \leq C \theta_2^{2-2\alpha} \int_{\theta_2 B} |d\psi|^2 d\mu,$$

$$\text{iii) } \left(\int_B |d\psi|^{p_+} d\mu \right)^{\frac{2}{p_+}} \leq C \int_{2B} |d\psi|^2 d\mu.$$

The Hölder regularity of harmonic function Proposition 3.6-i), is a classical consequence of the Harnack inequality (see for instance [GT01, Proof of Theorem 8.22]). The proof of [Car17, Proposition 5.5] implies that Proposition 3.6-ii) holds under Poincaré inequalities and the doubling condition. The reverse Hölder property for the gradient of harmonic functions Proposition 3.6-iii) is proven by Auscher and Coulhon [AC05, Subsection 2.1].

3.4.3. *Another equivalent condition for Theorem 3.1.* Similarly to Theorem 3.4, one obtains:

Theorem 3.7. *Let (M, g, μ) be a complete non-parabolic weighted Riemannian manifold satisfying the doubling condition $(D_{\kappa, \nu})$ and of the Poincaré inequalities (P_λ) and q be a non negative, locally integrable function then the following properties are equivalents*

- i) *There is a constant C_1 such that (12) holds.*
- ii) *$Q = \Delta^{-\frac{1}{2}} q$ is finite a.e. and there is a constant C_{ii} such that*

$$\Delta^{-\frac{1}{2}}(Q^2) \leq C_{ii} Q$$

- iii) *there is a constant C_{iii} such that for any geodesic ball B :*

$$\int_B \left| \Delta^{-\frac{1}{2}} (q \mathbf{1}_B) \right|^2(x) d\mu(x) \leq C_{iii} \int_B q d\mu$$

- iv) *$\tilde{Q}(x) = \int_0^{+\infty} \left(\int_{B(x,r)} q d\mu \right) dr$ is finite almost a.e. and there is a constant C_{iv} such that*

$$\int_0^{+\infty} \left(\int_{B(x,r)} \tilde{Q}^2 d\mu \right) dr \leq C_{iv} \tilde{Q}(x).$$

Moreover the constants $C_1, C_{ii}, C_{iii}, C_{iv}$ are mutually controlled.

Hint on the proof of Theorem 3.7: We introduce again the operator \mathcal{K} whose Schwartz kernel is given by

$$K(x, y) = \int_{d(x,y)}^{+\infty} \frac{dr}{\mu(B(x, r))}.$$

Then using the lower and upper Gaussian estimate for the heat kernel (DUE), Proposition A.4 yields that

$$\forall x \neq y \in M: c K(x, y) \leq \Delta^{-\frac{1}{2}}(x, y) \leq C K(x, y).$$

So that if (12) holds with constant C_1 then the operator $\mathcal{K}\sqrt{q}$ has a bounded extension to L_μ^2 and $\|\mathcal{K}\sqrt{q}\|_{L_\mu^2 \rightarrow L_\mu^2}^2 \leq C C_1$. And if the operator $\mathcal{K}\sqrt{q}$ has a bounded extension to L_μ^2 then (12) holds with constant $C_1 = c^{-1} \|\mathcal{K}\sqrt{q}\|_{L_\mu^2 \rightarrow L_\mu^2}^2$.

Hence Theorem A.3 **iv**) \Leftrightarrow **i**) implies the equivalence **iv**) \Leftrightarrow **i**) in Theorem 3.7. The estimates

$$c \tilde{Q} \leq Q \leq C \tilde{Q} \text{ and } c \mathcal{K}(\tilde{Q}^2) \leq \Delta^{-\frac{1}{2}} Q^2 \leq C \mathcal{K}(\tilde{Q}^2)$$

imply the equivalence **iv**) \Leftrightarrow **ii**) in Theorem 3.7. Similarly the equivalence **iii**) \Leftrightarrow **iv**) in Theorem 3.7 is a consequence of Theorem A.3 **iii**) \Leftrightarrow **iv**).

□

4. A STRONGER L^2 -WEIGHTED BOUNDEDNESS PROPERTY FOR THE HODGE PROJECTOR

4.1. **Muckenhoupt weight properties of equilibrium potential.** We recall:

Definition 4.1. *If (X, d, μ) is a measure metric space satisfying the doubling condition $(D_{\kappa, \nu})$, then a positive locally integrable function ω is said to be a A_1 -weight with constant C if for any ball $B \subset X$:*

$$\int_B \omega d\mu \leq C \inf_B \omega.$$

The following properties of A_1 -weight are classical [KK11, ST89]:

Proposition 4.2. *Let ω be a A_1 -weight with constant C on some measure metric space (X, d, μ) satisfying the doubling condition $(D_{\kappa, \nu})$. Then there is a constant D depending only on κ, ν and C such that for any ball $B \subset X$:*

- i) $\int_{2B} \omega d\mu \leq D \int_B \omega d\mu$,
- ii) $\inf_B \omega \leq \int_B \omega d\mu \leq C \inf_B \omega$,
- iii) $\inf_B \omega \leq D \inf_{2B} \omega$.

And we also have the following useful criteria for a weight to satisfy a reverse Hölder inequality

Corollary 4.3. *If (X, d, μ) is a measure metric space satisfying the doubling condition $(D_{\kappa, \nu})$ and if for some $r > 1$, ω^r is a A_1 -weight with constant C then ω is a reverse Hölder weight meaning that for any ball $B \subset X$:*

$$\left(\int_B \omega^r d\mu \right)^{\frac{1}{r}} \leq C^{1/r} \int_B \omega d\mu.$$

Our result concerning the weight properties of equilibrium potential is the following generalization of [MV95, Lemma 2.1]:

Proposition 4.4. *Let (M, g, μ) be a complete non-parabolic weighted Riemannian manifold satisfying the doubling condition $(D_{\kappa, \nu})$ and the Poincaré inequalities (P_λ) . Then for any $\tau \in [0, \nu/(\nu - 2))$, there is a positive constant C depending only on κ, ν, λ and τ such that for any bounded open set, its equilibrium potential h satisfies that h^τ is a A_1 -weight with constant C :*

$$\forall B \subset M: \int_B h^\tau d\mu \leq C \inf_B h^\tau.$$

Note that the non-parabolicity condition forces that $\nu > 2$ (remark 1.4). This proposition will be consequence of the same properties but only for Green functions.

Lemma 4.5. *Under the assumption of Proposition 4.4, for any $\tau \in [0, \nu/(\nu - 2))$, there is a positive constant C depending only on κ, ν, λ and τ such that for any $o \in M$ the Green function with pole at o satisfies that for any ball $B \subset M$:*

$$\int_B G_o^\tau d\mu \leq C \inf_B G_o^\tau.$$

Proof of Proposition 4.4 assuming Lemma 4.5. It is enough to show the result when $\tau > 1$. Let \mathcal{U} be a bounded open subset with equilibrium potential h and equilibrium measure $\nu_{\mathcal{U}}$:

$$h(x) = \int_M G(x, y) d\nu_{\mathcal{U}}(y).$$

Let B be a geodesic ball and let $f \in L^{\tau^*}(B)$ with $\frac{1}{\tau} + \frac{1}{\tau^*} = 1$ and $\int_B |f|^{\tau^*} d\mu = 1$. Then

$$\begin{aligned} \left| \int_B f h d\mu \right| &= \left| \int_M \left(\int_B G(x, y) f(x) d\mu(x) \right) d\nu_{\mathcal{U}}(y) \right| \\ &\leq \int_M \left(\int_B G^\tau(x, y) d\mu(x) \right)^{\frac{1}{\tau}} d\nu_{\mathcal{U}}(y) \\ &\leq C^{\frac{1}{\tau}} \int_M \left(\inf_{x \in B} G(x, y) \right) d\nu_{\mathcal{U}}(y) \\ &\leq C^{\frac{1}{\tau}} \inf_B h. \end{aligned}$$

□

Proof of Lemma 4.5. The estimate (DUE) and Proposition A.4 imply the following estimate of the Green kernel:

$$c \int_{d(x, y)}^{+\infty} \frac{r}{\mu(B(x, r))} dr \leq G(x, y) \leq C \int_{d(x, y)}^{+\infty} \frac{r}{\mu(B(x, r))} dr. \quad (17)$$

Notice that using the doubling condition one easily gets that

$$\int_s^{+\infty} \frac{r}{\mu(B(x, r))} dr \geq \int_s^{2s} \frac{r}{\mu(B(x, r))} dr \geq \frac{s^2}{\gamma \mu(B(x, s))},$$

so that for any $x, y \in M$:

$$G(x, y) \geq c \frac{d^2(x, y)}{\mu(B(x, d(x, y)))}. \quad (18)$$

Moreover if $y \in B(x, r)$ then $d(x, y) \leq r$ and

$$G(x, y) \geq c \int_r^{+\infty} \frac{r}{\mu(B(x, r))} dr \geq c \frac{r^2}{\gamma \mu(B(x, r))},$$

hence we deduce that

$$\inf_{y \in B(x, r)} G(x, y) \geq c \int_r^{+\infty} \frac{r}{\mu(B(x, r))} dr \geq c \frac{r^2}{\gamma \mu(B(x, r))}, \quad (19)$$

Let $\tau \in [1, \nu/(\nu-2))$ and $o \in M$ and $B = B(z, r)$. There are 3 cases to be considered

First case: $r \leq \mathbf{d}(o, \mathbf{z})/4$. In that case $x \mapsto G_o(x)$ is a positive harmonic function on $3B$ and the desired conclusion is then a direct consequence of the Harnack inequality

$$\sup_B G_o \leq C \inf_B G_o.$$

Second case: $o = \mathbf{z}$. Let $f \in L^{\tau^*}(B)$ with $\frac{1}{\tau} + \frac{1}{\tau^*} = 1$ and $\int_B |f|^{\tau^*} d\mu = 1$ then

$$\left| \int_B f G_o d\mu \right| \leq C \int_B \left(\int_{d(o, y)}^{+\infty} \frac{s}{\mu(B(o, s))} ds \right) |f|(y) d\mu(y) = C (\mathbf{I} + \mathbf{II})$$

with

$$\begin{aligned} \mathbf{I} &= \left(\int_r^{+\infty} \frac{s}{\mu(B(o, r))} ds \right) \int_B |f|(y) d\mu(y) \\ \text{and } \mathbf{II} &= \int_B \left(\int_{d(o, y)}^r \frac{s}{\mu(B(o, r))} ds \right) |f|(y) d\mu(y). \end{aligned}$$

Using that $\int_B |f|^{\tau^*} d\mu = 1$ and (19) one gets that

$$\mathbf{I} \leq \frac{1}{c} \inf_B G_o.$$

For the second term:

$$\begin{aligned} \mathbf{II} &= \int_B \left(\int_{d(o, y)}^r \frac{s}{\mu(B(o, s))} ds \right) |f|(y) d\mu(y). \\ &= \frac{1}{\mu(B)} \int_0^r s \left(\int_{B(o, s)} |f|(y) dy \right) ds \\ &\leq \frac{1}{\mu(B)} \int_0^r s \left(\frac{\mu(B(o, r))}{\mu(B(o, s))} \right)^{\frac{1}{\tau^*}} ds, \end{aligned}$$

where in the last line we used Hölder inequality and $\int_B |f|^{\tau^*} d\mu = 1$. Using the doubling condition $(D_{\kappa, \nu})$ one gets

$$\int_0^r s \left(\frac{\mu(B(o, r))}{\mu(B(o, s))} \right)^{\frac{1}{\tau^*}} ds \leq \kappa^{\frac{1}{\tau^*}} \int_0^r s \left(\frac{r}{s} \right)^{\frac{\nu}{\tau^*}} ds = \frac{\kappa^{\frac{1}{\tau^*}} \tau}{\nu - (\nu - 2)\tau} r^2.$$

Hence we deduce that

$$\left(\int_B G_o^\tau d\mu \right)^{\frac{1}{\tau}} \leq \frac{C}{c} \inf_B G_o + \frac{C \kappa^{\frac{1}{\tau^*}} \tau}{\nu - (\nu - 2)\tau} \frac{r^2}{\mu(B(o, r))}.$$

As (19) yields that $\frac{r^2}{\mu(B(o,r))} \leq C \inf_B G_o$, we get

$$\left(\int_B G_o^\tau d\mu \right)^{\frac{1}{\tau}} \leq C' \inf_B G_o.$$

Third case: $4r > d(o, z) > 0$. In that case, the result follows from $B(z, r) \subset B(o, 5r)$:

$$\begin{aligned} \int_B G_o^\tau d\mu &\leq \gamma^3 \int_{B(o, 5r)} G_o^\tau d\mu \\ &\leq \gamma^3 C \inf_{B(o, 5r)} G_o^\tau \text{ (using the result obtained in the second case)} \\ &\leq \gamma^3 C \inf_B G_o^\tau. \end{aligned}$$

□

4.2. Reverse Hölder for gradient of harmonic functions.

Lemma 4.6. *Let (M, g, μ) be a complete non-parabolic weighted Riemannian manifold satisfying the doubling condition $(D_{\kappa, \nu})$ and the Poincaré inequalities (P_λ) . There is some $q > 2$ and some positive constant C that depend only on κ, ν, λ such that for any $o \in M$ and any ball $B \subset M$ and any harmonic function $\psi: 3B \rightarrow \mathbb{R}$ satisfies the following reverse Hölder property:*

$$\int_B G_o(y) |d\psi|^q d\mu \leq C \inf_B G_o \left(\int_{2B} |d\psi|^2 d\mu \right)^{\frac{q}{2}}.$$

Proof. Proceeding as in the proof of Lemma 4.5 and provided $q \leq p_+$ where p_+ is given Proposition 3.6, it is enough to prove the result for balls centered at o . So assume that $B = B(o, r)$. Let $q > 2$ be such that for the constants $\alpha \in (0, 1]$ and $p_+ > 2$ given by Proposition 3.6:

$$q \leq p_+ \text{ and } q(1 - \alpha) < 2.$$

Using the Green kernel estimate (17) and proceeding as before, we get for any harmonic function $\psi: 3B \rightarrow \mathbb{R}$:

$$\int_B G_o(y) |d\psi|^q d\mu \leq C \inf_B G_o(y) \int_B |d\psi|^q d\mu + C \mathbf{II} \quad (20)$$

with

$$\begin{aligned} \mu(B) \mathbf{II} &= \int_0^r s \left(\int_{B(o, s)} |d\psi|^q d\mu \right) ds \\ &\leq \int_0^r s \left(\int_{B(o, 2s)} |d\psi|^2 d\mu \right)^{\frac{q}{2}} ds \text{ using Proposition 3.6.iii)} \\ &\leq \int_0^r s \left(\frac{r}{s} \right)^{q(1-\alpha)} ds \left(\int_{B(o, 2r)} |d\psi|^2 d\mu \right)^{\frac{q}{2}} \text{ using Proposition 3.6.ii)} \end{aligned}$$

Hence

$$\mathbf{II} \leq C \frac{r^2}{\mu(B)} \left(\int_{B(o, 2r)} |d\psi|^2 d\mu \right)^{\frac{q}{2}} \leq C \inf_B G_o \left(\int_{B(o, 2r)} |d\psi|^2 d\mu \right)^{\frac{q}{2}}, \quad (21)$$

where we have used (19):

$$\frac{r^2}{\mu(B)} \leq C \inf_B G_o.$$

Hence the inequality (20), the reverse Hölder inequality

$$\int_B |d\psi|^q d\mu \leq C \left(\int_{2B} |d\psi|^2 d\mu \right)^{\frac{q}{2}}$$

and (21) yields that

$$\int_B G_o(y) |d\psi|^q d\mu \leq C \inf_B G_o(y) \left(\int_{2B} |d\psi|^2 d\mu \right)^{\frac{q}{2}}. \quad (22)$$

□

This lemma has the following crucial consequence:

Proposition 4.7. *Let (M, g, μ) be a complete non-parabolic weighted Riemannian manifold satisfying the doubling condition $(D_{\kappa, \nu})$ and the Poincaré inequalities (P_λ) . There is some $q > 2$ and some positive constant C that depend only on κ, ν, λ such that for any equilibrium potential $h: M \rightarrow \mathbb{R}$, any $\tau \in [0, 1]$, any ball $B \subset M$ and any harmonic function $\psi: 3B \rightarrow \mathbb{R}$*

$$\frac{1}{\int_B h^\tau} \int_B h^\tau |d\psi|^q d\mu \leq C \left(\frac{1}{\int_{2B} h^\tau} \int_{2B} h^\tau |d\psi|^2 d\mu \right)^{\frac{q}{2}}.$$

Proof. Let $h: M \rightarrow \mathbb{R}$ be the equilibrium potential of some bounded open set $\mathcal{U} \subset M$, $\tau \in (0, 1]$, $B \subset M$ a geodesic ball and let $\psi: 3B \rightarrow \mathbb{R}$ be some harmonic function. Using Hölder inequality, we get

$$\int_B h^\tau |d\psi|^q d\mu \leq \left(\int_B h |d\psi|^q d\mu \right)^\tau \left(\int_B |d\psi|^q d\mu \right)^{1-\tau}.$$

If $\nu_{\mathcal{U}}$ is the equilibrium measure associated to \mathcal{U} , we have

$$\begin{aligned} \int_B h |d\psi|^q d\mu &= \int_M \left(\int_B G(x, y) |d\psi|^q(y) d\mu(y) \right) d\nu_{\mathcal{U}}(x) \\ &\leq C \int_M \inf_{y \in B} G(x, y) d\nu_{\mathcal{U}}(x) \left(\int_{2B} |d\psi|^2 d\mu \right)^{\frac{q}{2}} \\ &\leq C \inf_{y \in B} h(y) \left(\int_{2B} |d\psi|^2 d\mu \right)^{\frac{q}{2}}. \end{aligned}$$

So that

$$\begin{aligned} \int_B h^\tau |d\psi|^q d\mu &\leq C \inf_{x \in B} h^\tau(x) \left(\int_{2B} |d\psi|^2 d\mu \right)^{\frac{q}{2}} \\ &\leq C \left(\inf_{x \in B} h^\tau(x) \right) \left(\inf_{x \in 2B} h^\tau(x) \right)^{-\frac{q}{2}} \left(\int_{2B} h^\tau |d\psi|^2 d\mu \right)^{\frac{q}{2}}. \end{aligned}$$

Using the fact that h is a A_1 -weight and Proposition 4.2, we get that

$$\inf_{x \in B} h^\tau(x) \leq \int_B h^\tau d\mu \text{ and } \int_{2B} h^\tau d\mu \leq C \inf_{x \in 2B} h^\tau(x).$$

Hence the result. □

4.3. L^2 -weighted boundedness of the Hodge projector.

Theorem 4.8. *Let (M, g, μ) be a complete non-parabolic weighted Riemannian manifold satisfying the doubling condition $(D_{\kappa, \nu})$ and the Poincaré inequalities (P_λ) . There are constants $\tau_+ > 1$ and $C > 0$ both depending only on κ, ν, λ such that for any equilibrium potential $h: M \rightarrow \mathbb{R}$,*

$$\int_M |\Pi(\beta)|^2 h^{\tau_+} d\mu \leq C \int_M |\beta|^2 h^{\tau_+} d\mu. \quad (23)$$

Proof. Let $h: M \rightarrow \mathbb{R}$ be some equilibrium potential.

According to Auscher and Coulhon [AC05], there are $p_- < 2$ and a constant C both depending only on κ, ν, λ such that for any $r \in [p_-, p_-^*]$, the Hodge projector has a bounded extension to $L^r(T^*M)$ with

$$\|\Pi\|_{L^r \rightarrow L^r} \leq C.$$

Let $\delta := \frac{q-2}{q} \frac{1}{\nu-2}$ where q is given by Proposition 4.7 and let $p < 2$ be given by

$$\frac{1}{p} = \frac{\sqrt{1-\delta/2}}{2} + \frac{1-\sqrt{1-\delta/2}}{p_-}$$

and $\sigma \in (0, 1)$ given by

$$\sigma = 1 - \frac{\delta}{2} = \left(\sqrt{1 - \frac{\delta}{2}} \right)^2.$$

Interpolating [Ste56, Theorem 2] between the boundedness

$$\|\Pi\|_{L^{p_-} \rightarrow L^{p_-}} \leq C$$

and the weighted L^2 boundedness given by Theorem 2.8

$$\forall \beta \in C_c^\infty(T^*M): \int_M |\Pi(\beta)|^2 h^{\sqrt{\sigma}} d\mu \leq \left(\frac{1+\sqrt{\sigma}}{1-\sqrt{\sigma}} \right)^2 \int_M |\beta|^2 h^{\sqrt{\sigma}} d\mu,$$

we deduce the weighted L^p boundedness

$$\forall \beta \in C_c^\infty(T^*M): \int_M |\Pi(\beta)|^p h^\sigma d\mu \leq C_1 \int_M |\beta|^p h^\sigma d\mu.$$

Notice that Proposition 4.7 yields that for any ball $B \subset M$ and any $\beta \in L^2(T^*M)$ with support in $M \setminus 3B$ then

$$\left(\frac{1}{\int_B h^\sigma d\mu} \int_B |\Pi(\beta)|^q h^\sigma d\mu \right)^{\frac{1}{q}} \leq C_2 \left(\frac{1}{\int_{2B} h^\sigma d\mu} \int_{2B} |\Pi(\beta)|^p h^\sigma d\mu \right)^{\frac{1}{p}}.$$

The measure $h^\sigma d\mu$ is doubling hence we can use a result of Auscher and Martell [AM07, Theorem 3.14] and we know that if ω is a positive locally integrable weight satisfying for some positive constant D :

- ω is a A_2 -weight on $(M, d_g, h^\sigma d\mu)$: for any ball $B \subset M$ then

$$\left(\frac{1}{\int_B h^\sigma} \int_B \omega h^\sigma d\mu \right) \left(\frac{1}{\int_B h^\sigma} \int_B \omega^{-1} h^\sigma d\mu \right) \leq D,$$

- ω is a reverse Hölder weight of exponent $(q/2)^* = \frac{q}{q-2}$ on $(M, d_g, h^\sigma d\mu)$:

$$\left(\frac{1}{\int_B h^\sigma} \int_B \omega^{\frac{q}{q-2}} h^\sigma d\mu \right)^{1-\frac{2}{q}} \leq D \frac{1}{\int_B h^\sigma} \int_B \omega h^\sigma d\mu,$$

then for a constant C depending only on κ, ν, C_1, C_2 and D , we get the weighted L^2 boundedness of the Hodge projector:

$$\forall \beta \in C_c^\infty(T^*M): \int_M |\Pi(\beta)|^2 \omega h^\sigma d\mu \leq C \int_M |\beta|^2 \omega h^\sigma d\mu.$$

From Corollary 4.3, we know that $\omega = h^a$ satisfies these conditions when $h^{\sigma + \frac{q}{q-2}a}$ is a A_1 -weight on (M, d_g, μ) , hence it is the case when

$$\sigma + \frac{q}{q-2}a \leq \frac{\nu}{\nu-2}.$$

Considering $\omega = h^\delta$, then

$$h^\sigma \omega^{\frac{q}{q-2}} = h^{1 + (\frac{1}{2} + \frac{1}{q}) \frac{1}{\nu-2}}$$

is a A_{1-} weight on $(M, d_g, d\mu)$. Hence the result of Auscher and Martell implies the uniform weighted L^2 boundedness of the Hodge projector (23) for $\tau_+ = 1 + \delta/2$. \square

5. THE MAZ'YA-VERTBISKY PROBLEM

In this section we will prove Theorem A.

5.1. The direct implication. We start by the easiest implication and according to Maz'ya and Vertbisky, this idea comes for [CG76, Lemma2.1].

Proposition 5.1. *Let (M, g, μ) be a complete non-parabolic weighted Riemannian manifold and $\theta \in L^2_{loc}(T^*M)$ such that for some positive constant A :*

$$\forall \varphi \in C_c^\infty(M): \int_M |\theta|^2 \varphi^2 d\mu \leq A^2 \int_M |d\varphi|^2 d\mu$$

then the distribution $V = d_\mu^* \theta$ satisfies

$$\forall \varphi, \phi \in C_c^\infty(M): |\langle V, \varphi \phi \rangle| \leq 2A \|d\varphi\|_{L_\mu^2} \|d\phi\|_{L_\mu^2}.$$

Proof. By definition:

$$\langle V, \varphi \phi \rangle = \int_M \langle \theta, d(\varphi \phi) \rangle d\mu = \int_M \langle \theta, \phi d\varphi \rangle d\mu + \int_M \langle \theta, \varphi d\phi \rangle d\mu.$$

And using the Cauchy-Schwartz inequality and the hypothesis, one gets:

$$\begin{aligned} |\langle V, \varphi \phi \rangle| &\leq \|\phi \theta\|_{L_\mu^2} \|d\varphi\|_{L_\mu^2} + \|\varphi \theta\|_{L_\mu^2} \|d\phi\|_{L_\mu^2} \\ &\leq A \|d\phi\|_{L_\mu^2} \|d\varphi\|_{L_\mu^2} + A \|d\varphi\|_{L_\mu^2} \|d\phi\|_{L_\mu^2}. \end{aligned}$$

\square

5.2. In order to be able to prove a reciprocal, we start by the following notion

Definition 5.2. *If (M, g, μ) is a complete weighted Riemannian manifold, a non negative, locally integrable function ω is said to be a parabolic weight if there is a sequence $\chi_\ell \in C_c^\infty(M)$ such that*

$$\begin{cases} 0 \leq \chi_\ell \leq 1 & \text{every where on } M \\ \lim_{\ell \rightarrow +\infty} \chi_\ell = 1 & \text{uniformly on compact set} \\ \lim_{\ell \rightarrow +\infty} \int_M |d\chi_\ell|^2 \omega d\mu = 0 \end{cases} \quad (24)$$

Remark 5.3. *When ω is positive, this is equivalent to the parabolicity of the weighted Riemannian manifold $(M, g, \omega\mu)$. It is clear that if $\omega_1 \leq \omega_2$, then the parabolicity of ω_2 implies the parabolicity of ω_1 .*

Our first result is a refinement of [Car19, Proposition 2.27]:

Lemma 5.4. *If (M, g, μ) is a complete non-parabolic weighted Riemannian manifold satisfying the doubling condition $(D_{\kappa, \nu})$ and the Poincaré inequalities (P_λ) then for any $o \in M$, $\min\{G_o, 1\}$ is a parabolic weight.*

Proof. Recall 17:

$$C^{-1} \int_{d(x,o)}^{+\infty} \frac{r}{\mu(B(o,r))} dr \leq G_o(x) \leq C \int_{d(x,o)}^{+\infty} \frac{r}{\mu(B(o,r))} dr, \quad (25)$$

hence the non parabolicity implies the finiteness of integral :

$$\int_1^{+\infty} \frac{r}{\mu(B(o,r))} dr < +\infty$$

and that

$$\lim_{x \rightarrow +\infty} G_o(x) = 0.$$

We let $u: \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that

$$\begin{cases} u(t) = 0 & \text{if } t \leq 0 \\ u(t) = 1 & \text{if } t \geq 1 \\ |u'(t)| \leq 2 & \text{everywhere.} \end{cases}$$

and define

$$\chi_\ell(x) = u\left(\frac{\log(G_o(x)\ell^2)}{\log(\ell)}\right).$$

The estimates (25) implies that $\chi_\ell \mathcal{C}_c^\infty(M)$ and

$$\lim_{\ell \rightarrow +\infty} \chi_\ell = 1 \text{ uniformly on compact set.}$$

We have the following estimations

$$\begin{aligned} \int_M |d\chi_\ell|^2 \min\{G_o, 1\} d\mu &\leq 4 \log(\ell)^{-2} \int_{\{\ell^{-2} < G_o < \ell^{-1}\}} \frac{|dG_o|^2}{G_o} d\mu \\ &\leq 4 \log(\ell)^{-2} \int_{1/\ell^2}^{1/\ell} \left(\int_{\{G_o=x\}} |dG_o| \right) \frac{dx}{x} \\ &\leq 4 \log(\ell)^{-1}, \end{aligned}$$

where we have used that for almost every x

$$\int_{\{G_o=x\}} |dG_o| = 1.$$

Hence $\min\{G_o, 1\}$ is a parabolic weight. \square

We are now in position to finish the proof of our main result.

Proof of the reciprocal in Theorem A.: Let (M, g, μ) be a complete non-parabolic weighted Riemannian manifold satisfying the the doubling condition $(D_{\kappa, \nu})$ and the Poincaré inequalities (P_λ) . And let V be a distribution such that

$$\forall \varphi, \phi \in \mathcal{C}_c^\infty(M): |\langle V, \varphi\phi \rangle| \leq \mathbf{A} \|d\varphi\|_{L_\mu^2} \|d\phi\|_{L_\mu^2}.$$

We want to define a distribution $\theta \in \mathcal{C}^{-\infty}(T^*M)$ by the following relation: let $\beta \in \mathcal{C}_c^\infty(T^*M)$ and define $\varphi_\beta \in \mathring{W}^{1,2}(M)$ to be the solution of the equation

$$\Delta \varphi_\beta = d_\mu^* \beta$$

so that $\Pi(\beta) = d\varphi_\beta$ and let

$$\langle \theta, \beta \rangle = \langle V, \varphi_\beta \rangle.$$

We need to justify that such definition makes sense.

Let \mathcal{U} be a bounded open subset containing the support of β and let h be its equilibrium potential. Recall that by Proposition 3.6-i), there is some $\alpha \in (0, 1]$ such that we get the uniform Hölder regularity for harmonic functions.

Claim: if $\tau \in [1, 1 + 2\alpha/(\nu - 2)]$, then $h^{-\tau} \varphi_\beta^2$ is a parabolic weight.

Proof of the Claim. Let $o \in \mathcal{U}$ and $R > 0$ such that $\mathcal{U} \subset B(o, R)$. According to the maximum principle, we have that on $M \setminus \mathcal{U}$:

$$\frac{1}{\max_{\partial\mathcal{U}} G_o} G_o \leq h \leq \frac{1}{\min_{\partial\mathcal{U}} G_o} G_o. \quad (26)$$

Moreover as $\int_M d_\mu^* \beta d\mu = 0$, we have

$$\varphi_\beta(x) = \int_{B(o, R)} (G(x, y) - G(x, o)) d_\mu^* \beta(y) d\mu(y).$$

Now if $x \notin B(o, 6R)$ then $y \mapsto G(x, y)$ is a harmonic function on $B(o, d(o, x)/2)$ hence Proposition 3.6 implies that for any $y \in B(o, R)$:

$$\begin{aligned} |G(x, y) - G(x, o)| &\leq C \left(\frac{R}{d(o, x)} \right)^\alpha \sup_{z \in B(o, d(o, x)/3)} G(x, z) \\ &\leq C \left(\frac{R}{d(o, x)} \right)^\alpha G(x, o), \end{aligned} \quad (27)$$

where in the last inequality, we used the Harnack inequality given by Proposition 3.5 for the positive harmonic function $y \in B(o, 2d(o, x)/3) \mapsto G(x, y)$.

Using (18) and the doubling hypothesis, we also know that if $x \notin B(o, 6R)$ then

$$G(o, x) \geq C \frac{d^2(o, x)}{\mu(B(o, d(o, x)))} \geq C \frac{d^2(o, x)}{\mu(B(o, R))} \left(\frac{R}{d(o, x)} \right)^\nu = C \frac{R^\nu}{\mu(B(o, R))} \frac{1}{d^{\nu-2}(o, x)}. \quad (28)$$

The estimates (27) and (28) yields that outside $B(o, 6R)$:

$$|\varphi_\beta(x)| \leq C \left(\frac{\mu(B(o, R))}{R^2} \right)^{\frac{\alpha}{\nu-2}} G_o^{1+\frac{\alpha}{\nu-2}}(x)$$

and estimates (26) and Lemma 5.4 implies that if $\tau \in [1, 1 + 2\alpha/(\nu - 2)]$, then $h^{-\tau} \varphi_\beta^2$ is bounded by a parabolic weight, hence it is a parabolic weight. \square

Now let $\tau > 1$ be such that $\tau \leq \tau_+$ and $\tau \leq 1 + \frac{2\alpha}{\nu-2}$, where τ_+ is given by Theorem 4.8. By interpolation, the Hodge projector extends also continuously on $L^2(T^*M, h^{-\tau} d\mu)$, hence $h^{-\tau/2} d\varphi_\beta \in L^2$ with

$$\int_M h^{-\tau} |d\varphi_\beta|^2 d\mu \leq C \int_U |\beta|^2 d\mu, \quad (29)$$

where we use that $h = 1$ on $\text{supp } \beta$. The parabolicity of the weight $h^{-\tau} \varphi_\beta^2$ yields a sequence of good cut-off function $\chi_\ell \in C_c^\infty(M)$ such that $0 \leq \chi_\ell \leq 1$, (χ_ℓ) converges to the function 1 uniformly on compact sets of M and

$$\lim_{\ell \rightarrow +\infty} \int_M |d\chi_\ell|^2 h^{-\tau} \varphi_\beta^2 d\mu = 0.$$

Using the Hardy inequality (7), we get that

$$\begin{aligned} \frac{(1+\tau)^2}{4} \int_M \frac{|dh|^2}{h^2} \chi_\ell^2 \varphi_\beta^2 h^{-\tau} d\mu &\leq \int_M |d(\chi_\ell \varphi_\beta)|^2 h^{-\tau} d\mu \\ &\leq 2 \int_M |d\chi_\ell|^2 \varphi_\beta^2 h^{-\tau} d\mu + \int_M \chi_\ell^2 |d\varphi_\beta|^2 h^{-\tau} d\mu. \end{aligned}$$

Passing to the limit $\ell \rightarrow +\infty$ and using (29), we get that

$$\int_M \frac{|dh|^2}{h^2} \varphi_\beta^2 h^{-\tau} d\mu \leq \frac{4}{(1+\tau)^2} C \int_U |\beta|^2 d\mu \quad (30)$$

With

$$d(h^{-\tau/2} \varphi_\beta) = h^{-\tau/2} d\varphi_\beta - \frac{\tau}{2} h^{-\tau/2} \varphi_\beta \frac{dh}{h},$$

we get that $d(h^{-\tau/2} \varphi_\beta) \in L_\mu^2$ with

$$\int_M \left| d(h^{-\tau/2} \varphi_\beta) \right|^2 d\mu \leq C \int_U |\beta|^2 d\mu \quad (31)$$

where C depends only on the doubling and Poincaré constants. Using again the sequence of the cut-off function χ_ℓ , we easily obtain that $h^{-\tau/2} \varphi_\beta \in \mathring{W}^{1,2}(M)$ and it is the $\mathring{W}^{1,2}(M)$ -limit of the sequence $(\chi_\ell h^{-\tau/2} \varphi_\beta)_\ell$. Noticed that the assumptions

$$\forall \varphi, \phi \in \mathcal{C}_c^\infty(M): |\langle V, \varphi\phi \rangle| \leq \mathbf{A} \|d\varphi\|_{L_\mu^2} \|d\phi\|_{L_\mu^2},$$

implies that the bilinear form

$$(\varphi, \phi) \in \mathcal{C}_c^\infty(M) \times \mathcal{C}_c^\infty(M) \mapsto \langle V, \varphi\phi \rangle$$

extends continuously to a bilinear form on $\mathring{W}^{1,2}(M) \times \mathring{W}^{1,2}(M)$. As $h^{\tau/2}$ and $h^{-\tau/2}\varphi_\beta$ are both in $\mathring{W}^{1,2}(M)$, we obtain that

$$\begin{aligned} |\langle \theta, \beta \rangle| &= |\langle V, \varphi_\beta \rangle| \\ &= \left| \langle V, h^{\tau/2} h^{-\tau/2} \varphi_\beta \rangle \right| \\ &\leq \mathbf{A} \left\| dh^{\tau/2} \right\|_{L_\mu^2} \left\| d(h^{-\tau/2} \varphi_\beta) \right\|_{L_\mu^2} \end{aligned}$$

And using (31) together with Proposition 2.4, we deduce that

$$|\langle \theta, \beta \rangle| \leq \mathbf{A} \frac{\tau}{\sqrt{2\tau-1}} \text{cap}^{\frac{1}{2}}(\mathcal{U}) \sqrt{C} \|\beta\|_{L_\mu^2}.$$

This inequality being true for any $\beta \in \mathcal{C}_c^\infty(T^*\mathcal{U})$ and any bounded open subset \mathcal{U} , we deduce that θ is given by a locally square integrable 1-form and that for any bounded open subset \mathcal{U} :

$$\int_{\mathcal{U}} |\theta|^2 d\mu \leq C' \mathbf{A}^2 \text{cap}(\mathcal{U})$$

where C' depends only on the doubling and Poincaré constants. Then the conclusion of follows from the implication **iii**) \Rightarrow **ii**) in Theorem 3.1. \square

APPENDIX A. TRACE INEQUALITIES ON METRIC MEASURE SPACE

A.1. Setting. We consider a metric measure space (X, d, μ) that is doubling i.e. for any $x \in X$ and $r < R$:

$$\mu(B(x, R)) \leq \kappa \left(\frac{R}{r}\right)^\gamma \mu(B(x, r)), \quad (\mathbf{D}_{\kappa, \gamma})$$

and we will note $\gamma = \kappa 2^\gamma$ so that for any $x \in X$ and $r > 0$:

$$\mu(B(x, 2r)) \leq \gamma \mu(B(x, r)).$$

We introduce the notation:

$$V(x, r) := \mu(B(x, r)).$$

It is classical to prove that the doubling condition $(\mathbf{D}_{\kappa, \gamma})$ implies the following control on the volume of the balls:

$$\forall x, y \in X, \forall r > 0: V(x, r) \leq \kappa \left(1 + \frac{d(x, y)}{r}\right)^\gamma V(y, r).$$

We also assume that for some $x \in X$:

$$\int_1^{+\infty} \frac{d\tau}{V(x, \tau)} < +\infty.$$

In that case, it is easy to check that for any $y \in X$

$$\int_1^{+\infty} \frac{d\tau}{V(y, \tau)} < +\infty.$$

And we then introduce the operator

$$\mathcal{K}: L_c^1(X, \mu) \rightarrow L_{\text{loc}}^1(X, \mu)$$

defined by

$$\mathcal{K}f(x) = \int_0^{+\infty} \left(\int_{B(x,\tau)} f d\mu \right) d\tau$$

and whose Schwartz kernel is

$$K(x, y) = \int_{d(x,y)}^{+\infty} \frac{d\tau}{V(x, \tau)}.$$

The dual operator \mathcal{K}^* has Schwartz kernel given by

$$K^*(x, y) = K(y, x) = \int_{d(x,y)}^{+\infty} \frac{d\tau}{V(y, \tau)}.$$

The following lemma is a consequence of the doubling hypothesis:

Lemma A.1. *For any $x, y \in X$:*

$$\gamma^{-1} K(y, x) \leq K(x, y) \leq \gamma K(y, x).$$

Moreover we have

$$K_{1/2}(x, y) = \int_{\frac{1}{2}d(x,y)}^{+\infty} \frac{d\tau}{V(x, \tau)} \leq \gamma K(x, y).$$

The next estimate is merely an adaptation of [Ver08, Lemma 2.1]

Lemma A.2. *For any non negative $f \in L^1_c(X, \mu)$ and for any $x \in X$:*

$$|\mathcal{K}f|^2(x) \leq 2\gamma^3 (\mathcal{K}(f\mathcal{K}f))(x).$$

Proof. We start by the formula

$$|\mathcal{K}f|^2(x) = \int_{X \times X} K(x, y)K(x, z)f(y)f(z) d\mu(y)d\mu(z).$$

We write $X \times X = \mathcal{U}_x \cup \mathcal{V}_x$ where

$$\mathcal{U}_x := \{(y, z) \in X \times X, 2d(x, z) \geq d(y, z)\} \text{ and } \mathcal{V}_x := \{(y, z) \in X \times X, 2d(x, z) < d(y, z)\}.$$

Notice that when $(y, z) \in \mathcal{V}_x$ then

$$d(x, y) \geq d(y, z) - d(z, x) \geq \frac{1}{2}d(y, z),$$

hence

$$\mathcal{V}_x \subset \mathcal{W}_x = \{(y, z) \in X \times X, 2d(x, y) \geq d(y, z)\} = \{(y, z) \in X \times X, (z, y) \in \mathcal{U}_x\}.$$

Using Lemma A.1, one gets that for $(y, z) \in \mathcal{U}_x$:

$$K(x, z) \leq \gamma K(z, x) \leq \gamma K_{\frac{1}{2}}(z, y) \leq \gamma^2 K(z, y) \leq \gamma^3 K(y, z),$$

we obtain that:

$$\begin{aligned} \int_{\mathcal{U}_x} K(x, y)K(x, z)f(y)f(z) d\mu(y)d\mu(z) &\leq \gamma^3 \int_{\mathcal{U}_x} K(x, y)K(y, z)f(y)f(z) d\mu(y)d\mu(z) \\ &\leq \gamma^3 (\mathcal{K}(f\mathcal{K}f))(x). \end{aligned}$$

Using the same argument for the integral over $\mathcal{V}_x \subset \mathcal{W}_x$, we obtain the result. \square

A.2. Equivalent conditions for a Trace inequality. The following theorem is a generalisation of results by Maz'ya and Verbitsky [MV95] and of Kerman-Sawyer [KS86] (see also [PW03, SW92, SWZ96]) and our proof is inspired by the one provided by Verbitsky [Ver08]

Theorem A.3. *If (X, d, μ) is a metric measure space satisfying the doubling property $(D_{\kappa, \nu})$ and q is a non negative locally integrable function then the following are equivalent*

- i) *The operator $\mathcal{K}\sqrt{q}$ has a continuous extension $\mathcal{K}\sqrt{q}: L^2(X, \mu) \rightarrow L^2(X, \mu)$*
- ii) *$Q := \mathcal{K}(q)$ is finite μ -almost everywhere and the operator $\mathcal{K}Q$ has a continuous extension $\mathcal{K}Q: L^2(X, \mu) \rightarrow L^2(X, \mu)$,*
- iii) *$Q := \mathcal{K}(q)$ is finite μ -almost everywhere and there is a constant C such that*

$$\mathcal{K}(Q^2)(x) \leq C\mathcal{K}(q)(x), \mu \text{ a.e. } x \in X,$$

- iv) *there is a constant C' such that for any ball $B \subset X$:*

$$\int_B |\mathcal{K}(q \mathbf{1}_B)|^2 d\mu \leq C' \int_B q d\mu.$$

Moreover $\|\mathcal{K}\sqrt{q}\|_{L^2 \rightarrow L^2}^2$, $\|\mathcal{K}Q\|_{L^2 \rightarrow L^2}$ and the constants C, C' are mutually controlled.

Proof. Proof of i) \Rightarrow iv). We test the L^2 boundedness of $T = \mathcal{K}\sqrt{q}$ on $f := \sqrt{q} \mathbf{1}_B$ and get that

$$\int_B |\mathcal{K}(q \mathbf{1}_B)|^2 d\mu \leq \int_X |\mathcal{K}(q \mathbf{1}_B)|^2 d\mu(y) \leq \|T\|^2 \int_B q d\mu.$$

Hence iv) holds with $C' = \|T\|^2$.

Proof of ii) \Rightarrow i). We have that for any $f \in L^2(X, \mu)$:

$$\begin{aligned} \int_X q |\mathcal{K}(f)|^2 d\mu &\leq 2\gamma^3 \int_X q (\mathcal{K}(|f|\mathcal{K}|f|)) d\mu \\ &\leq 2\gamma^3 \langle \mathcal{K}^*(q)\mathcal{K}|f|, |f| \rangle_{L_\mu^2} \\ &\leq 2\gamma^4 \|\mathcal{K}q\|_{L_\mu^2} \|f\|_{L_\mu^2} \\ &\leq 2\gamma^5 \|\mathcal{K}q\|_{L^2 \rightarrow L^2} \|f\|_{L_\mu^2} \\ &\leq 2\gamma^5 \|\mathcal{K}q\|_{L^2 \rightarrow L^2} \|f\|_{L_\mu^2}^2. \end{aligned}$$

Using that $\int_X q |\mathcal{K}^*(f)|^2 d\mu \leq \gamma^2 \int_X q |\mathcal{K}(f)|^2 d\mu$ we obtain that

$$\|\mathcal{K}\sqrt{q}\|_{L^2 \rightarrow L^2}^2 = \|\sqrt{q}\mathcal{K}^*\|_{L^2 \rightarrow L^2}^2 \leq 2\gamma^7 \|\mathcal{K}q\|_{L^2 \rightarrow L^2}.$$

Proof of iii) \Rightarrow ii). Assuming that iii) holds and proceeding in the same way, for any $f \in L_c^2(X, \mu)$, we obtain the inequality

$$\begin{aligned} \int_X Q^2 |\mathcal{K}(f)|^2 d\mu &\leq 2\gamma^3 \langle \mathcal{K}^*(Q^2)\mathcal{K}|f|, |f| \rangle_{L_\mu^2} \\ &\leq 2\gamma^4 \langle \mathcal{K}(Q^2)\mathcal{K}|f|, |f| \rangle_{L_\mu^2}. \end{aligned}$$

Using the hypothesis $\mathcal{K}(Q^2) \leq CQ$ one gets that

$$\begin{aligned} \int_X Q^2(x) (\mathcal{K}|f|)^2(x) d\mu(x) &\leq 2\gamma^4 C \langle Q\mathcal{K}|f|, |f| \rangle_{L_\mu^2} \\ &\leq 2\gamma^4 C \|Q(\mathcal{K}|f|)\|_{L_\mu^2} \|f\|_{L_\mu^2}. \end{aligned}$$

As $f \in L_c^2(X, \mu)$ and $Q\mathcal{K}|f| \in L_{loc}^2$, the first inequality above implies that $Q(\mathcal{K}|f|) \in L_\mu^2$. Hence

$$\|Q(\mathcal{K}|f|)\|_{L_\mu^2} \leq 2\gamma^4 C \|f\|_{L_\mu^2}$$

and

$$\|\mathcal{K}Q\|_{L^2 \rightarrow L^2} = \|Q\mathcal{K}^*\|_{L^2 \rightarrow L^2} \leq \gamma \|\mathcal{K}\|_{L^2 \rightarrow L^2} \leq 2\gamma^5 C.$$

Proof of iv) \Rightarrow iii). This is the most complicated implication. Our goal is therefore to estimate the quantity:

$$\mathcal{K}(Q^2)(o) = \int_0^{+\infty} \left(\int_{B(o,r)} |\mathcal{K}(q)|^2(x) d\mu(x) \right) dr.$$

We decompose

$$\mathcal{K}(Q^2)(o) \leq 2\mathbf{I} + 2\mathbf{II}$$

where

$$\mathbf{I} = \int_0^{+\infty} \left(\int_{B(o,r)} |\mathcal{K}(\mathbf{1}_{B(o,2r)} q)|^2(x) d\mu(x) \right) dr$$

and

$$\mathbf{II} = \int_0^{+\infty} \left(\int_{B(o,r)} |\mathcal{K}(\mathbf{1}_{X \setminus B(o,2r)} q)|^2(x) d\mu(x) \right) dr.$$

Let's start by the estimation of \mathbf{I} , we easily get:

$$\int_{B(o,r)} |\mathcal{K}(\mathbf{1}_{B(o,2r)} q)|^2(x) d\mu(x) \leq \frac{V(o,2r)}{V(o,r)} \int_{B(o,2r)} |\mathcal{K}(\mathbf{1}_{B(o,2r)} q)|^2(x) d\mu(x)$$

and using the property *iv*) one gets that

$$\mathbf{I} \leq \gamma C' \int_0^{+\infty} \left(\int_{B(o,2r)} q(x) d\mu(x) \right) dr = \frac{1}{2} \gamma C' \mathcal{K}(q)(o). \quad (32)$$

For estimation of \mathbf{II} , we first need an estimate of $\int_B q$ for any ball $B = B(y, s)$. If $z, w \in B$, then $d(w, z) \leq 2s$ hence

$$K(z, w) \geq \int_{2s}^{3s} \frac{d\tau}{V(z, \tau)} \geq \frac{s}{V(z, 3s)} \geq \frac{s}{\gamma V(y, 3s)} \geq \frac{s}{\gamma^3 V(y, s)}.$$

Hence the condition *iv*) yields that

$$V(y, s) \left(\frac{s}{\gamma^3 V(y, s)} \right)^2 \left(\int_B q d\mu \right)^2 \leq C' \int_B q d\mu$$

so that

$$\int_B q d\mu \leq \gamma^6 C' \frac{1}{s^2}. \quad (33)$$

If $x \in B(o, r)$ and $\tau \leq r$ then $B(x, \tau) \subset B(o, 2r)$ so that

$$\mathcal{K}(\mathbf{1}_{X \setminus B(o, 2r)} q)(x) = \int_\tau^{+\infty} \left(\int_{B(x, \tau)} \mathbf{1}_{X \setminus B(o, 2r)} q d\mu \right) d\tau.$$

Similarly if $x \in B(o, r)$ and $\tau \geq r$ then $B(x, \tau) \subset B(o, 2\tau)$ and using $V(o, 2\tau) \leq \gamma^2 V(x, \tau)$ one gets that

$$\mathcal{K}(\mathbf{1}_{X \setminus B(o, 2r)} q)(x) \leq \gamma^2 \int_r^{+\infty} \left(\int_{B(o, 2\tau) \setminus B(o, 2r)} q d\mu \right) \frac{d\tau}{V(o, 2\tau)}. \quad (34)$$

We introduce $\Phi(\tau) = \int_{B(o, 2\tau)} q d\mu$, Using (33), we gets that

$$\Phi(\tau) \leq \gamma^6 C' \frac{1}{\tau^2}. \quad (35)$$

And from (34) we get the following estimates:

$$\begin{aligned}
\Pi &\leq \gamma^4 \int_{0 \leq r \leq \sigma, 0 \leq r \leq \tau} \Phi(\tau) \Phi(\sigma) d\tau d\sigma dr \\
&\leq 2\gamma^4 \int_{0 \leq r \leq \sigma \leq \tau} \Phi(\tau) \Phi(\sigma) d\tau d\sigma dr \\
&\leq 2\gamma^{10} C' \int_{0 \leq r \leq \sigma} \Phi(\sigma) \frac{1}{\sigma} d\sigma dr \\
&\leq 2\gamma^{10} C' \int_0^{+\infty} \Phi(\sigma) \left(\int_0^\sigma dr \right) \frac{d\sigma}{\sigma} \\
&\leq 2\gamma^{10} C' \int_0^{+\infty} \Phi(\sigma) d\sigma \\
&\leq 2\gamma^{10} C' \int_0^{+\infty} \left(\int_{B(o, 2\sigma)} q d\mu \right) d\sigma \\
&\leq \gamma^{10} C' \mathcal{K}(q)(o).
\end{aligned}$$

And this estimate together with (32) yields that if **iv**) holds then **iii**) holds with

$$C = (\gamma + \gamma^{10}) C'.$$

□

A.3. The next estimate provides a class of operators whose Schwartz kernels look like the same as the one of the operator \mathcal{K} studied previously.

Proposition A.4. *If (X, d, μ) is a metric measure space satisfying the doubling property $(D_{\kappa, \nu})$ then for any $D, s > 0$, there are positive constants c, C depending only on D, s and the doubling constants κ, ν such that for any $x, y \in X$:*

$$c \int_{d(x,y)}^{\infty} \frac{r^{2s-1}}{V(x,r)} dr \leq \int_0^{\infty} \frac{e^{-\frac{d^2(x,y)}{Dt}}}{V(x, \sqrt{t})} t^{s-1} dt \leq C \int_{d(x,y)}^{\infty} \frac{r^{2s-1}}{V(x,r)} dr.$$

Proof. Firstly, we easily get that

$$2e^{-\frac{1}{D}} \int_{d(x,y)}^{\infty} \frac{r^{2s-1}}{V(x,r)} dr \leq \int_{d^2(x,y)}^{\infty} \frac{e^{-\frac{d^2(x,y)}{Dt}}}{V(x, \sqrt{t})} t^{s-1} dt \leq 2 \int_{d(x,y)}^{\infty} \frac{r^{2s-1}}{V(x,r)} dr.$$

Secondly using the doubling assumption, we get

$$\begin{aligned}
\int_0^{d^2(x,y)} \frac{e^{-\frac{d^2(x,y)}{Dt}}}{V(x, \sqrt{t})} t^{s-1} dt &\leq \frac{\kappa}{V(x, 2d(x,y))} \int_0^{d^2(x,y)} e^{-\frac{d^2(x,y)}{Dt}} \left(\frac{2d(x,y)}{\sqrt{t}} \right)^\nu t^{s-1} dt \\
&\leq \frac{\kappa 2^\nu d^{2s}(x,y)}{V(x, 2d(x,y))} \int_0^1 e^{-\frac{1}{Dt}} t^{s-\nu/2-1} dt.
\end{aligned}$$

But we also have the lower estimate:

$$\begin{aligned}
\int_{d(x,y)}^{\infty} \frac{r^{2s-1}}{V(x,r)} dr &\geq \int_{d(x,y)}^{2d(x,y)} \frac{r^{2s-1}}{V(x,r)} dr \\
&\geq \frac{d^{2s}(x,y)}{2V(x, 2d(x,y))}.
\end{aligned}$$

Hence the result with

$$c = 2e^{-\frac{1}{D}} \quad \text{and} \quad C = 2 + \kappa 2^{\nu+1} \int_0^1 e^{-\frac{1}{Dt}} t^{s-\nu/2-1} dt.$$

□

REFERENCES

- [AC05] Pascal Auscher and Thierry Coulhon. Riesz transform on manifolds and Poincaré inequalities. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(3):531–555, 2005.
- [AM07] Pascal Auscher and José María Martell. Weighted norm inequalities, off-diagonal estimates and elliptic operators. I. General operator theory and weights. *Adv. Math.*, 212(1):225–276, 2007.
- [AM08] Pascal Auscher and José María Martell. Weighted norm inequalities, off-diagonal estimates and elliptic operators. IV. Riesz transforms on manifolds and weights. *Math. Z.*, 260(3):527–539, 2008.
- [Anc90] Alano Ancona. Théorie du potentiel sur les graphes et les variétés. In *École d'été de Probabilités de Saint-Flour XVIII—1988*, volume 1427 of *Lecture Notes in Math.*, pages 1–112. Springer, Berlin, 1990.
- [Car97] Gilles Carron. Inégalités de Hardy sur les variétés riemanniennes non-compactes. *J. Math. Pures Appl. (9)*, 76(10):883–891, 1997.
- [Car17] Gilles Carron. Riesz transform on manifolds with quadratic curvature decay. *Rev. Mat. Iberoam.*, 33(3):749–788, 2017.
- [Car19] Gilles Carron. Geometric inequalities for manifolds with Ricci curvature in the Kato class. *Ann. Inst. Fourier (Grenoble)*, 69(7):3095–3167, 2019.
- [CG76] Monique Combescure and Jean Ginibre. Spectral and scattering theory for the Schrödinger operator with strongly oscillating potentials. *Annales de l'I.H.P. Physique théorique*, 24(1):17–30, 1976.
- [Fef83] Charles L. Fefferman. The uncertainty principle. *Bull. Amer. Math. Soc. (N.S.)*, 9(2):129–206, 1983.
- [FP82] Charles L. Fefferman and Duong Hong Phong. Lower bounds for Schrödinger equations. In *Conference on Partial Differential Equations (Saint Jean de Monts, 1982)*, pages Conf. No. 7, 7. Soc. Math. France, Paris, 1982.
- [Gri91] Alexander Grigor'yan. The heat equation on noncompact Riemannian manifolds. *Mat. Sb.*, 182(1):55–87, 1991.
- [Gri94] Alexander Grigor'yan. Heat kernel upper bounds on a complete non-compact manifold. *Rev. Mat. Iberoamericana*, 10(2):395–452, 1994.
- [Gri99] Alexander Grigor'yan. Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 36(2):135–249, 1999.
- [GT01] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer Berlin Heidelberg, 2001.
- [KK11] Riikka Korte and Outi Elina Kansanen. Strong A_∞ -weights are A_∞ -weights on metric spaces. *Rev. Mat. Iberoam.*, 27(1):335–354, 2011.
- [KS86] Ronald Kerman and Eric T. Sawyer. The trace inequality and eigenvalue estimates for Schrödinger operators. *Ann. Inst. Fourier (Grenoble)*, 36(4):207–228, 1986.
- [Lan20] Maël Lansade. Lower bound of schrödinger operators on riemannian manifolds. *Journal of Spectral Theory*, to appear arXiv:2012.08841.
- [Maz64] Vladimir G. Maz'ya. On the theory of the higher-dimensional Schrödinger operator. *Izv. Akad. Nauk SSSR Ser. Mat.*, 28:1145–1172, 1964.
- [MV95] Vladimir G. Maz'ya and Igor E. Verbitsky. Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers. *Ark. Mat.*, 33(1):81–115, 1995.
- [MV02] Vladimir G. Maz'ya and Igor E. Verbitsky. The Schrödinger operator on the energy space: Boundedness and compactness criteria. *Acta Math.*, 188(2):263–302, 2002.
- [PW03] Carlos Pérez and Richard L. Wheeden. Potential operators, maximal functions, and generalizations of A_∞ . *Potential Anal.*, 19(1):1–33, 2003.
- [SC92] Laurent Saloff-Coste. A note on Poincaré, Sobolev, and Harnack inequalities. *Internat. Math. Res. Notices*, (2):27–38, 1992.
- [SC02] Laurent Saloff-Coste. *Aspects of Sobolev-type inequalities*, volume 289 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002.
- [Sch88] Martin Schechter. Weighted norm inequalities for potential operators. *Trans. Amer. Math. Soc.*, 308(1):57–68, 1988.
- [ST89] Jan-Olov Strömberg and Alberto Torchinsky. *Weighted Hardy spaces*, volume 1381 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.
- [Ste56] Elias M. Stein. Interpolation of linear operators. *Transactions of the American Mathematical Society*, 83(2):482–492, 1956.
- [SW92] E. Sawyer and R. L. Wheeden. Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. *Amer. J. Math.*, 114(4):813–874, 1992.
- [SWZ96] Eric T. Sawyer, Richard L. Wheeden, and Shiyong Zhao. Weighted norm inequalities for operators of potential type and fractional maximal functions. *Potential Anal.*, 5(6):523–580, 1996.
- [Ver08] Igor E. Verbitsky. Weighted norm inequalities with positive and indefinite weights. In *Perspectives in partial differential equations, harmonic analysis and applications. A volume in honor of Vladimir G. Maz'ya's 70th birthday*, pages 377–406. Providence, RI: American Mathematical Society (AMS), 2008.

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